Central and Class-Preserving Automorphisms of Finite p-Groups

Thesis

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Declaration of Authorship

I hereby declare that the work which is being presented in this thesis entitled ***“Central and Class-Preserving Automorphisms of Finite*** p***-Groups”*** submitted by me, for the award of the degree of Doctor of Philosophy in the School of Mathematics and Computer Applications, Thapar University, Patiala, is true and original record of my own independent and original research work carried out under the supervision of Dr. Deepak Gumber, Associate Professor, School of Mathematics and Computer Applications, Thapar University, Patiala, India. The matter embodied in this thesis has not been submitted in part or full to any other university or institute for the award of any degree in India or Abroad and that the ideas and references cited herein have been duly acknowledged.

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CERTIFICATE

This is to certify that the thesis *"Central and Class-Preserving Automorphisms of Finite p-Groups"* which is submitted by Mr. Mahak Sharma, in fulfilment of the requirement for the award of the degree of *Doctor of Philosophy* in the School of Mathematics and Computer Applications, Thapar University, Patiala, is a record of the candidate's own independent and original research work carried out by him under my supervision and guidance. The matter embodied in this thesis has not been submitted in part or full to any University or Institute for the award of any degree.

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October, 2014 *(Mahak Sharma)*

Dedicated
To
My
Family

Abstract

Let G be a finite p-group and let $\mathrm{Aut}(G)$ denote its full automorphism group. An automorphism α of G is called a class-preserving automorphism if, for each $x \in G$, there exists an element $g_x \in G$ such that $\alpha(x) = g_x^{-1}xg_x$; and is called an inner automorphism if, for all $x \in G$, there exists a fixed element $g \in G$ such that $\alpha(x) = g^{-1}xg$. The group $\mathrm{Inn}(G)$ of all inner automorphisms of G is a normal subgroup of the group $\mathrm{Aut_c}(G)$ of all class-preserving automorphisms of G. We denote the group $\mathrm{Aut_c}(G)/\mathrm{Inn}(G)$ of all class preserving outer automorphisms by $\mathrm{Out_c}(G)$. By $\mathrm{Aut}_{\mathrm{N}}^{\mathrm{M}}(G)$, we denote the set of all automorphisms of G which centralize G/M and N, where M and N are two normal subgroups of G. An automorphism α of G is called a central automorphism if $g^{-1}\alpha(g) \in Z(G)$ for all $g \in G$. Let $\mathrm{Aut_z}(G)$ denote the group of all central automorphisms of G and let $\mathrm{Aut}_{\mathrm{z}}^{\mathrm{z}}(G)$ denote the group of all those central automorphisms which fix the center of G element-wise.

The main object of the thesis is to study central and class-preserving automorphisms of a finite p-group, where p is a prime. More specifically, we study when $\mathrm{Aut_z}(G) = Z(\mathrm{Inn}(G))$ and when $\mathrm{Out_c}(G) \neq 1$. In chapter 2, we characterize all finite p-groups G of order $p^n (n \leq 6)$, where p is a prime for $n \leq 5$ and an odd prime for $n = 6$, such that the center of the inner automorphism group of G is equal to the group of central automorphisms of G. In chapter 3, we characterize all finite p-groups of order dividing p^7, where p is a prime for $n \leq 6$ and an odd prime for $n = 7$, such that $Z(\mathrm{Inn}(G)) = \mathrm{Aut}_{\mathrm{z}}^{\mathrm{z}}(G) < \mathrm{Aut_z}(G)$.

In chapter 4, we give necessary and sufficient conditions on a group of order p^5 such that G has a non-inner class-preserving automorphism. As a consequence, we give short and alternate proofs of results of Yadav [55, Section 5] and Kalra and Gumber [34, Theorem 4.2]. In chapter 5, we have decided the Hasse principle for some groups of order p^6, where p is an odd prime, using isoclinism families given by James [31], by giving alternate proofs of results of Narain and Karan [42] and Rai and Yadav [45]. We also classify some groups G of order p^6 such that G admits a non-inner automorphism of order p.

List of Research Papers

(1) Sharma, M. and Gumber, D. *Class-preserving automorphisms of some finite p-groups*, Proc. Indian Acad. Sci. (Math. Sci.) **(Accepted)**.

(2) Sharma, M. and Gumber, D. *On central automorphisms of finite p-groups*, Comm. Algebra, **41** (2013), 1117-1122.

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CHAPTER 1

Introduction and Basics

1.1 — Introduction

Let G be an arbitrary group and let $\mathrm{Aut}(G)$ denote the full automorphism group of G. Let G' and $Z(G)$ respectively denote the commutator subgroup and the center of G. An automorphism α of G is called a class-preserving automorphism if, for each $x \in G$, there exists an element $g_x \in G$ such that $\alpha(x) = g_x^{-1} x g_x$; and is called an inner automorphism if, for each $x \in G$, there exists a fixed element $g \in G$ such that $\alpha(x) = g^{-1}xg$. The group $\mathrm{Inn}(G)$ of all inner automorphisms of G is a normal subgroup of the group $\mathrm{Aut_c}(G)$ of all class-preserving automorphisms of G. We denote the group $\mathrm{Aut_c}(G)/\mathrm{Inn}(G)$ of all class preserving outer automorphisms by $\mathrm{Out_c}(G)$. By $\mathrm{Aut_N^M}(G)$, we denote the group of all automorphisms of G which centralize both G/M and N, where M and N are two normal subgroups of G. An automorphism α of G is called a central automorphism if it commutes with all inner automorphisms, or equivalently, if $g^{-1}\alpha(g) \in Z(G)$ for all $g \in G$. Let $\mathrm{Aut_z}(G)$ denote the group of all central automorphisms of G and let $\mathrm{Aut_z^z}(G)$ denote the group of all those central automorphisms which fix the center of G element-wise.

Throughout in the thesis, the letter p denotes an arbitrary prime.

The main object of the thesis is to study central and class-preserving automorphisms of a finite p-group, where p is a prime. It is interesting and natural to discuss the question: what are the necessary and sufficient conditions on a group G such that certain subgroups of $\mathrm{Aut}(G)$ are equal? Interest in the equality of two different automorphism groups dates back to 1908, when Hilton [25, p. 233] asked the following question: Whether a non-abelian group can have an abelian group of isomorphisms (automorphisms), that is, when $\mathrm{Aut}(G) = \mathrm{Aut}_{\mathrm{z}}(G)$? An affirmative answer to this question was given by Miller [39] in 1913. He constructed a non-abelian group G of order 64 for which $\mathrm{Aut}(G)$ was abelian of order 128. More examples of such finite 2-groups were constructed by Struik [49] in 1982, Curran [13] in 1987 and Jamali [30] in 2002.

The central automorphism group can be as large as possible when all automorphisms are central i.e. when $\mathrm{Aut}_{\mathrm{z}}(G) = \mathrm{Aut}(G)$, and can be as small as possible when $\mathrm{Aut}_{\mathrm{z}}(G) = Z(\mathrm{Inn}(G))$. If G is abelian, then $\mathrm{Inn}(G)$ is trivial and hence $\mathrm{Aut}_{\mathrm{z}}(G) = \mathrm{Aut}(G)$. If G is non-abelian and if $\mathrm{Aut}_{\mathrm{z}}(G) = \mathrm{Aut}(G)$, then since $\mathrm{Inn}(G)$ is abelian, G is a nilpotent group of class 2. So one can restrict attention to finite p-groups. Non-abelian p-groups G in which all automorphisms are central, that is when $\mathrm{Aut}_{\mathrm{z}}(G) = \mathrm{Aut}(G)$, have been well studied. If $\mathrm{Aut}(G)$ is abelian, then necessarily $\mathrm{Aut}_{\mathrm{z}}(G) = \mathrm{Aut}(G)$, and various authors have considered this situation (see for example [6, 16, 21, 22, 23, 26, 32, 40, 41]). If $\mathrm{Aut}(G)$ is non-abelian, even then all automorphisms may be central and this case has been explored, for example, in [12, 18, 29, 38].

In 2001, Curran and McCaughan [15] gave necessary and sufficient conditions for a finite p-group G such that $\mathrm{Aut_z}(G) = \mathrm{Inn}(G)$. They proved that for any finite p-group G, $\mathrm{Aut_z}(G) = \mathrm{Inn}(G)$ if and only if $G' = Z(G)$ and $Z(G)$ is cyclic. In 2007, Attar [3] established that for any finite p-group G, $\mathrm{Aut_z^z}(G) = \mathrm{Inn}(G)$ if and only if G is abelian or nilpotency class of G is 2 and $Z(G)$ is cyclic. In 2009, Yadav [56] gave necessary and sufficient conditions on a finite p-group G of nilpotency class 2 such that $\mathrm{Aut_z}(G) = \mathrm{Aut_z^z}(G)$. Attar [4] in 2012 and Jafari [28] in 2011 have generalized this result. They have given necessary and sufficient conditions on a finite p-group G of arbitrary nilpotency class such that each central automorphism of G fixes the center element-wise. In 2013, Azhdari and Malayeri [5] gave necessary and sufficient conditions on G such that $\mathrm{Aut_N^M}(G)$ is equal to $Z(\mathrm{Inn}(G))$, $\mathrm{Inn}(G)$, $\mathrm{Aut_z^z}(G)$ or $\mathrm{Aut_z}(G)$. Recently, Kalra and Gumber [34] and Yadav [57] have given necessary and sufficient conditions on a finite p-group G such that all central automorphisms are class preserving.

Consider now the opposite extreme, the case in which we are specially interested, when the central automorphism group is as small as possible. The bound is $Z(\mathrm{Inn}(G))$, because it is always contained in $\mathrm{Aut_z}(G)$. In 2004, Curran [14] studied finite p-groups G for which $\mathrm{Aut_z}(G) = Z(\mathrm{Inn}(G))$. He proved that for any finite p-group G, if $\mathrm{Aut_z}(G) = Z(\mathrm{Inn}(G))$, then $Z(G) \leq G'$ and $Z(\mathrm{Inn}(G))$ must not be cyclic. It follows that if G is a finite p-group of maximal class, then $\mathrm{Aut_z}(G) \neq Z(\mathrm{Inn}(G))$. Observe that if the nilpotency class of G is 2, then $Z(\mathrm{Inn}(G)) = \mathrm{Inn}(G)$ and thus, by the above mentioned result of Curran and McCaughan [15], $\mathrm{Aut_z}(G) = Z(\mathrm{Inn}(G))$ if and only if $G' = Z(G)$ and $Z(G)$ is cyclic.

Therefore, to characterize all finite p-groups G for which $\mathrm{Aut}_{\mathrm{z}}(G) = Z(\mathrm{Inn}(G))$, we can assume that the nilpotency class of G is bigger than 2 and G is not of maximal class. Of course, in such a case $|G| \geq p^5$. In chapter 2, we characterize all finite p-groups G of order $p^n (n \leq 6)$, where p is a prime for $n \leq 5$ and an odd prime for $n = 6$, such that the center of the inner automorphism group of G is equal to the group of central automorphisms of G. The results of this chapter have appeared in [47]. In chapter 3, we characterize all finite p-groups of order dividing p^7, where p is a prime for $n \leq 6$ and an odd prime for $n = 7$, such that $\mathrm{Aut}_{\mathrm{z}}^{\mathrm{z}}(G) = Z(\mathrm{Inn}(G))$.

In 1911, Burnside [9, Note B] posed the following question: Does there exist a finite group G such that G has a non-inner class-preserving automorphism? In 1913, Burnside [10] himself gave a positive answer to his question. He constructed a group G of order p^6, where p is an odd prime, such that $\mathrm{Aut}_{\mathrm{c}}(G) \neq \mathrm{Inn}(G)$. In the years 2000 and 2001, Kumar and Vermani [35, 36] proved that $\mathrm{Out}_{\mathrm{c}}(G) = 1$ for every finite non-abelian p-group G having a maximal subgroup which is cyclic. In particular, they proved that if G is an extra-special p-group or a group of order p^4, then $\mathrm{Out}_{\mathrm{c}}(G) = 1$. In 2002, Kumar and Vermani [37] found those finite non-abelian p-groups G of order p^m, $m > 4$, having a normal cyclic subgroup of order p^{m-2} but having no element of order p^{m-1} for which $\mathrm{Aut}_{\mathrm{c}}(G) = \mathrm{Inn}(G)$. In 2004, Fuma and Ninomiya [17] proved that for every non-abelian group of order p^m having a cyclic subgroup of order p^{m-2} but having no normal cyclic subgroup of order p^{m-2} and no element of order p^{m-1} , $\mathrm{Aut}_{\mathrm{c}}(G) = \mathrm{Inn}(G)$. It follows from [31] that all finite p-groups of order p^5, where p is an odd prime, are partitioned into ten isoclinism families; and from [20] that all finite 2-groups of order 2^5 are partitioned into eight isoclinism families. In 2008, Yadav [55] proved that if G and H are two finite non-abelian isoclinic groups, then $\mathrm{Aut_c}(G) \simeq \mathrm{Aut_c}(H)$. He then showed that $\mathrm{Out_c}(G) \neq 1$ for the groups $\Phi_7(1^5)$ and $\Phi_{10}(1^5)$ from the seventh and tenth families in [31], and concluded that if G is a finite p-group of order p^5, where p is an odd prime, then $\mathrm{Out_c}(G) \neq 1$ if and only if G is isoclinic to a group either in the seventh or in the tenth family. For more details about this problem, one can see the excellent survey article by Yadav [54]. Kalra and Gumber [34, Theorem 4.2], using the classifications, isoclinism families and presentations given by Hall and Senior [20] and Sag and Wamsley [46], have shown that if G is a finite 2-group of order 2^5, then $\mathrm{Out_c}(G) = 1$, except for the forty fourth and forty fifth groups from the sixth family in [20]. Recently, Narain and Karan [42] and Rai and Yadav [45] have classified groups of order p^6 for which $\mathrm{Out_c}(G) \neq 1$ using isoclinism families given by James [31], where p denotes an odd prime. A longstanding conjecture asserts that every finite non-abelian p-group admits a non-inner automorphism of order p. The conjecture has been verified in many cases and is still open in general case (see, for example, [1] for details and references). In chapter 4, we give necessary and sufficient conditions on a group of order p^5 such that G has a non-inner class-preserving automorphism. As a consequence, we give short and alternate proofs of results of Yadav [55, Section 5] and Kalra and Gumber [34, Theorem 4.2]. The results of this chapter will appear in [48]. In chapter 5, we have decided the Hasse principle for some groups of order p^6, where p is an odd prime, using isoclinism families given by James [31], by giving alternate proofs of results of Narain and Karan [42] and Rai and Yadav [45]. We also classify some groups G of order p^6 such that G admits a non-inner automorphism of order p.

1.2 — Basics

This section has been taken from thesis of Kalra [33]. In this section, we give some basic facts of group theory that are assumed in the foregoing chapters. The definitions and proofs of results presented here can be found in any standard book on group theory. We of course suppose a familiarity of more basic group theoretic terms and concepts like abelian, cyclic, coset, normal subgroup, factor or quotient group, homomorphism, isomorphism, direct product et cetera.

Let G be an arbitrary group and X be a subset of G. The intersection of the family of subgroups of G which contain X is a subgroup of G and is denoted by $\boldsymbol{\langle X\rangle}$. In other words, $\langle X\rangle$ is the smallest subgroup of G which contains X. The subgroup $\langle X\rangle$ is called the subgroup generated by X. If X is non-empty, then $\langle X\rangle$ contains every finite product of the type

$$x_1^{m_1}x_2^{m_2}\dots x_r^{m_r},\ \ r\geq 1,\ x_i\in X,\ m_i\in\mathbb{Z},$$

and conversely all such products form a subgroup of G containing X. It follows that $\langle X\rangle$ consists of all such products. A cyclic group is thus generated by a single element. We shall denote a cyclic group of order m by $\boldsymbol{C_m}$. The ***rank*** of a group G is the smallest cardinality of a generating set of G and is denoted by $\boldsymbol{d(G)}$. The least common multiple of the orders of the elements of a finite group G is called the ***exponent*** of G and is denoted by $\boldsymbol{exp(G)}$.

The ***commutator*** of two elements $a, b\in G$ is the element $\boldsymbol{[a,b]=a^{-1}b^{-1}ab}$

of G and the ***commutator subgroup*** or the ***derived subgroup*** $\boldsymbol{G'}$ of G is the subgroup of G generated by all commutators of G. It is easy to see that G' is a normal subgroup of G. If X and Y are two subsets of G, then we define $\boldsymbol{[X, Y] = \langle [x, y] | x \in X, y \in Y \rangle}$. Thus $[X, Y]$ is always a subgroup of G. For $x \in G$, $\boldsymbol{[x, G]}$ denotes the set of all commutators $[x, g]$, where $g \in G$. The following are well known commutator identities

$$[x, yz] = [x, z][x, y][x, y, z]; \quad [xy, z] = [x, z][x, z, y][y, z],$$

where $x, y, z \in G$ and will be frequently used in the thesis without any reference.

A series

$$1 = G_0 \leq G_1 \leq \ldots \leq G_l = G$$

of subgroups of G is called a ***normal series*** if each G_i is a normal subgroup of G. The normal series above is called a ***central series*** if for each i, $G_{i+1}/G_i \leq Z(G/G_i)$. Let $Z_0 = 1$ and let $Z_{i+1}/Z_i = Z(G/Z_i)$ for $i \geq 0$. Observe that Z_1 is the center of G and Z_{i+1}/Z_i, being the center of G/Z_i, is normal in G/Z_i and hence Z_{i+1} is normal in G for all $i \geq 0$. It follows that the series

$$1 = Z_0 \leq Z_1 \leq Z_2 \leq \ldots$$

is a central series of G. The subgroup Z_i is called the ***i-th center*** and this series is called the ***upper central series*** of G.

