SOME METHODS FOR SOLVING

FULLY FUZZY LINEAR SYSTEM OF EQUATIONS

Thesis submitted in partial fulfillment of the requirement for

The award of the degree of

Masters of Science

in

Mathematics and Computing

Submitted by

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JULY 2010

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DEDICATED

TO

GOD, MY PARENTS AND MY SUPERVISOR

CERTIFICATE

I hereby certify that the work which is being presented in the thesis entitled "**Some methods for solving fully fuzzy linear system of equations**" in partial fulfillment of the requirements for the award of degree of Master of Science, School of Mathematics and Computer Applications (SMCA), Thapar University, Patiala is an authentic record of my own work carried out under the supervision of **Dr. Amit Kumar**.

The matter presented in this thesis has not been submitted for the award of any other degree of this or any other university.

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This is to certify that the above statement made by the candidate is correct and true to the best of my knowledge.

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ACKNOWLEDGEMENT

First of all, I would like to thank the Almighty for granting perseverance. I would like to express my gratitude to **Dr. Amit Kumar, Assistant Professor, SMCA, Thapar University, Patiala,** for their patient guidance and support throughout this work. I was truly very fortunate to have the opportunity to work under him as a student. It was both an honor and a privilege to work with him. He also provides help in technical writing and presentation style and I found this guidance to be extremely valuable. I take this opportunity to express my sincere thanks to **Prof. S. S. Bhatia, Head SMCA, Thapar University, Patiala,** for their valuable support and help without which it would not have been possible for me to complete this work.

I am also thankful to **Miss Neetu Babbar**, Research Scholar, SMCA and all my friends who devoted their valuable time and helped me in all possible ways towards successful completion of this work. I do not find enough words with which I can express my feeling of thanks to the entire research scholars, faculty and staff of SMCA, Thapar University, Patiala, for their help, inspiration and moral support which went a long way in successful completion of my work. I thank all those who have contributed directly or indirectly to this work.

Lastly, and more importantly, I would like to thank my parents for their years of unyielding love and encouragement. They have always wanted the best for me and I admire my parent's determination and sacrifice to put me through college.

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ABSTRACT

System of simultaneous linear equations plays a major role in various areas such as mathematics, physics, statistics, engineering and social sciences. These are important for studying and solving a large proportion of the problems in many topics in applied mathematics. Usually in many applications all the system's parameters are represented by fuzzy rather than crisp numbers, and hence it is important to develop mathematical models and numerical procedures that would appropriately treat general fully fuzzy linear system and solve them.

This thesis is devoted to some methods for solving fully fuzzy linear systems (system of linear equations in which all the parameters are represented by fuzzy numbers).

The chapter-wise summary of the thesis is as follows:

Chapter 1 is introductory in nature. This chapter includes the basic concepts used throughout the work.

Chapter 2 presents brief review of the work done in the area of fuzzy linear system of equations.

In **Chapter 3**, matrix inversion method, Cramer's rule and *LU* decomposition method to solve the fully fuzzy linear systems are presented. To illustrate all the presented methods numerical examples are solved.

In **Chapter 4**, *LU* decomposition method, presented in chapter 3, is modified to show the advantages of the modified method. The numerical example, solved in chapter 3, is solved by the modified *LU* decomposition method and it is shown that the results obtained by both the methods are same while it is easy and less time

consuming to apply the modified LU decomposition method as compared to the LU decomposition method, presented in chapter 3.

In **Chapter 5** Jacobi and Gauss-Seidel methods are used to find the solution of fully fuzzy linear system of equations. To compare the methods a numerical example in chapter 3 is solved by using both the methods and it is shown that in number of iterations in Gauss-Jacobi method are more than numbers of iterations in Gauss-Seidel method.

Chapter 6, The methods presented in the previous chapter can be applied only for finding the solution of simultaneous fully fuzzy linear system of equations i.e the fully fuzzy linear system of equations in which numbers of fuzzy variables equals to number of equations. These methods can't be applied for solving non-simultaneous fully fuzzy linear system of equations. To overcome this shortcoming in this chapter a method is presented which can be applied for finding the solution of simultaneous as well as non-simultaneous fully fuzzy linear system of equations. To illustrate the presented method numerical examples are solved.

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Chapter 1

INTRODUCTION

There are several applications in science and engineering where relevant physical law(s) immediately produces a set of linear algebraic equations. A system of linear equations (or linear system) is a collection of linear equations involving the same set of variables. A general system of *m* linear equations with *n* unknowns may be written as:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &= b_m \end{aligned}$$

(1.1) In matrix form this system may be written as

$$A \otimes x = b$$

where A is an $m \times n$ matrix, x is a column vector with n entries, and b is a column vector with m entries.

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}, \qquad X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \qquad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

Definition 1.1

If all b_i are zero then the system of equations is said to be homogeneous and if at least one of b_i is not zero then it is said to be non-homogeneous.

Definition 1.2

The non-homogeneous system has a unique solution if and only if the determinant of A is non-zero.

Definition 1.3

The solution of a linear system is an assignment of values to the variables $x_1, x_2, ..., x_n$ such that each of the equations is satisfied. The set of all possible solutions is called the solution set.

A linear system of equations may behave in any one of three possible ways:

(i) The system has infinitely many solutions.

(ii) The system has a single unique solution.

(iii) The system has no solution.

If the system is consistent then we have either unique solution or infinitely many solutions of the non-homogeneous system. (1.1) has a unique solution if and only if determinant of A is non-zero i.e. $|A| \neq 0$ and the solution of the system may be written as $x = A^{-1}b$. The homogeneous system possesses only a trivial solution $x_1 = x_2 = \cdots = x_n = 0$ if $|A| \neq 0$. But it is interesting to note that a homogeneous system has also nontrivial solution provided $|A| = 0$ i.e. coefficient matrix is singular.

1.1 Methods to solve the system of linear equations

There are large numbers of methods to solve the system of linear equations and systems of three or four equations can be easily solved by hand using these methods, but computers are often used for larger systems.

These methods are generally divided into two categories:

(i) Direct method

(ii) Iterative method

There are several direct and iterative methods in the literature for solving the system of linear equations but in this thesis the following direct methods and iterative methods are used:

Direct methods

(i) Matrix inversion method

(ii) Cramer's rule

(iii) LU decomposition method

Iterative methods

(i) Gauss-Jacobi method

(ii) Gauss-Seidel method

1.2 Fuzzy linear system

For constructing a system of linear equations it is assumed that there is no uncertainty about any parameter of the system but in real life there may exist uncertainty about some or all parameters. In such a case it is better to represent all the parameters of system of linear equations by fuzzy sets (Zadeh, 1965).

Definition 1.4

Let X denote a universal set then a fuzzy subset $\tilde{A}$ of X is defined by its membership function

$$\mu_{\tilde{A}} : X \to [0,1]$$

which assigns a real number $\mu_{\tilde{A}}(x)$ in the interval [0, 1], to each element $x \in X$, where the value of $\mu_{\tilde{A}}(x)$ at x shows the grade of membership of x in $\tilde{A}$. A fuzzy subset $\tilde{A}$ can be characterized as a set of ordered pairs of element x and grade $\mu_{\tilde{A}}(x)$ and is often written as:

$$\tilde{A} = \left\{ \left(x, \mu_{\tilde{A}}(x) \right) \mid x \in X \right\}$$

Definition 1.5

A fuzzy set $\tilde{A}$ in X is said to be normal if there exist $x \in X$ such that $\mu_{\tilde{A}}(x) = 1$.

