FITTED MESH METHODS FOR THE NUMERICAL SOLUTIONS OF SINGULARLY PERTURBED PROBLEMS

A Thesis

submitted in fulfillment of the
requirement for the award of the degree of

Doctor of Philosophy
in
Mathematics

by

Vinod Kumar
(Regd. No. 950811014)

SCHOOL OF MATHEMATICS AND COMPUTER APPLICATIONS
THAPAR UNIVERSITY
PATIALA - 147004 (PUNJAB)
JUNE 2013

CANDIDATE'S DECLARATION

I hereby certify that the work being presented in the thesis entitled "**Fitted Mesh Methods for the Numerical Solutions of Singularly Perturbed Problems**" in fulfillment of the requirements for the award of the degree of Doctor of Philosophy, submitted in the School of Mathematics and Computer Applications of Thapar University, Patiala, is an authentic record of my own work carried out during a period from January, 2010 to June 2013 under the supervision of Dr. A. K. Lal and Dr. Rajesh. K. Bawa.

The matter embodied in this thesis has not been submitted by me for the award of any other degree.

Date: 26/06/13

(Vinod Kumar)

This is to certify that the above statement made by the candidate is correct to the best of my knowledge.

(A. K. Lal)
Associate Professor,
SMCA,
Thapar University, Patiala.

(Rajesh. K. Bawa)
Head,
Department of Computer Science,
Punjabi University, Patiala.

ACKNOWLEDGEMENTS

I acknowledge with a sense of deep and sincere gratitude, the meticulous guidance, ingenious discussions and steady encouragement rendered by **Dr. Rajesh. K. Bawa**, Head, Department of Computer Science, Punjabi University, Patiala and **Dr. A. K. Lal**, Associate Professor, Thapar University, Patiala . I owe the greatest debt of gratitude to them for suggesting the research problem followed by ever willing help and the freedom extended to me throughout the period of my research. I greatly appreciate their support and guidance not only in research but also in my day to day life.

I am profoundly obliged to **Dr. Rajesh Kumar**, Head, School of Mathematics and Computer Applications, Thapar University, Patiala for his encouragement and providing the necessary facilities needed for this work.

I take this opportunity to thank **Prof. K. K. Raina**, Director Thapar University, Patiala for providing me University's resources and facilities necessary to carry out this research work.

I am deeply thankful to **Dr. P. K. Bajpai**, Dean, Research & Sponsored Projects and **Dr. S. K. Mohapatra,** Dean, Academic affairs, Thapar University, Patiala, for their support and needful help during the various stages of my research work. I also take this opportunity to express my sincere thanks to all the faculty members and the staff of School of Mathematics and Computer Applications, Thapar University, Patiala for providing me constant encouragement.

I am also grateful to my friend Ankush Pathania for providing me the constant help, moral support and encouraging environment to take the challenges. Words are inadequate in paying regards to my respected parents.

Last but not the least, the encouragement given by my wife Pooja and son by adapting themselves to the nature of my work is a kind of love but for which i could not thought of this work at all. And above all, I thank and pay my regards to the Almighty for his love and blessings.

Date 26/06/13

(Vinod Kumar)

ABSTRACT

In the present thesis an attempt has been made to derive some simple and efficient numerical methods for solving singularly perturbed problems which are easy to implement and are not costly in terms of computer time also. It is observed that the numerical methods presented have been found to be efficient over the conventional methods and are at the same time, conceptually simple. We consider mainly the one dimensional singularly perturbed problems. Apart from the construction of methods, a full fledged theory for their convergence and error estimates is also presented. Numerical experiments are carried out extensively to support the theoretical results.

The thesis consists of seven chapters. A brief outline of the chapters is as follows:

Chapter 1

Chapter I of the thesis is introductory in nature which includes a brief survey of numerical methods of singularly perturbed problems. A summary of recent methods and work of succeeding chapters is also given.

Chapter 2

First order singularly perturbed initial value problems (SPIVP) and system of SPIVP are considered in this chapter.

We consider singularly perturbed initial value problem (IVP):

$$L_\epsilon u(x) \equiv \varepsilon u'(x) + b(x)u(x) = f(x), \qquad x \in \Omega = (0,1), \tag{0.0.1}$$

$$u(0) = A, \tag{0.0.2}$$

where A is a constant and $\varepsilon > 0$ is a small parameter, b and f are sufficiently smooth functions, such that $b(x) \geq \beta > 0$ on $\overline{\Omega} = [0,1]$. Under these assumptions, the IVP possesses a unique solution $u(x)$ [25].

System of first order singularly perturbed ordinary differential equations:

$$L_\epsilon \vec{u_\epsilon}(x) = \begin{pmatrix} \epsilon D & 0 & . & . & 0 \\ 0 & \epsilon D & . & . & 0 \\ . & . & . & . & . \\ . & . & . & . & . \\ 0 & 0 & . & . & \epsilon D \end{pmatrix} \vec{u_\epsilon}(x) + A(x)\vec{u_\epsilon}(x) = \vec{f}(x), \tag{0.0.3}$$

$$\vec{u_\epsilon}(0) = (u_{\epsilon,1}(0), u_{\epsilon,2}(0), ...u_{\epsilon,n}(0))^T, x \in (0,1],$$

where $\vec{u_\epsilon} = (u_{\epsilon,1}, u_{\epsilon,2}, ...u_{\epsilon,n})^T$, $A(x) = \begin{pmatrix} a_{11}(x) & a_{12}(x) & . & . & a_{1n}(x) \\ a_{21}(x) & a_{22}(x) & . & . & a_{2n}(x) \\ . & . & . & . & . \\ . & . & . & . & . \\ a_{n1}(x) & a_{n2}(x) & . & . & a_{nn}(x) \end{pmatrix}$

$\vec{f}(x) = (f_1(x), f_2(x), ..., f_n(x))^T$, D denotes $\frac{d}{dx}$ and $u_\epsilon \in C^{(1)}(\Omega), \Omega = (0,1)$. The singular perturbation parameter ϵ satisfies $0 < \epsilon \leq 1$. The functions $a_{ij}, f_i \in C^2(\overline{\Omega}), \overline{\Omega} = [0,1]$, $i,j = 1,2,...,n$ satisfies the following inequalities:

$$(i) \quad a_{ii} > \sum_{j=1, j\neq i}^{n} |a_{ij}(x)|, \quad i = 1(1)n$$

$$(ii) \quad a_{ij} < 0, \quad i,j = 1(1)n, i \neq j$$

In [39], a computational method is presented for a system of first order singularly perturbed ordinary differential equations. We proposed a second-order parameter-uniform convergent computational technique for them. The proposed technique is applied on an appropriate piecewise uniform Shishkin mesh. Numerical experiments are carried out on some test problems, confirming the robustness of the technique.

Chapter 3

In this chapter, we consider the following singularly perturbed delay differential equation in the interval $\overline{\Omega} = [0, m]$:

$$\varepsilon u'(x) + a(x)u(x) - b(x)u(x-1) = f(x), \qquad x \in \Omega, \tag{0.0.4}$$

$$u(x) = \varphi(x), \quad -1 < x \leq 0, \tag{0.0.5}$$

where $\Omega = (0, m] = \bigcup_{p=1}^{m} \Omega_p$, $\Omega_p = \{x : p-1 < x \leq p\}$, $p \geq 1$. $0 < \varepsilon \leq 1$ is the perturbation parameter, $a(x) \geq \beta > 0$, $b(x) > 0$, $f(x)$ and $\varphi(x)$ are given sufficiently smooth functions. The solution, $u(x)$, displays in general boundary layers on the right side of each point $x = p-1$ for small values of ε [2]. We extend the technique proposed in chapter 2 for (0.0.4-0.0.5).

Also, we consider the following singularly perturbed delay differential equation (DDE) in the interval $\overline{\Omega} = [0, 1]$:

$$L_\epsilon u(x) \equiv \varepsilon u''(x) + a(x)u'(x-\delta) + b(x)u(x) = f(x), \quad x \in \Omega = (0,1), \tag{0.0.6}$$

$$u(x) = \varphi(x), \quad -\delta \leq x \leq 0, \tag{0.0.7}$$

$$u(1) = \lambda, \tag{0.0.8}$$

where $0 < \varepsilon \ll 1$ is the small perturbation parameter and the delay parameter δ is such that $0 < \delta < 1$. The functions $a(x)$, $b(x)$, $f(x)$ and $\varphi(x)$ are sufficiently smooth functions and λ is a constant. It is also assumed that $b(x) \geq \beta > 0$, $\forall x \in \overline{\Omega}$. When $\delta = 0$, the equation (0.0.6-0.0.8) reduces to a singularly perturbed differential equation. Depending upon the sign of $a(x)$, i.e., if $a(x) > 0$ (or $a(x) < 0$), a boundary layer is located at left (or right) end of domain. The layer is maintained for sufficiently small δ with $\delta \neq 0$ and $\delta = o(\epsilon)$.

A parameter-uniform numerical scheme is proposed for (0.0.6-0.0.8). We first construct a difference scheme using cubic spline and then apply it on a layer resolving piecewise uniform mesh, known as the Shishkin mesh. In the boundary layer (inner) region, the mesh is fine, and the cubic spline scheme is stable. Whereas in the regular (outer) region, the mesh is coarse, and the cubic spline scheme is not stable. To obtain stability in the outer region, one has to restrict the step size in that region, but our aim is to propose an ϵ-uniform convergent numerical scheme. Therefore, for the outer region, instead of the cubic scheme we use the finite difference scheme, mainly for stability reasons. The newly obtained hybrid scheme is convergent independent of the singular perturbation parameter.

Chapter 4

Second order reaction-diffusion and convection-diffusion SPBVPs with continuous source term are considered in this chapter.

We consider the following class of SPBVPs:

$$L_\epsilon u(x) \equiv \epsilon u''(x) + a(x)u'(x) + b(x)u(x) = f(x), \qquad x \in \Omega = (0,1), \tag{0.0.9}$$

$$u(0) = p, \quad u(1) = q \qquad\qquad , \tag{0.0.10}$$

where $0 < \epsilon \ll 1$ is a small positive parameter, $a(x)$, $b(x)$ and $f(x)$ are sufficiently smooth functions, such that $a(x) \geq \beta > 0$ and $b(x) \geq 0$ on $\overline{\Omega} = [0,1]$. Under these assumptions, (0.0.9-0.0.10) possesses a unique solution $u(x) \in C^2(\overline{\Omega})$. In this chapter, we consider both the cases i.e. $a(x) = 0$ and $a(x) \neq 0$. We suggest an initial-value technique, in line of [123] for (0.0.9-0.0.10). The BVP is replaced with a suitable initial value problem (IVP) and a terminal value problem (TVP). The integration of these problems goes in the opposite directions, but each problem can be solved independently of the other. The IVP is of singularly perturbed type, whereas the TVP does not contain any small parameter. We propose an initial-value technique to solve these problems.

Chapter 5

Second order SPPs with discontinuous source term are considered in this chapter. A singularly perturbed convection-diffusion type second order ODE with a discontinuous source term is considered on the unit interval $\Omega = (0, 1)$. A single discontinuity is assumed to occur at a point $d \in \Omega$. This gives rise to an interior layer in the exact solution of the problem, in addition to the boundary layer at the outflow boundary point. Let $\Omega^- = (0, d)$ and $\Omega^+ = (d, 1)$ and denote the jump at d in any function with $[w](d) = w(d+) - w(d-)$. Consider the problem:

$$L_\epsilon u(x) \equiv \varepsilon u''(x) + a(x)u'(x) = f(x), \qquad x \in \Omega^- \cup \Omega^+, \tag{0.0.11}$$

$$u(0) = p, \quad u(1) = q, \tag{0.0.12}$$

where $0 < \epsilon \ll 1$ is a small parameter, $a(x)$ is a sufficiently smooth function on $\overline{\Omega} = [0, 1]$, such that $a(x) \geq \beta > 0$, $f(x)$ is a sufficiently smooth function on $\Omega^- \cup \Omega^+ \cup \{0, 1\}$, f and its derivatives have jump discontinuity at d. Because f is discontinuous at d, the solution u of (0.0.11-0.0.12) does not necessarily have a continuous second derivative at the point d. Thus, u need not belong to the class of functions $C^2(\Omega)$. But the first derivative of the solution exists and is continuous in Ω. Under these assumptions, (0.0.11-0.0.12) have a solution $u \in C^0(\overline{\Omega}) \cap C^1(\Omega) \cap C^2(\Omega^- \cup \Omega^+)$ [28].

In this chapter, we also discuss singularly perturbed reaction-diffusion type second order ODE with a discontinuous source term. To solve these type of problems, a modified initial value technique (MIVT) is proposed on an appropriate piecewise uniform Shishkin mesh. The MIVT is shown to be uniformly convergent with respect to the perturbation parameter.

Chapter 6

We treat the following system of two singularly perturbed convection-diffusion problems:

$$L_1\vec{u} \equiv -\epsilon u_1''(x) - a_1(x)u_1'(x) + b_{11}(x)u_1(x) + b_{12}(x)u_2(x) = f_1(x), \tag{0.0.13}$$

$$L_2\vec{u} \equiv -\mu u_2''(x) - a_2(x)u_2'(x) + b_{21}(x)u_1(x) + b_{22}(x)u_2(x) = f_2(x), \tag{0.0.14}$$

where $\vec{u} = (u_1, u_2)^T$, $x \in \Omega = (0,1)$ with the boundary conditions:

$$\vec{u}(0) = \begin{pmatrix} p \\ r \end{pmatrix}, \tag{0.0.15}$$

$$\vec{u}(1) = \begin{pmatrix} q \\ s \end{pmatrix}. \tag{0.0.16}$$

Without loss of generality, we shall assume that $0 < \epsilon \leq \mu \ll 1$. The functions $a_1(x)$, $a_2(x)$, $b_{11}(x)$, $b_{12}(x)$, $b_{21}(x)$, $b_{22}(x)$, $f_1(x)$, $f_2(x)$ are sufficiently smooth and satisfy the following inequalities:

$$\begin{aligned}
&(i) \quad a_1(x) \geq \alpha_1 > 0, \quad a_2(x) \geq \alpha_2 > 0, \\
&(ii) \quad b_{11}(x) > |b_{12}(x)|, \quad b_{22}(x) > |b_{21}(x)|, \\
&(iii) \quad b_{12}(x) < 0, \quad b_{21}(x) < 0,
\end{aligned}$$

$\forall x \in \overline{\Omega} = [0,1]$ and $u_1, u_2 \in C^2(\Omega) \cap C(\overline{\Omega})$.

Our main objective is to construct a modified initial value technique (MIVT) for (0.0.13-0.0.16) which is based on the underlying idea of AIVM [125]. First, in this technique, an asymptotic expansion approximation for the solution of the Boundary Value Problem (BVP) (0.0.13-0.0.16) has been constructed. Then, Initial Value Problems (IVPs) and Terminal Value Problems (TVPs) are formulated whose solutions are the terms of this asymptotic expansion. The IVPs are happened to be SPPs and therefore, they are solved by the proposed hybrid scheme. The MIVT displays uniform convergence with respect to the perturbation parameter ϵ. We deal with the case $0 < \epsilon = \mu \ll 1$. In this chapter we also consider the case when $a_1(x) = a_2(x) = 0$.

Chapter 7

Conclusions based on the present study are finally drawn in the concluding chapter 7. The significance of the present work and scope of future research work is also discussed in this concluding chapter.

Contents

LIST OF PUBLICATIONS

1. Rajesh K. Bawa, A. K. Lal, and Vinod Kumar, An ϵ uniform hybrid scheme for singularly perturbed delay differential equations. Appl. Math. Comp., 217(2011), pp.8216-8222. (Elsevier)

2. Vinod Kumar, Rajesh K. Bawa and A. K. Lal, A robust computational technique for a system of singularly perturbed convection-diffusion equations. Int. J. Comput. Methods, 9(4)(2013) 1350027-14. (World Scientific)

3. Vinod Kumar, Rajesh K. Bawa and A. K. Lal, A hybrid initial-value technique for singularly perturbed boundary value problems, Comput. Appl. Math., 32(1)(2013), pp.163-177. (Springer)

4. Vinod Kumar, Rajesh K. Bawa and A. K. Lal, A robust computational technique for a system of singularly perturbed reaction-diffusion equations, (Communicated).

5. Vinod Kumar, Rajesh K. Bawa and A. K. Lal, A modified initial value technique for singularly perturbed reaction-diffusion problems, (Communicated).

6. Vinod Kumar, Rajesh K. Bawa and A. K. Lal, A modified initial value technique for singularly perturbed problems of reaction-diffusion type with a discontinuous source term, (Communicated).

7. Vinod Kumar, Rajesh K. Bawa and A. K. Lal, A modified initial value technique for singularly perturbed problems of convection-diffusion type with a discontinuous source Term, (Communicated).

8. Vinod Kumar, Rajesh K. Bawa and A. K. Lal, Uniformly convergent numerical scheme for singularly perturbed delay differential equations, (Communicated).

List of Figures

List of Tables

ABBREVIATIONS

AIVM	Asymptotic Initial-Value Method
BVPs	Boundary Value Problems
BVT	Boundary Value Technique
EFFD	Exponentially Fitted Finite Difference
FEM	Finite Element Method
FMM	Fitted Mesh Method
FOM	Fitted Operator Method
IVPs	Initial Value Problems
IVT	Initial Value Technique
ODEs	Ordinary Differential Equations
SPPs	Singularly Perturbed Problems
TVPs	Terminal Value Problems

NOTATIONS AND TERMINOLOGY

NOTATIONS

x, t	Independent variables
$\vec{y}(x) = (y_1(x), y_2(x))$	Solution of a continuous problem for a system of ODEs
h_i	Mesh width on a non-uniform mesh
$\sigma, \sigma_1, \sigma_2, \sigma^-, \sigma^+$	Transition parameters on fitted meshes
N	An integer such that $\sum_{i=1}^{N} h_i$ is the length of the relevant interval
$\vec{Y}(x_i) = (Y_1(x_i), Y_2(x_i))$	Numerical solution corresponding to a continuous boundary value problem for a system of ODEs
L, L_1, L_2	Differential Operators
B_0, B_1	Boundary Operators
L^N, L_1^N, L_2^N	Difference Operators
B_0^N, B_1^N	Boundary Difference Operators
D^-	Backward Difference Operator
$\epsilon, \epsilon_1, \epsilon_2$	Singular Perturbation Parameters
C	Generic constants independent of x_i, N, ϵ, ϵ_1, ϵ_2
x_i	i^{th} node
$\lvert . \rvert$	Absolute value
$\lVert y \rVert_{\overline{\Omega}}$	$\max_{x \in \overline{\Omega}} \lvert y(x) \rvert$
$\lVert \vec{y} \rVert_{\overline{\Omega}}$	$\max\{\lVert y_1 \rVert, \lVert y_2 \rVert\}$
Ω	Open Interval (0,1)
$\overline{\Omega}$	Closed Interval [0,1]
Ω^-	Open Interval (0,d)
Ω^+	Open Interval (d,1)
$[w](d) = w(d_+) - w(d_-)$	Jump at $x = d$ in $w(x)$
$\overline{\Omega}_{\epsilon}^N$	Set of points in R
$C^K(\Omega), C^K(\overline{\Omega})$	k times continuously differential functions

TERMINOLOGY

Barrier Function: Function used, in conjunction with maximum principle, to estimate the error in an asymptotic approximation or numerical approximation.

Boundary Layer: A neighborhood of a boundary point in which the solution of a singularly perturbed problem may change rapidly.

Boundary Layer Function: Function with exponentially small values outside the boundary layer.

Convergence With Order p: $Y(x_i)$ converges to $y(x_i)$ as $h \to 0$ with order p if $|(y - Y)(x_i)| \leq Ch^p$, where C is the error constant.

Reduced Problem: Problem obtained from a singularly perturbed problem by setting $\epsilon = 0$ and omitting the appropriate boundary conditions.

Convection-Diffusion Type Problem: A problem whose order is reduced by one when ϵ is set equal to zero.

Reaction-Diffusion Type Problem: A problem whose order is reduced by two when ϵ is set equal to zero.

Exponentially Fitted Finite Difference Scheme: Difference scheme with a fitting factor containing exponential functions.

Less-severe (weak) Layer: A solution $y(x)$ of a BVP is said to have less severe layer at $x = a$ if a layer occurs for y’(x) at $x = a$ but not for $y(x)$.

Singularly Perturbed Problem: Problem depending on a parameter ϵ in such a way that the solution of the problem behaves non uniformly as the parameter tends to zero.

Uniform Boundedness: Boundedness independent of ϵ and μ.

Uniform Convergence With Order p: Convergence with order p, where p and the error constant are independent of singular perturbation parameters.

Chapter 1

Introduction

1.1 Introduction and background

Singularly perturbed problems (SPPs) have always played a prominent role in the theory of differential equations and in their applications to the physical world. Ever since Prandtl's work in the beginning of this century, singular perturbation techniques have been a traditional tool of fluid dynamics. These techniques entered into various other areas of application, where of course, the same terminology of 'boundary layer', 'interior layer', 'outer' and 'inner' were already in use. Just to arouse one's curiosity, we must mention its use in various applied areas such as fluid dynamics, plasticity, chemical reactor theory, nuclear reactor theory, plasma physics, aerodynamics, meteorology, oceanography, rarefied gas dynamics, diffraction theory, reaction-diffusion process, non-equilibrium and radiating flows, Navier-Stokes equations of fluid flow at high Reynolds number etc.

In the intensive development of science and technology, many practical problems, such as the mathematical boundary layer theory or approximation of solutions of various problems described by differential equations involving large or small parameters, become more complex. In some problems, the perturbations are operative over a very narrow region across which the dependent variable undergoes very rapid changes. These narrow regions frequently adjoin the boundaries of the domain of interest, owing to the fact that the small parameter multiplies the highest derivative. Consequently, they are usually referred to as boundary layers in fluid mechanics, edge layers in solid mechanics, skin layers in electrical applications, shock layers in fluid and solid mechanics, transition points in quantum mechanics and Stokes lines and surfaces in mathematics.

We give the brief definition of a singularly perturbed problem in its simplest and most widely used form. Consider a problem P_ϵ in some differential model, depending upon a small positive parameter ϵ, where $0 < \epsilon \ll 1$. Under some conditions, a solution $y_\epsilon(x)$ of the problem P_ϵ can be constructed by the well known method of perturbation that is as a power series in ϵ with first term y_0 being the solution of the problem P_0, which is obtained by putting ϵ equal to zero in the problem P_ϵ. Under the happiest circumstances, this perturbation method leads to altogether satisfactory results. This series can not often be pre-assumed to uniformly convergence, particularly for small values of ϵ, in the entire interval. When, such an expansion converges as $\epsilon \to 0$, uniformly in x, one speaks of 'Regular Perturbation Problem'. On the other hand, when $y_\epsilon(x)$ does not have a uniform limit in x as $\epsilon \to 0$, this straight forward perturbation method fails and as a consequence of the non-uniformity, one may miscalculate or even lose essential results. Such a problem is known as Singularly Perturbed Problem. A singularly perturbed problem is well defined as one in which no single asymptotic expansion is uniformly valid throughout the interval as $\epsilon \to 0$.

For more than two decades now, a great deal of research work on the qualitative and quantitative analysis of these problems both for ordinary and partial differential equations has been reported in the literature. But the major problem of obtaining accurate approximations to the solutions to these problems is still an open question. Traditional numerical methods which have been known to be effective for solving most problems that arise in applications have failed when applied to singularly perturbed problems. As a result, this area has attracted a keen interest amongst mathematicians today. Consequently, there are now a variety of methods for solving these kinds of problems.

Basically, the problem of inaccurate results of singularly perturbed problems has been associated with the perturbation parameter. This perturbation parameter prevents us from obtaining satisfactory numerical solutions. Most of the classical numerical methods are not effective for solving such problems because, as the singular perturbation parameter tends to zero [27], the errors in the numerical solutions increase and often becomes comparable in magnitude to the exact solution. Some of the methods which have been used in the solution of these problems include matched asymptotic expansions and averaging, difference schemes of methods of special condensing meshes, fitted difference schemes and exponentially fitted schemes.

1.2 Literature review

The development of numerical methods for solving SPPs started with methods aimed at solving ordinary differential equations, an account of which can be found in the first monograph on this subject by Doolan et al. [25]. A comprehensive historical account of numerical methods for SPPs can be found in Farrell et al. [27], Roos et al. [105]. The difficulty with standard numerical methods employing uniform meshes has been a lack of robustness with respect to the perturbation parameter ϵ [27, 79, 105]. Since the layer contracts as ϵ becomes smaller, the mesh needs to be refined substantially to grasp the dynamics within the diminishing layer. To overcome this problem two different routes to construct uniformly convergent schemes have been followed in recent years. These are: the fitted operator methods and fitted mesh methods. The first one has an advantage that it does not require the knowledge of location and width of the boundary layer. However, it is difficult to extend for higher dimensional problems. Whereas, the disadvantage of the second approach is the requirement of knowledge of location and width of boundary layer but it is gaining popularity because of simple piecewise uniform meshes like Shishkin meshes.

An comprehensive discussion of ϵ uniform fitted operator methods is given in [25, 27, 34].

Towards the use of the fitted mesh methods, tremendous works have been done so far. Various mesh selection strategies (either graded or piecewise uniform meshes) have been provided in the literature.

The history of layer-adapted meshes begin in 1969 with a paper by Bakhvalov [8] in the context of reaction-diffusion problems. Suppose, for a singularly perturbed BVP an exponential layer occurs at $x = 0$, so that $y = \exp(-\alpha x/\epsilon)$ for some fixed α, is present in the solution of the BVP. Bakhvalov's idea is to use an equidistant y-grid near $y = 1$ (which corresponds to $x = 0$), then to map this grid back to the $x - axis$ by means of such logarithmic functions are called Bakhvalov meshes.

Grids of Bakhvalov-type are uniformly spaced outside the layer(s) and are characterized by a gradual transition from the coarse to a very fine mesh at the layer(s). Vulanović [134, 135, 137] contributed many uniform convergence results on Bakhvalov meshes for turning point, nonlinear and other problems. To this class belong the meshes proposed by Liseikin and Yanenko [69, 70, 71] (quadratic function outside layer) and meshes generated by equidistribution of monitor functions which have been extensively studied by the group of Sloan and Mackenzie [12, 73, 94, 95], the graded mesh of Gartland [35] and its modification by Roos and Skalický [104]

A new impetus to the priori mesh approach was given by Shishkin [112]. The advent of the Shishkin meshes has fueled significant advances into the broad area of singularly perturbed differential equations.

Consider the following singularly perturbed BVP

$$-\epsilon u''(x) + a(x)u'(x) + b(x)u(x) = f(x), \qquad x \in \Omega = (0,1), \tag{1.2.1}$$

$$u(0) = A, \quad u(1) = B, \tag{1.2.2}$$

where $a(x)$, $b(x)$, and $f(x)$ are smooth functions such that $a(x) \geq \alpha > 0$, $b(x) \geq 0$ on $\overline{\Omega}$. He suggested for the BVP (1.2.1-1.2.2) that if $N/2$ grid points are placed equidistantly in each of the subintervals $[0, 1-\sigma]$ and $[1-\sigma, 1]$ with $\sigma = \min\{1/2, \frac{\epsilon}{\alpha}\ln N\}$, one can obtain the best results. Suppose that $\epsilon \ll N^{-1}$, which is reasonable in practice. Then σ is small. So, the Shishkin mesh is coarse on $[0, 1-\sigma]$ and fine on $[1-\sigma, 1]$.

The Shishkin mesh is remarkable in two ways, first it resolves part but not all of the boundary layer, yet still yields convergence that is uniform in ϵ, second, despite the fact that there is an abrupt change in mesh size, this does not destabilize the difference scheme. This mesh together with an appropriate difference scheme gives solution that are indeed robust with respect to the perturbation parameter ϵ. The Shishkin meshes are discussed in detail in [105, 115, 117] as well as in [103], where Roos surveyed results on layer adapted meshes. Simplicity is one of its commendable attributes. It is the simplicity that allows applications to many different types of problems specifically, the flexibility to tackle problems in higher dimensions [27, 79, 105].

The performance of the Shishkin meshes is however inferior to that of Bakhvalove meshes, which has prompted efforts to improve them while retaining some of their simplicity. Valanović [138] improved the Shishkin meshes by introducing the additional mesh transition points. Linß [62, 63] combines the ideas of Bakhvalove and Shishkin while Beckett and Mackengie [12] combine an equidistribution idea [26] with a Shishkin-type transition point.

Here, only a summery of some recent methods is presented. Obviously, a selection of techniques is done implicitly to present this summary, which is in some way relevant to our work. Mainly, we focus on those papers which are based on Shishkin meshes.

1.2.1 One dimensional problems with regular layers

Kadalbajoo and Gupta [43] constructed a B-spline collocation method on piecewise-uniform Shishkin mesh to solve a singularly perturbed convection-diffusion problem

$$L_\epsilon u(x) = -\epsilon u''(x) + a(x)u'(x) + b(x)u(x) = f(x), \qquad x \in \Omega = (0,1), \tag{1.2.3}$$

$$u(0) = A, \quad u(1) = B \tag{1.2.4}$$

$\epsilon > 0$ is a small parameter, a, b, and f, are smooth functions such that $a(x) \geq \alpha > 0$, $b(x) \geq \beta > 0$, $x \in \overline{\Omega} = [0,1]$. Authors derived the bounds for the derivative of the analytical solution by decomposing the solution into regular and singular parts. They shown that the present method is boundary layer resolving as well as second-order uniformly convergent in the maximum norm.

Clavero et al. [23] constructed higher order uniformly convergent finite difference methods on Shishkin meshes, using the HODIE (high order differences with identity expansion) technique for (1.2.3-1.2.4). The coefficients are determined by imposing that the local error operator associated with the methods was null on a vector space generated by polynomials and quasi-polynomials (products of polynomials by exponential functions). They proved that the methods constructed in this way are uniformly convergent of order $O(N^{-k}log^kN)$ with $k = 2, 3$, if a Shishkin mesh is used.

Natividad and Stynes [86] considered (1.2.3-1.2.4), and show that when simple upwinding is used on a piecewise-uniform Shishkin mesh, a version of Richardson extrapolation improves the accuracy of the computed solution from $O(N^{-1}lnN)$ to $O(N^{-2}ln^2N)$ in the discrete maximum norm.

Shahraki and Hosseini [108] considered (1.2.3-1.2.4). They proposed a finite difference scheme consisting of simple upwind scheme and central difference method on Shishkin mesh. They show numerically that the proposed scheme has a higher order convergence than the simple upwind scheme.

Patidar and Shikongo [91] designed and implemented some appropriate numerical finite difference scheme on Shishkin type mesh for a linear singularly perturbed boundary value problem

$$L_\epsilon u(x) = \epsilon u''(x) + a(x)u'(x) - b(x)u(x) = f(x), \qquad x \in \Omega = (0,1), \tag{1.2.5}$$

$$u(0) = A, \quad u(1) = B. \tag{1.2.6}$$

$\epsilon > 0$ is a small parameter, a, b, and f, are smooth functions such that $a(x) \geq \alpha > 0, b(x) \geq \beta > 0$, $x \in \overline{\Omega} = [0,1]$.

Awoke and Reddy [4] presented an exponentially fitted special second-order finite difference method for (1.2.5-1.2.6). A fitting factor is introduced in a tri-diagonal finite difference scheme and is obtained from the theory of singular perturbations. Thomas Algorithm is used to solve the system and its stability is investigated.

Awoke and Reddy [5] presented a method of a Terminal Boundary Condition for (1.2.5-1.2.6). In this method the original second order problem is divided in to two problems, an inner region and an outer region problem using a terminal point. Then, an implicit terminal boundary condition at the terminal point is determined from the outer region problem. The outer region problem with the implicit boundary condition is solved. Using the stretching transformation, the modified inner region problem is solved as a two-point boundary value problem. Finally, they combine the solutions of both the inner region and outer region problems to get the approximate solution of the original problem.

Surla et al. [120] also considered (1.2.5-1.2.6). They constructed a numerical method by using quadratic spline function as an approximation function in the collocation process on a piecewise uniform slightly modified Shishkin mesh. They proved point-wise convergence of order $O(N^{-2}ln^2N)$ inside the boundary layer and second order convergence elsewhere. Further, they approximate normalized flux and gave estimates of the error at the mesh points and between them.

Vigo-Aguiara and Natesan [133] considered the following singularly perturbed two-point boundary-value problem

$$L_\epsilon u(x) = \epsilon u''(x) + a(x)u'(x) - b(x)u(x) = f(x), \quad x \in \Omega = (0,1), \tag{1.2.7}$$

$$B_0 u(0) \equiv b(0)u(0) - a(0)u'(0) = -f(0), \quad B_1 u(1) \equiv u(1) = B. \tag{1.2.8}$$

$\epsilon > 0$ is a small parameter, a, b, and f are smooth functions such that $a(x) \geq a > 0$, $b(x) \geq b > 0$, $x \in \overline{\Omega} = [0,1]$. In this paper, author has developed two schemes to integrate singularly perturbed system of initial-value problems (IVPs); the first method is a combination of the classical finite difference scheme and the exponentially fitted difference (EFD) scheme of Doolan et al. [25], which is of order $O(h)$. The second method is derived from the schemes given in Vigo-Aguiar and Ferrndiz [131], is of order $O(h^2)$ and the accuracy of the method mainly depends on the initial-values. This scheme integrates exactly the differential equation with constant coefficients without local truncation error. By asymptotic approximation, it can reduce the number of iterations in the shooting technique. These schemes are having exponential weights to control the fast growth or decay in the exact solution, because of the presence of small parameter ϵ, and it avoids the use of very small step size relative to ϵ.

Vigo-Aguiara and Natesan [132] considered (1.2.7-1.2.8). They suggested an iterative non-overlapping domain decomposition method. They apply the classical finite difference scheme to determine the solution in all the sub-domains except in the boundary layer domain where they apply the exponentially difference scheme. The BVPs are independent in each sub-domain and one can use parallel computers to solve these BVPs. One of the characteristics of the method is that the number of processors available is a free parameter of the method.

Bawa and Natesan [11] considered (1.2.7-1.2.8). They proposed a hybrid scheme which combines the cubic spline scheme and the midpoint scheme in an appropriate manner. In the inner region, the convective term is approximated by three-point differences by spline approximation of solution at the mesh points, whereas in the outer region the midpoint approximations are used for convective term, and the classical central difference scheme is used for the diffusive term. The first-order derivative in the left boundary point is approximated by the cubic spline. This scheme is applied on the boundary layer resolving Shishkin mesh. They have shown that the hybrid scheme produces better results than the classical upwind finite difference scheme.

Ansari and Hegarty [3] considered

$$L_\epsilon u(x) = \epsilon u''(x) + a(x)u'(x) = f(x), \quad x \in \Omega = (0,1), \tag{1.2.9}$$

$$\beta_1 u(0) - \beta_2 u'(0) = A, \quad \gamma_1 u(1) + \gamma_2 u'(1) = B. \tag{1.2.10}$$

$\epsilon > 0$ is a small parameter, a and f, are smooth functions such that $a(x) \geq \alpha > 0, \beta_1, \beta_2 \geq 0,\ x \in \overline{\Omega} = [0,1]$. They solved this problem by employing standard upwind finite difference operators on Shishkin meshes and derived that the error estimate is bounded by $CN^{-1}lnN$.

Vulanović [136] solved the singularly perturbed problem

$$-\epsilon u'' - b(x)u' + c(x)u = f(x), \tag{1.2.11}$$

$$u(0) = \gamma_0, u(1) = \gamma_1, \text{or} - \epsilon u'(0) = \gamma_0, u(1) = \gamma_1. \tag{1.2.12}$$

The functions b, c, f are sufficiently smooth and $b(x) > \beta > 0$, $c(x) \geq 0$, while $0 < \epsilon \ll 1$. He obtained the second-order convergence uniform in ϵ due to the treatment of the boundary layer function, to a special non equidistant mesh which is dense in the layer, and to the use of a combination of central and mid-point finite-difference schemes.

Farrell et al. [31] considered the class of singularly perturbed quasi-linear Dirichlet problems

$$\epsilon u''(x) + b(x)u'(x) = f(x), \qquad x \in (0,1), \tag{1.2.13}$$

$$u(0) = 0, \quad u(1) = 1. \tag{1.2.14}$$

An upwind finite difference scheme is constructed on special piecewise uniform meshes, which are fitted to the boundary layers. The method is shown to be parameter uniform and an error estimate is established in the maximum norm

Stynes and Roos [115] considered the singularly perturbed boundary value problem

$$-\epsilon u''(x) + b(x)u'(x) = f(x), \qquad x \in (0,1), \tag{1.2.15}$$

$$u(0) = 0, \quad u(1) = 0. \tag{1.2.16}$$

A modified upwind scheme is applied. The scheme is analyzed on an arbitrary mesh. It is then analyzed on a Shishkin mesh and precise convergence bounds are obtained, which show that the scheme is superior to the standard upwind scheme. Also, a hybrid scheme consisting of midpoint upwind and central difference scheme on the same Shishkin mesh is proved to achieve even better convergence behavior.

Fröhner and Roos [33] considered (1.2.15-1.2.16). They discussed a defect correction method based on finite difference schemes on a Shishkin mesh. The proposed method combines the upwind difference operator with a central difference operator modified in several grid points. Authors proved almost second-order robust convergence of the scheme with respect to the perturbation parameter ϵ, in the discrete maximum norm.

Stynes and Tobiska [116] considered streamline diffusion finite element methods (SDFEM) for (1.2.15-1.2.16). The solution of this problem has a single boundary layer. To analyse the convergence of these methods, authors rewrite them as finite difference schemes. They first consider arbitrary meshes, then, in analysing the scheme on a Shishkin mesh, they consider two formulations on the fine part of the mesh: the usual streamline diffusion upwinding and the standard Galerkin method. They derived error estimates in the discrete L^∞-norm; in particular they give the first analysis that shows precisely how the error depends on the user-chosen parameter τ_0 specifying the mesh. When τ_0 is too small, the error becomes $O(1)$, but for τ_0 above a certain threshold value, the error is small and increases either linearly or quadratically as a function of τ_0. They conclude that the SDFEM/Galerkin method should always be used in preference to the SDFEM.

Roos [103] gives a survey on the application of Shishkin grids to convection-diffusion problems with dominant convection, further some new results and open problems are presented. The practical importance of these simple-structured grids lies in the possibility to resolve layers-the alternative technique of exponential fitting is not always successful.

Lenferink [61] considered numerical approximations to the solution of the problem

$$-\epsilon u''(x) + b(x)u'(x) = f(x), \qquad x \in [0,1], \tag{1.2.17}$$

$$u(0) = 0, \quad u(1) = 0. \tag{1.2.18}$$

where b and f are known, smooth functions. This problem is model of convection-diffusion phenomena with dominating convection. The solution usually contains a layer which is narrow and steep and difficult to treat numerically. A centered difference or finite element discretization is applied to a singularly perturbed, one-dimensional BVP (1.2.17-1.2.18). The discretization uses a piecewise equidistant mesh, a special mesh introduced by Shishkin. The coarse, equidistant mesh is used where the solution is smooth, while a very fine but still equidistant mesh is used to resolve most of the boundary layers. Author has the numerical approximation to the solution of (1.2.17-1.2.18), and formulated a theorem on the convergence of this approximation. Some proofs based on monotonicity properties are also stated. These properties are used in analyzing the point-wise convergence, uniform in ϵ at the even nodes of the coarse mesh, and on the fine mesh. On the coarse mesh, the convergence is of order $O(N^{-2})$, while the convergence of order $O(N^{-2}ln^2N)$ on the fine mesh has been found. Some lemma has been used directly to prove convergence at the odd numbered nodes of the coarse mesh. Discretization of (1.2.17-1.2.18) with a small amount of artificial viscosity makes the method more stable than a general mesh.

Miller et al. [78] gave a survey of their own results concerning difference methods for the numerical solution of singularly perturbed boundary value problems. The model problem

$$\epsilon u_{xx} + u_x = 0, \qquad \epsilon > 0, \tag{1.2.19}$$

$$u(0) = 0, \quad u(1) = 1. \tag{1.2.20}$$

and some simple (not exponentially fitted) corresponding difference schemes are considered for illustration. A class of piecewise-uniform meshes is proposed in which a finer mesh size h_1 is used for the leading N/2 grid points, and a coarser mesh size h_2 for the remaining N/2 grid points. They showed that the difference schemes considered are ϵ-uniform only if the transition point between the two mesh sizes, depends on N and ϵ in a specific way.

Fröhner et al. [32] considered the defect-correction method that combines a first-order upwind difference scheme with a second-order central difference scheme on a class of Shishkin type meshes for a model singularly perturbed convection-diffusion problem

$$-\epsilon u''(x) - (a(x)u(x))' + b(x)u(x) = f(x), \qquad x \in (0,1), \tag{1.2.21}$$

$$u(0) = 0, \quad u(1) = 0 \qquad . \tag{1.2.22}$$

Authors gave the first general proof of uniform second-order convergence (in the case of $\epsilon \leq N^{-1}$) of a defect-correction method based on simple upwinding and central differencing. As a corollary they derived error bounds for the gradient approximation of the upwind scheme.

Kopteva and Stynes [58] considered a singularly perturbed convection-diffusion two-point problem

$$-\epsilon u''(x) - (a(x)u(x))' = f(x), \qquad x \in (0,1), \tag{1.2.23}$$

$$u(0) = A, \quad u(1) = B. \tag{1.2.24}$$

They constructed an upwind conservative finite difference scheme on an arbitrary mesh and proved bounds, which are weighted by the small diffusion coefficient, on the errors in approximating the derivative of the true solution by divided differences of the computed solution. On a slightly less general mesh they proved unweighted bounds on these errors where the mesh is coarse. These bounds are then made more explicit for the particular cases of Shishkin and Bakhvalov meshes.

Kopteva [56] considered the singularly perturbed boundary value problem in the conservation form

$$L(u) = -\epsilon(p(x)u')' - (r(x)u)' + q(x)u = f(x), \qquad 0 < x < 1, \tag{1.2.25}$$

$$u(0) = g_0, \quad u(1) = g_1. \tag{1.2.26}$$

where $p(x) \geq p_0 =$ const. $> 0, r(x) \geq r_0 =$ const. > 0, $q(x) \geq 0$, $\epsilon \in (0,1]$ is a small parameter. He investigated the difference scheme with the central approximation of the convection term $r(x)u$ and studied convergence properties of this difference scheme on two layer-adopted meshes. On the logarithmically graded mesh (Bakhvalov-type mesh), he proved that the difference scheme converges uniformly in the perturbation parameter ϵ with a convergence rate $O(N^{-2})$. This estimate holds true on a modification of Bakhvalov- type mesh, where the requirement on the smoothness of the mesh generating function is omitted. He also showed that the difference scheme on the piecewise equidistant mesh (Shishkin-type mesh) converges uniformly with a convergence rate $O(N^{-2}ln^2N)$.

Cai and Liu [14] considered the conservative form of singularly perturbed ordinary differential equations

$$L(u) = \epsilon(p(x)u')' + (r(x)u)' + q(x)u = f(x), \qquad 0 < x < 1, \tag{1.2.27}$$

$$\alpha_1 u(0) - \beta_1 u'(0) = A, \quad \alpha_2 u(0) + \beta_2 u'(1) = B. \tag{1.2.28}$$

They constructed two kinds of schemes for these problems. One is a fitted mesh finite difference scheme (FMFDS), another kind is a class of conservative difference schemes (CCDS) with uniform mesh. These difference schemes are proved to be first-order uniformly convergent.

Linß [64] consider a quasi-linear convection diffusion problems on layer-adapted grids.

$$-\epsilon u''(x) - b(x,u)' + c(x,u) = 0, \qquad x \in (0,1), \tag{1.2.29}$$

$$u(0) = 0, \quad u(1) = 0. \tag{1.2.30}$$

He studied the convergence properties of an upwind difference scheme and a hybrid finite difference scheme comprises midpoint upwind difference operator and central difference operator. He derived sufficient conditions needed for uniform convergence of the methods. These conditions are easy to check and enable one to immediately deduce the rate of convergence. He proved first order uniform convergence of upwind difference scheme and almost second order uniform convergence of the hybrid difference scheme.

Vulanović [138] analyzed and compared the Bakhvalov and Shishkin discretization meshes for (1.2.29-1.2.30) He also generalized and improved the Shishkin meshes.

Kopteva and Linß [57] considered

$$-\epsilon u''(x) - b(x,u)' = f(x), \qquad x \in (0,1), \tag{1.2.31}$$

$$u(0) = 0, \quad u(1) = 0. \tag{1.2.32}$$

They constructed a standard central difference scheme on generalized Shishkin-type meshes to discretize the problem. They proved ϵ-uniform second-order convergence in the discrete L_∞ norm for the discussed problem.

Vulanović [139] highlighted the advantage of Shishkin meshes in constructing higher-order discretizations by considering

$$-\varepsilon^2 u''(x) - \mu a(x)u'(x) + c(x,u) = 0, \qquad x \in [0,1], \tag{1.2.33}$$

$$u(0) = p, \quad u(1) = q, \tag{1.2.34}$$

where $0 < \epsilon \leq 1, 0 \leq \mu \leq 1$.

Kadalbajoo and Yadaw [54] consider a B-spline collocation method for the following two-parameter singularly perturbed convection-diffusion boundary value problems:

$$-\varepsilon u''(x) + \mu a(x)u'(x) + b(x)u(x) = f(x), \qquad x \in (0,1), \tag{1.2.35}$$

$$u(0) = p, \quad u(1) = q, \tag{1.2.36}$$

where $0 < \epsilon \leq 1$, $0 \leq \mu \leq 1$, a, b and f are sufficiently smooth functions on $\overline{\Omega} = [0,1]$, such that $a(x) \geq \alpha > 0, b(x) \geq \beta > 0$. B-spline collocation method is used on a piecewise-uniform Shishkin mesh, which leads to a tridiagonal linear system. The convergence analysis is given and the method is shown to have uniform convergence of second order. Also this method produces a spline function which is usefull to obtain the solution at any point of the interval, whereas the finite difference method gives the solution only at selected nodal points.

Gracia et al. [38] constructed a monotone finite difference scheme on the Shishkin mesh for the two parameter problem

$$\varepsilon u''(x) + \mu a(x)u'(x) - b(x)u(x) = f(x), \qquad x \in (0,1), \tag{1.2.37}$$

$$u(0) = p, \quad u(1) = q, \tag{1.2.38}$$

where $0 < \epsilon \leq 1$, $0 \leq \mu \leq 1$, $a(x) \geq \alpha > 0$ and $b(x) \geq \beta > 0$. This problem encompasses both the reaction-diffusion problem when $\mu = 0$ and the convection-diffusion problem when $\mu = 1$. The finite difference operator in this paper is a combination of the central difference, mid-point and standard upwind difference operators. They established an asymptotic error bound of second-order in the maximum norm theoretically whose error constants are shown to be independent of both singular perturbation parameters.

Surla et al. [119] considered (1.2.37-1.2.38). They considerd a spline difference scheme on a piecewise uniform Shishkin mesh. They have shown that the discrete minimum principle holds for suitably chosen collocation points. Furthermore, bounds on the discrete counterparts of the layer functions are given.

O'Riordan et al. [88] considered (1.2.37-1.2.38). They examined transitions from convection-diffusion to reaction-diffusion and dealt both the cases of convection-diffusion ($\mu = 1$) and reaction-diffusion ($\mu = 0$). A numerical algorithm based on an upwind finite difference operator and an appropriate piecewise uniform mesh is constructed. Parameter-uniform error bounds for the numerical approximations are established.

Roos and Uzelac [106] considered (1.2.37-1.2.38) and solved using the streamline-diffusion finite element method (SDFEM) on a Shishkin mesh. They used mainly finite element tools for the analysis of the proposed method and proved that the pointwise error estimate is bounded by $C(N^{-1}lnN)^2$, independent of the perturbation parameters.

Kadalbajoo and Yadaw [55] presented a Ritz-Galerkin finite element method for the solution of two-parameter singularly perturbed boundary value problems similar to form (1.2.37-1.2.38). Due to these two small parameters boundary layers exist. To resolve the boundary layers a piece-wise uniform Shishkin mesh has been taken. It is relatively simple to collocate the solution at the mesh points. The results obtained using this method are more accurate than the stated existing method with same numbers of nodal points and gives the order of convergence to be almost two.

Cakir and Amiraliyev [15] consider the following singularly perturbed semilinear boundary value problem :

$$Lu := -\epsilon^2 u''(x) + \epsilon a(x) u'(x) - f(x,u) = 0, \quad x \in (0,l), \tag{1.2.39}$$

$$u(0) = A, \quad L_0 u := u(l) - \varphi(u(l_1)) = 0, \quad 0 < l_1 < l. \tag{1.2.40}$$

Author considered a uniform finite difference method on an S-mesh (Shishkin type mesh). They derived this approach on the basis of the method of integral identities using interpolating quadrature rules with the weight and remainder terms in integral form. The method is first order convergent in the discrete maximum norm, independent of the perturbation parameter except for a logarithmic factor.

Clavero et al. [22], considered the finite difference hybrid scheme constructed by Natesan et al. [83] for obtaining uniformly convergent global solution and uniformly convergent normalized flux for

$$Lu(x) \equiv -\epsilon u''(x) + b(x)u(x) = f(x), \qquad x \in D = (0,1), \tag{1.2.41}$$

$$u(0) = A, \quad u(1) = B. \tag{1.2.42}$$

where $\epsilon > 0$ is a small parameter and b, f are sufficiently smooth functions such that $b(x) \geq \beta > 0$ on $\overline{D} = [0,1]$. The global solution is obtained from the numerical solution at the mesh points of this scheme, having almost second-order uniform convergence at the nodal points when it is constructed on a piecewise uniform Shishkin mesh. Using a classical cubic spline, they defined the solution and the normalized flux on the entire domain. They proved that the uniform order of convergence of the global solution is the same as that of the hybrid scheme at the mesh points. In addition, the global normalized flux is also almost second-order uniformly convergent in the whole domain.

Rashidinia et. al. [101] used spline in compression to develop a class of methods which are second order and fourth order convergent for (1.2.41-1.2.42).

Bawa [9] proposed a parallel computational technique based on Numerov's scheme to solve the singularly perturbed self-adjoint boundary-value problem (BVP) (1.2.41-1.2.42). He divided the whole domain into three non-overlapping sub-domains, and corresponding subproblems are obtained by using zeroth-order approximations of the solution at the boundaries of these sub-problems. The sub-problems corresponding to boundary-layer regions are solved using Numerov's method after the introduction of suitable stretching variables and the solution of the reduced problem is taken as an approximate solution in the outer region. A numerical example is provided to show the efficiency and accuracy of the technique.

Natesan et al. [83] proposed a numerical scheme for (1.2.41-1.2.42) which is a combination of the cubic splines and the classical central difference scheme. The proposed scheme is applied on an appropriate piecewise uniform Shishkin mesh. They shows that, at each mesh point the method is uniformly convergent of second order and the normalized flux obtained via the cubic spline from the numerical solution is also uniformly convergent. They, also constructed the global solution using cubic splines, which is uniformly convergent in the boundary layer regions.

Mukesh and Rao [60] proposed a high order parameter robust finite difference method for

$$Lu(x) \equiv -\epsilon u''(x) + b(x)u(x) = f(x), \qquad x \in D = (0,1), \tag{1.2.43}$$

$$u(0) = 0, \quad u(1) = 0. \tag{1.2.44}$$

where $\epsilon > 0$ is a small parameter and b, f are sufficiently smooth functions such that $b(x) \geq \beta > 0$ on $\overline{D} = [0,1]$. The problem is discretized using a suitable combination of fourth order compact difference scheme and central difference scheme on generalized Shishkin mesh. The convergence analysis is given and the method is proved to be almost fourth order uniformly convergent in maximum norm with respect to singular perturbation parameter ϵ.

Bawa and Clavero [10] consider a one dimensional singularly perturbed reaction-diffusion equation (1.2.43-1.2.44). A modified Shishkin mesh is introduced and a higher order compact finite difference solution on this mesh is presented. Piece-wise cubic interpolants for both the exact solution and discrete solution are formulated. Thanks to the modified Shishkin mesh, the authors proved that the convergence is uniform in the sense that the convergence accuracy is the same for any value of the diffusion parameter ϵ. More precise, the convergence order analysis contains two principle results. The first result states that the method is, almost, of fourth convergence order. The second result states that the normalized flux of the piecewise cubic interpolant of the discrete solution approximates the normalized flux of the piecewise cubic interpolant of the exact solution by order three, almost everywhere, and by order four at mid-points of the mesh. The theoretical results are confirmed by numerical examples.

Natesan and Bawa [82] considered the following singularly perturbed reaction-diffusion boundary value problem (BVP):

$$L_\epsilon u(x) = -\epsilon u''(x) + b(x)u(x) = f(x), \quad x \in \Omega = (0,1), \tag{1.2.45}$$

$$\alpha_1 u(0) - \beta_1 u'(0) = A, \quad \alpha_2 u(1) + \beta_2 u'(1) = B. \tag{1.2.46}$$

where $\alpha_1, \beta_1, \alpha_2, \beta_2 > 0$ and $\epsilon > 0$ is a small parameter, b and f are smooth functions such that $b(x) \geq \beta > 0$, $x \in \overline{\Omega} = [0,1]$. They constructed a numerical method which involves both the cubic spline and classical finite difference schemes. They used cubic spline scheme only in the boundary layer regions and the classical finite difference scheme in the regular regions. The proposed scheme is applied on a piece-wise uniform Shishkin mesh and proved that scheme is second order convergent.

Kadalbajooand and Aggarwal [42], considered a class of self-adjoint singularly perturbed equations given by

$$-\epsilon(p(x)u')' + r(x)u = f(x), \qquad 0 < x < 1, \tag{1.2.47}$$

$$u(0) = A, \quad u(1) = B. \tag{1.2.48}$$

where $p(x) \geq p^* > 0, r(x) \geq r^* > 0, p'(x) \geq 0$, $\epsilon \in (0,1]$ is a small parameter. The authors used the fitted mesh technique to generate a piece-wise uniform mesh and then applied the B-spline method which leads to a tridiagonal linear system. The convergence analysis is given and the method is shown to have uniform convergence of second order.

Patidar [90] systematically described the idea of deriving some higher order finite difference schemes by considering the self-adjoint singularly perturbed two point boundary value problem (1.2.47-1.2.48). First he constructed a fourth-order finite difference scheme and showed that the method is ϵ-uniformly convergent on piecewise-uniform Shishkin mesh. Then, he gave a way to extend the idea of constructing the method of higher-order than four.

Linß [67] considered

$$-\epsilon^2 u''(x) + b(x)u = 0, \qquad \text{for } x \in (0,1), \tag{1.2.49}$$

$$u(0) = A, \quad u(1) = B. \tag{1.2.50}$$

The problem is discretized using a compact fourth order finite difference scheme. Although this discretization is not inverse monotone they are able to establish its maximum- norm stability and to prove its pointwise convergence on a Shishkin mesh. They have shown that the simple idea of fitting the coefficients to polynomials of a certain degree is successful in obtaining robust high-order approximations and that no combination with central differencing is necessary. In particular the scheme on a Shishkin mesh turns out to be of fourth order up to a logarithmic factor on a Shishkin mesh.

Vulanović [140] considered (1.2.49-1.2.50). Author illustrated that the discretization meshes of the Shishkin type are more suitable for high order finite-difference schemes than Bakhvalov-type meshes. He constructed the hybrid scheme and used a sixth-order five point difference scheme at most of the mesh points inside the boundary layers, whereas lower-order three-point schemes are used elsewhere. Author proved under certain conditions that this combined scheme is almost sixth-order ϵ-uniform accurate.

Kopteva and Stynes [59] considered nonlinear reaction-diffusion two-point boundary value problem with multiple solutions.

$$-\epsilon^2 u''(x) + b(x,u) = 0, \quad \text{for} \quad x \in (0,1), \tag{1.2.51}$$

$$u(0) = A, \quad u(1) = B. \tag{1.2.52}$$

Using dynamical systems techniques, they derived asymptotic properties of its discrete sub- and super-solutions. These properties are used to investigate the accuracy of solutions of a standard three-point difference scheme on layer-adapted meshes of Bakhvalov and Shishkin types. Author established second-order convergence (with, in the case of the Shishkin mesh, a logarithmic factor) in the discrete maximum norm, uniformly in ϵ for $\epsilon \leq CN^{-1}$.

Sun and Styne [117] considered a semilinear singularly perturbed reaction-diffusion problem of the type

$$-\epsilon^2 u''(x) + b(x,u) = 0, \qquad \text{for } x \in (0,1), \tag{1.2.53}$$

$$u(0) = u(1) = 0. \tag{1.2.54}$$

where ϵ is a perturbation parameter, with $0 < \epsilon \ll 1$. They considered a polynomial-based three-point difference scheme on a simple piece-wise equidistant mesh of Shishkin type. Existence and local uniqueness of a solution to the scheme are analysed. They showed that the scheme is almost fourth order accurate in the discrete maximum norm, uniformly in the perturbation parameter.

Rao and Kumar [98], considered (1.2.53-1.2.54). They used exponential spline difference scheme on the basis of splines in tension to discretize the problem on piecewise-uniform Shishkin mesh. They established almost second order uniform convergence in the discrete maximum norm. The same authors [99] developed a B-spline collocation method on a piecewise-uniform Shishkin mesh to solve a class of singularly perturbed semi-linear reaction-diffusion problems. The convergence analysis is given and the method is shown to be almost second-order convergent, uniformly with respect to the perturbation parameter ϵ in the maximum norm.