We define subgroups $\boldsymbol{\gamma_i(G)}$, $i \geq 1$, of G by setting

$$\gamma_1(G) = G, \quad \gamma_{i+1}(G) = [\gamma_i(G), G].$$

Observe that $\gamma_2(G) = G'$, each $\gamma_i(G)$ is normal in G and $\gamma_{i+1}(G) \leq \gamma_i(G)$. The series

$$G = \gamma_1(G) \geq G' \geq \dots \gamma_n(G) \geq \dots$$

is called the ***lower central series*** of G. If the lower central series of a group G terminates in a finite number of steps at 1, and if c is the least natural number such that $\gamma_{c+1}(G) = 1$, then G is called a ***nilpotent group*** of class c. The class of a nilpotent group is denoted by $\boldsymbol{cl(G)}$. Observe that if $cl(G) = 2$, then $G' \leq Z(G)$.

A ***maximal subgroup*** of G is a proper subgroup M such that there is no subgroup H of G with $M < H < G$. The intersection of all the maximal subgroups of G is the ***Frattini subgroup*** $\boldsymbol{\Phi(G)}$ of G. If G has no maximal subgroup, then we set $\Phi(G) = G$. An element a of G is called a ***non-generator*** of G if whenever $G = \langle a, X \rangle$, then $G = \langle X \rangle$. An interesting property of $\Phi(G)$ is that it is exactly the set of all non-generators of G. Two elements a and b of G are called ***conjugate*** if there exists an element g of G such that $b = g^{-1}ag$. It is easily seen that "conjugacy" is an equivalence relation on G and therefore it partitions G into equivalence classes. The equivalence class that contains the element a of G is called the ***conjugacy class*** of a and is denoted as $\boldsymbol{a^G}$. Let A be a non-empty subset of G. The set of elements of G which commute with every element of A is called the ***centralizer*** of A in G, and is denoted as $\boldsymbol{C_G(A)}$. If $A = \{a\}$ is singleton, then $C_G(\{a\})$ is simply denoted as $C_G(a)$. It is easy to see that $C_G(A)$ is a subgroup of G. For any $a \in G$, $|a^G| = |G|/|C_G(a)|$.

A finite group G is called a ***purely non-abelian*** group if it has no non-trivial abelian direct factor. If $Z(G)$ is cyclic or if, $Z(G) \leq \Phi(G)$, then G is purely non-abelian.

An isomorphism α of G to itself is called an ***automorphism*** of G. The set of all automorphisms of G is a group under the usual operation of composition of mappings. We call this group the full automorphism group of G and denote it by $\mathbf{Aut}(\boldsymbol{G})$.

Let A be an abelian group and let $\mathbf{Hom}(\boldsymbol{G}, \boldsymbol{A})$ denote the set of all homomorphisms of G into A. For $f, g \in \mathrm{Hom}(G, A)$, define $fg(x) = f(x)g(x)$. Then $\mathrm{Hom}(G, A)$ becomes an abelian group under this operation. If A, B, C are all finite abelian groups, then $\mathrm{Hom}(A, B \times C) \simeq \mathrm{Hom}(A, B) \times \mathrm{Hom}(A, C)$ and $\mathrm{Hom}(A, B) \simeq \mathrm{Hom}(B, A)$. Also, $\mathrm{Hom}(C_m, C_n) \simeq C_d$, where $d = \gcd(m, n)$.

For a fixed prime number p, a group G is called a $\boldsymbol{p}$**-group** if the order of every element of G is a power of p. For a p-group G, let $\boldsymbol{\Omega_1(G)}$= $\langle x \in G | x^p = 1\rangle$ and $\boldsymbol{G^p}$= $\langle x^p | x \in G\rangle$. The following are well known facts about p-groups: (i) $G^pG' = \Phi(G)$ and (ii) $G/\Phi(G)$ is elementary abelian of rank $d(G)$.

CHAPTER 2

Central Automorphism Group of Minimal Order

2.1 — Introduction

Let G be a finite p-group. By $Z(G), G', \mathrm{Aut}(G)$ and $\mathrm{Inn}(G)$, we denote the center, the commutator subgroup, the group of all automorphisms and the group of all inner automorphisms of G respectively. An automorphism α of G is called a central automorphism if α commutes with every inner automorphism, or equivalently, if $x^{-1}\alpha(x)$ lies in the centre of G for all x in G. The central automorphisms fix the commutator subgroup G' of G pointwise and form a normal subgroup $\mathrm{Aut_z}(G)$ of $\mathrm{Aut}(G)$. The group $\mathrm{Aut_z}(G)$ always lies between $Z(\mathrm{Inn}(G))$ and $\mathrm{Aut}(G)$. Curran [14] considered the case when $\mathrm{Aut_z}(G)$ is minimum possible, that is, when $\mathrm{Aut_z}(G) = Z(\mathrm{Inn}(G))$. He proved that for $\mathrm{Aut_z}(G)$ to be equal to $Z(\mathrm{Inn}(G))$, $Z(G)$ must be contained in G' and $Z(\mathrm{Inn}(G))$ must not be cyclic. These conditions are necessary but not sufficient because there are groups G for which $\mathrm{Aut_z}(G) \neq Z(\mathrm{Inn}(G))$ even if $Z(G)$ is contained in G' or $Z(\mathrm{Inn}(G))$ is not cyclic . For example, the group

$$\begin{aligned} G \;=\; & \langle a,b,c,d,e \mid [b,a]=c, [c,a]=d, [b,e]=d, [d,a]=1, [e,a]=1, [c,b]=1, \\ & [d,b]=1, [d,c]=1, [e,c]=1, [e,d]=1, a^p=b^p=c^p=d^p=e^p=1\rangle, \end{aligned}$$

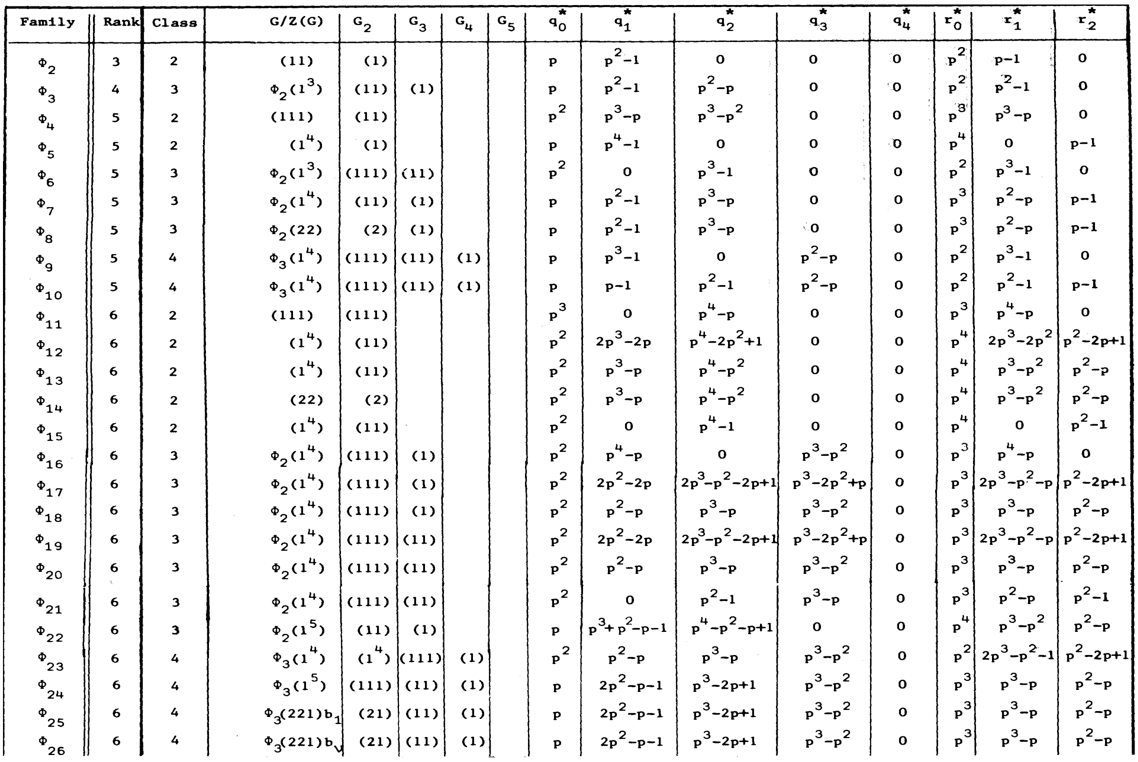
where p is an odd prime, has $|G| = p^5, |Z(G)| = p, |\Phi(G)| = |G'| = p^2, cl(G) = 3$, $Z(\mathrm{Inn}(G)) \simeq C_p \times C_p$ and $|\operatorname{Aut_z}(G)| = p^3$. Curran and McCaughan [15] proved that $\operatorname{Aut_z}(G) = \mathrm{Inn}(G)$ if and only if $G' = Z(G)$ and $Z(G)$ is cyclic. Observe that if the nilpotency class of G is 2, then $Z(\mathrm{Inn}(G)) = \mathrm{Inn}(G)$ and thus $\operatorname{Aut_z}(G) = Z(\mathrm{Inn}(G))$ if and only if $G' = Z(G)$ and $Z(G)$ is cyclic. Also observe that if G is of maximal class, then $|Z(\mathrm{Inn}(G))| = p$ and by [14] $\operatorname{Aut_z}(G) > Z(\mathrm{Inn}(G))$. Therefore, to characterize all finite p-groups G for which $\operatorname{Aut_z}(G) = Z(\mathrm{Inn}(G))$, we can assume that the nilpotency class of G is bigger than 2 and G is not of maximal class. Of course in such a case $|G| \geq p^5$.

In section 2.2, we prove two theorems which characterize all such groups of order p^5, where p is any prime, and of order p^6, where p is an odd prime. In section 2.3, as an application of the results obtained in section 2.2, we find all such groups from the list of groups of order p^n, where p is an odd prime and $4 \leq n \leq 6$, ordered into isoclinism families by James [31].

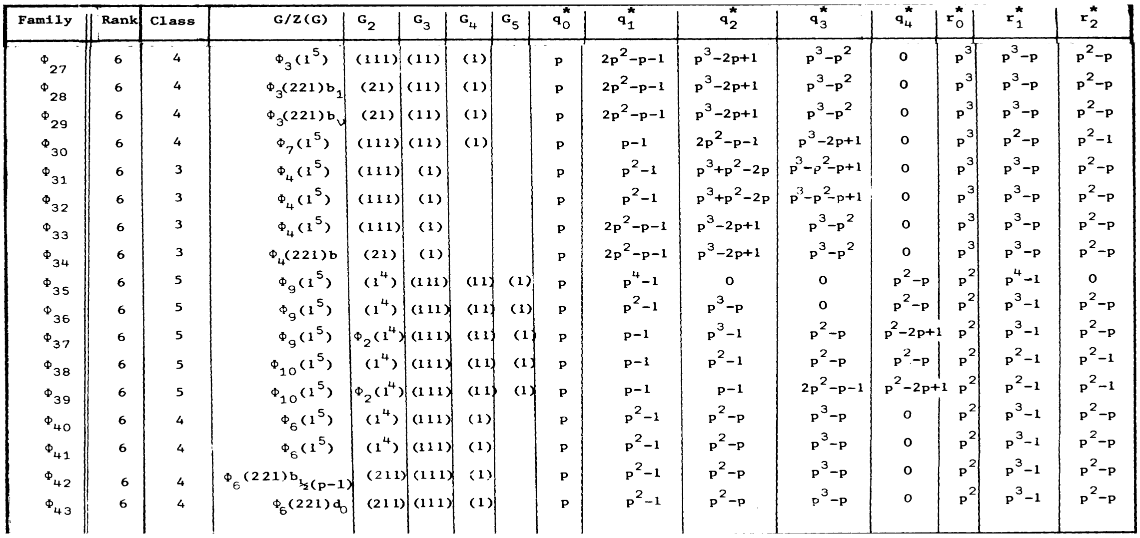
Let X be a finite group and let $\bar{X} = X/Z(X)$. Then commutation in X gives a well defined map $a_X : \bar{X} \times \bar{X} \to X'$ such that $a_X(xZ(X), yZ(Y)) = [x, y]$ for all $(x, y) \in X \times X$. Two finite groups G and H are called ***isoclinic***, as defined by Hall [19], if there exists an isomorphism ψ of the factor group $\bar{G} = G/Z(G)$ onto $\bar{H} = H/Z(H)$, and an isomorphism θ of the subgroup G' onto H' such that the following diagram is commutative

$$\begin{tikzcd}
\bar{G} \times \bar{G} \arrow[r, "a_G"] \arrow[d, "\psi \times \psi"'] & G' \arrow[d, "\theta"] \\
\bar{H} \times \bar{H} \arrow[r, "a_H"] & H'
\end{tikzcd}$$

The resulting pair (ψ, θ) is called an *isoclinism* of G onto H. Notice that isoclinism is an equivalence relation among finite groups. For the sake of convenience, the table containing information about the groups of order $p^n (n \leq 6)$ given by James [31], where p is an odd prime, is shown below:



Family	Rank	Class	G/Z(G)	G_2	G_3	G_4	G_5	q_0^*	q_1^*	q_2^*	q_3^*	q_4^*	r_0^*	r_1^*	r_2^*
Φ_2	3	2	(11)	(1)				p	p^2-1	0	0	0	p^2	$p-1$	0
Φ_3	4	3	$\Phi_2(1^3)$	(11)	(1)			p	p^2-1	p^2-p	0	0	p^2	p^2-1	0
Φ_4	5	2	(111)	(11)				p^2	p^3-p	p^3-p^2	0	0	p^3	p^3-p	0
Φ_5	5	2	(1^4)	(1)				p	p^4-1	0	0	0	p^4	0	$p-1$
Φ_6	5	3	$\Phi_2(1^3)$	(111)	(11)			p^2	0	p^3-1	0	0	p^2	p^3-1	0
Φ_7	5	3	$\Phi_2(1^4)$	(11)	(1)			p	p^2-1	p^3-p	0	0	p^3	p^2-p	$p-1$
Φ_8	5	3	$\Phi_2(22)$	(2)	(1)			p	p^2-1	p^3-p	0	0	p^3	p^2-p	$p-1$
Φ_9	5	4	$\Phi_3(1^4)$	(111)	(11)	(1)		p	p^3-1	0	p^2-p	0	p^2	p^3-1	0
Φ_{10}	5	4	$\Phi_3(1^4)$	(111)	(11)	(1)		p	$p-1$	p^2-1	p^2-p	0	p^2	p^2-1	$p-1$
Φ_{11}	6	2	(111)	(111)				p^3	0	p^4-p	0	0	p^3	p^4-p	0
Φ_{12}	6	2	(1^4)	(11)				p^2	$2p^3-2p$	p^4-2p^2+1	0	0	p^4	$2p^3-2p^2$	p^2-2p+1
Φ_{13}	6	2	(1^4)	(11)				p^2	p^3-p	p^4-p^2	0	0	p^4	p^3-p^2	p^2-p
Φ_{14}	6	2	(22)	(2)				p^2	p^3-p	p^4-p^2	0	0	p^4	p^3-p^2	p^2-p
Φ_{15}	6	2	(1^4)	(11)				p^2	0	p^4-1	0	0	p^4	0	p^2-1
Φ_{16}	6	3	$\Phi_2(1^4)$	(111)	(1)			p^2	p^4-p	0	p^3-p^2	0	p^3	p^4-p	0
Φ_{17}	6	3	$\Phi_2(1^4)$	(111)	(1)			p^2	$2p^2-2p$	$2p^3-p^2-2p+1$	p^3-2p^2+p	0	p^3	$2p^3-p^2-p$	p^2-2p+1
Φ_{18}	6	3	$\Phi_2(1^4)$	(111)	(1)			p^2	p^2-p	p^3-p	p^3-p^2	0	p^3	p^3-p	p^2-p
Φ_{19}	6	3	$\Phi_2(1^4)$	(111)	(11)			p^2	$2p^2-2p$	$2p^3-p^2-2p+1$	p^3-2p^2+p	0	p^3	$2p^3-p^2-p$	p^2-2p+1
Φ_{20}	6	3	$\Phi_2(1^4)$	(111)	(11)			p^2	p^2-p	p^3-p	p^3-p^2	0	p^3	p^3-p	p^2-p
Φ_{21}	6	3	$\Phi_2(1^4)$	(111)	(11)			p^2	0	p^2-1	p^3-p	0	p^3	p^2-p	p^2-1
Φ_{22}	6	3	$\Phi_2(1^5)$	(11)	(1)			p	p^3+p^2-p-1	p^4-p^2-p+1	0	0	p^4	p^3-p^2	p^2-p
Φ_{23}	6	4	$\Phi_3(1^4)$	(1^4)	(111)	(1)		p^2	p^2-p	p^3-p	p^3-p^2	0	p^2	$2p^3-p^2-1$	p^2-2p+1
Φ_{24}	6	4	$\Phi_3(1^5)$	(111)	(11)	(1)		p	$2p^2-p-1$	p^3-2p+1	p^3-p^2	0	p^3	p^3-p	p^2-p
Φ_{25}	6	4	$\Phi_3(221)b_1$	(21)	(11)	(1)		p	$2p^2-p-1$	p^3-2p+1	p^3-p^2	0	p^3	p^3-p	p^2-p
Φ_{26}	6	4	$\Phi_3(221)b_\nu$	(21)	(11)	(1)		p	$2p^2-p-1$	p^3-2p+1	p^3-p^2	0	p^3	p^3-p	p^2-p