.**Definition 1.6**

A fuzzy number $\tilde{A}$ is called positive (negative), denoted by $\tilde{A} > 0$ $(\tilde{A} < 0)$, if its membership function $\mu_{\tilde{A}}(x) = 0,\ \forall x < 0\ (\forall x > 0)$.

Definition 1.7

A fuzzy set with the following membership function

$$\mu_{\tilde{A}}(x) = \begin{cases} 1 - \dfrac{m-x}{\alpha}, & m - \alpha \le x < m,\ \alpha > 0, \\ 1 - \dfrac{x-m}{\beta}, & m \le x \le m + \beta,\ \beta > 0, \\ 0, & \text{otherwise.} \end{cases}$$

is named a triangular fuzzy number and is symbolically written as $\tilde{A} = (m, \alpha, \beta)$.

Definition 1.8

A triangular fuzzy number $\tilde{A} = (m, \alpha, \beta)$ is positive if and only if $m - \alpha \ge 0$

Definition 1.9

Two triangular fuzzy numbers $\tilde{A} = (m, \alpha, \beta)$ and $\tilde{B} = (n, \gamma, \delta)$ are said to be equal if and only if $m = n, \alpha = \gamma, \beta = \delta$.

Definition 1.10

A matrix $\tilde{A} = (\tilde{a}_{ij})$ is called a fuzzy matrix, if each element of $\tilde{A}$ is a fuzzy number.

A fuzzy matrix $\tilde{A}$ will be positive and denoted by $\tilde{A} > 0$ if each element of $\tilde{A}$ be positive. We may represent $n \times n$ fuzzy matrix $\tilde{A} = (\tilde{a}_{ij})_{n \times n}$ such that $\tilde{a}_{ij} = (a_{ij}, \alpha_{ij}, \beta_{ij})$ with new notation $\tilde{A} = (A, M, N)$, where $A = (a_{ij})$, $M = (\alpha_{ij})$ and $N = (\beta_{ij})$ are three $n \times n$ crisp matrices.

Definition 1.11

A square matrix $\tilde{A} = (\tilde{a}_{ij})$ is called an upper triangular matrix if

$$\tilde{a}_{ij} = \tilde{0} = (0,0,0), \qquad \forall i > j,$$

and a square matrix will be lower triangular matrix $\tilde{A} = (\tilde{a}_{ij})$ if

$$\tilde{a}_{ij} = \tilde{0} = (0,0,0), \qquad \forall i < j,$$

where $\tilde{0} = (0,0,0)$ as a zero triangular fuzzy number.

Definition 1.12

Consider the $n \times n$ fully fuzzy linear system of equations:

$$\begin{cases} (\tilde{a}_{11} \otimes \tilde{x}_1) \oplus (\tilde{a}_{12} \otimes \tilde{x}_2) \oplus \cdots \oplus (\tilde{a}_{1n} \otimes \tilde{x}_n) = \tilde{b}_1, \\ (\tilde{a}_{21} \otimes \tilde{x}_1) \oplus (\tilde{a}_{22} \otimes \tilde{x}_2) \oplus \cdots \oplus (\tilde{a}_{2n} \otimes \tilde{x}_n) = \tilde{b}_2, \\ \vdots \\ (\tilde{a}_{n1} \otimes \tilde{x}_1) \oplus (\tilde{a}_{n2} \otimes \tilde{x}_2) \oplus \cdots \oplus (\tilde{a}_{nn} \otimes \tilde{x}_n) = \tilde{b}_n. \end{cases}$$

The matrix form of the above equation

$$\tilde{A} \otimes \tilde{x} = \tilde{b}$$

where $\tilde{A} = (\tilde{a}_{ij}), 1 \le i, j \ge n$ is a $n \times n$ fuzzy matrix and $\tilde{x}_j, \tilde{b}_j \in F(R),$ where $F(R)$ is the set of all fuzzy numbers.

Definition 1.13

Let $\tilde{A} = (\tilde{a}_{ij})$ and $\tilde{B} = (\tilde{b}_{ij})$ be two $m \times n$ and $n \times p$ fuzzy matrices. We define $\tilde{A} \otimes \tilde{B} = \tilde{C} = (\tilde{c}_{ij})$ which is the $m \times p$ matrix where

$$(\tilde{C}_{ij}) = \sum_{k=1}^{n} \tilde{a}_{ik} \otimes \tilde{b}_{kj}$$

1.3 Arithmetic operations on fuzzy numbers

In this section arithmetic operations of triangular fuzzy numbers are presented.

Let $\tilde{A} = (m, \alpha, \beta)$ and $\tilde{B} = (n, \gamma, \delta)$ be two triangular fuzzy numbers then

(1) $\tilde{A} \oplus \tilde{B} = (m, \alpha, \beta) \oplus (n, \gamma, \delta) = (m+n, \alpha+\gamma, \beta+\delta)$.

(2) $-\tilde{A} = -(m, \alpha, \beta) = (-m, \beta, \alpha)$.

(3) If $\tilde{A} > 0$ and $\tilde{B} > 0$ then $(m, \alpha, \beta) \otimes (n, \gamma, \delta) \cong (mn, m\gamma + n\alpha, m\delta + n\beta)$.

(4) If λ is any scalar then $\lambda \otimes \tilde{A}$ is defined as

$$\lambda \otimes (m, \alpha, \beta) = \begin{cases} (\lambda m, \lambda\alpha, \lambda\beta), & \lambda \geq 0 \\ (\lambda m, -\lambda\beta, -\lambda\alpha), & \lambda < 0 \end{cases}$$

Chapter 2

LITERATURE REVIEW

Buckley and Qu (1990) presented necessary and sufficient conditions for some linear and quadratic equations to have a solution when the parameters are either real or complex fuzzy numbers. They showed that the solution will be a real or complex fuzzy number. Also they presented applications in chemistry, economics, finance and physics are presented for these types of equations.

Buckley and Qu (1991) presented a method to find the solution $\tilde{x}$ to the fuzzy matrix equation $\tilde{A} \otimes \tilde{x} = \tilde{b}$ when the elements in $\tilde{A}$ and $\tilde{b}$ are triangular fuzzy numbers. $\tilde{A}$ is square and non-singular matrix.

Zhao and Govind (1991) studied the algebraic equations involving generalized fuzzy numbers (which includes fuzzy numbers, fuzzy intervals, crisp numbers and interval numbers) with continuous membership functions.

Buckley (1992) proposed new solution procedure for solving fuzzy equations (linear, non-linear and differential) to the following three problems in economics and finance: (1) Leontief's input-output model (2) the internal rate of return and (3) a dynamic supply-demand model where price and supply are governed by a system of differential equations.

Friedman et al. (1998) investigated a general fuzzy system using the embedding approach. They derived the conditions for the existence of a unique fuzzy solution to $n \times n$ linear system of equations and designed a numerical procedure for calculating the solution. They illustrated the applicability of the proposed model with examples.

Rao and Chen (1998) proposed the solution of simultaneous linear equations that arise in the analysis of engineering systems involving fuzzy input parameters. In addition to a discussion on the existence of solution to the problem, they defined and formalized the numerical solution to the fuzzy linear system . The proposed methodology consists of three major aspects involving, computerized selection of fuzziness, implementation of fuzzy operations and development and execution of a search based algorithm.

Feuring et al. (1998) proposed a method to find solution to the fully fuzzified linear program where all the parameters and variables are fuzzy numbers. Firstly they changed the problem of maximizing a fuzzy number, the value of the objective function, into a multi-objective fuzzy linear programming problem. They designed an evolutionary algorithm to solve the fuzzy flexible program and applied this program to two applications to generate good solutions.