Sun and Stynes [118] considered (1.2.53-1.2.54) which may have multiple solutions. A simple central difference scheme on piecewise uniform Shishkin mesh is proposed. Asymptotic properties of solutions to this problem are discussed and analyzed. They proved that the proposed scheme is uniformly accurate of order $N^{-2} ln^2 N$, in the discrete maximum norm.

Cen [18] considered a quasi-linear singularly perturbed boundary value problem

$$\epsilon u'(x) + f(x,u,\lambda) = 0, \qquad x \in (0,1], \tag{1.2.55}$$

$$u(0) = A, \quad u(1) = B. \tag{1.2.56}$$

He discretized the considered problem using a hybrid difference scheme on Shishkin-type meshes. Hybrid difference scheme uses a midpoint difference scheme whenever the local mesh size allows to do this without losing stability, but employs an upwind difference scheme away from the boundary layer. Author established second order ϵ-uniform error estimate in discrete maximum norm. The hybrid difference scheme is the modification of the difference scheme in [64, 115].

Valarmathi and Ramanujam [130] considered the following singularly perturbed third-order ODEs of the reaction-diffusion type:

$$-\varepsilon u^{iii}(x) + b(x)u' + c(x)y(x) = f(x), \quad x \in D = (0,1), \tag{1.2.57}$$

$$u(0) = p, \quad u'(0) = q, \quad u'(1) = \gamma, \tag{1.2.58}$$

Authors used the same numerical techniques discussed in [109] to solve these problems.

Shanthi and Ramanujam [109] considered the following fourth-order ODE with a small positive parameter multiplying the highest derivative

$$-\varepsilon u^{iv}(x) - a(x)u'''(x) + b(x)u'' - c(x)y(x) = -f(x), \quad x \in D = (0,1), \tag{1.2.59}$$

$$u(0) = p, \quad u(1) = q, \quad u''(0) = -\gamma, \quad u''(1) = -s. \tag{1.2.60}$$

Authors first find the zeroth order asymptotic approximation expansion of the solution of the weakly coupled system. Then, the system is decoupled by replacing the first component of the solution by its zero order asymptotic approximation expansion of the solution in the second equation. Then the second equation is solved by the fitted operator method (FOM), fitted mesh method (FMM) and boundary value technique (BVT). Error estimates are derived for both the problems.

Shanthi and Ramanujam [96] considered the following fourth-order ODE with a small positive parameter multiplying the highest derivative

$$-\varepsilon u^{iv}(x)+b(x)u''(x)+c(x)y(x)=f(x),\quad x\in D=(0,1), \tag{1.2.61}$$

$$u(0)=p,\quad u'(0)=q,\quad u''(0)=\gamma,\quad u''(1)=s, \tag{1.2.62}$$

They transformed this boundary value problem into an equivalent weakly coupled system of two first order ordinary differential equations subject to suitable initial conditions and one second order singularly perturbed ordinary differential equations subject to suitable boundary conditions. Then they first find a zeroth order asymptotic approximation of the solution of the weakly coupled system. Then the system is decoupled by replacing the first component of the solution by its zeroth order asymptotic approximation of the solution in the second order equation. Then the second order equation is solved separately by three methods namely fitted operator method (FOM), fitted mesh method and boundary value technique (BVT).

Rashidinia and Ghasemi [100] develop a smooth approximation based on B-spline functions to compute the numerical solution of the nonlinear two-point BVPs of the form :

$$L_m(u)\ \equiv u^m(x)-\phi(x,u(x),u'(x),...,u^{m-1}(x))=0,\quad a\leq x\leq b, 1\leq m\leq 6, \tag{1.2.63}$$

$$B_m(u)=\textstyle\sum_{j=0}^{m-1}(\alpha_{ij}u^j(a)+\beta_{ij}u^j(b))=b_i,\quad 0\leq i\leq m-1. \tag{1.2.64}$$

where α_{ij}, β_{ij} and b_i are given real constants. A numerical method based on B-spline is developed to solve the general nonlinear two-point boundary value problems up to order 6. The standard formulation of sextic spline for the solution of boundary value problems leads to non-optimal approximations. In order to derive higher orders of accuracy, high order perturbations of the problem are generated and applied to construct the numerical algorithm. The error analysis and convergence properties of the method are studied via Greens function approach.

Jator and Li [41] derived a third derivative method (TDM) with continuous coefficients and used to obtain a main and additional methods, which are simultaneously applied to provide all approximations on the entire interval for initial and boundary value problems of the form $y'' = f(x, y, y')$. The convergence analysis of the method is discussed. An algorithm involving the TDMs is developed and equipped with an automatic error estimate based on the double mesh principle.

Jator [40] considered a linear multistep method (LMM) with continuous coefficients and directly applied to solve second order initial value problems (IVPs). The method is derived through the interpolation and collocation procedures by the matrix inverse approach. The continuous method is used to obtain Multiple Finite Difference Methods (MFDMs) (each of order 6) which are combined as simultaneous numerical integrators to provide a direct solution to IVPs over subintervals which do not overlap. The convergence of the MFDMs is discussed by conveniently representing the MFDMs as a block method and verifying that the block method is zero-stable and consistent.

1.2.2 System of singularly perturbed differential equations

Valanarasu and Ramanujam [125] considered a system of singularly perturbed second-order ordinary differential equations:

$$\begin{aligned} L\vec{u}(x) \equiv & \begin{pmatrix} -\epsilon\frac{d^2}{dx^2} & 0 \\ 0 & -\epsilon\frac{d^2}{dx^2} \end{pmatrix} \vec{u}(x) + \begin{pmatrix} -a_1(x)\frac{d}{dx} & 0 \\ 0 & -a_2(x)\frac{d}{dx} \end{pmatrix} \vec{u}(x) + B\vec{u}(x) \\ & = \vec{f}(x), \quad x \,\varepsilon\, \Omega, \qquad\qquad (1.2.65) \\ \vec{u}(0) = & \begin{pmatrix} p \\ r \end{pmatrix} \quad \vec{u}(1) = \begin{pmatrix} q \\ s \end{pmatrix}, \end{aligned}$$

where ϵ is a small parameter,

$$\vec{u}(x) = \begin{pmatrix} u_1(x) \\ u_2(x) \end{pmatrix}, \quad B = \begin{pmatrix} b_{11}(x) & b_{12}(x) \\ b_{21}(x) & b_{22}(x) \end{pmatrix}, \quad \vec{f}(x) = \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix},$$

the functions $a_1(x), a_2(x), b_{12}(x), b_{21}(x), b_{22}(x), f_1(x), f_2(x)$ are sufficiently smooth and satisfy the inequalities

$$\begin{aligned} & a_1(x) \geq \alpha_1 > 0, \quad a_2(x) \geq \alpha_2 > 0, \\ & b_{11}(x) > \mid b_{12}(x) \mid, \quad b_{22}(x) \geq \mid b_{21}(x) \mid, \\ & b_{12}(x) \leq 0, \quad b_{21}(x) \leq 0 \quad \forall\, x \,\varepsilon\, \overline{\Omega}, \end{aligned}$$

with $\Omega = (0,1), \overline{\Omega} = [0,1]$ and $u_1, u_2 \,\varepsilon\, C^2(\Omega) \cap C(\overline{\Omega})$.
An asymptotic expansion approximation to the solution of boundary value problem BVP is constructed by using the basic idea of WKB method. Then, they formulate initial value problems (IVPs) and a system of terminal value problems (TVPs). The IVPs are happened to be singularly perturbed problems, which are then solved by using exponentially fitted finite-difference schemes [25]. The necessary error estimates are derived and examples are provided to illustrate the method.
Z. Cen [16] proposed a parameter uniform finite difference method for the following system of equations

$$L_1\vec{u} \equiv -\epsilon u_1''(x) - a_1(x)u_1'(x) + b_{11}(x)u_1(x) + b_{12}(x)u_2(x) = f_1(x), \tag{1.2.66}$$

$$L_2\vec{u} \equiv -\mu u_2''(x) - a_2(x)u_2'(x) + b_{21}(x)u_1(x) + b_{22}(x)u_2(x) = f_2(x), \tag{1.2.67}$$

where $\vec{u} = (u_1, u_2)^T$, $x \in \Omega = (0,1)$ with the boundary conditions:

$$\vec{u}(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

$$\vec{u}(1) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

$0 < \epsilon \leq \mu \ll 1$. The functions $a_1(x)$, $a_2(x)$, $b_{11}(x)$, $b_{12}(x)$, $b_{21}(x)$, $b_{22}(x)$, $f_1(x)$, $f_2(x)$ are sufficiently smooth and satisfy the following inequalities:

$$(i) \quad a_1(x) \geq \alpha > 0, \quad a_2(x) \geq \alpha > 0, \tag{1.2.68}$$

$$(ii) \quad \min\{b_{11}(x) + b_{12}(x), b_{21}(x) + b_{22}(x)\} > \beta > 0, x \in \overline{\Omega}. \tag{1.2.69}$$

$\forall x \in \overline{\Omega} = [0,1]$ and $u_1, u_2 \in C^2(\Omega) \cap C(\overline{\Omega})$. He proposed a upwind difference scheme on a piecewise uniform Shishkin mesh The scheme is almost first order uniformly convergent with respect to small parameter.

Priyadharshini et al. [93] considered

$$L_1\vec{u} \equiv -\epsilon u_1''(x) + a_1(x)u_1'(x) + b_{11}(x)u_1(x) + b_{12}(x)u_2(x) = f_1(x), \tag{1.2.70}$$

$$L_2\vec{u} \equiv -\mu u_2''(x) + a_2(x)u_2'(x) + b_{21}(x)u_1(x) + b_{22}(x)u_2(x) = f_2(x), \tag{1.2.71}$$

where $\vec{u} = (u_1, u_2)^T$, $x \in \Omega = (0,1)$ with the boundary conditions:

$$\vec{u}(0) = \begin{pmatrix} p \\ q \end{pmatrix},$$

$$\vec{u}(1) = \begin{pmatrix} r \\ s \end{pmatrix}.$$

$0 < \epsilon = \mu \ll 1$. The functions $a_1(x)$, $a_2(x)$, $b_{11}(x)$, $b_{12}(x)$, $b_{21}(x)$, $b_{22}(x)$, $f_1(x)$, $f_2(x)$ are sufficiently smooth and satisfy the following inequalities:

$$(i) \quad a_1(x) \geq \alpha_1 > 0, \quad a_2(x) \geq \alpha_2 > 0, \tag{1.2.72}$$

$$(ii) \quad b_{12}(x) \leq 0, \quad b_{21}(x) \leq 0, \tag{1.2.73}$$

$$(iii) \quad b_{11}(x) + b_{12}(x) \geq 0, \quad b_{21}(x) + b_{22}(x) > 0, x \in \overline{\Omega}. \tag{1.2.74}$$

$\forall x \in \overline{\Omega} = [0,1]$ and $u_1, u_2 \in C^2(\Omega) \cap C(\overline{\Omega})$. They proposed two hybrid difference schemes on the Shishkin mesh. The first hybrid scheme uses central finite difference scheme in the fine mesh region and mid-point difference scheme in the coarse region. The second hybrid difference scheme uses cubic spline in the fine mesh region and the mid-point difference scheme in the coarse mesh region. Both the schemes are almost second order convergent in the supremum norm independent of the diffusion parameter.

O'Riordan and Stynes [89] examined a system of two coupled singularly perturbed convection-diffusion ordinary differential equations

$$\epsilon u_1''(x) + a_{11}(x)u_1'(x) + a_{12}(x)u_2'(x) = f_1(x) \quad on\,(0,1), \tag{1.2.75}$$

$$\epsilon u_2''(x) + a_{21}(x)u_2'(x) + a_{22}(x)u_2'(x) = f_2(x) \quad on\,(0,1), \tag{1.2.76}$$

$$u_j(0) = d_{j,0}, \quad u_j(1) = d_{j,1}, \quad for\, j = 0,1. \tag{1.2.77}$$

The problem does not satisfy a conventional maximum principle. Authors decomposed its solution into regular and layer components and bounds on the derivatives of these components are established that show explicitly their dependence on the small parameter. Then, they constructed a numerical method consisting of simple upwinding and an appropriate piecewise-uniform Shishkin mesh and show first order convergent, uniformly in the small parameter.

Bellew and O'Riordan [13] constructed a numerical method for the following system

$$\epsilon u_1''(x) + a_{11}(x)u_1'(x) = f_1(x), \quad x \in \Omega, \tag{1.2.78}$$

$$\mu u_2''(x) + a_{22}(x)u_2'(x) + a_{21}(x)u_1'(x) = f_2(x), \quad x \in \Omega, \tag{1.2.79}$$

They assume that $0 < \epsilon \ll 1, 0 < \mu \ll 1$. The functions $a_{11}(x)$, $a_{21}(x)$, $a_{22}(x)$, $f_1(x)$, $f_2(x)$ $\in C^3(\Omega)$ and satisfy the following inequality:

$$(i) \quad a_{11}(x) \geq \alpha_1 > 0, \quad a_{22}(x) \geq \alpha_2 > 0,$$

$\forall x \in \overline{\Omega} = [0,1]$ and $u_1(0), u_1(1), u_2(0), u_2(1)$ are given constants. The numerical method is composed of an upwind finite difference operator which is applied on a piece-wise uniform Shishkin mesh. The numerical approximations are shown to converge to the continuous solutions uniformly with respect to two singular perturbation parameters.

Matthews et al. [77] considered a system of singularly perturbed reaction-diffusion boundary value problems:

$$\begin{aligned} L_{\epsilon_1,\epsilon_2}\vec{u}(x) \equiv \begin{pmatrix} -\epsilon_1 \frac{d^2}{dx^2} & 0 \\ 0 & -\epsilon_2 \frac{d^2}{dx^2} \end{pmatrix} \vec{u}(x) + A(x)\vec{u}(x) = \vec{f}(x), \quad x \in \Omega, \\ \vec{u}(0) = \begin{pmatrix} p \\ r \end{pmatrix} \quad \vec{u}(1) = \begin{pmatrix} q \\ s \end{pmatrix}, \end{aligned} \tag{1.2.80}$$

where

$$\vec{u}(x) = \begin{pmatrix} u_1(x) \\ u_2(x) \end{pmatrix}, \quad A = \begin{pmatrix} a_{11}(x) & a_{12}(x) \\ a_{21}(x) & a_{22}(x) \end{pmatrix}, \quad \vec{f}(x) = \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix}, \tag{1.2.81}$$

the functions $a_{11}(x), a_{12}(x), a_{21}(x), a_{22}(x), f_1(x), f_2(x)$ are sufficiently smooth and satisfy the inequalities

$$\begin{aligned} a_{11}(x) >\mid a_{12}(x) \mid, \quad a_{22}(x) \geq\mid a_{21}(x) \mid, \\ a_{12}(x) \leq 0, \quad a_{21}(x) \leq 0 \quad \forall\, x\, \varepsilon\, \overline{\Omega}, \end{aligned}$$

with $0 < \epsilon_1, \epsilon_2 \leq 1$.

They provided a method for (1.2.80-1.2.81) on a Shishkin mesh using classical finite difference scheme. They examined the case, when $\epsilon_1 = \epsilon, \epsilon_2 = 1$ and establish second order parameter uniform convergence. Similar system was studied by Natesan and Deb [84]. They proposed a uniformly convergent second order hybrid scheme on a Shishkin mesh. The hybrid scheme is the combination of the cubic spline and central difference scheme. In the boundary layer region cubic spline is used whereas in the regular layer region central difference scheme is used. Nonlinear system of equations have been handled by the proposed scheme after linearization.

Madden and Stynes [75] considered (1.2.80-1.2.81). They examined the case when $0 < \epsilon_1 \leq \epsilon_2 \ll 1$. The solutions to the system exhibit boundary layers that overlap and interact. They analysed the structure of these layers, and this leads to the construction of a piecewise-uniform mesh that is a variant of the usual Shishkin mesh. On this mesh, central difference scheme is proved to be almost first-order accurate, uniformly in both the small parameters.

Linß and Madden [68] constructed a numerical method for the following system

$$-\epsilon^2 u_1''(x) + a_{1,1}(x)u_1(x) + a_{1,2}(x)u_2(x) = f_1(x), \quad x \,\varepsilon\, \Omega, \tag{1.2.82}$$

$$-\mu^2 u_2''(x) + a_{2,2}(x)u_2(x) + a_{2,1}(x)u_1(x) = f_2(x), \quad x \,\varepsilon\, \Omega, \tag{1.2.83}$$

$$u_1(0) = u_2(0) = u_1(1) = u_2(1) = 0. \tag{1.2.84}$$

A central difference scheme on layer-adapted piece-wise uniform meshes is used to solve the system numerically. They have shown that the scheme is almost second-order convergent, uniformly in both perturbation parameters.

Amiraliyev [1] considered the following singularly perturbed non-linear system of initial boundary value problem

$$\varepsilon u_1' + f_1(x, u_1, u_2) = 0, \quad 0 < x \leq l, \tag{1.2.85}$$

$$-u_2'' + f_2(x, u_1, u_2) = 0, \quad 0 < x < l, \tag{1.2.86}$$

$$u_1(0) = u, \quad u_2(0) = A, \quad u_2(l) = B. \tag{1.2.87}$$

Author derived a method based on using finite elements with piece-wise constant and piecewise linear basis functions and appropriate quadrature formulae with the remainder term in integral form. In the boundary layer, author introduced the non-uniform mesh, which is constructed by using the estimates of derivatives of the exact solution and the analysis of the local truncation error and proved that the solution converges point-wise independently of the singular perturbation parameter.

Hemavathi et al. [39] gave a numerical method for a system of first order singularly perturbed ordinary differential equations of the form:

$$L_\epsilon \vec{u_\epsilon}(x) = \begin{pmatrix} \epsilon D & 0 & . & . & 0 \\ 0 & \epsilon D & . & . & 0 \\ . & . & . & . & . \\ . & . & . & . & . \\ 0 & 0 & . & . & \epsilon D \end{pmatrix} \vec{u_\epsilon}(x) + A(x)\vec{u_\epsilon}(x) = \vec{f}(x), \tag{1.2.88}$$

$$\vec{u_\epsilon}(0) = (u_{\epsilon,1}(0), u_{\epsilon,2}(0), ...u_{\epsilon,n}(0))^T, x \in (0,1],$$

where $\vec{u_\epsilon} = (u_{\epsilon,1}, u_{\epsilon,2}, ...u_{\epsilon,n})^T$, $A(x) = \begin{pmatrix} a_{11}(x) & a_{12}(x) & . & . & a_{1n}(x) \\ a_{21}(x) & a_{22}(x) & . & . & a_{2n}(x) \\ . & . & . & . & . \\ . & . & . & . & . \\ a_{n1}(x) & a_{n2}(x) & . & . & a_{nn}(x) \end{pmatrix}$,

$\vec{f}(x) = (f_1(x), f_2(x), ..., f_n(x))^T$, D denotes $\frac{d}{dx}$ and $u_\epsilon \in C^{(1)}(\Omega), \Omega = (0,1)$. The singular perturbation parameter ϵ satisfies $0 < \epsilon \leq 1$. The functions $a_{ij}, f_i \in C^2(\overline{\Omega}), \overline{\Omega} = [0,1]$, $i, j = 1, 2, ..., n$ satisfies the following inequalities:

$$(i) \quad a_{ii} > \sum_{j=1, j\neq i}^{n} |a_{ij}(x)|, \quad i = 1(1)n. \tag{1.2.89}$$

$$(ii) \quad a_{ij} < 0, \quad i, j = 1(1)n, i \neq j.$$

They discretised (1.2.88-1.2.89) by using a fitted mesh method composed of a classical finite difference operator on a piece-wise uniform fitted mesh and shows that it gives essentially first order parameter uniform convergence in the maximum norm. Numerical results are also presented in support of the theory.

Cen et al. [21] considered following system of singularly perturbed initial value problems

$$\varepsilon u_1'(x) + f_1(x, u_1, u_2) = 0, \quad 0 < x \leq 1, \tag{1.2.90}$$

$$\varepsilon u_2'(x) + f_2(x, u_1, u_2) = 0, \quad 0 < x \leq 1, \tag{1.2.91}$$

$$u_1(0) = A, \quad u_2(0) = B, \tag{1.2.92}$$

where $\varepsilon > 0$ is a small parameter. On Shishkin mesh, they constructed a hybrid finite difference scheme which is almost second order accurate.

Cen et al. [20] considered following system of singularly perturbed initial value problems

$$\varepsilon_1 u_1'(x) + f_1(x, u_1, u_2) = 0, \quad 0 < x \leq 1, \tag{1.2.93}$$

$$\varepsilon_2 u_2'(x) + f_2(x, u_1, u_2) = 0, \quad 0 < x \leq 1, \tag{1.2.94}$$

$$u_1(0) = A, \quad u_2(0) = B, \tag{1.2.95}$$

where $0 < \varepsilon_1 \leq \varepsilon_2 \ll 1$. On Shishkin mesh, they constructed a hybrid finite difference scheme which is almost second order accurate uniformly in both small parameters. The hybrid difference scheme is the modification of the difference scheme in [64, 115].

1.2.3 One dimensional problems with interior and turning point layers

Valanarasu and Ramanujam [127] suggested an asymptotic initial value method to solve the following class of problems:

$$L_\epsilon u(x) \equiv \varepsilon u''(x) + a(x)u'(x) = f(x), \qquad x \in \Omega^- \cup \Omega^+, \tag{1.2.96}$$

$$u(0) = p, \quad u(1) = q, \tag{1.2.97}$$

where $\varepsilon > 0$ is a small parameter, $a(x)$ is sufficiently smooth function on $\overline{\Omega} = [0, 1]$, such that $a(x) \geq \beta > 0$, $f(x)$ is sufficiently smooth function on $\Omega^- \cup \Omega^+ \cup \{0, 1\}$, f and its derivatives have jump discontinuity at d. Because f is discontinuous at d, the solution u of (1.2.96-1.2.97) does not necessarily have a continuous second derivative at the point d. Thus u does not belong to the class of functions $C^2(\Omega)$. But the first derivative of the solution exists and is continuous in Ω. Under these assumptions, the SPPs (1.2.96-1.2.97) have a solution $u \in C^0(\overline{\Omega}) \cap C^1(\Omega) C^2(\Omega^- \cup \Omega^+)$ [28]. The method has been constructed by using the basic ideas of well known perturbation method WKB. Then some initial value problems and terminal value problems are constructed such that their solutions are the terms of this asymptotic expansion. These initial value problems are happened to be singularly perturbed problems and therefore fitted mesh method (Shishkin mesh) is used to solve these problems. For the scheme optimal error bound is of order $O(N^{-1} lnN)$.

Similar equations were considered by Cen [17]. He uses the central difference scheme in the outer region and an upwind scheme in the boundary layer region. The scheme is applied on a Shishkin mesh and is proved of second order convergent in the maximum norm independent of the diffusion parameter.

Priyadharshini and Ramanujam [92] considered (1.2.96-1.2.97) and used the hybrid scheme [17] on a Shishkin mesh. On this hybrid scheme, an ϵ uniform convergent numerical approximation to the scaled first derivative of the solution is generated. The main theoretical result is the ϵ-uniform convergence in the supremum norm of the approximations generated by the finite difference method.

Farrell et al. [28] examined (1.2.96-1.2.97). They constructed a numerical method by using the standard upwind finite difference method on a piecewise uniform mesh of Shishkin type, which is fitted to the interior and boundary layers and proved almost first order uniform convergence in the perturbation parameter. They proved a comparison principle for the discrete problem and introduced a decomposition of the discrete solution, which enabled them to prove the ϵ-uniform convergence in the global maximum norm of the approximations generated by the finite difference method.

Farrell et al. [29] considered the same problem discussed in [28], having a discontinuity only in the convection coefficient $a(x)$. The interior layers generated here are strong, while the interior layers in the case of [28] were weak. They used the standard upwind finite difference method on Shishkin mesh to discretize the problem and proved ϵ-uniform convergence in the global maximum norm.

Roos and Zarin [107] consider singularly perturbed convection-diffusion problem with a point source given by

$$L_\epsilon u(x) \equiv -\varepsilon u''(x) + b(x)u'(x) + c(x)u(x) = f(x) + \delta(x-d),\ x \in \Omega^- \cup \Omega^+ \quad (1.2.98)$$

$$u(0) = 0, \quad u(1) = 0, \quad (1.2.99)$$

where $\varepsilon > 0$ is a small parameter, $\Omega = (0,1)$, $d \in \Omega, \Omega^- = (0,d)$, $\Omega^+ = (d,1)$ and $\delta(x-d)$ denotes the dirac delta function. $b, c, f$are sufficiently smooth function on $\overline{\Omega} = [0,1]$, such that $b(x) \geq \beta > 0$ and $c(x) \geq 0$. For this problem, solution u exhibits an exponential interior layer at $x = d$ and the standard exponential boundary layer at $x = 1$. The problem is solved by using the streamline-diffusion finite element method on a class of Shishkin-type meshes. Authors proved that the method is almost optimal with second order of ϵ-uniform convergence in the maximum norm.

Linß [65] considerd

$$L_\epsilon u(x) \equiv -\varepsilon u''(x) - (b(x)u(x))' + c(x)u(x) = f(x) + \delta(x-d),\; x \in \Omega^- \cup \Omega^+ \tag{1.2.100}$$

$$u(0) = 0, \quad u(1) = 0, \tag{1.2.101}$$

where $\varepsilon > 0$ is a small parameter, $\Omega = (0,1), d \in \Omega, \Omega^- = (0,d),\ \Omega^+ = (d,1)$ and $\delta(x-d)$ denotes the dirac delta function. $b, c, f$are sufficiently smooth function on $\overline{\Omega} = [0,1]$, such that $b(x) \geq \beta_1 > 0$ on $(0,d)$, $b(x) \geq \beta_2 > 0$ on $(d,1)$, and set $\beta = \min\{\beta_1, \beta_2\}$. For this problem, solution u exhibits an exponential boundary layer at the outflow boundary $x = 0$ and an internal layer at $x = d$. He solved the problem numerically by using two upwind difference schemes on general meshes and studies the convergence. Then these results are applied to two standard layer-adapted meshes and proved ϵ-uniform convergence in the discrete maximum norm on these Shishkin and Bakhvalov meshes.

Consider the problem:

$$L_\epsilon u(x) \equiv -\varepsilon u''(x) + a(x)u(x) = f(x), \qquad x \in \Omega^- \cup \Omega^+, \tag{1.2.102}$$

$$u(0) = p, \quad u(1) = q, \tag{1.2.103}$$

where $0 < \epsilon \ll 1$ is a small parameter, $a(x)$ is a sufficiently smooth function on $\overline{\Omega} = [0,1]$, such that $a(x) \geq \beta > 0$, $f(x)$ is a sufficiently smooth function on $\Omega^- \cup \Omega^+ \cup \{0,1\}$, f and its derivatives have jump discontinuity at d. Because f is discontinuous at d, the solution u of (1.2.102-1.2.103) does not necessarily have a continuous second derivative at the point d. Thus, u need not belong to the class of functions $C^2(\Omega)$. But the first derivative of the solution exists and is continuous in Ω. Under these assumptions, (1.2.102-1.2.103) have a solution $u \in C^0(\overline{\Omega}) \cap C^1(\Omega) \cap C^2(\Omega^- \cup \Omega^+)$ [30].

Valanarasu and Ramanujam [126] proposed an asymptotic initial value method (AIVM) to solve (1.2.102-1.2.103) on an appropriate Shishkin mesh which is of first order convergence.

Miller et al. [80] constructed a numerical method for (1.2.102-1.2.103) arising in the context of models of simple semiconductor devices. The method comprises a standard finite difference operator on a non-standard piecewise-uniform fitted mesh, which generates numerical solutions that converge parameter-uniformly in the maximum norm. The mesh is fitted to the boundary and interior layers that occur in the solution of the problem.

Shanti et al. [111] considered the following ODE having two parameters with a discontinuous source term

$$\varepsilon u''(x) + \mu a(x)u'(x) - b(x)u(x) = f(x), \qquad x \in \Omega^- \cup \Omega^+, \tag{1.2.104}$$

$$u(0) = p, \quad u(1) = q, \tag{1.2.105}$$

where $0 < \epsilon \leq 1$, $0 \leq \mu \leq 1$, a and b are sufficiently smooth functions on $\overline{\Omega} = [0, 1]$, such that $a(x) \geq \alpha > 0$, $b(x) \geq \beta > 0$, $f(x)$ is a sufficiently smooth function on $\Omega^- \cup \Omega^+ \cup \{0, 1\}$, f and its derivatives have jump discontinuity at d. Under these assumptions, (1.2.104-1.2.105) have a solution $u \in C^0(\overline{\Omega}) \cap C^1(\Omega) \cap C^2(\Omega^- \cup \Omega^+)$. An appropriate piecewise uniform mesh is constructed, and classical upwind finite difference schemes are used on this mesh to obtain the discrete system of equations. Parameter uniform error bounds for the numerical approximations are established.

Shanthi and Ramanujam [128] developed a computational method to solve singularly perturbed boundary value problems for third-order equations of the type

$$-\varepsilon u'''(x) + a(x)u''(x) + b(x)u' + c(x)y(x) = f(x), \quad x \in \Omega^- \bigcup \Omega^+, \tag{1.2.106}$$

$$u(0) = p, \quad u'(0) = q, u'(1) = r, \tag{1.2.107}$$

where $0 < \epsilon \leq 1$, $a(x)$, $b(x)$ and $c(x)$ are sufficiently smooth functions on $\overline{\Omega} = [0, 1]$, such that $a(x) \leq -\alpha$, $b(x) \geq 0$, $0 \geq c(x) \geq -\gamma$, $f(x)$ is a sufficiently smooth function on $\Omega^- \cup \Omega^+ \cup \{0, 1\}$,

f and its derivatives have jump discontinuity at d. Under these assumptions, (1.2.106-1.2.107) have a solution $u \in C^1(\overline{\Omega}) \cap C^2(\Omega) \cap C^3(\Omega^- \cup \Omega^+)$. The BVP (1.2.106-1.2.107) is reduced to a weakly coupled system consisting of one first order ordinary differential equation with a suitable initial condition and one second order singularly perturbed ODE subject to boundary conditions. In order to solve this system, a computational method is suggested. First, in this method, they find the zero order asymptotic expansion approximation of the solution of the weakly coupled system. Then, the system is decoupled by replacing the first component of the solution by its zero order asymptotic expansion approximation of the solution in the second equation. After that the second equation is solved by a finite difference method on Shishkin mesh (a fitted mesh method). Similar technique was applied by the same authors in [129] for the case $a(x) = 0$.

Shanthi and Ramanujam [97] developed a computational method to solve singularly perturbed boundary value problems for fourth-order equations of the type

$$-\varepsilon u^{iv}(x) + a(x)u'''(x) + b(x)u''(x) - c(x)u(x) = -f(x), \quad x \in \Omega^- \bigcup \Omega^+, \tag{1.2.108}$$

$$u(0) = p, \quad u(1) = q, u''(0) = -\gamma, \quad u''(1) = -s, \tag{1.2.109}$$

where $\Omega^- \bigcup \Omega^+ = (0, d) \bigcup (d, 1)$, $d \in \Omega = (0, 1)$. Further they assumed that $f(x)$ is a sufficiently smooth function on $\Omega^- \cup \Omega^+ \cup \{0, 1\}$, f and its derivatives have jump discontinuity at d. In general this discontinuity gives rise to an interior layer in the second derivative of the exact solution of the problem. More precisely the boundary layer is weak for the second derivative and stronger for third derivative. They transformed the problem into a system of weakly coupled systems of two second order ODEs, one without the parameter and the other with the parameter ϵ, and suitable boundary conditions. Then authors first find the zero-order asymptotic approximation expansion of the solution of the weakly coupled system. Then the system is decoupled by replacing the first component of the solution by its zero-order asymptotic approximation expansion of the solution in the second equation. Then the second equation is solved by the numerical method which is constructed for this problem which involves Shishkin mesh. Similar technique was applied by the same authors in [110] for the case $a(x) = 0$.

Babu and Ramanujam [7] considered singularly perturbed boundary value problems for third and fourth order ordinary differential equations with discontinuous source term and a small positive parameter multiplying the highest derivative. Because of the type of boundary conditions imposed on these equations these problems can be transformed into weakly coupled systems. In this system, the first equation does not have the small parameter but the second contains it. Authors presented a computational method named as ”an asymptotic finite element method” for solving these systems. In this method they first find an zero order asymptotic approximation to the solution and then the system is decoupled by replacing the first component of the solution by this approximation in the second equation. Then they solved the second equation independently by a fitted mesh finite element method (FEM).

Tamilselvan and Ramanujam [121] proposed a parameter uniform finite difference method for the following system of equations

$$L_1\vec{u} \equiv -\epsilon u_1''(x) + a_1(x)u_1'(x) + b_{11}(x)u_1(x) + b_{12}(x)u_2(x) = f_1(x), \qquad (1.2.110)$$

$$L_2\vec{u} \equiv -\epsilon u_2''(x) + a_2(x)u_2'(x) + b_{21}(x)u_1(x) + b_{22}(x)u_2(x) = f_2(x), \qquad (1.2.111)$$

where $\vec{u} = (u_1, u_2)^T$, $x \in \Omega = (0, 1)$ with the boundary conditions:

$$\vec{u}(0) = \begin{pmatrix} p \\ r \end{pmatrix},$$

$$\vec{u}(1) = \begin{pmatrix} q \\ s \end{pmatrix}.$$

Without loss of generality, we shall assume that $0 < \epsilon = \mu \ll 1$. The functions $a_1(x)$, $a_2(x)$, $b_{11}(x)$, $b_{12}(x)$, $b_{21}(x)$, $b_{22}(x)$, $f_1(x)$, $f_2(x)$ are sufficiently smooth and satisfy the following inequalities:

$$(i) \quad a_1(x) \geq \alpha_1 > 0, \forall x \in \Omega^- \quad a_2(x) \leq -\alpha_2 < 0, \forall x \in \Omega^+, \tag{1.2.112}$$

$$(ii) \quad b_{11}(x) > |b_{12}(x)|, \quad b_{22}(x) > |b_{21}(x)|, \tag{1.2.113}$$

$$(iii) \quad b_{12}(x) < 0, \quad b_{21}(x) < 0, \tag{1.2.114}$$

$\forall x \in \overline{\Omega} = [0,1]$ and $u_1, u_2 \in C^0(\overline{\Omega}) \cap C^1(\Omega) \cap C^2(\Omega^- \cup \Omega^+)$. They constructed a numerical method based on finite difference scheme and Shishkin mesh for singularly perturbed two second order weakly coupled system of ordinary differential equations with discontinuous convection coefficients. They derived an error estimate to show that the method is uniformly convergent with respect to the singular perturbation parameter. Similar technique was applied by the same authors [122] for the case when $a_1(x) = a_2(x) = 0$.

Kadalbajoo and Gupta [44] proposed a numerical scheme to solve a singularly perturbed two-point boundary value problem

$$L_\epsilon u(x) = \epsilon u''(x) + a(x)u'(x) - b(x)u(x) = f(x), \quad x \in \Omega = (-1,1), \tag{1.2.115}$$

$$u(-1) = A, \quad u(1) = B, \tag{1.2.116}$$

with a turning point at $x = 0$. The problem exhibits twin boundary layers at both the end points. The scheme comprises B-spline collocation method on a non-uniform mesh of Shishkin type. Authors established asymptotic bounds for the derivative of the analytical solution. They show that the present method is of second-order accurate in the discrete maximum norm. Spline collocation methods are more economical and straight forward to use, since they require no numerical integrations as in finite element method or Galerkin approximate. Also, B-spline collocation method in solving differential equations leads to banded matrices with a small number of bands, as opposed to the full matrices one obtains using (say), polynomials, trigonometric functions, and other well-known non-piecewise approximates. Moreover, the present method does not require any information about the asymptotic approximation of the solution and is easy to implement. In fact, they proved that the B-spline collocation method provides uniform convergence in ϵ.

Natesan et al. [85] presented a classical finite-difference scheme on piecewise-uniform Shishkin mesh to solve (1.2.115-1.2.116). They proved that the proposed method is almost first order uniformly accurate in the perturbation parameter ϵ.

Linß [66] gave a classification and a useful survey of layer-adapted meshes for stationary convection-dominated one and two-dimensional convection-diffusion problems. In this survey paper, theoretical results are presented for a number of standard numerical schemes that demonstrate that the use of properly layer-adapted meshes yield robust methods. Author also reviewed a number of techniques used in the convergence analysis of these methods

1.2.4 Singularly perturbed delay differential equations

Kadalbajoo and Sharma [51] considered singularly perturbed delay differential difference problem of the convection-diffusion type

$$\varepsilon y''(x) + a(x)y'(x-\delta) + b(x)y(x) = f(x), \quad x \in (0,1), \tag{1.2.117}$$

$$y(x) = \phi(x) \quad \text{on} \quad -\delta \leq x \leq 0, \quad y(1) = \gamma. \tag{1.2.118}$$

In this paper, authors constructed an upwind finite difference scheme on piecewise uniform Shishkin mesh and established a parameter robust error estimate. The proposed difference scheme is compared with the classical finite difference scheme. They have shown that the standard upwind difference scheme on the uniform mesh works nicely till $h < \epsilon$ but does not behave uniformly to the singular perturbation parameter ϵ as the condition $h < \epsilon$ is violated. But the proposed fitted mesh method is parameter uniform and works nicely independent of the mesh parameter h and the singular perturbation parameter ϵ.

Kadalbajoo and Ramesh [52] considered (1.2.117-1.2.118). They analysed three difference operators: a simple upwind scheme, midpoint upwind scheme and a hybrid scheme, respectively, on a Shishkin mesh to approximate the solution of the problem. The hybrid algorithm uses central difference in the boundary layer region and midpoint upwind scheme outside in the boundary layer. Authors concluded that the hybrid scheme gives better accuracy.

Kadalbajoo and Kumar [45] considered (1.2.117-1.2.118). The fitted mesh technique is employed to generate a piecewise-uniform mesh, condensed in the neighborhood of the boundary layers. B-spline collocation method is used with fitted mesh. They show that the method has almost second-order parameter-uniform convergence.

Awoke and Reddy [6] presented an asymptotic fitted approach to solve (1.2.117-1.2.118). In this approach, the singularly perturbed delay differential equations is modified by approximating the term containing negative shift using Taylor series expansion. After approximating the coefficient of the second derivative of the new equation, they introduced a fitting parameter. The three term recurrence relation obtained is solved using Thomas algorithm.

Kadalbajoo and Sharma [49] considered following class of singularly perturbed non-linear differential equations with negative shift:

$$\varepsilon y^{''} = f(x, y(x), y'(x-\delta)), \quad x \in (0, b), \tag{1.2.119}$$

$$y(x) = \phi(x) \quad \text{on} \quad -\delta \leq x \leq 0, \quad y(b) = \gamma, \tag{1.2.120}$$

where ϵ is a small singular perturbation parameter, $0 < \epsilon \ll 1$ and δ is the shift. The solution $y(x)$ of the BVP must be continuous on $[0, b]$ and continuously differentiable on $(0, b)$. Three difficulties are encountered when handling such problems: (i) the presence of non linearity; (ii) the presence of terms containing shifts; (iii) the presence of a singular perturbation parameter. To resolve the first difficulty, a quasi-linearization process is used to linearize the non-linear differential equation. After applying the quasi-linearization process, a sequence of linearized problems is obtained. This sequence converges quadratically to the solution of the original non-linear problem. To resolve the second difficulty, Taylor series is used to tackle the terms containing shifts, provided that the shifts are of a small order of the singular perturbation parameter. When shift is of high order of singular perturbation, a special type of mesh is used. Finally, to resolve the third difficulty, a piece-wise uniform mesh which is dense in the boundary layer region and coarse in the outer region is used and the derivatives are approximated using standard finite-difference operators. The difference scheme obtained is shown to be parameter uniform by establishing the parameter uniform error estimates.

Kadalbajoo and Kumar [46] considered following class of singularly perturbed non-linear differential equations with negative shift:

$$\varepsilon y^{''} = f(x, y(x), y'(x-\delta)), \quad x \in [0,1], \tag{1.2.121}$$

$$y(x) = \phi(x) \quad \text{on} \quad -\delta \leq x \leq 0, \quad y(1) = \gamma, \tag{1.2.122}$$

where ϵ is a small singular perturbation parameter, $0 < \epsilon \ll 1$ and δ is the shift of $o(\epsilon)$. The B-spline collocation method is used on a piecewise uniform Shishkin mesh. Taylor series is used to tackle the term containing shift. The proposed method is accurate of order almost two.

Kadalbajoo and Ramesh [53] considered second-order singularly perturbed differential difference equation with negative shift δ of the form

$$-\varepsilon y''(x) + a(x)y'(x-\delta) + b(x)y(x-\delta) + c(x)u(x) = f(x), \quad x \in (0,1), \tag{1.2.123}$$

$$y(x) = \phi(x) \text{ on } \quad -\delta \leq x \leq 0, \quad y(1) = \gamma. \tag{1.2.124}$$

They discussed the same three finite difference schemes considered in [52] with the grid adaptation strategies. They examined grid redistribution by solving the considered problem using hybrid method for some values of ϵ and N and gets improved accuracy by this grid adaptation strategy with almost the same computational cost.

Kadalbajoo and Sharma [50] developed a numerical approach, based on the finite-difference method, for solving a mathematical model arising from a model of neuronal variability. The mathematical modelling of the determination of the expected time for generation of action potentials in nerve cells by random synaptic inputs in dendrites includes a general BVP for a singularly perturbed differential difference equation with small shifts. The numerical method comprises a standard upwind finite difference operator on a fitted piecewise-uniform mesh which is condensed in the boundary layers. They first approximate the terms containing small shift by Taylor series and then apply the fitted mesh method, provided shifts are of $o(\epsilon)$. The convergence of the difference scheme is almost linear, but it converges independently of the singular perturbation parameter. Several numerical examples are solved to demonstrate the efficiency of the numerical scheme and to show the effect of a small shift on the behaviour of the solution.

Kadalbajoo and Patidar [47] considered six possible sub-problems of the general singular perturbation problems of the form

$$\pm\varepsilon y''(x) + a(x)y'(x) + b(x)y(x) = f(x), \qquad x \in (0,1), \tag{1.2.125}$$

$$y(0) = A, \quad y(1) = B. \tag{1.2.126}$$

In this paper authors constructed a very simple and direct parameter uniform fitted mesh finite difference method of $O(N^{-2}ln^2N)$ on piecewise-uniform Shishkin mesh, without using any acceleration of convergence techniques of Richardsons extrapolation or defect correction type.

Cen [19] considered

$$\varepsilon y^{'} = f(x, y(x), y(x-1)), \quad x \in (0,T], \tag{1.2.127}$$

$$y(x) = \phi(x) \text{ on } -1 \leq x \leq 0. \tag{1.2.128}$$

On Shishkin mesh, author constructed a hybrid finite difference scheme which is almost second order accurate uniformly in both small parameters. The scheme uses a midpoint difference method whenever the local mesh size allows us to do this without losing stability, but employs an upwind difference method away from the boundary layer. The hybrid difference scheme is the modification of the difference scheme in [64, 115].

Amiraliyev and Erdogan [2] considered the following singularly perturbed delay differential equation:

$$\varepsilon u'(x) + a(x)u(x) + b(x)u(x-r) = f(x), \qquad x \in \Omega, \tag{1.2.129}$$

$$u(x) = \varphi(x), \quad -r < x \leq 0, \tag{1.2.130}$$

where $\Omega = (0,T] = \bigcup_{p=1}^{m} \Omega_p, \Omega_p = \{x : r_{p-1} < x \leq r_p\}, 1 \leq p \leq m$ and $r_s = sr$, for $0 \leq s \leq m$. $0 < \varepsilon \leq 1$ is the perturbation parameter, $a(x) \geq \beta > 0$, $b(x) > 0$, $f(x)$ and $\varphi(x)$ are given sufficiently smooth functions. The solution, $u(x)$, displays in general boundary layers on the right side of each point $x = r_s$ for small values of ε. They discretize (1.2.129-1.2.130) using a numerical method, which is composed of an implicit finite difference scheme on piecewise-uniform Shishkin-meshes on each time subinterval. The difference scheme is shown to converge to the continuous solution uniformly with respect to the perturbation parameter.

1.3 Summary of thesis

In the present work, we consider mainly the one dimensional singularly perturbed problems. Methods are devised for initial value problems, boundary value problems with smooth data, boundary value problems with non-smooth data, system of boundary value problems and delay differential equations. Apart from the construction of methods, a full fledged theory for their convergence and error estimates is also presented. Numerical experiments are carried out extensively to support the theoretical results and comparisons with exiting methods is also done.

Chapter-wise summery of the work presented in the subsequent chapters of this thesis is as follows:

Chapter 2

First order singularly perturbed initial value problems (SPIVP) and system of SPIVP are considered in this chapter.

We consider singularly perturbed initial value problem (IVP):

$$L_\epsilon u(x) \equiv \varepsilon u'(x) + b(x)u(x) = f(x), \qquad x \in \Omega = (0,1), \tag{1.3.1}$$

$$u(0) = A, \tag{1.3.2}$$

where A is a constant and $\varepsilon > 0$ is a small parameter, b and f are sufficiently smooth functions, such that $b(x) \geq \beta > 0$ on $\overline{\Omega} = [0,1]$. Under these assumptions, the IVP possesses a unique solution $u(x)$ [25].

System of first order singularly perturbed ordinary differential equations:

$$L_\epsilon \vec{u_\epsilon}(x) = \begin{pmatrix} \epsilon D & 0 & . & . & 0 \\ 0 & \epsilon D & . & . & 0 \\ . & . & . & . & . \\ . & . & . & . & . \\ 0 & 0 & . & . & \epsilon D \end{pmatrix} \vec{u_\epsilon}(x) + A(x)\vec{u_\epsilon}(x) = \vec{f}(x), \tag{1.3.3}$$

$$\vec{u_\epsilon}(0) = (u_{\epsilon,1}(0), u_{\epsilon,2}(0), ...u_{\epsilon,n}(0))^T, x \in (0,1],$$

where $\vec{u_\epsilon} = (u_{\epsilon,1}, u_{\epsilon,2}, ...u_{\epsilon,n})^T$, $A(x) = \begin{pmatrix} a_{11}(x) & a_{12}(x) & . & . & a_{1n}(x) \\ a_{21}(x) & a_{22}(x) & . & . & a_{2n}(x) \\ . & . & . & . & . \\ . & . & . & . & . \\ a_{n1}(x) & a_{n2}(x) & . & . & a_{nn}(x) \end{pmatrix}$

$\vec{f}(x) = (f_1(x), f_2(x), ..., f_n(x))^T$, D denotes $\frac{d}{dx}$ and $u_\epsilon \in C^{(1)}(\Omega), \Omega = (0,1)$. The singular perturbation parameter ϵ satisfies $0 < \epsilon \leq 1$. The functions

$a_{ij}, f_i \in C^2(\overline{\Omega}), \overline{\Omega} = [0,1]$, $i, j = 1, 2, ..., n$ satisfies the following inequalities:

$$(i) \quad a_{ii} > \sum_{j=1, j\neq i}^{n} |a_{ij}(x)|, \quad i = 1(1)n$$

$$(ii) \quad a_{ij} < 0, \quad i, j = 1(1)n, i \neq j$$

In [39], a computational method is presented for a system of first order singularly perturbed ordinary differential equations. We proposed a second-order parameter-uniform convergent computational technique for them. The proposed technique is applied on an appropriate piecewise uniform Shishkin mesh. Numerical experiments are carried out on some test problems, confirming the robustness of the technique.

Chapter 3

In this chapter, we consider the following singularly perturbed delay differential equation in the interval $\overline{\Omega} = [0, m]$:

$$\varepsilon u'(x) + a(x)u(x) - b(x)u(x-1) = f(x), \qquad x \in \Omega, \tag{1.3.4}$$

$$u(x) = \varphi(x), \quad -1 < x \leq 0, \tag{1.3.5}$$

where $\Omega = (0, m] = \bigcup_{p=1}^{m} \Omega_p$, $\Omega_p = \{x : p - 1 < x \leq p\}$, $p \geq 1$. $0 < \varepsilon \leq 1$ is the perturbation parameter, $a(x) \geq \beta > 0$, $b(x) > 0$, $f(x)$ and $\varphi(x)$ are given sufficiently smooth functions. The solution, $u(x)$, displays in general boundary layers on the right side of each point $x = p - 1$ for small values of ε [2]. We extend the technique proposed in chapter 2 for (1.3.4-1.3.5).

Also, we consider the following singularly perturbed delay differential equation (DDE) in the interval $\overline{\Omega} = [0,1]$:

$$L_\epsilon u(x) \equiv \varepsilon u''(x) + a(x)u'(x-\delta) + b(x)u(x) = f(x), \quad x \in \Omega = (0,1), \tag{1.3.6}$$

$$u(x) = \varphi(x), \quad -\delta \le x \le 0, \tag{1.3.7}$$

$$u(1) = \lambda, \tag{1.3.8}$$

where $0 < \varepsilon \ll 1$ is the small perturbation parameter and the delay parameter δ is such that $0 < \delta < 1$. The functions $a(x)$, $b(x)$, $f(x)$ and $\varphi(x)$ are sufficiently smooth functions and λ is a constant. It is also assumed that $b(x) \ge \beta > 0$, $\forall x \in \overline{\Omega}$. When $\delta = 0$, the equation (1.3.6-1.3.8) reduces to a singularly perturbed differential equation. Depending upon the sign of $a(x)$, i.e., if $a(x) > 0$ (or $a(x) < 0$), a boundary layer is located at left (or right) end of domain. The layer is maintained for sufficiently small δ with $\delta \neq 0$ and $\delta = o(\epsilon)$.

A parameter-uniform numerical scheme is proposed for (1.3.6-1.3.8). We first construct a difference scheme using cubic spline and then apply it on a layer resolving piecewise uniform mesh, known as the Shishkin mesh. In the boundary layer (inner) region, the mesh is fine, and the cubic spline scheme is stable. Whereas in the regular (outer) region, the mesh is coarse, and the cubic spline scheme is not stable. To obtain stability in the outer region, one has to restrict the step size in that region, but our aim is to propose an ϵ-uniform convergent numerical scheme. Therefore, for the outer region, instead of the cubic scheme we use the finite difference scheme, mainly for stability reasons. The newly obtained hybrid scheme is convergent independent of the singular perturbation parameter.

Chapter 4

Second order reaction-diffusion and convection-diffusion SPBVPs with continuous source term are considered in this chapter.

We consider the following class of SPBVPs:

$$L_\epsilon u(x) \equiv \epsilon u''(x) + a(x)u'(x) + b(x)u(x) = f(x), \qquad x \in \Omega = (0,1), \tag{1.3.9}$$

$$u(0) = p, \quad u(1) = q \qquad\qquad\qquad , \tag{1.3.10}$$

where $0 < \epsilon \ll 1$ is a small positive parameter, $a(x)$, $b(x)$ and $f(x)$ are sufficiently smooth functions, such that $a(x) \geq \beta > 0$ and $b(x) \geq 0$ on $\overline{\Omega} = [0,1]$. Under these assumptions, (1.3.9-1.3.10) possesses a unique solution $u(x) \in C^2(\overline{\Omega})$. In this chapter, we consider both the cases i.e. $a(x) = 0$ and $a(x) \neq 0$. We suggest an initial-value technique, in line of [123] for (1.3.9-1.3.10). The BVP is replaced with a suitable initial value problem (IVP) and a terminal value problem (TVP). The integration of these problems goes in the opposite directions, but each problem can be solved independently of the other. The IVP is of singularly perturbed type, whereas the TVP does not contain any small parameter. We propose an initial-value technique to solve these problems.

Chapter 5

Second order SPPs with discontinuous source term are considered in this chapter. A singularly perturbed convection-diffusion type second order ODE with a discontinuous source term is considered on the unit interval $\Omega = (0,1)$. A single discontinuity is assumed to occur at a point $d \in \Omega$. This gives rise to an interior layer in the exact solution of the problem, in addition to the boundary layer at the outflow boundary point. Let $\Omega^- = (0,d)$ and $\Omega^+ = (d,1)$ and denote the jump at d in any function with $[w](d) = w(d+) - w(d-)$. Consider the problem:

$$L_\epsilon u(x) \equiv \varepsilon u''(x) + a(x)u'(x) = f(x), \qquad x \in \Omega^- \cup \Omega^+, \tag{1.3.11}$$

$$u(0) = p, \quad u(1) = q, \tag{1.3.12}$$

where $0 < \epsilon \ll 1$ is a small parameter, $a(x)$ is a sufficiently smooth function on $\overline{\Omega} = [0,1]$, such that $a(x) \geq \beta > 0$, $f(x)$ is a sufficiently smooth function on $\Omega^- \cup \Omega^+ \cup \{0,1\}$, f and its derivatives have jump discontinuity at d. Because f is discontinuous at d, the solution u of (1.3.11-1.3.12) does not necessarily have a continuous second derivative at the point d. Thus, u need not belong to the class of functions $C^2(\Omega)$. But the first derivative of the solution exists and is continuous in Ω. Under these assumptions, (1.3.11-1.3.12) have a solution $u \in C^0(\overline{\Omega}) \cap C^1(\Omega) \cap C^2(\Omega^- \cup \Omega^+)$ [28].

In this chapter, we also discuss singularly perturbed reaction-diffusion type second order ODE with a discontinuous source term. To solve these type of problems, a modified initial value technique (MIVT) is proposed on an appropriate piecewise uniform Shishkin mesh. The MIVT is shown to be uniformly convergent with respect to the perturbation parameter.

Chapter 6

We treat the following system of two singularly perturbed convection-diffusion problems:

$$L_1\vec{u} \equiv -\epsilon u_1''(x) - a_1(x)u_1'(x) + b_{11}(x)u_1(x) + b_{12}(x)u_2(x) = f_1(x), \tag{1.3.13}$$

$$L_2\vec{u} \equiv -\mu u_2''(x) - a_2(x)u_2'(x) + b_{21}(x)u_1(x) + b_{22}(x)u_2(x) = f_2(x), \tag{1.3.14}$$

where $\vec{u} = (u_1, u_2)^T$, $x \in \Omega = (0, 1)$ with the boundary conditions:

$$\vec{u}(0) = \begin{pmatrix} p \\ r \end{pmatrix}, \tag{1.3.15}$$

$$\vec{u}(1) = \begin{pmatrix} q \\ s \end{pmatrix}. \tag{1.3.16}$$

Without loss of generality, we shall assume that $0 < \epsilon \leq \mu \ll 1$. The functions $a_1(x)$, $a_2(x)$, $b_{11}(x)$, $b_{12}(x)$, $b_{21}(x)$, $b_{22}(x)$, $f_1(x)$, $f_2(x)$ are sufficiently smooth and satisfy the following inequalities:

$$
\begin{aligned}
&(i) \quad a_1(x) \geq \alpha_1 > 0, \quad a_2(x) \geq \alpha_2 > 0, \\
&(ii) \quad b_{11}(x) > |b_{12}(x)|, \quad b_{22}(x) > |b_{21}(x)|, \\
&(iii) \quad b_{12}(x) < 0, \quad b_{21}(x) < 0,
\end{aligned}
$$

$\forall x \in \overline{\Omega} = [0, 1]$ and $u_1, u_2 \in C^2(\Omega) \cap C(\overline{\Omega})$.

Our main objective is to construct a modified initial value technique (MIVT) for (1.3.13-1.3.16) which is based on the underlying idea of AIVM [125]. First, in this technique, an asymptotic expansion approximation for the solution of the Boundary Value Problem (BVP) (1.3.13-1.3.16) has been constructed. Then, Initial Value Problems (IVPs) and Terminal Value Problems (TVPs) are formulated whose solutions are the terms of this asymptotic expansion. The IVPs are happened to be SPPs and therefore, they are solved by the proposed hybrid scheme. The MIVT displays uniform convergence with respect to the perturbation parameter ϵ. We deal with the case $0 < \epsilon = \mu \ll 1$. In this chapter we also consider the case when $a_1(x) = a_2(x) = 0$.

Chapter 7

Conclusions based on the present study are finally drawn in the concluding chapter 7. The significance of the present work and scope of future research work is also discussed in this concluding chapter.

Chapter 2

Singularly Perturbed Initial Value Problems

In many areas of application, notably fluid mechanics, electrical networks, chemical reactions and control theory, singularly perturbed initial value problems arise which have a narrow region known as 'initial layer' near the initial point where solution changes very rapidly. The use of the conventional numerical methods for singularly perturbed initial value problems require a very fine mesh in the initial layer region, which makes these methods quite demanding on the computer time. Also, the problem may become ill-posed numerically when mesh size gets too small. In this chapter, we propose a hybrid scheme to solve a class of initial value problems and then the scheme is generalized for the system of initial value problems.

The outline of this chapter is as follows: Section 2.1 deals with the singularly perturbed initial value problems. System of first order singularly perturbed differential equations are next discussed in Section 2.2. Some numerical results computed from the proposed schemes are presented in Section 2.3. Certain conclusions based on the present analysis are finally presented in Section 2.4.