Family	Rank	Class	G/Z(G)	G_2	G_3	G_4	G_5	q_0^*	q_1^*	q_2^*	q_3^*	q_4^*	r_0^*	r_1^*	r_2^*
Φ_{27}	6	4	$\Phi_3(1^5)$	(111)	(11)	(1)		p	$2p^2-p-1$	p^3-2p+1	p^3-p^2	0	p^3	p^3-p	p^2-p
Φ_{28}	6	4	$\Phi_3(221)b_1$	(21)	(11)	(1)		p	$2p^2-p-1$	p^3-2p+1	p^3-p^2	0	p^3	p^3-p	p^2-p
Φ_{29}	6	4	$\Phi_3(221)b_\nu$	(21)	(11)	(1)		p	$2p^2-p-1$	p^3-2p+1	p^3-p^2	0	p^3	p^3-p	p^2-p
Φ_{30}	6	4	$\Phi_7(1^5)$	(111)	(11)	(1)		p	$p-1$	$2p^2-p-1$	p^3-2p+1	0	p^3	p^2-p	p^2-1
Φ_{31}	6	3	$\Phi_4(1^5)$	(111)	(1)			p	p^2-1	p^3+p^2-2p	p^3-p^2-p+1	0	p^3	p^3-p	p^2-p
Φ_{32}	6	3	$\Phi_4(1^5)$	(111)	(1)			p	p^2-1	p^3+p^2-2p	p^3-p^2-p+1	0	p^3	p^3-p	p^2-p
Φ_{33}	6	3	$\Phi_4(1^5)$	(111)	(1)			p	$2p^2-p-1$	p^3-2p+1	p^3-p^2	0	p^3	p^3-p	p^2-p
Φ_{34}	6	3	$\Phi_4(221)b$	(21)	(1)			p	$2p^2-p-1$	p^3-2p+1	p^3-p^2	0	p^3	p^3-p	p^2-p
Φ_{35}	6	5	$\Phi_9(1^5)$	(1^4)	(111)	(11)	(1)	p	p^4-1	0	0	p^2-p	p^2	p^4-1	0
Φ_{36}	6	5	$\Phi_9(1^5)$	(1^4)	(111)	(11)	(1)	p	p^2-1	p^3-p	0	p^2-p	p^2	p^3-1	p^2-p
Φ_{37}	6	5	$\Phi_9(1^5)$	$\Phi_2(1^4)$	(111)	(11)	(1)	p	$p-1$	p^3-1	p^2-p	p^2-2p+1	p^2	p^3-1	p^2-p
Φ_{38}	6	5	$\Phi_{10}(1^5)$	(1^4)	(111)	(11)	(1)	p	$p-1$	p^2-1	p^2-p	p^2-p	p^2	p^2-1	p^2-1
Φ_{39}	6	5	$\Phi_{10}(1^5)$	$\Phi_2(1^4)$	(111)	(11)	(1)	p	$p-1$	$p-1$	$2p^2-p-1$	p^2-2p+1	p^2	p^2-1	p^2-1
Φ_{40}	6	4	$\Phi_6(1^5)$	(1^4)	(111)	(1)		p	p^2-1	p^2-p	p^3-p	0	p^2	p^3-1	p^2-p
Φ_{41}	6	4	$\Phi_6(1^5)$	(1^4)	(111)	(1)		p	p^2-1	p^2-p	p^3-p	0	p^2	p^3-1	p^2-p
Φ_{42}	6	4	$\Phi_6(221)b_{\frac{1}{2}(p-1)}$	(211)	(111)	(1)		p	p^2-1	p^2-p	p^3-p	0	p^2	p^3-1	p^2-p
Φ_{43}	6	4	$\Phi_6(221)d_0$	(211)	(111)	(1)		p	p^2-1	p^2-p	p^3-p	0	p^2	p^3-1	p^2-p

If G and H are two isoclinic groups, then the nilpotency class of G is equal to the nilpotency class of H and the terms in the lower central series of G are isomorphic to the respective terms of the lower central series of H. More precisely, one can say that the commutator structure of G is similar to the commutator structure of H.

The following well known results will be used quite frequently in the thesis without any reference.

Theorem 2.1.1 ([2, Theorem 1]) *If G is a purely non-abelian finite group, then* $|\operatorname{Aut_z}(G)| = |\operatorname{Hom}(G/G', Z(G))|$.

Lemma 2.1.2 [14, Corollaries 3.7, 3.8] *Let G be a finite non-abelian p-group such that* $\operatorname{Aut_z}(G) = Z(\operatorname{Inn}(G))$. *Then $Z(G) \leq G'$ and $Z(\operatorname{Inn}(G))$ is not cyclic.*

A p-group G is said to be regular if for any two elements x, y in G, there is an element z in the commutator subgroup H' of the subgroup $H = \langle x, y\rangle$ of G, such that $x^p y^p = (xy)^p z^p$.

Lemma 2.1.3 [7, Theorem 7.2(e)] *If G is a regular p-group, then for all $x, y \in G$ and $i, j \geq 0$,*

$$[x^{p^i}, y^{p^j}] = 1 \text{ if and only if } [x, y]^{p^{i+j}} = 1.$$

Lemma 2.1.4 [43, Theorem 2] *Let G be a non-abelian p-group of order p^4. Then one of the following holds:*

(*a*) $|Z(G)| = p^2, |G'| = p$ *and* $G' \leq Z(G)$, *and*

(*b*) $|Z(G)| = p$, $|G'| = p^2$ *and* $Z(G) \leq G'$.

Lemma 2.1.5 ([41, Lemma 0.4]) *Let G be a finite nilpotent group of class 2. Then $\exp(G') = \exp(G/Z(G))$ and in the decomposition of $G/Z(G)$ into a direct product of cyclic groups, at least two factors of maximal order must occur.*

2.2 Characterization

In this section, we characterize all groups of order p^5, where p is any prime, and all groups of order p^6, where p is an odd prime, such that $\mathrm{Aut_z}(G) = Z(\mathrm{Inn}(G))$. The characteristics of such groups turn out to be the same in both the cases. We begin with the following simple lemma.

Lemma 2.2.1 *Let G be a finite p-group of order p^4 and rank 2. If $|G'| = p$, then $Z(G) = \Phi(G)$ and $|Z(G)| = p^2$.*

Proof. Observe that $|\Phi(G)| = p^2$ and $cl(G) = 2$. Therefore by lemma 2.1.5, $\exp(G/Z(G)) = \exp(G') = p$ and hence $Z(G) = \Phi(G)$. □

Theorem 2.2.2 *Let G be a finite p-group such that $|G| = p^5$ and $cl(G) = 3$. Then $\mathrm{Aut_z}(G) = Z(\mathrm{Inn}(G))$ if and only if $d(G) = 2$ and $|Z(G)| = p$.*

Proof. First suppose that $d(G) = 2$ and $|Z(G)| = p$. Then G is purely non-abelian and thus $|\mathrm{Aut_z}(G)| = |\mathrm{Hom}(G/G', Z(G))| = p^2$. Since $Z(\mathrm{Inn}(G)) \leq \mathrm{Aut_z}(G)$, $|Z(\mathrm{Inn}(G))| = p$ or p^2. If $|Z(\mathrm{Inn}(G))| = p$, then $G/Z(G)$ is a 2-generated non abelian group of order p^4 having nilpotency class 2. Thus

$$|Z(G/Z(G))| = |Z(\mathrm{Inn}(G))| = p = |(G/Z(G))'|,$$

which is a contradiction to Lemma 2.2.1. Therefore $|Z(\mathrm{Inn}(G))| = p^2$ and hence $\mathrm{Aut_z}(G) = Z(\mathrm{Inn}(G))$.

Conversely suppose that $\mathrm{Aut_z}(G) = Z(\mathrm{Inn}(G))$. Then $Z(G) < G'$ by Lemma 2.1.2 and the fact that $cl(G) = 3$. Therefore G is purely non-abelian and $\mathrm{Inn}(G)$ is a non-abelian group of order p^4, because if $|\mathrm{Inn}(G)| = p^3$, then $|Z(\mathrm{Inn}(G))| = p$ which is a contradiction to Lemma 2.1.2. Thus $|Z(G)| = p$ and $|Z(\mathrm{Inn}(G))| = p^2$. Then

$$|\mathrm{Aut_z}(G)| = |\mathrm{Hom}(G/G', Z(G))| = |Z(\mathrm{Inn}(G))| = p^2$$

implies that $d(G) = 2$. □

Theorem 2.2.3 *Let G be a finite p-group, p an odd prime, such that $|G| = p^6$ and $cl(G) = 3$ or 4. Then* $\mathrm{Aut_z}(G) = Z(\mathrm{Inn}(G))$ *if and only if* $d(G) = 2$ *and* $|Z(G)| = p$.

Proof. If $d(G) = 2$ and $|Z(G)| = p$, then as in above theorem, $|\mathrm{Aut_z}(G)| = p^2$. We proceed to prove that $|Z(\mathrm{Inn}(G))| = p^2$. On the contrary suppose that $|Z(\mathrm{Inn}(G))| = p$. First suppose that $cl(G) = 3$. Let $H = G/Z(G)$. Since the nilpotency class of G is 3, the nilpotency class of H is 2. Thus $H' = Z(H) = Z(\mathrm{Inn}(G))$ is of order p. This implies that $\exp(H') = \exp(H/Z(H)) = p$ and therefore H is an extra-special p-group, which is a contradiction to the fact that $d(H) = 2$. Next suppose that $cl(G) = 4$. Let $H = G/Z(G)$ and let $K = H/Z(H)$. Then $|K| = p^4$, $cl(K) = 2$ and $\exp(K/Z(K)) = \exp(K')$. Now $|Z(K)| \neq p$ because if $|Z(K)| = p$, then $d(K) = 3$. Thus $|Z(K)| = p^2$ and hence $|K'| = p$ by lemma 2.1.4. It then follows that $|H'| = p^2$, $|G'| = p^3$ and $|\gamma_3(G)| = p^2$. Let $\{x, y\}$ be a minimal generating set for H. If H' is elementary abelian, then

$$[x^p, y] = [x, y]^{x^{p-1}}[x, y]^{x^{p-2}} \dots [x, y]^x[x, y] = [x, y]^p[x, y, x]^{p(p-1)/2} = 1.$$

Thus x^p and similarly y^p is in $Z(H) < H'$. Therefore $\Phi(H) = H'$, a contradiction to the fact that $d(H) = 2$. If H' is cyclic, then $|\Phi(H'))| = p$. But

$$\Phi(H') = \Phi(G'/Z(G)) = \Phi(G')/Z(G),$$

because $Z(G)$ being of order p is contained in $\Phi(G')$. This implies that $|\Phi(G')| = p^2$. It then follows that G' is cyclic and therefore G is regular. Suppose

$$G/G' \simeq C_{p^2} \times C_p \simeq \langle xG' \rangle \times \langle yG' \rangle.$$

Then $\{x, y\}$ is a minimal generating set for G, $G' = \langle [x, y] \rangle$ and $x^{p^2}, y^p \in G'$ but $x^p \notin G'$. Since the nilpotency class of G is 4, $\gamma_3(G) \leq Z_2(G)$. But

$$|Z_2(G)/Z(G)| = |Z(\mathrm{Inn}(G))| = p = |Z(G)|$$

implies that $|Z_2(G)| = p^2 = |\gamma_3(G)|$ and thus $\gamma_3(G) = Z_2(G)$. If $y^p \in Z_2(G) = \gamma_3(G)$, then $[y^p, x] \in Z(G)$ and since G is regular,

$$1 = [y^p, x]^p = [y^{p^2}, x] = [y, x]^{p^2},$$

a contradiction to the fact that $|G'| = p^3$. Therefore $y^p \in G' - \gamma_3(G)$, $G' = \langle y^p \rangle$ and $|y| = p^4$. Thus $x^{p^2} \in \langle y \rangle$ and therefore $[x^{p^2}, y] = [x, y]^{p^2} = 1$, which is again a contradiction to the fact that $|G'| = p^3$. Hence $\mathrm{Aut_z}(G) = Z(\mathrm{Inn}(G))$ is of order p^2.

Conversely suppose that $Z(\mathrm{Inn}(G)) = \mathrm{Aut_z}(G)$. Then $Z(G) < G'$ by Lemma 2.1.2 and the fact that $cl(G) > 2$. Thus G is purely non-abelian and $G/Z(G)$ is a non-abelian group of order at most p^5. Therefore $p^2 \leq |Z(\mathrm{Inn}(G))| \leq p^3$, because $Z(\mathrm{Inn}(G))$ cannot be cyclic by Lemma 2.1.2. The possibility of G being rank 4 is immediately ruled out because if $d(G) = 4$, then

$$|\,\mathrm{Aut_z}(G)| = |\mathrm{Hom}(C_p \times C_p \times C_p \times C_p, C_p)| = p^4 \neq |Z(\mathrm{Inn}(G))|.$$

If $d(G) = 3$, then $|\operatorname{Aut_z}(G)| \geq p^3$ and therefore by assumption, $|\operatorname{Aut_z}(G)| = |Z(\operatorname{Inn}(G))| = p^3$. Also $|Z(G)| = p$, for if $|Z(G)| = p^2$, then $|Z(\operatorname{Inn}(G))| = p^2$. Let $H = G/Z(G)$. Then $|H/Z(H)| = p^2$ and hence $|H'| = p$. It then follows that $|G'| = p^2$, $|\gamma_3(G)| = p$ and therefore $cl(G) = 3$. Suppose that

$$G/G' \simeq C_p \times C_p \times C_{p^2} \simeq \ \langle xG' \rangle \times \langle yG' \rangle \times \langle zG' \rangle.$$

Then $G = \ \langle x, y, z \rangle$ and $x^p, y^p, z^{p^2} \in G'$ but $z^p \notin G'$. Since $|G/Z_2(G)| = p^2$, one of x, y and z lies in $Z_2(G)$. If $z \in Z_2(G)$, then $z^p \in Z(G) \leq G'$, which is a contradiction. We can therefore, without any loss of generality, assume that $x \in Z_2(G)$. Then $[x, y], [x, z] \in Z(G)$ and $[z^p, x] = 1$. If G' is elementary abelian, then $[z^p, y] = [z, y]^p[z, y, z]^{p(p-1)/2} = 1$. Thus $z^p \in Z(G)$, again a contradiction. Now suppose that G' is cyclic. Let $M = \ \langle y, z, \Phi(G) \rangle$. Then M is a maximal subgroup of G and $G = MZ_2(G)$. It follows from [8, Theorem 1.3] that $\gamma_i(G) = \gamma_i(M)$ for all $i \geq 2$. Since $Z(G) \leq M$, $Z(G) \leq Z(M)$ and $C_G(M) = Z(M)$. We prove that $Z(G) = Z(M)$. Since $|M| = p^5$, order of $Z(M)$ lies between p and p^3. If $|Z(M)| = p^2$ or p^3, then since $M' = G'$ is cyclic, it follows from [7, Prop. 21.20] that

$$p^3 \geq |M/Z(M)| \geq |M'|^2 = p^4,$$

a contradiction and thus $|Z(M)| = p = |Z(G)|$. This proves that $Z(M) = Z(G)$. Since G' is cyclic, $\Omega_1(G') = Z(G)$ and thus $C_G(C_G(M)) = M$ [11, Theorem C]. Now from the facts that $C_G(M) = Z(M)$, $Z(M) = Z(G)$ and $C_G(C_G(M)) = M$, it follows that $C_G(Z(G)) = M$, a final contradiction to $d(G) = 3$. Hence $d(G)$=2 and therefore, because of assumption, $|Z(G)| = p$ or p^2. We prove that $|Z(G)| = p$. If $|Z(G)| = p^2$, then $|Z(\operatorname{Inn}(G))| = p^2$ by Lemma 2.1.4 and therefore $|G'/Z(G)| = p$.

This implies that $|G'| = p^3$ and $G/G' \simeq C_{p^2} \times C_p$. Thus

$$|\operatorname{Aut_z}(G)| = |\operatorname{Hom}(C_{p^2} \times C_p, Z(G))| \geq p^3 > |Z(\operatorname{Inn}(G))|,$$

a contradiction to the assumption. This proves the " *if* " part of the theorem. □

2.3 — Application

In this section, we use the classification of all groups of order p^n, where p is an odd prime and $5 \leq n \leq 6$, given by James [31]. It follows from [31] that all finite p-groups of order p^5 are partitioned into ten isoclinism families; and all finite p-groups of order p^6 are partitioned into forty three isoclinism families. As an application of our results, we find those groups G of order p^5 and p^6 for which $\operatorname{Aut_z}(G) = Z(\operatorname{Inn}(G))$. We shall mainly use the information concerning these groups given in §4.1 and presentations of these groups given in §4.5 and §4.6 of [31]. The i-th family is denoted as Φ_i.