Allahviranloo (2004) proposed algorithms for solving ‘fuzzy system of linear equations’. He discussed the schemes based on the iterative Jacobi and Gauss-Seidel methods in detail.

Allahviranloo (2005) transformed the Gauss-Seidel iterative method for solving fuzzy system of linear equations to the successive over relaxation method.

Dehghan and Hashemi (2006) investigated a general fuzzy linear system of equations and then extended several well-known numerical algorithms such as Richardson, Jacobi, Jacobi Over Relaxation, Gauss-Seidel, Successive Over Relaxation, Accelerated Over Relaxation, Symmetric and Unsymmetrical Successive Over Relaxation and Extrapolated Modified Aitken for solving fuzzy linear system of equations.

Dehghan et al. (2006) extended the Adomain decomposition method for fully fuzzy linear system of equations. For find a solution of the system $\tilde{A} \otimes \tilde{x} = \tilde{b}$, where $\tilde{A}$ and $\tilde{b}$ are respectively a fuzzy matrix and fuzzy vector.

Mosleh et al. (2007) proposed the solution of the fully fuzzy linear system $Ax + b = Cx + d$, where A and C are fuzzy matrices, b and d are fuzzy vectors.

Dehghan et al. (2007) employed Dubois and Prade's approximate arithmetic operators on LR fuzzy numbers for finding a positive fuzzy vector $\tilde{x}$ which satisfies $\tilde{A} \otimes \tilde{x} = \tilde{b}$, where $\tilde{A}$ and $\tilde{b}$ are a fuzzy matrix and a fuzzy vector, respectively. Fully fuzzy linear system of equations is transformed and proposed iterative techniques such as Richardson, Jacobi, Jacobi over relaxation, Gauss–Seidel, Successive Over Relaxation, Accelerated Over Relaxation, Symmetric and Unsymmetrical Successive Over Relaxation and Extrapolated Modified Aitken are used for solving fully fuzzy linear system of equations. In addition, the methods of Newton, Quasi-Newton and Conjugate Gradient are proposed from non-linear programming for solving a fully fuzzy linear system of equations.

Nasseri et al. (2008) used a certain decomposition of the coefficient matrix of the fully fuzzy linear system of equations to obtain a simple algorithm for solving these systems. The new algorithm can solve fully fuzzy linear system of equations in a smaller computing process.

Allahviranloo et al. (2008) presented a method to solve fully fuzzy linear system of equations. They replaced the $n \times n$ fully fuzzy linear system of equations by three $n \times n$ crisp linear system of equations. They calculated its homomorphic solution in canonical trapezoidal form based on three $n \times n$ crisp linear solutions associated with three parameters, value, ambiguity and fuzziness.

Gao and Zhang (2009) proposed a unified iterative scheme to solve general fully fuzzy linear system of equations $\tilde{A} \otimes \tilde{x} = \tilde{b}$ in which all parameters are LR fuzzy numbers. By this iterative scheme, they presented Gradient iterative algorithm and Least-squares iterative algorithm for solving non-square fully fuzzy linear system of equations.

Mikaeilvand and Allahviranloo (2009) found from the definition of extended operations on fuzzy numbers, subtraction and division of fuzzy numbers are not the inverse operations to addition and multiplication, respectively. Hence for solving equations or system of equations, methods without using inverse operators must be used. They proposed a novel method to find the non-zero solutions of fully fuzzy linear system of equationss. They splitted system's parameters to two groups of non-positives and non-negatives by solving one multi objective linear program and employing embedding method to transform $n \times n$ fully fuzzy linear system of equations to $2n \times 2n$ parametric form linear system of equations and hence, transformed operations on fuzzy numbers to operations on functions. Finally, they used numerical examples to illustrate their approach.

Mosleh et al. (2009) discussed a new decomposition of a non-singular fuzzy matrix, the symmetric times triangular decomposition. By this decomposition, every non-singular fuzzy matrix can be represented as a product of a fuzzy symmetric matrix S and a fuzzy triangular matrix T.

Rahgooy et al. (2009) modeled the fuzzy complex linear equations in the field of circuit analysis and then solved it by an existing method.

Nasseri and Zahmatkesh (2010) concentrated on solving fully fuzzy linear system of equations of equations and proposed Huang method for computing a non-negative solution of the fully fuzzy linear system of equations.

Liu (2010) proposed an approach to solve fully fuzzy linear system of equations that is found to give more accurate than the iterative Jacobi and Gauss-Seidel methods on approximating the solution of fully fuzzy linear system of equations.

Chapter 3

SOME DIRECT METHODS FOR SOLVING FULLY FUZZY LINEAR SYSTEM OF EQUATIONS

In this chapter, matrix inversion method, Cramer's rule and LU decomposition method to solve $\tilde{A}\otimes\tilde{x}=\tilde{b},$ where $\tilde{A}$ is a fuzzy matrix and $\tilde{x}$ and $\tilde{b}$ are fuzzy vectors with appropriate sizes are presented.

3.1 Fully fuzzy linear system and equations

Consider the $n\times n$ fully fuzzy linear system of equations:

$$\begin{cases} (\tilde{a}_{11}\otimes\tilde{x}_1)\oplus(\tilde{a}_{12}\otimes\tilde{x}_2)\oplus\cdots\oplus(\tilde{a}_{1n}\otimes\tilde{x}_n) = \tilde{b}_1, \\ (\tilde{a}_{21}\otimes\tilde{x}_1)\oplus(\tilde{a}_{22}\otimes\tilde{x}_2)\oplus\cdots\oplus(\tilde{a}_{2n}\otimes\tilde{x}_n) = \tilde{b}_2, \\ \vdots \\ (\tilde{a}_{n1}\otimes\tilde{x}_1)\oplus(\tilde{a}_{n2}\otimes\tilde{x}_2)\oplus\cdots\oplus(\tilde{a}_{nn}\otimes\tilde{x}_n) = \tilde{b}_n. \end{cases} \tag{3.1}$$

The matrix form of the above system is

$$\tilde{A}\otimes\tilde{x}=\tilde{b} \tag{3.2}$$

where the coefficient matrix $\tilde{A}=(\tilde{a}_{ij}),\ 1\le i,j\le n$ is a $n\times n$ fuzzy matrix and $\tilde{x}_i,\tilde{b}_i,\ 1\le i\le n$ are fuzzy vectors. This system is called fully fuzzy linear system.