2.1 Initial value problems

Consider the following singularly perturbed initial value problem (SPIVP):

$$L_\epsilon y(x) \equiv \epsilon y'(x) + b(x)y(x) = g(x), \quad x \in \Omega, \tag{2.1.1}$$

$$y(0) = A, \tag{2.1.2}$$

where A is a constant and $\epsilon > 0$ is a small parameter, b and g are sufficiently smooth functions, such that $b(x) \geq \beta > 0$ on $\overline{\Omega}$. Under these assumptions, the SPIVP possesses a unique solution $y(x)$ [25].

In order to solve the SPIVP, we propose a hybrid scheme which is the combination of the Trapezoidal method and a backward difference scheme. The hybrid scheme not only retains the oscillation free behavior of the backward difference scheme but also retains the higher order of convergence of Trapezoidal method.

Notation: Throughout the thesis, the constant C is assumed to be a generic positive constant that may take different values in the different formulas, but is always independent of N and ϵ. $||.||$ denotes the maximum norm over $\overline{\Omega}$.

2.1.1 Proposed hybrid scheme

Let the mesh points of $\overline{\Omega} = [0,1]$ be

$$x_0 = 0,\, x_i = \sum_{k=0}^{i-1} h_k,\, h_k = x_{k+1} - x_k,\, x_N = 1,\; i = 1, 2, \ldots, N-1. \tag{2.1.3}$$

In order to solve (2.1.1-2.1.2), we propose the following hybrid scheme:

$$L_\epsilon^N Y_i \equiv \begin{cases} \epsilon D^- Y_i + \dfrac{b_{i-1}Y_{i-1} + b_i Y_i}{2} = \dfrac{g_{i-1} + g_i}{2} & 0 < i \leq N/2, \\ \\ \epsilon D^- Y_i + b_i Y_i = g_i, \quad N/2 < i \leq N, \end{cases} \tag{2.1.4}$$

$$Y_0 = A, \tag{2.1.5}$$

where $D^- Y_i = \frac{Y_i - Y_{i-1}}{h_{i-1}}$, $b_i = b(x_i), g_i = g(x_i)$.

2.1.2 Piece-wise uniform Shishkin mesh

It is known that on an equidistant mesh, no scheme can attain convergence at all mesh points uniformly in ϵ, unless its coefficients have an exponential property. The hybrid scheme defined above, in order to be $\epsilon-$uniform convergent, we will use the Shishkin mesh. Shishkin mesh is attractive because of its simplicity and is also adequate for handling a wide variety of singularly perturbed problems [79]. The basic idea behind this mesh is to divide $\overline{\Omega}$ into $[0, \sigma]$ and $[\sigma, 1]$, where σ is a transition point (a function of N and ϵ) and place $N/2$ mesh points in the region $[0, \sigma]$ known as "inner region " where the solution varies fastly and place remaining $N/2$ mesh points in the region $[\sigma, 1]$ called "outer region" where the solution varies slowly. The transition point σ which separates the fine and coarse portions of the mesh is obtained by:

$$\sigma = \min\left\{\frac{1}{2}, \sigma_0 \epsilon \ln N\right\}, \tag{2.1.6}$$

where $\sigma_0 \geq \frac{2}{\beta}$. Further, we denote the mesh size in the region $[0, \sigma]$ by $h = \frac{2\sigma}{N}$ and in $(\sigma, 1]$ by $H = \frac{2(1-\sigma)}{N}$.

2.1.3 Convergence analysis

In this sub-section, we derive an ϵ-uniform second-order error estimates.

Theorem 2.1.1 *Let $y(x)$ and Y_i be respectively the solutions of (2.1.1-2.1.2) and (2.1.4-2.1.5). Then, the local truncation error satisfies the following bounds :*

$$\begin{aligned}
|L_\epsilon^N(Y_i - y(x_i))| &\leq CN^{-2}\sigma_0^2 \ln^2 N, \quad \textit{for} \quad 0 < i \leq N/2, \\
|L_\epsilon^N(Y_i - y(x_i))| &\leq C(N^{-1}\epsilon + N^{-\beta\sigma_0}), \quad \textit{for} \quad N/2 < i \leq N, \quad \textit{and} \quad H \leq \epsilon, \\
|L_\epsilon^N(Y_i - y(x_i))| &\leq C(N^{-2} + N^{-\beta\sigma_0}), \quad \textit{for} \quad N/2 < i \leq N, \quad \textit{and} \quad H > \epsilon.
\end{aligned}$$

Proof. We distinguish several cases depending on the location of the mesh points. Firstly, we state the bound for the derivatives of the continuous solution i.e., the solution $y(x)$ of (2.1.1-2.1.2) satisfies the following bound [25]

$$|y^{(k)}(x)| \leq C\left[1 + \epsilon^{-k}\exp(-\beta x/\epsilon)\right]. \tag{2.1.7}$$

By using the usual Taylor series expansion for $x_i \in (0, \sigma]$, we get

$$|L_\epsilon^N(Y_i - y(x_i))| \leq C\epsilon h^2 |y'''(\xi)|, \quad \text{for} \quad 0 < i \leq N/2, \tag{2.1.8}$$

for some point ξ, $x_{i-1} \leq \xi \leq x_i$.

First, we consider the case when the mesh is uniform. In this case, $\sigma = 1/2$ and $\epsilon^{-1} \leq C\sigma_0 \ln N$.

Using the bound (2.1.7), we have

$$\begin{aligned}|L_\varepsilon^N(Y_i - y(x_i))| &\leq C\varepsilon h^2\left[1+\varepsilon^{-3}\exp(-\beta\xi/\varepsilon)\right],\\ &\leq CN^{-2}\sigma_0^2 ln^2 N, \quad \text{for} \quad 0 < i \leq N/2. \qquad (2.1.9)\end{aligned}$$

Secondly, we consider the case when the mesh is non-uniform, that is $h = 2N^{-1}\sigma_0\epsilon lnN$ on the above bound and bounding the exponential function by a constant, we have

$$|L_\epsilon^N(Y_i - y(x_i))| \leq CN^{-2}\sigma_0^2 ln^2 N, \quad \text{for} \quad 0 < i \leq N/2. \qquad (2.1.10)$$

Again, using the usual Taylor series expansion for $x_i \in (\sigma, 1]$, we get

$$|L_\epsilon^N(Y_i - y(x_i))| \leq C\epsilon H|y''(\xi)|, \quad \text{for} \quad N/2 < i \leq N. \qquad (2.1.11)$$

Note that the above expression for the truncation error in the outer region can also be represented as

$$|L_\epsilon^N(Y_i - y(x_i))| = \frac{\epsilon}{h_{i-1}}R_1(x_i, x_{i-1}, y), \qquad (2.1.12)$$

where $R_n(a,p,g) = \dfrac{1}{n!}\displaystyle\int_a^p (p-\xi)^n g^{(n+1)}(\xi)d\xi$ denotes the remainder term obtained from Taylor expansion in an integral form.

We discuss two cases: First, if $H < \epsilon$, from (2.1.11), we obtain

$$\begin{aligned}|L_\epsilon^N(Y_i - y(x_i))| &\leq C\epsilon H|y''(\xi)|,\\ &\leq C[H\epsilon + H\epsilon^{-1}\exp(-\beta x_i/\epsilon)],\\ &\leq C[N^{-1}\epsilon + N^{-\beta\sigma_0}]. \qquad (2.1.13)\end{aligned}$$

Secondly, if $H \geq \epsilon$, then using the bounds of the derivatives of $y(x)$ from (2.1.7), one can obtain the following

$$|L_\epsilon^N(Y_i - y(x_i))| \leq C\left(H\epsilon + \int_{x_i-1}^{x_i}(x_i-\xi)\epsilon^{-2}\exp(-\beta\xi/\epsilon)d\xi\right). \tag{2.1.14}$$

Integrating by parts, we get

$$\begin{aligned}\int_{x_i-1}^{x_i}(x_i-\xi)\epsilon^{-2}\exp(-\beta\xi/\epsilon)d\xi \quad &\leq C\left(H\epsilon + \int_{x_i-1}^{x_i}\epsilon^{-1}\exp(-\beta\xi/\epsilon)d\xi\right),\\ &\leq C\left[H\epsilon + N^{-\beta\sigma_0}\right]. \end{aligned} \tag{2.1.15}$$

Since $H < 2N^{-1}$ and $\epsilon \leq H$, we get

$$|L_\epsilon^N(Y_i - y(x_i))| \;\leq\; C\left(N^{-2} + N^{-\beta\sigma_0}\right). \tag{2.1.16}$$

Combining all the previous results, we obtain the required truncation error. Hence, we obtain the required result. ∎

Theorem 2.1.2 *Let $y(x)$ be the solution of the IVP (2.1.1-2.1.2) and Y_i be the numerical solution obtained from the hybrid scheme (2.1.4-2.1.5). Then, for sufficiently large N, and $N^{-1}\sigma_0 \ln N\beta^* < 1$, where $\beta^* = \max_{0\leq i\leq N} b(x_i)$, we have,*

$$|Y_i - y(x_i)| \;\leq\; C\left[N^{-2}\ln^2 N + N^{-1}\epsilon + N^{-\beta\sigma_0}\right], \quad \forall x_i \in \overline{\Omega}.$$

Proof. Let $B_i^- = (2 - \rho_i b_i)$, $B_i^+ = (2 + \rho_i b_i)$ and $b_i^+ = (1 + \rho_i b_i)$, where $\rho_i = \frac{h_i}{\epsilon}$.
The solution of the scheme (2.1.4-2.1.5) can be expressed as:
For $0 < i \leq N/2$

$$Y_i = \frac{\prod_{j=0}^{i-1} B_j^-}{\prod_{j=1}^{i} B_j^+} Y_0 + \frac{\rho_i \prod_{j=1}^{i-1} B_j^-}{\prod_{j=1}^{i} B_j^+}(g_0+g_1) + \frac{\rho_i \prod_{j=2}^{i-1} B_j^-}{\prod_{j=2}^{i} B_j^+}(g_1+g_2) + ... + \frac{\rho_i}{B_i^+}(g_{i-1}+g_i), \quad (2.1.17)$$

and for $N/2 < i \leq N$

$$Y_i = \frac{1}{\prod_{j=N/2+1}^{i} b_j^+} Y_{N/2} + \frac{\rho_i}{\prod_{j=N/2+1}^{i} b_j^+} g_{N/2+1} + \frac{\rho_i}{\prod_{j=N/2+2}^{i} b_j^+} g_{N/2+2} + ... + \frac{\rho_i}{b_i^+} g_i. \quad (2.1.18)$$

Clearly, $B_i^+\,'s$ and $b_i^+\,'s$ are non-negative.

For $B_i^- > 0$, $0 < i \leq N/2$, we have

$$B_i^- = 2 - \rho_i b_i = 2 - \frac{h_i b_i}{\epsilon}. \quad (2.1.19)$$

Since $h_i = 2N^{-1}\sigma_0 \epsilon ln N$ and $b_i \leq \beta^*$, we have $B_i^- > 0$. So, the solution satisfies the discrete maximum principle and hence there are no oscillations.

Defining the discrete barrier function

$$\phi_i = C\left[N^{-2}\ln^2 N + N^{-1}\epsilon + N^{-\beta\sigma_0}\right]. \quad (2.1.20)$$

Now, Choosing C sufficiently large, and using the discrete maximum principle, it is easier to see that

$$L_\epsilon^N(\phi_i \pm (Y_i - y(x_i))) \geq 0 \quad (2.1.21)$$

equivalently,

$$L_\epsilon^N(\phi_i) \geq |Y_i - y(x_i)| \quad (2.1.22)$$

Therefore, it follows that

$$|Y_i - y(x_i)| \;\leq\; |\phi_i|, \quad \forall x_i \in \overline{\Omega}. \quad (2.1.23)$$

Thus, we have the required ϵ-uniform error bound. ∎

Remark 2.1.3 *In Theorem 2.1.1, One can notice that the truncation error is of order $N^{-\beta\sigma_o}$ for $H > \epsilon$. It is assumed that $\beta\sigma_o \geq 2$ and we are interested in the case $\epsilon \leq N^{-1}$. Also, we obtain the error bound of order $N^{-1}\varepsilon$ only in outer region for the case $H < \epsilon$, which is not the practical case. With these points, we conclude that the order of convergence is almost two (up to a logarithmic factor). Our numerical results given in Section 2.3 reveal the same behavior.*

2.2 System of first order differential equations

In this section, we present a computational technique for a system of first order singularly perturbed ordinary differential equations of the form:

$$L_\epsilon \vec{u_\epsilon}(x) = \begin{pmatrix} \epsilon D & 0 & . & . & 0 \\ 0 & \epsilon D & . & . & 0 \\ . & . & . & . & . \\ . & . & . & . & . \\ 0 & 0 & . & . & \epsilon D \end{pmatrix} \vec{u_\epsilon}(x) + A(x)\vec{u_\epsilon}(x) = \vec{f}(x), \tag{2.2.1}$$

$$\vec{u_\epsilon}(0) = (u_{\epsilon,1}(0), u_{\epsilon,2}(0), ...u_{\epsilon,n}(0))^T, x \in (0,1], \tag{2.2.2}$$

where $\vec{u_\epsilon} = (u_{\epsilon,1}, u_{\epsilon,2}, ...u_{\epsilon,n})^T$, $A(x) = \begin{pmatrix} a_{11}(x) & a_{12}(x) & . & . & a_{1n}(x) \\ a_{21}(x) & a_{22}(x) & . & . & a_{2n}(x) \\ . & . & . & . & . \\ . & . & . & . & . \\ a_{n1}(x) & a_{n2}(x) & . & . & a_{nn}(x) \end{pmatrix}$

$\vec{f}(x) = (f_1(x), f_2(x), ..., f_n(x))^T$, D denotes $\frac{d}{dx}$ and $u_\epsilon \in C^{(1)}(\Omega), \Omega = (0,1)$. The singular perturbation parameter ϵ satisfies $0 < \epsilon \leq 1$. The functions $a_{ij}, f_i \in C^2(\overline{\Omega}), \overline{\Omega} = [0,1], \quad i,j =$

$1, 2, ..., n$ satisfies the following inequalities:

$$(i) \quad a_{ii} > \sum_{j=1, j\neq i}^{n} |a_{ij}(x)|, \quad i = 1(1)n \tag{2.2.3}$$

$$(ii) \quad a_{ij} < 0, \quad i, j = 1(1)n, \ i \neq j \tag{2.2.4}$$

We introduce the positive number,

$$\beta = \min\left\{\sum_{j=1}^{n} a_{1j}, \sum_{j=1}^{n} a_{2j}, ..., \sum_{j=1}^{n} a_{nj}\right\}. \tag{2.2.5}$$

In [39], a computational method is presented for a system of first order singularly perturbed ordinary differential equations. In the present work, we propose a hybrid scheme for (2.2.1-2.2.2) on piece-wise uniform Shishkin mesh.

2.2.1 Proposed hybrid scheme

Let the mesh points of $\overline{\Omega} = [0, 1]$ be

$$x_0 = 0, \ x_i = \sum_{k=0}^{i-1} h_k, h_k = x_{k+1} - x_k, \ x_N = 1, \ i = 1, 2, ..., N-1. \tag{2.2.6}$$

We define the following hybrid scheme for the equation (2.2.1-2.2.2)

$$L_\epsilon^N \vec{U}_\epsilon(x_j) \equiv \begin{cases} Q\vec{U}_\epsilon(x_j) + \frac{A(x_{j-1})\vec{U}_\epsilon(x_{j-1}) + A(x_j)\vec{U}_\epsilon(x_j)}{2} = \frac{\vec{f}(x_{j-1}) + \vec{f}(x_j)}{2}, 0 < j \leq \frac{N}{2}, \\ \\ Q\vec{U}_\epsilon(x_j) + A(x_j)\vec{U}_\epsilon(x_j) = \vec{f}(x_j), \quad \frac{N}{2} < j \leq N, \end{cases} \tag{2.2.7}$$

$$\vec{U}_\epsilon(0) = \vec{u_\epsilon}(0). \tag{2.2.8}$$

where $\vec{U_\epsilon} = (U_{\epsilon,1}, U_{\epsilon,2}, ...U_{\epsilon,n})^T,\ Q = \begin{pmatrix} \epsilon D^- & 0 & . & . & 0 \\ 0 & \epsilon D^- & . & . & 0 \\ . & . & . & . & . \\ . & . & . & . & . \\ 0 & 0 & . & . & \epsilon D^- \end{pmatrix}$,

and $D^- U_{\epsilon,i}(x_j) = \frac{U_{\epsilon,i}(x_j) - U_{\epsilon,i}(x_{j-1})}{h_{j-1}}$, j=1 (1) N, i= 1(1)n.

2.2.2 Piece-wise uniform Shishkin mesh

As mentioned in sub-section 2.1.2, in this mesh, the domain $\overline{\Omega}$ is divided into two sub-intervals as $\overline{\Omega} = [0, \sigma] \cup (\sigma, 1]$ for some σ such that $0 \leq \sigma \leq \frac{1}{2}$ and defined as in (2.1.6).

2.2.3 Convergence analysis

In this sub-section, we present some analytical results available in the literature, which include a maximum principle, uniform stability and estimates of the derivatives of the solution. These results will be used to derive error bounds for the derivatives of the solution.

Lemma 2.2.1 *[39] Consider the system of initial value problems (2.2.1-2.2.2). Then* $u_{\epsilon,i}(0) \geq 0, i = 1, ..., n$ *and* $(L_\epsilon u_\epsilon)_i(x) \geq 0$ *for all* $x \in (0, 1], i = 1, ..., n$ *imply that* $u_{\epsilon,i}(x) \geq 0, i = 1, ..., n$ *for all* $x \in \overline{\Omega}$.

Proof. Define a test function $s(x) = (s_1(x), s_2(x), \ldots, s_n(x))^T$ by $s_i(x) \equiv 1,\ i = 1, \ldots, n$. Then it is clear that $s_i(x) > 0$, $\forall\, x \in \overline{\Omega}$ and $i = 1, \ldots, n$, using (2.2.3),

$$(L_\epsilon(s))_i(x) = \epsilon s_i'(x) + \sum_{j=1}^{n} a_{ij}(x) s_j(x) = \sum_{j=1}^{n} a_{ij}(x) > 0, \quad \text{for } i = 1, \ldots, n. \tag{2.2.9}$$

Now assume that the lemma is not true and introduce the quantity

$$\xi = \max_{i} \{\max_{x \in \overline{\Omega}}(\frac{-u_{\epsilon,i}}{s_i})\}. \tag{2.2.10}$$

Since the lemma is not true, there exist $x^* \in \Omega_0$ such that $u_{\epsilon,i}(x^*) < 0$ for at least one i. Clearly $\xi > 0$ and $(u_{\epsilon,i} + \xi s_i)(x) \geq 0 \,\forall\, i = 1, \ldots, n$ and for all $x \in \overline{\Omega}$. Furthermore there exist a point $x_0 \in \overline{\Omega}$ such that $(u_{\epsilon,k} + \xi s_k)(x_0) = 0$ for some k. This means that $(u_{\epsilon,k} + \xi s_k)$ attains a minimum at $x = x_0$. Then

$$\begin{aligned} 0 &< (L_\epsilon(u_\epsilon + \xi s))_k(x_0), \\ &= \epsilon(u_{\epsilon,k} + \xi s_k)'(x_0) + \sum_{j=1, j\neq k}^{n} a_{kj}(x_0)(u_{\epsilon,j} + \xi s_j)(x_0), \\ &\leq 0, \quad \text{since } (u_{\epsilon,k} + \xi s_k)'(x_0) = 0, \end{aligned} \tag{2.2.11}$$

if $x_0 \in \Omega$ and $(u_{\epsilon,k} + \xi s_k)'(x_0) < 0$ if $x_0 = 1$, a contradiction.

Hence, it can be concluded that $u_{\epsilon,i}(x) \geq 0, i = 1, ..., n$ for all $x \in \overline{\Omega}$.

Using this maximum principle lemma, a uniform stability bound is obtained in the following lemma. ∎

Lemma 2.2.2 *[39] If $u_\epsilon(x)$ is the solution of system of initial value problems (2.2.1-2.2.2), then*

$$\| u_\epsilon(x) \| \leq C \max \{\| u_\epsilon(0) \|, \| L_\epsilon u_\epsilon(x) \|\} \textit{ for all } x \in \overline{\Omega}. \tag{2.2.12}$$

where C is a constant independent of x and ϵ

Proof. Defining barrier functions $\Psi^{\pm}(x) = (\Psi_1^{\pm}(x), \Psi_2^{\pm}(x), \ldots, \Psi_n^{\pm}(x))^T$ by

$$\Psi_i^{\pm}(x) = M \pm u_{\epsilon,i}(x), \quad \text{for } i = 1, ..., n, \tag{2.2.13}$$

where $M = C' \max\{\parallel u_\epsilon(0) \parallel, \parallel L_\epsilon u_\epsilon(x) \parallel\}$ where C' is a constant.

It is easy to check that $\Psi_i^{\pm}(x) \geq 0, \quad$ for $i = 1, ..., n$, and

$$(L_\epsilon \Psi^{\pm})_i(x) = M \sum_{j=1}^{n} a_{ij}(x) \pm f_i(x) \geq 0, \quad \text{if } C' \geq \frac{1}{\beta} \quad \text{for } i = 1, ..., n. \tag{2.2.14}$$

Then the desired stability bound follows from Lemma 2.2.1 ∎

Lemma 2.2.3 *[39] If $u_\epsilon(x)$ is the solution of system of initial value problems (2.2.1-2.2.2), then $u_\epsilon(x)$ for i= 1,2,...,n satisfy*

$$\mid u^k_{\epsilon,i}(x) \mid \leq C(1 + \epsilon^{-k} e^{-\beta x/\epsilon}) \quad \textit{for}\, i = 1, 2, ..., n \tag{2.2.15}$$

for $0 \leq k \leq 2$, $x \in \overline{\Omega}$ and C is a constant independent of x and ϵ

Proof. The result is true for $k = 0$ by Lemma 2.2.2. We now verify the result for $k = 1$. Consider the equation given by (2.2.1-2.2.2).

$$(L_\epsilon(u_\epsilon))_i = \epsilon u'_{\epsilon,i} + \sum_{j=1}^{n} a_{ij} u_{\epsilon,j} = f_i, \quad \text{for } i = 1, \ldots, n. \tag{2.2.16}$$

Differentiating the above equations once we obtain

$$\epsilon u''_{\epsilon,i} + \sum_{j=1}^{n} a_{ij} u^{'}_{\epsilon,j} = f^{'}_i - \sum_{j=1}^{n} a^{'}_{ij} u_{\epsilon,j}, \quad \text{for } i = 1, \ldots, n. \tag{2.2.17}$$

This implies that

$$(L_\epsilon u^{'}_\epsilon)_i(x) \leq C, \quad \text{for } i = 1, \ldots, n. \tag{2.2.18}$$

From (2.2.1-2.2.2) and Lemma 2.2.2, it is easy to check that

$$\mid u^{'}_{\epsilon,i}(x) \mid \leq C\epsilon^{-1} \quad \text{for}\, i = 1, 2, ..., n \tag{2.2.19}$$

We introduce the barrier functions

$$\Psi_i^{\pm}(x) = C^{'}(1+\epsilon^{-1}e^{-\beta x/\epsilon}) \pm u^{'}_{\epsilon,i}(x) \quad \text{for}\, i = 1,2,...,n, \tag{2.2.20}$$

where C' is a constant independent of x and ϵ. Then

$$(L_\epsilon\Psi^{\pm})_i(x) = C^{'}\epsilon^{-1}e^{-\beta x/\epsilon}(\sum_{j=1}^{n} a_{ij}(x) - \beta) + C'\sum_{j=1}^{n} a_{ij}(x) \pm C \geq 0, \quad \text{for } i = 1,...,n. \tag{2.2.21}$$

and

$$\Psi_i^{\pm}(0) = C'(1+\epsilon^{-1}) \pm u'_{\epsilon,i}(0) \geq 0, \quad \text{for } i = 1,...,n, \tag{2.2.22}$$

Therefor, by Lemma 2.2.1, $\Psi^{\pm}(x) \geq 0$, for all $x \in \overline{\Omega}$ or

$$\mid u^{'}_{\epsilon,i}(x) \mid \leq C((1+\epsilon^{-1}e^{-\beta x/\epsilon})) \quad \text{for}\, i = 1,2,...,n, \tag{2.2.23}$$

where C is a constant independent of x and ϵ. Proceeding on similar lines, we can also verify the result for $k = 2$, which completes the proof of the lemma. ∎

Lemma 2.2.4 *Consider the discrete IVP (2.2.7-2.2.8). Then* $U_{\epsilon,i}(0) \geq 0, i = 1,...,n$ *and* $(L_\epsilon^N U_\epsilon)_i(x_j) \geq 0$ *for all* $x \in (0,1]$, *i=1,...,n, j=1,2,...,N imply that* $U_{\epsilon,i}(x_j) \geq 0, i = 1,...,n, j = 0,1,...,N.$

Proof. The proof is obtained by arguments analogous to those used in proving Lemma 2.2.1.
An immediate consequence of this lemma is the discrete stability bound given by

Lemma 2.2.5 *If* $U_\epsilon(x_j)$ *is any mesh function, then*

$$\parallel U_\epsilon(x_j) \parallel \leq C \max\{\parallel U_\epsilon(0) \parallel, \parallel L_\epsilon^N U_\epsilon(x_j) \parallel\}, \quad j = 1,2,...,N. \tag{2.2.24}$$

where C *is a constant independent of* x *and* ϵ

Generalization of results of Theorem 2.1.1 and Theorem 2.1.2 for system of IVPs leads to the following results.

Theorem 2.2.6 *Let $u_{\epsilon,i}(x_j)$ and $U_{\epsilon,i}(x_j)$, $\quad i = 1, ..., n, j = 0, 1, ..., N$ be respectively the solutions of (2.2.1-2.2.2) and (2.2.7-2.2.8). Then, the local truncation error satisfies the following bounds:*

$$\begin{aligned}
|L_\varepsilon^N(U_{\epsilon,i}(x_j) - u_{\epsilon,i}(x_j)| &\leq CN^{-2}\sigma_0^2 \ln^2 N, \quad \textit{for} \quad 0 < j \leq N/2, \\
|L_\varepsilon^N(U_{\epsilon,i}(x_j) - u_{\epsilon,i}(x_j)| &\leq C(N^{-1}\varepsilon + N^{-\beta\sigma_0}), \quad \textit{for} \quad N/2 < j \leq N, \quad \textit{and} \quad H \leq \varepsilon, \\
|L_\varepsilon^N(U_{\epsilon,i}(x_j) - u_{\epsilon,i}(x_j)| &\leq C(N^{-2} + N^{-\beta\sigma_0}), \quad \textit{for} \quad N/2 < j \leq N, \quad \textit{and} \quad H > \varepsilon.
\end{aligned}$$

Theorem 2.2.7 *Let $\vec{u_\epsilon} = (u_{\epsilon,1}, u_{\epsilon,2}, ...u_{\epsilon,n})^T$ be the solution of the IVP (2.2.1-2.2.2) and $\vec{U_\epsilon} = (U_{\epsilon,1}, U_{\epsilon,2}, ...U_{\epsilon,n})^T$ be the numerical solution obtained from the difference scheme as given in (2.2.7-2.2.8). Then, we have,*

$$\| \vec{U_\epsilon}(x_j) - \vec{u_\epsilon}(x_j) \| \leq C\left[N^{-2}\ln^2 N + N^{-1}\varepsilon + N^{-\beta\sigma_0}\right], \quad \forall x_j \in \overline{\Omega}.$$

Remark 2.2.8 *In Theorem 2.2.6, one can notice that the truncation error is of order $N^{-\beta\sigma_o}$ for $H > \epsilon$. It is assumed that $\beta\sigma_o \geq 2$ and we are interested in the case $\epsilon \leq N^{-1}$. Also, we obtain the error bound of order $N^{-1}\varepsilon$ only in outer region for the case $H < \epsilon$, which is not the practical case. With these points, we conclude that the order of convergence is almost two (up to a logarithmic factor). Our numerical results given in Section 2.3 reveal the same behavior.*

2.3 Numerical experiments and discussions

Before implementing the proposed scheme to any example, we introduce in brief some definitions and notations.

Let $u(x_i)$ be the exact solution and U_i^N is the numerical solution obtained by using N mesh intervals in the domain $\overline{\Omega}^N$.

Then, If the exact solution is known, we calculate maximum point-wise error by

$$E_\epsilon^N = \max_{x_i \epsilon \overline{\Omega}^N} \{|u(x_i) - U_i^N|\}, \tag{2.3.1}$$

If exact solution is not available, then to obtain the maximum point-wise errors and rates of convergence, we use double mesh principle. By following the idea of Sun and Styne [117], we modify the Shishkin mesh. We calculate the numerical solutions U^N on $\overline{\Omega}_\epsilon^N$ and $\widetilde{U}^N$on the mesh $\widetilde{\Omega}_\epsilon^N$, where the transition parameter σ is altered slightly to $\widetilde{\sigma} = \min\{\frac{1}{2}, \sigma_0 \epsilon \ln \frac{N}{2}\}$. Then, for $i = 0, 1...N$, the ith point of the mesh $\overline{\Omega}^N$ coincides with the $(2i)$th point of the mesh $\overline{\Omega}^{2N}$. As a result, the transition point does not move, when N is changed to $2N$. Hence, use of interpolation for double mesh principle can be avoided. The double mesh difference is defined as

$$E_\epsilon^N = \max_{x_i \in \overline{\Omega}_\epsilon^N} \{|U^N(x_i) - \widetilde{U}^{2N}(x_i)|\}, \tag{2.3.2}$$

where $U^N(x_i)$ and $\widetilde{U}^{2N}(x_i)$ denote respectively the numerical solutions obtained by using N and $2N$ mesh intervals.

In addition, the rate of convergence is calculated as:

$$p_\epsilon^N = \frac{lnE_\epsilon^N - lnE_\epsilon^{2N}}{ln2}. \tag{2.3.3}$$

To illustrate the predicted theory, following examples are provided here. Computational results are given in the form of tables. The results are presented with maximum point-wise errors for various values of ϵ and N. We have also computed the computational order of convergence which has been shown in the same table along with maximum errors. In all the cases, we take $\sigma_0 = 2$.

Example 2.3.1 *Consider the following Problem:*

$$\begin{aligned} \epsilon u' + u &= g(x) + \epsilon g'(x), \qquad x \in [0,1], \\ u(0) = 3, \quad g(x) &= sin(0.1x) + 2. \end{aligned}$$

The exact solution of this problem is

$$u(x) = g(x) + (u(0) - g(0))e^{-x/\epsilon}.$$

The numerical results by the proposed scheme are given in Table 2.1 and Figure 2.1.

Example 2.3.2 *Consider the following Problem:*

$$\begin{aligned} \epsilon u' + (1+x)u = 11.5x^3 - 2x^2 - 10.5x + 3, \quad x \in [0,1], \\ u(0) = 0. \end{aligned}$$

The numerical results for the present example are shown in Table 2.2.

Example 2.3.3 *Consider the following Problem:*

$$\begin{aligned} \epsilon u' + u &= g(x) + \epsilon g'(x), \qquad x \in [0,1] \\ u(0) = 10, \quad g(x) &= 10 - (10+x)e^{-x}. \end{aligned}$$

The exact solution of this problem is

$$u(x) = g(x) + (u(0) - g(0))e^{-x/\epsilon}.$$

The numerical results by proposed scheme are in Table 2.3.

Example 2.3.4 *Consider the following Problem:*

$$\epsilon u' + u = x^2, \quad x \in [0,1]$$
$$u(0) = 1.$$

The exact solution of this problem is

$$u(x) = x^2 - 2\epsilon x + 2\epsilon^2 + (1 - 2\epsilon^2)e^{-x/\epsilon}.$$

The maximum errors and rate of convergence of this example are given in Table 2.4.

Example 2.3.5 *Consider the following Problem:*

$$\epsilon u_1'(x) + (2+x)u_1(x) - u_2(x) = 1 + x,$$
$$\epsilon u_2'(x) - (1+x)u_1(x) + (2+x)u_2(x) = x, \quad x \in (0,1]$$
$$u_1(0) = 1, \quad u_2(0) = 0.5.$$

Table 2.5 displays the results for this example.

Example 2.3.6 *Consider the following Problem:*

$$\epsilon u_1'(x) + 2u_1(x) - (1 + \tfrac{x}{2})u_2(x) = 1,$$
$$\epsilon u_2'(x) - u_1(x) + (2+2x)u_2(x) = x + 2, \quad x \in (0,1]$$
$$u_1(0) = 1, \quad u_2(0) = 1.5.$$

Numerical results computed by proposed scheme are given in Table 2.6.

2.4 Conclusions

We have proposed a robust computational technique for solving singularly perturbed initial value problems. It is observed that although backward difference scheme satisfy discrete maximum principle in whole domain $[0, 1]$, but its order is one (up to a logarithmic factor). We can get order two (up to a logarithmic factor) by applying Trapezoidal scheme in $[0, 1]$ with proper choice of σ_0, but it results in small oscillations. Hence the solution is not stable unless mesh size is very small even in outer region, where a coarse mesh is enough to give satisfactory result. Since these oscillations are so small that one gets the impression that Trapezoidal scheme has second order (up to a logarithmic factor) convergence. We have overcome this difficulty in the proposed scheme by taking into account a proper combination of Trapezoidal scheme and backward difference scheme. The graphs plotted in Figure 2.2 are convergent curves in the maximum norm at nodal points for the different values of ϵ for Examples (2.3.1-2.3.6).

Figure 2.1: Graph of the numerical solution of the Example 2.3.1 for $\epsilon = 10^{-7}$ with $N = 16$

Table 2.1: Maximum point-wise errors and rates of convergence by proposed method for Example 2.3.1.

ε	Number of mesh points						
	16	32	64	128	256	512	1024
2^{-4}	1.47488E-02	5.82269E-03	1.92913E-03	4.79823E-04	1.19803E-04	2.99413E-05	7.48473E-06
	1.341	1.594	2.007	2.002	2.000	2.000	
2^{-6}	1.47488E-02	5.82269E-03	2.08475E-03	7.05251E-04	2.30149E-04	7.28171E-05	2.24765E-05
	1.341	1.482	1.564	1.616	1.660	1.696	
2^{-8}	1.47488E-02	5.82269E-03	2.08475E-03	7.05251E-04	2.30149E-04	7.28171E-05	2.24765E-05
	1.341	1.482	1.564	1.616	1.660	1.696	
2^{-10}	1.47488E-02	5.82269E-03	2.08475E-03	7.05251E-04	2.30149E-04	7.28171E-05	2.24765E-05
	1.341	1.482	1.564	1.616	1.660	1.696	
2^{-12}	1.47488E-02	5.82269E-03	2.08475E-03	7.05251E-04	2.30149E-04	7.28171E-05	2.24765E-05
	1.341	1.482	1.564	1.616	1.660	1.696	
2^{-14}	1.47488E-02	5.82269E-03	2.08475E-03	7.05251E-04	2.30149E-04	7.28171E-05	2.24765E-05
	1.341	1.482	1.564	1.616	1.660	1.696	
2^{-16}	1.47488E-02	5.82269E-03	2.08475E-03	7.05251E-04	2.30149E-04	7.28171E-05	2.24765E-05
	1.341	1.482	1.564	1.616	1.660	1.696	
2^{-18}	1.47488E-02	5.82269E-03	2.08475E-03	7.05251E-04	2.30149E-04	7.28171E-05	2.24765E-05
	1.341	1.482	1.564	1.616	1.660	1.696	
2^{-20}	1.47488E-02	5.82269E-03	2.08475E-03	7.05251E-04	2.30149E-04	7.28171E-05	2.24765E-05
	1.341	1.482	1.564	1.616	1.660	1.696	
2^{-22}	1.47488E-02	5.82269E-03	2.08475E-03	7.05251E-04	2.30149E-04	7.28171E-05	2.24765E-05
	1.341	1.482	1.564	1.616	1.660	1.696	
2^{-24}	1.47488E-02	5.82269E-03	2.08475E-03	7.05251E-04	2.30149E-04	7.28171E-05	2.24765E-05
	1.341	1.482	1.564	1.616	1.660	1.696	
2^{-26}	1.47488E-02	5.82269E-03	2.08475E-03	7.05251E-04	2.30149E-04	7.28171E-05	2.24765E-05
	1.341	1.482	1.564	1.616	1.660	1.696	
2^{-28}	1.47488E-02	5.82269E-03	2.08475E-03	7.05251E-04	2.30149E-04	7.28171E-05	2.24765E-05
	1.341	1.482	1.564	1.616	1.660	1.696	
2^{-30}	1.47488E-02	5.82269E-03	2.08475E-03	7.05251E-04	2.30149E-04	7.28171E-05	2.24765E-05
	1.341	1.482	1.564	1.616	1.660	1.696	
2^{-32}	1.47488E-02	5.82269E-03	2.08475E-03	7.05251E-04	2.30149E-04	7.28171E-05	2.24765E-05
	1.341	1.482	1.564	1.616	1.660	1.696	
2^{-34}	1.47488E-02	5.82269E-03	2.08475E-03	7.05251E-04	2.30149E-04	7.28171E-05	2.24765E-05
	1.341	1.482	1.564	1.616	1.660	1.696	
2^{-36}	1.47488E-02	5.82269E-03	2.08475E-03	7.05251E-04	2.30149E-04	7.28171E-05	2.24765E-05
	1.341	1.482	1.564	1.616	1.660	1.696	
2^{-38}	1.47488E-02	5.82269E-03	2.08475E-03	7.05251E-04	2.30149E-04	7.28171E-05	2.24765E-05
	1.341	1.482	1.564	1.616	1.660	1.696	
2^{-40}	1.47488E-02	5.82269E-03	2.08475E-03	7.05251E-04	2.30149E-04	7.28171E-05	2.24765E-05
	1.341	1.482	1.564	1.616	1.660	1.696	

Table 2.2: Maximum point-wise errors and rates of convergence by proposed method for Example 2.3.2.

ε	Number of mesh points						
	16	32	64	128	256	512	1024
2^{-4}	3.91763E-02	1.53295E-02	5.09915E-03	1.76956E-03	8.90635E-04	4.46867E-04	2.23832E-04
	1.354	1.588	1.527	0.990	0.995	0.997	
2^{-6}	3.48629E-02	1.36753E-02	4.88807E-03	1.65201E-03	5.38815E-04	2.29599E-04	1.10164E-04
	1.350	1.484	1.565	1.616	1.231	1.059	
2^{-8}	3.39766E-02	1.32960E-02	4.74555E-03	1.60354E-03	5.23064E-04	1.65465E-04	5.10700E-05
	1.354	1.486	1.565	1.616	1.660	1.696	
2^{-10}	3.37579E-02	1.32029E-02	4.71071E-03	1.59170E-03	5.19215E-04	1.64251E-04	5.06969E-05
	1.354	1.487	1.565	1.616	1.660	1.696	
2^{-12}	3.37034E-02	1.31797E-02	4.70205E-03	1.58876E-03	5.18258E-04	1.63949E-04	5.06041E-05
	1.355	1.487	1.565	1.616	1.660	1.696	
2^{-14}	3.36897E-02	1.31739E-02	4.69989E-03	1.58803E-03	5.18019E-04	1.63874E-04	5.05809E-05
	1.355	1.487	1.565	1.616	1.660	1.696	
2^{-16}	3.36863E-02	1.31725E-02	4.69935E-03	1.58784E-03	5.17959E-04	1.63855E-04	5.05751E-05
	1.355	1.487	1.565	1.616	1.660	1.696	
2^{-18}	3.36855E-02	1.31721E-02	4.69921E-03	1.58780E-03	5.17944E-04	1.63850E-04	5.05737E-05
	1.355	1.487	1.565	1.616	1.660	1.696	
2^{-20}	3.36853E-02	1.31720E-02	4.69918E-03	1.58778E-03	5.17941E-04	1.63849E-04	5.05733E-05
	1.355	1.487	1.565	1.616	1.660	1.696	
2^{-22}	3.36852E-02	1.31720E-02	4.69917E-03	1.58778E-03	5.17940E-04	1.63849E-04	5.05732E-05
	1.355	1.487	1.565	1.616	1.660	1.696	
2^{-24}	3.36852E-02	1.31720E-02	4.69917E-03	1.58778E-03	5.17940E-04	1.63849E-04	5.05732E-05
	1.355	1.487	1.565	1.616	1.660	1.696	
2^{-26}	3.36852E-02	1.31720E-02	4.69917E-03	1.58778E-03	5.17940E-04	1.63849E-04	5.05732E-05
	1.355	1.487	1.565	1.616	1.660	1.696	
2^{-28}	3.36852E-02	1.31720E-02	4.69917E-03	1.58778E-03	5.17940E-04	1.63849E-04	5.05732E-05
	1.355	1.487	1.565	1.616	1.660	1.696	
2^{-30}	3.36852E-02	1.31720E-02	4.69917E-03	1.58778E-03	5.17940E-04	1.63849E-04	5.05732E-05
	1.355	1.487	1.565	1.616	1.660	1.696	
2^{-32}	3.36852E-02	1.31720E-02	4.69917E-03	1.58778E-03	5.17940E-04	1.63849E-04	5.05732E-05
	1.355	1.487	1.565	1.616	1.660	1.696	
2^{-34}	3.36852E-02	1.31720E-02	4.69917E-03	1.58778E-03	5.17940E-04	1.63849E-04	5.05732E-05
	1.355	1.487	1.565	1.616	1.660	1.696	
2^{-36}	3.36852E-02	1.31720E-02	4.69917E-03	1.58778E-03	5.17940E-04	1.63849E-04	5.05732E-05
	1.355	1.487	1.565	1.616	1.660	1.696	
2^{-38}	3.36852E-02	1.31720E-02	4.69917E-03	1.58778E-03	5.17940E-04	1.63849E-04	5.05732E-05
	1.355	1.487	1.565	1.616	1.660	1.696	
2^{-40}	3.36852E-02	1.31720E-02	4.69917E-03	1.58778E-03	5.17940E-04	1.63849E-04	5.05732E-05
	1.355	1.487	1.565	1.616	1.660	1.696	

Table 2.3: Maximum point-wise errors and rates of convergence by proposed method for Example 2.3.3.

ε	Number of mesh points						
	16	32	64	128	256	512	1024
2^{-4}	1.47454E-01 1.340	5.82116E-02 1.594	1.92858E-02 2.007	4.79686E-03 2.002	1.19769E-03 2.000	2.99328E-04 1.210	1.29396E-04
2^{-6}	1.47488E-01 1.341	5.82266E-02 1.482	2.08474E-02 1.564	7.05247E-03 1.616	2.30148E-03 1.660	7.28167E-04 1.696	2.24764E-04
2^{-8}	1.47488E-01 1.341	5.82269E-02 1.482	2.08474E-02 1.564	7.05251E-03 1.616	2.30149E-03 1.660	7.28171E-04 1.696	2.24765E-04
2^{-10}	1.47488E-01 1.341	5.82269E-02 1.482	2.08475E-02 1.564	7.05251E-03 1.616	2.30149E-03 1.660	7.28171E-04 1.696	2.24765E-04
2^{-12}	1.47488E-01 1.341	5.82269E-02 1.482	2.08475E-02 1.564	7.05251E-03 1.616	2.30149E-03 1.660	7.28171E-04 1.696	2.24765E-04
2^{-14}	1.47488E-01 1.341	5.82269E-02 1.482	2.08475E-02 1.564	7.05251E-03 1.616	2.30149E-03 1.660	7.28171E-04 1.696	2.24765E-04
2^{-16}	1.47488E-01 1.341	5.82269E-02 1.482	2.08475E-02 1.564	7.05251E-03 1.616	2.30149E-03 1.660	7.28171E-04 1.696	2.24765E-04
2^{-18}	1.47488E-01 1.341	5.82269E-02 1.482	2.08475E-02 1.564	7.05251E-03 1.616	2.30149E-03 1.660	7.28171E-04 1.696	2.24765E-04
2^{-20}	1.47488E-01 1.341	5.82269E-02 1.482	2.08475E-02 1.564	7.05251E-03 1.616	2.30149E-03 1.660	7.28171E-04 1.696	2.24765E-04
2^{-22}	1.47488E-01 1.341	5.82269E-02 1.482	2.08475E-02 1.564	7.05251E-03 1.616	2.30149E-03 1.660	7.28171E-04 1.696	2.24765E-04
2^{-24}	1.47488E-01 1.341	5.82269E-02 1.482	2.08475E-02 1.564	7.05251E-03 1.616	2.30149E-03 1.660	7.28171E-04 1.696	2.24765E-04
2^{-26}	1.47488E-01 1.341	5.82269E-02 1.482	2.08475E-02 1.564	7.05251E-03 1.616	2.30149E-03 1.660	7.28171E-04 1.696	2.24765E-04
2^{-28}	1.47488E-01 1.341	5.82269E-02 1.482	2.08475E-02 1.564	7.05251E-03 1.616	2.30149E-03 1.660	7.28171E-04 1.696	2.24765E-04
2^{-30}	1.47488E-01 1.341	5.82269E-02 1.482	2.08475E-02 1.564	7.05251E-03 1.616	2.30149E-03 1.660	7.28171E-04 1.696	2.24765E-04
2^{-32}	1.47488E-01 1.341	5.82269E-02 1.482	2.08475E-02 1.564	7.05251E-03 1.616	2.30149E-03 1.660	7.28171E-04 1.696	2.24765E-04
2^{-34}	1.47488E-01 1.341	5.82269E-02 1.482	2.08475E-02 1.564	7.05251E-03 1.616	2.30149E-03 1.660	7.28171E-04 1.696	2.24765E-04
2^{-36}	1.47488E-01 1.341	5.82269E-02 1.482	2.08475E-02 1.564	7.05251E-03 1.616	2.30149E-03 1.660	7.28171E-04 1.696	2.24765E-04
2^{-38}	1.47488E-01 1.341	5.82269E-02 1.482	2.08475E-02 1.564	7.05251E-03 1.616	2.30149E-03 1.660	7.28171E-04 1.696	2.24765E-04
2^{-40}	1.47488E-01 1.341	5.82269E-02 1.482	2.08475E-02 1.564	7.05251E-03 1.616	2.30149E-03 1.660	7.28171E-04 1.696	2.24765E-04

Table 2.4: Maximum point-wise errors and rates of convergence by proposed method for Example 2.3.4.

ε	Number of mesh points						
	16	32	64	128	256	512	1024
2^{-4}	1.46336E-02 1.340	5.77720E-03 1.594	1.91406E-03 1.971	4.88085E-04 1.000	2.44066E-04 1.000	1.22039E-04 1.000	6.10205E-05
2^{-6}	1.47416E-02 1.340	5.81985E-03 1.482	2.08373E-03 1.564	7.04906E-04 1.616	2.30039E-04 1.660	7.27815E-05 1.696	2.39072E-05
2^{-8}	1.47484E-02 1.340	5.82251E-03 1.482	2.08468E-03 1.564	7.05229E-04 1.616	2.30142E-04 1.660	7.28149E-05 1.696	2.24758E-05
2^{-10}	1.47488E-02 1.340	5.82268E-03 1.482	2.08474E-03 1.564	7.05249E-04 1.616	2.30149E-04 1.660	7.28169E-05 1.696	2.24765E-05
2^{-12}	1.47488E-02 1.340	5.82269E-03 1.482	2.08474E-03 1.564	7.05251E-04 1.616	2.30149E-04 1.660	7.28171E-05 1.696	2.24765E-05
2^{-14}	1.47488E-02 1.340	5.82269E-03 1.482	2.08474E-03 1.564	7.05251E-04 1.616	2.30149E-04 1.660	7.28171E-05 1.696	2.24765E-05
2^{-16}	1.47488E-02 1.340	5.82269E-03 1.482	2.08474E-03 1.564	7.05251E-04 1.616	2.30149E-04 1.660	7.28171E-05 1.696	2.24765E-05
2^{-18}	1.47488E-02 1.340	5.82269E-03 1.482	2.08474E-03 1.564	7.05251E-04 1.616	2.30149E-04 1.660	7.28171E-05 1.696	2.24765E-05
2^{-20}	1.47488E-02 1.340	5.82269E-03 1.482	2.08474E-03 1.564	7.05251E-04 1.616	2.30149E-04 1.660	7.28171E-05 1.696	2.24765E-05
2^{-22}	1.47488E-02 1.340	5.82269E-03 1.482	2.08474E-03 1.564	7.05251E-04 1.616	2.30149E-04 1.660	7.28171E-05 1.696	2.24765E-05
2^{-24}	1.47488E-02 1.340	5.82269E-03 1.482	2.08474E-03 1.564	7.05251E-04 1.616	2.30149E-04 1.660	7.28171E-05 1.696	2.24765E-05
2^{-26}	1.47488E-02 1.340	5.82269E-03 1.482	2.08474E-03 1.564	7.05251E-04 1.616	2.30149E-04 1.660	7.28171E-05 1.696	2.24765E-05
2^{-28}	1.47488E-02 1.340	5.82269E-03 1.482	2.08474E-03 1.564	7.05251E-04 1.616	2.30149E-04 1.660	7.28171E-05 1.696	2.24765E-05
2^{-30}	1.47488E-02 1.340	5.82269E-03 1.482	2.08474E-03 1.564	7.05251E-04 1.616	2.30149E-04 1.660	7.28171E-05 1.696	2.24765E-05
2^{-32}	1.47488E-02 1.340	5.82269E-03 1.482	2.08474E-03 1.564	7.05251E-04 1.616	2.30149E-04 1.660	7.28171E-05 1.696	2.24765E-05
2^{-34}	1.47488E-02 1.340	5.82269E-03 1.482	2.08474E-03 1.564	7.05251E-04 1.616	2.30149E-04 1.660	7.28171E-05 1.696	2.24765E-05
2^{-36}	1.47488E-02 1.340	5.82269E-03 1.482	2.08474E-03 1.564	7.05251E-04 1.616	2.30149E-04 1.660	7.28171E-05 1.696	2.24765E-05
2^{-38}	1.47488E-02 1.340	5.82269E-03 1.482	2.08474E-03 1.564	7.05251E-04 1.616	2.30149E-04 1.660	7.28171E-05 1.696	2.24765E-05
2^{-40}	1.47488E-02 1.340	5.82269E-03 1.482	2.08474E-03 1.564	7.05251E-04 1.616	2.30149E-04 1.660	7.28171E-05 1.696	2.24765E-05

Table 2.5: Maximum point-wise errors and rates of convergence by proposed method for Example 2.3.5.

ε	Number of mesh points						
	16	32	64	128	256	512	1024
2^{-4}	1.24880E-02 1.428	4.63986E-03 1.815	1.31901E-03 2.018	3.25680E-04 2.016	8.05314E-05 2.002	2.01092E-05 1.728	6.06851E-06
2^{-6}	1.26784E-02 1.414	4.75778E-03 1.722	1.44266E-03 1.569	4.86310E-04 1.622	1.58032E-04 1.660	4.99932E-05 1.698	1.54045E-05
2^{-8}	1.27293E-02 1.411	4.78790E-03 1.719	1.45453E-03 1.574	4.88392E-04 1.619	1.58993E-04 1.662	5.02567E-05 1.698	1.54884E-05
2^{-10}	1.27422E-02 1.410	4.79547E-03 1.718	1.45750E-03 1.576	4.88919E-04 1.618	1.59234E-04 1.662	5.03231E-05 1.698	1.55096E-05
2^{-12}	1.27454E-02 1.410	4.79737E-03 1.718	1.45824E-03 1.576	4.89051E-04 1.618	1.59295E-04 1.662	5.03397E-05 1.698	1.55149E-05
2^{-14}	1.27462E-02 1.410	4.79784E-03 1.718	1.45843E-03 1.576	4.89084E-04 1.618	1.59310E-04 1.662	5.03439E-05 1.698	1.55162E-05
2^{-16}	1.27465E-02 1.410	4.79796E-03 1.718	1.45848E-03 1.576	4.89092E-04 1.618	1.59313E-04 1.662	5.03449E-05 1.698	1.55165E-05
2^{-18}	1.27465E-02 1.410	4.79799E-03 1.718	1.45849E-03 1.576	4.89094E-04 1.618	1.59314E-04 1.662	5.03452E-05 1.698	1.55166E-05
2^{-20}	1.27465E-02 1.410	4.79800E-03 1.718	1.45849E-03 1.576	4.89095E-04 1.618	1.59315E-04 1.662	5.03452E-05 1.698	1.55166E-05
2^{-22}	1.27465E-02 1.410	4.79800E-03 1.718	1.45849E-03 1.576	4.89095E-04 1.618	1.59315E-04 1.662	5.03452E-05 1.698	1.55166E-05
2^{-24}	1.27465E-02 1.410	4.79800E-03 1.718	1.45849E-03 1.576	4.89095E-04 1.618	1.59315E-04 1.662	5.03452E-05 1.698	1.55166E-05
2^{-26}	1.27465E-02 1.410	4.79800E-03 1.718	1.45849E-03 1.576	4.89095E-04 1.618	1.59315E-04 1.662	5.03452E-05 1.698	1.55166E-05
2^{-28}	1.27465E-02 1.410	4.79800E-03 1.718	1.45849E-03 1.576	4.89095E-04 1.618	1.59315E-04 1.662	5.03452E-05 1.698	1.55166E-05
2^{-30}	1.27465E-02 1.410	4.79800E-03 1.718	1.45849E-03 1.576	4.89095E-04 1.618	1.59315E-04 1.662	5.03452E-05 1.698	1.55166E-05
2^{-32}	1.27465E-02 1.410	4.79800E-03 1.718	1.45849E-03 1.576	4.89095E-04 1.618	1.59315E-04 1.662	5.03452E-05 1.698	1.55166E-05
2^{-34}	1.27465E-02 1.410	4.79800E-03 1.718	1.45849E-03 1.576	4.89095E-04 1.618	1.59315E-04 1.662	5.03452E-05 1.698	1.55166E-05
2^{-36}	1.27465E-02 1.410	4.79800E-03 1.718	1.45849E-03 1.576	4.89095E-04 1.618	1.59315E-04 1.662	5.03452E-05 1.698	1.55166E-05
2^{-38}	1.27465E-02 1.410	4.79800E-03 1.718	1.45849E-03 1.576	4.89095E-04 1.618	1.59315E-04 1.662	5.03452E-05 1.698	1.55166E-05
2^{-40}	1.27465E-02 1.410	4.79800E-03 1.718	1.45849E-03 1.576	4.89095E-04 1.618	1.59315E-04 1.662	5.03452E-05 1.698	1.55166E-05

Table 2.6: Maximum point-wise errors and rates of convergence by proposed method for Example 2.3.6.

ε	Number of mesh points						
	16	32	64	128	256	512	1024
2^{-4}	1.20248E-02 1.426	4.47551E-03 1.818	1.26928E-03 2.016	3.13923E-04 2.016	7.76156E-05 2.003	1.93698E-05 2.001	4.84031E-06
2^{-6}	1.25640E-02 1.413	4.71723E-03 1.721	1.43128E-03 1.570	4.81908E-04 1.621	1.56677E-04 1.661	4.95531E-05 1.698	1.52696E-05
2^{-8}	1.27007E-02 1.410	4.77780E-03 1.719	1.45169E-03 1.575	4.87296E-04 1.619	1.58655E-04 1.662	5.01471E-05 1.698	1.54548E-05
2^{-10}	1.27351E-02 1.410	4.79295E-03 1.718	1.45679E-03 1.576	4.88645E-04 1.618	1.59150E-04 1.662	5.02957E-05 1.698	1.55012E-05
2^{-12}	1.27437E-02 1.410	4.79674E-03 1.718	1.45807E-03 1.576	4.88982E-04 1.618	1.59273E-04 1.662	5.03329E-05 1.698	1.55128E-05
2^{-14}	1.27458E-02 1.410	4.79768E-03 1.718	1.45839E-03 1.576	4.89067E-04 1.618	1.59304E-04 1.662	5.03422E-05 1.698	1.55157E-05
2^{-16}	1.27463E-02 1.410	4.79792E-03 1.718	1.45847E-03 1.576	4.89088E-04 1.618	1.59312E-04 1.662	5.03445E-05 1.698	1.55164E-05
2^{-18}	1.27465E-02 1.410	4.79798E-03 1.718	1.45849E-03 1.576	4.89093E-04 1.618	1.59314E-04 1.662	5.03451E-05 1.698	1.55166E-05
2^{-20}	1.27465E-02 1.410	4.79799E-03 1.718	1.45849E-03 1.576	4.89094E-04 1.618	1.59315E-04 1.662	5.03452E-05 1.698	1.55166E-05
2^{-22}	1.27465E-02 1.410	4.79800E-03 1.718	1.45849E-03 1.576	4.89095E-04 1.618	1.59315E-04 1.662	5.03452E-05 1.698	1.55166E-05
2^{-24}	1.27465E-02 1.410	4.79800E-03 1.718	1.45849E-03 1.576	4.89095E-04 1.618	1.59315E-04 1.662	5.03452E-05 1.698	1.55166E-05
2^{-26}	1.27465E-02 1.410	4.79800E-03 1.718	1.45849E-03 1.576	4.89095E-04 1.618	1.59315E-04 1.662	5.03452E-05 1.698	1.55166E-05
2^{-28}	1.27465E-02 1.410	4.79800E-03 1.718	1.45849E-03 1.576	4.89095E-04 1.618	1.59315E-04 1.662	5.03452E-05 1.698	1.55166E-05
2^{-30}	1.27465E-02 1.410	4.79800E-03 1.718	1.45849E-03 1.576	4.89095E-04 1.618	1.59315E-04 1.662	5.03452E-05 1.698	1.55166E-05
2^{-32}	1.27465E-02 1.410	4.79800E-03 1.718	1.45849E-03 1.576	4.89095E-04 1.618	1.59315E-04 1.662	5.03452E-05 1.698	1.55166E-05
2^{-34}	1.27465E-02 1.410	4.79800E-03 1.718	1.45849E-03 1.576	4.89095E-04 1.618	1.59315E-04 1.662	5.03452E-05 1.698	1.55166E-05
2^{-36}	1.27465E-02 1.410	4.79800E-03 1.718	1.45849E-03 1.576	4.89095E-04 1.618	1.59315E-04 1.662	5.03452E-05 1.698	1.55166E-05
2^{-38}	1.27465E-02 1.410	4.79800E-03 1.718	1.45849E-03 1.576	4.89095E-04 1.618	1.59315E-04 1.662	5.03452E-05 1.698	1.55166E-05
2^{-40}	1.27465E-02 1.410	4.79800E-03 1.718	1.45849E-03 1.576	4.89095E-04 1.618	1.59315E-04 1.662	5.03452E-05 1.698	1.55166E-05

(a)(b)
ExEx-
amam-
pleple
2.12.2

(c)(d)
ExEx-
amam-
pleple
2.32.4

(e)(f)
ExEx-
amam-
pleple
2.52.6

Figure 2.2: Loglog plots of N Vs. maximum error.