Theorem 2.3.1 *If* $|G| = p^5$ *and* $cl(G) = 3$*, then* $\operatorname{Aut_z}(G) = Z(\operatorname{Inn}(G))$ *if and only if* G *is isomorphic to* $\Phi_8(32)$*.*

Proof. There are only 2 isoclinism families *viz.* Φ_7 and Φ_8 which consist of groups G such that $|Z(G)| = p$ and $cl(G) = 3$. The family Φ_7 consists of groups G with $G/Z(G) \simeq \Phi_2(1^4)$. Thus $G/Z(G)$ is of exponent p. Since $|G'| = p^2$, it follows that $Z(G) < G'$ and G/G' is an elementary abelian group of order p^3. Thus $d(G) = 3$ and hence $\operatorname{Aut_z}(G) \neq Z(\operatorname{Inn}(G))$ by Theorem 2.2.2. The family Φ_8 consists of only one group *viz.*

$$\Phi_8(32) = \langle \alpha_1, \alpha_2, \beta \mid [\alpha_1, \alpha_2] = \beta = \alpha_1^p,\ [\alpha_1, \beta] = 1,\ [\alpha_2, \beta] = 1, \beta^{p^2} = \alpha_2^{p^2} = 1 \rangle$$

which is of rank 2 because $\beta = [\alpha_1, \alpha_2] \in (\Phi_8(32))' \leq \Phi(\Phi_8(32))$ and hence $\mathrm{Aut_z}(\Phi_8(32)) = Z(\mathrm{Inn}(\Phi_8(32)))$ by Theorem 2.2.2.

□

Theorem 2.3.2 *If* $|G| = p^6$ *and* $cl(G) = 3$ *or* 4, *then* $\mathrm{Aut_z}(G) = Z(\mathrm{Inn}(G))$ *if and only if* G *is isomorphic to one of the groups in the isoclinism families* $\Phi_{25}, \Phi_{26}, \Phi_{28}, \Phi_{29}$ *and* $\Phi_{40} - \Phi_{43}$.

Proof. There are 16 isoclinism families *viz.* $\Phi_{22}, \Phi_{24} - \Phi_{34}$ and $\Phi_{40} - \Phi_{43}$ which consist of groups G for which $|Z(G)| = p$ and $cl(G) = 3$ or 4. If G is any group from the families $\Phi_{22}, \Phi_{24}, \Phi_{27}$ and $\Phi_{30} - \Phi_{33}$, then $Z(G) < G'$, $|G'| = p^2$ or p^3 and $G/Z(G)$ is of exponent p. Therefore G is of rank 3 or 4 and hence $\mathrm{Aut_z}(G) \neq Z(\mathrm{Inn}(G))$ by Theorem 2.2.3. The family Φ_{34} consists of groups G such that $|G'| = p^3$. From the presentations of these groups, it follows that they are generated by 6 elements $\alpha, \alpha_1, \alpha_2, \beta_1, \beta_2, \gamma$ such that $\beta_1, \beta_2, \gamma, \alpha^p, \alpha_1^p, \alpha_2^p \in G'$. This implies that $G^p \leq G'$. Thus $d(G) = 3$ and hence $\mathrm{Aut_z}(G) \neq Z(\mathrm{Inn}(G))$ by Theorem 2.2.3. Any group G of the families Φ_{25}, Φ_{26}, Φ_{28} and Φ_{29} is generated by 5 elements $\alpha, \alpha_1, \alpha_2, \alpha_3, \alpha_4$ such that $\alpha_2, \alpha_3, \alpha_4 \in G'$. Thus $d(G) = 2$ and hence $\mathrm{Aut_z}(G) = Z(\mathrm{Inn}(G))$ by Theorem 2.2.3. If G is any group from the families $\Phi_{40} - \Phi_{43}$, then $|G'| = p^4$ and hence $d(G) = 2$. Thus $\mathrm{Aut_z}(G) = Z(\mathrm{Inn}(G))$ by Theorem 2.2.3. □

CHAPTER 3

Central Automorphisms Fixing the Center Elementwise

3.1 — Introduction

The purpose of this chapter is to find finite p-groups G for which

$$Z(\mathrm{Inn}(G)) = \mathrm{Aut}_{\mathrm{z}}^{\mathrm{z}}(G) < \mathrm{Aut}_{\mathrm{z}}(G). \tag{3.1}$$

Observe that $Z(\mathrm{Inn}(G)) \leq \mathrm{Aut}_{\mathrm{z}}^{\mathrm{z}}(G) \leq \mathrm{Aut}_{\mathrm{z}}(G)$ and that $Z(\mathrm{Inn}(G)) = \mathrm{Aut}_{\mathrm{z}}^{\mathrm{z}}(G)$ if $Z(\mathrm{Inn}(G)) = \mathrm{Aut}_{\mathrm{z}}(G)$. In section 3.2, we prove that there is no finite p-group G of order dividing p^6, satisfying (3.1); and we prove that a group G of order p^7, where p is an odd prime, satisfies (3.1) if and only if $Z(G) \simeq C_{p^2}$, $|G'| = p^4$ and the nilpotency class of G is 4. Observe that if $Z(G) \leq G'$, then $\mathrm{Aut}_{\mathrm{z}}^{\mathrm{z}}(G) = \mathrm{Aut}_{\mathrm{z}}(G)$. Therefore, hereafter, in our proofs of theorems, we shall assume that $Z(G)$ is not contained in G'. Attar [3] proved that $\mathrm{Aut}_{\mathrm{z}}^{\mathrm{z}}(G) = \mathrm{Inn}(G)$ if and only if $G' = Z(G)$ and $Z(G)$ is cyclic. If G is of nilpotency class 2, then $\mathrm{Inn}(G) = Z(\mathrm{Inn}(G))$; it follows that if $\mathrm{Aut}_{\mathrm{z}}^{\mathrm{z}}(G) = Z(\mathrm{Inn}(G))$, then $G' = Z(G)$, a contradiction to our assumption. It is, therefore, also necessary to assume that G is of nilpotency class at least 3. It follows that G is not of maximal class, $|Z(G)| \geq p^2$ and $|G|$ is at least p^5.

It follows from the results of Curran [14, Theorem 2.3] and Adney and Yen [2, Theorem 1] that for any non-abelian group G,

$$\operatorname{Aut}_{\mathrm{z}}^{\mathrm{z}}(G) \simeq \operatorname{Hom}(G/G'Z(G), Z(G)),$$

and if G is purely non-abelian and finite, then

$$|\operatorname{Aut}_{\mathrm{z}}(G)| = |\operatorname{Hom}(G/G', Z(G))|.$$

We start with the following useful lemma.

Lemma 3.1.1 [27, Satz 7.8, p. 306] *Let G be a finite p-group. If $Z(G')$ is cyclic, then so is G'.*

3.2 — Main Results

Lemma 3.2.1 *Let G be a finite non-abelian p-group. Then $G/G'Z(G)$ is not cyclic.*

Proof. Suppose that $G/G'Z(G)$ is cyclic. Then, since $G' \leq \Phi(G)$, it follows that G has a generating set $\{a_1, a_2, ..., a_n\}$, where $a_i \in Z(G)$ for all $i > 1$. It follows that G is abelian. □

As explained in the introduction, we shall assume that $Z(G)$ is not contained in G' and the nilpotency class of G is at least 3.

Theorem 3.2.2 *If G is of coclass 2, then G does not satisfy* (3.1).

Proof. Let $|G| = p^n$. Observe that $|Z(G)| = p^2$ and $G/Z(G)$ is a maximal class group of order p^{n-2}. It follows that $|Z(\operatorname{Inn}(G))| = p$, $|(G/Z(G))'| = p^{n-4}$ and hence $|\operatorname{Aut}_{\mathrm{z}}^{\mathrm{z}}(G)| = |\operatorname{Hom}(G/G'Z(G), Z(G))| \geq p^2 > |Z(\operatorname{Inn}(G))|$. □

Theorem 3.2.3 *There does not exist a group of order dividing p^6 satisfying* (3.1).

Proof. The result follows from Theorem 3.2.2 if $|G| = p^5$. Therefore suppose that $|G| = p^6$. Observe that either (i) the nilpotency class of G is 4 and $|Z(G)| = p^2$ or (ii) the nilpotency class of G is 3 and $|Z(G)| = p^2$ or p^3. The result follows again from Theorem 3.2.2 in case (i). Assume that the nilpotency class of G is 3 and $|Z(G)| = p^3$. Then $G/Z(G)$ is an extra-special p-group and hence $|Z(\mathrm{Inn}(G))| = |(G/Z(G))'| = p$. It follows that $|G'Z(G)| = p^4$ and hence

$$|\operatorname{Aut}_{\mathrm{z}}^{\mathrm{z}}(G)| = |\operatorname{Hom}(G/G'Z(G), Z(G))| \geq p^2 > |Z(\mathrm{Inn}(G))|.$$

Finally assume that the nilpotency class of G is 3 and $|Z(G)| = p^2$. Then $G/Z(G)$ is a class 2 group of order p^4; consequently $|Z(\mathrm{Inn}(G))| = p^2$, $|(G/Z(G))'| = p$ by Lemma 2.1.4 and hence $G'Z(G)$ is an abelian normal subgroup of G of order p^3. It follows from Lemma 3.2.1 that $G/G'Z(G)$ is not cyclic and hence $|\operatorname{Aut}_{\mathrm{z}}^{\mathrm{z}}(G)| > p^2 = |Z(\mathrm{Inn}(G))|$. □

Theorem 3.2.4 *A group G of order p^7, where p is an odd prime, satisfies* (3.1) *if and only if $Z(G) \simeq C_{p^2}$, $|G'| = p^4$ and the nilpotency class of G is* 4.

Proof. First suppose that $Z(G) \simeq C_{p^2}$, $|G'| = p^4$ and the nilpotency class of G is 4. It implies that G is purely non-abelian and hence

$$|\operatorname{Aut}_{\mathrm{z}}^{\mathrm{z}}(G)| = |\operatorname{Hom}(G/G'Z(G), Z(G))| = p^2 < p^3 = |\operatorname{Hom}(G/G', Z(G))| = |\operatorname{Aut}_{\mathrm{z}}(G)|.$$

Now $G/Z(G)$ is a group of order p^5 and of nilpotency class 3. Therefore $|Z(\mathrm{Inn}(G))| = p$ or p^2. Assume that $|Z(\mathrm{Inn}(G))| = p$. Then $|Z_2(G)| = p^3$. Since G is of nilpotency class 4 and $Z(G)$ is not contained in G', $\gamma_3(G) < Z_2(G)$ and hence $|\gamma_3(G)| = p^2$.

Observe that G/G' is isomorphic to $C_{p^2} \times C_p$ or $C_p \times C_p \times C_p$. The first case is not possible by [8, Theorem 1.5(iii)]. In the second case, $G' = \Phi(G)$. We can choose a minimal generating set $\{x, y, z\}$ of G in which one of the generators, say z, is in $Z(G)$. Then $G' = \langle [x, y], \gamma_3(G)\rangle$, and hence $G'/\gamma_3(G)$ is a cyclic group of order p^2. But this is a contradiction to [8, Theorem 1.5(i)]. It follows that $|Z(\mathrm{Inn}(G))| = p^2$ and hence G satisfies (3.1).

Conversely suppose that G satisfies (3.1). In view of theorem 3.2.2, nilpotency class of G is either 3 or 4. First assume that G is of nilpotency class 3. Then G' is abelian and therefore

$$d(\mathrm{Aut}_{\mathrm{z}}^{\mathrm{z}}(G)) \geq d(G/G'Z(G)) \geq 2$$

by Lemma 3.2.1. It follows that $p^2 \leq |Z(\mathrm{Inn}(G))| \leq p^3$ and hence $p^2 \leq |Z(G)| \leq p^3$. If $|Z(G)| = p^3$, then $|Z(\mathrm{Inn}(G))| = p^2$, $|(G/Z(G))'| = p$ by Lemma 2.1.4 and thus $|G'Z(G)| = p^4$. Since $G/G'Z(G)$ is not cyclic,

$$|\mathrm{Aut}_{\mathrm{z}}^{\mathrm{z}}(G)| = |\mathrm{Hom}(G/G'Z(G), Z(G))| \geq p^3,$$

which is a contradiction to (3.1). We therefore suppose that $|Z(G)| = p^2$. First consider the case when $|Z(\mathrm{Inn}(G))| = p^2$. Observe that $|G'| < p^4$, because if $|G'| \geq p^4$, then $|Z_2(G)| = |Z(G)G'| \geq p^5$ and hence $|Z(\mathrm{Inn}(G))| \geq p^3$. If $|G'| = p^2$ or p^3, then $G/G'Z(G)$ is not cyclic by Lemma 3.2.1. Thus

$$|\mathrm{Aut}_{\mathrm{z}}^{\mathrm{z}}(G)| = |\mathrm{Hom}(G/G'Z(G), Z(G))| \geq p^3,$$

a contradiction to (3.1). Next consider the case when $|Z(\mathrm{Inn}(G))| = p^3$. Let $H = G/Z(G)$. Then $|H| = p^5$, $|H/Z(H)| = p^2$ and the nilpotency class of H is 2. Thus

$|H'| = p$ by [52, Theorem 2.1(i)]. Therefore $|G'Z(G)| = p^3$ and hence $|G'| = p^2$. Now

$$|\operatorname{Aut}_{\mathrm{z}}^{\mathrm{z}}(G)| = |\operatorname{Hom}(G/G'Z(G), Z(G))| = |Z(\operatorname{Inn}(G))| = p^3,$$

which is possible when $Z(G) \simeq C_{p^2}$ and $G/G'Z(G) \simeq C_{p^3} \times C_p$. Let

$$G/G'Z(G) \simeq \langle xG'Z(G) \rangle \times \langle yG'Z(G) \rangle.$$

Then $x^{p^3}, y^p \in G'Z(G)$ but $x^p, x^{p^2} \notin G'Z(G)$. If G' is elementary abelian, then for any $g \in G$,

$$[x^p, g] = [x, g]^{x^{p-1}}[x, g]^{x^{p-2}} \dots [x, g]^x [x, g] = [x, g]^p [x, g, x]^{p(p-1)/2} = 1$$

implies that $x^p \in Z(G)$; and if G' is cyclic, then G is regular and thus $1 = [x, g]^{p^2} = [x^{p^2}, g]$ implies that $x^{p^2} \in Z(G)$. In both the cases, we get a contradiction.

We next assume that the nilpotency class of G is 4. Observe that $|G'| < p^5$, because if $|G'| = p^5$, then $d(G) = 2$ and one of the generators x, y of G will lie in $Z(G)$ and thus $G' = \langle [x, y], \gamma_3(G) \rangle = \gamma_3(G)$, which is not so. We now divide the proof in three cases depending on the order $|Z(\operatorname{Inn}(G))|$ of $Z(\operatorname{Inn}(G))$. If $|Z(\operatorname{Inn}(G))| = p^3$, then $|Z_2(G)| \geq p^5$, which is not possible in a p-group of nilpotency class 4. Next suppose that $|Z(\operatorname{Inn}(G))| = p$. Then $G/Z(G)G' \simeq C_p$ and thus $|Z(G)G'| = p^6$. Now if $|Z(G)| = p^2$, then $|G'| = p^5$, not possible. If $|Z(G)| = p^3$, then $G/Z(G)$ has nilpotency class 3 and order p^4 and thus $|(G/Z(G))'| = p^2$ by Lemma 2.1.4, which gives $|Z(G)G'| = p^5$, a contradiction. If $|Z(G)| = p^4$, then $G/Z(G)$ has nilpotency class 3 and order p^3 and thus$|(G/Z(G))'| = p$ implies that $|Z(G)G'| = p^5$, a contradiction. Finally suppose that $|Z(\operatorname{Inn}(G))| = p^2$. In this case $G/Z(G)G' \simeq C_p \times C_p$ or C_p or C_{p^2} or C_{p^3} or C_{p^4}. Observe that $|Z(G)| = p^2$, because if $|Z(G)| = p^3$

or p^4, then $|Z_2(G)| = p^5$ or p^6, which is not possible in a p-group of nilpotency class 4. We claim that $Z(G) \simeq C_{p^2}$ and $|G'| = p^4$. If $Z(G) \simeq C_p \times C_p$, then $G/Z(G)G'$ is cyclic and thus G' is not abelian by Lemma 3.2.1. Since $|G'| < p^5$, $|G'| = p^4$ by Lemma 3.1.1. Therefore $G/Z(G)G' \simeq C_{p^2}$. If $d(G) = 3$, then $\exp(G/G') = p$ and if $d(G) = 2$, then $G'Z(G) = \Phi(G)$, which is a contradiction to $\exp(G/G'Z(G)) = p^2$. It follows that $Z(G) \simeq C_{p^2}$. If $|G'| = p^2$ or p^3, then since $|\operatorname{Aut_z}(G)| = p^2$, $G/G'Z(G) \simeq C_{p^3}$ or C_{p^4}, which is not possible by Lemma 3.2.1 and Lemma 3.1.1. This proves our claim. $\square$

We next give an example of a group G of order 3^7 which satisfies equation (3.1). This group G has ID 131 among the groups of order 2187 in SmallGroup library of GAP [51]. Consider

$$\begin{aligned} G \;=\; & \langle \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7 \mid [\alpha_1, \alpha_2] = \alpha_3^{-1}, [\alpha_1, \alpha_3] = \alpha_5^{-1}, [\alpha_1, \alpha_4] = 1, \\ & [\alpha_1, \alpha_5] = 1, [\alpha_1, \alpha_6] = 1, [\alpha_1, \alpha_7] = 1, [\alpha_2, \alpha_3] = \alpha_6^{-1}, [\alpha_2, \alpha_4] = 1, [\alpha_2, \alpha_5] = 1, \\ & [\alpha_2, \alpha_6] = 1, [\alpha_2, \alpha_7] = 1, [\alpha_3, \alpha_4] = 1, [\alpha_3, \alpha_5] = 1, [\alpha_3, \alpha_6] = 1, [\alpha_3, \alpha_7] = 1, \\ & [\alpha_4, \alpha_5] = 1, [\alpha_4, \alpha_6] = 1, [\alpha_4, \alpha_7] = 1, [\alpha_5, \alpha_6] = 1, [\alpha_5, \alpha_7] = 1, [\alpha_6, \alpha_7] = 1, \\ & \alpha_1^3 = \alpha_4, \alpha_3^3 = \alpha_7^{-1}, \alpha_4^3 = \alpha_7, \alpha_1^{27} = \alpha_2^3 = \alpha_3^9 = \alpha_4^9 = \alpha_5^3 = \alpha_6^3 = \alpha_7^3 = 1 \rangle. \end{aligned}$$

It can be checked that $Z(G) \simeq \langle \alpha_4 \rangle \simeq C_9$, $G' \simeq \langle \alpha_3, \alpha_5, \alpha_6, \alpha_7 \rangle$, $\alpha_4 \in \Phi(G) - G'$, $d(G) = 2$, nilpotency class of G is 4, $Z(G) \cap G' \simeq \langle \alpha_7 \rangle \simeq C_3$, $|G'| = 81$, $Z(G) \nsubseteq G'$, $|\operatorname{Aut}_{\mathrm{z}}^{\mathrm{z}}(G)| = |\operatorname{Hom}(G/G'Z(G), Z(G))| = |\operatorname{Hom}(C_p \times C_p, C_{p^2})| = p^2$ and $|Z_2(G)/Z(G)| = p^2$.