We want to find the non-negative solution of fully fuzzy linear system $\tilde{A}\otimes\tilde{x}=\tilde{b}$ where $\tilde{A}=(A,M,N)\ge 0,\ \tilde{b}=(b,g,h)\ge 0,$ and $\tilde{x}=(x,y,z)\ge 0.$

So we have

$$\left(A, M, N\right) \otimes \left(x, y, z\right) = \left(b, g, h\right) \tag{3.3}$$

3.2 Some direct methods

In this section some direct methods to solve the equation (3.1) is presented:

3.2.1 Matrix inversion method

Using section 1.3, equation (3.3) may be written as

$$(Ax, Ay + Mx, Az + Nx) = (b, g, h) \tag{3.4}$$

Using definition 1.10, we have

$$\begin{aligned} Ax &= b, \\ Ay + Mx &= g, \\ Az + Nx &= h. \end{aligned} \tag{3.5}$$

i.e

$$\begin{aligned} Ax &= b, \\ Ay &= g - Mx \\ Az &= h - Nx. \end{aligned} \tag{3.6}$$

Assuming A as a non-singular matrix, equation (3.6) may be written as

$$\begin{aligned} &x = A^{-1}b, \\ &y = A^{-1}g - A^{-1}Mx, \\ &z = A^{-1}h - A^{-1}Nx. \end{aligned} \tag{3.7}$$

The fuzzy solution (x, y, z) can be easily obtained by using the values of x, y and $z.$

Example 3.1

Consider the following fully fuzzy linear system and solve it by direct method:

$$\begin{aligned} (5,1,1) \otimes (x_1, y_1, z_1) \ \oplus \ (6,1,2) \otimes (x_2, y_2, z_2) &= (50,10,17) \\ (7,1,0) \otimes (x_1, y_1, z_1) \ \oplus \ (4,0,1) \otimes (x_2, y_2, z_2) &= (48,5,7) \end{aligned}$$

So we have,

$$A=\begin{bmatrix}5 & 6\\ 7 & 4\end{bmatrix} \qquad b=\begin{bmatrix}50\\ 48\end{bmatrix} \qquad x=\begin{bmatrix}x_1\\ x_2\end{bmatrix}$$

$$\begin{bmatrix}5 & 6\\ 7 & 4\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}50\\ 48\end{bmatrix} \quad \Rightarrow \begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}4\\ 5\end{bmatrix}$$

Similarly by equations (3.7),

$$\begin{bmatrix}5 & 6\\ 7 & 4\end{bmatrix}\begin{bmatrix}y_1\\ y_2\end{bmatrix}=\begin{bmatrix}10\\ 5\end{bmatrix}-\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}\begin{bmatrix}4\\ 5\end{bmatrix}\Rightarrow\begin{bmatrix}y_1\\ y_2\end{bmatrix}=\begin{bmatrix}\frac{1}{11}\\ \frac{1}{11}\end{bmatrix}$$

and

$$\begin{bmatrix}5 & 6\\ 7 & 4\end{bmatrix}\begin{bmatrix}z_1\\ z_2\end{bmatrix}=\begin{bmatrix}17\\ 7\end{bmatrix}-\begin{bmatrix}1 & 2\\ 0 & 1\end{bmatrix}\begin{bmatrix}4\\ 5\end{bmatrix}\Rightarrow\begin{bmatrix}z_1\\ z_2\end{bmatrix}=\begin{bmatrix}0\\ \frac{1}{2}\end{bmatrix}$$

Thus the solution is

$$\tilde{x}=\begin{bmatrix}(4,\frac{1}{11},0)\\ (5,\frac{1}{11},\frac{1}{2})\end{bmatrix}$$

3.2.2 Cramer's rule

For solving fully fuzzy linear system (3.1) with this method,

$$x_i=\frac{\det(A^{(i)})}{\det(A)}, \qquad i=1,2,\dots,n,$$

where $A^{(i)}$ denotes the matrix obtained from A by replacing its i^{th} column by b. Then using solution x, we have

$$y_i=\frac{\det(A'^{(i)})}{\det(A)}, \qquad i=1,2,\dots,n,$$

$$z_i = \frac{\det(A''^{(i)})}{\det(A)}, \qquad i = 1, 2, \ldots, n,$$

where $A'^{(i)}$ and $A''^{(i)}$ denotes matrix obtained from A by replacing its i^{th} column by $h - Mx$ and $g - Nx$, respectively.

Example 3.2

Consider the following fully fuzzy linear system of equations and solve it using Cramer's rule:

$$\begin{bmatrix} (4,3,2) & (5,2,1) & (3,0,3) \\ (7,4,3) & (10,6,5) & (2,1,1) \\ (6,2,2) & (7,1,2) & (15,5,4) \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{x}_3 \end{bmatrix} = \begin{bmatrix} (71,54,76) \\ (118,115,129) \\ (155,89,151) \end{bmatrix}$$

Here $A = \begin{bmatrix} 4 & 5 & 3 \\ 7 & 10 & 2 \\ 6 & 7 & 15 \end{bmatrix} \quad b = \begin{bmatrix} 71 \\ 118 \\ 155 \end{bmatrix}$

$$\det(A) = 4\begin{vmatrix} 10 & 2 \\ 7 & 15 \end{vmatrix} - 5\begin{vmatrix} 7 & 2 \\ 6 & 15 \end{vmatrix} + 3\begin{vmatrix} 7 & 10 \\ 6 & 7 \end{vmatrix}$$

$$= 4(150 - 14) - 5(105 - 12) + 3(49 - 60)$$

$$= 4 \times 136 - 5 \times 93 - 3 \times 11$$

$$= 544 - 465 - 33 = 46$$

$$A^{(1)} = \begin{bmatrix} 71 & 5 & 3 \\ 118 & 10 & 2 \\ 155 & 7 & 15 \end{bmatrix}, \; A^{(2)} = \begin{bmatrix} 4 & 71 & 3 \\ 7 & 118 & 2 \\ 6 & 155 & 15 \end{bmatrix} \; A^{(3)} = \begin{bmatrix} 4 & 5 & 71 \\ 7 & 10 & 118 \\ 6 & 7 & 155 \end{bmatrix}$$

Determinant of each of above matrix is

$$\det(A^{(1)}) = 184$$
$$\det(A^{(2)}) = 368$$
$$\det(A^{(3)}) = 230$$

Thus we have, $x_1 = \dfrac{184}{46} = 4, \quad x_2 = \dfrac{368}{46} = 8 \quad$ and $\quad x_3 = \dfrac{230}{46} = 5$

$$x = \begin{bmatrix} 4 \\ 8 \\ 5 \end{bmatrix}$$

From the given problem, we have

$$h - Mx = \begin{bmatrix} 54 \\ 115 \\ 89 \end{bmatrix} - \begin{bmatrix} 3 & 2 & 0 \\ 4 & 6 & 1 \\ 2 & 1 & 5 \end{bmatrix}\begin{bmatrix} 4 \\ 8 \\ 5 \end{bmatrix} = \begin{bmatrix} 54 \\ 115 \\ 89 \end{bmatrix} - \begin{bmatrix} 28 \\ 69 \\ 41 \end{bmatrix} = \begin{bmatrix} 26 \\ 46 \\ 48 \end{bmatrix}$$

$$g - Nx = \begin{bmatrix} 76 \\ 129 \\ 151 \end{bmatrix} - \begin{bmatrix} 2 & 1 & 3 \\ 3 & 5 & 1 \\ 2 & 2 & 4 \end{bmatrix}\begin{bmatrix} 4 \\ 8 \\ 5 \end{bmatrix} = \begin{bmatrix} 76 \\ 129 \\ 151 \end{bmatrix} - \begin{bmatrix} 31 \\ 57 \\ 44 \end{bmatrix} = \begin{bmatrix} 45 \\ 72 \\ 107 \end{bmatrix}$$