Chapter 3

Singularly Perturbed Delay Differential Equations

Delay differential equations are of sufficient importance in modelling real life phenomena to merit the attention of computational scientists. The computation of their solutions has been a great challenge and has been of great importance due to the appearance of such equations in the mathematical modeling of processes in various applications, where they provide the best approximation of the observed phenomena. Problems of this type occur where the future depends not only on the immediate present, but also on the past history of the system under consideration.

A delay differential equation is of the retarded type if the delay argument does not occur in the highest order derivative term. If we restrict this class to a class in which the highest derivative term is multiplied by a small parameter, then we get singularly perturbed delay differential equations of the retarded type. These equations govern in the study of an ”Optical Bistable Device” [24] and in a variety of models for physiological processes or diseases [74]. Such a problem also appears to describe the so-called human pupil-light reflex [72]. In this chapter two hybrid difference schemes are constructed , one is for first order singularly perturbed delay differential equations and other is for second order singularly perturbed delay differential equations.

This chapter is organized as follows: Section 3.1 deals with the first order singularly perturbed delay differential equations. Second order singularly perturbed delay differential equations are discussed in Section 3.2. Some numerical results computed from the proposed schemes are next presented in Section 3.3. Certain conclusions based on the analysis are finally summarized in Section 3.4.

3.1 First order delay differential equations

In this section, we consider the following singularly perturbed delay differential equation in the interval $\overline{\Omega} = [0, m]$:

$$\varepsilon u'(x) + a(x)u(x) - b(x)u(x-1) = f(x), \quad x \in \Omega, \tag{3.1.1}$$

$$u(x) = \varphi(x), \quad -1 < x \leq 0, \tag{3.1.2}$$

where $\Omega = (0, m] = \bigcup_{p=1}^{m} \Omega_p$, $\Omega_p = \{x : p-1 < x \le p\}$, $p \ge 1$ and $0 < \varepsilon \le 1$ is the perturbation parameter, $a(x) \ge \beta > 0$, $b(x) > 0$, $f(x)$ and $\varphi(x)$ are given sufficiently smooth functions. The solution, $u(x)$, displays in general boundary layers on the right side of each point $x = p - 1$ for small values of ε [2].

Recently, Amiraliyev and Erdogan [2] have given almost a first order uniformly convergent scheme for solving the problems of the form (3.1.1-3.1.2). They solved the problem (3.1.1-3.1.2) by employing a backward difference scheme on a non-uniform mesh which consists of the special piecewise uniform meshes on each time interval.

3.1.1 Discretization and mesh

Let $\overline{w}_{N_0}$ be a nonuniform mesh on $\bar{\Omega}$.

$$\overline{w}_{N_0} = \{0 = x_0 < x_1 < ...... < x_{N_0} = m, h_i = x_i - x_{i-1}\} \tag{3.1.3}$$

which contains N mesh points at each subinterval Ω_p $(1 \le p \le m)$

$$w_{N,p} = \{x_i : (p-1)N + 1 \le i \le pN\}, \quad 1 \le p \le m, \tag{3.1.4}$$

and consequently $w_{N_0} = \bigcup_{p=1}^{m} w_{N,p}$.

To simplify the notation, we set $g_i = g(x_i)$ for any function $g(x)$. Using a standard discretization for $u^{'}(x)$ in (x_{i-1}, x_i), we define the following scheme for the approximation of (3.1.1-3.1.2) as:

$$L_\epsilon^N U_i \equiv \begin{cases} \varepsilon D^- U_i + \frac{a_{i-1}U_{i-1} + a_i U_i}{2} - \frac{b_{i-1}U_{i-1-N} + b_i U_{i-N}}{2} = \frac{f_{i-1} + f_i}{2}, \quad (p-1)N < i \le \frac{(2p-1)N}{2}, \\ \\ \varepsilon D^- U_i + a_i U_i - b_i U_{i-N} = f_i, \quad \frac{(2p-1)N}{2} < i \le pN, \end{cases} \tag{3.1.5}$$

$$U_i = \varphi_i, \quad -N < i \le 0, \tag{3.1.6}$$

where $D^-U_i = \frac{U_i - U_{i-1}}{h_i}$.

3.1.2 Piece-wise uniform Shishkin mesh

We divide Ω_p into $[p-1, \sigma_p]$ and $[\sigma_p, p]$, where σ_p is a transition point (a function of N and ε) and place $N/2$ points of the mesh in the region $[p-1, \sigma_p]$ and place remaining $N/2$ mesh points in the region $[\sigma_p, p]$. The transition point σ_p which separates the fine and coarse portions of the mesh is obtained by:

$$\sigma_p = p - 1 + \sigma, \tag{3.1.7}$$

where $\sigma = \min\left\{\frac{1}{2}, \sigma_0 \varepsilon \ln N\right\}$ and $\sigma_0 \geq \frac{2}{\beta}$.

Further, we denote the mesh size in the region $[p-1, \sigma_p]$ by $h = \frac{2\sigma}{N}$ and in $(\sigma_p, p]$ by $H = \frac{2(1-\sigma)}{N}$.

3.1.3 Convergence analysis

In this section, we derive an ϵ-uniform second-order error estimates.

Proposition 3.1.1 *Let $u(x)$ and U_i be respectively the solutions of (3.1.1-3.1.2) and (3.1.5-3.1.6). Then, the local truncation error satisfies the following bounds:*

$$\begin{aligned}
|L_\varepsilon^N(U_i - u(x_i))| &\leq CN^{-2}\sigma_0^2 \ln^2 N, \quad \textit{for} \quad (p-1)N < i \leq \frac{(2p-1)N}{2}, \\
|L_\varepsilon^N(U_i - u(x_i))| &\leq C(N^{-1}\varepsilon + N^{-\beta\sigma_0}), \quad \textit{for} \quad \frac{(2p-1)N}{2} < i \leq pN, \quad \textit{and} \quad H \leq \varepsilon, \\
|L_\varepsilon^N(U_i - u(x_i))| &\leq C(N^{-2} + N^{-\beta\sigma_0}), \quad \textit{for} \quad \frac{(2p-1)N}{2} < i \leq pN, \quad \textit{and} \quad H > \varepsilon.
\end{aligned}$$

Proof. We distinguish several cases depending on the location of the mesh points. Firstly, we state the bound for the derivatives of the continuous solution i.e. the solution $u(x)$ of the IVP (3.1.1-3.1.2) satisfies the following bound

$$|u^{(k)}(x)| \leq C\left[1 + \varepsilon^{-k} \exp(-\beta x/\varepsilon)\right]. \tag{3.1.8}$$

By using the usual Taylor series expansion for $x_i \in (p-1, \sigma_p]$, we get

$$|L_\varepsilon^N(U_i - u(x_i))| \leq C\varepsilon h^2 |u'''(\xi)|, \quad \text{for} \quad (p-1)N < i \leq \frac{(2p-1)N}{2}, \tag{3.1.9}$$

for some point ξ, $x_{i-1} \leq \ \xi \leq x_i$.

First we consider the case when the mesh is uniform. In this case, $\sigma = 1/2$ and $\epsilon^{-1} \leq C\sigma_0 \ln N$. Using the bound (3.1.8), we have

$$\begin{aligned} |L_\varepsilon^N(U_i - u(x_i))| \quad & \leq C\varepsilon h^2 \left[1 + \varepsilon^{-3} \exp(-\beta\xi/\varepsilon)\right], \\ & \leq CN^{-2}\sigma_0^2 ln^2 N, \quad \text{for} \quad (p-1)N < i \leq \frac{(2p-1)N}{2}. \end{aligned} \tag{3.1.10}$$

Secondly, we consider the case when the mesh is non-uniform, that is $h = 2N^{-1}\sigma_0\varepsilon lnN$ on the above bound and bounding the exponential function by a constant, we have

$$|L_\varepsilon^N(U_i - u(x_i))| \leq CN^{-2}\sigma_0^2 ln^2 N, \quad \text{for} \quad (p-1)N < i \leq \frac{(2p-1)N}{2}. \tag{3.1.11}$$

Again, using the usual Taylor series expansion for $x_i \in (\sigma_p, p]$, we get

$$|L_\varepsilon^N(U_i - u(x_i))| \leq C\varepsilon H |u''(\xi)|, \quad \text{for} \quad \frac{(2p-1)N}{2} < i \leq pN. \tag{3.1.12}$$

Note that the above expression for truncation error in the outer region can also be represented as

$$|L_\varepsilon^N(U_i - u(x_i))| = \frac{\varepsilon}{h_i} R_1(x_{i-1}, x_i, u), \tag{3.1.13}$$

where $R_n(a,p,g) = \dfrac{1}{n!}\displaystyle\int_a^p (p-\xi)^n g^{(n+1)}(\xi) d\xi$ denotes the remainder term obtained from Taylor expansion in an integral form.

We discuss two cases: First, if $H < \varepsilon$, from (3.1.12), we obtain

$$\begin{aligned} |L_\varepsilon^N(U_i - u(x_i))| &\leq C\varepsilon H|u''(\xi)|, \\ &\leq C[H\varepsilon + H\varepsilon^{-1}\exp(-\beta x_i/\varepsilon)], \\ &\leq C[N^{-1}\varepsilon + N^{-\beta\sigma_0}]. \end{aligned} \quad (3.1.14)$$

Secondly, if $H \geq \varepsilon$, then using the bounds of the derivatives of $u(x)$ from (3.1.8), one can obtain the following

$$|L_\varepsilon^N(U_i - u(x_i))| \leq C\left(H\varepsilon + \int_{x_{i-1}}^{x_i}(x_i - \xi)\varepsilon^{-2}\exp(-\beta\xi/\varepsilon)d\xi\right). \quad (3.1.15)$$

Integrating by parts, we get

$$\begin{aligned} \int_{x_{i-1}}^{x_i}(x_i - \xi)\varepsilon^{-2}\exp(\beta\xi/\varepsilon)d\xi \quad &\leq C\left(H\varepsilon + \int_{x_{i-1}}^{x_i}\varepsilon^{-1}\exp(-\beta\xi/\varepsilon)d\xi\right), \\ &\leq C\left[N^{-1}\varepsilon + N^{-\beta\sigma_0}\right].[8pt] \end{aligned} \quad (3.1.16)$$

Since $H < 2N^{-1}$ and $\varepsilon \leq H$, we get

$$|L_\varepsilon^N(U_i - u(x_i))| \leq C\left(N^{-2} + N^{-\beta\sigma_0}\right). \quad (3.1.17)$$

Combining all the previous results, we obtain the required truncation error.

Lemma 3.1.2 *Let U_i be the numerical solution obtained from the difference scheme as given in (3.1.5-3.1.6). Then, for sufficiently large N and $N^{-1}\sigma_0 \ln N\beta^* < 1$, where $\beta^* = \max\limits_{0\leq i\leq N} a(x_i)$, we have, $U_i \geq 0\, for\, i = 0, 1, 2, ...N$ and it satisfies discrete maximum principle.*

Proof. Let $B_i^- = (2 - \rho_i a_i)$, $B_i^+ = (2 + \rho_i a_i)$ and $b_i^+ = (1 + \rho_i a_i)$, where $\rho_i = \frac{h_i}{\varepsilon}$.
The solution of the scheme (3.1.5-3.1.6) can be expressed as:

For $(p-1)N < i \leq \frac{(2p-1)N}{2}$

$$U_i = \frac{\prod_{j=0}^{i-1} B_j^-}{\prod_{j=1}^{i} B_j^+} U_0 + \frac{\rho_i \prod_{j=1}^{i-1} B_j^-}{\prod_{j=1}^{i} B_j^+}(f_0+f_1+b_0U_{-N}+b_1U_{1-N}) + \frac{\rho_i \prod_{j=2}^{i-1} B_j^-}{\prod_{j=2}^{i} B_j^+}(f_1+f_2+b_1U_{1-N}+b_2U_{2-N})+ \ldots$$
$$+ \frac{\rho_i}{B_i^+}(f_{i-1}+f_i+b_{i-1}U_{i-1-N}+b_iU_{i-N}) \quad (3.1.18)$$

and for $\frac{(2p-1)N}{2} < i \leq pN$,

$$U_i = \frac{1}{\prod_{j=N/2+1}^{i} b_j^+} U_{N/2} + \frac{\rho_i}{\prod_{j=N/2+1}^{i} b_j^+}(f_{N/2+1}+b_1U_{N/2+1-N}) + \frac{\rho_i}{\prod_{j=N/2+2}^{i} b_j^+}(f_{N/2+2}+b_2U_{N/2+2-N})+ \ldots$$
$$+ \frac{\rho_i}{b_i^+}(f_i+b_iU_{i-N}). \quad (3.1.19)$$

Clearly, B_i^+, b_i^+ are non-negative. Also, B_i^- is non-negative for $(p-1)N < i \leq \frac{(2p-1)N}{2}$. Hence, we obtain the required result . So, the solution satisfies the discrete maximum principle.

Theorem 3.1.3 *Let $u(x)$ be the solution of the IVP (3.1.1-3.1.2) and U_i be the numerical solution obtained from the difference scheme as given in (3.1.5-3.1.6). Then,*

$$|U_i - u(x_i)| \leq C\left[N^{-2}\ln^2 N + N^{-1}\varepsilon + N^{-\beta\sigma_0}\right], \quad \forall x_i \in \overline{\Omega}.$$

Proof.

Defining the discrete barrier function

$$\phi_i = C\left[N^{-2}\ln^2 N + N^{-1}\varepsilon + N^{-\beta\sigma_0}\right]. \quad (3.1.20)$$

Now, Choosing C sufficiently large, and using the discrete maximum principle, it is easy to see that,

$$L_\epsilon^N(\phi_i \pm (U_i - u(x_i))) \geq 0, \quad (3.1.21)$$

equivalently,

$$L_\epsilon^N(\phi_i) \geq |U_i - u(x_i)|. \tag{3.1.22}$$

Therefore, it follows that

$$|U_i - u(x_i)| \leq |\phi_i|, \quad \forall x_i \in \overline{\Omega}. \tag{3.1.23}$$

Thus, we have the required ε-uniform error bound. ∎

Remark 3.1.4 *In Theorem 3.1.1, One can notice that the truncation error is of order $N^{-\beta\sigma_o}$ for $H > \epsilon$. It is assumed that $\beta\sigma_o \geq 2$ and we are interested in the case $\epsilon \leq N^{-1}$. Also, we obtain the error bound of order $N^{-1}\varepsilon$ only in outer region for the case $H < \epsilon$, which is not the practical case. With these points, we conclude that the order of convergence is almost two (up to a logarithmic factor). Our numerical results and comparison with [2] given in Section 3.3 reveal the same behavior.*

3.2 Second order delay differential equations

In this section, we consider the following singularly perturbed delay differential equation (DDE) in the interval $\overline{\Omega} = [0,1]$:

$$L_\epsilon u(x) \equiv \varepsilon u''(x) + a(x)u'(x-\delta) + b(x)u(x) = f(x), \quad x \in \Omega = (0,1), \tag{3.2.1}$$

$$u(x) = \varphi(x), \quad -\delta \leq x \leq 0, \tag{3.2.2}$$

$$u(1) = \lambda, \tag{3.2.3}$$

where $0 < \varepsilon \ll 1$ is the small perturbation parameter and the delay parameter δ is such that $0 < \delta < 1$. The functions $a(x)$, $b(x)$, $f(x)$ and $\varphi(x)$ are sufficiently smooth functions and λ is a constant. It is also assumed that $b(x) \geq \beta > 0$, $\forall x \in \overline{\Omega}$. When $\delta = 0$, the equation (3.2.1-3.2.3) reduces to a singularly perturbed differential equation. Depending upon the sign of $a(x)$, i.e., if $a(x) > 0$ (or $a(x) < 0$), a boundary layer is located at left (or right) end of domain. The layer is maintained for sufficiently small δ with $\delta \neq 0$ and $\delta = o(\epsilon)$. These equations govern in many mathematical models of biophysics and mechanics where delay term plays an important role [114], models for physiological processes or disease [141, 74]. The DDEs provide more realistic models than the conventional singularly perturbed differential equations. For example, in population dynamics, these small parameters display time-lag or after-effect and hence play an important role in modelling real-life phenomena.

Kadalbajoo and Sharma [48] used a fitted-mesh method on a Shishkin mesh to study linear ordinary differential-difference equations with both positive and negative shifts, that is, with both delay and advance, using first-order accurate Taylor series expansion of the terms with shifts. For a singularly perturbed convection-diffusion problem, Roos and Stynes [115] presented a numerical method that is composed of the central difference scheme and a mid-point scheme defined on the Shishkin mesh. Bawa and Natesan [11] have analyzed a hybrid scheme for convection-dominated boundary value problems that uses cubic spline in the fine mesh region and the mid-point upwind scheme in the coarse mesh region. Motivated by these works, a hybrid difference scheme is proposed to approximate the solution of singularly perturbed delay differential equation.

First, we construct a difference scheme using cubic spline. And then, we apply this scheme on a Shishkin mesh. In the boundary layer (inner) region, the mesh is fine, and the cubic spline scheme is stable. Whereas in the regular (outer) region, the mesh is coarse, and the cubic spline scheme is not stable. To obtain stability in the outer region, one has to restrict the step size in that region, but our aim is to propose an ϵ-uniform convergent numerical scheme. Therefor, for the outer region, instead of the cubic scheme we use the finite difference scheme, mainly for stability reasons. The newly obtained hybrid scheme is convergent independent of the singular perturbation parameter.

Assumption: We shall assume that $\epsilon \leq CN^{-1}$ as is generally the case for discretization of convection-dominated problems [115].

3.2.1 Maximum principle and stability result

We consider the case when delay δ is $o(\epsilon)$ and uses Taylor series to tackle the delay argument.

$$u'(x-\delta) \approx u'(x) - \delta u''(x). \tag{3.2.4}$$

Using (3.2.4) in the problem (3.2.1-3.2.3), we get

$$L_\epsilon u(x) \equiv (\varepsilon - \delta a(x))u''(x) + a(x)u'(x) + b(x)u(x) = f(x), \quad x \in \Omega = (0,1), \tag{3.2.5}$$

$$u(0) = \varphi(0), \tag{3.2.6}$$

$$u(1) = \lambda, \tag{3.2.7}$$

Without loss of generality, here we assume that $a(x) \geq 2\alpha > 0$ and $(\epsilon - \delta a(x)) > 0 \, \forall \, x \in [0,1]$. Under these assumptions, the solution of the problem (3.2.5-3.2.7) has a unique solution and it exhibits layer behavior on the left side of the domain.

Now, we present some analytical results which include a maximum principle, uniform stability and estimates of the derivatives of the solution. These results will be used to derive error bounds for the derivatives of the solution.

Lemma 3.2.1 *Let v be a smooth function satisfying $v(0) \geq 0$, $v(1) \geq 0$ and $L_\epsilon v(x) \geq 0$, $x \in \Omega$, then $v(x) \geq 0$ in $\overline{\Omega}$.*

Proof. Let $x^* \in \overline{\Omega}$ be such that $v(x^*) = \min_{x \in \overline{\Omega}} v(x)$ and assume that $v(x^*) < 0$. Clearly $x^*\{0,1\}$ and $v'(x^*) = 0$ and $v''(x^*) \geq 0$. Now consider

$$L_\epsilon v(x^*) \equiv (\varepsilon - \delta a(x^*))v''(x^*) + a(x^*)v'(x^*) + b(x^*)v(x^*) < 0, \tag{3.2.8}$$

which contradicts our assumption. Hence $v(x) \geq 0$ in $\overline{\Omega}$.

An immediate consequence of the maximum principle is the following stability estimate. ∎

Lemma 3.2.2 *If u is the solution of the boundary value problem (3.2.1-3.2.3) then*

$$||u|| \leq \frac{1}{\beta}||f|| + \max\{|u(0)|, |\lambda|\}.$$

Proof. Let us consider the following barrier function

$$\Psi^{\pm}(x) = \frac{1}{\beta}||f|| + \max\{|u(0)|, |\lambda|\} \pm u(x). \tag{3.2.9}$$

It is easy to show that $\Psi^{\pm}(x)$ is non-negative at $x = 0, 1$. Now from (3.2.5-3.2.7)

$$\begin{aligned} L_\epsilon \Psi^{\pm}(x) &= (\varepsilon - \delta a(x))(\Psi^{\pm}(x))'' + a(x)(\Psi^{\pm}(x))' + b(x)(\Psi^{\pm}(x)) && (3.2.10)\\ &= \left[\frac{1}{\beta}||f|| + \max\{|u(0)|, |\lambda|\}\right] \pm L_\epsilon u(x) && (3.2.11)\\ &\geq [||f|| \pm f(x)] + b(x)\ \max\{|u(0)|, |\lambda|\} \geq 0. && (3.2.12) \end{aligned}$$

Thus by applying the maximum principle we conclude that $\Psi^{\pm}(x) \geq 0,\ \forall x \in \overline{\Omega}$, which is the required result.

Lemma 3.2.3 *[81] Let $0 \leq k \leq 5$ be a positive integer. Assume that $f \in C^k[0,1]$, then the derivatives $u^{(k)}$ of the solution u of (3.2.1-3.2.3) satisfy the following bound:*

$$|u^{(k)}| \leq C\left[1 + (\epsilon - \delta\alpha)^{-k} exp\left(\frac{-\alpha(1-x)}{\epsilon - \delta\alpha}\right)\right].$$

where C depends upon $||a||, ||a'||, ||b||, ||b'||$ and on the boundary conditions.

3.2.2 Discrete problem

In this sub-section, first we derive a cubic spline scheme on a variable mesh. Let the mesh points of $\overline{\Omega}$ be

$$x_0 = 0,\ x_i = \sum_{k=0}^{i-1} h_k,\ h_k = x_{k+1} - x_k,\ x_N = 1,\ i = 1, 2, \ldots, N-1, \tag{3.2.13}$$

then, for given values $u(x_0), u(x_1), \cdots, u(x_N)$ of a function $u(x)$ at the nodal points $x_0, x_1, \ldots, x_N$, there exists an interpolating cubic spline $S(x)$ with the following properties:

(i) $S(x)$ coincides with a polynomial of degree three on each subinterval $[x_i, x_{i+1}],\ i = 0, \ldots, N-1$;

(ii) $S(x) \in C^2[0,1]$;

(iii) $S(x_i) = u(x_i),\ i = 0, \ldots, N$.

It is well known that the cubic spline can be written in the form

$$\begin{aligned} S(x) = {} & \frac{(x_{i+1}-x)^3}{6h_i} M_i + \left(u(x_i) - \frac{h_i^2}{6} M_i\right)\left(\frac{x_{i+1}-x}{h_i}\right) + \frac{(x-x_i)^3}{6h_i} M_{i+1} \\ & + \left(u(x_{i+1}) - \frac{h_i^2}{6} M_{i+1}\right)\left(\frac{x-x_i}{h_i}\right), \quad x_i \le x \le x_{i+1},\ i = 0, \cdots, N-1, \end{aligned} \tag{3.2.14}$$

where $M_i = S''(x_i),\ i = 0, \cdots, N$. From the basic properties of splines, it should satisfy the following 'condition of continuity':

$$\frac{h_{i-1}}{6} M_{i-1} + \left(\frac{h_i + h_{i-1}}{3}\right) M_i + \frac{h_i}{6} M_{i+1} = \left(\frac{u(x_{i+1}) - u(x_i)}{h_i}\right) - \left(\frac{u(x_i) - u(x_{i-1})}{h_{i-1}}\right),\ i = 1, \cdots, N-1. \tag{3.2.15}$$

The continuity condition given above ensures the continuity of the first order derivatives of the spline $S(x)$ at the interior nodes. Then, substituting

$$p_j M_j = -a(x_j) u'(x_j) - b(x_j) u(x_j) + f(x_j), \quad j = i, i \pm 1, \tag{3.2.16}$$

in (3.2.15), we get the following system which gives the approximations $u_1, u_2, \cdots u_{N-1}$ of the solution $u(x)$ at $x_1, x_2, \cdots x_{N-1}$:

$$\begin{aligned}
&\left[\frac{3p_{i-1}p_ip_{i+1}}{h_{i-1}(h_i+h_{i-1})} - \frac{p_ip_{i+1}(2h_{i-1}+h_i)}{2(h_i+h_{i-1})^2}a_{i-1} - \frac{p_{i-1}p_{i+1}h_i}{h_{i-1}(h_i+h_{i-1})}a_i\right.\\
&\left.+ \frac{p_{i-1}p_ih_i^2}{2h_{i-1}(h_i+h_{i-1})^2}a_{i+1} + \frac{p_ip_{i+1}h_{i-1}}{2(h_i+h_{i-1})}b_{i-1}\right]u_{i-1}\\
&+\left[\frac{-3p_{i-1}p_ip_{i+1}}{h_ih_{i-1}} + \frac{p_ip_{i+1}}{2h_i}a_{i-1} + \frac{p_{i-1}p_{i+1}(h_i-h_{i-1})}{h_ih_{i-1}}a_i - \frac{p_{i-1}p_i}{2h_{i-1}}a_{i+1} + p_{i-1}p_{i+1}b_i\right]u_i\\
&+\left[\frac{+3p_{i-1}p_ip_{i+1}}{h_i(h_i+h_{i-1})} - \frac{p_ip_{i+1}h_{i-1}^2}{2h_i(h_i+h_{i-1})^2}a_{i-1} + \frac{p_{i-1}p_{i+1}h_{i-1}}{h_i(h_i+h_{i-1})}a_i\right.\\
&\left.+\frac{p_{i-1}p_i(2h_i+h_{i-1})}{2(h_i+h_{i-1})^2}a_{i+1} + \frac{h_ip_{i-1}p_i}{2(h_i+h_{i-1})}b_{i+1}\right]u_{i+1}\\
&= \left[\frac{p_ip_{i+1}h_{i-1}}{2(h_i+h_{i-1})}\right]f_{i-1} + p_{i-1}p_{i+1}f_i + \left[\frac{p_{i-1}p_ih_i}{2(h_i+h_{i-1})}\right]f_{i+1},
\end{aligned} \tag{3.2.17}$$

where $p_i = \epsilon - \delta a(x_i)$. Note that $u_0 = \varphi(0)$ and $u_N = \lambda$ are the natural discretizations of the Dirichlet boundary conditions.

3.2.3 Piece-wise uniform Shishkin mesh

As mentioned in sub-section 2.1.2, in this mesh, the domain $\overline{\Omega}$ is divided into two sub-intervals as $\overline{\Omega} = [0, \sigma] \cup (\sigma, 1]$ for some σ such that $0 \le \sigma \le \frac{1}{2}$, defined as

$$\sigma = \min\{\frac{1}{2}, \frac{2(\epsilon - \delta\alpha)}{\beta}\ln N\}. \tag{3.2.18}$$

It is obvious that the mesh is uniform when $\sigma = 1/2$. Further, we denote the mesh size in the region $[0, \sigma)$ by $h = \frac{2\sigma}{N}$ and in $[\sigma, 1]$ by $H = \frac{2(1-\sigma)}{N}$.

The hybrid scheme

As pointed out in Section 3.2, the cubic spline scheme loses its stability in the outer region, where the meshes are coarse. In order to retain its stability, we use the midpoint scheme [115] in the outer region. This combination yields the following uniformly-stable second-order scheme:

$$L_\epsilon^N u_i \equiv \; r_i^- u_{i-1} + r_i^c u_i + r_i^+ u_{i+1} = q_i^- f_{i-1} + q_i^c f_i + q_i^+ f_{i+1}, \quad i = 1, \cdots, N-1, \; . \tag{3.2.19}$$

along with boundary conditions $u_0 = \varphi(0)$ and $u_N = \lambda$. For $i = 1, \cdots, N/2 - 1$, the coefficients are given by

$$\left\{ \begin{aligned}
r_i^- &= \frac{3p_{i-1}p_ip_{i+1}}{h_{i-1}(h_i + h_{i-1})} - \frac{p_ip_{i+1}(2h_{i-1} + h_i)}{2(h_i + h_{i-1})^2} a_{i-1} - \frac{p_{i-1}p_{i+1}h_i}{h_{i-1}(h_i + h_{i-1})} a_i \\
&\qquad + \frac{p_{i-1}p_ih_i^2}{2h_{i-1}(h_i + h_{i-1})^2} a_{i+1} + \frac{p_ip_{i+1}h_{i-1}}{2(h_i + h_{i-1})} b_{i-1}; \\
r_i^c &= \frac{-3p_{i-1}p_ip_{i+1}}{h_ih_{i-1}} + \frac{p_ip_{i+1}}{2h_i} a_{i-1} + \frac{p_{i-1}p_{i+1}(h_i - h_{i-1})}{h_ih_{i-1}} a_i - \frac{p_{i-1}p_i}{2h_{i-1}} a_{i+1} + p_{i-1}p_{i+1}b_i; \\
r_i^+ &= \frac{+3p_{i-1}p_ip_{i+1}}{h_i(h_i + h_{i-1})} - \frac{p_ip_{i+1}h_{i-1}^2}{2h_i(h_i + h_{i-1})^2} a_{i-1} + \frac{p_{i-1}p_{i+1}h_{i-1}}{h_i(h_i + h_{i-1})} a_i + \frac{p_{i-1}p_i(2h_i + h_{i-1})}{2(h_i + h_{i-1})^2} a_{i+1} \\
&\qquad + \frac{h_ip_{i-1}p_i}{2(h_i + h_{i-1})} b_{i+1}; \\
q_i^- &= \frac{p_ip_{i+1}h_{i-1}}{2(h_i + h_{i-1})}; \quad q_i^c = p_{i-1}p_{i+1}; \quad q_i^+ = \frac{p_{i-1}p_ih_i}{2(h_i + h_{i-1})},
\end{aligned} \right. . \tag{3.2.20}$$

and for $i = N/2, \cdots, N-1$

$$\left\{ \begin{aligned}
r_i^- &= \frac{2p_i}{h_{i-1}(h_i + h_{i-1})}; \quad r_i^c = \frac{-2p_i}{h_{i-1}(h_i + h_{i-1})} - \frac{2p_i}{h_i(h_i + h_{i-1})} - \frac{a_{i+1} + a_i}{2h_i} + \frac{b_{i+1} + b_i}{2}; \\
r_i^+ &= \frac{2p_i}{h_i(h_i + h_{i-1})} + \frac{a_{i+1} + a_i}{2h_i} + \frac{b_{i+1} + b_i}{2}; \\
q_i^- &= 0; \quad q_i^c = 0.5; \quad q_i^+ = 0.5.
\end{aligned} \right. \tag{3.2.21}$$

Note that the stiffness matrix of the newly modified hybrid scheme (3.2.19) is an M-matrix.

3.2.4 Truncation error

Now, we derive the truncation error for the proposed difference scheme (3.2.19).

For $i = 1, \cdots, N/2$, the truncation error of the hybrid scheme is given by

$$\tau_{i,u} = [r_i^- u(x_{i-1}) + r_i^c u(x_i) + r_i^+ u(x_{i+1})] - [q_i^- f(x_{i-1}) + q_i^c f(x_i) + q_i^+ f(x_{i+1})]. \quad (3.2.22)$$

Using the differential equation (3.2.1) for f in the above expression, we get

$$\begin{aligned}\tau_{i,u} = \left[r_i^- u(x_{i-1}) + r_i^c u(x_i) + r_i^+ u(x_{i+1})\right] - [q_i^- \left(p_{i-1} u''(x_{i-1}) + a_{i-1} u'(x_{i-1}) + b_{i-1} u(x_{i-1})\right) \\ + q_i^c \left(p_i u''(x_i) + a_i u'(x_i) + b_i u(x_i)\right) + q_i^+ \left(p_{i+1} u''(x_{i+1}) + a_{i+1} u'(x_{i+1}) + b_{i+1} u(x_{i+1})\right)].\end{aligned} \quad (3.2.23)$$

Now, making use of the Taylor series expansion, we have

$$u(x_{i-1}) = u(x_i) - h_{i-1} u'(x_i) + \frac{h_{i-1}^2}{2!} u''(x_i) - \frac{h_{i-1}^3}{3!} u^{(iii)}(x_i) + \frac{h_{i-1}^4}{4!} u^{(iv)}(x_i) + \cdots,$$

and

$$u(x_{i+1}) = u(x_i) + h_i u'(x_i) + \frac{h_i^2}{2!} u''(x_i) + \frac{h_i^3}{3!} u^{(iii)}(x_i) + \frac{h_i^4}{4!} u^{(iv)}(x_i) + \cdots$$

Using these values of $u(x_{i-1}),\ u(x_{i+1})$ in (3.2.23), we have

$$\tau_{i,u} = T_{0,i} u(x_i) + T_{1,i} u'(x_i) + T_{2,i} u''(x_i) + T_{3,i} u^{(iii)}(x_i) + T_{4,i} u^{(iv)}(x_i) + \text{h.o.t.}, \quad (3.2.24)$$

where

$$
\begin{aligned}
T_{0,i} &= r_i^- + r_i^c + r_i^+ - (q_i^- b_{i-1} + q_i^c b_i + q_i^+ b_{i+1}),\\[2ex]
T_{1,i} &= -h_{i-1} r_i^- + h_i r_i^+ - (q_i^- a_{i-1} + q_i^c a_i + q_i^+ a_{i+1}) + (h_{i-1} q_i^- b_{i-1} - h_i q_i^+ b_{i+1}),\\[2ex]
T_{2,i} &= \frac{h_{i-1}^2}{2!} r_i^- + \frac{h_i^2}{2!} r_i^+ - (p_{i-1} q_i^- + p_i q_i^c + p_{i+1} q_i^+) + (h_{i-1} q_i^- a_{i-1} - h_i q_i^+ a_{i+1})\\
&\quad - \left(\frac{h_{i-1}^2}{2!} b_{i-1} q_i^- + \frac{h_i^2}{2!} b_{i+1} q_i^+\right),\\[2ex]
T_{3,i} &= -\frac{h_{i-1}^3}{3!} r_i^- + \frac{h_i^3}{3!} r_i^+ + (p_{i-1} q_i^- h_{i-1} - p_{i+1} q_i^+ h_i) - \left(\frac{h_{i-1}^2}{2!} q_i^- a_{i-1} + \frac{h_i^2}{2!} q_i^+ a_{i+1}\right)\\
&\quad + \left(\frac{h_{i-1}^3}{3!} q_i^- b_{i-1} - \frac{h_i^3}{3!} q_i^+ b_{i+1}\right),\\[2ex]
T_{4,i} &= \frac{h_{i-1}^4}{4!} r_i^- + \frac{h_i^4}{4!} r_i^+ - \left(\frac{h_{i-1}^2}{2!} q_i^- p_{i-1} + \frac{h_i^2}{2!} q_i^+ p_{i+1}\right) + \left(\frac{h_{i-1}^3}{3!} q_i^- a_{i-1} - \frac{h_i^3}{3!} q_i^+ a_{i+1}\right)\\
&\quad - \left(\frac{h_{i-1}^4}{4!} q_i^- b_{i-1} + \frac{h_i^4}{4!} q_i^+ b_{i+1}\right).
\end{aligned}
$$

It can be easily seen that

$$
T_{0,i} = T_{1,i} = T_{2,i} = T_{3,i} = 0, \quad T_{4,i} = -3 p_{i-1} p_i p_{i+1} \left(\frac{h_i^3 + h_{i-1}^3}{h_i + h_{i-1}}\right) \left[\frac{1}{4!} - \frac{1}{2!6}\right].
$$

Thus, we have

$$
\tau_{i,u} = -3 p_{i-1} p_i p_{i+1} \left(\frac{h_i^3 + h_{i-1}^3}{h_i + h_{i-1}}\right) \left[\frac{1}{4!} - \frac{1}{2!6}\right] u^{(iv)}(x_i) + O(N^{-3}). \tag{3.2.25}
$$

For $i = N/2, \cdots, N/2 - 1$, we can proceed in a similar manner to show that

$$
\tau_{i,u} = -p_i \left(\frac{h_i - h_{i-1}}{3}\right) u^{(iii)}(x_i) + \frac{2p_i}{4!} \left(\frac{h_i^3 + h_{i-1}^3}{h_i + h_{i-1}}\right) u^{(iv)}(x_i) + O(N^{-3}). \tag{3.2.26}
$$

Using the bounds of the solution obtained in Section 3.2, we have the following proposition.

Proposition 3.2.4 *Let $u(x)$, $x \in \overline{\Omega}$ be the solution of (3.2.1-3.2.3) and let $U(x_i)$, $x \in \overline{\Omega}^N$ be respectively the solutions of (3.2.19). Then, we have:*

$$\sup_{0<\epsilon\leq 1} ||U - u||_{\Omega^N} \ \leq \ CN^{-2}(\ln N)^3.$$

3.3 Numerical experiments and discussions

For first order delay differential equations, maximum point-wise errors and rates of convergence are calculated as follows: Maximum point-wise errors are calculated as:

$$E_\varepsilon^{N,p} = \max_{x_i \varepsilon w_{N,p}} \{|u(x_i) - U_i^{N,p}|\},\ p = 1, 2, \tag{3.3.1}$$

where $u(x_i)$ is the exact solution and $U_i^{N,p}$ is the numerical solution obtained by using N mesh intervals in the domain Ω_p. The rates of convergence are calculated as:

$$r^{N,p} = \frac{lnE_\varepsilon^{N,p} - lnE_\varepsilon^{2N,p}}{ln2},\ p = 1, 2. \tag{3.3.2}$$

For second order delay differential equations, maximum point-wise errors and rates of convergence are calculated as follows:

If the exact solution of the example is not available, then to obtain the maximum point-wise errors and rates of convergence, we use double mesh principle. We calculate the numerical solutions U^N on $\overline{\Omega}_\epsilon^N$ and $\widetilde{U}^N$on the mesh $\widetilde{\Omega}_\epsilon^N$, where the transition parameter σ is altered slightly to $\widetilde{\sigma} = min\{1/2, \frac{2(\epsilon-\delta\alpha)}{\beta} \ln N/2\}$. The double mesh difference is defined as

$$E_\epsilon^N = \max_{x_i \in \overline{\Omega}_\epsilon^N} \{|U_i^N - \widetilde{U}_i^{2N}|\}, \tag{3.3.3}$$

where U_i^N and $\widetilde{U}_i^{2N}$ denote respectively the numerical solutions obtained by using N and $2N$ mesh intervals. The rates of convergence are calculated as:

$$p_\epsilon^N = \frac{lnE_\epsilon^N - lnE_\epsilon^{2N}}{ln2}. \tag{3.3.4}$$

Example 3.3.1 *Consider the following Problem:*

$$\varepsilon u'(x) + u(x) = \tfrac{1}{2}u(x-1), \quad x \in (0,\infty),$$
$$u(x) = 2, \quad -1 \le x \le 0.$$

The exact solution for $0 \le x \le 2$ is given by

$$u(x) = \begin{cases} 1 + e^{-x/\varepsilon}, & x \in [0,1], \\ \tfrac{1}{2} + e^{-x/\varepsilon} + [\tfrac{1}{2} + \tfrac{x-1}{2\varepsilon}]e^{-(x-1)/\varepsilon}, & x \in (1,2]. \end{cases}$$

We present the maximum point-wise errors obtained for different values of ϵ and N by our method in Table 3.1 & Table 3.3 and maximum point-wise errors obtained by Amiraliyev & Erdogan [2] for different values of ϵ and N in Table 3.2 & Table 3.4. Tables 3.1 & 3.3 show that the proposed hybrid scheme is parameter uniform and works nicely independent of the mesh parameter h and the singular perturbation parameter ϵ. Fig.3.1 gives comparison of maximum point-wise error of the proposed method and Amiraliyev & Erdogan [2] for different values of ϵ on Ω_1.

Example 3.3.2 *Consider the following Problem:*

$$\epsilon u''(x) + u'(x-\delta) - u(x) = 0,$$
$$u(x) = 1, \quad -\delta < x \le 0,$$
$$u(1) = 1.$$

The exact solution for $0 \le x \le 2$ is given by

$$u(x) = \frac{(1-exp(m_2))exp(m_1x) - (1-exp(m_1))exp(m_2x)}{exp(m_1) - exp(m_2)},$$

where

$$m_1 = \frac{-1+\sqrt{1+4(\epsilon-\delta)}}{2(\epsilon-\delta)} \text{ and } m_2 = \frac{-1-\sqrt{1+4(\epsilon-\delta)}}{2(\epsilon-\delta)}.$$

The numerical results by the proposed scheme are given in Table 3.5.

Example 3.3.3 *Consider the following Problem:*

$$\epsilon u''(x) + exp(-x)u'(x-\delta) - xu(x) = 0,$$

$$u(x) = 1, \quad -\delta < x \leq 0,$$

$$u(1) = 1.$$

The numerical results by proposed scheme are in Table 3.6.

3.4 Conclusions

In this chapter two hybrid difference schemes are constructed , one is for first order singularly perturbed delay differential equations and other is for second order singularly perturbed delay differential equation. These hybrid schemes are of almost second order convergent upto a logarithmic factor. Numerical experiments show that maximum point-wise errors and rate of convergence are in agrement with the theoretical results. The graphs plotted in Figure 3.2 are convergent curves in the maximum norm at nodal points for the different values of ϵ for Examples (3.3.1-3.3.3).

The proposed scheme gives uniform convergence and accurate solutions to singularly perturbed delay differential equations over a wide range of ϵ and δ. The numerical results demonstrate that the method is robust, i.e., converges for all ϵ and δ with the condition δ is $O(\epsilon)$.

Figure 3.1: Comparison of maximum point-wise error of the proposed method and Amiraliyev and Erdogan [2] for different values of ϵ on Ω_1 for Example 3.3.1.

(a) Example 3.1 on Ω_1

(b) Example 3.2 on Ω_2

(c) Example 3.3

(d) Example 3.4

Figure 3.2: Loglog plots of N Vs. maximum error.

Table 3.1: Maximum point-wise errors and rates of convergence by proposed method on Ω_1 for Example 3.3.1.

ε	Number of mesh points						
	16	32	64	128	256	512	1024
2^{-4}	1.47488E-02	5.82269E-03	1.92913E-03	4.7982E-04	1.19803E-04	2.99413E-05	7.48472E-06
	1.34	1.59	2.01	2.00	2.00	2.00	
2^{-8}	1.47488E-02	5.82269E-03	2.08474E-03	7.05251E-04	2.30149E-04	7.28171E-05	2.24765E-05
	1.34	1.48	1.56	1.62	1.66	1.70	
2^{-10}	1.47488E-02	5.82269E-03	2.08474E-03	7.05251E-04	2.30149E-04	7.28171E-05	2.24765E-05
	1.34	1.48	1.56	1.62	1.66	1.70	
2^{-12}	1.47488E-02	5.82269E-03	2.08474E-03	7.05251E-04	2.30149E-04	7.28171E-05	2.24765E-05
	1.34	1.48	1.56	1.62	1.66	1.70	
2^{-14}	1.47488E-02	5.82269E-03	2.08474E-03	7.05251E-04	2.30149E-04	7.28171E-05	2.24765E-05
	1.34	1.48	1.56	1.62	1.66	1.70	
2^{-16}	1.47488E-02	5.82269E-03	2.08474E-03	7.05251E-04	2.30149E-04	7.28171E-05	2.24765E-05
	1.34	1.48	1.56	1.62	1.66	1.70	
2^{-18}	1.47488E-02	5.82269E-03	2.08474E-03	7.05251E-04	2.30149E-04	7.28171E-05	2.24765E-05
	1.34	1.48	1.56	1.62	1.66	1.70	
2^{-20}	1.47488E-02	5.82269E-03	2.08474E-03	7.05251E-04	2.30149E-04	7.28171E-05	2.24765E-05
	1.34	1.48	1.56	1.62	1.66	1.70	
2^{-22}	1.47488E-02	5.82269E-03	2.08474E-03	7.05251E-04	2.30149E-04	7.28171E-05	2.24765E-05
	1.34	1.48	1.56	1.62	1.66	1.70	
2^{-24}	1.47488E-02	5.82269E-03	2.08474E-03	7.05251E-04	2.30149E-04	7.28171E-05	2.24765E-05
	1.34	1.48	1.56	1.62	1.66	1.70	
2^{-26}	1.47488E-02	5.82269E-03	2.08474E-03	7.05251E-04	2.30149E-04	7.28171E-05	2.24765E-05
	1.34	1.48	1.56	1.62	1.66	1.70	
2^{-28}	1.47488E-02	5.82269E-03	2.08474E-03	7.05251E-04	2.30149E-04	7.28171E-05	2.24765E-05
	1.34	1.48	1.56	1.62	1.66	1.70	
2^{-30}	1.47488E-02	5.82269E-03	2.08474E-03	7.05251E-04	2.30149E-04	7.28171E-05	2.24765E-05
	1.34	1.48	1.56	1.62	1.66	1.70	
2^{-32}	1.47488E-02	5.82269E-03	2.08474E-03	7.05251E-04	2.30149E-04	7.28171E-05	2.24765E-05
	1.34	1.48	1.56	1.62	1.66	1.70	
2^{-34}	1.47488E-02	5.82269E-03	2.08474E-03	7.05251E-04	2.30149E-04	7.28171E-05	2.24765E-05
	1.34	1.48	1.56	1.62	1.66	1.70	
2^{-36}	1.47488E-02	5.82269E-03	2.08474E-03	7.05251E-04	2.30149E-04	7.28171E-05	2.24765E-05
	1.34	1.48	1.56	1.62	1.66	1.70	
2^{-38}	1.47488E-02	5.82269E-03	2.08474E-03	7.05251E-04	2.30149E-04	7.28171E-05	2.24765E-05
	1.34	1.48	1.56	1.62	1.66	1.70	
2^{-40}	1.47488E-02	5.82269E-03	2.08474E-03	7.05251E-04	2.30149E-04	7.28171E-05	2.24765E-05
	1.34	1.48	1.56	1.62	1.66	1.70	

Table 3.2: Maximum-point wise errors and rates of convergence by Amiraliyev and Erdogan [2] on Ω_1 for Example 3.3.1.

ε	Number of mesh points						
	16	32	64	128	256	512	1024
2^{-4}	5.59992E-02	3.66214E-02	2.26994E-02	1.35156E-02	7.82744E-03	4.43734E-03	2.47624E-03
	0.61	0.69	0.75	0.79	0.82	0.84	
2^{-8}	5.59992E-02	3.66214E-02	2.26994E-02	1.35156E-02	7.82744E-03	4.43734E-03	2.47624E-03
	0.61	0.69	0.75	0.79	0.82	0.84	
2^{-10}	5.59992E-02	3.66214E-02	2.26994E-02	1.35156E-02	7.82744E-03	4.43734E-03	2.47624E-03
	0.61	0.69	0.75	0.79	0.82	0.84	
2^{-12}	5.59992E-02	3.66214E-02	2.26994E-02	1.35156E-02	7.82744E-03	4.43734E-03	2.47624E-03
	0.61	0.69	0.75	0.79	0.82	0.84	
2^{-14}	5.59992E-02	3.66214E-02	2.26994E-02	1.35156E-02	7.82744E-03	4.43734E-03	2.47624E-03
	0.61	0.69	0.75	0.79	0.82	0.84	
2^{-16}	5.59992E-02	3.66214E-02	2.26994E-02	1.35156E-02	7.82744E-03	4.43734E-03	2.47624E-03
	0.61	0.69	0.75	0.79	0.82	0.84	
2^{-18}	5.59992E-02	3.66214E-02	2.26994E-02	1.35156E-02	7.82744E-03	4.43734E-03	2.47624E-03
	0.61	0.69	0.75	0.79	0.82	0.84	
2^{-20}	5.59992E-02	3.66214E-02	2.26994E-02	1.35156E-02	7.82744E-03	4.43734E-03	2.47624E-03
	0.61	0.69	0.75	0.79	0.82	0.84	
2^{-22}	5.59992E-02	3.66214E-02	2.26994E-02	1.35156E-02	7.82744E-03	4.43734E-03	2.47624E-03
	0.61	0.69	0.75	0.79	0.82	0.84	
2^{-24}	5.59992E-02	3.66214E-02	2.26994E-02	1.35156E-02	7.82744E-03	4.43734E-03	2.47624E-03
	0.61	0.69	0.75	0.79	0.82	0.84	
2^{-26}	5.59992E-02	3.66214E-02	2.26994E-02	1.35156E-02	7.82744E-03	4.43734E-03	2.47624E-03
	0.61	0.69	0.75	0.79	0.82	0.84	
2^{-28}	5.59992E-02	3.66214E-02	2.26994E-02	1.35156E-02	7.82744E-03	4.43734E-03	2.47624E-03
	0.61	0.69	0.75	0.79	0.82	0.84	
2^{-30}	5.59992E-02	3.66214E-02	2.26994E-02	1.35156E-02	7.82744E-03	4.43734E-03	2.47624E-03
	0.61	0.69	0.75	0.79	0.82	0.84	
2^{-32}	5.59992E-02	3.66214E-02	2.26994E-02	1.35156E-02	7.82744E-03	4.43734E-03	2.47624E-03
	0.61	0.69	0.75	0.79	0.82	0.84	
2^{-34}	5.59992E-02	3.66214E-02	2.26994E-02	1.35156E-02	7.82744E-03	4.43734E-03	2.47624E-03
	0.61	0.69	0.75	0.79	0.82	0.84	
2^{-36}	5.59992E-02	3.66214E-02	2.26994E-02	1.35156E-02	7.82744E-03	4.43734E-03	2.47624E-03
	0.61	0.69	0.75	0.79	0.82	0.84	
2^{-38}	5.59992E-02	3.66214E-02	2.26994E-02	1.35156E-02	7.82744E-03	4.43734E-03	2.47624E-03
	0.61	0.69	0.75	0.79	0.82	0.84	
2^{-40}	5.59992E-02	3.66214E-02	2.26994E-02	1.35156E-02	7.82744E-03	4.43734E-03	2.47624E-03
	0.61	0.69	0.75	0.79	0.82	0.84	

Table 3.3: Maximum point-wise errors and rates of convergence by proposed method on Ω_2 for Example 3.3.1.

ε	Number of mesh points						
	16	32	64	128	256	512	1024
2^{-4}	1.04734E-02 1.53	3.63796E-03 1.59	1.20634E-03 2.00	3.00847E-04 2.00	7.50886E-05 2.00	1.87759E-05 2.00	4.69502E-06
2^{-8}	1.04716E-02 1.53	3.63761E-03 1.47	1.31343E-03 1.56	4.44358E-04 1.62	1.44454E-04 1.66	4.56719E-05 1.70	1.40874E-05
2^{-10}	1.04716E-02 1.53	3.63761E-03 1.47	1.31343E-03 1.56	4.44358E-04 1.62	1.44454E-04 1.66	4.56719E-05 1.70	1.40874E-05
2^{-12}	1.04716E-02 1.53	3.63761E-03 1.47	1.31343E-03 1.56	4.44358E-04 1.62	1.44454E-04 1.66	4.56719E-05 1.70	1.40874E-05
2^{-14}	1.04716E-02 1.53	3.63761E-03 1.47	1.31343E-03 1.56	4.44358E-04 1.62	1.44454E-04 1.66	4.56719E-05 1.70	1.40874E-05
2^{-16}	1.04716E-02 1.53	3.63761E-03 1.47	1.31343E-03 1.56	4.44358E-04 1.62	1.44454E-04 1.66	4.56719E-05 1.70	1.40874E-05
2^{-18}	1.04716E-02 1.53	3.63761E-03 1.47	1.31343E-03 1.56	4.44358E-04 1.62	1.44454E-04 1.66	4.56719E-05 1.70	1.40874E-05
2^{-20}	1.04716E-02 1.53	3.63761E-03 1.47	1.31343E-03 1.56	4.44358E-04 1.62	1.44454E-04 1.66	4.56719E-05 1.70	1.40874E-05
2^{-22}	1.04716E-02 1.53	3.63761E-03 1.47	1.31343E-03 1.56	4.44358E-04 1.62	1.44454E-04 1.66	4.56719E-05 1.70	1.40874E-05
2^{-24}	1.04716E-02 1.53	3.63761E-03 1.47	1.31343E-03 1.56	4.44358E-04 1.62	1.44454E-04 1.66	4.56720E-05 1.70	1.40875E-05
2^{-26}	1.04716E-02 1.53	3.63761E-03 1.47	1.31343E-03 1.56	4.44358E-04 1.62	1.44454E-04 1.66	4.56719E-05 1.70	1.40874E-05
2^{-28}	1.04716E-02 1.53	3.63761E-03 1.47	1.31343E-03 1.56	4.44358E-04 1.62	1.44454E-04 1.66	4.56719E-05 1.70	1.40874E-05
2^{-30}	1.04716E-02 1.53	3.63761E-03 1.47	1.31343E-03 1.56	4.44358E-04 1.62	1.44454E-04 1.66	4.56719E-05 1.70	1.40874E-05
2^{-32}	1.04717E-02 1.53	3.63761E-03 1.47	1.31346E-03 1.56	4.44353E-04 1.62	1.44452E-04 1.66	4.56644E-05 1.70	1.40862E-05
2^{-34}	1.04716E-02 1.53	3.63761E-03 1.47	1.31343E-03 1.56	4.44358E-04 1.62	1.44454E-04 1.66	4.56719E-05 1.70	1.40874E-05
2^{-36}	1.04716E-02 1.53	3.63761E-03 1.47	1.31343E-03 1.56	4.44358E-04 1.62	1.44454E-04 1.66	4.56719E-05 1.70	1.40874E-05
2^{-38}	1.04716E-02 1.53	3.63761E-03 1.47	1.31343E-03 1.56	4.44358E-04 1.62	1.44454E-04 1.66	4.56719E-05 1.70	1.40874E-05
2^{-40}	1.04716E-02 1.53	3.63761E-03 1.47	1.31343E-03 1.56	4.44358E-04 1.62	1.44454E-04 1.66	4.56719E-05 1.70	1.40874E-05

Table 3.4: Maximum point-wise errors and rates of convergence by Amiraliyev and Erdogan [2] on Ω_2 for Example 3.3.1.