CHAPTER 4

Class-Preserving Automorphisms of Groups of order p^5

4.1 — Introduction

Let G be a finite group. An automorphism α of G is called a class-preserving automorphism if, for each element $x \in G$, there exists an element $g_x \in G$ such that $\alpha(x) = g_x^{-1} x g_x$; and is called an inner automorphism if, for all $x \in G$, there exists a fixed element $g \in G$ such that $\alpha(x) = g^{-1}xg$. The group $\operatorname{Inn}(G)$ of all inner automorphisms of G is a normal subgroup of the group $\operatorname{Aut_c}(G)$ of all class-preserving automorphisms of G. We denote the group $\operatorname{Aut_c}(G)/\operatorname{Inn}(G)$ of all class-preserving outer automorphisms by $\operatorname{Out_c}(G)$.

In section 4.2, we give necessary and sufficient conditions on a group of order p^5 such that G has a non-inner class-preserving automorphism. We would like to mention here that in proving our results, we are not using any of the available classifications or presentations of the groups. As a consequence, we give short and alternate proofs of results of Yadav [55, Section 5] and Kalra and Gumber [34, Theorem 4.2].

Yadav [55] proved that if G and H are two finite non-abelian isoclinic groups, then $\mathrm{Aut_c}(G) \simeq \mathrm{Aut_c}(H)$. He then showed that $\mathrm{Out_c}(G) \neq 1$ for the groups $\Phi_7(1^5)$ and $\Phi_{10}(1^5)$ from the seventh and tenth families in [31], and concluded that if G is a finite p-group of order p^5, where p is an odd prime, then $\mathrm{Out_c}(G) \neq 1$ if and only if G is isoclinic to a group either in the seventh or in the tenth family. Kalra and Gumber [34, Theorem 4.2], using the classifications, isoclinism families and presentations given by Hall and Senior [20] and Sag and Wamsley [46], have shown that if G is a finite 2-group of order 2^5, then $\mathrm{Out_c}(G) = 1$, except for the forty fourth and forty fifth groups from the sixth family in [20]. In section 4.3, as a consequence of the results proved in section 4.2, we find all finite p-groups G of order p^5 for which $\mathrm{Out_c}(G) \neq 1$ from the list of all such groups given by Hall and Senior [20] (for $p = 2$) and given by James [31] (for odd primes p).

Let H be a non-trivial proper normal subgroup of a finite p-group G. The pair (G, H) is called a ***Camina pair*** if $H \subseteq [y, G]$ for all $y \in G - H$; and G is called a ***Camina group*** if (G, G') is a Camina pair.

We shall use the following known results without further reference.

Proposition 4.1.1 ([24, Proposition 14.4]) *Let G be a finite group having an abelian normal subgroup A with cyclic quotient G/A. Then class-preserving automorphisms of G are inner automorphisms.*

Lemma 4.1.2 ([7, Lemma 36.5 (a)]) *If a finite p-group G is a two-generated group of class 2, then G' is cyclic.*

4.2 — Main Results

We start with the following crucial technical lemma. The lemma can be of independent interest also.

Lemma 4.2.1 *Let* G *be any group such that* $\mathrm{Out_c}(G/Z(G))$ *is trivial. Then* $\mathrm{Aut_c}(G) = (\mathrm{Aut_c}(G) \cap \mathrm{Aut_z}(G))\,\mathrm{Inn}(G)$. *In addition, if* G *is finite, then*

$$|\mathrm{Aut_c}(G)| = \frac{|\mathrm{Aut_c}(G) \cap \mathrm{Aut_z}(G)|\,|\mathrm{Inn}(G)|}{|Z(\mathrm{Inn}(G))|}.$$

Proof. If $\alpha \in \mathrm{Aut_c}(G)$, then it induces a class-preserving automorphism, say $\overline{\alpha}$, on $G/Z(G)$ given by $\overline{\alpha}(xZ(G)) = \alpha(x)Z(G)$ for all $x \in G$. Since $\mathrm{Out_c}(G/Z(G)) = 1$, $\alpha(x)Z(G) = a^{-1}xaZ(G)$ for a fixed $a \in G$. Therefore, for each $x \in G$, there exists an element $z_x \in Z(G)$ such that $\alpha(x) = a^{-1}xaz_x = a^{-1}(xz_x)a$. Define a map $\beta : G \to G$ by $\beta(x) = a\alpha(x)a^{-1}$ for all $x \in G$. It is easy to see that $\beta \in \mathrm{Aut_c}(G) \cap \mathrm{Aut_z}(G)$ and $\alpha = i_a\beta$, where i_a denotes the inner automorphism of G given by conjugation with a. It thus follows that $\mathrm{Aut_c}(G) = (\mathrm{Aut_c}(G) \cap \mathrm{Aut}_z(G))\,\mathrm{Inn}(G)$. □

Proposition 4.2.2 *Let* G *be a finite non-abelian* p*-group of order* p^5 *such that* $|Z(G)| \geq p^2$. *Then* $\mathrm{Out_c}(G) = 1$.

Proof. If $|Z(G)| \geq p^3$, then G has a maximal abelian subgroup and hence $\mathrm{Out_c}(G) = 1$. We therefore suppose that $|Z(G)| = p^2$. Then $cl(G)$ is either 2 or 3. First suppose that $cl(G) = 2$. Then $G' \leq Z(G)$. If G' is cyclic, then $\mathrm{Out_c}(G) = 1$. Therefore suppose that $G' = Z(G) \simeq C_p \times C_p$. Then $\exp(G/Z(G)) = \exp(G') = p$. It follows that $Z(G) = G' = \Phi(G)$ and hence $d(G) = 3$. If $G = \langle a, b, c\rangle$, then $G' = \langle [a, b], [a, c], [b, c]\rangle$. We can assume that

$$[b, c] = [a, b]^m [a, c]^n$$

for some m, n, $0 \leq m, n \leq p-1$. Set $u := ba^{-n}$ and $v := ca^m$. Then

$$[u, v] = [ba^{-n}, ca^m] = [b, c][b, a]^m[a, c]^{-n} = 1.$$

It follows that $\langle u, v, G' \rangle$ is a maximal abelian subgroup of G and hence $\mathrm{Out_c}(G) = 1$.

Now suppose that $cl(G) = 3$. Then G' is abelian of order p^2 or p^3. First assume that $|G'| = p^2$. Since $G/C_G(G')$ is isomorphic to a subgroup of $\mathrm{Aut}(G')$, $[G : C_G(G')] \leq p$ and hence $|C_G(G')| = p^4$. The subgroup $G'Z(G)$ is of order p^3 and is contained in $C_G(G')$. It follows that $C_G(G') = \langle a, G'Z(G) \rangle$, where $a \in C_G(G') - G'Z(G)$, is a maximal abelian subgroup of G and hence $\mathrm{Out_c}(G) = 1$. Next assume that $|G'| = p^3$. Then $G' = \Phi(G)$ and $d(G) = 2$. Let $G = \langle a, b \rangle$ and let $w := [a, b], u := [a, w]$ and $v := [b, w]$. Then $G' = \langle w, \gamma_3(G) \rangle$ and $\gamma_3(G) = \langle u, v \rangle$. Since

$$[a, b]^p \equiv [a, b^p] \equiv 1 \pmod{\gamma_3(G)},$$

$|G'/\gamma_3(G)| = p$ and hence $|\gamma_3(G)| = p^2$. If $|h^G| = p^3$ for some $h \in G - G'$, then $|C_G(h)| = p^2$ and hence $h \in Z(G)$, which is absurd. Since $|x^G| \neq p$, it follows that $|x^G| = p^2$ for all $x \in G - G'$. Then $|\mathrm{Aut_c}(G)| \leq p^4$. Let $|\mathrm{Aut_c}(G)| = p^4$. Then, for any $x, y \in G$, there exists an $\alpha \in \mathrm{Aut_c}(G)$ such that $\alpha(a) = x^{-1}ax$ and $\alpha(b) = y^{-1}by$. In particular, there exists an $\alpha \in \mathrm{Aut_c}(G)$ such that $\alpha(a) = b^{-1}ab$ and $\alpha(b) = a^{-1}ba$. Now $(ab)^{-1}\alpha(ab) \in [ab, G]$. But

$$(ab)^{-1}\alpha(ab) = b^{-1}a^{-1}\alpha(a)\alpha(b) = b^{-1}wbw^{-1} = [w, b] = v^{-1}.$$

Thus $v^{-1} \in [ab, G]$. Let $v^{-1} = [ab, h]$, where $h \in G$. Observe that

$$uv = [a, w][b, w] = [ab, w]$$

and therefore

$$u = (uv)v^{-1} = [ab, w][ab, h] = [ab, wh] \in [ab, G].$$

Thus $\gamma_3(G) \leq [ab, G]$. But since

$$|[ab, G]| = |(ab)^G| = p^2,$$

$\gamma_3(G) = [ab, G]$. Thus

$$w^{-1} = [b, a] = [ab, a] \in [ab, G] = \gamma_3(G).$$

This is a contradiction and hence $\operatorname{Out_c}(G) = 1$. $\square$

Theorem 4.2.3 *Let G be a finite non-abelian p-group of order p^5. Then* $\operatorname{Out_c}(G) \neq 1$ *if and only if* $|Z(G)| = p$, $Z(G) < G'$ *and either (i)* $cl(G) = 3$ *and* $d(G) = 3$ *or (ii)* $cl(G) = 4$ *and* $Z(G) \subseteq [x, G]$ *for all* $x \in G - G'$.

Proof. First suppose that $\operatorname{Out_c}(G) \neq 1$. Then $|Z(G)| = p$ by Proposition 4.2.2 and hence $Z(G) < G'$. It follows that G is purely non-abelian and $cl(G)$ is either 3 or 4. Suppose that $cl(G) = 3$. Since $|G'|$=p^2 or p^3,

$$|\operatorname{Aut_z}(G)| = |\operatorname{Hom}(G/G', Z(G))| \leq p^3$$

by [2, Theorem 1]. Also, since $G/Z(G)$ is a class 2 group of order p^4, $|Z(\operatorname{Inn}(G))| = p^2$. It now follows by Lemma 4.2.1 that

$$|\operatorname{Aut_c}(G)| = p^2 |\operatorname{Aut_c}(G) \cap \operatorname{Aut_z}(G)| \leq p^5.$$

Since $|\operatorname{Aut_c}(G)| > |\operatorname{Inn}(G)| = p^4$, $|\operatorname{Aut_z}(G)| = p^3$ and hence $d(G) = 3$. Next suppose that $cl(G) = 4$. Then G is of maximal class and hence $|G'| = p^3$, $|Z(\operatorname{Inn}(G))| = p$, $d(G) = 2$ and $|\operatorname{Aut_z}(G)| = |\operatorname{Hom}(C_p \times C_p, C_p)| = p^2$. Thus

$$|\operatorname{Aut_c}(G)| = p^3 |\operatorname{Aut_c}(G) \cap \operatorname{Aut_z}(G)| \leq p^5,$$

by Lemma 4.2.1. It follows that $|\operatorname{Aut_c}(G) \cap \operatorname{Aut_z}(G)| = p^2$ and hence $\operatorname{Aut_z}(G) \leq \operatorname{Aut_c}(G)$. Let $a \in G - G'$ and let $G = \langle a, b \rangle$. Suppose that all central automorphisms of G fix a. Then each $1 \neq \alpha \in \operatorname{Aut_z}(G)$ would have to move b. Since $|\operatorname{Aut_z}(G)| = p^2$, this would require that $|Z(G)| = p^2$, which is not so. Thus, there exists a central automorphism α such that $\alpha(a) = az$ for some $1 \neq z \in Z(G)$. On the other hand, since α is class-preserving, $\alpha(a) = g^{-1}ag$ for some $g \in G$. It follows that $z = [a, g]$ and, since $z^n = [a, g^n]$ for all $n \geq 1$, $Z(G) \subseteq [x, G]$ for all $x \in G - G'$.

To prove the converse, we first suppose that $|Z(G)| = p$, $Z(G) < G'$, $cl(G) = 4$ and $Z(G) \subseteq [x, G]$ for all $x \in G - G'$. Since G is of maximal class, $|Z(\operatorname{Inn}(G))| = p$, $|G'| = p^3$, $d(G) = 2$ and hence $|\operatorname{Aut_z}(G)| = |\operatorname{Hom}(C_p \times C_p, C_p)| = p^2$. Also, since $Z(G) \subseteq [x, G]$ for all $x \in G - G'$, $\operatorname{Aut_z}(G) \leq \operatorname{Aut_c}(G)$. It follows by Lemma 4.2.1 that $|\operatorname{Aut_c}(G)| = p^5 > |\operatorname{Inn}(G)|$. Next suppose that $|Z(G)| = p$, $Z(G) < G'$ and $d(G) = 3$. Then $cl(G) = 3$, $|G'| = |\Phi(G)| = p^2$ and hence

$$|\operatorname{Aut_z}(G)| = |\operatorname{Hom}(G/G', Z(G))| = p^3.$$

It follows from [50, Theorems 4.7 and 5.1] that $(G, Z(G))$ is a Camina pair. Thus $Z(G) \subseteq [x, G]$ for all $x \in G - Z(G)$ and hence $\operatorname{Aut_z}(G) \leq \operatorname{Aut_c}(G)$. Since $|G/Z(G)| = p^4$ and $cl(G/Z(G)) = 2$, $|Z(\operatorname{Inn}(G))| = p^2$ and thus $|\operatorname{Aut_c}(G)| = p^5 > |\operatorname{Inn}(G)|$ by Lemma 4.2.1. This completes the proof of the theorem. □

4.3 — Applications

As a consequence of Theorem 4.2.3 , we can now obtain the following results of Yadav [55, Section 5] and Kalra and Gumber [34, Theorem 4.2]. It follows from [31] that all finite p-groups of order p^5, where p is an odd prime, are partitioned into ten isoclinism families; and from [20] that all finite 2-groups of order 2^5 are partitioned into eight isoclinism families. The i-th family is denoted as Φ_i. We would like to remark here that the derived groups of all groups in the tenth family of [31] are elementary abelian for $p \geq 5$, but, for $p = 3$, the derived groups are not elementary abelian because $|\alpha_2| = 9$. It follows that, for $p = 3$, the groups

$$\begin{aligned}
H &= \Phi_{10}(1^5) = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \alpha_4 \mid [\alpha_1, \alpha] = \alpha_2,\ [\alpha_2, \alpha] = \alpha_3,\ [\alpha_3, \alpha] = \alpha_4, \\
&\quad [\alpha_4, \alpha] = 1,\ [\alpha_1, \alpha_2] = \alpha_4, \alpha^3 = 1,\ [\alpha_1, \alpha_3] = 1,\ [\alpha_1, \alpha_4] = 1,\ [\alpha_2, \alpha_3] = 1, \\
&\quad [\alpha_2, \alpha_4] = 1,\ [\alpha_3, \alpha_4] = 1,\ \alpha_1^3\alpha_2^3\alpha_3 = 1,\ \alpha_2^3\alpha_3^3\alpha_4 = 1,\ \alpha_3^3\alpha_4^3 = 1,\ \alpha_4^3 = 1\rangle, \\
\\
H_1 &= \Phi_{10}(2111)a_0 = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \alpha_4 \mid [\alpha_1, \alpha] = \alpha_2,\ [\alpha_2, \alpha] = \alpha_3,\ [\alpha_3, \alpha] = \alpha_4, \\
&\quad [\alpha_4, \alpha] = 1,\ [\alpha_1, \alpha_2] = \alpha_4 = \alpha^3,\ [\alpha_1, \alpha_3] = 1,\ [\alpha_1, \alpha_4] = 1,\ [\alpha_2, \alpha_3] = 1, \\
&\quad [\alpha_2, \alpha_4] = 1,\ [\alpha_3, \alpha_4] = 1,\ \alpha_1^3\alpha_2^3\alpha_3 = 1,\ \alpha_2^3\alpha_3^3\alpha_4 = 1,\ \alpha_3^3\alpha_4^3 = 1,\ \alpha_4^3 = 1\rangle, \\
\\
H_2 &= \Phi_{10}(2111)a_2 = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \alpha_4 \mid [\alpha_1, \alpha] = \alpha_2,\ [\alpha_2, \alpha] = \alpha_3,\ [\alpha_3, \alpha] = \alpha_4, \\
&\quad [\alpha_4, \alpha] = 1,\ [\alpha_1, \alpha_2]^g = \alpha_4^g = \alpha^3,\ [\alpha_1, \alpha_3] = 1,\ [\alpha_1, \alpha_4] = 1,\ [\alpha_2, \alpha_3] = 1, \\
&\quad [\alpha_2, \alpha_4] = 1,\ [\alpha_3, \alpha_4] = 1,\ \alpha_1^3\alpha_2^3\alpha_3 = 1,\ \alpha_2^3\alpha_3^3\alpha_4 = 1,\ \alpha_3^3\alpha_4^3 = 1,\ \alpha_4^3 = 1\rangle \\
&\quad = \langle a, b, c | a^3 = b^9 = c^9 = 1, [b, c] = c^3, [a, c] = b^3, [b, a] = c\rangle
\end{aligned}$$

are not in the tenth family.