Now, $\quad A'^{(1)} = \begin{bmatrix} 26 & 5 & 3 \\ 46 & 10 & 2 \\ 48 & 7 & 15 \end{bmatrix} \quad A'^{(2)} = \begin{bmatrix} 4 & 26 & 3 \\ 7 & 46 & 2 \\ 6 & 48 & 15 \end{bmatrix} \quad A'^{(3)} = \begin{bmatrix} 4 & 5 & 26 \\ 7 & 10 & 46 \\ 6 & 7 & 48 \end{bmatrix}$

$$\det(A'^{(1)}) = 92$$
$$\det(A'^{(2)}) = 138$$
$$\det(A'^{(3)}) = 46$$

$$y_1 = \frac{92}{46} = 2, \quad y_2 = \frac{138}{46} = 3 \quad \text{and} \quad y_3 = \frac{46}{46} = 1$$

i.e., $$y = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$$

Similarly, $A''^{(1)} = \begin{bmatrix} 45 & 5 & 3 \\ 72 & 10 & 2 \\ 107 & 7 & 15 \end{bmatrix}$, $A''^{(2)} = \begin{bmatrix} 4 & 45 & 3 \\ 7 & 72 & 2 \\ 6 & 107 & 15 \end{bmatrix}$, $A''^{(3)} = \begin{bmatrix} 4 & 5 & 45 \\ 7 & 10 & 72 \\ 6 & 7 & 107 \end{bmatrix}$

$$\det(A''^{(1)}) = 92$$
$$\det(A''^{(2)}) = 230$$
$$\det(A''^{(3)}) = 184$$

$$z_1 = \frac{92}{46} = 2, \quad z_2 = \frac{230}{46} = 5 \quad \text{and} \quad z_3 = \frac{184}{46} = 4$$

$$z = \begin{bmatrix} 2 \\ 5 \\ 4 \end{bmatrix}$$

Thus the solution of this problem is

$$\tilde{x} = \begin{bmatrix} (4,2,2) \\ (8,3,5) \\ (5,1,4) \end{bmatrix}$$

3.2.3 *LU* decomposition method

In the *LU* decomposition method, the coefficients matrix of the linear system of equations is factored into the product of two triangular matrices. This method is frequently used to solve a large system of equations. Consider the system of equations (3.2), where A is a non-singular matrix. We first of all write the matrix A as the product of a lower triangular matrix L and an upper triangular matrix U in the form

$$\tilde{A} = \tilde{L} \otimes \tilde{U} \text{,}$$

where $\tilde{A} = (A, M, N)$, $\tilde{L} = (L_1, L_2, L_3)$, and $\tilde{U} = (U_1, U_2, U_3)$.

So we have

$$(A, M, N) = (L_1, L_2, L_3) \otimes (U_1, U_2, U_3)$$

$$(A, M, N) = (L_1U_1, L_1U_2 + L_2U_1, L_1U_3 + L_3U_1)$$

$$\Rightarrow \quad A = L_1U_1, \tag{3.8}$$
$$M = L_1U_2 + L_2U_1, \tag{3.9}$$
$$N = L_1U_3 + L_3U_1. \tag{3.10}$$

In order to find a unique solution we either take all the diagonal elements of L as 1 or all the diagonal elements of U as 1. For $U_{ii} = 1,\ i = 1, 2, ..., n$, the method is called the Crout's LU decomposition method and for $L_{ii} = 1,\ i = 1, 2, ..., n$, it is called Doolittle's method. Here we will use Doolittle method.

Firstly, we calculate L_1 and U_1 such that $A = L_1U_1$, where L_1 is a lower triangular crisp matrix, having the diagonal of 1's and U_1 is an upper triangular crisp matrix with the general diagonal.

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ l_{21} & 1 & 0 & \cdots & 0 \\ l_{31} & l_{32} & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ l_{n1} & l_{n2} & l_{n3} & \cdots & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} & \cdots & u_{1n} \\ 0 & u_{22} & u_{23} & \cdots & u_{2n} \\ 0 & 0 & u_{33} & \cdots & u_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & u_{nn} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix}$$

which amount to n^2 equations in the n^2 unknowns l_{ij} and u_{ij}. The computations run as follows:

$$u_{1j} = a_{1j}, \qquad j = 1, 2, ..., n. \tag{3.11}$$

$$l_{i1}u_{11} = a_{i1} \Rightarrow l_{i1} = \frac{a_{i1}}{u_{11}}, \qquad i = 1, 2, ..., n. \tag{3.12}$$

Continuing in a recursive way for $r = 2,3,...,n,$ we alternatively find the rows of U_1 and corresponding columns of L_1 to be

$$u_{rj} = a_{rj} - \sum_{k=1}^{r-1} l_{rk} u_{kj}, \qquad j = r, r+1, ..., n, \tag{3.13}$$

Each row follows by the corresponding column of L_1

$$l_{ir} = \frac{a_{ir} - \sum_{k=1}^{r-1} l_{ik} u_{kr}}{u_{rr}} \qquad i = r, r+1, ..., n, \tag{3.14}$$

We set the diagonals of L_2 and L_3 to be consisting of 0*'s* not 1*'s*.

By (3.9), and $l_2 = (l'_{ij})$ with diagonals of 0*'s* and $U_2 = (u'_{ij})$ we may write

$$m_{ij} = \sum_{k=1}^{n} l_{ik} u'_{kj}, \quad 1 \le i, j \le n, \ l'_{ii} = 0. \tag{3.15}$$

With L_1 and U_1 in hand, we can continue our approach to the second step for L_2 and U_2 in the following way:

$$u'_{1j} = m_{ij}, \qquad j = 1, 2, ..., n, \tag{3.16}$$

$$l'_{i1} = \frac{m_{i1} - l_{i1} u'_{11}}{u_{11}}, \qquad i = 1, 2, ..., n \tag{3.17}$$

Continuing in a recursive way for $r = 2,3,...,n,$ we alternatively find the rows of U_1 and corresponding columns of L_1 to be

$$\begin{aligned} u'_{rj} &= m_{rj} - \sum_{k=1}^{r-1} l_{rk} u'_{kj}, \qquad j = r, r+1, ..., n, \\ l'_{ir} &= \frac{m_{ir} - \sum_{k=1}^{r-1} l'_{ik} u_{kr} - \sum_{k=1}^{r} l_{ik} u'_{kr}}{u_{rr}} \qquad i = r, r+1, ..., n, \end{aligned} \tag{3.18}$$

Similarly by (3.10), and $L_3 = (l''_{ij})$ and $U_3 = (u''_{ij})$ we may write

$$n_{ij} = \sum_{k=1}^{n} l_{ik} u''_{kj} + l''_{ik} u_{kj}, \quad 1 \le i, j \le n. \tag{3.19}$$

We can continue our approach to the second step for finding L_3 and U_3 in the following way:

$$\begin{aligned} u''_{1j} &= n_{1j}, \quad j = 1, 2, \ldots, n, \\ l''_{i1} &= \frac{n_{i1} - l_{i1} u''_{11}}{u_{11}}, \quad i = 1, 2, \ldots, n. \end{aligned} \tag{3.20}$$

Finally we alternatively find the rows of U_3 and the corresponding columns of L_3 for $r = 2, 3, \ldots, n$ to be

$$\begin{aligned} u''_{rj} &= n_{rj} - \sum_{k=1}^{r-1} (l_{rk} u''_{kj} + l''_{rk} u_{kj}), \qquad j = r, r+1, \ldots, n, \\ l''_{ir} &= \frac{n_{ir} - \sum_{k=1}^{r-1} l''_{ik} u_{kr} - \sum_{k=1}^{r} l_{ik} u''_{kr}}{u_{rr}} \qquad i = r, r+1, \ldots, n, \end{aligned} \tag{3.21}$$

The solution to the problem $\tilde{A} \otimes \tilde{x} = \tilde{b}$ could be found by a two step triangular solve process.

$$\begin{aligned} & \tilde{A} \otimes \tilde{x} = \tilde{b} \\ & \tilde{L} \otimes \tilde{U} \otimes \tilde{x} = \tilde{b} \\ \Rightarrow \quad & \tilde{L} \otimes \tilde{x}' = \tilde{b} \\ \text{where} \quad & \tilde{U} \otimes \tilde{x} = \tilde{x}' \end{aligned} \tag{3.22}$$

By solving system (3.22), we get the solution $\tilde{x}$.