ε	Number of mesh points						
	16	32	64	128	256	512	1024
2^{-4}	3.17312E-02	1.53850E-02	9.53417E-03	5.68254E-03	3.28873E-03	1.86398E-03	1.04008E-03
	1.04	0.69	0.75	0.79	0.82	0.84	
2^{-8}	2.35909E-02	1.53846E-02	9.53410E-03	5.68252E-03	3.28872E-03	1.86397E-03	1.04009E-03
	0.62	0.69	0.75	0.79	0.82	0.84	
2^{-10}	2.35909E-02	1.53846E-02	9.53410E-03	5.68252E-03	3.28872E-03	1.86397E-03	1.04009E-03
	0.62	0.69	0.75	0.79	0.82	0.84	
2^{-12}	2.35909E-02	1.53846E-02	9.53410E-03	5.68252E-03	3.28872E-03	1.86397E-03	1.04009E-03
	0.62	0.69	0.75	0.79	0.82	0.84	
2^{-14}	2.35909E-02	1.53846E-02	9.53410E-03	5.68252E-03	3.28872E-03	1.86397E-03	1.04009E-03
	0.62	0.69	0.75	0.79	0.82	0.84	
2^{-16}	2.35909E-02	1.53846E-02	9.53410E-03	5.68252E-03	3.28872E-03	1.86397E-03	1.04009E-03
	0.62	0.69	0.75	0.79	0.82	0.84	
2^{-18}	2.35909E-02	1.53846E-02	9.53410E-03	5.68252E-03	3.28872E-03	1.86397E-03	1.04009E-03
	0.62	0.69	0.75	0.79	0.82	0.84	
2^{-20}	2.35909E-02	1.53846E-02	9.53410E-03	5.68252E-03	3.28872E-03	1.86397E-03	1.04009E-03
	0.62	0.69	0.75	0.79	0.82	0.84	
2^{-22}	2.35909E-02	1.53846E-02	9.53410E-03	5.68252E-03	3.28872E-03	1.86397E-03	1.04009E-03
	0.62	0.69	0.75	0.79	0.82	0.84	
2^{-24}	2.35909E-02	1.53846E-02	9.53410E-03	5.68252E-03	3.28872E-03	1.86397E-03	1.04008E-03
	0.62	0.69	0.75	0.79	0.82	0.84	
2^{-26}	2.35909E-02	1.53846E-02	9.53410E-03	5.68252E-03	3.28872E-03	1.86397E-03	1.04009E-03
	0.62	0.69	0.75	0.79	0.82	0.84	
2^{-28}	2.35909E-02	1.53846E-02	9.53410E-03	5.68252E-03	3.28872E-03	1.86397E-03	1.04009E-03
	0.62	0.69	0.75	0.79	0.82	0.84	
2^{-30}	2.35909E-02	1.53846E-02	9.53410E-03	5.68252E-03	3.28872E-03	1.86397E-03	1.04009E-03
	0.62	0.69	0.75	0.79	0.82	0.84	
2^{-32}	2.35909E-02	1.53846E-02	9.53410E-03	5.68252E-03	3.28872E-03	1.86397E-03	1.04009E-03
	0.62	0.69	0.75	0.79	0.82	0.84	
2^{-34}	2.35909E-02	1.53846E-02	9.53410E-03	5.68251E-03	3.28872E-03	1.86397E-03	1.04008E-03
	0.62	0.69	0.75	0.79	0.82	0.84	
2^{-36}	2.35909E-02	1.53846E-02	9.53410E-03	5.68252E-03	3.28872E-03	1.86397E-03	1.04009E-03
	0.62	0.69	0.75	0.79	0.82	0.84	
2^{-38}	2.35909E-02	1.53846E-02	9.53410E-03	5.68252E-03	3.28872E-03	1.86397E-03	1.04009E-03
	0.62	0.69	0.75	0.79	0.82	0.84	
2^{-40}	2.35909E-02	1.53846E-02	9.53410E-03	5.68252E-03	3.28872E-03	1.86397E-03	1.04009E-03
	0.62	0.69	0.75	0.79	0.82	0.84	

Table 3.5: Maximum point-wise errors and rates of convergence for Example 3.3.2 by the proposed scheme, $\delta = 0.5\epsilon$.

ε	Number of mesh points						
	16	32	64	128	256	512	1024
2^{-4}	4.41692E-02	1.61911E-02	5.07722E-03	1.24860E-03	3.13676E-04	7.98974E-05	2.07595E-05
	1.448	1.673	2.024	1.993	1.973	1.944	
2^{-6}	4.39646E-02	1.59659E-02	5.44108E-03	1.81221E-03	5.90556E-04	1.87243E-04	5.81083E-05
	1.461	1.553	1.586	1.618	1.657	1.688	
2^{-8}	4.38879E-02	1.58928E-02	5.41865E-03	1.80178E-03	5.86463E-04	1.85445E-04	5.73285E-05
	1.465	1.552	1.589	1.619	1.661	1.694	
2^{-10}	4.38666E-02	1.58730E-02	5.41196E-03	1.79856E-03	5.85052E-04	1.84784E-04	5.70149E-05
	1.467	1.552	1.589	1.620	1.663	1.696	
2^{-12}	4.38611E-02	1.58680E-02	5.41022E-03	1.79771E-03	5.84673E-04	1.84604E-04	5.69291E-05
	1.467	1.552	1.590	1.620	1.663	1.697	
2^{-14}	4.38598E-02	1.58667E-02	5.40978E-03	1.79749E-03	5.84576E-04	1.84558E-04	5.69071E-05
	1.467	1.552	1.590	1.621	1.663	1.697	
2^{-16}	4.38594E-02	1.58664E-02	5.40967E-03	1.79744E-03	5.84552E-04	1.84547E-04	5.69015E-05
	1.467	1.552	1.590	1.621	1.663	1.697	
2^{-18}	4.38593E-02	1.58664E-02	5.40964E-03	1.79742E-03	5.84546E-04	1.84544E-04	5.69001E-05
	1.467	1.552	1.590	1.621	1.663	1.697	
2^{-20}	4.38593E-02	1.58663E-02	5.40963E-03	1.79742E-03	5.84544E-04	1.84543E-04	5.68998E-05
	1.467	1.552	1.590	1.621	1.663	1.697	
2^{-22}	4.38593E-02	1.58663E-02	5.40963E-03	1.79742E-03	5.84544E-04	1.84543E-04	5.68997E-05
	1.467	1.552	1.590	1.621	1.663	1.697	
2^{-24}	4.38593E-02	1.58663E-02	5.40963E-03	1.79742E-03	5.84544E-04	1.84543E-04	5.68997E-05
	1.467	1.552	1.590	1.621	1.663	1.697	
2^{-26}	4.38593E-02	1.58663E-02	5.40963E-03	1.79742E-03	5.84542E-04	1.84541E-04	5.68980E-05
	1.467	1.552	1.590	1.621	1.663	1.697	
2^{-28}	4.38593E-02	1.58663E-02	5.40963E-03	1.79742E-03	5.84543E-04	1.84543E-04	5.68993E-05
	1.467	1.552	1.590	1.621	1.663	1.697	
2^{-30}	4.38593E-02	1.58663E-02	5.40963E-03	1.79742E-03	5.84544E-04	1.84543E-04	5.68996E-05
	1.467	1.552	1.590	1.621	1.663	1.697	
2^{-32}	4.38593E-02	1.58663E-02	5.40963E-03	1.79742E-03	5.84544E-04	1.84543E-04	5.68997E-05
	1.467	1.552	1.590	1.621	1.663	1.697	
2^{-34}	4.38593E-02	1.58663E-02	5.40963E-03	1.79742E-03	5.84544E-04	1.84543E-04	5.68997E-05
	1.467	1.552	1.590	1.621	1.663	1.697	
2^{-36}	4.38593E-02	1.58663E-02	5.40963E-03	1.79742E-03	5.84544E-04	1.84543E-04	5.68997E-05
	1.467	1.552	1.590	1.621	1.663	1.697	
2^{-38}	4.38593E-02	1.58663E-02	5.40963E-03	1.79742E-03	5.84544E-04	1.84543E-04	5.68997E-05
	1.467	1.552	1.590	1.621	1.663	1.697	
2^{-40}	4.38593E-02	1.58663E-02	5.40963E-03	1.79742E-03	5.84544E-04	1.84543E-04	5.68997E-05
	1.467	1.552	1.590	1.621	1.663	1.697	

Table 3.6: Maximum point-wise errors and rates of convergence for Example 3.3.3 by the proposed scheme, $\delta = 0.5\epsilon$.

ε	Number of mesh points						
	16	32	64	128	256	512	1024
2^{-4}	3.36532E-02 1.535	1.16139E-02 1.628	3.75797E-03 1.846	1.04546E-03 1.683	3.25532E-04 1.472	1.17336E-04 1.088	5.52111E-05
2^{-6}	3.51561E-02 1.492	1.24958E-02 1.507	4.39748E-03 1.516	1.53726E-03 1.504	5.42047E-04 1.486	1.93501E-04 1.437	7.14649E-05
2^{-8}	3.44919E-02 1.492	1.22591E-02 1.546	4.19948E-03 1.568	1.41645E-03 1.564	4.79140E-04 1.582	1.60068E-04 1.566	5.40473E-05
2^{-10}	3.41868E-02 1.495	1.21286E-02 1.564	4.10139E-03 1.586	1.36611E-03 1.605	4.49236E-04 1.640	1.44165E-04 1.655	4.57889E-05
2^{-12}	3.40992E-02 1.496	1.20896E-02 1.570	4.07241E-03 1.592	1.35115E-03 1.617	4.40348E-04 1.657	1.39641E-04 1.686	4.34135E-05
2^{-14}	3.40765E-02 1.496	1.20795E-02 1.571	4.06486E-03 1.593	1.34724E-03 1.621	4.38026E-04 1.661	1.38473E-04 1.694	4.27917E-05
2^{-16}	3.40708E-02 1.496	1.20769E-02 1.571	4.06295E-03 1.593	1.34626E-03 1.621	4.37439E-04 1.661	1.38177E-04 1.696	4.26379E-05
2^{-18}	3.40694E-02 1.496	1.20763E-02 1.571	4.06247E-03 1.593	1.34601E-03 1.621	4.37292E-04 1.661	1.38103E-04 1.697	4.25998E-05
2^{-20}	3.40690E-02 1.496	1.20761E-02 1.571	4.06235E-03 1.593	1.34595E-03 1.621	4.37255E-04 1.661	1.38085E-04 1.697	4.25902E-05
2^{-22}	3.40689E-02 1.496	1.20761E-02 1.571	4.06232E-03 1.593	1.34593E-03 1.621	4.37246E-04 1.661	1.38080E-04 1.697	4.25878E-05
2^{-24}	3.40689E-02 1.496	1.20761E-02 1.571	4.06231E-03 1.593	1.34593E-03 1.621	4.37243E-04 1.661	1.38079E-04 1.697	4.25872E-05
2^{-26}	3.40689E-02 1.496	1.20761E-02 1.571	4.06231E-03 1.593	1.34593E-03 1.621	4.37243E-04 1.661	1.38079E-04 1.697	4.25871E-05
2^{-30}	3.40689E-02 1.496	1.20761E-02 1.571	4.06231E-03 1.593	1.34593E-03 1.621	4.37243E-04 1.661	1.38078E-04 1.697	4.25870E-05
2^{-32}	3.40689E-02 1.496	1.20761E-02 1.571	4.06231E-03 1.593	1.34593E-03 1.621	4.37243E-04 1.661	1.38078E-04 1.697	4.25870E-05
2^{-34}	3.40689E-02 1.496	1.20761E-02 1.571	4.06231E-03 1.593	1.34593E-03 1.621	4.37243E-04 1.661	1.38078E-04 1.697	4.25870E-05
2^{-36}	3.40689E-02 1.496	1.20761E-02 1.571	4.06231E-03 1.593	1.34593E-03 1.621	4.37243E-04 1.661	1.38078E-04 1.697	4.25870E-05
2^{-38}	3.40689E-02 1.496	1.20761E-02 1.571	4.06231E-03 1.593	1.34593E-03 1.621	4.37243E-04 1.661	1.38078E-04 1.697	4.25870E-05
2^{-40}	3.40689E-02 1.496	1.20761E-02 1.571	4.06231E-03 1.593	1.34593E-03 1.621	4.37243E-04 1.661	1.38078E-04 1.697	4.25870E-05

Chapter 4

Singularly Perturbed Boundary Value Problems with Smooth Data

Singularly perturbed boundary value problems (SPBVPs) occur in many areas of engineering and applied mathematics. In many practical problems the coefficient of the second derivative is small as compared to the coefficient of the first derivative. Examples of these are heat transport problems with large peclet numbers, Navier-Stokes flows with large Reynold numbers etc. Because of the presence of 'Boundary layer', difficulties are experienced in solving problems of above type using numerical methods with uniform mesh. In order to get a good approximation, a fine mesh is required in the boundary layer region. In general, finding numerical solution of a boundary value problem is more difficult than finding numerical solution of corresponding initial value problem. Therefore it is better to convert the second order problem into an asymptotic equivalent first order problem, wherever possible.

The outline of this chapter is as follows: Section 4.1 deals with convection-diffusion equations. Reaction-diffusion equations are taken in Section 4.2. Some numerical results computed from the proposed schemes are presented in Section 4.3. Certain conclusions based on the analysis are finally presented in Section 4.4.

4.1 Convection-diffusion problems

In this section, we consider the following class of SPBVPs:

$$L_\epsilon u(x) \equiv \epsilon u''(x) + a(x)u'(x) + b(x)u(x) = f(x), \quad x \in \Omega = (0,1), \tag{4.1.1}$$

$$u(0) = p, \quad u(1) = q, \tag{4.1.2}$$

where $0 < \epsilon \ll 1$ is a small positive parameter, $a(x)$, $b(x)$ and $f(x)$ are sufficiently smooth functions, such that $a(x) \geq \beta > 0$ and $b(x) \geq 0$ on $\overline{\Omega} = [0,1]$. Under these assumptions, (4.1.1-4.1.2) possesses a unique solution $u(x) \in C^2(\overline{\Omega})$ with a boundary layer of width $O(\epsilon)$ at $x = 0$.

Rencently, some researchers solved (4.1.1-4.1.2) by applying the techniques, which are based on the idea of replacing a two point boundary value problem by two suitable initial-value problems. For example, Gasparo and Macconi [36] considered a semi-linear ordinary differential equation which was integrated to obtain a first-order ordinary differential equation and considered both the inner and outer solutions. A similar matching idea of combining the reduced problem and a WKB approximation has also been employed by Gasparo and Macconi [37] for linear and semi-linear SPBVPs. These matching ideas are based on the work of Robert [102]. Robert's idea has been extended by Valanarasu and Ramanujam [123]. These authors used the combination of Euler method and exponentially fitted method. The main disadvantage of this method is that, it works only for those values of N (number of mesh points) which are of same order as of ϵ (perturbation parameter). Also, its theoretical order of convergence is only one.

We suggest a modified initial-value technique (MIVT), in line of [37] for (4.1.1-4.1.2). The BVP (4.1.1-4.1.2) is replaced with a suitable initial value problem (IVP) and a terminal value problem (TVP). The integration of these problems goes in the opposite directions, but each problem can be solved independently of the other. The IVP is of singularly perturbed type, whereas the TVP does not contain any small parameter. We solve the IVP by a hybrid scheme.

Assumption: We shall assume that $\epsilon \leq CN^{-1}$ as is generally the case for discretization of convection-dominated problem [115].

4.1.1 Asymptotic expansion approximation

It is well known that, by using the fundamental idea of WKB [87], an asymptotic expansion approximation for the solution of SPBVP (4.1.1-4.1.2) is given by:

$$u_{as}(x) = u_R(x) + (p - u_R(0))v(x) + O(\epsilon), \tag{4.1.3}$$

where $u_R(x)$ is the solution of the reduced problem

$$a(x)u'_R(x) + b(x)u_R(x) = f(x), \quad x \in \overline{\Omega}, \tag{4.1.4}$$

$$u_R(1) = q, \tag{4.1.5}$$

and $v(x)$ is defined on $\overline{\Omega}$ by

$$v(x) = \exp\left\{-\int_0^x [\frac{a(s)}{\epsilon} - \frac{b(s)}{a(s)}]ds\right\}. \tag{4.1.6}$$

Also, $v(x)$ satisfies the following initial value problem

$$\epsilon a(x)v'(x) + (a^2(x) - \epsilon b(x))v(x) = 0, \quad x \in \overline{\Omega}, \tag{4.1.7}$$

$$v(0) = 1. \tag{4.1.8}$$

Theorem 4.1.1 *[25] The zeroth order asymptotic expansion approximation u_{as} satisfies the inequality*

$$|(u - u_{as})(x)| \leq C\epsilon, \quad x \in \overline{\Omega},$$

where u(x) is the solution of BVP (4.1.1-4.1.2).

Theorem 4.1.2 *[79] Let u(x) be the solution of BVP (4.1.1-4.1.2). Then:*

$$|u^{(k)}(x)| \leq C[1 + \epsilon^{-k}e^{\frac{-\beta x}{\epsilon}}], \quad x \in \overline{\Omega},\ k = 1, 2....$$

4.1.2 Description of method

In this sub-section, we describe the MIVT to solve (4.1.1-4.1.2)

1. Solve the TVP (4.1.4-4.1.5) by using the Trapezoidal Method. Let $U_0(x_i)$ be its solution.

2. Solve the IVP (4.1.7-4.1.8) by using the hybrid scheme (2.1.4-2.1.5) described on Shishkin mesh in sub-section 2.1.2. Let $V(x_i)$ be its solution.

3. Compute solution of (4.1.1-4.1.2) as

$$U_i = U_0(x_i) + (p - u_R(0))V(x_i), \quad x \in \overline{\Omega}. \tag{4.1.9}$$

Now, an error estimate is derived for the numerical solution obtained by the MIVT.

Theorem 4.1.3 *Let $u(x)$ be the solution of the BVP (4.1.1-4.1.2) and $U(x_i)$ be the numerical solution obtained from MIVT, then we have*

$$|(U - u)(x_i)|_{\overline{\Omega}_\epsilon^N} \leq C\left[N^{-2}\ln^2 N + N^{-1}\epsilon + N^{-\beta\sigma_0} + \epsilon\right].$$

Proof. Theorem 2.1.2, when applied to the IVP (4.1.7-4.1.8) yields

$$|(V - v)(x_i)| \leq C\left[N^{-2}\ln^2 N + N^{-1}\epsilon + N^{-\beta\sigma_0}\right], \quad x \in \overline{\Omega}. \tag{4.1.10}$$

From the definitions of $u_{as}(x)$, $U(x_i)$ and above inequality, we have

$$|(u_{as} - U)(x_i)| \leq C\left[N^{-2}\ln^2 N + N^{-1}\epsilon + N^{-\beta\sigma_0}\right], \quad \text{for } x_i \in \Omega_\epsilon^N. \tag{4.1.11}$$

From Theorem 4.1.1, we have

$$|(u - u_{as})(x)| \leq C\epsilon, \quad x \in \overline{\Omega}. \tag{4.1.12}$$

The desired estimate follows from the inequalities (4.1.11) and (4.1.12).

4.2 Reaction-diffusion problems

In this section, we consider the following class of singularly perturbed reaction-diffusion problems:

$$\begin{aligned} L_\epsilon u(x) &\equiv -\epsilon u''(x) + a(x)u(x) = f(x), \quad x \in \Omega = (0,1), && (4.2.1) \\ u(0) &= p, \quad u(1) = q, && (4.2.2) \end{aligned}$$

where $0 < \epsilon \ll 1$ is a small positive parameter, $a(x)$ and $f(x)$ are sufficiently smooth functions, such that $a(x) \geq \beta > 0$ on $\overline{\Omega} = [0,1]$. Under these assumptions (4.2.1-4.2.2) possesses a unique solution $u(x) \in C^2(\Omega) \bigcap C(\overline{\Omega})$.

We suggest a modified initial-value technique (MIVT), in line of [37] for (4.2.1-4.2.2). The BVP (4.2.1-4.2.2) is replaced with a suitable initial value problem (IVP) and a terminal value problem (TVP). The integration of these problems goes in the opposite directions, but each problem can be solved independently of the other. The IVP and the TVP are of singularly perturbed type. We solve these IVP and TVP by a hybrid scheme.

4.2.1 Asymptotic expansion approximation

An asymptotic expansion approximation for the solution of BVP (4.2.1-4.2.2) can be constructed as

$$u_{as}(x) = u_R(x) + v(x) + O(\sqrt{\epsilon}), \tag{4.2.3}$$

where $u_R(x)$ is the solution of the reduced problem of (4.2.1-4.2.2) and is given by

$$\begin{aligned} a(x)u_R(x) &= f(x), \quad x \in [0,1), && (4.2.4) \\ u_R(1) &= q, && (4.2.5) \end{aligned}$$

and $v(x)$ is given by

$$v(x) = [p - u_R(0)] \left[\frac{a(0)}{a(x)}\right]^{1/4} v_L(x) + [q - u_R(1)] \left[\frac{a(1)}{a(x)}\right]^{1/4} v_R(x). \tag{4.2.6}$$

$v_L(x)$ is a "left boundary layer correction" and $v_R(x)$ is a "right boundary layer correction" are defined as

$$v_L(x) = \exp\left\{-\int_0^x \sqrt{\frac{a(s)}{\epsilon}}ds\right\}, \tag{4.2.7}$$

$$v_R(x) = \exp\left\{-\int_x^1 \sqrt{\frac{a(s)}{\epsilon}}ds\right\}. \tag{4.2.8}$$

It is easy to verify that $v_L(x)$ and $v_R(x)$ satisfy the following IVP and TVP respectively:

$$\sqrt{\epsilon}v_L'(x) + \sqrt{a(x)}v_L(x) = 0, \tag{4.2.9}$$

$$v_L(0) = 1, \tag{4.2.10}$$

$$\sqrt{\epsilon}v_R'(x) - \sqrt{a(x)}v_R(x) = 0, \tag{4.2.11}$$

$$v_R(1) = 1. \tag{4.2.12}$$

Theorem 4.2.1 *[123] The zeroth order asymptotic expansion approximation u_{as} satisfies the inequality*

$$|(u - u_{as})(x)| \leq C\sqrt{\epsilon},$$

where $u(x)$ is the solution of BVP (4.2.1-4.2.2).

4.2.2 Proposed scheme

A fitted mesh method for the problem (4.2.1-4.2.2) is now introduced. On $\overline{\Omega}$ a piece-wise uniform mesh of N mesh intervals is constructed as follows. The domain $\overline{\Omega}$ is sub-divided into three sub-intervals as

$$\overline{\Omega} = [0,\sigma]\bigcup(\sigma, 1-\sigma]\bigcup(1-\sigma, 1], \tag{4.2.13}$$

for some σ that satisfies $0 < \sigma \leq 1/4$. On $[0,\sigma]$ and $[1-\sigma, 1]$ a uniform mesh with $N/4$ mesh-intervals is placed, while $[\sigma, 1-\sigma]$ has a uniform mesh with $N/2$ mesh intervals. It is obvious that mesh is uniform when $\sigma = 1/4$. It is fitted to the problem by choosing σ to be the function of N and ϵ and

$$\sigma = \min\left\{1/4, \sigma_0\sqrt{\epsilon}\ln N\right\},$$

where $\sigma_0 \geq 2/\sqrt{\beta}$. Applying hybrid scheme (2.1.4-2.1.5) for (4.2.9-4.2.10), we get

$$L_{\epsilon}^{N}V_{L,i} \equiv \begin{cases} \varepsilon D^{-}V_{L,i} + \dfrac{\sqrt{a_{i-1}}V_{L,i-1} + \sqrt{a_i}V_{L,i}}{2} = 0, \quad 0 < i \leq N/4,\ 3N/4 < i \leq N, \\ \\ \varepsilon D^{-}V_{L,i} + \sqrt{a_i}V_{L,i} = 0, \quad N/4 < i \leq 3N/4, \end{cases} \tag{4.2.14}$$

$$V_{L,0} = 1. \tag{4.2.15}$$

Similarly, we can formulate the hybrid scheme for (4.2.11-4.2.12).

4.2.3 Description of method

In this sub-section, we describe the MIVT to solve the (SPBVP) (4.2.1-4.2.2)

Step 1. Solve the IVP (4.2.9-4.2.10) by using the hybrid scheme (4.2.14-4.2.15) described on Shishkin mesh in Section 4.2.2. Let $V_{L,i}$ be its solution.

Step 2. Solve the TVP (4.2.11-4.2.12) by using the hybrid scheme. Let $V_{R,i}$ be its solution.

Step 3. Define the mesh function U_i as

$$U_i = u_R(x_i) + [p - u_R(0)] \left[\frac{a(0)}{a(x)}\right]^{1/4} V_{L,i} + [q - u_R(1)] \left[\frac{a(1)}{a(x)}\right]^{1/4} V_{R,i}. \tag{4.2.16}$$

Now, an error estimate is derived for the numerical solution obtained by the MIVT.

Theorem 4.2.2 *Let $u(x)$ be the solution of the BVP (4.2.1-4.2.2) and U_i be the numerical solution obtained by MIVT, then we have*

$$|u(x_i) - U_i| \, \leq \, C \left[N^{-2} \ln^2 N + N^{-1}\varepsilon + N^{-\sqrt{\beta}\sigma_0} + \sqrt{\epsilon}\right].$$

Proof. Theorem 2.1.1, when applied to the IVPs (4.2.9-4.2.10), (4.2.11-4.2.12) yields

$$|v_L(x_i) - V_{L,i}| \, \leq \, C \left[N^{-2} \ln^2 N + N^{-1}\varepsilon + N^{-\sqrt{\beta}\sigma_0}\right], \quad 0 \leq x_i \leq 1, \tag{4.2.17}$$

$$|v_R(x_i) - V_{R,i}| \, \leq \, C \left[N^{-2} \ln^2 N + N^{-1}\varepsilon + N^{-\sqrt{\beta}\sigma_0}\right], \quad 0 \leq x_i \leq 1. \tag{4.2.18}$$

From the definitions of $u_{as}(x)$, U_i and above inequalities, we have

$$|u_{as}(x_i) - U_i| \, \leq \, C \left[N^{-2} \ln^2 N + N^{-1}\varepsilon + N^{\sqrt{\beta}\sigma_0}\right], \quad \text{for}\, x_i \in \Omega_\epsilon^N. \tag{4.2.19}$$

From theorem 4.2.1, we have

$$|u(x_i) - u_{as}(x_i)| \leq C\sqrt{\epsilon}, \quad x \, \in \overline{\Omega}. \tag{4.2.20}$$

The desired estimate follows from the inequalities (4.2.19) and (4.2.20).

4.3 Numerical experiments and discussions

For convection-diffusion problems, maximum point-wise errors and rates of convergence are calculated as described in Section 2.3. For reaction-diffusion problems, maximum point-wise errors and rates of convergence are calculated as follows:

We calculate the numerical solutions U^N on $\overline{\Omega}_\epsilon^N$ and $\widetilde{U}^N$on the mesh $\widetilde{\Omega}_\epsilon^N$, where the transition parameter σ is altered slightly to $\widetilde{\sigma} = \min\{1/4, \sigma_0\sqrt{\epsilon}\ln\frac{N}{2}\}$. Then, for $i = 0, 1...N$, the ith point of the mesh $\overline{\Omega}^N$ coincides with the $(2i)$th point of the mesh $\overline{\Omega}^{2N}$. The double mesh difference is defined as

$$E_\epsilon^N = \max_{x_i \epsilon \overline{\Omega}^N}\{|U_i^N - \widetilde{U}_i^{2N}|\},$$

where U_i^N and $\widetilde{U}_i^{2N}$, respectively denote the numerical solutions obtained by using N and $2N$ mesh intervals. The rates of convergence are calculated as:

$$p_\epsilon^N = \frac{lnE_\epsilon^N - lnE_\epsilon^{2N}}{ln2}.$$

Example 4.3.1 *Consider the BVP:*

$$\epsilon u''(x) + u'(x) = 0, \quad x \in (0,1),$$
$$u(0) = 0, \quad u(1) = 1.$$

The exact solution of this problem is

$$u(x) = \frac{1 - e^{-x/\epsilon}}{1 - e^{-1/\epsilon}}.$$

Results are given in Table 4.1, for various values of N and ϵ.

Example 4.3.2 *Consider the non-homogeneous BVP:*

$$\epsilon u''(x) + u'(x) = 1 + 2x, \quad x \in (0,1),$$
$$u(0) = 0, \quad u(1) = 1.$$

The exact solution of this problem is

$$u(x) = x(x+1-2\epsilon) + (2\epsilon - 1)\frac{1 - e^{-x/\epsilon}}{1 - e^{-1/\epsilon}}.$$

The numerical results for the present example are shown in Table 4.2.

Example 4.3.3 *Consider the problem:*

$$\epsilon u''(x) + (1+x)^2 u'(x) + 2(1+x)u = \tfrac{1}{2}e^{-x/2}[(1+x)(3-x) + \tfrac{\epsilon}{2}], \quad x \in (0,1),$$
$$u(0) = 0, \quad u(1) = e^{-1/2} - e^{-7/3\epsilon}.$$

The exact solution of this problem is

$$u(x) = e^{-x/2} - e^{-x(x^2+3x+3)/3\epsilon}.$$

The numerical results are given in Table 4.3.

Example 4.3.4 *Consider the following Problem:*

$$-\epsilon u''(x) + (1 + x^2 + cos(x))u(x) = x^{4.5} + sin(x), x \in (0,1),$$
$$u(0) = 1, \quad u(1) = 1.$$

The numerical results by proposed scheme are in Table 4.4.

Example 4.3.5 *Consider the following Problem:*

$$-\epsilon u''(x) + (e^x + sin(x) - x - x^3)u(x) = cos(x) + x^2 - e^x + 1, x \in (0,1),$$

$$u(0) = 0, \quad u(1) = 2.$$

Table 4.5 displays the results for this example.

4.4 Conclusions

We have proposed a robust computational technique for solving the singularly perturbed boundary value problems. This chapter demonstrates, the effectiveness of the Shishkin mesh by modifying the initial value technique [37] in a very simple way so that higher order, almost second order of convergence can be achieved with no restrictions on values of h and ϵ. Numerical results were presented which are in agreement with the theoretical results. Our method is easier to apply and more effective in the sense of solution errors. From, the numerical results shown in tables, we conclude that the proposed MIVT works nicely independent of the mesh parameter h and the perturbation parameter ϵ. Also, the proposed scheme is of almost second order convergence upto a logarithmic factor. The graphs plotted in Figure 4.1 are convergent curves in the maximum norm at nodal points for the different values of ϵ for Examples (4.3.1-4.3.5).

Table 4.1: Maximum point-wise errors and rates of convergence by the proposed technique for Example 4.3.1.

ε	Number of mesh points						
	16	32	64	128	256	512	1024
10^{-6}	1.47488E-02 1.341	5.82269E-03 1.482	2.08475E-03 1.564	7.05251E-04 1.616	2.30149E-04 1.660	7.28171E-05 1.696	2.24765E-05
10^{-8}	1.47488E-02 1.341	5.82269E-03 1.482	2.08475E-03 1.564	7.05251E-04 1.616	2.30149E-04 1.660	7.28171E-05 1.696	2.24765E-05
10^{-10}	1.47488E-02 1.341	5.82269E-03 1.482	2.08475E-03 1.564	7.05251E-04 1.616	2.30149E-04 1.660	7.28171E-05 1.696	2.24765E-05
10^{-12}	1.47488E-02 1.341	5.82269E-03 1.482	2.08475E-03 1.564	7.05251E-04 1.616	2.30149E-04 1.660	7.28171E-05 1.696	2.24765E-05
10^{-14}	1.47488E-02 1.341	5.82269E-03 1.482	2.08475E-03 1.564	7.05251E-04 1.616	2.30149E-04 1.660	7.28171E-05 1.696	2.24765E-05
10^{-16}	1.47488E-02 1.341	5.82269E-03 1.482	2.08475E-03 1.564	7.05251E-04 1.616	2.30149E-04 1.660	7.28171E-05 1.696	2.24765E-05
10^{-18}	1.47488E-02 1.341	5.82269E-03 1.482	2.08475E-03 1.564	7.05251E-04 1.616	2.30149E-04 1.660	7.28171E-05 1.696	2.24765E-05
10^{-20}	1.47488E-02 1.341	5.82269E-03 1.482	2.08475E-03 1.564	7.05251E-04 1.616	2.30149E-04 1.660	7.28171E-05 1.696	2.24765E-05
10^{-22}	1.47488E-02 1.341	5.82269E-03 1.482	2.08475E-03 1.564	7.05251E-04 1.616	2.30149E-04 1.660	7.28171E-05 1.696	2.24765E-05
10^{-24}	1.47488E-02 1.341	5.82269E-03 1.482	2.08475E-03 1.564	7.05251E-04 1.616	2.30149E-04 1.660	7.28171E-05 1.696	2.24765E-05
10^{-26}	1.47488E-02 1.341	5.82269E-03 1.482	2.08475E-03 1.564	7.05251E-04 1.616	2.30149E-04 1.660	7.28171E-05 1.696	2.24765E-05
10^{-28}	1.47488E-02 1.341	5.82269E-03 1.482	2.08475E-03 1.564	7.05251E-04 1.616	2.30149E-04 1.660	7.28171E-05 1.696	2.24765E-05
10^{-30}	1.47488E-02 1.341	5.82269E-03 1.482	2.08475E-03 1.564	7.05251E-04 1.616	2.30149E-04 1.660	7.28171E-05 1.696	2.24765E-05
10^{-32}	1.47488E-02 1.341	5.82269E-03 1.482	2.08475E-03 1.564	7.05251E-04 1.616	2.30149E-04 1.660	7.28171E-05 1.696	2.24765E-05
10^{-34}	1.47488E-02 1.341	5.82269E-03 1.482	2.08475E-03 1.564	7.05251E-04 1.616	2.30149E-04 1.660	7.28171E-05 1.696	2.24765E-05
10^{-36}	1.47488E-02 1.341	5.82269E-03 1.482	2.08475E-03 1.564	7.05251E-04 1.616	2.30149E-04 1.660	7.28171E-05 1.696	2.24765E-05
10^{-38}	1.47488E-02 1.341	5.82269E-03 1.482	2.08475E-03 1.564	7.05251E-04 1.616	2.30149E-04 1.660	7.28171E-05 1.696	2.24765E-05
10^{-40}	1.47488E-02 1.341	5.82269E-03 1.482	2.08475E-03 1.564	7.05251E-04 1.616	2.30149E-04 1.660	7.28171E-05 1.696	2.24765E-05

Table 4.2: Maximum point-wise errors and rates of convergence by the proposed technique for Example 4.3.2.

ε	Number of mesh points						
	16	32	64	128	256	512	1024
10^{-6}	1.47498E-02 1.341	5.82385E-03 1.481	2.08604E-03 1.562	7.06559E-04 1.610	2.31442E-04 1.643	7.40984E-05 1.641	2.37525E-05
10^{-8}	1.47488E-02 1.341	5.82270E-03 1.482	2.08476E-03 1.564	7.05264E-04 1.616	2.30162E-04 1.660	7.28299E-05 1.695	2.24892E-05
10^{-10}	1.47488E-02 1.341	5.82269E-03 1.482	2.08475E-03 1.564	7.05251E-04 1.616	2.30149E-04 1.660	7.28172E-05 1.696	2.24766E-05
10^{-12}	1.47488E-02 1.341	5.82269E-03 1.482	2.08475E-03 1.564	7.05251E-04 1.616	2.30149E-04 1.660	7.28172E-05 1.696	2.24765E-05
10^{-14}	1.47488E-02 1.341	5.82269E-03 1.482	2.08475E-03 1.564	7.05251E-04 1.616	2.30149E-04 1.660	7.28172E-05 1.696	2.24765E-05
10^{-16}	1.47488E-02 1.341	5.82269E-03 1.482	2.08475E-03 1.564	7.05251E-04 1.616	2.30149E-04 1.660	7.28172E-05 1.696	2.24765E-05
10^{-18}	1.47488E-02 1.341	5.82269E-03 1.482	2.08475E-03 1.564	7.05251E-04 1.616	2.30149E-04 1.660	7.28172E-05 1.696	2.24765E-05
10^{-20}	1.47488E-02 1.341	5.82269E-03 1.482	2.08475E-03 1.564	7.05251E-04 1.616	2.30149E-04 1.660	7.28172E-05 1.696	2.24765E-05
10^{-22}	1.47488E-02 1.341	5.82269E-03 1.482	2.08475E-03 1.564	7.05251E-04 1.616	2.30149E-04 1.660	7.28172E-05 1.696	2.24765E-05
10^{-24}	1.47488E-02 1.341	5.82269E-03 1.482	2.08475E-03 1.564	7.05251E-04 1.616	2.30149E-04 1.660	7.28172E-05 1.696	2.24765E-05
10^{-26}	1.47488E-02 1.341	5.82269E-03 1.482	2.08475E-03 1.564	7.05251E-04 1.616	2.30149E-04 1.660	7.28172E-05 1.696	2.24765E-05
10^{-28}	1.47488E-02 1.341	5.82269E-03 1.482	2.08475E-03 1.564	7.05251E-04 1.616	2.30149E-04 1.660	7.28172E-05 1.696	2.24765E-05
10^{-30}	1.47488E-02 1.341	5.82269E-03 1.482	2.08475E-03 1.564	7.05251E-04 1.616	2.30149E-04 1.660	7.28172E-05 1.696	2.24765E-05
10^{-32}	1.47488E-02 1.341	5.82269E-03 1.482	2.08475E-03 1.564	7.05251E-04 1.616	2.30149E-04 1.660	7.28172E-05 1.696	2.24765E-05
10^{-34}	1.47488E-02 1.341	5.82269E-03 1.482	2.08475E-03 1.564	7.05251E-04 1.616	2.30149E-04 1.660	7.28172E-05 1.696	2.24765E-05
10^{-36}	1.47488E-02 1.341	5.82269E-03 1.482	2.08475E-03 1.564	7.05251E-04 1.616	2.30149E-04 1.660	7.28172E-05 1.696	2.24765E-05
10^{-38}	1.47488E-02 1.341	5.82269E-03 1.482	2.08475E-03 1.564	7.05251E-04 1.616	2.30149E-04 1.660	7.28172E-05 1.696	2.24766E-05
10^{-40}	1.47488E-02 1.341	5.82269E-03 1.482	2.08475E-03 1.564	7.05251E-04 1.616	2.30149E-04 1.660	7.28172E-05 1.696	2.24765E-05

Table 4.3: Maximum point-wise errors and rates of convergence by the proposed technique for Example 4.3.3.

ε	Number of mesh points						
	16	32	64	128	256	512	1024
10^{-6}	1.48957E-02 1.345	5.86336E-03 1.485	2.09535E-03 1.567	7.07278E-04 1.621	2.30000E-04 1.673	7.21320E-05 1.736	2.16598E-05
10^{-8}	1.48966E-02 1.345	5.86424E-03 1.484	2.09622E-03 1.566	7.08138E-04 1.617	2.30856E-04 1.661	7.29855E-05 1.697	2.25116E-05
10^{-10}	1.48966E-02 1.345	5.86425E-03 1.484	2.09623E-03 1.566	7.08146E-04 1.617	2.30864E-04 1.661	7.29941E-05 1.697	2.25202E-05
10^{-12}	1.48966E-02 1.345	5.86425E-03 1.484	2.09623E-03 1.566	7.08146E-04 1.617	2.30864E-04 1.661	7.29942E-05 1.697	2.25202E-05
10^{-14}	1.48966E-02 1.345	5.86425E-03 1.484	2.09623E-03 1.566	7.08146E-04 1.617	2.30864E-04 1.661	7.29942E-05 1.697	2.25202E-05
10^{-16}	1.48966E-02 1.345	5.86425E-03 1.484	2.09623E-03 1.566	7.08146E-04 1.617	2.30864E-04 1.661	7.29942E-05 1.697	2.25202E-05
10^{-18}	1.48966E-02 1.345	5.86425E-03 1.484	2.09623E-03 1.566	7.08146E-04 1.617	2.30864E-04 1.661	7.29942E-05 1.697	2.25202E-05
10^{-20}	1.48966E-02 1.345	5.86425E-03 1.484	2.09623E-03 1.566	7.08146E-04 1.617	2.30864E-04 1.661	7.29942E-05 1.697	2.25202E-05
10^{-22}	1.48966E-02 1.345	5.86425E-03 1.484	2.09623E-03 1.566	7.08146E-04 1.617	2.30864E-04 1.661	7.29942E-05 1.697	2.25202E-05
10^{-24}	1.48966E-02 1.345	5.86425E-03 1.484	2.09623E-03 1.566	7.08146E-04 1.617	2.30864E-04 1.661	7.29942E-05 1.697	2.25202E-05
10^{-26}	1.48966E-02 1.345	5.86425E-03 1.484	2.09623E-03 1.566	7.08146E-04 1.617	2.30864E-04 1.661	7.29942E-05 1.697	2.25202E-05
10^{-28}	1.48966E-02 1.345	5.86425E-03 1.484	2.09623E-03 1.566	7.08146E-04 1.617	2.30864E-04 1.661	7.29942E-05 1.697	2.25202E-05
10^{-30}	1.48966E-02 1.345	5.86425E-03 1.484	2.09623E-03 1.566	7.08146E-04 1.617	2.30864E-04 1.661	7.29942E-05 1.697	2.25202E-05
10^{-32}	1.48966E-02 1.345	5.86425E-03 1.484	2.09623E-03 1.566	7.08146E-04 1.617	2.30864E-04 1.661	7.29942E-05 1.697	2.25202E-05
10^{-34}	1.48966E-02 1.345	5.86425E-03 1.484	2.09623E-03 1.566	7.08146E-04 1.617	2.30864E-04 1.661	7.29942E-05 1.697	2.25202E-05
10^{-36}	1.48966E-02 1.345	5.86425E-03 1.484	2.09623E-03 1.566	7.08146E-04 1.617	2.30864E-04 1.661	7.29942E-05 1.697	2.25202E-05
10^{-38}	1.48966E-02 1.345	5.86425E-03 1.484	2.09623E-03 1.566	7.08146E-04 1.617	2.30864E-04 1.661	7.29942E-05 1.697	2.25202E-05
10^{-40}	1.48966E-02 1.345	5.86425E-03 1.484	2.09623E-03 1.566	7.08146E-04 1.617	2.30864E-04 1.661	7.29942E-05 1.697	2.25202E-05

Table 4.4: Maximum point-wise errors and rates of convergence by the proposed technique for Example 4.3.4.

ε	Number of mesh points						
	16	32	64	128	256	512	1024
10^{-4}	3.67280E-02	1.48586E-02	4.76355E-03	1.53136E-03	4.86575E-04	1.53356E-04	4.73044E-05
	1.306	1.641	1.637	1.654	1.666	1.697	
10^{-6}	3.65811E-02	1.48069E-02	4.74749E-03	1.52640E-03	4.85007E-04	1.52863E-04	4.71527E-05
	1.305	1.641	1.637	1.654	1.666	1.697	
10^{-8}	3.65664E-02	1.48017E-02	4.74589E-03	1.52590E-03	4.84850E-04	1.52813E-04	4.71375E-05
	1.305	1.641	1.637	1.654	1.666	1.697	
10^{-10}	3.65649E-02	1.48012E-02	4.74573E-03	1.52585E-03	4.84835E-04	1.52808E-04	4.71360E-05
	1.305	1.641	1.637	1.654	1.666	1.697	
10^{-12}	3.65648E-02	1.48011E-02	4.74571E-03	1.52584E-03	4.84833E-04	1.52808E-04	4.71358E-05
	1.305	1.641	1.637	1.654	1.666	1.697	
10^{-14}	3.65647E-02	1.48011E-02	4.74571E-03	1.52584E-03	4.84833E-04	1.52808E-04	4.71358E-05
	1.305	1.641	1.637	1.654	1.666	1.697	
10^{-16}	3.65647E-02	1.48011E-02	4.74571E-03	1.52584E-03	4.84833E-04	1.52808E-04	4.71358E-05
	1.305	1.641	1.637	1.654	1.666	1.697	
10^{-18}	3.65647E-02	1.48011E-02	4.74571E-03	1.52584E-03	4.84833E-04	1.52808E-04	4.71358E-05
	1.305	1.641	1.637	1.654	1.666	1.697	
10^{-20}	3.65647E-02	1.48011E-02	4.74571E-03	1.52584E-03	4.84833E-04	1.52808E-04	4.71358E-05
	1.305	1.641	1.637	1.654	1.666	1.697	
10^{-22}	3.65647E-02	1.48011E-02	4.74571E-03	1.52584E-03	4.84833E-04	1.52808E-04	4.71358E-05
	1.305	1.641	1.637	1.654	1.666	1.697	
10^{-24}	3.65647E-02	1.48011E-02	4.74571E-03	1.52584E-03	4.84833E-04	1.52808E-04	4.71358E-05
	1.305	1.641	1.637	1.654	1.666	1.697	
10^{-26}	3.65647E-02	1.48011E-02	4.74571E-03	1.52584E-03	4.84833E-04	1.52808E-04	4.71358E-05
	1.305	1.641	1.637	1.654	1.666	1.697	
10^{-28}	3.65647E-02	1.48011E-02	4.74571E-03	1.52584E-03	4.84833E-04	1.52808E-04	4.71358E-05
	1.305	1.641	1.637	1.654	1.666	1.697	
10^{-30}	3.65647E-02	1.48011E-02	4.74571E-03	1.52584E-03	4.84833E-04	1.52808E-04	4.71358E-05
	1.305	1.641	1.637	1.654	1.666	1.697	
10^{-32}	3.65647E-02	1.48011E-02	4.74571E-03	1.52584E-03	4.84833E-04	1.52808E-04	4.71358E-05
	1.305	1.641	1.637	1.654	1.666	1.697	
10^{-34}	3.65647E-02	1.48011E-02	4.74571E-03	1.52584E-03	4.84833E-04	1.52808E-04	4.71358E-05
	1.305	1.641	1.637	1.654	1.666	1.697	
10^{-36}	3.65647E-02	1.48011E-02	4.74571E-03	1.52584E-03	4.84833E-04	1.52808E-04	4.71358E-05
	1.305	1.641	1.637	1.654	1.666	1.697	
10^{-38}	3.65647E-02	1.48011E-02	4.74571E-03	1.52584E-03	4.84833E-04	1.52808E-04	4.71358E-05
	1.305	1.641	1.637	1.654	1.666	1.697	
10^{-40}	3.65647E-02	1.48011E-02	4.74571E-03	1.52584E-03	4.84833E-04	1.52808E-04	4.71358E-05
	1.305	1.641	1.637	1.654	1.666	1.697	

Table 4.5: Maximum point-wise errors and rates of convergence by the proposed technique for Example 4.3.5.

ε	Number of mesh points						
	16	32	64	128	256	512	1024
10^{-4}	1.78088E-01 1.411	6.69655E-02 1.694	2.06991E-02 1.562	7.00979E-03 1.625	2.27267E-03 1.662	7.18157E-04 1.698	2.21396E-04
10^{-6}	1.78934E-01 1.411	6.72665E-02 1.694	2.07889E-02 1.562	7.03977E-03 1.625	2.28235E-03 1.662	7.21221E-04 1.698	2.22339E-04
10^{-8}	1.79022E-01 1.412	6.72969E-02 1.694	2.07980E-02 1.562	7.04281E-03 1.625	2.28333E-03 1.662	7.21530E-04 1.698	2.22434E-04
10^{-10}	1.79030E-01 1.412	6.73000E-02 1.694	2.07989E-02 1.562	7.04311E-03 1.625	2.28343E-03 1.662	7.21561E-04 1.698	2.22444E-04
10^{-12}	1.79031E-01 1.412	6.73003E-02 1.694	2.07990E-02 1.562	7.04314E-03 1.625	2.28344E-03 1.662	7.21564E-04 1.698	2.22445E-04
10^{-14}	3.65647E-02 1.305	1.48011E-02 1.641	4.74571E-03 1.637	1.52584E-03 1.654	4.84833E-04 1.666	1.52808E-04 1.697	4.71358E-05
10^{-16}	3.65647E-02 1.305	1.48011E-02 1.641	4.74571E-03 1.637	1.52584E-03 1.654	4.84833E-04 1.666	1.52808E-04 1.697	4.71358E-05
10^{-18}	3.65647E-02 1.305	1.48011E-02 1.641	4.74571E-03 1.637	1.52584E-03 1.654	4.84833E-04 1.666	1.52808E-04 1.697	4.71358E-05
10^{-20}	3.65647E-02 1.305	1.48011E-02 1.641	4.74571E-03 1.637	1.52584E-03 1.654	4.84833E-04 1.666	1.52808E-04 1.697	4.71358E-05
10^{-22}	3.65647E-02 1.305	1.48011E-02 1.641	4.74571E-03 1.637	1.52584E-03 1.654	4.84833E-04 1.666	1.52808E-04 1.697	4.71358E-05
10^{-24}	3.65647E-02 1.305	1.48011E-02 1.641	4.74571E-03 1.637	1.52584E-03 1.654	4.84833E-04 1.666	1.52808E-04 1.697	4.71358E-05
10^{-26}	3.65647E-02 1.305	1.48011E-02 1.641	4.74571E-03 1.637	1.52584E-03 1.654	4.84833E-04 1.666	1.52808E-04 1.697	4.71358E-05
10^{-28}	3.65647E-02 1.305	1.48011E-02 1.641	4.74571E-03 1.637	1.52584E-03 1.654	4.84833E-04 1.666	1.52808E-04 1.697	4.71358E-05
10^{-30}	3.65647E-02 1.305	1.48011E-02 1.641	4.74571E-03 1.637	1.52584E-03 1.654	4.84833E-04 1.666	1.52808E-04 1.697	4.71358E-05
10^{-32}	3.65647E-02 1.305	1.48011E-02 1.641	4.74571E-03 1.637	1.52584E-03 1.654	4.84833E-04 1.666	1.52808E-04 1.697	4.71358E-05
10^{-34}	3.65647E-02 1.305	1.48011E-02 1.641	4.74571E-03 1.637	1.52584E-03 1.654	4.84833E-04 1.666	1.52808E-04 1.697	4.71358E-05
10^{-36}	3.65647E-02 1.305	1.48011E-02 1.641	4.74571E-03 1.637	1.52584E-03 1.654	4.84833E-04 1.666	1.52808E-04 1.697	4.71358E-05
10^{-38}	3.65647E-02 1.305	1.48011E-02 1.641	4.74571E-03 1.637	1.52584E-03 1.654	4.84833E-04 1.666	1.52808E-04 1.697	4.71358E-05
10^{-40}	3.65647E-02 1.305	1.48011E-02 1.641	4.74571E-03 1.637	1.52584E-03 1.654	4.84833E-04 1.666	1.52808E-04 1.697	4.71358E-05

(a) Example 4.1 (b) Example 4.2

(c) Example 4.3 (d) Example 4.4

(e) Example 4.5

Figure 4.1: Loglog plots of N Vs. maximum error.

Chapter 5

Singularly Perturbed Boundary Value Problems with Non-Smooth Data

The outline of this chapter is as follows: Section 5.1 deals with convection-diffusion type second order ODEs with a discontinuous source term. Reaction-diffusion type second order ODEs with a discontinuous source term are taken in Section 5.2. Some numerical results computed from the proposed schemes are presented in Section 5.3. Certain conclusions based on the analysis are finally presented in Section 5.4.

5.1 Convection-diffusion problems with non-smooth data

In this section, a singularly perturbed convection-diffusion type second order ODE with a discontinuous source term is considered on the unit interval $\Omega = (0,1)$. A single discontinuity is assumed to occur at a point $d \in \Omega$. This gives rise to an interior layer in the exact solution of the problem, in addition to the boundary layer at the outflow boundary point. Let $\Omega^- = (0,d)$ and $\Omega^+ = (d,1)$ and denote the jump at d in any function with $[w](d) = w(d+) - w(d-)$. Consider the problem:

$$L_\epsilon u(x) \equiv \varepsilon u''(x) + a(x)u'(x) = f(x), \quad x \in \Omega^- \cup \Omega^+, \tag{5.1.1}$$

$$u(0) = p, \quad u(1) = q, \tag{5.1.2}$$

where $0 < \epsilon \ll 1$ is a small parameter, $a(x)$ is a sufficiently smooth function on $\overline{\Omega} = [0,1]$, such that $a(x) \geq \beta > 0$, $f(x)$ is a sufficiently smooth function on $\Omega^- \cup \Omega^+ \cup \{0,1\}$, f and its derivatives have jump discontinuity at d. Because f is discontinuous at d, the solution u of (5.1.1-5.1.2) does not necessarily have a continuous second derivative at the point d. Thus, u need not belong to the class of functions $C^2(\Omega)$. But the first derivative of the solution exists and is continuous in Ω. Under these assumptions, (5.1.1-5.1.2) have a solution $u \in C^0(\overline{\Omega}) \cap C^1(\Omega) \cap C^2(\Omega^- \cup \Omega^+)$ [28].

In [127], Valanarasu and Ramanujam proposed an asymptotic initial value method (AIVM) to solve (5.1.1-5.1.2) on an appropriate Shishkin mesh which is of first order convergence. In this section, our main objective is to construct a modified initial value technique (MIVT) for (5.1.1-5.1.2) which is based on the underlying idea of AIVM [127].

First, in this technique, an asymptotic expansion approximation for the solution of the boundary value problem (BVP) (5.1.1-5.1.2) has been constructed. Then, the initial value problems (IVPs) and the terminal value problems (TVPs) are formulated whose solutions are the terms of this asymptotic expansion. The IVPs are happened to be SPPs and therefore, they are solved by the proposed hybrid scheme.

Assumption: We shall assume that $\epsilon \leq CN^{-1}$ as is generally the case for discretization of convection-dominated problems [115].

5.1.1 Maximum principle and stability result

In this sub-section, a maximum principle and a stability result are presented for the BVP (5.1.1-5.1.2).

Theorem 5.1.1 *[28] (Maximum Principle). Suppose that* $u \in C^0(\overline{\Omega}) \cap C^1(\Omega) \cap C^2(\Omega^- \cup \Omega^+)$ *satisfies*

$$u(0) \geq 0, \quad u(1) \geq 0,$$
$$L_\epsilon u(x) \leq 0, \quad \forall x \in \Omega^- \cup \Omega^+,$$

and $[u'](d) \leq 0, \, then \quad u(x) \geq 0, \, \forall x \in \overline{\Omega}.$

Theorem 5.1.2 *[127] (Stability Result). If* $u \in U$*, then*

$$||u||_{\overline{\Omega}} \leq \max\{|u(0)|, |u(1)|, ||L_\epsilon u||_{\Omega^- \cup \Omega^+}\}.$$

5.1.2 Asymptotic expansion approximation

An asymptotic expansion approximation for the solution of BVP (5.1.1-5.1.2) can be constructed as

$$u(x) = u_{as}(x) + O(\epsilon), \tag{5.1.3}$$

where $u_{as}(x) = u_0(x)+v_0(x)+w_0(x)$, $u_0(x)$ is the solution of the reduced problem of (5.1.1-5.1.2) given by

$$\begin{aligned} a(x)u_0'(x) &= f(x), \quad x \in \Omega^- \cup \Omega^+, && (5.1.4) \\ u_0(1) &= q, \quad u_0 \in C^0(\overline{\Omega}) \cap C^1(\Omega^- \cup \Omega^+). && (5.1.5) \end{aligned}$$

$v_0(x)$ and $w_0(x)$ are the "left boundary layer corrections" given by

$$v_0(x) = [u(0) - u_0(x)]v(x), \quad x \in \overline{\Omega}, \tag{5.1.6}$$

and

$$w_0(x) \equiv \begin{cases} \epsilon\frac{[u_0'(d+)-u_0'(d-)]}{a(d)}, & x \in \Omega^- \cup \{0\}, \\ \\ \epsilon\frac{[u_0'(d+)-u_0'(d-)]}{a(d)}w(x), & x \in \Omega^+ \cup \{d, 1\}, \end{cases} \tag{5.1.7}$$

where

$$v(x) = \exp\left\{-\int_0^x \frac{a(s)}{\varepsilon}ds\right\}, \tag{5.1.8}$$

and

$$w(x) = \exp\left\{-\int_d^x \frac{a(s)}{\varepsilon}ds\right\}. \tag{5.1.9}$$

It is easy to verify that $v(x)$ and $w(x)$ satisfy the following IVPs respectively.

$$\varepsilon v'(x) + a(x)v(x) = 0, \quad x \in \Omega \cup \{1\}, \tag{5.1.10}$$

$$v(0) = 1. \tag{5.1.11}$$

$$\varepsilon w'(x) + a(x)w(x) = 0, \quad x \in \Omega^+ \cup \{1\}, \tag{5.1.12}$$

$$w(d) = 1. \tag{5.1.13}$$

Theorem 5.1.3 *[127] The function u_{as} defined by*

$$u_{as}(x) = u_0(x) + v_0(x) + w_0(x), \quad x \in \overline{\Omega},$$

satisfies the inequality

$$|(u - u_{as})(x)| \leq C\epsilon, \quad x \in \overline{\Omega},$$

where $u(x)$ is the solution of (5.1.1-5.1.2).