Theorem 4.3.1 *Let G be a finite p-group of order p^5,where p is an odd prime. Then* $\mathrm{Out_c}(G) \neq 1$ *if and only if G is isomorphic to one of the groups in Φ_7 or one of the groups in Φ_{10} for $p \geq 5$ or to H, H_1 or H_2.*

Proof. If G is any group from the first six families, then either $cl(G) < 3$ or $|Z(G)| > p$. There are two isoclinism families Φ_7 and Φ_8 consisting of groups of class 3; and there are two isoclinism families Φ_9 and Φ_{10} consisting of groups of class 4. The only group in eighth family is

$$\Phi_8(32) = \langle \alpha_1, \alpha_2, \beta \mid [\alpha_1, \alpha_2] = \beta = \alpha_1^p,\ [\alpha_1, \beta] = 1,\ [\alpha_2, \beta] = 1,\ \beta^{p^2} = 1,\ \alpha_2^{p^2} = 1\rangle$$

with $d(\Phi_8(32)) = 2$. Any group G in the seventh family is generated by α, α_1, β, and it is easy to see that the p-th power of any of these generators is either 1 or is in G'. It thus follows that $|\Phi(G)| = |G'|=p^2$ and consequently $d(G) = 3$. Any group G in the ninth family is minimally generated by α and α_1 with abelian commutator subgroup $G' = \langle [\alpha_1, \alpha]\gamma_3(G)\rangle = \langle \alpha_2, \gamma_3(G)\rangle$ and $\gamma_3(G)=\langle [\alpha_1, \alpha_2], [\alpha, \alpha_2], Z(G)\rangle = \langle \alpha_3^{-1}, Z(G)\rangle$. Thus α_1 commutes with G' and hence $|\alpha_1^G| = p$. It is easy to see that any element g of G is of the form $g'\alpha_1^k\alpha^j$, where $0 \leq j, k \leq p-1$ and $g' \in G'$, and hence $[\alpha_1, g] = [\alpha_1, g'\alpha_1^k\alpha^j]= [\alpha_1, \alpha^j] \in G' - \gamma_3(G)$. It follows that $Z(G)$ is not contained in $[x, G]$ for all $x \in G - G'$. Let G be any group in the tenth family. Any element $g \in G - G'$ is of the form $g = g'\alpha_1^l\alpha^m$, where $0 \leq l, m \leq p-1$ and $g' \in G'$. If $m \neq 0$, then $[g, \alpha_3]= [\alpha^m, \alpha_3] \in Z(G)$. Let $m = 0$ and $l \neq 0$. It is easy to see, by induction, that $[\alpha_1^l, \alpha_2] = [\alpha_1, \alpha_2]^l z_1z_2...z_{l-1}$, where $z_1, z_2, ..., z_{l-1} \in Z(G)$. Then

$$[g, \alpha_2] = [g'\alpha_1^l, \alpha_2] = [\alpha_1^l, \alpha_2] = [\alpha_1, \alpha_2]^l \in Z(G),$$

because $[\alpha_1, \alpha_2] \in Z(G)$. It follows that $Z(G) \subseteq [x, G]$ for all $x \in G - G'$. □

There are, in all, fifty one groups of order 32. Sag and Wamsley [46] have given minimal presentations of these groups. As mentioned in [46], the groups are in the same order in [46] and [20]. We denote the i-th group as G_i.

Theorem 4.3.2 *Let G be a finite group of order 32. Then* $\mathrm{Out_c}(G) \neq 1$ *if and only if either G is isomorphic to G_{44} or isomorphic to G_{45}.*

Proof. The first forty three groups are divided amongst the first five families. The families Φ_1, Φ_2, Φ_4 and Φ_5 contain groups of class ≤ 2. The third family contains ten groups $G_{23} - G_{32}$ of class 3, and each of these groups has center of order 4. The sixth family contains groups G_{44} and G_{45}. Both of these groups are of class 3, rank 3, and with centres of order 2. The seventh family contains groups $G_{46} - G_{48}$ of class

3 and rank 2. The last family contains 2-generated groups $G_{49} - G_{51}$ of maximal class. It is clear from [46] that in each of these groups, one generator, say x, is of order 16 and hence $|x^G| = 2$. If y is another generator, then $y^{-1}xy$ is respectively x^{15}, x^7 and x^{15} in the group G_{49},G_{50} and G_{51}. The center of each of these groups is $\{1, x^8\}$. For any element g in these groups , if $[x, g] = x^8$, then $g^{-1}xg = x^9$, which is not so. $\square$

CHAPTER 5

Non-Inner Automorphisms of order p and Class-Preserving Automorphisms

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5.1 — Introduction

Let G be a finite group. A map $f : G \to G$ is called a cocycle of G if $f(xy) = f(x)xf(y)x^{-1}$ for all $x, y \in G$. A cocycle f of G is called a local coboundary if for each $x \in G$, there exists an element $a_x \in G$ such that $f(x) = a_x^{-1}xa_xx^{-1}$; and is called a global coboundary, or simply a coboundary, if for all $x \in G$, there exists a fixed element $a \in G$ such that $f(x) = a^{-1}xax^{-1}$. A group G is said to enjoy the Hasse principle if every local coboundary of G is a coboundary. It is known that a finite group G enjoys the Hasse principle if and only if every class-preserving automorphism of G is an inner automorphism [44].

In section 5.2, we have decided the Hasse principle for some groups of order p^6, where p is an odd prime, using isoclinism families given by James [31], by giving alternate proofs of results of Narain and Karan [42] and Rai and Yadav [45]. We also classify some groups G of order p^6 such that G admits a non-inner automorphism of order p.

We first state some very useful results.

Proposition 5.1.1 ([35, Proposition 2.2]) *Let G be a direct product of its subgroups H and K. Then G enjoys the Hasse principle if and only if both H and K enjoy the Hasse principle.*

Lemma 5.1.2 ([55, Lemma 2.2]) *Let G be a finite p-group such that $Z(G) \subseteq [x, G]$ for all $x \in G - G'$. Then $|\operatorname{Aut}_c(G)| \geq |\operatorname{Aut}_z(G)||G/Z_2(G)|$.*

Lemma 5.1.3 ([55, Theorem 4.1]) *Let G and H be two finite isoclinic groups. Then $\operatorname{Aut}_c(G) \simeq \operatorname{Aut}_c(H)$.*

Lemma 5.1.4 ([55, Corollary 3.6]) *Let G be a finite p-group of class 2 such that G' is cyclic. Then $\operatorname{Out}_c(G) = 1$.*

Lemma 5.1.5 ([34, Theorem 5.2]) *A non-abelian group G of order p^6, where p is an odd prime, is a Camina group of class 2 if and only if $G \in \Phi_{15}$.*

Lemma 5.1.6 ([34, Theorem 4.4]) *Let G be a non-abelian group of order p^6. Then $\operatorname{Aut}_c(G) = \operatorname{Aut}_z(G)$ if and only if either $G' = Z(G)$ and $Z(G)$ is cyclic or G is a Camina p-group of nilpotency class 2.*

5.2 — Non-inner automorphisms of order p and class-preserving automorphisms

In this section, we use the classification of groups of order p^6, where p is an odd prime, given by James [31]. Each group of order p^6 in James [31] is presented in terms of certain generators and relations. Missing commutators are trivial, that is, all relations of the form $[a, b] = 1$ (with a, b generators) are excluded from the presentation. Throughout ν denotes the smallest positive integer which is a non-quadratic residue (mod p) and $\mathbf{g}$ denotes the smallest positive integer which is a primitive root (mod p). In the presentation of groups, the symbol $\alpha_{i+1}^{(p)}$ denotes $\alpha_{i+1}^{p}\alpha_{i+2}^{\binom{p}{2}} \dots \alpha_{i+k}^{\binom{p}{k}} \dots \alpha_{i+p}$, where i is a positive integer. We shall pick one group G from each isoclinism family and check whether $\mathrm{Out_c}(G)$ is trivial or not. If G and H are two isoclinic groups, then $\mathrm{Inn}(G) \simeq \mathrm{Inn}(H)$, $G' \simeq H'$ and by lemma 5.1.3, we have $\mathrm{Aut_c}(G) \simeq \mathrm{Aut_c}(H)$.

Theorem 5.2.1 *Let G be a group from one of the families Φ_1 to Φ_{10}. Then* $\mathrm{Out_c}(G) \neq 1$ *if and only if G is isoclinic to one of the groups in the family Φ_7 or Φ_{10} .*

Proof. The result follows from [55, Theorem 5.5] and Lemma 5.1.3. □

Theorem 5.2.2 *If G is any group from family Φ_{12}, then* $\mathrm{Out_c}(G) = 1$.

Proof. Consider the group $G = \Phi_{12}(2211)b = \Phi_2(21) \times \Phi_2(21)$. Since $\Phi_2(21)$ is a group of order p^3, it enjoys the Hasse principle. The result then follows by lemma 5.1.1. □

Theorem 5.2.3 *If G is any group from family Φ_{14}, then* $\mathrm{Out_c}(G) = 1$.

Proof. Consider the group

$$G = \Phi_{14}(222) = \langle \alpha_1, \alpha_2, \beta \mid [\alpha_1, \alpha_2] = \beta,\ [\alpha_1, \beta] = 1,\ [\alpha_2, \beta] = 1,\ \alpha_1^{p^2} = \alpha_2^{p^2} = \beta^{p^2} = 1 \rangle.$$

Since G' is cyclic and the nilpotency class of G is 2, the result follows by lemma 5.1.4. □

Theorem 5.2.4 *If G is any group from family Φ_{15}, then* $\mathrm{Out_c}(G) \neq 1$.

Proof. Consider the group $G = \Phi_{15}(1^6) = \langle \alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta_1, \beta_2 \mid [\alpha_1, \alpha_2] = \beta_1, [\alpha_1, \alpha_3] = \beta_2, [\alpha_3, \alpha_4] = \beta_1, [\alpha_2, \alpha_4] = \beta_2^{\mathbf{g}}, [\alpha_1, \alpha_4] = 1, [\alpha_1, \beta_1] = 1, [\alpha_1, \beta_2] = 1, [\alpha_2, \alpha_3] = 1, [\alpha_2, \beta_1] = 1, [\alpha_2, \beta_2] = 1, [\alpha_3, \beta_1] = 1, [\alpha_3, \beta_2] = 1, [\alpha_4, \beta_1] = 1, [\alpha_4, \beta_2] = 1, [\beta_1, \beta_2] = 1, \alpha_1^p = \alpha_2^p = \alpha_3^p = \alpha_4^p = \beta_1^p = \beta_2^p = 1 \rangle$. Observe that $Z(G) = G' \simeq C_p \times C_p$. It is easy to see that $\Phi(G) = G^pG' = G'$ and every element $g \in G$ can be written as $g = g'\alpha_1^i\alpha_2^j\alpha_3^k\alpha_4^l$, where $0 \leq i, j, k, l \leq p-1$ and $g' \in G'$. Since $|G'| = p^2$, $d(G) = 4$. By Lemma 5.1.5, G is a Camina p-group of nilpotency class 2 and thus $\mathrm{Aut_z}(G) = \mathrm{Aut_c}(G)$ by Lemma 5.1.6. Now $\mathrm{Aut_z}(G) = \mathrm{Aut_z^z}(G) = \mathrm{Hom}(G/Z(G)G', Z(G)) = \mathrm{Hom}(G/G', Z(G)) = \mathrm{Hom}(C_p \times C_p \times C_p \times C_p, C_p \times C_p) = \mathrm{Aut_c}(G)$ implies that $|\mathrm{Aut_c}(G)| = p^8$. Hence $\mathrm{Out_c}(G) \neq 1$ and G has a non-inner central automorphism of order p. □

Theorem 5.2.5 *If G is any group from family Φ_{16}, then* $\mathrm{Out_c}(G) = 1$.

Proof. Consider the group $G = \Phi_{16}(1^6) = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \beta, \gamma \mid [\alpha_1, \alpha] = \alpha_2, [\alpha_2, \alpha] = \alpha_3, [\beta, \alpha] = \gamma, [\alpha_3, \alpha] = 1, [\gamma, \alpha] = 1, [\alpha_2, \alpha_1] = 1, [\alpha_3, \alpha_1] = 1, [\beta, \alpha_1] = 1, [\gamma, \alpha_1] = 1, [\alpha_3, \alpha_2] = 1, [\beta, \alpha_2] = 1, [\gamma, \alpha_2] = 1, [\beta, \alpha_3] = 1, [\gamma, \alpha_3] = 1, [\gamma, \beta] = 1, \alpha_1^{(p)} = \alpha^p = \alpha_2^p = \alpha_3^p = \beta^p = \gamma^p = 1 \rangle$. Observe that $Z(G) = \langle \alpha_3, \gamma \rangle \simeq C_p \times C_p \leq G'$. It is easy to see that $\Phi(G) = G^pG' = G'$ and every element $g \in G$ can be written as $g = g'\alpha^k\alpha_1^l\beta^m$, where for $p \geq 5$, $0 \leq k, l, m \leq p-1$, and for $p = 3$, $0 \leq k, m \leq p-1$, $0 \leq l \leq p^2 - 1$ and $g' \in G'$. Since $|G'| = p^3$, $d(G) = 3$. Now G has an abelian maximal subgroup $\langle G', \alpha_1, \beta \rangle$ of order p^5 and hence $\mathrm{Out_c}(G) = 1$ by Lemma 4.1.1. Also observe that $\mathrm{Aut_z}(G) = \mathrm{Aut_z^z}(G) = \mathrm{Hom}(G/Z(G)G', Z(G)) = \mathrm{Hom}(G/G', Z(G)) = \mathrm{Hom}(C_p \times C_p \times C_p, C_p \times C_p)$ and thus G has a non-inner central automorphism of order p. $\square$

Theorem 5.2.6 *If G is any group from family Φ_{21}, then* $\mathrm{Out_c}(G) \neq 1$.

Proof. It follows from [53, Proposition 5.8] that G is isoclinic to an almost Camina group and $|\mathrm{Aut_c}(G)| = p^8$. Therefore $\mathrm{Out_c}(G) \neq 1$. $\square$

Theorem 5.2.7 *If G is any group from family Φ_{22}, then* $\mathrm{Out_c}(G) = 1$.

Proof. Consider the group $G = \Phi_{22}(1^6) = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2 \mid [\alpha_1, \alpha] = \alpha_2, [\alpha_2, \alpha] = \alpha_3, [\beta_1, \beta_2] = \alpha_3, [\alpha_3, \alpha] = 1, [\beta_1, \alpha] = 1, [\beta_2, \alpha] = 1, [\alpha_2, \alpha_1] = 1, [\alpha_3, \alpha_1] = 1, [\beta_1, \alpha_1] = 1, [\beta_2, \alpha_1] = 1, [\alpha_3, \alpha_2] = 1, [\beta_1, \alpha_2] = 1, [\beta_2, \alpha_2] = 1, [\beta_1, \alpha_3] = 1, [\beta_2, \alpha_3] = 1, \alpha^p = \alpha_1^{(p)} = \alpha_2^p = \alpha_3^p = \beta_1^p = \beta_2^p = 1\rangle$. Observe that $Z(G) = \langle \alpha_3 \rangle \simeq C_p < G'$. It is easy to see that $\Phi(G) = G^pG' = G'$ and every element $g \in G$ can be written as $g = g'\alpha^i\alpha_1^j\beta_1^k\beta_2^l$, where for $p \geq 5$, $0 \leq i, j, k, l \leq p-1$, and for $p = 3$, $0 \leq i, k, l \leq p-1$, $0 \leq j \leq p^2 - 1$ and $g' \in G'$. Since $|G'| = p^2$, $d(G) = 4$. Since $C_G(\alpha) = \langle \alpha, \alpha_3, \beta_1, \beta_2 \rangle$ is of order p^4; and $C_G(\alpha_1) = \langle \alpha_1, \alpha_2, \alpha_3, \beta_1, \beta \rangle$, $C_G(\beta_1) = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \beta_1 \rangle$ and $C_G(\beta_2) = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \beta_2 \rangle$ are of order p^5,

$$|\mathrm{Aut_c}(G)| \leq |\alpha^G|\, |\alpha_1^G|\, |\beta_1^G|\, |\beta_2^G| = p^2 p\, p\, p = p^5.$$

Hence $\mathrm{Out_c}(G) = 1$. It follows by Lemma 4.2.1 that G has a non-inner central automorphism; and since $\mathrm{Aut_z}(G) = \mathrm{Aut_z^z}(G) = \mathrm{Hom}(G/Z(G)G', Z(G)) = \mathrm{Hom}(G/G', Z(G)) = \mathrm{Hom}(C_p \times C_p \times C_p \times C_p, C_p)$, G has a non-inner central automorphism of order p. $\square$

Theorem 5.2.8 *If G is any group from family Φ_{24}, then* $\mathrm{Out_c}(G) \neq 1$.