Example 3.3

Let $$\tilde{A} = \begin{bmatrix} (6,1,4) & (5,2,2) & (3,2,1) \\ (12,8,20) & (14,12,15) & (8,8,10) \\ (24,10,34) & (32,30,30) & (20,19,24) \end{bmatrix}$$

be a fuzzy matrix

Here

$$A = \begin{bmatrix} 6 & 5 & 3 \\ 12 & 14 & 8 \\ 24 & 32 & 20 \end{bmatrix},\ M = \begin{bmatrix} 1 & 2 & 2 \\ 8 & 12 & 8 \\ 10 & 30 & 19 \end{bmatrix},\ N = \begin{bmatrix} 4 & 2 & 1 \\ 20 & 15 & 10 \\ 34 & 30 & 24 \end{bmatrix}$$

From equations (3.11), (3.12), (3.13) and (3.14), we can find the elements of L_1 and U_1.

$$u_{11} = a_{11} = 6, \qquad u_{12} = a_{12} = 5, \qquad u_{13} = a_{13} = 3.$$

$$l_{11} = \frac{a_{11}}{u_{11}} = \frac{6}{6} = 1, \qquad l_{21} = \frac{a_{21}}{u_{11}} = \frac{12}{6} = 2, \qquad l_{31} = \frac{a_{31}}{u_{11}} = \frac{24}{6} = 4.$$

$$u_{22} = a_{22} - l_{21}u_{12} = 14 - 2\times 5 = 4,$$

$$u_{23} = a_{23} - l_{21}u_{13} = 8 - 2\times 3 = 2$$

$$l_{22} = \frac{a_{22} - l_{21}u_{12}}{u_{22}} = \frac{14 - 2\times 5}{4} = 1,$$

$$l_{32} = \frac{a_{32} - l_{31}u_{12}}{u_{22}} = \frac{32 - 4\times 5}{4} = 3$$

$$u_{33} = a_{33} - l_{31}u_{13} - l_{32}u_{23} = 20 - 4\times 3 - 3\times 2 = 2,$$

$$l_{33} = 1$$

Therefore we have,

$$L_1 = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 4 & 3 & 1 \end{bmatrix} \qquad U_1 = \begin{bmatrix} 6 & 5 & 3 \\ 0 & 4 & 2 \\ 0 & 0 & 2 \end{bmatrix}$$

From equations (3.15), (3.16), (3.17) and (3.18) we can obtained L_2 and U_2,

$$L_2 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 2 & 0 \end{bmatrix} \qquad U_2 = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 3 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Similarly, from (3.19), (3.20) & (3.21), we get

$$L_3 = \begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 3 & 1 & 0 \end{bmatrix} \qquad U_3 = \begin{bmatrix} 4 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{bmatrix}$$

Thus LU decomposition method of fuzzy matrix $\tilde{A}$ is

$$\tilde{A} = \tilde{L} \otimes \tilde{U} = \begin{bmatrix} (1,0,0) & (0,0,0) & (0,0,0) \\ (2,1,2) & (1,0,0) & (0,0,0) \\ (4,1,3) & (3,2,1) & (1,0,0) \end{bmatrix} \otimes \begin{bmatrix} (6,1,4) & (5,2,2) & (3,2,1) \\ (0,0,0) & (4,3,1) & (2,1,2) \\ (0,0,0) & (0,0,0) & (2,1,3) \end{bmatrix}$$

Now we solve the fully fuzzy linear system of equations $\tilde{L}\tilde{x}' = \tilde{b}$ i.e,

$$\begin{bmatrix} (1,0,0) & (0,0,0) & (0,0,0) \\ (2,1,2) & (1,0,0) & (0,0,0) \\ (4,1,3) & (3,2,1) & (1,0,0) \end{bmatrix} \begin{bmatrix} \tilde{x}' \\ \tilde{y}' \\ \tilde{z}' \end{bmatrix} = \begin{bmatrix} (58,30,60) \\ (142,139,257) \\ (316,297,514) \end{bmatrix}$$

Using the Cramer's rule, we easily compute $\tilde{x}'$ in the following form:

$$\tilde{x}' = \begin{bmatrix} (58,30,60) \\ (26,21,21) \\ (6,4,11) \end{bmatrix}$$

Now we solve the fully fuzzy linear system of equations $\tilde{U}\tilde{x} = \tilde{x}'$ i.e,

$$\begin{bmatrix} (6,1,4) & (5,2,2) & (3,2,1) \\ (0,0,0) & (4,3,1) & (2,1,2) \\ (0,0,0) & (0,0,0) & (2,1,3) \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{x}_3 \end{bmatrix} = \begin{bmatrix} (58,30,60) \\ (26,21,21) \\ (6,4,11) \end{bmatrix}$$

Again using the Cramer's rule, we may compute the solution $\tilde{x}$ in the following form:

$$\tilde{x} = \begin{bmatrix} (4,1,3) \\ (5,0.5,2) \\ (3,0.5,1) \end{bmatrix}$$

which is required solution.

3.3 Conclusion

In this chapter the fully fuzzy linear system of equations, i.e. fuzzy linear system of equations with fuzzy coefficients involving fuzzy variables are investigated and some direct methods are applied for solving these systems. It may be useful in analysis of physical systems and other topics in real engineering problems. The proposed schemes for finding the positive solution of the same systems, when parameters are positive are quite satisfactory.

Chapter 4

MODIFIED *LU* DECOMPOSITION METHOD FOR SOLVING FULLY FUZZY LINEAR SYSTEM OF EQUATIONS

In this chapter, the LU decomposition method, presented in chapter 3, is modified. To show the advantages of the modified method, the numerical example, solved in chapter 3, is solved by the modified LU decomposition method and it is shown that the results obtained by both the methods are same while it is easy and less time consuming to apply the modified LU decomposition method as compared to the LU decomposition method, presented in chapter 3.

4.1 Modified *LU* decomposition method

Consider fully fuzzy linear system of equations

$$\tilde{A} \otimes \tilde{x} = \tilde{b}, \tag{4.1}$$

Where $\tilde{A} = (A, M, N)$, $\tilde{x} = (x, y, z)$, $\tilde{b} = (b, h, g)$

Let $\tilde{A} = \tilde{L} \otimes \tilde{U}$ where $\tilde{L} = (L, 0, 0)$ and $\tilde{U} = (U_1, U_2, U_3)$. Here, we consider $\tilde{L} = (L, 0, 0)$ instead of $\tilde{L} = (L_1, L_2, L_3)$ because our aim is to find a lower triangular fuzzy matrix $\tilde{L}$ with the diagonal of fuzzy identity number in LR fuzzy multiplication which is $\tilde{1} = (1,0,0)$, not $\tilde{1} = (1,1,1)$. So, $\tilde{A} = \tilde{L} \otimes \tilde{U}$ now becomes

$$(A, M, N) = (L_1, 0, 0) \otimes (U_1, U_2, U_3)$$

$$\Rightarrow \qquad (A, M, N) = (L_1U_1, L_1U_2, L_1U_3) \tag{4.2}$$

So, from equations (4.1) and (4.2), we have

$$(L_1U_1, L_1U_2, L_1U_3) \otimes (x, y, z) = (b, h, g)$$

By using section 1.3, we have

$$(L_1U_1x, L_1U_2x + L_1U_1y, L_1U_3x + L_1U_1z) = (b, h, g)$$

$$\begin{aligned} &L_1U_1 = b, \\ &L_1U_2x + L_1U_1y = h, \\ &L_1U_3 + L_1U_1z = g, \end{aligned}$$

and therefore,

$$\begin{aligned} &x = U_1^{-1}L_1^{-1}b, \\ &y = U_1^{-1}L_1^{-1}(h - L_1U_2x), \\ &z = U_1^{-1}L_1^{-1}(g - L_1U_3x). \end{aligned} \tag{4.3}$$