Proof. We have

$$|L(u - u_{as})(x)| = |f(x) - L(u_0(x) + v_0(x) + w_0(x))|, \quad x \in \Omega^- \cup \Omega^+, \tag{5.1.14}$$

On Ω^-

$$\begin{aligned} |L(u - u_{as})(x)| &= |f(x) - L(\varepsilon u_0''(x) + a(x)u_0'(x) + \varepsilon v_0''(x) + a(x)v_0'(x) \\ &\quad + \varepsilon w_0''(x) + a(x)w_0'(x))|, \\ &\leq C\epsilon \left[1 + \epsilon^{-1} \exp\left(\frac{-\beta x}{2\epsilon}\right)\right]. \end{aligned} \tag{5.1.15}$$

On Ω^+

$$|L(u - u_{as})(x)| \leq C\epsilon \left[1 + \epsilon^{-1} \exp\left(\frac{-\beta x}{2\epsilon}\right) + \exp\left(\frac{-\beta(x-d)}{2\epsilon}\right)\right]. \tag{5.1.16}$$

Define$\phi^{\pm}(x)$ as

$$\phi^{\pm}(x) = \begin{cases} C_1\epsilon\left(\frac{1}{2} - \frac{x}{8} + \frac{d}{8}\right) + C_2\epsilon\exp\left(\frac{-\beta x}{2\epsilon}\right) + \\ C_2\epsilon \pm (u(x) - u_{as}(x)), \quad x \in \Omega^- \cup \{0\}, \\ \\ C_1\epsilon\left(\frac{1}{2} - \frac{x}{8} + \frac{d}{8}\right) + C_2\epsilon\exp\left(\frac{-\beta(x-d)}{2\epsilon}\right) + C_2\epsilon\exp\left(\frac{-\beta x}{2\epsilon}\right) \\ \pm(u(x) - u_{as}(x)), \quad x \in \Omega^+ \cup \{d, 1\}, \end{cases} \tag{5.1.17}$$

where the positive numbers C_1 and C_2 are to be chosen suitably. We have

$$\phi^{\pm}(0) = C_1\left(\frac{1}{2} + \frac{d}{8}\right) + 2C_2\epsilon \mp \epsilon\frac{[u_0'(d+) - u_0'(d-)]}{a(d)} \geq 0 \tag{5.1.18}$$

and

$$\begin{aligned} \phi^{\pm}(1) = C_1\left(\frac{1}{4} + \frac{d}{4}\right) + C_2\epsilon\exp\left(\frac{-\beta(1-d)}{2\epsilon}\right) + C_2\epsilon\exp\left(\frac{-\beta d}{2\epsilon}\right) \\ \mp\left\{[u(0) - u_0(0)]v(1) + \epsilon w(1)\frac{[u_0'(d+) - u_0'(d-)]}{a(d)}\right\} \geq 0, \end{aligned} \tag{5.1.19}$$

by proper choice of C$_1$ and C_2. Also on Ω^- and Ω^+ respectively we have

$$\begin{aligned} L\phi^{\pm}(x) &\leq -\frac{C_1\epsilon\beta}{8} - \frac{C_2\beta^2}{4}\exp\left(\frac{-\beta x}{2\epsilon}\right) \pm C\epsilon\left[1 + \epsilon^{-1}\exp\left(\frac{-\beta x}{2\epsilon}\right)\right] \\ &\leq 0, \end{aligned} \tag{5.1.20}$$

and

$$\begin{aligned} L\phi^{\pm}(1) &\leq -\frac{C_1\epsilon\beta}{4} - \frac{C_2\beta^2}{4}\exp\left(\frac{-\beta d}{2\epsilon}\right) - \frac{C_2\beta^2}{4}\exp\left(\frac{-\beta(x-d)}{2\epsilon}\right) \\ &\quad \pm C\epsilon\left[1 + \epsilon^{-1}\exp\left(\frac{-\beta x}{2\epsilon}\right) + \exp\left(\frac{-\beta(x-d)}{2\epsilon}\right)\right] \\ &\leq 0, \end{aligned} \tag{5.1.21}$$

by proper choice of C$_1$ and C_2. Furthermore $[\phi^{\pm'}](d) = \frac{-C_1\epsilon}{8} - \frac{C_2\beta}{2} \leq 0$. Applying Theorem 5.1.1, we get

$$|(u - u_{as})(x)| \leq C\epsilon, \quad x \in \overline{\Omega}. \tag{5.1.22}$$

5.1.3 Discretization and mesh

A fitted mesh method for the BVP (5.1.1-5.1.2) is now described. On $\Omega^- \bigcup \Omega^+$, a piecewise uniform mesh of N mesh intervals is constructed as follows. The domain $\overline{\Omega}$ is sub-divided into four subintervals

$$[0, \sigma_1] \cup [\sigma_1, d] \cup [d, d + \sigma_2] \cup [d + \sigma_2, 1]$$

for some σ_1, σ_2 that satisfy $0 < \sigma_1 \leq \frac{d}{2}, 0 < \sigma_2 \leq \frac{1-d}{2}$. On each sub-interval, a uniform mesh with $N/4$ mesh intervals is placed. The interior points of the mesh are denoted by

$$x_0 = 0,\, x_i = \sum_{k=0}^{i-1} h_k,\, h_k = x_{k+1} - x_k, x_N = 1, i = 1, 2, ..., N - 1.$$

Clearly, $x_{\frac{N}{2}} = d$ and $\overline{\Omega}_\epsilon^N = \{x_i\}_0^N$. Note that this is a uniform mesh when $\sigma_1 = \frac{d}{2}$ and $\sigma_2 = \frac{1-d}{2}$. It is fitted to (5.1.1-5.1.2) by choosing σ_1 and σ_2 to be the following functions of N and ϵ

$$\sigma_1 = \min\left\{\frac{d}{2}, \sigma_0\epsilon \ln N\right\}, \quad \sigma_2 = \min\left\{\frac{1-d}{2}, \sigma_0\epsilon \ln N\right\},$$

where $\sigma_0 \geq \frac{2}{\beta}$. Further, we denote the mesh size in the regions $[0, \sigma_1]$ by $h_1 = 4\frac{\sigma_1}{N}$, in $[\sigma_1, d]$ by $H_1 = 4\frac{(d-\sigma_1)}{N}$, in $[d, d + \sigma_2]$ by $h_2 = 4\frac{\sigma_2}{N}$ and in $[d + \sigma_2, 1]$ by $H_2 = 4\frac{(1-d-\sigma_2)}{N}$.

Applying hybrid scheme (2.1.4-2.1.5) for (5.1.10-5.1.11), we get :

$$L_\epsilon^N Y_i \equiv \begin{cases} \varepsilon D^- Y_i + \frac{a_{i-1}Y_{i-1} + a_i Y_i}{2} = 0 \quad 0 < i \leq N/4,\, N/2 < i \leq 3N/4, \\ \\ \varepsilon D^- Y_i + a_i Y_i = 0, \quad N/4 < i \leq N/2,\, 3N/4 < i \leq N, \end{cases} \tag{5.1.23}$$

$$Y_0 = A, \tag{5.1.24}$$

where $D^- Y_i = \frac{Y_i - Y_{i-1}}{x_i - x_{i-1}}$ and $a_i = a(x_i)$. Similarly, we can formulate the hybrid scheme for (5.1.12)-(5.1.13).

5.1.4 Description of method

In this sub-section, we describe the MIVT to solve the (SPBVP) (5.1.1-5.1.2)

Step 1. Solve the TVP (5.1.4-5.1.5) by using the Trapezoidal method. Let $U_0(x_i)$ be its solution.

Step 2. Solve the IVP (5.1.10-5.1.11) by using the hybrid scheme (5.1.23-5.1.24) described on Shishkin mesh in Section 5.1.3. Let $V(x_i)$ be its solution.

Step 3. Solve the IVP (5.1.12-5.1.13) by using the hybrid scheme. Let $W(x_i)$ be its solution.

Step 4. Define mesh function $U(x_i)$ as

$$U(x_i) = \begin{cases} U_0(x_i) + [u(0) - u_0(x)]V(x_i) + \epsilon \frac{[u_0'(d+) - u_0'(d-)]}{a(d)} & 0 \le x_i \le d, \\ \\ U_0(x_i) + [u(0) - u_0(x)]V(x_i) + \epsilon \frac{[u_0'(d+) - u_0'(d-)]}{a(d)} W(x_i), & d < x_i \le 1. \end{cases} \tag{5.1.25}$$

5.1.5 An error estimate

Theorem 5.1.4 *Let $y(x)$ and Y_i be respectively the solutions of (5.1.10-5.1.11) and (5.1.23-5.1.24). Then, the truncation error satisfies the following bounds:*

$$\begin{aligned}
|L_\varepsilon^N(Y_i - y(x_i))| &\leq CN^{-2}\sigma_0^2 \ln^2 N, \textit{ for} \quad 0 < i \leq N/4,\ N/2 < i \leq 3N/4,\\
|L_\varepsilon^N(Y_i - y(x_i))| &\leq C(N^{-1}\varepsilon + N^{-\beta\sigma_0}), \textit{ for} \quad N/4 < i \leq N/2 \textit{ and } H_1 \leq \varepsilon,\\
|L_\varepsilon^N(Y_i - y(x_i))| &\leq C(N^{-2} + N^{-\beta\sigma_0}), \textit{ for} \quad N/4 < i \leq N/2 \textit{ and } H_1 > \varepsilon,\\
|L_\varepsilon^N(Y_i - y(x_i))| &\leq C(N^{-1}\varepsilon + N^{-\beta\sigma_0}), \textit{ for} \quad 3N/4 < i \leq N \textit{ and } H_2 \leq \varepsilon,\\
|L_\varepsilon^N(Y_i - y(x_i))| &\leq C(N^{-2} + N^{-\beta\sigma_0}), \textit{ for} \quad 3N/4 < i \leq N \textit{and } H_2 > \varepsilon.
\end{aligned}$$

Proof. Following the method of proof given in Theorem2.1.1, we can prove the present theorem.

Theorem 5.1.5 *Let $y(x)$ be the solution of the IVP (5.1.10-5.1.11) and Y_i be the numerical solution obtained from the hybrid scheme as given in (5.1.23-5.1.24). Then, for sufficiently large N, and $N^{-1}\sigma_0 \ln N\beta^* < 1$, where $\beta^* = \max_{0\leq i\leq N} a(x_i)$, we have,*

$$|Y_i - y(x_i)| \leq C\left[N^{-2}\ln^2 N + N^{-1}\varepsilon + N^{-\beta\sigma_0}\right], \quad \forall x_i \in \overline{\Omega}.$$

Proof. Let $B_i^- = (2 - \rho_i a_i)$, $B_i^+ = (2 + \rho_i a_i)$ and $b_i^+ = (1 + \rho_i a_i)$, where $\rho_i = \frac{h_i}{\epsilon}$.

The solution of the scheme (5.1.23-5.1.24) can be expressed as:

For $0 < i \leq N/4$

$$Y_i = \frac{\prod_{j=0}^{i-1} B_j^-}{\prod_{j=1}^{i} B_j^+} Y_0 + \frac{\rho_i \prod_{j=1}^{i-1} B_j^-}{\prod_{j=1}^{i} B_j^+}(g_0 + g_1) + \frac{\rho_i \prod_{j=2}^{i-1} B_j^-}{\prod_{j=2}^{i} B_j^+}(g_1 + g_2) + ... + \frac{\rho_i}{B_i^+}(g_{i-1} + g_i). \quad (5.1.26)$$

For $N/4 < i \leq N/2$

$$Y_i = \frac{1}{\prod_{j=N/4+1}^{i} b_j^+} Y_{N/4} + \frac{\rho_i}{\prod_{j=N/4+1}^{i} b_j^+} g_{N/4+1} + \frac{\rho_i}{\prod_{j=N/4+2}^{i} b_j^+} g_{N/4+2} + ... + \frac{\rho_i}{b_i^+} g_i. \quad (5.1.27)$$

For $N/2 < i \leq 3N/4$

$$Y_i = \frac{\prod_{j=N/2}^{i-1} B_j^-}{\prod_{j=N/2+1}^{i} B_j^+} Y_{N/2} + \frac{\rho_i \prod_{j=N/2+1}^{i-1} B_j^-}{\prod_{j=N/2+1}^{i} B_j^+} (g_{N/2} + g_{N/2+1}) + \frac{\rho_i \prod_{j=N/2+2}^{i-1} B_j^-}{\prod_{j=N/2+2}^{i} B_j^+} (g_{N/2+1} + g_{N/2+2}) + \\ ... + \frac{\rho_i}{B_i^+}(g_{i-1} + g_i). \tag{5.1.28}$$

and for $3N/4 < i \leq N$

$$Y_i = \frac{1}{\prod_{j=3N/4+1}^{i} b_j^+} Y_{3N/4} + \frac{\rho_i}{\prod_{j=3N/4+1}^{i} b_j^+} g_{3N/4+1} + \frac{\rho_i}{\prod_{j=3N/4+2}^{i} b_j^+} g_{3N/4+2} + ... + \frac{\rho_i}{b_i^+} g_i. \tag{5.1.29}$$

Clearly, $B_i^+{'s}$ and $b_i^+{'s}$ are non-negative.

For $B_i^- > 0$, $0 < i \leq N/4$ and $N/2 < i \leq 3N/4$, we have

$$B_i^- = 2 - \rho_i b_i = 2 - \frac{h_i a_i}{\epsilon}. \tag{5.1.30}$$

Since $h_i = 2N^{-1}\sigma_0 \epsilon lnN$ and $a_i \leq \beta^*$, we have $B_i^- > 0$.

So, the solution satisfies the discrete maximum principle and hence there are no oscillations.

Defining the discrete barrier function

$$\phi_i = C\left[N^{-2}\ln^2 N + N^{-1}\epsilon + N^{-\beta\sigma_0}\right]. \tag{5.1.31}$$

Now, by choosing C sufficiently large, and using the discrete maximum principle, it is easier to see that,

$$L_\epsilon^N(\phi_i \pm (Y_i - y(x_i))) \geq 0, \tag{5.1.32}$$

equivalently,

$$L_\epsilon^N(\phi_i) \geq |Y_i - y(x_i)|. \tag{5.1.33}$$

Therefore, it follows that

$$|Y_i - y(x_i)| \leq |\phi_i|, \quad \forall x_i \in \overline{\Omega}. \tag{5.1.34}$$

Thus, we have the required ϵ-uniform error bound. ∎

Remark 5.1.6 *In Theorem 5.1.4, One can notice that the truncation error is of order $N^{-\beta\sigma_o}$ for $H_1 > \epsilon$, and $H_2 > \epsilon$. It is assumed that $\beta\sigma_o \geq 2$ and we are interested in the case $\epsilon \leq N^{-1}$. Also, we obtain the error bound of order $N^{-1}\varepsilon$ only in outer region for the case $H_1 < \epsilon$, and $H_2 < \epsilon$, which is not the practical case. With these points, we conclude that the order of convergence is almost two (up to a logarithmic factor). Our numerical results given in Section 5.3 reveal the same behavior.*

Now, an error estimate is derived for the numerical solution obtained by the MIVT.

Theorem 5.1.7 *Let $u(x)$ be the solution of the BVP (5.1.1-5.1.2) and $U(x_i)$ be the numerical solution obtained by MIVT, then we have*

$$|(U-u)(x_i)|_{\overline{\Omega}_\epsilon^N} \leq C\left[N^{-2}\ln^2 N + N^{-1}\varepsilon + N^{-\beta\sigma_0} + \epsilon\right].$$

Proof. Theorem 5.1.5, when applied to the IVPs (5.1.10-5.1.11), (5.1.12-5.1.13) yields

$$|(V-v)(x_i)| \leq C\left[N^{-2}\ln^2 N + N^{-1}\varepsilon + N^{-\beta\sigma_0}\right], \quad 0 \leq x_i \leq 1, \tag{5.1.35}$$

$$|(W-w)(x_i)| \leq C\left[N^{-2}\ln^2 N + N^{-1}\varepsilon + N^{-\beta\sigma_0}\right], \quad d \leq x_i \leq 1. \tag{5.1.36}$$

From the definitions of $u_{as}(x)$, $U(x_i)$ and above inequalities, we have

$$|(u_{as}-U)(x_i)| \leq C\left[N^{-2}\ln^2 N + N^{-1}\varepsilon + N^{-\beta\sigma_0}\right], \quad \text{for}\, x_i \in \Omega_\epsilon^N. \tag{5.1.37}$$

From Theorem 5.1.3, we have

$$|(u-u_{as})(x)| \leq C\epsilon, \quad x \in \overline{\Omega}. \tag{5.1.38}$$

The desired estimate follows from the inequalities (5.1.37) and (5.1.38).

5.2 Reaction-diffusion problems with non-smooth data

In this section, a singularly perturbed reaction-diffusion type second order ODE with a discontinuous source term [30] is considered on the unit interval $\Omega = (0, 1)$. A single discontinuity is assumed to occur at a point $d \in \Omega$. This gives rise to an interior layer in the exact solution of the problem, in addition to the boundary layer at the outflow boundary point. Let $\Omega^- = (0, d)$ and $\Omega^+ = (d, 1)$ and denote the jump at d in any function with $[w](d) = w(d+) - w(d-)$. Consider the problem:

$$L_\epsilon u(x) \equiv -\varepsilon u''(x) + a(x)u(x) = f(x), \quad x \in \Omega^- \cup \Omega^+, \tag{5.2.1}$$

$$u(0) = p, \quad u(1) = q, \tag{5.2.2}$$

where $0 < \epsilon \ll 1$ is a small parameter, $a(x)$ is a sufficiently smooth function on $\overline{\Omega} = [0, 1]$, such that $a(x) \geq \beta > 0$, $f(x)$ is a sufficiently smooth function on $\Omega^- \cup \Omega^+ \cup \{0, 1\}$, f and its derivatives have jump discontinuity at d. Because f is discontinuous at d, the solution u of (5.2.1-5.2.1) does not necessarily have a continuous second derivative at the point d. Thus, u need not belong to the class of functions $C^2(\Omega)$. But the first derivative of the solution exists and is continuous in Ω. Under these assumptions, (5.2.1-5.2.1) have a solution $u \in C^0(\overline{\Omega}) \cap C^1(\Omega) \cap C^2(\Omega^- \cup \Omega^+)$ [30].

In [126], Valanarasu and Ramanujam proposed an asymptotic initial value method (AIVM) to solve (5.2.1-5.2.2) on an appropriate Shishkin mesh which is of first order convergence. In this section, our main objective is to construct a modified initial value technique (MIVT) for (5.2.1-5.2.2) which is based on the underlying idea of AIVM [126].

First, in this technique, an asymptotic expansion approximation for the solution of the Boundary Value Problem (BVP) (5.2.1-5.2.2) has been constructed. Then, the Initial Value Problems (IVPs) and the Terminal Value Problems (TVPs) are formulated whose solutions are the terms of this asymptotic expansion. The IVPs and TVPs are happened to be SPPs and therefore, they are solved by the hybrid scheme. The MIVT displays uniform convergence with respect to the perturbation parameter ϵ and the order of convergence is almost two (up to a logarithmic factor).

5.2.1 Maximum principle and stability result

In this sub-section, a maximum principle and a stability result are presented for the BVP (5.2.1-5.2.2).

Theorem 5.2.1 *[30] (Maximum Principle). Suppose that* $u \in C^0(\overline{\Omega}) \cap C^2(\Omega^- \cup \Omega^+)$ *satisfies*

$$u(0) \geq 0, \quad u(1) \geq 0,$$
$$L_\epsilon u(x) \geq 0, \quad \forall\, x \in \Omega^- \cup \Omega^+,$$

and $[u'](d) \leq 0,\ then \quad u(x) \geq 0,\ \forall x \in \overline{\Omega}.$

Theorem 5.2.2 *[126] (Stability Result). If* $u \in U$*, then*

$$||u||_{\overline{\Omega}} \leq C\ \max\{|u(0)|, |u(1)|, ||L_\epsilon u||_{\Omega^- \cup \Omega^+}\}.$$

5.2.2 Asymptotic expansion approximation

An asymptotic expansion approximation for the solution of BVP (5.2.1-5.2.2) can be constructed as

$$u(x) = u_{as}(x) + O(\sqrt{\epsilon}), \tag{5.2.3}$$

where $u_{as}(x) = u_0(x) + v_0(x) + w_0(x)$, $u_0(x)$ is the solution of the reduced problem of (5.2.1-5.2.2) given by

$$a(x)u_0(x) = f(x), \quad x \in \Omega^- \cup \Omega^+, \tag{5.2.4}$$

and let $v_0(x)$ be the "left boundary layer correction" given by

$$v_0(x) \equiv \begin{cases} k_1[a(x)]^{-1/4} v_l(x), & x \in \Omega^-, \\ k_2[a(x)]^{-1/4} v_r(x), & x \in \Omega^+, \end{cases} \tag{5.2.5}$$

where

$$v_l(x) = \exp\left\{ -\int_0^x \sqrt{\frac{a(s)}{\varepsilon}} ds \right\}, \tag{5.2.6}$$

$$v_r(x) = \exp\left\{-\int_d^x \sqrt{\frac{a(s)}{\varepsilon}}ds\right\}, \tag{5.2.7}$$

and $w_0(x)$ be the ”right boundary layer correction” given by

$$w_0(x) \equiv \begin{cases} k_3[a(x)]^{-1/4}w_l(x), & x \in \Omega^-, \\ \\ k_4[a(x)]^{-1/4}w_r(x), & x \in \Omega^+, \end{cases} \tag{5.2.8}$$

where

$$w_l(x) = exp\left\{-\int_x^d \sqrt{\frac{a(s)}{\varepsilon}}ds\right\}, \tag{5.2.9}$$

and

$$w_r(x) = exp\left\{-\int_x^1 \sqrt{\frac{a(s)}{\varepsilon}}ds\right\}. \tag{5.2.10}$$

The constants k_1, k_2, k_3, k_4 are determined by imposing the following boundary and continuity conditions:

$$u_0(0) + v_0(0) + w_0(0) = u(0), \quad u_0(1) + v_0(1) + w_0(1) = u(1), \tag{5.2.11}$$

$$u_0(d-) + v_0(d-) + w_0(d-) = u_0(d+) + v_0(d+) + w_0(d+), \tag{5.2.12}$$

$$u_0'(d-) + v_0'(d-) + w_0'(d-) = u_0'(d+) + v_0'(d+) + w_0'(d+). \tag{5.2.13}$$

These are given by

$$k_1 = [u(0) - u_0(0)][a(0)]^{1/4} - k_3 v_l(d), \tag{5.2.14}$$

$$k_3 = \frac{k_2(1 - w_r(d)w_r(d)) + k_{31} + k_{32} + k_{33}}{1 - v_l(d)v_l(d)}, \tag{5.2.15}$$

$$k_2 = \frac{2\sqrt{\frac{a(d)}{\epsilon}}(k_{11} - k_{12}) + k_{13} + k_{14} + k_{15} + k_{16}}{2\sqrt{\frac{a(d)}{\epsilon}}(1 - v_l(d)v_l(d)w_r(d)w_r(d))}, \tag{5.2.16}$$

$$k_4 = [u(1) - u_0(1)][a(1)]^{1/4} - k_2 w_r(d), \tag{5.2.17}$$

$$k_{11} = [u(0) - u_0(0)][a(0)]^{1/4} v_l(d), \tag{5.2.18}$$

$$k_{12} = [u(1) - u_0(1)][a(1)]^{1/4} w_r(d)v_l(d)v_l(d), \tag{5.2.19}$$

$$k_{13} = [u_0(d-) - u_0(d+)][a(d)]^{1/4} \left\{ \frac{a'(d)}{4a(d)} + \sqrt{\frac{a(d)}{\epsilon}} \right\} v_l(d) v_l(d), \tag{5.2.20}$$

$$k_{14} = -[u_0(d-) - u_0(d+)][a(d)]^{1/4} \left\{ \frac{a'(d)}{4a(d)} - \sqrt{\frac{a(d)}{\epsilon}} \right\}, \tag{5.2.21}$$

$$k_{15} = -[u_0'(d-) - u_0'(d+)][a(d)]^{1/4}, \tag{5.2.22}$$

$$k_{16} = [u_0'(d-) - u_0'(d+)][a(d)]^{1/4} v_l(d) v_l(d), \tag{5.2.23}$$

$$k_{31} = [u(1) - u_0(1)][a(1)]^{1/4} w_r(d), \tag{5.2.24}$$

$$k_{32} = -[u(0) - u_0(0)][a(0)]^{1/4} v_l(d), \tag{5.2.25}$$

$$k_{33} = -[u_0(d-) - u_0(d+)][a(d)]^{1/4}. \tag{5.2.26}$$

It is easy to verify that $v_l(x)$, $v_r(x)$, $w_l(x)$and $w_r(x)$ satisfy the following IVPs and TVPs respectively:

$$\sqrt{\varepsilon} v_l'(x) + \sqrt{a(x)} v_l(x) = 0, \quad x \in \Omega^- \cup \{d\}, \tag{5.2.27}$$

$$v_l(0) = 1. \tag{5.2.28}$$

$$\sqrt{\varepsilon} v_r'(x) + \sqrt{a(x)} v_r(x) = 0, \quad x \in \Omega^+ \cup \{1\}, \tag{5.2.29}$$

$$v_r(d) = 1. \tag{5.2.30}$$

$$\sqrt{\varepsilon} w_l'(x) - \sqrt{a(x)} w_l(x) = 0, \quad x \in \Omega^- \cup \{0\}, \tag{5.2.31}$$

$$w_l(d) = 1. \tag{5.2.32}$$

$$\sqrt{\varepsilon} w_r'(x) - \sqrt{a(x)} w_r(x) = 0, \quad x \in \Omega^+ \cup \{d\}, \tag{5.2.33}$$

$$w_r(1) = 1. \tag{5.2.34}$$

Theorem 5.2.3 *[126] The function u_{as} defined by*

$$u_{as}(x) = \begin{cases} u_0(x) + v_0(x) + w_0(x), & x \in \Omega^- \bigcup \Omega^+, \\ \\ u_0(d-) + v_0(d-) + w_0(d-), & x = d, \end{cases}$$

satisfies the inequality

$$|(u - u_{as})(x)| \leq C\sqrt{\epsilon}, \quad x \in \overline{\Omega},$$

where $u(x)$ is the solution of (5.2.1-5.2.2).

Proof. It is easy to verify that

$$(u - u_{as})(0) = 0 \qquad (u - u_{as})(1) = 0. \tag{5.2.35}$$

$$|L(u - u_{as})(x)| = |f(x) - L(u_0(x) + v_0(x) + w_0(x))|, \quad x \in \Omega^- \cup \Omega^+, \tag{5.2.36}$$

On Ω^-, we have

$$\begin{aligned} |L(u - u_{as})(x)| &= |f(x) - (-\varepsilon u_0''(x) + a(x)u_0(x) - \varepsilon v_0''(x) + a(x)v_0(x) - \varepsilon w_0''(x) + a(x)w_0(x))|, \\ &= |\varepsilon u_0''(x) + k_1 \left\{ \frac{5\epsilon[a'(x)]^2}{16[a(x)]^{9/4}} - \frac{\epsilon a''(x)}{4[a(x)]^{5/4}} \right\} v_l(x) \\ &\quad + k_3 \left\{ \frac{5\epsilon[a'(x)]^2}{16[a(x)]^{9/4}} - \frac{\epsilon a''(x)}{4[a(x)]^{5/4}} \right\} w_l(x)| \\ &\leq C\sqrt{\epsilon} + C\sqrt{\epsilon} \left(\exp\left\{ -\int_0^x \sqrt{\frac{a(s)}{\varepsilon}} ds \right\} + \exp\left\{ -\int_x^d \sqrt{\frac{a(s)}{\varepsilon}} ds \right\} \right) \\ &\leq C\sqrt{\epsilon}. \end{aligned} \tag{5.2.37}$$

Similarly it can be shown that, on Ω^+

$$\begin{aligned} |L(u - u_{as})(x)| &\leq C\sqrt{\epsilon} + C\sqrt{\epsilon} \left(\exp\left\{ -\int_d^x \sqrt{\frac{a(s)}{\varepsilon}} ds \right\} + \exp\left\{ -\int_x^1 \sqrt{\frac{a(s)}{\varepsilon}} ds \right\} \right) \\ &\leq C\sqrt{\epsilon}. \end{aligned} \tag{5.2.38}$$

Furthermore, $[u'_{as}](d) = 0$. From the stability result, we conclude that

$$|(u - u_{as})(x)| \leq C\sqrt{\epsilon}, \quad x \in \overline{\Omega}. \tag{5.2.39}$$

5.2.3 Proposed scheme

A fitted mesh method for the BVP (5.2.1-5.2.2) is now described. On $\Omega^- \bigcup \Omega^+$ a piecewise uniform mesh of N mesh intervals is constructed as follows. The interval $\overline{\Omega}^-$ is subdivided into the three subintervals.

$$[0, \sigma_1], \quad [\sigma_1, d - \sigma_1] \quad and\, [d - \sigma_1, d] \tag{5.2.40}$$

for some σ_1 that satisfies $0 < \sigma_1 \leq d/4$. On $[0, \sigma_1]$ and $[d - \sigma_1, d]$ a uniform mesh with $N/8$ mesh-intervals is placed, while $[\sigma_1, d - \sigma_1]$ has a uniform mesh with $N/4$ mesh intervals. The subintervals $[d, d + \sigma_2]$,$[d + \sigma_2, 1 - \sigma_2]$, $[1 - \sigma_2, 1]$ of $\overline{\Omega}^+$ are treated analogously for some σ_2 satisfying $0 < \sigma_2 \leq (1 - d)/4$. The interior points of the mesh are denoted by

$$x_0 = 0,\, x_i = \sum_{k=0}^{i-1} h_k,\, h_k = x_{k+1} - x_k, x_N = 1, i = 1, 2, ..., N - 1.$$

Clearly, $x_{\frac{N}{2}} = d$ and $\overline{\Omega}_\epsilon^N = \{x_i\}_0^N$. Note that this is a uniform mesh when $\sigma_1 = \frac{d}{4}$ and $\sigma_2 = \frac{1-d}{4}$. It is fitted to the problem by choosing σ_1 and σ_2 to be the following functions of N and ϵ

$$\sigma_1 = \min\left\{\frac{d}{4}, \sigma_0\sqrt{\epsilon}\ln N\right\}, \quad \sigma_2 = \min\left\{\frac{1-d}{4}, \sigma_0\sqrt{\epsilon}\ln N\right\},$$

where $\sigma_0 \geq \frac{2}{\sqrt{\beta}}$. Applying hybrid scheme (2.1.4-2.1.5) for (5.2.27-5.2.28), we get

$$L_\epsilon^N V_l(x_i) \equiv \begin{cases} \varepsilon D^- V_l(x_i) + \dfrac{\sqrt{a(x_{i-1})}V_l(x_{i-1}) + \sqrt{a(x_i)}V_l(x_i)}{2} = 0,\, 0 < i \leq N/8,\, 3N/8 < i \leq N/2, \\ \\ \varepsilon D^- V_l(x_i) + \sqrt{a(x_i)}V_l(x_i) = 0, \quad N/8 < i \leq 3N/8, \end{cases} \tag{5.2.41}$$

$$V_l(0) = 1. \tag{5.2.42}$$

Similarly, we can formulate the hybrid scheme for (5.2.29-5.2.30), (5.2.31-5.2.32), (5.2.33)-(5.2.34).

5.2.4 Description of method

In this sub-section, we describe the MIVT to solve the (SPBVP) (5.2.1)-(5.2.2)

Step 1. Solve the IVPs (5.2.27-5.2.28) and (5.2.29-5.2.30) by using the hybrid scheme (5.2.41-5.2.42). Let $V_l(x_i)$and $V_r(x_i)$ be the solutions respectively.

Step 2. Solve the TVPs (5.2.31-5.2.32) and (5.2.33-5.2.34) by using the hybrid scheme (5.2.41-5.2.42) after transforming into IVPs. Let $W_l(x_i)$and $W_r(x_i)$ be the solutions respectively.

Step 3. Define mesh function $U(x_i)$ as

$$U(x_i) = \begin{cases} u_0(x_i) + k_1[a(x_i)]^{-1/4}V_l(x_i) + k_3[a(x_i)]^{-1/4}W_l(x_i), & 0 \le x_i < d \\ \dfrac{f(x_{N/2-1})}{a(x_{N/2})} + k_1[a(x_{N/2})]^{-1/4}V_l(x_{N/2}) + k_3[a(x_{N/2})]^{-1/4}W_l(x_{N/2}), & x_{N/2} = d, \\ u_0(x_i) + k_2[a(x_i)]^{-1/4}V_r(x_i) + k_4[a(x_i)]^{-1/4}W_r(x_i), & d < x_i \le 1. \end{cases} \tag{5.2.43}$$

Now, an error estimate is derived for the numerical solution obtained by the MIVT.

Theorem 5.2.4 *Let $u(x)$ be the solution of the BVP (5.2.1-5.2.2) and $U(x_i)$ be the numerical solution obtained by MIVT, then we have*

$$|(U-u)(x_i)|_{\overline{\Omega}_\epsilon^N} \le C\left[N^{-2}\ln^2 N + N^{-1}\varepsilon + N^{-\sqrt{\beta}\sigma_0} + \sqrt{\epsilon}\right].$$

Proof. Theorem 2.1.2, when applied to the IVPs (5.2.27-5.2.28), (5.2.29-5.2.30), (5.2.31-5.2.32) and (5.2.33-5.2.34) yields

$$|(V_l - v_l)(x_i)| \le C\left[N^{-2}\ln^2 N + N^{-1}\varepsilon + N^{-\sqrt{\beta}\sigma_0}\right], \quad 0 \le x_i \le d, \tag{5.2.44}$$

$$|(V_r - v_r)(x_i)| \le C\left[N^{-2}\ln^2 N + N^{-1}\varepsilon + N^{-\sqrt{\beta}\sigma_0}\right], \quad d \le x_i \le 1, \tag{5.2.45}$$

$$|(W_l - w_l)(x_i)| \le C\left[N^{-2}\ln^2 N + N^{-1}\varepsilon + N^{-\sqrt{\beta}\sigma_0}\right], \quad 0 \le x_i \le d, \tag{5.2.46}$$

$$|(W_r - w_r)(x_i)| \le C\left[N^{-2}\ln^2 N + N^{-1}\varepsilon + N^{-\sqrt{\beta}\sigma_0}\right], \quad d \le x_i \le 1. \tag{5.2.47}$$

From the definitions of $u_{as}(x)$, $U(x_i)$ and above inequalities, we have

$$|(u_{as} - U)(x_i)| \le C\left[N^{-2}\ln^2 N + N^{-1}\varepsilon + N^{-\sqrt{\beta}\sigma_0}\right], \quad \text{for } x_i \in \Omega_\epsilon^N. \tag{5.2.48}$$

From Theorem 5.2.3, we have

$$|(u - u_{as})(x)| \le C\sqrt{\epsilon}, \quad x \in \overline{\Omega}. \tag{5.2.49}$$

The desired estimate follows from the inequalities (5.2.48) and (5.2.49).

5.3 Numerical experiments and discussions

To illustrate the predicted theory, following examples are provided here. Computational results are given in the form of tables. The results are presented with maximum point-wise errors for various values of ϵ and N. We have also computed the computational order of convergence which has been shown in the same table along with maximum errors. In all the cases, we take $\sigma_0 = 2$.

Example 5.3.1 *Consider the singularly perturbed BVP with a constant coefficient and a discontinuous source term:*

$$\begin{aligned} &\varepsilon u''(x) + u'(x) = f(x), \quad x \in \Omega^- \cup \Omega^+, \\ &u(0) = 1, \quad u(1) = 0, \end{aligned}$$

where

$$f(x) = \begin{cases} 0.6, & x \le 0.5 \\ -0.7, & x > 0.5. \end{cases}$$

The exact solution of this example is not available. Therefore, to obtain the maximum point-wise errors and rates of convergence, we use double mesh principle. By following the idea of Sun and Styne [117], we modify the Shishkin mesh. We calculate the numerical solution U^N on $\overline{\Omega}_\epsilon^N$ and the numerical solution $\widetilde{U}^N$on the mesh $\widetilde{\Omega}_\epsilon^N$, where the transition parameter σ_1 is altered slightly to $\widetilde{\sigma}_1 = \min\{\frac{d}{4}, \sigma_0\epsilon\ln\frac{N}{2}\}$ and , σ_2 is altered slightly to $\widetilde{\sigma}_2 = \min\{\frac{1-d}{4}, \sigma_0\epsilon\ln\frac{N}{2}\}$. Note that this slightly altered value of σ_1 and σ_2 will ensure that the positions of transition points remain the same in meshes $\overline{\Omega}_{\epsilon}^{N}$ and $\widetilde{\Omega}_{\epsilon}^{2N}$. The double mesh difference is defined as

$$E_{\epsilon}^{N} = \max_{x_i \in \overline{\Omega}_{\epsilon}^{N}} \{|U^{N}(x_i) - \widetilde{U}^{2N}(x_i)|\}, \tag{5.3.1}$$

where $U^{N}(x_i)$ and $\widetilde{U}^{2N}(x_i)$ denote respectively the numerical solutions obtained by using N and $2N$ mesh intervals. The rates of convergence are calculated as:

$$p_{\epsilon}^{N} = \frac{lnE_{\epsilon}^{N} - lnE_{\epsilon}^{2N}}{ln2}. \tag{5.3.2}$$

The numerical results by the proposed scheme have been given in Table 5.1.

Example 5.3.2 *Consider the singularly perturbed BVP with a variable coefficient and a discontinuous source term:*

$$\begin{aligned} &\varepsilon u''(x) + (x+1)u'(x) = f(x), \quad x \in \Omega^{-} \cup \Omega^{+}, \\ &u(0) = 1, \quad u(1) = 0, \end{aligned}$$

where

$$f(x) = \begin{cases} (x+1)^2, & x \leq 0.5 \\ x, & x > 0.5. \end{cases}$$

The numerical results by the proposed scheme have been given in Table 5.3.

Example 5.3.3 *Consider the singularly perturbed BVP with a constant coefficient and a discontinuous source term:*

$$\begin{aligned} &-\varepsilon u''(x) + u(x) = f(x), \quad x \in \Omega^{-} \cup \Omega^{+}, \\ &u(0) = 0, \quad u(1) = 0, \end{aligned}$$

where

$$f(x) = \begin{cases} 0.7, & x \leq 0.5 \\ -0.6, & x > 0.5. \end{cases}$$

We calculate the numerical solution $\widetilde{U}^N$on the mesh $\widetilde{\Omega}_\epsilon^N$, where the transition parameter σ_1 is altered slightly to $\widetilde{\sigma}_1 = min\{\frac{d}{4}, \sigma_0\sqrt{\epsilon}\ln\frac{N}{2}\}$ and , σ_2 is altered slightly to $\widetilde{\sigma}_2 = min\{\frac{1-d}{4}, \sigma_0\sqrt{\epsilon}\ln\frac{N}{2}\}$. Note that this slightly altered value of σ_1 and σ_2 will ensure that the positions of transition points remain the same in meshes $\overline{\Omega}_\epsilon^N$ and $\widetilde{\Omega}_\epsilon^{2N}$. The double mesh difference is defined as

$$E_\epsilon^N = \max_{x_i\in\overline{\Omega}_\epsilon^N}\{|U^N(x_i) - \widetilde{U}^{2N}(x_i)|\}, \tag{5.3.3}$$

where $U^N(x_i)$ and $\widetilde{U}^{2N}(x_i)$ denote respectively the numerical solutions obtained by using N and $2N$ mesh intervals. The rates of convergence are calculated as:

$$p_\epsilon^N = \frac{lnE_\epsilon^N - lnE_\epsilon^{2N}}{ln2}. \tag{5.3.4}$$

The numerical results by the proposed scheme have been given in Table 5.5.

5.4 Conclusions

We have proposed a robust computational technique for solving the singularly perturbed boundary value problems with non-smooth data. This chapter demonstrates, the effectiveness of the Shishkin mesh by modifying the initial value technique [37] in a very simple way so that higher order, almost second order of convergence can be achieved with no restrictions on values of h and ϵ. Numerical results were presented which are in agreement with the theoretical results. Our method is easier to apply and more effective in the sense of solution errors. From, the numerical results shown in tables, we conclude that the proposed MIVT works nicely independent of the mesh parameter h and the perturbation parameter ϵ. Also, the proposed scheme is of almost second order convergence upto a logarithmic factor. The graphs plotted in Figure 5.2 are convergent curves in the maximum norm at nodal points for the different values of ϵ for Examples (5.3.1-5.3.3).

Figure 5.1: Comparison of proposed scheme and Valanarasu's scheme [127] for different values of ϵ of Example 5.3.1

Table 5.1: Maximum point-wise errors and rates of convergence by the proposed technique for Example 5.3.1.

ε	Number of mesh points						
	16	32	64	128	256	512	1024
10^{-2}	5.15247E-02 1.501	1.82052E-02 1.573	6.12007E-03 1.595	2.02616E-03 1.623	6.58036E-04 1.663	2.07735E-04 1.697	6.40580E-05
10^{-4}	5.15247E-02 1.501	1.82052E-02 1.573	6.12007E-03 1.595	2.02616E-03 1.623	6.58036E-04 1.663	2.07735E-04 1.697	6.40580E-05
10^{-6}	5.15247E-02 1.501	1.82052E-02 1.573	6.12007E-03 1.595	2.02616E-03 1.623	6.58036E-04 1.663	2.07735E-04 1.697	6.40580E-05
10^{-8}	5.15247E-02 1.501	1.82052E-02 1.573	6.12007E-03 1.595	2.02616E-03 1.623	6.58036E-04 1.663	2.07735E-04 1.697	6.40580E-05
10^{-10}	5.15247E-02 1.501	1.82052E-02 1.573	6.12007E-03 1.595	2.02616E-03 1.623	6.58036E-04 1.663	2.07735E-04 1.697	6.40580E-05
10^{-12}	5.15247E-02 1.501	1.82052E-02 1.573	6.12007E-03 1.595	2.02616E-03 1.623	6.58036E-04 1.663	2.07735E-04 1.697	6.40580E-05
10^{-14}	5.15247E-02 1.501	1.82052E-02 1.573	6.12007E-03 1.595	2.02616E-03 1.623	6.58036E-04 1.663	2.07735E-04 1.697	6.40580E-05
10^{-16}	5.15247E-02 1.501	1.82052E-02 1.573	6.12007E-03 1.595	2.02616E-03 1.623	6.58036E-04 1.663	2.07735E-04 1.697	6.40580E-05
10^{-18}	5.15247E-02 1.501	1.82052E-02 1.573	6.12007E-03 1.595	2.02616E-03 1.623	6.58036E-04 1.663	2.07735E-04 1.697	6.40580E-05
10^{-20}	5.15247E-02 1.501	1.82052E-02 1.573	6.12007E-03 1.595	2.02616E-03 1.623	6.58036E-04 1.663	2.07735E-04 1.697	6.40580E-05
10^{-22}	5.15247E-02 1.501	1.82052E-02 1.573	6.12007E-03 1.595	2.02616E-03 1.623	6.58036E-04 1.663	2.07735E-04 1.697	6.40580E-05
10^{-24}	5.15247E-02 1.501	1.82052E-02 1.573	6.12007E-03 1.595	2.02616E-03 1.623	6.58036E-04 1.663	2.07735E-04 1.697	6.40580E-05
10^{-26}	5.15247E-02 1.501	1.82052E-02 1.573	6.12007E-03 1.595	2.02616E-03 1.623	6.58036E-04 1.663	2.07735E-04 1.697	6.40580E-05
10^{-28}	5.15247E-02 1.501	1.82052E-02 1.573	6.12007E-03 1.595	2.02616E-03 1.623	6.58036E-04 1.663	2.07735E-04 1.697	6.40580E-05
10^{-30}	5.15247E-02 1.501	1.82052E-02 1.573	6.12007E-03 1.595	2.02616E-03 1.623	6.58036E-04 1.663	2.07735E-04 1.697	6.40580E-05
10^{-32}	5.15247E-02 1.501	1.82052E-02 1.573	6.12007E-03 1.595	2.02616E-03 1.623	6.58036E-04 1.663	2.07735E-04 1.697	6.40580E-05
10^{-34}	5.15247E-02 1.501	1.82052E-02 1.573	6.12007E-03 1.595	2.02616E-03 1.623	6.58036E-04 1.663	2.07735E-04 1.697	6.40580E-05
10^{-36}	5.15247E-02 1.501	1.82052E-02 1.573	6.12007E-03 1.595	2.02616E-03 1.623	6.58036E-04 1.663	2.07735E-04 1.697	6.40580E-05
10^{-38}	5.15247E-02 1.501	1.82052E-02 1.573	6.12007E-03 1.595	2.02616E-03 1.623	6.58036E-04 1.663	2.07735E-04 1.697	6.40580E-05
10^{-40}	5.15247E-02 1.501	1.82052E-02 1.573	6.12007E-03 1.595	2.02616E-03 1.623	6.58036E-04 1.663	2.07735E-04 1.697	6.40580E-05

Table 5.2: Maximum point-wise errors and rates of convergence by AIVM [127] for Example 5.3.1.

ε	Number of mesh points						
	16	32	64	128	256	512	1024
10^{-2}	4.24490E-02 0.514	2.97248E-02 0.604	1.95566E-02 0.691	1.21113E-02 0.754	7.18324E-03 0.798	4.13269E-03 0.829	2.32630E-03
10^{-4}	4.24490E-02 0.514	2.97248E-02 0.604	1.95566E-02 0.691	1.21113E-02 0.754	7.18324E-03 0.798	4.13269E-03 0.829	2.32630E-03
10^{-6}	4.24490E-02 0.514	2.97248E-02 0.604	1.95566E-02 0.691	1.21113E-02 0.754	7.18324E-03 0.798	4.13269E-03 0.829	2.32630E-03
10^{-8}	4.24490E-02 0.514	2.97248E-02 0.604	1.95566E-02 0.691	1.21113E-02 0.754	7.18324E-03 0.798	4.13269E-03 0.829	2.32630E-03
10^{-10}	4.24490E-02 0.514	2.97248E-02 0.604	1.95566E-02 0.691	1.21113E-02 0.754	7.18324E-03 0.798	4.13269E-03 0.829	2.32630E-03
10^{-12}	4.24490E-02 0.514	2.97248E-02 0.604	1.95566E-02 0.691	1.21113E-02 0.754	7.18324E-03 0.798	4.13269E-03 0.829	2.32630E-03
10^{-14}	4.24490E-02 0.514	2.97248E-02 0.604	1.95566E-02 0.691	1.21113E-02 0.754	7.18324E-03 0.798	4.13269E-03 0.829	2.32630E-03
10^{-16}	4.24490E-02 0.514	2.97248E-02 0.604	1.95566E-02 0.691	1.21113E-02 0.754	7.18324E-03 0.798	4.13269E-03 0.829	2.32630E-03
10^{-18}	4.24490E-02 0.514	2.97248E-02 0.604	1.95566E-02 0.691	1.21113E-02 0.754	7.18324E-03 0.798	4.13269E-03 0.829	2.32630E-03
10^{-20}	4.24490E-02 0.514	2.97248E-02 0.604	1.95566E-02 0.691	1.21113E-02 0.754	7.18324E-03 0.798	4.13269E-03 0.829	2.32630E-03
10^{-22}	4.24490E-02 0.514	2.97248E-02 0.604	1.95566E-02 0.691	1.21113E-02 0.754	7.18324E-03 0.798	4.13269E-03 0.829	2.32630E-03
10^{-24}	4.24490E-02 0.514	2.97248E-02 0.604	1.95566E-02 0.691	1.21113E-02 0.754	7.18324E-03 0.798	4.13269E-03 0.829	2.32630E-03
10^{-26}	4.24490E-02 0.514	2.97248E-02 0.604	1.95566E-02 0.691	1.21113E-02 0.754	7.18324E-03 0.798	4.13269E-03 0.829	2.32630E-03
10^{-28}	4.24490E-02 0.514	2.97248E-02 0.604	1.95566E-02 0.691	1.21113E-02 0.754	7.18324E-03 0.798	4.13269E-03 0.829	2.32630E-03
10^{-30}	4.24490E-02 0.514	2.97248E-02 0.604	1.95566E-02 0.691	1.21113E-02 0.754	7.18324E-03 0.798	4.13269E-03 0.829	2.32630E-03
10^{-32}	4.24490E-02 0.514	2.97248E-02 0.604	1.95566E-02 0.691	1.21113E-02 0.754	7.18324E-03 0.798	4.13269E-03 0.829	2.32630E-03
10^{-34}	4.24490E-02 0.514	2.97248E-02 0.604	1.95566E-02 0.691	1.21113E-02 0.754	7.18324E-03 0.798	4.13269E-03 0.829	2.32630E-03
10^{-36}	4.24490E-02 0.514	2.97248E-02 0.604	1.95566E-02 0.691	1.21113E-02 0.754	7.18324E-03 0.798	4.13269E-03 0.829	2.32630E-03
10^{-38}	4.24490E-02 0.514	2.97248E-02 0.604	1.95566E-02 0.691	1.21113E-02 0.754	7.18324E-03 0.798	4.13269E-03 0.829	2.32630E-03
10^{-40}	4.24490E-02 0.514	2.97248E-02 0.604	1.95566E-02 0.691	1.21113E-02 0.754	7.18324E-03 0.798	4.13269E-03 0.829	2.32630E-03

Table 5.3: Maximum point-wise errors and rates of convergence by the proposed technique for Example 5.3.2.

ε	Number of mesh points						
	16	32	64	128	256	512	1024
10^{-2}	9.75351E-02	3.44198E-02	1.15947E-02	3.83708E-03	1.24725E-03	3.93639E-04	1.21412E-04
	1.503	1.570	1.595	1.621	1.664	1.697	
10^{-4}	9.94714E-02	3.51727E-02	1.18262E-02	3.91605E-03	1.27193E-03	4.01567E-04	1.23836E-04
	1.500	1.572	1.595	1.622	1.663	1.697	
10^{-6}	9.94908E-02	3.51802E-02	1.18285E-02	3.91683E-03	1.27217E-03	4.01646E-04	1.23860E-04
	1.500	1.572	1.595	1.622	1.663	1.697	
10^{-8}	9.94910E-02	3.51803E-02	1.18285E-02	3.91684E-03	1.27217E-03	4.01647E-04	1.23861E-04
	1.500	1.572	1.595	1.622	1.663	1.697	
10^{-10}	9.94910E-02	3.51803E-02	1.18285E-02	3.91684E-03	1.27217E-03	4.01647E-04	1.23861E-04
	1.500	1.572	1.595	1.622	1.663	1.697	
10^{-12}	9.94910E-02	3.51803E-02	1.18285E-02	3.91684E-03	1.27217E-03	4.01647E-04	1.23861E-04
	1.500	1.572	1.595	1.622	1.663	1.697	
10^{-14}	9.94910E-02	3.51803E-02	1.18285E-02	3.91684E-03	1.27217E-03	4.01647E-04	1.23861E-04
	1.500	1.572	1.595	1.622	1.663	1.697	
10^{-16}	9.94910E-02	3.51803E-02	1.18285E-02	3.91684E-03	1.27217E-03	4.01647E-04	1.23861E-04
	1.500	1.572	1.595	1.622	1.663	1.697	
10^{-18}	9.94910E-02	3.51803E-02	1.18285E-02	3.91684E-03	1.27217E-03	4.01647E-04	1.23861E-04
	1.500	1.572	1.595	1.622	1.663	1.697	
10^{-20}	9.94910E-02	3.51803E-02	1.18285E-02	3.91684E-03	1.27217E-03	4.01647E-04	1.23861E-04
	1.500	1.572	1.595	1.622	1.663	1.697	
10^{-22}	9.94910E-02	3.51803E-02	1.18285E-02	3.91684E-03	1.27217E-03	4.01647E-04	1.23861E-04
	1.500	1.572	1.595	1.622	1.663	1.697	
10^{-24}	9.94910E-02	3.51803E-02	1.18285E-02	3.91684E-03	1.27217E-03	4.01647E-04	1.23861E-04
	1.500	1.572	1.595	1.622	1.663	1.697	
10^{-26}	9.94910E-02	3.51803E-02	1.18285E-02	3.91684E-03	1.27217E-03	4.01647E-04	1.23861E-04
	1.500	1.572	1.595	1.622	1.663	1.697	
10^{-28}	9.94910E-02	3.51803E-02	1.18285E-02	3.91684E-03	1.27217E-03	4.01647E-04	1.23861E-04
	1.500	1.572	1.595	1.622	1.663	1.697	
10^{-30}	9.94910E-02	3.51803E-02	1.18285E-02	3.91684E-03	1.27217E-03	4.01647E-04	1.23861E-04
	1.500	1.572	1.595	1.622	1.663	1.697	
10^{-32}	9.94910E-02	3.51803E-02	1.18285E-02	3.91684E-03	1.27217E-03	4.01647E-04	1.23861E-04
	1.500	1.572	1.595	1.622	1.663	1.697	
10^{-34}	9.94910E-02	3.51803E-02	1.18285E-02	3.91684E-03	1.27217E-03	4.01647E-04	1.23861E-04
	1.500	1.572	1.595	1.622	1.663	1.697	
10^{-36}	9.94910E-02	3.51803E-02	1.18285E-02	3.91684E-03	1.27217E-03	4.01647E-04	1.23861E-04
	1.500	1.572	1.595	1.622	1.663	1.697	
10^{-38}	9.94910E-02	3.51803E-02	1.18285E-02	3.91684E-03	1.27217E-03	4.01647E-04	1.23861E-04
	1.500	1.572	1.595	1.622	1.663	1.697	
10^{-40}	9.94910E-02	3.51803E-02	1.18285E-02	3.91684E-03	1.27217E-03	4.01647E-04	1.23861E-04
	1.500	1.572	1.595	1.622	1.663	1.697	

Table 5.4: Maximum point-wise errors and rates of convergence by AIVM [127] for Example 5.3.2.

ε	Number of mesh points						
	16	32	64	128	256	512	1024
10^{-2}	9.30311E-02	6.28909E-02	4.01992E-02	2.46206E-02	1.44705E-02	8.26954E-03	4.63196E-03
	0.565	0.646	0.707	0.767	0.807	0.836	
10^{-4}	9.48341E-02	6.40592E-02	4.08571E-02	2.50225E-02	1.47023E-02	8.40026E-03	4.70449E-03
	0.565	0.646	0.707	0.767	0.807	0.836	
10^{-6}	9.48528E-02	6.40715E-02	4.08641E-02	2.50269E-02	1.47048E-02	8.40172E-03	4.70529E-03
	0.565	0.646	0.707	0.767	0.807	0.836	
10^{-8}	9.48530E-02	6.40716E-02	4.08642E-02	2.50269E-02	1.47048E-02	8.40173E-03	4.70530E-03
	0.565	0.646	0.707	0.767	0.807	0.836	
10^{-10}	9.48530E-02	6.40716E-02	4.08642E-02	2.50269E-02	1.47048E-02	8.40173E-03	4.70530E-03
	0.565	0.646	0.707	0.767	0.807	0.836	
10^{-12}	9.48530E-02	6.40716E-02	4.08642E-02	2.50269E-02	1.47048E-02	8.40173E-03	4.70530E-03
	0.565	0.646	0.707	0.767	0.807	0.836	
10^{-14}	9.48530E-02	6.40716E-02	4.08642E-02	2.50269E-02	1.47048E-02	8.40173E-03	4.70530E-03
	0.565	0.646	0.707	0.767	0.807	0.836	
10^{-16}	9.48530E-02	6.40716E-02	4.08642E-02	2.50269E-02	1.47048E-02	8.40173E-03	4.70530E-03
	0.565	0.646	0.707	0.767	0.807	0.836	
10^{-18}	9.48530E-02	6.40716E-02	4.08642E-02	2.50269E-02	1.47048E-02	8.40173E-03	4.70530E-03
	0.565	0.646	0.707	0.767	0.807	0.836	
10^{-20}	9.48530E-02	6.40716E-02	4.08642E-02	2.50269E-02	1.47048E-02	8.40173E-03	4.70530E-03
	0.565	0.646	0.707	0.767	0.807	0.836	
10^{-22}	9.48530E-02	6.40716E-02	4.08642E-02	2.50269E-02	1.47048E-02	8.40173E-03	4.70530E-03
	0.565	0.646	0.707	0.767	0.807	0.836	
10^{-24}	9.48530E-02	6.40716E-02	4.08642E-02	2.50269E-02	1.47048E-02	8.40173E-03	4.70530E-03
	0.565	0.646	0.707	0.767	0.807	0.836	
10^{-26}	9.48530E-02	6.40716E-02	4.08642E-02	2.50269E-02	1.47048E-02	8.40173E-03	4.70530E-03
	0.565	0.646	0.707	0.767	0.807	0.836	
10^{-28}	9.48530E-02	6.40716E-02	4.08642E-02	2.50269E-02	1.47048E-02	8.40173E-03	4.70530E-03
	0.565	0.646	0.707	0.767	0.807	0.836	
10^{-30}	9.48530E-02	6.40716E-02	4.08642E-02	2.50269E-02	1.47048E-02	8.40173E-03	4.70530E-03
	0.565	0.646	0.707	0.767	0.807	0.836	
10^{-32}	9.48530E-02	6.40716E-02	4.08642E-02	2.50269E-02	1.47048E-02	8.40173E-03	4.70530E-03
	0.565	0.646	0.707	0.767	0.807	0.836	
10^{-34}	9.48530E-02	6.40716E-02	4.08642E-02	2.50269E-02	1.47048E-02	8.40173E-03	4.70530E-03
	0.565	0.646	0.707	0.767	0.807	0.836	
10^{-36}	9.48530E-02	6.40716E-02	4.08642E-02	2.50269E-02	1.47048E-02	8.40173E-03	4.70530E-03
	0.565	0.646	0.707	0.767	0.807	0.836	
10^{-38}	9.48530E-02	6.40716E-02	4.08642E-02	2.50269E-02	1.47048E-02	8.40173E-03	4.70530E-03
	0.565	0.646	0.707	0.767	0.807	0.836	
10^{-40}	9.48530E-02	6.40716E-02	4.08642E-02	2.50269E-02	1.47048E-02	8.40173E-03	4.70530E-03
	0.565	0.646	0.707	0.767	0.807	0.836	

Table 5.5: Maximum point-wise errors and rates of convergence by the proposed technique for Example 5.3.3.