Proof. Consider the group $G = \Phi_{24}(1^6) = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta \mid [\alpha_1, \alpha] = \alpha_2, [\alpha_2, \alpha] = \alpha_3, [\alpha_3, \alpha] = \alpha_4, [\alpha_1, \beta] = \alpha_4, [\alpha_4, \alpha] = 1, [\beta, \alpha] = 1, [\alpha_2, \alpha_1] = 1, [\alpha_3, \alpha_1] = 1, [\alpha_4, \alpha_1] =$

$1, [\alpha_3, \alpha_2] = 1, [\alpha_4, \alpha_2] = 1, [\beta, \alpha_2] = 1, [\alpha_4, \alpha_3] = 1, [\beta, \alpha_3] = 1, [\beta, \alpha_4] = 1, \alpha^p = \alpha_1^{(p)} = \alpha_2^{(p)} = \alpha_3^{(p)} = \alpha_4^p = \beta^p = 1\rangle$. Observe that $Z(G) = \langle \alpha_4 \rangle$ and $G' = \langle \alpha_2, \alpha_3, \alpha_4 \rangle$. It is easy to see that $\Phi(G) = G^pG' = G'$ and every element $g \in G$ can be written as $g = g'\alpha^k\alpha_1^l\beta^m$, where for $p \geq 5$, $0 \leq k, l, m \leq p-1$, and for $p = 3$, $0 \leq k, m \leq p-1$, $0 \leq l \leq p^2 - 1$ and $g' \in G'$. Since $|G'| = p^3$, $d(G) = 3$. We claim that $Z(G) \subseteq [g, G]$ for all $g \in G - G'$. Suppose that $k, l, m \neq 0$. Then

$$\begin{aligned}
[g, \alpha_3] &= [g'\alpha^k\alpha_1^l\beta^m, \alpha_3] \\
&= [g', \alpha_3]\ [g', \alpha_3, \alpha^k\alpha_1^l\beta^m]\ [\alpha^k\alpha_1^l\beta^m, \alpha_3] \\
&= [\alpha^k\alpha_1^l\beta^m, \alpha_3] \\
&= [\alpha^k, \alpha_3]\ [\alpha^k, \alpha_3, \alpha_1^l\beta^m]\ [\alpha_1^l\beta^m, \alpha_3] \\
&= [\alpha^k, \alpha_3]\ [[\alpha^k, \alpha_3], \alpha_1^l\beta^m]\ [\alpha_1^l, \alpha_3]\ [\alpha_1^l, \alpha_3, \beta^m]\ [\beta^m, \alpha_3] \\
&= [\alpha, \alpha_3]^k[[\alpha, \alpha_3]^k, \alpha_1^l\beta^m] \\
&= \alpha_4^{-k}[\alpha_4^{-k}, \alpha_1^l\beta^m] \\
&= \alpha_4^{-k} \in Z(G).
\end{aligned}$$

Similarly, considering all other possible cases on k, l, m (zero or non-zero), and since $|Z(G)| = p$, we can show that $Z(G) \subseteq [g, G]$ for all $g \in G - G'$. It follows that $\mathrm{Aut_z}(G) < \mathrm{Aut_c}(G)$. Also Observe that

$$\begin{aligned}
\mathrm{Aut_z}(G) &= \mathrm{Aut_z^z}(G) = \mathrm{Hom}(G/Z(G)G', Z(G)) \\
&= \mathrm{Hom}(G/G', Z(G)) = \mathrm{Hom}(C_p \times C_p \times C_p, C_p).
\end{aligned}$$

By Lemma 4.2.1,

$$|\mathrm{Aut_c}(G)| = p^3 |\mathrm{Aut_c}(G) \cap \mathrm{Aut_z}(G)| = p^3p^3 = p^6.$$

Hence $\mathrm{Out_c}(G) \neq 1$ and G has a non-inner central automorphism of order p. $\square$

Theorem 5.2.9 *If G is any group from one of the families Φ_{25} or Φ_{26} or Φ_{28} or Φ_{29} or $\Phi_{40} - \Phi_{43}$, then* $\mathrm{Out}_{\mathrm{c}}(G) = 1$.

Proof. By Theorem 2.3.2, $|\operatorname{Aut}_{\mathrm{z}}(G)| = |Z(\operatorname{Inn}(G))| = p^2$. It now follows from Lemma 4.2.1, that

$$|\operatorname{Aut}_{\mathrm{c}}(G)| = p^3|\operatorname{Aut}_{\mathrm{c}}(G) \cap \operatorname{Aut}_{\mathrm{z}}(G)| = p^3p^2 = p^5 = |\operatorname{Inn}(G)|,$$

and hence $\mathrm{Out}_{\mathrm{c}}(G) = 1$. $\square$

Theorem 5.2.10 *If G is any group from family $\Phi_{31} - \Phi_{34}$, then* $\mathrm{Out}_{\mathrm{c}}(G) = 1$.

Proof. Consider the group $G = \Phi_{31}(1^6) = \langle \alpha, \alpha_1, \alpha_2, \beta_1, \beta_2, \gamma \mid [\alpha_1, \alpha] = \beta_1, [\alpha_2, \alpha] = \beta_2, [\alpha_1, \beta_1] = \gamma, [\alpha_2, \beta_2] = \gamma^y, [\beta_1, \alpha] = 1, [\beta_2, \alpha] = 1, [\gamma, \alpha] = 1, [\alpha_2, \alpha_1] = 1, [\beta_2, \alpha_1] = 1, [\gamma, \alpha_1] = 1, [\beta_1, \alpha_2] = 1, [\gamma, \alpha_2] = 1, [\beta_2, \beta_1] = 1, [\gamma, \beta_1] = 1, [\gamma, \beta_2] = 1, \alpha^p = \alpha_1^p = \alpha_2^p = \beta_1^p = \beta_2^p = \gamma^p = 1\rangle$, where $y = \nu^x$. Observe that $|Z(G)| = p$, nilpotency class of G is 3, $\Phi(G) = G^pG' = G'$, and every element $g \in G$ can be written as $g'\alpha^i\alpha_1^j\alpha_2^k$, where $0 \le i, j, k \le p-1$ and $g' \in G'$. Since $|G'| = p^3$, $d(G) = 3$ and thus $\operatorname{Aut}_{\mathrm{z}}(G) = \operatorname{Aut}_{\mathrm{z}}^{\mathrm{z}}(G) = \operatorname{Hom}(G/Z(G)G', Z(G)) = \operatorname{Hom}(G/G', Z(G)) = \operatorname{Hom}(C_p \times C_p \times C_p, C_p)$. It is easy to see that the map $\tau : G \to G$ defined by $\tau(\alpha) = \alpha\gamma$, $\tau(\alpha_1) = \alpha_1$, $\tau(\alpha_2) = \alpha_2$, $\tau(\beta_1) = \beta_1$, $\tau(\beta_2) = \beta_2$, $\tau(\gamma) = \gamma$ is an automorphism of G. Observe that

$$\begin{aligned} g^{-1}\tau(g) &= (g'\alpha^i\alpha_1^j\alpha_2^k)^{-1}\tau(g'\alpha^i\alpha_1^j\alpha_2^k) \\ &= \alpha_2^{-k}\alpha_1^{-j}\alpha^{-i}(g')^{-1}\tau(g')\tau(\alpha^i)\tau(\alpha_1^j)\tau(\alpha_2^k) \\ &= \alpha_2^{-k}\alpha_1^{-j}\alpha^{-i}(g')^{-1}g'(\alpha\gamma)^i\alpha_1^j\alpha_2^k \\ &= \gamma^i \in Z(G). \end{aligned}$$

It follows that τ is a central automorphism. We claim that τ is not a class-preserving automorphism. On the contrary, suppose that τ is class-preserving. Then $\tau(\alpha) =$

$g^{-1}\alpha g = \alpha\gamma$ for some $g \in G$ and thus $[g, \alpha] = \gamma^{-1} \in Z(G)$. If $g = g'\alpha^i\alpha_1^j\alpha_2^k$ for some $g' \in G'$ and $0 \leq i, j, k \leq p-1$, then observe that

$$\begin{aligned}
[g, \alpha] &= [g'\alpha^i\alpha_1^j\alpha_2^k, \alpha] \\
&= [g', \alpha]\ [g', \alpha, \alpha^i\alpha_1^j\alpha_2^k]\ [\alpha^i\alpha_1^j\alpha_2^k, \alpha] \\
&= [\alpha^i\alpha_1^j\alpha_2^k, \alpha] \\
&= [\alpha_1^j\alpha_2^k, \alpha] \\
&= [\alpha_1^j, \alpha]\ [\alpha_1^j, \alpha, \alpha_2^k]\ [\alpha_2^k, \alpha].
\end{aligned}$$

It follows that

$$\begin{aligned}
1 &= [\,[g, \alpha], \alpha_1] \\
&= [\,[\alpha_1^j, \alpha]\ [\alpha_1^j, \alpha, \alpha_2^k]\ [\alpha_2^k, \alpha], \alpha_1] \\
&= [\,[\alpha_1^j, \alpha]\ [\alpha_1^j, \alpha, \alpha_2^k], \alpha_1]\ [\,[\alpha_1^j, \alpha]\ [\alpha_1^j, \alpha, \alpha_2^k], \alpha_1, [\alpha_2^k, \alpha]\,]\ [\,[\alpha_2^k, \alpha], \alpha_1] \\
&= [\,[\alpha_1^j, \alpha], \alpha_1]\ [\,[\alpha_2^k, \alpha], \alpha_1] \\
&= [\alpha_1^j, \alpha, \alpha_1]\ [\alpha_2^k, \alpha, \alpha_1].
\end{aligned}$$

It is easy to see, by induction, that $[\alpha_1^j, \alpha] = \beta_1^j z$ and $[\alpha_2^k, \alpha] = \beta_2^k z_1$ for some $z, z_1 \in Z(G)$. Therefore

$$\begin{aligned}
1 &= [\alpha_1^j, \alpha, \alpha_1]\ [\alpha_2^k, \alpha, \alpha_1] \\
&= [\beta_1^j z, \alpha_1]\ [\beta_2^k z_1, \alpha_1] \\
&= [\beta_1, \alpha_1]^j [\beta_2^k, \alpha_1] \\
&= \gamma^{-j},
\end{aligned}$$

which is a contradiction unless $j = 0$. Observe that if both j and k are 0, then $[g, \alpha] = [g'\alpha^i, \alpha] = 1$, which is not so. Now if $j = 0$ and $k \neq 0$, then

$$[g, \alpha] = [g'\alpha^i\alpha_2^k, \alpha] = [\alpha_2^k, \alpha] = \beta_2^k z_1.$$

It follows that $\beta_2^k \in Z(G)$, a contradiction because $\beta_2 \in G' - Z(G)$. Therefore, there exists a central automorphism which is not class-preserving. Since $|Z(\mathrm{Inn}(G))| = p^2$

and $|\operatorname{Aut}_{\mathrm{z}}(G)| = p^3$, it follows by Lemma 4.2.1 that

$$|\operatorname{Aut}_{\mathrm{c}}(G)| = p^3 |\operatorname{Aut}_{\mathrm{c}}(G) \cap \operatorname{Aut}_{\mathrm{z}}(G)| = p^3 p^2 = p^5 = |\operatorname{Inn}(G)|.$$

Hence $\operatorname{Out}_{\mathrm{c}}(G) = 1$ and G has a non-inner central automorphism of order p.

Similarly, we can prove that $\operatorname{Out}_{\mathrm{c}}(G) = 1$ if G is any group from family $\Phi_{32} - \Phi_{34}$. □

Theorem 5.2.11 *If G is any group from family Φ_{35}, then* $\operatorname{Out}_{\mathrm{c}}(G) = 1$.

Proof. Consider the group $G = \Phi_{35}(1^6) = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5 \mid [\alpha_1, \alpha] = \alpha_2, [\alpha_2, \alpha] = \alpha_3, [\alpha_3, \alpha] = \alpha_4, [\alpha_4, \alpha] = \alpha_5, [\alpha_1, \alpha_2] = \alpha_5, [\alpha_5, \alpha] = 1, [\alpha_3, \alpha_1] = 1, [\alpha_4, \alpha_1] = 1, [\alpha_5, \alpha_1] = 1, [\alpha_3, \alpha_2] = 1, [\alpha_4, \alpha_2] = 1, [\alpha_5, \alpha_2] = 1, [\alpha_4, \alpha_3] = 1, [\alpha_5, \alpha_3] = 1, [\alpha_5, \alpha_4] = 1, \alpha^p = \alpha_1^{(p)} = \alpha_2^{(p)} = \alpha_3^{(p)} = \alpha_4^{(p)} = \alpha_5^{(p)} = 1 \rangle$. Since $\langle G', \alpha_1 \rangle$ is a maximal abelian subgroup of order p^5, $\operatorname{Out}_{\mathrm{c}}(G) = 1$ by Lemma 4.1.1. Also, since G is of maximal class, $|Z(\operatorname{Inn}(G))| = p$ and $\operatorname{Aut}_{\mathrm{z}}(G) = \operatorname{Aut}_{\mathrm{z}}^{\mathrm{z}}(G) = \operatorname{Hom}(G/Z(G)G', Z(G)) = \operatorname{Hom}(G/G', Z(G)) = \operatorname{Hom}(C_p \times C_p, C_p)$. Hence G has a non-inner central automorphism of order p. □

Theorem 5.2.12 *If G is any group from family Φ_{36}, then* $\operatorname{Out}_{\mathrm{c}}(G) \neq 1$.

Proof. Let G be the group $\Phi_{36}(1^6) = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5 \mid [\alpha_1, \alpha] = \alpha_2, [\alpha_2, \alpha] = \alpha_3, [\alpha_3, \alpha] = \alpha_4, [\alpha_4, \alpha] = \alpha_5, [\alpha_1, \alpha_2] = \alpha_5, [\alpha_5, \alpha] = 1, [\alpha_3, \alpha_1] = 1, [\alpha_4, \alpha_1] = 1, [\alpha_5, \alpha_1] = 1, [\alpha_3, \alpha_2] = 1, [\alpha_4, \alpha_2] = 1, [\alpha_5, \alpha_2] = 1, [\alpha_4, \alpha_3] = 1, [\alpha_5, \alpha_3] = 1, [\alpha_5, \alpha_4] = 1, \alpha^p = \alpha_1^{(p)} = \alpha_2^{(p)} = \alpha_3^{(p)} = \alpha_4^{(p)} = \alpha_5^{(p)} = 1 \rangle$. Observe that because G is of maximal class, therefore $d(G) = 2$, $|Z(G)| = p$, $|Z(\operatorname{Inn}(G))| = p$ and $\operatorname{Aut}_{\mathrm{z}}(G) = \operatorname{Aut}_{\mathrm{z}}^{\mathrm{z}}(G) = \operatorname{Hom}(G/Z(G)G', Z(G)) = \operatorname{Hom}(G/G', Z(G)) = \operatorname{Hom}(C_p \times C_p, C_p)$. We claim that $Z(G) \subseteq [g, G]$ for all $g \in G - G'$. It is easy to see that every element $g \in G$

can be written as $g = g'\alpha_1^k\alpha^l$, where for $p \geq 7$, $0 \leq k, l \leq p-1$; for $p = 5$, $0 \leq l \leq p-1$, $0 \leq k \leq p^2-1$, and for $p = 3$, $0 \leq l \leq p-1$, $0 \leq k \leq p^3-1$ and $g' \in G'$. Let $k, l \neq 0$. Then

$$[g, \alpha_4] = [g'\alpha_1^k\alpha^l, \alpha_4] = [\alpha_1^k, \alpha_4]\,[\alpha^l, \alpha_4] = [\alpha^l, \alpha_4] = \alpha_5^{-l} \in Z(G).$$

Next let $k \neq 0$ and $l = 0$. Then

$$[g, \alpha_2] = [g'\alpha_1^k\alpha^l, \alpha_2] = [\alpha_1^k, , \alpha_2]\,[\alpha^l, \alpha_2] = \alpha_5^k \in Z(G).$$

Finally, let $l \neq 0$ and $k = 0$. Then

$$[g, \alpha_4] = [g'\alpha^l, \alpha_4] = [\alpha^l, \alpha_4] = \alpha_5^{-l} \in Z(G).$$

Since $|Z(G)| = p$, it follows that $Z(G) \subseteq [g, G]$ for all $g \in G - G'$ and thus $\mathrm{Aut_z}(G) < \mathrm{Aut_c}(G)$. By Lemma 4.2.1,

$$|\mathrm{Aut_c}(G)| = p^4|\mathrm{Aut_c}(G) \cap \mathrm{Aut_z}(G)| = p^4p^2 = p^6.$$

Hence $\mathrm{Out_c}(G) \neq 1$ and G has a non-inner central automorphism of order p. □

Theorem 5.2.13 *If G is any group from family Φ_{38}, then* $\mathrm{Out_c}(G) \neq 1$.

Proof. Considering the group $G = \Phi_{38}(1^6) = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5 \mid [\alpha_1, \alpha] = \alpha_2, [\alpha_2, \alpha] = \alpha_3, [\alpha_3, \alpha] = \alpha_4, [\alpha_4, \alpha] = \alpha_5, [\alpha_1, \alpha_2] = \alpha_4\alpha_5^{-1}, [\alpha_1, \alpha_3] = \alpha_5, , [\alpha_5, \alpha] = 1, [\alpha_4, \alpha_1] = 1, [\alpha_5, \alpha_1] = 1, [\alpha_3, \alpha_2] = 1, [\alpha_4, \alpha_2] = 1, [\alpha_5, \alpha_2] = 1, [\alpha_4, \alpha_3] = 1, [\alpha_5, \alpha_3] = 1, [\alpha_5, \alpha_4] = 1, \alpha^p = \alpha_1^{(p)} = \alpha_2^p = \alpha_3^p = \alpha_4^p = \alpha_5^p = 1\rangle$, we can show, as in Theorem 5.2.12, that $\mathrm{Aut_z}(G) < \mathrm{Aut_c}(G)$. It follows by Lemma 5.1.2 that

$$|\mathrm{Aut_c}(G)| \geq p^2|G/Z_2(G)| = p^2p^4 = p^6$$

and hence $\mathrm{Out_c}(G) \neq 1$ and G has a non-inner central automorphism of order p. □

Theorem 5.2.14 *If G is any group from family Φ_{30}, then* $\mathrm{Out_c}(G) \neq 1$.