Example 4.1

Consider the following fully fuzzy linear system of equations

$$\begin{bmatrix} (6,1,4) & (5,2,2) & (3,2,1) \\ (12,8,20) & (14,12,15) & (8,8,10) \\ (24,10,34) & (32,30,30) & (20,19,24) \end{bmatrix} \begin{bmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{z} \end{bmatrix} = \begin{bmatrix} (58,30,60) \\ (142,139,257) \\ (316,297,514) \end{bmatrix}$$

$$A = \begin{bmatrix} 6 & 5 & 3 \\ 12 & 14 & 8 \\ 24 & 32 & 20 \end{bmatrix}, \; M = \begin{bmatrix} 1 & 2 & 2 \\ 8 & 12 & 8 \\ 10 & 30 & 19 \end{bmatrix}, \; N = \begin{bmatrix} 4 & 2 & 1 \\ 20 & 15 & 10 \\ 34 & 30 & 24 \end{bmatrix}$$

Firstly, we obtain LU – decomposition for matrix A as follow:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

$$u_{11} = a_{11} = 6, \qquad u_{12} = a_{12} = 5, \qquad u_{13} = a_{13} = 3$$

$$l_{21} = \frac{a_{21}}{u_{11}} = \frac{12}{6} = 2, \qquad l_{31} = \frac{a_{31}}{u_{11}} = \frac{24}{6} = 4.$$

$$u_{22} = a_{22} - l_{21}u_{12} = 14 - 2\times 5 = 4, \qquad u_{23} = a_{23} - l_{21}u_{13} = 8 - 2\times 3 = 2$$

$$l_{32} = \frac{a_{32} - l_{31}u_{12}}{u_{22}} = \frac{32 - 4\times 5}{4} = 3,$$

$$u_{33} = a_{33} - l_{31}u_{13} - l_{32}u_{23} = 20 - 4\times 3 - 3\times 2 = 2,$$

Hence we have,

$$L_1 = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 4 & 3 & 1 \end{bmatrix}, \qquad U_1 = \begin{bmatrix} 6 & 5 & 3 \\ 0 & 4 & 2 \\ 0 & 0 & 2 \end{bmatrix}$$

So we can obtain U_2 and U_3 as follows:

$$L_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 2 & -3 & 1 \end{bmatrix}, \qquad U_1^{-1} = \frac{1}{48}\begin{bmatrix} 8 & -10 & -2 \\ 0 & 12 & -12 \\ 0 & 0 & 24 \end{bmatrix}$$

$$U_2 = L_1^{-1}M = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 2 & -3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 2 \\ 8 & 12 & 8 \\ 10 & 30 & 19 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 2 \\ 6 & 8 & 4 \\ -12 & -2 & -1 \end{bmatrix}$$

$x = U_1^{-1}L_1^{-1}b,$

$$U_1^{-1}L_1^{-1} = \frac{1}{48}\begin{bmatrix} 8 & -10 & -2 \\ 0 & 12 & -12 \\ 0 & 0 & 24 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 2 & -3 & 1 \end{bmatrix} = \frac{1}{48}\begin{bmatrix} 24 & -4 & -2 \\ -48 & 48 & -12 \\ 48 & -72 & 24 \end{bmatrix}$$

$$x = \frac{1}{48}\begin{bmatrix} 24 & -4 & -2 \\ -48 & 48 & -12 \\ 48 & -72 & 24 \end{bmatrix}\begin{bmatrix} 58 \\ 142 \\ 316 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 3 \end{bmatrix}$$

$y = U_1^{-1}L_1^{-1}(h - L_1U_2x),$

$$L_1U_2 = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 4 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 \\ 6 & 8 & 4 \\ -12 & -2 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 2 \\ 8 & 12 & 8 \\ 10 & 30 & 19 \end{bmatrix}$$

$$L_1U_2x = \begin{bmatrix} 1 & 2 & 2 \\ 8 & 12 & 8 \\ 10 & 30 & 19 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \\ 3 \end{bmatrix} = \begin{bmatrix} 20 \\ 116 \\ 247 \end{bmatrix}$$

$$h - L_1U_2x = \begin{bmatrix} 30 \\ 139 \\ 297 \end{bmatrix} - \begin{bmatrix} 20 \\ 116 \\ 247 \end{bmatrix} = \begin{bmatrix} 10 \\ 23 \\ 50 \end{bmatrix}$$

$$y = \frac{1}{48} \begin{bmatrix} 24 & -4 & -2 \\ -48 & 48 & -12 \\ 48 & -72 & 24 \end{bmatrix} \begin{bmatrix} 10 \\ 23 \\ 50 \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$$

$z = U_1^{-1}L_1^{-1}(g - L_1U_3x),$

$$L_1U_3 = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 4 & 3 & 1 \end{bmatrix} \begin{bmatrix} 4 & 2 & 1 \\ -12 & 11 & 8 \\ 18 & -11 & -4 \end{bmatrix} = \begin{bmatrix} 4 & 2 & 1 \\ 20 & 15 & 10 \\ 34 & 30 & 24 \end{bmatrix}$$

$$L_1U_3x = \begin{bmatrix} 4 & 2 & 1 \\ 20 & 15 & 10 \\ 34 & 30 & 24 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \\ 3 \end{bmatrix} = \begin{bmatrix} 29 \\ 185 \\ 358 \end{bmatrix}$$

$$g - L_1U_2x = \begin{bmatrix} 60 \\ 257 \\ 514 \end{bmatrix} - \begin{bmatrix} 29 \\ 185 \\ 358 \end{bmatrix} = \begin{bmatrix} 31 \\ 72 \\ 156 \end{bmatrix}$$

$$z = \frac{1}{48} \begin{bmatrix} 24 & -4 & -2 \\ -48 & 48 & -12 \\ 48 & -72 & 24 \end{bmatrix} \begin{bmatrix} 31 \\ 72 \\ 156 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

Therefore,

$$\tilde{x} = (4,1,3), \qquad \tilde{y} = (5,\frac{1}{2},1), \qquad \tilde{z} = (3,\frac{1}{2},1)$$

4.2 Conclusion

LU decomposition method, presented in chapter 3, is modified and the advantage of modified *LU* decomposition method, presented in chapter 3 is shown.

Chapter 5

SOME ITERATIVE METHODS FOR SOLVING FULLY FUZZY LINEAR SYSTEM OF EQUATIONS

In the previous chapters, we have presented some direct methods for solving fully fuzzy linear system of equations. In this chapter, two iterative methods (Gauss-Jacobi, Gauss-Seidel) are presented to find the solution of fully fuzzy linear system of equations. To explain the method, a numerical example solved in chapter 3 is solved by using these two methods and it is shown that the results obtained by the direct methods and iterative methods are same.