ϵ	Number of mesh points						
	16	32	64	128	256	512	1024
10^{-4}	1.36308E-01	5.93797E-02	2.03813E-02	6.08372E-03	1.96321E-03	6.14671E-04	1.89030E-04
	1.199	1.543	1.744	1.632	1.675	1.701	
10^{-6}	1.36308E-01	5.93797E-02	2.03813E-02	6.08372E-03	1.96321E-03	6.14671E-04	1.89030E-04
	1.199	1.543	1.744	1.632	1.675	1.701	
10^{-8}	1.36308E-01	5.93797E-02	2.03813E-02	6.08372E-03	1.96321E-03	6.14671E-04	1.89030E-04
	1.199	1.543	1.744	1.632	1.675	1.701	
10^{-10}	1.36308E-01	5.93797E-02	2.03813E-02	6.08372E-03	1.96321E-03	6.14671E-04	1.89030E-04
	1.199	1.543	1.744	1.632	1.675	1.701	
10^{-12}	1.36308E-01	5.93797E-02	2.03813E-02	6.08372E-03	1.96321E-03	6.14671E-04	1.89030E-04
	1.199	1.543	1.744	1.632	1.675	1.701	
10^{-14}	1.36308E-01	5.93797E-02	2.03813E-02	6.08372E-03	1.96321E-03	6.14671E-04	1.89030E-04
	1.199	1.543	1.744	1.632	1.675	1.701	
10^{-16}	1.36308E-01	5.93797E-02	2.03813E-02	6.08372E-03	1.96321E-03	6.14671E-04	1.89030E-04
	1.199	1.543	1.744	1.632	1.675	1.701	
10^{-18}	1.36308E-01	5.93797E-02	2.03813E-02	6.08372E-03	1.96321E-03	6.14671E-04	1.89030E-04
	1.199	1.543	1.744	1.632	1.675	1.701	
10^{-20}	1.36308E-01	5.93797E-02	2.03813E-02	6.08372E-03	1.96321E-03	6.14671E-04	1.89030E-04
	1.199	1.543	1.744	1.632	1.675	1.701	
10^{-22}	1.36308E-01	5.93797E-02	2.03813E-02	6.08372E-03	1.96321E-03	6.14671E-04	1.89030E-04
	1.199	1.543	1.744	1.632	1.675	1.701	
10^{-24}	1.36308E-01	5.93797E-02	2.03813E-02	6.08372E-03	1.96321E-03	6.14671E-04	1.89030E-04
	1.199	1.543	1.744	1.632	1.675	1.701	
10^{-26}	1.36308E-01	5.93797E-02	2.03813E-02	6.08372E-03	1.96321E-03	6.14671E-04	1.89030E-04
	1.199	1.543	1.744	1.632	1.675	1.701	
10^{-28}	1.36308E-01	5.93797E-02	2.03813E-02	6.08372E-03	1.96321E-03	6.14671E-04	1.89030E-04
	1.199	1.543	1.744	1.632	1.675	1.701	
10^{-30}	1.36308E-01	5.93797E-02	2.03813E-02	6.08372E-03	1.96321E-03	6.14671E-04	1.89030E-04
	1.199	1.543	1.744	1.632	1.675	1.701	
10^{-32}	1.36308E-01	5.93797E-02	2.03813E-02	6.08372E-03	1.96321E-03	6.14671E-04	1.89030E-04
	1.199	1.543	1.744	1.632	1.675	1.701	
10^{-34}	1.36308E-01	5.93797E-02	2.03813E-02	6.08372E-03	1.96321E-03	6.14671E-04	1.89030E-04
	1.199	1.543	1.744	1.632	1.675	1.701	
10^{-36}	1.36308E-01	5.93797E-02	2.03813E-02	6.08372E-03	1.96321E-03	6.14671E-04	1.89030E-04
	1.199	1.543	1.744	1.632	1.675	1.701	
10^{-38}	1.36308E-01	5.93797E-02	2.03813E-02	6.08372E-03	1.96321E-03	6.14671E-04	1.89030E-04
	1.199	1.543	1.744	1.632	1.675	1.701	
10^{-40}	1.36308E-01	5.93797E-02	2.03813E-02	6.08372E-03	1.96321E-03	6.14671E-04	1.89030E-04
	1.199	1.543	1.744	1.632	1.675	1.701	

Table 5.6: Maximum point-wise errors and rates of convergence by AIVM [127] for Example 5.3.3.

ε	Number of mesh points						
	16	32	64	128	256	512	1024
10^{-4}	4.91645E-02 0.489	3.50415E-02 0.473	2.52464E-02 0.623	1.63931E-02 0.702	1.00745E-02 0.769	5.91362E-03 0.810	3.37241E-03
10^{-6}	4.91627E-02 0.489	3.50415E-02 0.473	2.52464E-02 0.623	1.63931E-02 0.702	1.00745E-02 0.769	5.91362E-03 0.810	3.37241E-03
10^{-8}	4.91627E-02 0.489	3.50415E-02 0.473	2.52464E-02 0.623	1.63931E-02 0.702	1.00745E-02 0.769	5.91362E-03 0.810	3.37241E-03
10^{-10}	4.91627E-02 0.489	3.50415E-02 0.473	2.52464E-02 0.623	1.63931E-02 0.702	1.00745E-02 0.769	5.91362E-03 0.810	3.37241E-03
10^{-12}	4.91627E-02 0.489	3.50415E-02 0.473	2.52464E-02 0.623	1.63931E-02 0.702	1.00745E-02 0.769	5.91362E-03 0.810	3.37241E-03
10^{-14}	4.91627E-02 0.489	3.50415E-02 0.473	2.52464E-02 0.623	1.63931E-02 0.702	1.00745E-02 0.769	5.91362E-03 0.810	3.37241E-03
10^{-16}	4.91627E-02 0.489	3.50415E-02 0.473	2.52464E-02 0.623	1.63931E-02 0.702	1.00745E-02 0.769	5.91362E-03 0.810	3.37241E-03
10^{-18}	4.91627E-02 0.489	3.50415E-02 0.473	2.52464E-02 0.623	1.63931E-02 0.702	1.00745E-02 0.769	5.91362E-03 0.810	3.37241E-03
10^{-20}	4.91627E-02 0.489	3.50415E-02 0.473	2.52464E-02 0.623	1.63931E-02 0.702	1.00745E-02 0.769	5.91362E-03 0.810	3.37241E-03
10^{-22}	4.91627E-02 0.489	3.50415E-02 0.473	2.52464E-02 0.623	1.63931E-02 0.702	1.00745E-02 0.769	5.91362E-03 0.810	3.37241E-03
10^{-24}	4.91627E-02 0.489	3.50415E-02 0.473	2.52464E-02 0.623	1.63931E-02 0.702	1.00745E-02 0.769	5.91362E-03 0.810	3.37241E-03
10^{-26}	4.91627E-02 0.489	3.50415E-02 0.473	2.52464E-02 0.623	1.63931E-02 0.702	1.00745E-02 0.769	5.91362E-03 0.810	3.37241E-03
10^{-28}	4.91627E-02 0.489	3.50415E-02 0.473	2.52464E-02 0.623	1.63931E-02 0.702	1.00745E-02 0.769	5.91362E-03 0.810	3.37241E-03
10^{-30}	4.91627E-02 0.489	3.50415E-02 0.473	2.52464E-02 0.623	1.63931E-02 0.702	1.00745E-02 0.769	5.91362E-03 0.810	3.37241E-03
10^{-32}	4.91627E-02 0.489	3.50415E-02 0.473	2.52464E-02 0.623	1.63931E-02 0.702	1.00745E-02 0.769	5.91362E-03 0.810	3.37241E-03
10^{-34}	4.91627E-02 0.489	3.50415E-02 0.473	2.52464E-02 0.623	1.63931E-02 0.702	1.00745E-02 0.769	5.91362E-03 0.810	3.37241E-03
10^{-36}	4.91627E-02 0.489	3.50415E-02 0.473	2.52464E-02 0.623	1.63931E-02 0.702	1.00745E-02 0.769	5.91362E-03 0.810	3.37241E-03
10^{-38}	4.91627E-02 0.489	3.50415E-02 0.473	2.52464E-02 0.623	1.63931E-02 0.702	1.00745E-02 0.769	5.91362E-03 0.810	3.37241E-03
10^{-40}	4.91627E-02 0.489	3.50415E-02 0.473	2.52464E-02 0.623	1.63931E-02 0.702	1.00745E-02 0.769	5.91362E-03 0.810	3.37241E-03

(a) Example 5.1

(b) Example 5.2

(c) Example 5.3

Figure 5.2: Loglog plots of N Vs. maximum error.

Chapter 6

System of Singularly Perturbed Boundary Value Problems with Smooth Data

Boundary value problems for systems of singularly perturbed differential equation often occur, for examples, in modeling and analysis of heat and mass transfer processes when the thermal conductivity and diffusion coefficients are small and (or) the rate of reaction is large, the processes of mathematical modeling of chemical reactions with point sources taken into account. Other applications include equations of predator-prey population dynamics, $p-n$ junction in semiconductor devices, optimum control problems in certain resistance-capacitor electrical circuits, etc.

The outline of this chapter is as follows: Section 6.1 deals with system of convection-diffusion singularly perturbed differential equations. System of reaction-diffusion singularly perturbed differential equations are taken in Section 6.2. Some numerical results computed from the proposed schemes are presented in Section 6.3. Certain conclusions based on the analysis are finally presented in Section 6.4.

6.1 System of convection-diffusion problems

In this chapter, we treat the following system of two singularly perturbed convection-diffusion problems:

$$L_1\vec{u} \equiv -\epsilon u_1''(x) - a_1(x)u_1'(x) + b_{11}(x)u_1(x) + b_{12}(x)u_2(x) = f_1(x), \tag{6.1.1}$$

$$L_2\vec{u} \equiv -\mu u_2''(x) - a_2(x)u_2'(x) + b_{21}(x)u_1(x) + b_{22}(x)u_2(x) = f_2(x), \tag{6.1.2}$$

where $\vec{u} = (u_1, u_2)^T$, $x \in \Omega = (0,1)$ with the boundary conditions:

$$\vec{u}(0) = \begin{pmatrix} p \\ r \end{pmatrix}, \tag{6.1.3}$$

$$\vec{u}(1) = \begin{pmatrix} q \\ s \end{pmatrix}. \tag{6.1.4}$$

Without loss of generality, we shall assume that $0 < \epsilon \leq \mu \ll 1$. The functions $a_1(x)$, $a_2(x)$, $b_{11}(x)$, $b_{12}(x)$, $b_{21}(x)$, $b_{22}(x)$, $f_1(x)$, $f_2(x)$ are sufficiently smooth and satisfy the following inequalities:

$$(i) \qquad a_1(x) \geq \alpha_1 > 0, \quad a_2(x) \geq \alpha_2 > 0, \tag{6.1.5}$$

$$(ii) \qquad b_{11}(x) > |b_{12}(x)|, \quad b_{22}(x) > |b_{21}(x)|, \tag{6.1.6}$$

$$(iii) \qquad b_{12}(x) < 0, \quad b_{21}(x) < 0, \tag{6.1.7}$$

$\forall x \in \overline{\Omega} = [0, 1]$ and $u_1, u_2 \in C^2(\Omega) \cap C(\overline{\Omega})$.

In this section, our main objective is to construct a modified initial value technique (MIVT) for (6.1.1-6.1.4) which is based on the underlying idea of AIVM [125]. First, in this technique, an asymptotic expansion approximation for the solution of the Boundary Value Problem (BVP) (6.1.1-6.1.4) has been constructed. Then, Initial Value Problems (IVPs) and Terminal Value Problems (TVPs) are formulated whose solutions are the terms of this asymptotic expansion. The IVPs are happened to be SPPs and therefore, they are solved by the proposed hybrid scheme. The MIVT displays uniform convergence with respect to the perturbation parameter ϵ. Here we deal with the case $0 < \epsilon = \mu \ll 1$

Assumption: We shall assume that $\epsilon \leq CN^{-1}$ as is generally the case for discretization of convection-dominated problems [115].

6.1.1 Maximum principle and stability result

In this sub-section, a maximum principle and a stability result are presented for the BVP (6.1.1-6.1.4).

Lemma 6.1.1 *Consider the system of BVP (6.1.1-6.1.4). If $L_1\vec{y} \geq 0$, $L_2\vec{y} \geq 0$ in Ω and $\vec{y}(0) \geq \vec{0}$, $\vec{y}(1) \geq \vec{0}$, then $\vec{y}(x) \geq \vec{0}$ in $\overline{\Omega}$.*

Proof. Let $y_1(p) = \min_{\overline{\Omega}} y_1(x)$ and $y_2(q) = \min_{\overline{\Omega}} y_2(x)$. Assume without loss of generality that $y_1(p) \leq y_2(q)$. Assume also that $y_1(p) < 0$, which will lead to a contradiction. Note that $p \neq 0, 1$ and $y_1'(p) = 0$, $y_1''(p) \geq 0$.

$$\begin{aligned} L_1\vec{y} &= -\epsilon y_1''(p) - a_1(p)y_1'(p) + b_{11}(p)y_1(p) + b_{12}(p)y_2(p), && (6.1.8) \\ &= -\epsilon y_1''(p) + (b_{11}(p) + b_{12}(p))y_1(p) + (y_2(p) - y_1(p))b_{12}(p) < 0. && (6.1.9) \end{aligned}$$

which contradicts the hypotheses of the lemma.

An immediate consequence is the following result.

Lemma 6.1.2 *If $\vec{y}(x)$ is the solution of BVP (6.1.1-6.1.4), then*

$$\parallel \vec{y}(x) \parallel \leq C \max\left\{ \parallel \vec{y}(0) \parallel, \parallel \vec{y}(1) \parallel, \max_{x\in\overline{\Omega}} \mid L_1\vec{y} \mid, \max_{x\in\overline{\Omega}} \mid L_2\vec{y} \mid \right\} \textit{for all } x \in \overline{\Omega},$$

where C is a constant independent of x and ϵ.

6.1.2 Asymptotic expansion approximation

An asymptotic expansion approximation for the solution of BVP (6.1.1-6.1.4) can be constructed as

$$\vec{u}_{as}(x) = \vec{u}_R(x) + \vec{v}(x) + O(\epsilon), \tag{6.1.10}$$

where $\vec{u}_R(x) = \begin{pmatrix} u_{R1}(x) \\ u_{R2}(x) \end{pmatrix}$ is the solution of the reduced problem of (6.1.1-6.1.4) and is given by

$$\begin{aligned} -a_1(x)u'_{R1}(x) + b_{11}(x)u_{R1}(x) + b_{12}(x)u_{R2}(x) &= f_1(x), & (6.1.11)\\ -a_2(x)u'_{R2}(x) + b_{21}(x)u_{R1}(x) + b_{22}(x)u_{R2}(x) &= f_2(x), \quad x \in [0,1), & (6.1.12)\end{aligned}$$

$$\vec{u}_R(1) = \begin{pmatrix} q \\ s \end{pmatrix}. \tag{6.1.13}$$

$\vec{v}(x) = \begin{pmatrix} v_1(x) \\ v_2(x) \end{pmatrix}$ is a ”left boundary layer correction” given by $v_1(x) = [p - u_{R1}(0)]v_{L1}(x)$,

$$v_{L1}(x) = \exp\left\{ -\int_0^x \left(\frac{a_1(s)}{\varepsilon} + \frac{[b_{11}(s) + b_{12}(s)]}{a_1(s)} \right) ds \right\}, \tag{6.1.14}$$

and $v_2(x) = [r - u_{R2}(0)]v_{L2}(x)$,

$$v_{L2}(x) = \exp\left\{ -\int_0^x \left(\frac{a_2(s)}{\varepsilon} + \frac{[b_{21}(s) + b_{22}(s)]}{a_2(s)} \right) ds \right\}. \tag{6.1.15}$$

It is easy to verify that $v_{L1}(x)$ and $v_{L2}(x)$ satisfy the following IVPs respectively.

$$\varepsilon v'_{L1}(x) + \frac{a_1^2(x) + \epsilon[b_{11}(x) + b_{12}(x)]}{a_1(x)} v_{L1}(x) = 0, \tag{6.1.16}$$

$$v_{L1}(0) = 1, \tag{6.1.17}$$

$$\varepsilon v'_{L2}(x) + \frac{a_2^2(x) + \epsilon[b_{21}(x) + b_{22}(x)]}{a_2(x)} v_{L2}(x) = 0, \tag{6.1.18}$$

$$v_{L2}(0) = 1. \tag{6.1.19}$$

Theorem 6.1.3 *[125] The zeroth order asymptotic expansion approximation $\vec{u}_{as}$ satisfies the inequality*

$$\| (\vec{u} - \vec{u}_{as})(x) \| \leq C\epsilon,$$

where $\vec{u}(x)$ is the solution of BVP (6.1.1-6.1.4).

Proof. We have

$$\begin{aligned} L((\vec{u} - \vec{u}_{as})(x)) &= \begin{pmatrix} \epsilon u''_{R1}(x) \\ \epsilon u''_{R2}(x) \end{pmatrix} \\ &- \begin{pmatrix} [p - u_{R1}(0)](-\epsilon v''_{L1}(x) - a_1(x)v'_{L1}(x) + b_{11}(x)v_{L1}(x) + b_{12}(x)v_{L2}(x)) \\ [r - u_{R2}(0)](-\epsilon v''_{L2}(x) - a_2(x)v'_{L2}(x) + b_{21}(x)v_{L1}(x) + b_{22}(x)v_{L2}(x)) \end{pmatrix}. \end{aligned}$$

It can be proved that

$$L((\vec{u} - \vec{u}_{as})(x)) \leq C_1\epsilon + C_2 \begin{pmatrix} \exp\left(-\left[\frac{\alpha}{\epsilon} + \frac{\beta}{\alpha}\right] x\right) \\ \exp\left(-\left[\frac{\alpha}{\epsilon} + \frac{\beta}{\alpha}\right] x\right) \end{pmatrix} \tag{6.1.20}$$

and

$$L((\vec{u} - \vec{u}_{as})(x)) \geq -C_1\epsilon - C_2 \begin{pmatrix} \exp\left(-\left[\frac{\alpha}{\epsilon} + \frac{\beta}{\alpha}\right] x\right) \\ \exp\left(-\left[\frac{\alpha}{\epsilon} + \frac{\beta}{\alpha}\right] x\right) \end{pmatrix}. \tag{6.1.21}$$

where $\beta = \min_{\overline{\Omega}}\{b_{11}(x) + b_{12}(x), b_{21}(x) + b_{22}(x)\}$ and $\alpha = \min_{\overline{\Omega}}\{a_1(x), a_2(x)\}$.

Define the barrier functions $\vec{\phi}^{\pm}(x)$as

$$\vec{\phi}^{\pm}(x) = C_4\epsilon \begin{pmatrix} 1-\frac{x}{2} \\ 1-\frac{x}{2} \end{pmatrix} + C_4\epsilon \begin{pmatrix} \exp\left(-\frac{1}{2}\left[\frac{\alpha}{\epsilon}+\frac{\beta}{\alpha}\right]x\right) \\ \exp\left(-\frac{1}{2}\left[\frac{\alpha}{\epsilon}+\frac{\beta}{\alpha}\right]x\right) \end{pmatrix} \pm (\vec{u}-\vec{u}_{as})(x), \tag{6.1.22}$$

where the positive number C_4 is to be chosen suitably. We have

$$\vec{\phi}^{\pm}(0) = C_4\epsilon \begin{pmatrix} 2 \\ 2 \end{pmatrix} \geq \vec{0} \tag{6.1.23}$$

and

$$\begin{aligned} \vec{\phi}^{\pm}(1) &= C_4\epsilon \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} + C_4\epsilon \begin{pmatrix} \exp\left(-\frac{1}{2}\left[\frac{\alpha}{\epsilon}+\frac{\beta}{\alpha}\right]\right) \\ \exp\left(-\frac{1}{2}\left[\frac{\alpha}{\epsilon}+\frac{\beta}{\alpha}\right]\right) \end{pmatrix} \\ &\mp \begin{pmatrix} [p-u_{R1}(0)]\exp\left\{-\int_0^1\left(\frac{a_1(x)}{\varepsilon}+\frac{[b_{11}(x)+b_{12}(x)]}{a_1(x)}\right)dx\right\} \\ [q-u_{R2}(0)]\exp\left\{-\int_0^1\left(\frac{a_2(x)}{\varepsilon}+\frac{[b_{21}(x)+b_{22}(x)]}{a_2(x)}\right)dx\right\} \end{pmatrix} \geq \vec{0} \end{aligned} \tag{6.1.24}$$

by a proper choice of C_4. Furthermore,

$$\begin{aligned} L\vec{\phi}^{+}(x) &= \begin{pmatrix} -\epsilon\frac{d^2}{dx^2} & 0 \\ 0 & -\epsilon\frac{d^2}{dx^2} \end{pmatrix} \vec{\phi}^{+}(x) + \begin{pmatrix} -a_1(x)\frac{d}{dx} & 0 \\ 0 & -a_2(x)\frac{d}{dx} \end{pmatrix} \vec{\phi}^{+}(x) \\ &+ \begin{pmatrix} b_{11}(x) & b_{12}(x) \\ b_{21}(x) & b_{22}(x) \end{pmatrix} \vec{\phi}^{+}(x) + L((\vec{u}-\vec{u}_{as})(x)) \\ &\geq \epsilon \begin{pmatrix} \frac{C_4\alpha}{2}-C_1 \\ \frac{C_4\alpha}{2}-C_1 \end{pmatrix} + \begin{pmatrix} \left(\frac{C_4\alpha^2}{4}-C_2\right)\exp\left(-\frac{1}{2}\left[\frac{\alpha}{\epsilon}+\frac{\beta}{\alpha}\right]x\right) \\ \left(\frac{C_4\alpha^2}{4}-C_2\right)\exp\left(-\frac{1}{2}\left[\frac{\alpha}{\epsilon}+\frac{\beta}{\alpha}\right]x\right) \end{pmatrix} \\ &+ \begin{pmatrix} C_4\beta\epsilon(1-\frac{x}{2}) \\ C_4\beta\epsilon(1-\frac{x}{2}) \end{pmatrix} + \begin{pmatrix} C_4\beta\epsilon(1-\frac{\beta\epsilon}{4\alpha^2})\exp\left(-\frac{1}{2}\left[\frac{\alpha}{\epsilon}+\frac{\beta}{\alpha}\right]\right) \\ C_4\beta\epsilon\left(1-\frac{\beta\epsilon}{4\alpha^2}\right)\exp\left(-\frac{1}{2}\left[\frac{\alpha}{\epsilon}+\frac{\beta}{\alpha}\right]\right) \end{pmatrix} \\ &\geq \vec{0}, \quad x \in \Omega \end{aligned} \tag{6.1.25}$$

by a proper choice c_4. Similarly, it can be shown that $L\vec{\phi}^-(x) \geq \vec{0}$. Applying Lemma 6.1.1 we get

$$\vec{\phi}^{\pm}(x) \geq \vec{0}, \quad \text{for all } x \in \Omega. \tag{6.1.26}$$

Therefore,

$$\begin{pmatrix} | u_1(x) - u_{1as}(x) | \\ | u_2(x) - u_{2as}(x) | \end{pmatrix} \leq C_4\epsilon \begin{pmatrix} 1 - \frac{x}{2} \\ 1 - \frac{x}{2} \end{pmatrix} + C_4\epsilon \begin{pmatrix} \exp\left(-\frac{1}{2}\left[\frac{\alpha}{\epsilon} + \frac{\beta}{\alpha}\right]x\right) \\ \exp\left(-\frac{1}{2}\left[\frac{\alpha}{\epsilon} + \frac{\beta}{\alpha}\right]x\right) \end{pmatrix} \tag{6.1.27}$$

Hence,

$$\| (\vec{u} - \vec{u}_{as})(x) \| \leq C\epsilon. \tag{6.1.28}$$

6.1.3 Description of method

In this sub-section, we describe the MIVT to solve the (SPBVP) (6.1.1-6.1.4)

Step 1. Solve the TVP (6.1.11-6.1.12) by using a Trapezoidal method. Let $\vec{u}_{R,i} = \begin{pmatrix} u_{R1,i} \\ u_{R2,i} \end{pmatrix}$ be its solution.

Step 2. Solve the IVP (6.1.16-6.1.17) by using the hybrid scheme (2.1.4-2.1.5) described on Shishkin mesh in sub-section 2.1.2. Let $v_{L1,i}$ be its solution.

Step 3. Solve the IVP (6.1.18-6.1.19) by using the hybrid scheme. Let $v_{L2,i}$ be its solution.

Step 4. Define mesh function $\vec{U}_i$ as

$$\vec{U}_i = \begin{pmatrix} u_{1,i} \\ u_{2,i} \end{pmatrix} = \begin{pmatrix} u_{R1,i} \\ u_{R2,i} \end{pmatrix} + \begin{pmatrix} [p - u_{R1}(0)]v_{L1,i} \\ [r - u_{R2}(0)]v_{L2,i} \end{pmatrix}. \tag{6.1.29}$$

6.1.4 An error estimate

In this sub-section, an error estimate is derived for the numerical solution obtained by the MIVT.

Theorem 6.1.4 *Let $\vec{u}(x)$ be the solution of the BVP (6.1.1-6.1.4) and $\vec{U}_i$ be the numerical solution obtained by MIVT, then we have*

$$\| \vec{u}(x_i) - \vec{U}_i \| \leq C \left[N^{-2} \ln^2 N + N^{-1}\varepsilon + N^{-\beta\sigma_0} + \epsilon \right].$$

Proof. Theorem 2.1.2, when applied to the IVPs (6.1.16-6.1.17), (6.1.18-6.1.19) yields

$$|v_{L1}(x_i) - v_{L1,i}| \leq C \left[N^{-2} \ln^2 N + N^{-1}\varepsilon + N^{-\beta\sigma_0} \right], \quad 0 \leq x_i \leq 1, \tag{6.1.30}$$

$$|v_{L2}(x_i) - v_{L2,i}| \leq C \left[N^{-2} \ln^2 N + N^{-1}\varepsilon + N^{-\beta\sigma_0} \right], \quad 0 \leq x_i \leq 1. \tag{6.1.31}$$

From the definitions of $\vec{u}_{as}(x)$, $\vec{U}_i$ and the above inequalities, we have

$$\| \vec{u}_{as}(x_i) - \vec{U}_i \| \leq C \left[N^{-2} \ln^2 N + N^{-1}\varepsilon + N^{-\beta\sigma_0} \right], \quad \text{for } x_i \in \Omega_\epsilon^N. \tag{6.1.32}$$

From Theorem 6.1.3, we have

$$\| \vec{u}(x_i) - \vec{u}_{as}(x_i) \| \leq C\epsilon, \quad x \in \overline{\Omega}. \tag{6.1.33}$$

The desired estimate follows from the inequalities (6.1.32) and (6.1.33).

6.2 System of reaction-diffusion problems

In this section, we treat the following system of two singularly perturbed reaction-diffusion equations:

$$L_1\vec{u} \equiv -\epsilon u_1''(x) + a_{11}(x)u_1(x) + a_{12}(x)u_2(x) = f_1(x), \tag{6.2.1}$$

$$L_2\vec{u} \equiv -\mu u_2''(x) + a_{21}(x)u_1(x) + a_{22}(x)u_2(x) = f_2(x), \tag{6.2.2}$$

where $\vec{u} = (u_1, u_2)^T$, $x \in \Omega = (0,1)$ with the boundary conditions:

$$\vec{u}(0) = \begin{pmatrix} p \\ r \end{pmatrix}, \quad \vec{u}(1) = \begin{pmatrix} q \\ s \end{pmatrix}. \tag{6.2.3}$$

Without loss of generality, we shall assume that $0 < \epsilon \leq \mu \leq 1$. The functions $a_{11}(x)$, $a_{12}(x)$, $a_{21}(x)$, $a_{22}(x)$, $f_1(x)$, $f_2(x)$ are sufficiently smooth and satisfy the following inequalities:

$$(i) \qquad a_{11}(x) > |a_{12}(x)|, \quad a_{22}(x) > |a_{21}(x)|, \quad x \in \overline{\Omega} = [0,1] \tag{6.2.4}$$

$$(ii) \qquad a_{12}(x) \leq 0, \quad a_{21}(x) \leq 0, \quad x \in \overline{\Omega}. \tag{6.2.5}$$

Shishkin [113] classifies three separate cases for a system of two singularly perturbed reaction-diffusion problems with diffusion coefficients ϵ, μ: (i) $0 < \epsilon = \mu \ll 1$ (ii) $0 < \epsilon \ll \mu = 1$ and (iii) ϵ, μ arbitrary. Matthews et al. [76] consider case (i), showing that a standard finite difference scheme is uniformly convergent on a fitted piecewise uniform mesh. They establish first-order convergence up to a logarithmic factor in the discrete maximum norm. The same authors have also obtained a similar result for case (ii), which they haven strengthened to show almost second-order convergence [77]. Madden and Stynes [**?**] obtained almost first-order convergence for the general case (iii). For case (ii), Natesan and Deb [84] developed a numerical method which is a combination of cubic spline and a finite difference scheme. Valanarasu and Ramanujam [124] proposed an asymptotic initial value method (AIVM) to solve (6.2.1-6.2.3), whose theoretical order of convergence is one.

We construct a modified initial value technique (MIVT) for (6.2.1-6.2.3) which is based on the underlying idea of AIVM [124]. The aim of the present study is to improve the order of convergence to almost second order (up to a logarithmic factor) for case (i), i.e., for $0 < \epsilon = \mu \ll 1$.

First, in this technique, an asymptotic expansion approximation for the solution of the Boundary Value Problem (BVP) (6.2.1-6.2.3) has been constructed. Then, the Initial Value Problems (IVPs) and the Terminal Value Problems (TVPs) are formulated whose solutions are the terms of this asymptotic expansion. The IVPs and TVPs are happened to be the SPPs and therefore, they are solved by the proposed hybrid scheme.

6.2.1 Maximum principle and stability result

In this sub-section, a maximum principle and a stability result are presented for the BVP (6.2.1-6.2.3).

Lemma 6.2.1 *Consider the system of BVP (6.2.1-6.2.3). If* $L_1\vec{y} \geq 0,\ L_2\vec{y} \geq 0$ *in* Ω *and* $\vec{y}(0) \geq \vec{0}$, $\vec{y}(1) \geq \vec{0}$, *then* $\vec{y}(x) \geq \vec{0}$ *in* $\overline{\Omega}$.

Proof. Let $y_1(p) = \min_{\overline{\Omega}} y_1(x)$ and $y_2(q) = \min_{\overline{\Omega}} y_2(x)$. Assume without loss of generality that $y_1(p) \leq y_2(q)$. Assume also that $y_1(p) < 0$, which will lead to a contradiction. Note that $p \neq 0, 1$ and $y_1''(p) \geq 0$.

$$\begin{aligned} L_1\vec{y} &= -\epsilon y_1''(p) + a_{11}(p)y_1(p) + a_{12}(p)y_2(p), && (6.2.6) \\ &= -\epsilon y_1''(p) + (a_{11}(p) + a_{12}(p))y_1(p) + (y_2(p) - y_1(p))a_{12}(p) < 0. && (6.2.7) \end{aligned}$$

which contradicts the hypotheses of the lemma.

An immediate consequence is the following result.

Lemma 6.2.2 *If $\vec{y}(x)$ is the solution of BVP (6.2.1-6.2.3), then*

$$\| \vec{y}(x) \| \leq \frac{1}{\gamma} \| \vec{f} \| + \| \vec{y}(0) \| + \| \vec{y}(1) \|,$$

$$\textit{where } \gamma = \min_{\Omega}\{a_{11}(x) + a_{12}(x), a_{21}(x) + a_{22}(x)\}.$$

6.2.2 Asymptotic expansion approximation

An asymptotic expansion approximation for the solution of BVP (6.2.1-6.2.3) can be constructed as

$$\vec{u}_{as}(x) = \vec{u}_R(x) + \vec{v}(x) + O(\sqrt{\epsilon}), \tag{6.2.8}$$

where $\vec{u}_R(x) = \begin{pmatrix} u_{R1}(x) \\ u_{R2}(x) \end{pmatrix}$ is the solution of the reduced problem of (6.2.1-6.2.3) and is given by

$$a_{11}(x)u_{R1}(x) + a_{12}(x)u_{R2}(x) = f_1(x), \tag{6.2.9}$$

$$a_{21}(x)u_{R1}(x) + a_{22}(x)u_{R2}(x) = f_2(x), \quad x \in [0, 1), \tag{6.2.10}$$

and $\vec{v}(x) = \begin{pmatrix} v_1(x) \\ v_2(x) \end{pmatrix}$ is given by

$$v_1(x) = [p - u_{R1}(0)] \left[\frac{a_{11}(0) + a_{12}(0)}{a_{11}(x) + a_{12}(x)}\right]^{\frac{1}{4}} v_{L1}(x) + [q - u_{R1}(1)] \left[\frac{a_{11}(1) + a_{12}(1)}{a_{11}(x) + a_{12}(x)}\right]^{\frac{1}{4}} w_{R1}(x), \tag{6.2.11}$$

$$v_2(x) = [r - u_{R2}(0)] \left[\frac{a_{21}(0) + a_{22}(0)}{a_{21}(x) + a_{22}(x)}\right]^{\frac{1}{4}} v_{L2}(x) + [s - u_{R2}(1)] \left[\frac{a_{21}(1) + a_{22}(1)}{a_{21}(x) + a_{22}(x)}\right]^{\frac{1}{4}} w_{R2}(x) \tag{6.2.12}$$

$\vec{v}_L(x) = \begin{pmatrix} v_{L1}(x) \\ v_{L2}(x) \end{pmatrix}$ is a ”left boundary layer correction” and $\vec{w}_R(x) = \begin{pmatrix} w_{R1}(x) \\ w_{R2}(x) \end{pmatrix}$ is a ”right boundary layer correction” are defined as

$$v_{L1}(x) = exp\left\{-\int_0^x \sqrt{\frac{[a_{11}(s) + a_{12}(s)]}{\epsilon}} ds\right\}, \tag{6.2.13}$$

$$v_{L2}(x) = exp\left\{-\int_0^x \sqrt{\frac{[a_{21}(s) + a_{22}(s)]}{\epsilon}} ds\right\}, \tag{6.2.14}$$

$$w_{R1}(x) = exp\left\{-\int_x^1 \sqrt{\frac{[a_{11}(s) + a_{12}(s)]}{\epsilon}} ds\right\}, \tag{6.2.15}$$

$$w_{R2}(x) = exp\left\{-\int_x^1 \sqrt{\frac{[a_{21}(s) + a_{22}(s)]}{\epsilon}} ds\right\}. \tag{6.2.16}$$

It is easy to verify that $v_{L1}(x)$, $v_{L2}(x)$, $w_{R1}(x)$ and $w_{R2}(x)$ satisfy the following IVPs and TVPs respectively:

$$\sqrt{\epsilon} v'_{L1}(x) + \sqrt{[a_{11}(x) + a_{12}(x)]} v_{L1}(x) = 0, \tag{6.2.17}$$

$$v_{L1}(0) = 1, \tag{6.2.18}$$

$$\sqrt{\epsilon} v'_{L2}(x) + \sqrt{[a_{21}(x) + a_{22}(x)]} v_{L2}(x) = 0, \tag{6.2.19}$$

$$v_{L2}(0) = 1, \tag{6.2.20}$$

$$\sqrt{\epsilon}w'_{R1}(x) - \sqrt{[a_{11}(x) + a_{12}(x)]}w_{R1}(x) = 0, \tag{6.2.21}$$

$$w_{R1}(1) = 1, \tag{6.2.22}$$

and

$$\sqrt{\epsilon}w'_{R2}(x) - \sqrt{[a_{21}(x) + a_{22}(x)]}w_{R2}(x) = 0, \tag{6.2.23}$$

$$w_{R2}(1) = 1. \tag{6.2.24}$$

Theorem 6.2.3 *[124] The zeroth order asymptotic expansion approximation $\vec{u}_{as}$ satisfies the inequality*

$$\| (\vec{u} - \vec{u}_{as})(x) \| \leq C\sqrt{\epsilon},$$

where $\vec{u}(x)$ is the solution of BVP (6.2.1-6.2.3).

Proof.

Define the barrier functions $\vec{\phi}^{\pm}(x)$ as

$$\vec{\phi}^{\pm}(x) = C_1\sqrt{\epsilon}\begin{pmatrix}1\\1\end{pmatrix} + C_1\sqrt{\epsilon}\begin{pmatrix}\exp\left(-x\sqrt{\frac{\alpha}{\epsilon}}\right)\\ \exp\left(-x\sqrt{\frac{\alpha}{\epsilon}}\right)\end{pmatrix} + \begin{pmatrix}\exp\left(-(1-x)\sqrt{\frac{\alpha}{\epsilon}}\right)\\ \exp\left(-(1-x)\sqrt{\frac{\alpha}{\epsilon}}\right)\end{pmatrix} \pm (\vec{u} - \vec{u}_{as})(x), \tag{6.2.25}$$

where the positive number C_1 is to be chosen suitably. We have

$$\vec{\phi}^{\pm}(0) = C_1\sqrt{\epsilon}\begin{pmatrix}2\\2\end{pmatrix} + C\sqrt{\epsilon}\begin{pmatrix}\exp\left(\sqrt{\frac{-\alpha}{\epsilon}}\right)\\ \exp\left(\sqrt{\frac{-\alpha}{\epsilon}}\right)\end{pmatrix} \pm \vec{w}_R(0) \geq \vec{0} \tag{6.2.26}$$

and

$$\vec{\phi}^{\pm}(1) = C_1\sqrt{\epsilon}\begin{pmatrix}2\\2\end{pmatrix} + C_1\sqrt{\epsilon}\begin{pmatrix}\exp\left(\sqrt{\frac{-\alpha}{\epsilon}}\right)\\ \exp\left(\sqrt{\frac{-\alpha}{\epsilon}}\right)\end{pmatrix} \pm \vec{v}_L(1) \geq \vec{0} \tag{6.2.27}$$

by a proper choice of C_1. Furthermore,

$$\begin{aligned}
L\vec{\phi}^{+}(x) &= \begin{pmatrix} -\epsilon\frac{d^2}{dx^2} & 0 \\ 0 & -\epsilon\frac{d^2}{dx^2} \end{pmatrix} \vec{\phi}^{+}(x) + \begin{pmatrix} a_{11}(x) & a_{12}(x) \\ a_{21}(x) & a_{22}(x) \end{pmatrix} \vec{\phi}^{+}(x) + +L((\vec{u} - \vec{u}_{as})(x))[12pt] \\
&\geq \sqrt{\epsilon}\begin{pmatrix} C_1(a_{11}(x) + a_{12}(x)) - C \\ C_1(a_{21}(x) + a_{22}(x)) - C \end{pmatrix} + \begin{pmatrix} (C_1\sqrt{\epsilon}(a_{11}(x) + a_{12}(x) - \alpha) - C)\exp\left(-x\sqrt{\frac{\alpha}{\epsilon}}\right) \\ (C_1\sqrt{\epsilon}(a_{21}(x) + a_{22}(x) - \alpha) - C)\exp\left(-x\sqrt{\frac{\alpha}{\epsilon}}\right) \end{pmatrix} \\
&\quad + \begin{pmatrix} (C_1\sqrt{\epsilon}(a_{11}(x) + a_{12}(x) - \alpha) - C)\exp\left(-(1-x)\sqrt{\frac{\alpha}{\epsilon}}\right) \\ (C_1\sqrt{\epsilon}(a_{21}(x) + a_{22}(x) - \alpha) - C)\exp\left(-(1-x)\sqrt{\frac{\alpha}{\epsilon}}\right) \end{pmatrix} \\
&\geq \vec{0}, \quad x \in \Omega \qquad (6.2.28)
\end{aligned}$$

by a proper choice C_1. Similarly, it can be shown that $L\vec{\phi}^{-}(x) \geq \vec{0}$. Applying Lemma 6.2.1 we get

$$\vec{\phi}^{\pm}(x) \geq \vec{0}, \quad \text{for all } x \in \Omega. \qquad (6.2.29)$$

Therefore,

$$\mid (\vec{u} - \vec{u}_{as})(x) \mid \leq \vec{\phi}^{\pm}(x) \leq C\sqrt{\epsilon}, \quad \text{for all } x \in \Omega. \qquad (6.2.30)$$

It is obvious that

$$\mid (\vec{u} - \vec{u}_{as})(x) \mid \leq C\sqrt{\epsilon}, \quad \text{for } x = 0 \text{ and } x = 1. \qquad (6.2.31)$$

Hence,

$$\parallel (\vec{u} - \vec{u}_{as})(x) \parallel \leq C\sqrt{\epsilon}. \qquad (6.2.32)$$

6.2.3 Proposed scheme

A fitted mesh method for the problem (6.2.1-6.2.3) is now introduced. On $\overline{\Omega}$ a piecewise uniform mesh of N mesh intervals is constructed as given in sub-section 4.2.2. Applying hybrid scheme (2.1.4-2.1.5) for (6.2.17-6.2.18), we get

$$L_\epsilon^N V_{L1,i} \equiv \begin{cases} \varepsilon D^- V_{L1,i} + \dfrac{\sqrt{a_{11,i-1}+a_{12,i-1}}V_{L1,i-1} + \sqrt{a_{11,i}+a_{12,i}}V_{L1,i}}{2} = 0, \quad 0 < i \le N/4,\ 3N/4 < i \le N, \\ \\ \varepsilon D^- V_{L1,i} + \sqrt{a_{11,i}+a_{12,i}}V_{L1,i} = 0, \quad N/4 < i \le 3N/4, \end{cases} \tag{6.2.33}$$

$$V_{L1,0} = 1. \tag{6.2.34}$$

Similarly, we can formulate the hybrid scheme for (6.2.19-6.2.20), (6.2.21-6.2.22) and (6.2.23-6.2.24).

6.2.4 Description of method

In this sub-section, we describe the MIVT to solve (6.2.1-6.2.3)

Step 1. Solve the IVP (6.2.17-6.2.18) by using the hybrid scheme described on Shishkin mesh in Section 6.2.3. Let $V_{L1,i}$ be its solution.

Step 2. Solve the IVP (6.2.19-6.2.20) by using the hybrid scheme. Let $V_{L2,i}$ be its solution.

Step 3. Solve the TVP (6.2.21-6.2.22) by using the hybrid scheme. Let $W_{R1,i}$ be its solution.

Step 4. Solve the TVP (6.2.23-6.2.24) by using the hybrid scheme. Let $W_{R2,i}$ be its solution.

Step 5. Define mesh function $\vec{U_i}$ as

$$\vec{U_i} = \begin{pmatrix} U_{1,i} \\ U_{2,i} \end{pmatrix} = \begin{pmatrix} u_{R1,i} \\ u_{R2,i} \end{pmatrix} + \begin{pmatrix} [p - u_{R1}(0)] \left[\dfrac{a_{11}(0)+a_{12}(0)}{a_{11}(x_i)+a_{12}(x_i)}\right]^{\frac{1}{4}} V_{L1,i} \\ [r - u_{R2}(0)] \left[\dfrac{a_{21}(0)+a_{22}(0)}{a_{21}(x_i)+a_{22}(x_i)}\right]^{\frac{1}{4}} V_{L2,i} \end{pmatrix}$$
$$+ \begin{pmatrix} [q - u_{R1}(1)] \left[\dfrac{a_{11}(1)+a_{12}(1)}{a_{11}(x_i)+a_{12}(x_i)}\right]^{\frac{1}{4}} W_{R1,i} \\ [s - u_{R2}(1)] \left[\dfrac{a_{21}(1)+a_{22}(1)}{a_{21}(x_i)+a_{22}(x_i)}\right]^{\frac{1}{4}} W_{R2,i} \end{pmatrix}. \tag{6.2.35}$$

Now, an error estimate is derived for the numerical solution obtained by the MIVT.

Theorem 6.2.4 *Let $\vec{u}(x)$ be the solution of the BVP (6.2.1-6.2.3) and $\vec{U}_i$ be the numerical solution obtained by MIVT, then we have*

$$\parallel \vec{U}_i - \vec{u}(x_i) \parallel \leq \; C \left[N^{-2} \ln^2 N + N^{-1} \varepsilon + N^{-\sqrt{\beta}\sigma_0} + \sqrt{\epsilon} \right].$$

Proof. Theorem 2.1.1, when applied to the IVPs (6.2.17-6.2.18), (6.2.19-6.2.20) and TVPs (6.2.21-6.2.22), (6.2.23-6.2.24) yields

$$|V_{L1,i} - v_{L1}(x_i)| \leq C \left[N^{-2} \ln^2 N + N^{-1} \varepsilon + N^{-\sqrt{\beta}\sigma_0} \right], 0 \leq x_i \leq 1, \tag{6.2.36}$$

$$|V_{L2,i} - v_{L2}(x_i)| \leq C \left[N^{-2} \ln^2 N + N^{-1} \varepsilon + N^{-\sqrt{\beta}\sigma_0} \right], 0 \leq x_i \leq 1, \tag{6.2.37}$$

$$|W_{R1,i} - w_{R1}(x_i)| \leq C \left[N^{-2} \ln^2 N + N^{-1} \varepsilon + N^{-\sqrt{\beta}\sigma_0} \right], 0 \leq x_i \leq 1, \tag{6.2.38}$$

$$|W_{R2,i} - w_{R2}(x_i)| \leq C \left[N^{-2} \ln^2 N + N^{-1} \varepsilon + N^{-\sqrt{\beta}\sigma_0} \right], 0 \leq x_i \leq 1. \tag{6.2.39}$$

From the definitions of $\vec{u}_{as}(x)$, $\vec{U}_i$ and above inequalities, we have

$$\parallel \vec{u}_{as}(x_i) - \vec{U}_i \parallel \leq \; C \left[N^{-2} \ln^2 N + N^{-1} \varepsilon + N^{-\sqrt{\beta}\sigma_0} \right], \text{for } x_i \in \Omega_\epsilon^N. \tag{6.2.40}$$

From theorem 6.2.3, we have

$$\parallel \vec{u}(x_i) - \vec{u}_{as}(x_i) \parallel \leq C\sqrt{\epsilon}, x \in \overline{\Omega}. \tag{6.2.41}$$

The desired estimate follows from the inequalities (6.2.40) and (6.2.41).

6.3 Numerical experiments and discussions

To illustrate the predicted theory, following examples are provided here. Computational results are given in the form of tables. The results are presented with maximum point-wise errors for various values of ϵ and N.

We have also computed the computational order of convergence which has been shown in the same table along with maximum errors. In all the cases, we take $\sigma_0 = 2$.

Example 6.3.1 *Consider the following Problem:*

$$
\begin{aligned}
-\epsilon u_1''(x) - u_1'(x) + 3u_1(x) - u_2(x) &= 2,\\
-\epsilon u_2''(x) - u_2'(x) - 1u_1(x) + 3u_2(x) &= 3, \quad x \in (0,1],\\
\vec{u}(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \vec{u}(1) &= \begin{pmatrix} 0 \\ 0 \end{pmatrix}.
\end{aligned}
$$

The exact solution of this problem is

$$
\begin{aligned}
u_1(x) = \frac{9}{8} + \left[\frac{exp(m2) - 1}{(exp(m1) - exp(m2))}\right] exp(m1)x + \frac{5}{4}\left[\frac{1 - exp(m1)}{(exp(m1) - exp(m2))}\right] exp(m2)x\\
- \frac{1}{8}\left[\frac{exp(m4) - 1}{(exp(m3) - exp(m4))}\right] exp(m3)x - \frac{1}{8}\left[\frac{1 - exp(m3)}{(exp(m3) - exp(m4))}\right] exp(m4)x,
\end{aligned}
$$

$$
\begin{aligned}
u_2(x) = \frac{11}{8} + \frac{5}{4}\left[\frac{exp(m2) - 1}{(exp(m1) - exp(m2))}\right] exp(m1)x + \left[\frac{1 - exp(m1)}{(exp(m1) - exp(m2))}\right] exp(m2)x\\
+ \frac{1}{8}\left[\frac{exp(m4) - 1}{(exp(m3) - exp(m4))}\right] exp(m3)x + \frac{1}{8}\left[\frac{1 - exp(m3)}{(exp(m3) - exp(m4))}\right] exp(m4)x,
\end{aligned}
$$

where $m1 = (-1 + \sqrt{1 + 8\epsilon})/2\epsilon$, $m2 = (-1 - \sqrt{1 + 8\epsilon})/2\epsilon$, $m3 = (-1 + \sqrt{1 + 16\epsilon})/2\epsilon$, $m4 = (-1 - \sqrt{1 + 16\epsilon})/2\epsilon$.

Since the exact solution is known, we calculate maximum point-wise error by

$$
E_{\epsilon,j}^N = \max_{x_i \epsilon \overline{\Omega}^N} \{|u_j(x_i) - U_{j,i}^N|\}, \quad j = 1, 2, \tag{6.3.1}
$$

where $u_j(x_i)$ is the exact solution and $U_{j,i}^N$ is the numerical solution obtained by using N mesh intervals in the domain $\overline{\Omega}^N$. In addition, the rate of convergence is calculated as:

$$p_{\epsilon,j}^{N}=\frac{lnE_{\epsilon,j}^{N}-lnE_{\epsilon,j}^{2N}}{ln2},\quad j=1,2. \tag{6.3.2}$$

Tables 6.1 and 6.2 display, respectively, the maximum point-wise errors for u_1 and u_2 for several values of ϵ and N.

Example 6.3.2 *Consider the following Problem:*

$$\begin{aligned}
&-\epsilon u_1''(x)-u_1'(x)+4u_1(x)-u_2(x)=4,\\
&-\epsilon u_2''(x)-u_2'(x)-1u_1(x)+4u_2(x)=2,\quad x\in(0,1],\\
&\vec{u}(0)=\begin{pmatrix}1\\1\end{pmatrix},\vec{u}(1)=\begin{pmatrix}0\\0\end{pmatrix}.
\end{aligned}$$

The exact solution of this problem is

$$\begin{aligned}
u_1(x)=\frac{6}{5}-&\left[\frac{1}{(exp(m1)-exp(m2))}\right]exp(m1)x+\left[\frac{1}{(exp(m1)-exp(m2))}\right]exp(m2)x\\
&-\frac{1}{5}\left[\frac{1-exp(m4)}{(exp(m3)-exp(m4))}\right]exp(m3)x-\frac{1}{5}\left[\frac{exp(m3)-1}{(exp(m3)-exp(m4))}\right]exp(m4)x,
\end{aligned}$$

$$\begin{aligned}
u_2(x)=\frac{4}{5}-&\left[\frac{1}{(exp(m1)-exp(m2))}\right]exp(m1)x+\left[\frac{1}{(exp(m1)-exp(m2))}\right]exp(m2)x\\
&+\frac{1}{5}\left[\frac{1-exp(m4)}{(exp(m3)-exp(m4))}\right]exp(m3)x+\frac{1}{5}\left[\frac{exp(m3)-1}{(exp(m3)-exp(m4))}\right]exp(m4)x,
\end{aligned}$$

where $m1=(-1+\sqrt{1+12\epsilon})/2\epsilon$, $m2=(-1-\sqrt{1+12\epsilon})/2\epsilon$, $m3=(-1+\sqrt{1+20\epsilon})/2\epsilon$, $m4=(-1-\sqrt{1+20\epsilon})/2\epsilon$.

Maximum point-wise errors and rate of convergence for u_1 and u_2 are given in Table 6.3 and 6.4 respectively.

Example 6.3.3 *Consider the following Problem:*

$$
\begin{gathered}
-\epsilon u_1''(x) + 3u_1(x) - u_2(x) = 2, \\
-\epsilon u_2''(x) - u_1(x) + 3u_2(x) = 3, x \in (0, 1], \\
\vec{u}(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \vec{u}(1) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.
\end{gathered}
$$

We calculate the numerical solution U^N on $\overline{\Omega}_\epsilon^N$ and the numerical solution $\widetilde{U}^N$on the mesh $\widetilde{\Omega}_\epsilon^N$, where the transition parameter σ is altered slightly to $\widetilde{\sigma} = min\{\frac{1}{4}, \sigma_0 \epsilon \ln \frac{N}{2}\}$. Note that this slightly altered value of σ will ensure that the positions of transition points remain the same in meshes $\overline{\Omega}_\epsilon^N$ and $\widetilde{\Omega}_\epsilon^{2N}$. The double mesh difference is defined as

$$
E_{\epsilon,j}^N = \max_{x_i \epsilon \overline{\Omega}^N} \{|U_{i,j}^N - \widetilde{U}_{i,j}^{2N}|\}, \quad j = 1, 2, \tag{6.3.3}
$$

where U_i^N and $\widetilde{U}_i^{2N}$ respectively denote the numerical solutions obtained by using N and $2N$ mesh intervals. The rates of convergence are calculated as:

$$
p_{\epsilon,j}^N = \frac{lnE_{\epsilon,j}^N - lnE_{\epsilon,j}^{2N}}{ln2}, \quad j = 1, 2. \tag{6.3.4}
$$

Tables 6.5 and 6.6 display, respectively, the maximum point-wise errors for u_1 and u_2 for several values of ϵ and N.

Example 6.3.4 *Consider the following Problem:*

$$
\begin{gathered}
-\epsilon u_1''(x) + 2(x+1)^2 u_1(x) - (x^3+1) u_2(x) = 2e^x, \\
-\epsilon u_2''(x) - 2cos(\pi x/4) u_1(x) + 2.2e^{-x+1} u_2(x) = 10x + 1, \\
x \in (0, 1], \; \vec{u}(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \vec{u}(1) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.
\end{gathered}
$$

Maximum point-wise errors and rate of convergence for u_1 and u_2 are given in Table 6.7 and 6.8 respectively.

6.4 Conclusions

We have proposed a robust computational technique for solving the system of two singularly perturbed boundary value problems. This chapter demonstrates, the effectiveness of the Shishkin mesh by modifying the initial value technique [125] in a very simple way so that higher order, almost second order of convergence can be achieved with no restrictions on values of h and ϵ. Our method is easier to apply and more effective in the sense of solution errors. From rates of convergence one can conclude that the present method is of second order convergence up to a logarithmic factor. Nonlinear system of equations have been handled by the present technique after linearization. The graphs plotted in Figures (6.1-6.2) are convergent curves in the maximum norm at nodal points for the different values of ϵ for Examples (6.3.1-6.3.4).

Table 6.1: Maximum point-wise errors and rates of convergence of u_1 for Example 6.3.1.

ϵ	Number of mesh points						
	16	32	64	128	256	512	1024
2^{-14}	1.51668E-02 1.377	5.83844E-03 1.480	2.09231E-03 1.533	7.22980E-04 1.502	2.55289E-04 1.319	1.02351E-04 1.259	5.52680E-05 1.211
2^{-16}	1.51395E-02 1.380	5.81643E-03 1.491	2.06894E-03 1.564	6.99588E-04 1.591	2.32182E-04 1.566	7.84239E-05 1.405	2.96184E-05 1.332
2^{-18}	1.51327E-02 1.381	5.81093E-03 1.494	2.06310E-03 1.572	6.93741E-04 1.615	2.26406E-04 1.640	7.26602E-05 1.617	2.36894E-05 1.608
2^{-20}	1.51310E-02 1.381	5.80956E-03 1.495	2.06164E-03 1.574	6.92279E-04 1.622	2.24962E-04 1.659	7.12281E-05 1.679	2.22513E-05 1.698
2^{-22}	1.51306E-02 1.381	5.80921E-03 1.495	2.06127E-03 1.575	6.91914E-04 1.623	2.24601E-04 1.664	7.08701E-05 1.695	2.18961E-05 1.715
2^{-24}	1.51305E-02 1.381	5.80913E-03 1.495	2.06118E-03 1.575	6.91822E-04 1.624	2.24510E-04 1.665	7.07806E-05 1.699	2.18075E-05 1.728
2^{-26}	1.51304E-02 1.381	5.80911E-03 1.495	2.06116E-03 1.575	6.91799E-04 1.624	2.24488E-04 1.666	7.07582E-05 1.700	2.17854E-05 1.728
2^{-28}	1.51304E-02 1.381	5.80910E-03 1.495	2.06115E-03 1.575	6.91792E-04 1.624	2.24480E-04 1.666	7.07510E-05 1.700	2.17782E-05 1.728
2^{-30}	1.51304E-02 1.381	5.80910E-03 1.495	2.06115E-03 1.575	6.91792E-04 1.624	2.24480E-04 1.666	7.07509E-05 1.700	2.17781E-05 1.728
2^{-32}	1.51304E-02 1.381	5.80910E-03 1.495	2.06115E-03 1.575	6.91792E-04 1.624	2.24480E-04 1.666	7.07508E-05 1.700	2.17780E-05 1.728
2^{-34}	1.51304E-02 1.381	5.80910E-03 1.495	2.06115E-03 1.575	6.91792E-04 1.624	2.24480E-04 1.666	7.07508E-05 1.700	2.17780E-05 1.728
2^{-36}	1.51304E-02 1.381	5.80910E-03 1.495	2.06115E-03 1.575	6.91792E-04 1.624	2.24480E-04 1.666	7.07508E-05 1.700	2.17780E-05 1.728
2^{-38}	1.51304E-02 1.381	5.80910E-03 1.495	2.06115E-03 1.575	6.91792E-04 1.624	2.24480E-04 1.666	7.07508E-05 1.700	2.17780E-05 1.728
2^{-40}	1.51304E-02 1.381	5.80910E-03 1.495	2.06115E-03 1.575	6.91792E-04 1.624	2.24480E-04 1.666	7.07508E-05 1.700	2.17780E-05 1.728

(a)(b)
ExEx-
amam-
pleple
6.16.1
forfor
u_1 u_2

(c)(d)
ExEx-
amam-
pleple
6.26.2
forfor
u_1 u_2

Figure 6.1: Loglog plots of N Vs. maximum error.

Table 6.2: Maximum point-wise errors and rates of convergence of u_2 for Example 6.3.1.