Proof. Consider the group $G = \Phi_{30}(1^6) = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta \mid [\alpha_1, \alpha] = \alpha_2, [\alpha_2, \alpha] = \alpha_3, [\alpha_3, \alpha] = \alpha_4, [\alpha_1, \beta] = \alpha_3, [\alpha_2, \beta] = \alpha_4, [\alpha_4, \alpha] = 1, [\beta, \alpha] = 1, [\alpha_2, \alpha_1] = 1, [\alpha_3, \alpha_1] = 1, [\alpha_4, \alpha_1] = 1, [\alpha_3, \alpha_2] = 1, [\alpha_4, \alpha_2] = 1, [\alpha_4, \alpha_3] = 1, [\beta, \alpha_3] = 1, [\beta, \alpha_4] = 1, \alpha^p = \alpha_1^{(p)} = \alpha_2^{(p)} = \alpha_3^p = \alpha_4^p = \beta^p = 1\rangle$. Observe that $d(G) = 3, Z(G) = \langle \alpha_4 \rangle$, nilpotency class of G is 4, $\Phi(G) = G^pG' = G'$, and every element $g \in G$ can be written as $g'\alpha^i\alpha_1^j\beta^k$, where for $p \geq 5$, $0 \leq i, j, k \leq p-1$, and for $p = 3$, $0 \leq i, k \leq p-1$, $0 \leq j \leq p^2-1$ and $g' \in G'$. It is easy to see that the map $\tau : G \to G$ defined by $\tau(\alpha) = \alpha$, $\tau(\alpha_1) = \alpha_1$, $\tau(\alpha_2) = \alpha_2$, $\tau(\alpha_3) = \alpha_3$, $\tau(\alpha_4) = \alpha_4$, $\tau(\beta) = \beta\alpha_4$ is an automorphism of G. Observe that

$$
\begin{aligned}
g^{-1}\tau(g) &= (g'\alpha^i\alpha_1^j\beta^k)^{-1}\tau(g'\alpha^i\alpha_1^j\beta^k) \\
&= \beta^{-k}\alpha_1^{-j}\alpha^{-i}(g')^{-1}\tau(g')\tau(\alpha^i)\tau(\alpha_1^j)\tau(\beta^k) \\
&= \beta^{-k}\alpha_1^{-j}\alpha^{-i}(g')^{-1}g'\alpha^i\alpha_1^j(\beta\alpha_4)^k \\
&= \alpha_4^k \in Z(G).
\end{aligned}
$$

It follows that τ is a central automorphism. We claim that τ is a class-preserving automorphism. Observe that

$$
\begin{aligned}
[g, \alpha_3^l] &= [g'\alpha^i\alpha_1^j\beta^k, \alpha_3^l] \\
&= [g', \alpha_3^l]\ [g', \alpha_3^l, \alpha^i\alpha_1^j\beta^k]\ [\alpha^i\alpha_1^j\beta^k, \alpha_3^l] \\
&= [\alpha^i\alpha_1^j\beta^k, \alpha_3^l] \\
&= [\alpha^i, \alpha_3^l] \\
&= \alpha_4^{-il},
\end{aligned}
$$

where $0 \leq l \leq p-1$. Let i be non-zero modulo p, then $Z(G) \leq [g, G]$. Thus τ maps g to a conjugate of g.

Now, if $i \equiv 0 \pmod{p}$, then

$$\begin{aligned}
[g, \alpha_2^r] &= [g'\alpha_1^j\beta^k, \alpha_2^r] \\
&= [g', \alpha_2^r]\,[g', \alpha_2^r, \alpha_1^j\beta^k]\,[\alpha_1^j\beta^k, \alpha_2^r] \\
&= [\alpha_1^j\beta^k, \alpha_2^r] \\
&= [\beta^k, \alpha_2^r] \\
&= \alpha_4^{-kr},
\end{aligned}$$

where $0 \leq r \leq p-1$. It implies that if k is non zero modulo p, then τ maps g to a conjugate of g. Finally, if $k \equiv 0 \pmod{p}$, then $\tau(g'\alpha_1^j) = g'\alpha_1^j$. Thus for every $g \in G$, $\tau(g)$ is a conjugate of g. Hence τ is a class- preserving automorphism.

Next, suppose that τ is an inner automorphism. Since τ is a central automorphism , it is induced by some element of $Z_2(G)$. It is easy to see that $Z_2(G) = \langle \alpha_3, \alpha_4 \rangle$. Let us assume that τ is induced by α_3^s, where $0 \leq s \leq p-1$, then

$$\tau_{\alpha_3^s}(\beta) = \alpha_3^{-s}\beta\alpha_3^s = \beta,$$

a contradiction to the definition of τ. Thus τ is a non-inner class-preserving automorphism. Hence $\mathrm{Out_c}(G) \neq 1$. □

Theorem 5.2.15 *If G is any group from family* Φ_{37}, *then* $\mathrm{Out_c}(G) = 1$.

Proof. Let G be the group $\Phi_{37}(1^6) = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5 \mid [\alpha_1, \alpha] = \alpha_2, [\alpha_2, \alpha] = \alpha_3, [\alpha_3, \alpha] = \alpha_4, [\alpha_2, \alpha_3] = \alpha_5, [\alpha_3, \alpha_1] = \alpha_5, [\alpha_4, \alpha_1] = \alpha_5, [\alpha_5, \alpha] = 1, [\alpha_2, \alpha_1] = 1, [\alpha_4, \alpha] = 1, [\alpha_5, \alpha_1] = 1, [\alpha_4, \alpha_2] = 1, [\alpha_5, \alpha_2] = 1, [\alpha_4, \alpha_3] = 1, [\alpha_5, \alpha_3] = 1, [\alpha_5, \alpha_4] = 1, \alpha^p = \alpha_1^p = \alpha_2^p = \alpha_3^p = \alpha_4^p = \alpha_5^p = 1 \rangle$. Observe that $d(G) = 2, Z(G) = \langle \alpha_5 \rangle$, nilpotency class of G is 5, $\Phi(G) = G^pG' = G'$, and every element $g \in G$ can be written as $g'\alpha^i\alpha_1^j$, where $0 \leq i, j \leq p-1$ and $g' \in G'$. It is easy to see that the map $\tau : G \to G$

defined by $\tau(\alpha) = \alpha\alpha_5$, $\tau(\alpha_1) = \alpha_1$, $\tau(\alpha_2) = \alpha_2$, $\tau(\alpha_3) = \alpha_3$, $\tau(\alpha_4) = \alpha_4$, $\tau(\alpha_5) = \alpha_5$ is an automorphism of G. Observe that

$$\begin{aligned} g^{-1}\tau(g) &= (g'\alpha^i\alpha_1^j)^{-1}\tau(g'\alpha^i\alpha_1^j) \\ &= \alpha_1^{-j}\alpha^{-i}(g')^{-1}\tau(g')\tau(\alpha^i)\tau(\alpha_1^j) \\ &= \alpha_1^{-j}\alpha^{-i}(g')^{-1}g'(\alpha\alpha_5)^i\alpha_1^j \\ &= \alpha_5^i \in Z(G). \end{aligned}$$

It follows that τ is a central automorphism. We claim that τ is not a class-preserving automorphism. On the contrary, suppose that τ is class-preserving. Then $\tau(\alpha) = g^{-1}\alpha g = \alpha\alpha_5$ for some $g \in G$ and thus $[g, \alpha] = \alpha_5^{-1} \in Z(G)$. If $g = g'\alpha^i\alpha_1^j = \alpha_2^k\alpha_3^l\alpha_4^r\alpha_5^s\alpha^i\alpha_1^j$ for some $g' \in G'$ and $0 \leq k, l, r, s, i, j \leq p-1$, then observe that

$$\begin{aligned} [g, \alpha] &= [\alpha_2^k\alpha_3^l\alpha_4^r\alpha_5^s\alpha^i\alpha_1^j, \alpha] \\ &= [\alpha_2^k\alpha_3^l\alpha_4^r\alpha_5^s\alpha^i, \alpha]\ [\alpha_2^k\alpha_3^l\alpha_4^r\alpha_5^s\alpha^i, \alpha, \alpha_1^j]\ [\alpha_1^j, \alpha] \\ &= [\alpha_2^k\alpha_3^l\alpha_4^r\alpha_5^s, \alpha][\alpha_2^k\alpha_3^l\alpha_4^r\alpha_5^s, \alpha, \alpha_1^j]\ [\alpha_1^j, \alpha] \\ &= [\alpha_2^k\alpha_3^l, \alpha][\alpha_2^k\alpha_3^l, \alpha, \alpha_1^j]\ [\alpha_1^j, \alpha] \\ &= [\alpha_2^k, \alpha][\alpha_3^l, \alpha]\ [\alpha_2^k\alpha_3^l, \alpha, \alpha_1^j][\alpha_1^j, \alpha] \end{aligned}$$

It is easy to see, by induction, that $[\alpha_1^j, \alpha] = \alpha_2^j$, $[\alpha_2^k, \alpha] = \alpha_3^k\alpha_5^{k(k-1)/2}$,$[\alpha_3^l, \alpha] = \alpha_4^l$ and $[\alpha_2^k\alpha_3^l, \alpha, \alpha_1^j] = \alpha_5^{kj}\alpha_5^{lj}$. Therefore

$$[g, \alpha] = \alpha_3^k\alpha_5^{k(k-1)/2}\alpha_4^l\alpha_5^{kj}\alpha_5^{lj}\alpha_2^j.$$

It follows that

$$\begin{aligned} 1 &= [\,[g,\alpha],\alpha] \\ &= [\alpha_3^k\alpha_5^{k(k-1)/2}\alpha_4^l\alpha_5^{kj}\alpha_5^{lj}\alpha_2^j,\alpha] \\ &= [\alpha_3^k\alpha_4^l\alpha_2^j,\alpha] \\ &= [\alpha_3^k\alpha_4^l,\alpha][\alpha_3^k\alpha_4^l,\alpha,\alpha_2^j][\alpha_2^j,\alpha] \\ &= [\alpha_3^k,\alpha][\alpha_3^k,\alpha,\alpha_4^l][\alpha_4^l,\alpha][\alpha_3^k\alpha_4^l,\alpha,\alpha_2^j][\alpha_2^j,\alpha] \\ &= \alpha_4^k[\alpha_2^j,\alpha] \\ &= \alpha_4^k\alpha_3^j\alpha_5^{j(j-1)/2}, \end{aligned}$$

which is possible only when $k = j = 0$. It implies that $[g,\alpha] = \alpha_4^l = \alpha_5^{-1}$, which is not so. Thus, there exists a central automorphism which is not class-preserving. Since $|Z(\mathrm{Inn}(G))| = p$ and $|\operatorname{Aut_z}(G)| = p^2$, it follows by Lemma 4.2.1 that

$$|\operatorname{Aut_c}(G)| = p^4|\operatorname{Aut_c}(G)\cap\operatorname{Aut_z}(G)| = p^4p = p^5 = |\operatorname{Inn}(G)|.$$

Hence $\operatorname{Out_c}(G) = 1$ and G has a non-inner central automorphism of order p. □

Theorem 5.2.16 *If G is any group from family Φ_{39}, then* $\operatorname{Out_c}(G) \neq 1$.

Proof. Let G be the group $\Phi_{39}(1^6) = \langle \alpha,\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5 \mid [\alpha_1,\alpha] = \alpha_2, [\alpha_2,\alpha] = \alpha_3, [\alpha_3,\alpha] = \alpha_4, [\alpha_1,\alpha_2] = \alpha_4, [\alpha_2,\alpha_3] = \alpha_5, [\alpha_3,\alpha_1] = \alpha_5, [\alpha_4,\alpha_1] = \alpha_5, [\alpha_4,\alpha] = 1, [\alpha_5,\alpha] = 1, [\alpha_5,\alpha_1] = 1, [\alpha_4,\alpha_2] = 1, [\alpha_5,\alpha_2] = 1, [\alpha_4,\alpha_3] = 1, [\alpha_5,\alpha_3] = 1, [\alpha_5,\alpha_4] = 1, \alpha^p = \alpha_1^p = \alpha_2^p = \alpha_3^p = \alpha_4^p = \alpha_5^p = 1\rangle$. Observe that every element $g \in G$ can be written as $g = g'\alpha^k\alpha_1^l = \alpha_2^q\alpha_3^r\alpha_4^s\alpha_5^t\alpha^k\alpha_1^l$, where $0 \leq q,r,s,t,k,l \leq p-1$ and $\alpha_2^q\alpha_3^r\alpha_4^s\alpha_5^t = g' \in G'$. The center is generated by α_5. It is easy to see that the map $\tau : G \to G$ defined by $\tau(\alpha) = \alpha$, $\tau(\alpha_1) = \alpha_1\alpha_4\alpha_5$, $\tau(\alpha_2) = \alpha_2$, $\tau(\alpha_3) = \alpha_3$, $\tau(\alpha_4) = \alpha_4$, $\tau(\alpha_5) = \alpha_5$ is an automorphism of G. We claim that τ is a non-inner class-preserving automorphism of G. Observe that

$$\begin{aligned}\tau(g) &= \tau(\alpha_2^q\alpha_3^r\alpha_4^s\alpha_5^t\alpha^k\alpha_1^l)\\ &= \alpha_2^q\alpha_3^r\alpha_4^s\alpha_5^t\alpha^k(\alpha_1\alpha_4\alpha_5)^l\\ &= \alpha_2^q\alpha_3^r\alpha_4^s\alpha_5^t\alpha^k\alpha_1^l\alpha_4^l\alpha_5^{l(l+1)/2}\\ &= g'\alpha^k\alpha_1^l\alpha_4^l\alpha_5^{l(l+1)/2}\end{aligned}$$

Consider

$$\begin{aligned}[g'\alpha^k\alpha_1^l, \alpha_3^{r_1}\alpha_4^{s_1}] &= [\alpha_2^q\alpha_3^r\alpha_4^s\alpha_5^t\alpha^k\alpha_1^l, \alpha_3^{r_1}\alpha_4^{s_1}]\\ &= [\alpha_2^q\alpha^k\alpha_1^l, \alpha_3^{r_1}\alpha_4^{s_1}]\\ &= \alpha_4^{-kr_1}\alpha_5^{qr_1-lr_1-ls_1-lkr_1},\end{aligned}$$

where $0 \le r_1, s_1 \le p-1$. Let l, k be non-zero modulo p, then there exist r_1 and s_1 such that

$$-kr_1 \equiv l \pmod{p}$$

and

$$qr_1 - lr_1 - ls_1 - lkr_1 \equiv l(l+1)/2 \pmod{p}.$$

Now, if l is non-zero modulo p and $k \equiv 0 \pmod{p}$, then we have

$$\begin{aligned}[g'\alpha_1^l, \alpha_2^{q_1}\alpha_3^{r_1}] &= [\alpha_2^q\alpha_3^r\alpha_4^s\alpha_5^t\alpha_1^l, \alpha_2^{q_1}\alpha_3^{r_1}]\\ &= [\alpha_2^q\alpha_3^r\alpha_1^l, \alpha_2^{q_1}\alpha_3^{r_1}]\\ &= [\alpha_2^q\alpha_3^r, \alpha_2^{q_1}\alpha_3^{r_1}][\alpha_2^q\alpha_3^r, \alpha_2^{q_1}\alpha_3^{r_1}, \alpha_1^l][\alpha_1^l, \alpha_2^{q_1}\alpha_3^{r_1}]\\ &= \alpha_4^{lq_1}\alpha_5^{qr_1-rq_1-lr_1+q_1l(l-1)/2},\end{aligned}$$

where $0 \le q_1, r_1 \le p-1$. Since l is non-zero modulo p, there exist q_1 and r_1 such that

$$lq_1 \equiv l \pmod{p}$$

and

$$qr_1 - rq_1 - lr_1 + q_1 l(l-1)/2 \equiv l(l+1)/2 \pmod p.$$

Finally, if k is non-zero modulo p and $l \equiv 0 \pmod p$, then $\tau(g'\alpha^k) = g'\alpha^k$, which is a conjugate of $g'\alpha^k$. Therefore τ is a class-preserving automorphism of G.

Next, suppose that τ is an inner automorphism. Then τ is induced by some element of $Z_3(G)$ since $g^{-1}\tau(g) \in Z_2(G)$. It is easy to see that $Z_3(G) = \langle \alpha_3, \alpha_4, \alpha_5 \rangle$. Let us assume that τ is induced by $\alpha_3^r\alpha_4^s$, then

$$\tau_{\alpha_3^r\alpha_4^s}(\alpha) = \alpha_3^{-r}\alpha\alpha_3^r = \alpha,$$

which is possible when $r \equiv 0 \pmod p$. Then $\tau_{\alpha_4^s}(\alpha_1) = \alpha_4^{-s}\alpha_1\alpha_4^s = \alpha_1\alpha_5^{-t}$, a contradiction to definition of τ . Thus τ is a non-inner class-preserving automorphism. Hence $\mathrm{Out_c}(G) \neq 1$. □

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