ϵ	Number of mesh points						
	16	32	64	128	256	512	1024
2^{-14}	1.90171E-02 1.378	7.31525E-03 1.486	2.61141E-03 1.550	8.91935E-04 1.550	3.04635E-04 1.437	1.12488E-04 1.406	6.56963E-05 1.368
2^{-16}	1.89946E-02 1.380	7.29923E-03 1.493	2.59404E-03 1.569	8.74565E-04 1.604	2.87602E-04 1.606	9.44886E-05 1.510	3.31797E-05 1.470
2^{-18}	1.89890E-02 1.380	7.29523E-03 1.494	2.58970E-03 1.573	8.70225E-04 1.619	2.83346E-04 1.650	9.02605E-05 1.650	2.87526E-05 1.675
2^{-20}	1.89876E-02 1.380	7.29422E-03 1.495	2.58862E-03 1.575	8.69140E-04 1.622	2.82282E-04 1.662	8.92104E-05 1.687	2.76979E-05 1.694
2^{-22}	1.89872E-02 1.380	7.29397E-03 1.495	2.58834E-03 1.575	8.68869E-04 1.623	2.82016E-04 1.665	8.89479E-05 1.697	2.74393E-05 1.720
2^{-24}	1.89872E-02 1.380	7.29391E-03 1.495	2.58828E-03 1.575	8.68801E-04 1.624	2.81950E-04 1.665	8.88822E-05 1.699	2.73747E-05 1.725
2^{-26}	1.89871E-02 1.380	7.29390E-03 1.495	2.58826E-03 1.575	8.68784E-04 1.624	2.81933E-04 1.666	8.88658E-05 1.700	2.73585E-05 1.728
2^{-28}	1.89871E-02 1.380	7.29389E-03 1.495	2.58825E-03 1.575	8.68778E-04 1.624	2.81927E-04 1.666	8.88601E-05 1.700	2.73529E-05 1.728
2^{-30}	1.89871E-02 1.380	7.29389E-03 1.495	2.58825E-03 1.575	8.68778E-04 1.624	2.81927E-04 1.666	8.88603E-05 1.700	2.73531E-05 1.728
2^{-32}	1.89871E-02 1.380	7.29389E-03 1.495	2.58825E-03 1.575	8.68778E-04 1.624	2.81928E-04 1.666	8.88603E-05 1.700	2.73531E-05 1.728
2^{-34}	1.89871E-02 1.380	7.29389E-03 1.495	2.58825E-03 1.575	8.68778E-04 1.624	2.81928E-04 1.666	8.88604E-05 1.700	2.73531E-05 1.728
2^{-36}	1.89871E-02 1.380	7.29389E-03 1.495	2.58825E-03 1.575	8.68778E-04 1.624	2.81928E-04 1.666	8.88604E-05 1.700	2.73531E-05 1.728
2^{-38}	1.89871E-02 1.380	7.29389E-03 1.495	2.58825E-03 1.575	8.68778E-04 1.624	2.81928E-04 1.666	8.88604E-05 1.700	2.73531E-05 1.728
2^{-40}	1.89871E-02 1.380	7.29389E-03 1.495	2.58825E-03 1.575	8.68778E-04 1.624	2.81928E-04 1.666	8.88604E-05 1.700	2.73531E-05 1.728

(a)(b)
ExEx-
amam-
pleple
6.36.3
forfor
u_1 u_2

(c)(d)
ExEx-
amam-
pleple
6.46.4
forfor
u_1 u_2

Figure 6.2: Loglog plots of N Vs. maximum error.

Table 6.3: Maximum point-wise errors and rates of convergence of u_1 for Example 6.3.2.

ϵ	Number of mesh points						
	16	32	64	128	256	512	1024
2^{-14}	3.64477E-03	1.42368E-03	4.92862E-04	1.50878E-04	4.97356E-05	1.37356E-05	4.97356E-06
	1.331	1.449	1.474	1.547	1.488	1.568	1.580
2^{-16}	2.19741E-03	8.68935E-04	3.12899E-04	1.07579E-04	3.67927E-05	1.22742E-05	4.08738E-06
	1.338	1.474	1.540	1.548	1.584	1.586	1.605
2^{-18}	2.19604E-03	8.67331E-04	3.10984E-04	1.05635E-04	3.48930E-05	1.14688E-05	3.76876E-06
	1.340	1.480	1.558	1.598	1.605	1.606	1.638
2^{-20}	2.19570E-03	8.66930E-04	3.10506E-04	1.05149E-04	3.44192E-05	1.09950E-05	3.50140E-06
	1.341	1.481	1.562	1.611	1.646	1.651	1.660
2^{-22}	2.19562E-03	8.66830E-04	3.10386E-04	1.05028E-04	3.43007E-05	1.08787E-05	3.38382E-06
	1.341	1.482	1.563	1.614	1.657	1.685	1.695
2^{-24}	2.19560E-03	8.66805E-04	3.10356E-04	1.04998E-04	3.42711E-05	1.08496E-05	3.35544E-06
	1.341	1.482	1.564	1.615	1.659	1.693	1.720
2^{-26}	2.19559E-03	8.66799E-04	3.10349E-04	1.04990E-04	3.42637E-05	1.08424E-05	3.34834E-06
	1.341	1.482	1.564	1.615	1.660	1.695	1.725
2^{-28}	2.19559E-03	8.66797E-04	3.10347E-04	1.04988E-04	3.42617E-05	1.08404E-05	3.34645E-06
	1.341	1.482	1.564	1.616	1.660	1.696	1.725
2^{-30}	2.19559E-03	8.66796E-04	3.10346E-04	1.04987E-04	3.42611E-05	1.08398E-05	3.34584E-06
	1.341	1.482	1.564	1.616	1.660	1.696	1.725
2^{-32}	2.19559E-03	8.66797E-04	3.10346E-04	1.04987E-04	3.42612E-05	1.08399E-05	3.34594E-06
	1.341	1.482	1.564	1.616	1.660	1.696	1.725
2^{-34}	2.19559E-03	8.66797E-04	3.10346E-04	1.04987E-04	3.42612E-05	1.08399E-05	3.34594E-06
	1.341	1.482	1.564	1.616	1.660	1.696	1.725
2^{-36}	2.19559E-03	8.66797E-04	3.10346E-04	1.04987E-04	3.42612E-05	1.08399E-05	3.34594E-06
	1.341	1.482	1.564	1.616	1.660	1.696	1.725
2^{-38}	2.19559E-03	8.66797E-04	3.10346E-04	1.04987E-04	3.42612E-05	1.08399E-05	3.34594E-06
	1.341	1.482	1.564	1.616	1.660	1.696	1.725
2^{-40}	2.19559E-03	8.66797E-04	3.10346E-04	1.04987E-04	3.42612E-05	1.08399E-05	3.34594E-06
	1.341	1.482	1.564	1.616	1.660	1.696	1.725

Table 6.4: Maximum point-wise errors and rates of convergence of u_2 for Example 6.3.2.

ϵ	Number of mesh points						
	16	32	64	128	256	512	1024
2^{-14}	3.64477E-03	1.42368E-03	4.92862E-04	1.50878E-04	4.97356E-05	1.37356E-05	4.97356E-06
	1.356	1.530	1.708	1.601	1.571	1.466	1.410
2^{-16}	3.65934E-03	1.44086E-03	5.11664E-04	1.68866E-04	5.10515E-05	1.56361E-05	5.04361E-06
	1.345	1.494	1.599	1.726	1.707	1.632	1.528
2^{-18}	3.66298E-03	1.44515E-03	5.16366E-04	1.73626E-04	5.56318E-05	1.65485E-05	5.10218E-06
	1.342	1.485	1.572	1.642	1.749	1.697	1.670
2^{-20}	3.66389E-03	1.44623E-03	5.17541E-04	1.74815E-04	5.67837E-05	1.77044E-05	5.19685E-06
	1.341	1.483	1.566	1.622	1.681	1.768	1.785
2^{-22}	3.66412E-03	1.44650E-03	5.17835E-04	1.75113E-04	5.70795E-05	1.79934E-05	5.48670E-06
	1.341	1.482	1.564	1.617	1.666	1.713	1.745
2^{-24}	3.66417E-03	1.44656E-03	5.17908E-04	1.75187E-04	5.71535E-05	1.80661E-05	5.55972E-06
	1.341	1.482	1.564	1.616	1.662	1.700	1.734
2^{-26}	3.66419E-03	1.44658E-03	5.17927E-04	1.75206E-04	5.71720E-05	1.80845E-05	5.57797E-06
	1.341	1.482	1.564	1.616	1.661	1.697	1.725
2^{-28}	3.66419E-03	1.44658E-03	5.17931E-04	1.75211E-04	5.71767E-05	1.80892E-05	5.58265E-06
	1.341	1.482	1.564	1.616	1.660	1.696	1.725
2^{-30}	3.66419E-03	1.44659E-03	5.17933E-04	1.75212E-04	5.71780E-05	1.80905E-05	5.58392E-06
	1.341	1.482	1.564	1.616	1.660	1.696	1.725
2^{-32}	3.66419E-03	1.44659E-03	5.17933E-04	1.75212E-04	5.71781E-05	1.80906E-05	5.58402E-06
	1.341	1.482	1.564	1.616	1.660	1.696	1.725
2^{-34}	3.66419E-03	1.44659E-03	5.17933E-04	1.75212E-04	5.71781E-05	1.80906E-05	5.58405E-06
	1.341	1.482	1.564	1.616	1.660	1.696	1.725
2^{-36}	3.66419E-03	1.44659E-03	5.17933E-04	1.75212E-04	5.71781E-05	1.80906E-05	5.58405E-06
	1.341	1.482	1.564	1.616	1.660	1.696	1.725
2^{-38}	3.66419E-03	1.44659E-03	5.17933E-04	1.75212E-04	5.71781E-05	1.80906E-05	5.58405E-06
	1.341	1.482	1.564	1.616	1.660	1.696	1.725
2^{-40}	3.66419E-03	1.44659E-03	5.17933E-04	1.75212E-04	5.71781E-05	1.80906E-05	5.58405E-06
	1.341	1.482	1.564	1.616	1.660	1.696	1.725

Table 6.5: Maximum point-wise errors and rates of convergence of u_1 for Example 6.3.3.

ε	Number of mesh points						
	16	32	64	128	256	512	1024
10^{-4}	2.94547E-01	1.14899E-01	3.56373E-02	1.18109E-02	3.82798E-03	1.20389E-03	3.71125E-04
	1.358	1.689	1.593	1.625	1.669	1.698	
10^{-6}	2.94547E-01	1.14899E-01	3.56373E-02	1.18109E-02	3.82798E-03	1.20389E-03	3.71125E-04
	1.358	1.689	1.593	1.625	1.669	1.698	
10^{-8}	2.94547E-01	1.14899E-01	3.56373E-02	1.18109E-02	3.82798E-03	1.20389E-03	3.71125E-04
	1.358	1.689	1.593	1.625	1.669	1.698	
10^{-10}	2.94547E-01	1.14899E-01	3.56373E-02	1.18109E-02	3.82798E-03	1.20389E-03	3.71125E-04
	1.358	1.689	1.593	1.625	1.669	1.698	
10^{-12}	2.94547E-01	1.14899E-01	3.56373E-02	1.18109E-02	3.82798E-03	1.20389E-03	3.71125E-04
	1.358	1.689	1.593	1.625	1.669	1.698	
10^{-14}	2.94547E-01	1.14899E-01	3.56373E-02	1.18109E-02	3.82798E-03	1.20389E-03	3.71125E-04
	1.358	1.689	1.593	1.625	1.669	1.698	
10^{-16}	2.94547E-01	1.14899E-01	3.56373E-02	1.18109E-02	3.82798E-03	1.20389E-03	3.71125E-04
	1.358	1.689	1.593	1.625	1.669	1.698	
10^{-18}	2.94547E-01	1.14899E-01	3.56373E-02	1.18109E-02	3.82798E-03	1.20389E-03	3.71125E-04
	1.358	1.689	1.593	1.625	1.669	1.698	
10^{-20}	2.94547E-01	1.14899E-01	3.56373E-02	1.18109E-02	3.82798E-03	1.20389E-03	3.71125E-04
	1.358	1.689	1.593	1.625	1.669	1.698	
10^{-22}	2.94547E-01	1.14899E-01	3.56373E-02	1.18109E-02	3.82798E-03	1.20389E-03	3.71125E-04
	1.358	1.689	1.593	1.625	1.669	1.698	
10^{-24}	2.94547E-01	1.14899E-01	3.56373E-02	1.18109E-02	3.82798E-03	1.20389E-03	3.71125E-04
	1.358	1.689	1.593	1.625	1.669	1.698	
10^{-26}	2.94547E-01	1.14899E-01	3.56373E-02	1.18109E-02	3.82798E-03	1.20389E-03	3.71125E-04
	1.358	1.689	1.593	1.625	1.669	1.698	
10^{-28}	2.94547E-01	1.14899E-01	3.56373E-02	1.18109E-02	3.82798E-03	1.20389E-03	3.71125E-04
	1.358	1.689	1.593	1.625	1.669	1.698	
10^{-30}	2.94547E-01	1.14899E-01	3.56373E-02	1.18109E-02	3.82798E-03	1.20389E-03	3.71125E-04
	1.358	1.689	1.593	1.625	1.669	1.698	
10^{-32}	2.94547E-01	1.14899E-01	3.56373E-02	1.18109E-02	3.82798E-03	1.20389E-03	3.71125E-04
	1.358	1.689	1.593	1.625	1.669	1.698	
10^{-34}	2.94547E-01	1.14899E-01	3.56373E-02	1.18109E-02	3.82798E-03	1.20389E-03	3.71125E-04
	1.358	1.689	1.593	1.625	1.669	1.698	
10^{-36}	2.94547E-01	1.14899E-01	3.56373E-02	1.18109E-02	3.82798E-03	1.20389E-03	3.71125E-04
	1.358	1.689	1.593	1.625	1.669	1.698	
10^{-38}	2.94547E-01	1.14899E-01	3.56373E-02	1.18109E-02	3.82798E-03	1.20389E-03	3.71125E-04
	1.358	1.689	1.593	1.625	1.669	1.698	
10^{-40}	2.94547E-01	1.14899E-01	3.56373E-02	1.18109E-02	3.82798E-03	1.20389E-03	3.71125E-04
	1.358	1.689	1.593	1.625	1.669	1.698	

Table 6.6: Maximum point-wise errors and rates of convergence of u_2 for Example 6.3.3.

ε	Number of mesh points						
	16	32	64	128	256	512	1024
10^{-4}	2.94547E-01	1.14899E-01	3.56373E-02	1.18109E-02	3.82798E-03	1.20389E-03	3.71125E-04
	1.358	1.689	1.593	1.625	1.669	1.698	
10^{-6}	2.94547E-01	1.14899E-01	3.56373E-02	1.18109E-02	3.82798E-03	1.20389E-03	3.71125E-04
	1.358	1.689	1.593	1.625	1.669	1.698	
10^{-8}	2.94547E-01	1.14899E-01	3.56373E-02	1.18109E-02	3.82798E-03	1.20389E-03	3.71125E-04
	1.358	1.689	1.593	1.625	1.669	1.698	
10^{-10}	2.94547E-01	1.14899E-01	3.56373E-02	1.18109E-02	3.82798E-03	1.20389E-03	3.71125E-04
	1.358	1.689	1.593	1.625	1.669	1.698	
10^{-12}	2.94547E-01	1.14899E-01	3.56373E-02	1.18109E-02	3.82798E-03	1.20389E-03	3.71125E-04
	1.358	1.689	1.593	1.625	1.669	1.698	
10^{-14}	2.94547E-01	1.14899E-01	3.56373E-02	1.18109E-02	3.82798E-03	1.20389E-03	3.71125E-04
	1.358	1.689	1.593	1.625	1.669	1.698	
10^{-16}	2.94547E-01	1.14899E-01	3.56373E-02	1.18109E-02	3.82798E-03	1.20389E-03	3.71125E-04
	1.358	1.689	1.593	1.625	1.669	1.698	
10^{-18}	2.94547E-01	1.14899E-01	3.56373E-02	1.18109E-02	3.82798E-03	1.20389E-03	3.71125E-04
	1.358	1.689	1.593	1.625	1.669	1.698	
10^{-20}	2.94547E-01	1.14899E-01	3.56373E-02	1.18109E-02	3.82798E-03	1.20389E-03	3.71125E-04
	1.358	1.689	1.593	1.625	1.669	1.698	
10^{-22}	2.94547E-01	1.14899E-01	3.56373E-02	1.18109E-02	3.82798E-03	1.20389E-03	3.71125E-04
	1.358	1.689	1.593	1.625	1.669	1.698	
10^{-24}	2.94547E-01	1.14899E-01	3.56373E-02	1.18109E-02	3.82798E-03	1.20389E-03	3.71125E-04
	1.358	1.689	1.593	1.625	1.669	1.698	
10^{-26}	2.94547E-01	1.14899E-01	3.56373E-02	1.18109E-02	3.82798E-03	1.20389E-03	3.71125E-04
	1.358	1.689	1.593	1.625	1.669	1.698	
10^{-28}	2.94547E-01	1.14899E-01	3.56373E-02	1.18109E-02	3.82798E-03	1.20389E-03	3.71125E-04
	1.358	1.689	1.593	1.625	1.669	1.698	
10^{-30}	2.94547E-01	1.14899E-01	3.56373E-02	1.18109E-02	3.82798E-03	1.20389E-03	3.71125E-04
	1.358	1.689	1.593	1.625	1.669	1.698	
10^{-32}	2.94547E-01	1.14899E-01	3.56373E-02	1.18109E-02	3.82798E-03	1.20389E-03	3.71125E-04
	1.358	1.689	1.593	1.625	1.669	1.698	
10^{-34}	2.94547E-01	1.14899E-01	3.56373E-02	1.18109E-02	3.82798E-03	1.20389E-03	3.71125E-04
	1.358	1.689	1.593	1.625	1.669	1.698	
10^{-36}	2.94547E-01	1.14899E-01	3.56373E-02	1.18109E-02	3.82798E-03	1.20389E-03	3.71125E-04
	1.358	1.689	1.593	1.625	1.669	1.698	
10^{-38}	2.94547E-01	1.14899E-01	3.56373E-02	1.18109E-02	3.82798E-03	1.20389E-03	3.71125E-04
	1.358	1.689	1.593	1.625	1.669	1.698	
10^{-40}	2.94547E-01	1.14899E-01	3.56373E-02	1.18109E-02	3.82798E-03	1.20389E-03	3.71125E-04
	1.358	1.689	1.593	1.625	1.669	1.698	

Table 6.7: Maximum point-wise errors and rates of convergence of u_1 for Example 6.3.4.

ε	Number of mesh points						
	16	32	64	128	256	512	1024
10^{-4}	6.57432E-01 1.128	3.00718E-01 1.471	1.08497E-01 1.799	3.11750E-02 1.674	9.77281E-03 1.693	3.02225E-03 1.749	8.99222E-04
10^{-6}	6.55359E-01 1.125	3.00433E-01 1.465	1.08804E-01 1.784	3.15870E-02 1.658	1.00113E-02 1.668	3.15045E-03 1.706	9.65973E-04
10^{-8}	6.55157E-01 1.125	3.00405E-01 1.465	1.08835E-01 1.783	3.16281E-02 1.656	1.00351E-02 1.666	3.16325E-03 1.701	9.72645E-04
10^{-10}	6.55137E-01 1.125	3.00402E-01 1.465	1.08838E-01 1.783	3.16322E-02 1.656	1.00375E-02 1.666	3.16453E-03 1.701	9.73312E-04
10^{-12}	6.55135E-01 1.125	3.00402E-01 1.465	1.08838E-01 1.783	3.16326E-02 1.656	1.00377E-02 1.666	3.16466E-03 1.701	9.73379E-04
10^{-14}	6.55135E-01 1.125	3.00402E-01 1.465	1.08838E-01 1.783	3.16326E-02 1.656	1.00377E-02 1.666	3.16466E-03 1.701	9.73379E-04
10^{-16}	6.55135E-01 1.125	3.00402E-01 1.465	1.08838E-01 1.783	3.16326E-02 1.656	1.00377E-02 1.666	3.16466E-03 1.701	9.73379E-04
10^{-18}	6.55135E-01 1.125	3.00402E-01 1.465	1.08838E-01 1.783	3.16326E-02 1.656	1.00377E-02 1.666	3.16466E-03 1.701	9.73379E-04
10^{-20}	6.55135E-01 1.125	3.00402E-01 1.465	1.08838E-01 1.783	3.16326E-02 1.656	1.00377E-02 1.666	3.16466E-03 1.701	9.73379E-04
10^{-22}	6.55135E-01 1.125	3.00402E-01 1.465	1.08838E-01 1.783	3.16326E-02 1.656	1.00377E-02 1.666	3.16466E-03 1.701	9.73379E-04
10^{-24}	6.55135E-01 1.125	3.00402E-01 1.465	1.08838E-01 1.783	3.16326E-02 1.656	1.00377E-02 1.666	3.16466E-03 1.701	9.73379E-04
10^{-26}	6.55135E-01 1.125	3.00402E-01 1.465	1.08838E-01 1.783	3.16326E-02 1.656	1.00377E-02 1.666	3.16466E-03 1.701	9.73379E-04
10^{-28}	6.55135E-01 1.125	3.00402E-01 1.465	1.08838E-01 1.783	3.16326E-02 1.656	1.00377E-02 1.666	3.16466E-03 1.701	9.73379E-04
10^{-30}	6.55135E-01 1.125	3.00402E-01 1.465	1.08838E-01 1.783	3.16326E-02 1.656	1.00377E-02 1.666	3.16466E-03 1.701	9.73379E-04
10^{-32}	6.55135E-01 1.125	3.00402E-01 1.465	1.08838E-01 1.783	3.16326E-02 1.656	1.00377E-02 1.666	3.16466E-03 1.701	9.73379E-04
10^{-34}	6.55135E-01 1.125	3.00402E-01 1.465	1.08838E-01 1.783	3.16326E-02 1.656	1.00377E-02 1.666	3.16466E-03 1.701	9.73379E-04
10^{-36}	6.55135E-01 1.125	3.00402E-01 1.465	1.08838E-01 1.783	3.16326E-02 1.656	1.00377E-02 1.666	3.16466E-03 1.701	9.73379E-04
10^{-38}	6.55135E-01 1.125	3.00402E-01 1.465	1.08838E-01 1.783	3.16326E-02 1.656	1.00377E-02 1.666	3.16466E-03 1.701	9.73379E-04
10^{-40}	6.55135E-01 1.125	3.00402E-01 1.465	1.08838E-01 1.783	3.16326E-02 1.656	1.00377E-02 1.666	3.16466E-03 1.701	9.73379E-04

Table 6.8: Maximum point-wise errors and rates of convergence of u_2 for Example 6.3.4.

ε	Number of mesh points						
	16	32	64	128	256	512	1024
10^{-4}	4.60874E-01 1.368	1.78595E-01 1.631	5.76521E-02 1.547	1.97354E-02 1.530	6.83615E-03 1.469	2.46891E-03 1.370	9.55156E-04
10^{-6}	4.61430E-01 1.393	1.75721E-01 1.692	5.43734E-02 1.602	1.79112E-02 1.613	5.85455E-03 1.647	1.86935E-03 1.659	5.91967E-04
10^{-8}	4.61487E-01 1.395	1.75436E-01 1.699	5.40469E-02 1.608	1.77292E-02 1.623	5.75642E-03 1.667	1.81307E-03 1.693	5.60807E-04
10^{-10}	4.61493E-01 1.396	1.75408E-01 1.699	5.40143E-02 1.609	1.77110E-02 1.624	5.74661E-03 1.668	1.80791E-03 1.696	5.57848E-04
10^{-12}	4.61494E-01 1.396	1.75405E-01 1.699	5.40110E-02 1.609	1.77092E-02 1.624	5.74563E-03 1.669	1.80739E-03 1.697	5.57552E-04
10^{-14}	4.61494E-01 1.396	1.75405E-01 1.699	5.40110E-02 1.609	1.77092E-02 1.624	5.74563E-03 1.669	1.80739E-03 1.697	5.57552E-04
10^{-16}	4.61494E-01 1.396	1.75405E-01 1.699	5.40110E-02 1.609	1.77092E-02 1.624	5.74563E-03 1.669	1.80739E-03 1.697	5.57552E-04
10^{-18}	4.61494E-01 1.396	1.75405E-01 1.699	5.40110E-02 1.609	1.77092E-02 1.624	5.74563E-03 1.669	1.80739E-03 1.697	5.57552E-04
10^{-20}	4.61494E-01 1.396	1.75405E-01 1.699	5.40110E-02 1.609	1.77092E-02 1.624	5.74563E-03 1.669	1.80739E-03 1.697	5.57552E-04
10^{-22}	4.61494E-01 1.396	1.75405E-01 1.699	5.40110E-02 1.609	1.77092E-02 1.624	5.74563E-03 1.669	1.80739E-03 1.697	5.57552E-04
10^{-24}	4.61494E-01 1.396	1.75405E-01 1.699	5.40110E-02 1.609	1.77092E-02 1.624	5.74563E-03 1.669	1.80739E-03 1.697	5.57552E-04
10^{-26}	4.61494E-01 1.396	1.75405E-01 1.699	5.40110E-02 1.609	1.77092E-02 1.624	5.74563E-03 1.669	1.80739E-03 1.697	5.57552E-04
10^{-28}	4.61494E-01 1.396	1.75405E-01 1.699	5.40110E-02 1.609	1.77092E-02 1.624	5.74563E-03 1.669	1.80739E-03 1.697	5.57552E-04
10^{-30}	4.61494E-01 1.396	1.75405E-01 1.699	5.40110E-02 1.609	1.77092E-02 1.624	5.74563E-03 1.669	1.80739E-03 1.697	5.57552E-04
10^{-32}	4.61494E-01 1.396	1.75405E-01 1.699	5.40110E-02 1.609	1.77092E-02 1.624	5.74563E-03 1.669	1.80739E-03 1.697	5.57552E-04
10^{-34}	4.61494E-01 1.396	1.75405E-01 1.699	5.40110E-02 1.609	1.77092E-02 1.624	5.74563E-03 1.669	1.80739E-03 1.697	5.57552E-04
10^{-36}	4.61494E-01 1.396	1.75405E-01 1.699	5.40110E-02 1.609	1.77092E-02 1.624	5.74563E-03 1.669	1.80739E-03 1.697	5.57552E-04
10^{-38}	4.61494E-01 1.396	1.75405E-01 1.699	5.40110E-02 1.609	1.77092E-02 1.624	5.74563E-03 1.669	1.80739E-03 1.697	5.57552E-04
10^{-40}	4.61494E-01 1.396	1.75405E-01 1.699	5.40110E-02 1.609	1.77092E-02 1.624	5.74563E-03 1.669	1.80739E-03 1.697	5.57552E-04

Chapter 7

Concluding Observations

In this chapter, we have critically reviewed in brief the work done in the earlier chapters of this thesis and have drawn certain conclusions. In the present thesis we have primarily investigated the one dimensional singularly perturbed problems. Methods are devised for initial value problems, boundary value problems with smooth data, boundary value problems with non-smooth data, system of boundary value problems and delay differential equations. Apart from the construction of methods, a full fledged theory for their convergence and error estimates is also presented. Numerical experiments are carried out extensively to support the theoretical results. Our main conclusions are as follows.

7.1 Significance of present Work

In Chapter II, we have proposed a robust computational technique for solving singularly perturbed initial value problems. It is observed that although backward difference scheme satisfies discrete maximum principle in whole domain $[0, 1]$, but its order is one (up to a logarithmic factor). We can get order two (up to a logarithmic factor) by applying Trapezoidal scheme in $[0, 1]$ with proper choice of σ_0, but it results in small oscillations. Hence the solution is not stable unless mesh size is very small even in outer region, where a coarse mesh is enough to give satisfactory result. Since these oscillations are so small that one gets the impression that Trapezoidal scheme has second order (up to a logarithmic factor) convergence. We have overcome this difficulty in the proposed scheme by taking into account a proper combination of Trapezoidal scheme and backward difference scheme.

In Chapter III, two hybrid difference schemes are constructed , one is for first order singularly perturbed delay differential equations and other is for second order singularly perturbed delay differential equation. These hybrid schemes are of almost second order convergent upto a logarithmic factor. The proposed schemes gives uniform convergence and accurate solutions to singularly perturbed delay differential equations over a wide range of ϵ and δ. The numerical results demonstrate that the method is robust, i.e., converges for all ϵ and δ with the condition δ is $O(\epsilon)$. For second order singularly perturbed delay differential equation, we have constructed a difference scheme using cubic spline. And then, we apply this scheme on a Shishkin mesh. In the boundary layer (inner) region, the mesh is fine, and the cubic spline scheme is stable. Whereas in the regular (outer) region, the mesh is coarse, and the cubic spline scheme is not stable. To obtain stability in the outer region, one has to restrict the step size in that region, but our aim is to propose an ϵ-uniform convergent numerical scheme. Therefore, for the outer region, instead of the cubic scheme we use the finite difference scheme, mainly for stability reasons. The newly obtained hybrid scheme is convergent independent of the singular perturbation parameter.

Furthermore in Chapter IV, we have proposed a robust computational technique for solving the singularly perturbed boundary value problems. This chapter demonstrates the effectiveness of the Shishkin mesh by modifying the initial value technique [37] in a very simple way so that higher order, almost second order of convergence can be achieved with no restrictions on values of h and ϵ. Our method is easier to apply and more effective in the sense of solution errors. From, the numerical results shown in tables, we conclude that the proposed MIVT works nicely independent of the mesh parameter h and the perturbation parameter ϵ. Also, the proposed scheme is of almost second order convergence upto a logarithmic factor. The ideas developed here are extended in Chapter V and Chapter VI to solve a singularly perturbed boundary value problem with non-smooth data and system of two boundary value problems. Our method is easier to apply and more effective in the sense of solution errors. From rates of convergence one can conclude that the present method is of second order convergence up to a logarithmic factor.

7.2 Scope for Future Work

In the present thesis, some numerical methods based on finite difference scheme and Shiskin mesh are constructed for a class of singularly perturbed problems. Non-linear system of equations can be treated by methods presented in this thesis using Newton's method of quasi-linearization. These methods can also be applied to partial differential equations. This will be taken as a project work in the future. In this thesis only non-turning point problems are considered. Hence, the methods proposed in the thesis can also be extended to turning point problems.

For a successful application of any fitted mesh method, a significant amount of a priori information about the presence, location, height and width of the layers is required. Therefore, to obviate the need for unrealistic amounts of a priori information about the solution, it is necessary to develop adaptive algorithms which can automatically detect the presence and thickness of the boundary layers.

References

[1] Amiraliyev G. M., The convergence of a finite difference method on layer-adapted mesh for a singularly perturbed system, Appl. Math. Comput., 162(2005), pp.1023-1034.

[2] Amiraliyev G. M., Erdogan F., Uniform numerical method for singularly perturbed delay differential equations, Comput. Math. Appl. 53(2007), pp.1251-1259.

[3] Ansari A. R., Hegarty A. F., Numerical solution of a convection diffusion problem with Robin boundary conditions, J. Comput. Appl. Math., 156(2003), pp.221-238.

[4] Awoke A., Reddy Y. N., An exponentially fitted special second-order finite difference method for solving singular perturbation problems, Appl. Math. Comput., 190(2007), pp.1767-1782.

[5] Awoke A., Reddy Y. N., Terminal boundary condition for singularly perturbed two-point boundary value problems, Neural Parallel Sci. Comput., 16(2008), pp.435-448.

[6] Awoke A., Reddy Y. N., An asymptotic-fitted method for solving singularly perturbed delay differential equations, Appl. Math., 3(2012), pp.895-902.

[7] Babu A. R., Ramanujam N., An asymptotic finite element method for singularly perturbed third and fourth order ordinary differential equations with discontinuous source term, Appl. Math. Comput., 191(2007), pp.372-380.

[8] Bakhvalov N. S., Towards optimization of methods for solving boundary value problems in the presence of boundary layers, Zh. Vychisl. Mat. Mat. Fiz., 9(1969), pp.841-859. (In Russian)

[9] Bawa, R. K., A parallel approach for self-adjoint singular perturbation problems using Numerovs scheme, Int. J. Comput. Math., 84(3)(207), pp.317323.

[10] Bawa, R. K., Clavero C., Higher order global solution and normalized flux for singularly perturbed reaction-diffusion problems, Appl. Math. Comput., 216(7)(2010), pp.2058-2068.

[11] Bawa Rajesh K., Natesan S., An efficient hybrid numerical scheme for convection-dominated boundary value problems, Int. J. Comput. Math., 86(2)(2009), pp.261-273.

[12] Beckett G., Mackenzie J. A., Convergence analysis of finite difference approximations on equidistributed grids to a singularly perturbed boundary value problem, Appl. Numer. Math., 35(2)(2000), pp.87-109.

[13] Bellew S., O'Riordan E., A parameter robust numerical method for a system of two singularly perturbed convection-diffusion equations, Appl. Numer. Math., 51(2004), pp.171-186.

[14] Cai X., Liu F., Uniform convergence difference schemes for singularly perturbed mixed boundary problems, J. Comput. Appl. Math., 166(2004), pp.31-54.

[15] Cakir M., Amiraliyev G. M., Numerical solution of a singularly perturbed three-point boundary value problem, Int. J. Comput. Math., 84(10)(2007), pp.1465-1481.

[16] Cen Z., Parameter-uniform finite difference scheme for a system of coupled singularly perturbed convection-diffusion equations, Int. J. Comput. Math., 82(2005), pp.177-192.

[17] Cen Z., A hybrid difference scheme for a singularly perturbed convection-diffusion problem with discontinuous convection coefficient, Appl. Math. Comput., 169(2005), pp.689-699.

[18] Cen Z., A second-order difference scheme for a parameterized singular perturbation problem, J. Comput. Appl. Math., 221(2008), pp.174-182.

[19] Cen Z., A second-order finite difference scheme for a class of singularly perturbed delay differential equations, Int. J. Comput. Math., 87(1)(2010), pp.173-185.

[20] Cen Z., Aimin Xu, Anbo Le, A second-order hybrid finite difference scheme for a system of singularly perturbed initial value problems, J. Comput. Appl. Math., 234(2010), pp.3445-3457.

[21] Cen Z., Chen Jingfeng, Lifeng X., A second-order hybrid finite difference scheme for a system of coupled singularly perturbed initial value problems, Int. J. Nonlinear Sci., 8(2)(2009), pp.148-154.

[22] Clavero C., Bawa Rajesh K., Natesan S., A robust second-order numerical method for global solution and global normalized flux of singularly perturbed self-adjoint boundary-value problems, Int. J. Comput. Math., 86(10)(2009), pp.1731-1745.

[23] Clavero C., Gracia J. L., Lisbona F., High order methods on Shishkin meshes for singular perturbation problems of convectiondiffusion type, Numer. Algor., 22(1999), pp.7397.

[24] Derstine M. W., Gibbs H. M., Hopf F. A., Kaplan D. L., Bifurcation gap in a hybrid optical system. Phy. Rev. A26(1982), pp.3720-3722.

[25] Doolan E. P., Miller J. J. H., Schilders W. H. A., Uniform Numerical Methods For Problems With Initial And Boundary Layers, Boole Press, Dublin, 1980.

[26] Dorfi E. A., Drury L. O'C., Simple adaptive grids for 1-D initial value problems, J. Comput. Phys., 69(1987), pp.175-195.

[27] Farrell P. A., Hegarty A. F., Miller J. J. H., O'Riordan E., and Shishkin G. I., Robust Computational Techniques for Boundary Layers, Chapman & Hall/CRC Press, 2000.

[28] Farrell P. A., Hegarty A. F., Miller J. J. H., O'Riordan E., and Shishkin G. I., Singularly perturbed convection diffusion problems with boundary and weak interior layers, J. Comput. Appl. Math, 166(2004), pp.133-151.

[29] Farrell P. A., Hegarty, A. F., Miller J. J. H., O'Riordan E., and Shishkin G. I., Global maximum norm parameter-uniform numerical method for a singularly perturbed convection diffusion problem with discontinuous convection coefficient, Math. Comput. Modelling., 40(2004), pp.1375-1392.

[30] Farrell P. A., Miller J. J. H., O'Riordan E., and Shishkin G. I, Singularly perturbed differential equations with discontinuous source terms, In: J. J. H. Miller, G. I. Shishkin, L. Vulkov (eds.) Proceedings of Meeting on Analytical and Numerical Methods for Convection-Dominated and Singularly Perturbed Problems, Lozenetz, Bulgaria, 1998. pp. 23-32. Nova Science, New York (2000).

[31] Farrell P. A., O'Riordan E., Miller J. J. H., Shishkin, G. I., Parameter-uniform fitted mesh method for quasilinear differential equations with boundary layers, Comput. Methods Appl. Math., 1(2)(2001), pp.154-172.

[32] Fröhner A., Linß T., Roos H. G., Defect correction on Shishkin-type meshes, Numer. Algorithm, 26(2001), pp.281-299.

[33] Fröhner A., Roos H. G., The ϵ-uniform convergence of a defect correction method on a Shishkin mesh, Appl. Numer. Math., 37(2001), pp.79-94.

[34] Gartland E. C., Uniform high-order difference schemes for a singularly perturbed two point boundary value problem, Math. Comp., 48(1987), pp.551-564.

[35] Gartland E. C., Graded-mesh difference schemes for singularly perturbed two-point boundary value problems, Math. Comp., 51(184)(1988), pp.631-657.

[36] Gasparo M. G., Macconi M., New initial-value method for singularly perturbed boundary-value problems, J. Optim. Theory Applic., (63)(1989), pp.213-224.

[37] Gasparo M. G., Macconi M., Initial-value methods for second-order singularly perturbed boundary-value problems, J. Optim. Theory Applic., (66)(1990), pp.197-210.

[38] Gracia J. L., O'Riordan E., Pickett M. L., A parameter robust second order numerical method for a singularly perturbed two-parameter problem, Appl. Numer. Math., 56(2006), pp.962-980.

[39] Hemavathi S., Bhuvaneswari T., Valarmathi S., Miller J. J. H., A parameter uniform numerical method for a system of singularly perturbed ordinary differential equations, Appl. Math. Comput., 191(2007), pp.1-11.

[40] Jator S. N., A sixth order linear multistep method for the direct solution of $y'' = f(x, y, y')$, Int. J. Pure Appl. Math., 40(4)(2007), pp. 457-472.

[41] Jator S. N., Li J., An algorithm for second order initial and boundary value problems with an automatic error estimate based on a third derivative method, Numer. Algor., 59(2012), pp.333-346.

[42] Kadalbajoo M. K., Aggarwal V. K., Fitted mesh B-spline collocation method for solving self adjoint singularly perturbed boundary value problems, Appl. Math. Comp., 161(2005), pp.973-987.

[43] Kadalbajoo M. K., Gupta V., Numerical solution of singularly perturbed convection-diffusion problem using parameter uniform B-spline collocation method, J. Math. Anal. Appl., 355(1)(2009), pp.439-452.

[44] Kadalbajoo M. K., Gupta V., A parameter uniform B-spline collocation method for solving singularly perturbed turning point problem having twin boundary layers, Int. J. Comput. Math., 87(14)(2010), pp.3218-3235.

[45] Kadalbajoo M. K., Kumar D., Fitted mesh B-spline collocation method for singularly perturbed differential-difference equations with small delay, Appl. Math. Comput., 204(2008), pp.90-98.

[46] Kadalbajoo M. K., Kumar D., A computational method for singularly perturbed nonlinear differential-difference equations with small shift, Appl. Math. Modelling, 34(2010), pp.2584-2596.

[47] Kadalbajoo M. K., Patidar K. C., ϵ-Uniformly convergent fitted mesh finite difference methods for general singular perturbation problems, Appl. Math. Comput., 179(2006), pp.248-266.

[48] Kadalbajoo M. K., Sharma K. K., ϵ uniform fitted mesh method for singularly perturbed differential-difference equations: Mixed type of shifts with layer behavior, Int. J. Comput. Math. 81(1)(2004), pp.49-62.

[49] Kadalbajoo M. K., Sharma K. K., Numerical treatment for singularly perturbed non-linear differential-difference equations with negative shifts, Nonlinear Anal., 63(5-7)(2005), pp.1909-1924.

[50] Kadalbajoo M. K., Sharma K. K., Numerical treatment of a mathematical model arising from a model of neuronal variability, J. Math. Anal. Appl., 307(2005), pp.606-627.

[51] Kadalbajoo M. K., Sharma K. K., Parameter-uniform fitted mesh method for singularly perturbed delay differential equations with layer behavior, Electron. Trans. Numer. Anal., 23(2006), pp.180-201.

[52] Kadalbajoo M. K., Ramesh V. P., Hybrid method for numerical solution of singularly perturbed delay differential equations, Appl. Math. Comput., 187(2007), pp.797-814.

[53] Kadalbajoo M. K., Ramesh V. P., Numerical methods on Shishkin mesh for singularly perturbed delay differential equations with a grid adaptation strategy, Appl. Math. Comput., 188(2007), pp.1816-1831.

[54] Kadalbajoo M. K., Yadaw A. S., B-spline collocation method for a two-parameter singularly perturbed convection-diffusion boundary value problems, Appl. Math. Comput., 201(2008), pp.504-513.

[55] Kadalbajoo M. K., Yadaw A. S., Parameter-uniform Ritz-Galerkin finite element method for two-parameter singularly perturbed boundary value problems, Int. J. Pure Appl. Math., 55(2009), pp.287-300.

[56] Kopteva N. V., On the convergence uniform with respect to the small parameter of a scheme with central difference on refined grids , Comput. Math. Math. Phys., 39(10)(1999), pp.1594-1610.

[57] Kopteva N., Linß T., Uniform second-order pointwise convergence of a central difference approximation for a quasilinear convection-diffusion problem, J. Comput. Appl. Math., 137(2001), pp.257-267.

[58] Kopteva N., Stynes M., Approximation of derivatives in a convection-diffusion two-point boundary value problem, Appl. Numer. Math., 39(2001), pp.47-60.

[59] Kopteva N., Stynes M., Numerical analysis of a singularly perturbed nonlinear reaction-diffusion problem with multiple solutions, Appl. Numer. Math., 51(2004), pp.273-288.

[60] Kumar M., Rao S. C. S., High order parameter robust mnumerical method for singularly perturbed reaction-diffusion problems, Appl. Math. Comput., 216(2010), pp. 1036-1046.

[61] Lenferink W., Point-wise convergence of approximations to a convection-diffusion equation on a Shishkin mesh, Appl. Numer. Math., 32(2000), pp.69-86.

[62] Linß T., An upwind difference scheme on a novel Shishkin-type mesh for a linear convection-diffusion problem, J. Comput. Appl. Math., (1999), pp.93-104.

[63] Linß T., Analysis of a Galerkin finite element method on a Bakhvalov-Shishkin mesh for a linear convection-diffusion problem, IMA J. Numer. Anal., 20(4)(2000), pp.621-632.

[64] Linß T., Sufficient conditions for uniform convergence on layer-adapted grids, Appl. Numer. Math., 37 (2001), pp.241-255.

[65] Linß T., Finite difference schemes for convection-diffusion problems with a concentrated source and a discontinuous field, Comput. Methods Appl. Math., 2(1)(2002), pp.41-49.

[66] Linß T., Layer-adapted meshes for convection-diffusion problems, Comput. Methods Appl. Mech. Eng., 192(2003), pp.1061-1105.

[67] Linß T., Robust convergence of a compact fourth-order finite difference scheme for reaction-diffusion problems, Numer. Math., 111(2008), pp.239-249.

[68] Linß T., Madden N., Accurate solution of a system of coupled singularly perturbed reaction-diffusion equations, Computing, 73 (2004), pp.121-133.

[69] Liseikin V. D., The use of special transformations in the numerical solution of boundary layer problems, Comput. Math. Math. Phys., 30(1)(1990), pp.43-53.

[70] Liseikin V. D., Layer resolving grids and transformations for singular parturbation problems. VSP BV, Utrecht, 2001.

[71] Liseikin V. D., Yanenko N. N., On the numerical solution of equations with interior and exterior boundary layers on a nonuniform mesh. In BAIL III, Proc. 3rd Int. Conf. Boundary and interior layers, pp. 68-80, Dublin/Ireland, 1984.

[72] Longtin A., Milton J., Complex oscillations in the human pupil light reflex with mixed and delayed feedback, Math. Biosci. 90(1988), pp.183-199.

[73] Mackenzie J. A., Uniform convergence analysis of an upwind finite-difference approximation of a convection-diffusion boundary value problem on an adaptive grid, IMA J. Numer. Anal., 19(2)(1999), pp.233-249.

[74] Mackey M. C., Glass L., Oscillations and chaos in physiological control systems, Science., 197(1977), pp.287-289.

[75] Madden N., Stynes M., A uniformly convergent numerical method for a coupled system of two singularly perturbed linear reaction-diffusion problems, IMA J. Numer. Anal., 23(2003), pp.627-644.

[76] Matthews S., Miller J. J. H., O'Riordan E., Shishkin G. I., Parameter-robust numerical methods for a system of reaction-diffusion problems with boundary layers, in: G. I. Shishkin, J. J. H. Miller, L.

Vulkov (Eds.), Analytical and numerical methods for convection-dominated and singularly perturbed problems. New York: Nova Science Publishers; USA, 2000, pp.219-224.

[77] Matthews S., O'Riordan E., Shishkin G. I., A numerical method for a system of singularly perturbed reaction-diffusion equations, J. Comput. Appl. Math., 145(2002), pp.151-166.

[78] Miller J. J. H., O'Riordan E., Shishkin G. I., On piecewise-uniform meshes for upwind and central-difference operators for solving singularly perturbed problems, IMA J. Numer. Anal., 15(1)(1995), pp.89-99.

[79] Miller J. J. H., O'Riordan E., Shishkin G. I., Fitted Numerical Methods For Singular Perturbation Problems, World Scientific, 1996

[80] Miller J. J. H., O'Riordan E., Shishkin G. I., Wang S., A parameter-uniform Schwarz method for a singularly perturbed reaction-diffusion problem with an interior layer, Appl. Numer. Math., 35(2000), pp.323-337.

[81] Mohapatra J., Natesan S., Uniformly convergent second order numerical method for singularly perturbed delay differential equations, Neural Parallel Sci. Comput. 16(2008), pp.353-370.

[82] Natesan S., Bawa K. Rajesh, Second order numerical scheme for singularly perturbed reaction-diffusion robin problems, J. Numer. Anal. Indust. Appl. Math., 2(3-4)(2007), pp.177-192.

[83] Natesan S., Bawa K. Rajesh, Clavero C., Uniformly convergent compact numerical scheme for the normalized flux of singularly perturbed reaction-diffusion problems., Int. J. Inform. Syst. Sci., 3(2)(2007), pp.207-221.

[84] Natesan S., Deb B. S., A robust computational method for singularly perturbed coupled system of reaction-diffsion boundary value problems, Appl. Math. Comput., 188(2007), pp.353-364.

[85] Natesan S., Jaykumar J., Vigo-Aguiar J., Parameter uniform numerical method for singularly perturbed turning point problems exhibiting boundary layers, J. Comput. Appl. Math., 158(2003), pp.121-134.

[86] Natividad M. C., Stynes M., Richardson extrapolation for a convection-diffusion problem using a Shishkin mesh, Appl. Numer. Math., 45(2003), pp.315-329.

[87] Nayfeh A. H., Introduction to Perturbation Methods, Wiley, New York, 1981.

[88] O'Riordan E., Pickett M. L., Shishkin G. I., Singularly perturbed problems modeling reaction-convection-diffusion processes, Comput. Methods Appl. Math., 3(3)(2003), pp.424-442.

[89] O'Riordan E., Stynes M., Numerical analysis of a strongly coupled system of two singularly perturbed convection-diffusion problems, Adv. Comput. Math., 30(2009), pp.101-121.

[90] Patidar K. C., High order parameter uniform numerical method for singular perturbation problems, Appl. Math. Comput., 188(2007), pp.720-733.

[91] Patidar K. C., Shikongo A., The application of asymptotic analysis for developing reliable numerical method for a model singular perturbation problem, J. Numer. Anal. Indust. Appl. Math., 2(3-4)(2007), pp.193-207.

[92] Priyadharshini R. M., Ramanujam N., Approximation of derivative to a singularly perturbed second order ordinary differential equation with discontinuous convection coefficient using hybrid difference scheme, Int. J. comput. Math., 86(8)(2009), pp.1355-1365.

[93] Priyadharshini R. M., Ramanujam N., Tamilselvan A. Hybrid Difference schemes for a system of singularly perturbed convection-diffusion equations, J. Appl. Math. and Informatics, 27(5-6)(2009), pp.1001-1015.

[94] Qiu Y., Sloan D. M., Analysis of difference approximations to a singularly perturbed two-point boundary value problem on an adaptively generated grid, J. Comput. Appl. Math., 101(1-2)(1999), pp.1-25.

[95] Qiu Y., Sloan D. M., Tang T., Numerical solution of a singularly pertured two-point boundary valuel problem using equidistribution: Analysis of convergence, J. Comput. Appl. Math., 116(1)(2000), pp.121-143.

[96] Ramanujam N., Shanthi V., Computational methods for reaction-diffusion problems for fourth order ordinary differential equations with a small parameter at the highest derivative, Appl. Math. Comput., 147(2004), pp.97-113.

[97] Ramanujam N., Shanthi V., Asymptotic numerical method for boundary value problems for singularly perturbed fourth-order ordinary differential equations with a weak interior layer, Appl. Math. Comput., 172(2006), pp.252-266.

[98] Rao S. C. S., Kumar M., Parameter-uniformly convergent exponential spline difference scheme for singularly perturbed semilinear reaction-diffusion problems, Nonlinear Anal., 71(2009), pp.e1579-e1588.

[99] Rao S. C. S., Kumar S., Kumar M., A Parameter-uniform B-spline collocation method for singularly perturbed semi linear reaction-diffusion problems, J. Optim. Theory Appl., 71(2009) 146 (2010), pp.795-809.

[100] Rashidinia J., Ghasemi M., B-spline collocation for solution of two-point boundary value problems, J. Comput. Appl. Math., 235(2011), pp.2325-2342.

[101] Rashidinia J., Ghasemi M., Mahmoodi Z., Spline approach to the solution of a singularly-perturbed boundary-value problems, Appl. Math. Comput., 189(2007), pp.72-78.

[102] Roberts S. M., A boundary value technique for singular perturbation problems, J. Math. Anal. Applic., (87)(1982), pp.489-508.

[103] Roos H. G., Layer-adapted grids for singular perturbation problems, ZAMM Z. Angew. Math. Mech., 78(5)(1998), pp.291-309.

[104] Roos H. G., Skalický T., A comparison of the finite element method on Shishkin and Gartland-type meshes for convection-diffusion problems, J. Comput. Appl. Math. CWI Q., 10(3-4)(1997), pp.277-300.

[105] Roos H. G., Stynes M., and Tobiska L., Numerical Methods for Singularly Perturbed Differential Equations. Springer, Berlin, 1996.

[106] Roos H. G., Uzelac Z., The SDFEM for a convection-diffusion problem with two small parameters, Comput. Methods Appl. Math., 3(3)(2003), pp.443-458.

[107] Roos H. G., Zarin H., The streamline-diffusion method for a convection-diffusion problem with a point source, J. Comput. Appl. Math., 150(2003), pp.109-128.

[108] Shahraki M. M., Hosseini S. M., Comparison of a higher order method and the simple upwind and non-monotone methods for singularly perturbed boundary value problems, Appl. Math. Comput., 182(2006), pp.460-473.

[109] Shanthi V., Ramanujam N., Asymptotic numerical methods for singularly perturbed fourth order ordinary differential equations of convection diffusion type, Appl. Math. Comput., 133(2002), pp.559-579.

[110] Shanthi V., Ramanujam N., An asymptotic numerical method for fourth order singular perturbation problrms with a discontinuous source term, Int. J. Comput. Math., 85(7)(2008), pp.1147-1159.

[111] Shanthi V., Ramanujam N., Natesan S., Fitted mesh method for singularly perturbed reaction-convection-diffusion problems with boundary and interior layers, J. Appl. Math. Comput., 22(1-2)(2006), pp.49-65.

[112] Shishkin G. I., A difference scheme for a singularly perturbed parabolic equation with a discontinuous boundary condition, Zh. Vychisl. Mat. Mat. Fiz., 28(1988), pp.1679-1692 (in Russian).

[113] Shishkin G. I., Mesh approximatin of singularly perturbed boundary-value problems for systems of elliptic and parabolic equations, Comput. Maths. Math. Phys. (35)(4)(1995), pp.429-446.

[114] Stein R. B., Some models of neuronal variability, Biophysical Journal, 7(1967), pp.37-68.

[115] Stynes M., Roos H.G., The midpoint upwind scheme, Appl. Numer. Math., 23 (1997), pp.361-374.

[116] Stynes M., Tobiska L., A finite difference analysis of a streamline diffusion method on a Shishkin mesh, Numer. Algorithms, 18(1998), pp.337-360.

[117] Sun Guangfu, Stynes Martin., An almost fourth order uniformly convergent difference scheme for a semilinear singularly perturbed reaction-diffusion problem, Numer. Math., 70(1995), pp.487-500.

[118] Sun G. F., Stynes M., A uniformly convergent method for a singularly perturbed semilinear reaction-diffusion problem with multiple solutions, Math.Comp., 65(215)(1996), pp.1085-1109.

[119] Surla K., Teofanov L., Uzelac Z., The structure of spline collocation matrix for singularly perturbation problems with two small parameters, Novi Sad J. Math., 35(1)(2005), pp.41-48.

[120] Surla K., Teofanov L., Uzelac Z., A robust layer-resolving spline collocation method for a convection-diffusion problem, Appl. Math. Comput., 208(2009), pp.76-89.

[121] Tamilselvan A., Ramanujam N., A parameter uniform numerical method for a system of singularly perturbed convection-diffusion equations with discontinuous convection coefficients, Int. J. Comput. Math., iFirst(2009), pp.1-15.

[122] Tamilselvan A., Ramanujam N., Shanthi V., A numerical method for singularly perturbed weakly coupled system of two second order ordinary differential equations with discontinuous source term, J. Comput. Appl. Math., 202(2007), pp.203-216.

[123] Valanarasu T., Ramanujam N., Asymptotic intial value method for singularly perturbedboundary value problems for second order ordinary differential equations, Appl. Math. Comput, (116)(2003), pp.167-182.

[124] Valanarasu T., Ramanujam N., An asymptotic initial-value method for boundary value problems for a system of singularly perturbed second-order ordinary differential equations, Appl. Math. Comput. (147)(2004), pp.227-240.

[125] Valanarasu T., Ramanujam N., Asymptotic initial-value method for a system of singularly perturbed second-order ordinary differential equations of convection-diffusion type, Int. J. Comput. Math., 81(11)(2004), pp.13811393.

[126] Valanarasu T., Ramanujam N., An Asymptotic initial value method for second order singular perturbation problems of reaction-diffusion type with discontinuous source term, J. Optim. Theory. Appl., 133(3)(2007), pp.371-383.

[127] Valanarasu T., Ramanujam N., An Asymptotic intial value method for second order singular perturbation problems of convection-diffusion type with a discontinuous source term, J. Optim. Theory Appl., 23(2007), pp.141-152.

[128] Valanarasu T., Ramanujam N., An asymptotic numerical method for singularly perturbed third order ordinary differential equations with a weak interior layer, Int. J. comput. Math., 84(3)(2007), pp.333-346.

[129] Valanarasu T., Ramanujam N., Asymptotic numerical method for singularly perturbed third order ordinary differential equations with a discontinuous source term, Novi Sad J. Math., 37(2)(2007), pp.41-57.

[130] Valarmathi S., Ramanujam N., An asymptotic numerical fitted mesh method for singularly perturbed third order ordinary differential equations of reaction-diffusion type, Appl. Math. Comput., 132 (2002), pp.87-104.

[131] Vigo-Aguiar J., Ferrándiz J. M., A general procedure for the adaptation of multistep algorithms to the integration of oscillatory problems, SIAM J. Numer. Anal., 35(4)(1998), pp.1684-1708.

[132] Vigo-Aguiar J., Natesan S., A parallel boundary value technique for singularly perturbed two-point boundary value problems, J. Supercomputing, 27(2004), pp.195-206.

[133] Vigo-Aguiar J., Natesan S., An efficient numerical method for singular perturbation problems, J. Comput. Appl. Math., 192(2006), pp.132-141.

[134] Vulanović R., On a numerical solution of a type of singularly perturbed boundary value problem by using a special discretization mesh, Zb. Rad., Prir.-Mat. Fak., Univ. Novom Sadu, Ser. Mat., 13(1983), pp.187-201.

[135] Vulanović R., Mesh Construction for Discretization of Singularly Perturbed Boundary Value Problems, Ph. D. thesis, University of Novi Sad, 1986.

[136] Vulanović R., High order monotone schemes for a nonlinear singular perturbation problem, ZAMM Z. Angew. Math. Mech., 68(5)(1988), pp.T428-T430.

[137] Vulanović R., A uniform numerical method for quasilinear singular perturbation problems without turning points, Computing, 41(1-2)(1989), pp.97-106.

[138] Vulanović R., A priori meshes for singularly perturbed quasilinear two-point boundary value problems, IMA J. Numer. Anal., 21(1)(2001), pp.349-366.

[139] Vulanović R., Special meshes and higher order schemes for singularly perturbed boundary value problems, Novi Sad J. Math., 31(1)(2001), pp.1-7.

[140] Vulanović R., An almost sixth-order finite-difference method for semilinear singular perturbation problems, Comput. Methods Appl. Math., 4(3)(2004), pp.368-383.

[141] Wazewska-Czyzewska M., Lasota A., Mathematical models of the red cell system, Ma. Stos., 6(1976), pp.25-40.