5.1 Iterative methods

Iterative methods are those which start with an initial approximation and which, by applying a suitably chosen algorithm, lead to successively better approximations. Even if the process converges, we can only hope to obtain an approximate solution by iterative methods. Iterative methods vary with the algorithm chosen and in their rates of convergence. Some iterative method may actually diverge, others may converge so slowly that they are computationally useless. Merits of iterative methods are their simplicity and uniformity of operations to be performed, which make them well suited for use on computer.

5.1.1 Gauss-Jacobi method

To solve fully fuzzy linear system of equations $\tilde{A}\otimes\tilde{x}=\tilde{b},$ we have already discussed an approach in chapter 3. According to which the positive vectors x, y and z can be found by solving following linear system of equations synchronously:

$$\begin{cases} Ax=b \\ Ay+Mx=g \\ Az+Nx=h \end{cases} \tag{5.1}$$

In this section, we employ iterative methods for obtaining solution to above problem i.e. we use Gauss-Jacobi or Gauss-Seidel method to solve $\tilde{A}\otimes\tilde{x}=\tilde{b}.$

System (5.1) can be written as

$$\begin{cases} a_{i1}x_1+a_{i2}x_2+\ldots+a_{in}x_n=b_i \\ \left(a_{i1}y_1+a_{i2}y_2+\ldots+a_{in}y_n\right)+\left(m_{i1}x_1+m_{i2}x_2+\ldots+m_{in}x_n\right)=g_i, \qquad 1\le i\le n \\ \left(a_{i1}z_1+a_{i2}z_2+\ldots+a_{in}z_n\right)+\left(n_{i1}x_1+n_{i2}x_2+\ldots+n_{in}x_n\right)=h_i \end{cases} \tag{5.2}$$

Using above equations, we can find

$$\begin{cases} a_{ii}x_i=b_i-\sum\limits_{j=1,\,j\neq i}^{n}a_{ij}x_j \\ a_{ii}y_i=g_i-\left(\sum\limits_{j=1,\,j\neq i}^{n}a_{ij}y_j+\sum\limits_{j=1}^{n}m_{ij}x_j\right), \qquad 1\le i\le n \\ a_{ii}z_i=h_i-\left(\sum\limits_{j=1,\,j\neq i}^{n}a_{ij}z_j+\sum\limits_{j=1}^{n}n_{ij}x_j\right) \end{cases} \tag{5.3}$$

hence

$$
\begin{cases}
x_i = \dfrac{1}{a_{ii}}\left(b_i - \sum\limits_{j=1,\, j\neq i}^{n} a_{ij}x_j\right) \\
y_i = \dfrac{1}{a_{ii}}\left(g_i - \left(\sum\limits_{j=1,\, j\neq i}^{n} a_{ij}y_j + \sum\limits_{j=1}^{n} m_{ij}x_j\right)\right) \qquad 1 \le i \le n \\
z_i = \dfrac{1}{a_{ii}}\left(h_i - \left(\sum\limits_{j=1,\, j\neq i}^{n} a_{ij}z_j + \sum\limits_{j=1}^{n} n_{ij}x_j\right)\right)
\end{cases}
\tag{5.4}
$$

This can be re-written as

$$
\begin{cases}
x_i = -\dfrac{1}{a_{ii}}\sum\limits_{j=1,\, j\neq i}^{n} a_{ij}x_j + \dfrac{b_i}{a_{ii}} \\
y_i = -\dfrac{1}{a_{ii}}\left(\sum\limits_{j=1,\, j\neq i}^{n} a_{ij}y_j + \sum\limits_{j=1}^{n} m_{ij}x_j\right) + \dfrac{g_i}{a_{ii}} \qquad 1 \le i \le n \\
z_i = -\dfrac{1}{a_{ii}}\left(\sum\limits_{j=1,\, j\neq i}^{n} a_{ij}z_j + \sum\limits_{j=1}^{n} n_{ij}x_j\right) + \dfrac{h_i}{a_{ii}}
\end{cases}
\tag{5.5}
$$

Equation (5.5) can be written in compact form or matrix form as:

$$
\begin{bmatrix} x \\ y \\ z \end{bmatrix} = J \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \gamma
\tag{5.6}
$$

where J is called the iteration matrix and γ is a vector.

To solve system (5.6), we can make initial approximation $X^{(0)}$ of the solution vector and substitute into right hand side of equation (5.6). The solution of equation (5.6) will give a vector $X^{(1)}$, which is better approximation to the solution than $X^{(0)}$. We continue in this manner until the successive iteration $X^{(k)}$ converges to the solution up to desired accuracy, which suggests the following iterative process as the Gauss-Jacobi method for solving a fully fuzzy linear system of equations:

$$
\begin{cases}
x_i^{(k+1)} = \dfrac{1}{a_{ii}}\left(b_i - \displaystyle\sum_{j=1,\, j\neq i}^{n} a_{ij}x_j^{(k)}\right) & (5.7)\\[2ex]
y_i^{(k+1)} = \dfrac{1}{a_{ii}}\left(g_i - \left(\displaystyle\sum_{j=1,\, j\neq i}^{n} a_{ij}y_j^{(k)} + \sum_{j=1}^{n} m_{ij}x_j^{(k)}\right)\right), \qquad 1 \le i \le n & (5.8)\\[2ex]
z_i^{(k+1)} = \dfrac{1}{a_{ii}}\left(h_i - \left(\displaystyle\sum_{j=1,\, j\neq i}^{n} a_{ij}z_j^{(k)} + \sum_{j=1}^{n} n_{ij}x_j^{(k)}\right)\right) & (5.9)
\end{cases}
$$

In general, we can write this sequence as

$$
\begin{bmatrix} x^{(k+1)} \\ y^{(k+1)} \\ z^{(k+1)} \end{bmatrix} = J \begin{bmatrix} x^{(k)} \\ y^{(k)} \\ z^{(k)} \end{bmatrix} + \gamma, \qquad (k \ge 0) \tag{5.10}
$$

where J is called the iteration matrix of the iterative method and γ is a vector. $x^{(k+1)}$ and $x^{(k)}$ denote solution at k^{th} and $(k+1)^{th}$ iteration respectively.

Equation (5.7) can be written in the matrix form as:

$$
\begin{bmatrix} x_1^{(k+1)} \\ x_2^{(k+1)} \\ \cdot\cdot \\ \cdot\cdot \\ x_n^{(k+1)} \end{bmatrix} = -\begin{bmatrix} \frac{1}{a_{11}} & & & & \\ & \frac{1}{a_{22}} & & & \\ & & \frac{1}{a_{33}} & & \\ & & & \cdot\cdot & \\ & & & & \frac{1}{a_{nn}} \end{bmatrix} \left\{ \begin{bmatrix} 0 & a_{12} & a_{13} & \cdot\cdot & a_{1n} \\ a_{21} & 0 & a_{23} & \cdot\cdot & a_{2n} \\ a_{31} & a_{32} & 0 & \cdot\cdot & a_{3n} \\ \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot \\ a_{n1} & a_{n2} & a_{n3} & \cdot\cdot & 0 \end{bmatrix} \begin{bmatrix} x_1^{(k)} \\ x_2^{(k)} \\ x_{13}^{(k)} \\ \cdot\cdot \\ x_n^{(k)} \end{bmatrix} - \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \cdot\cdot \\ b_n \end{bmatrix} \right\}
$$

or

$$
x^{(k+1)} = -D_A^{-1}\left(L_A + U_A\right)x^{(k)} + D_A^{-1}b
$$

Similarly, equation (5.8) and (5.9) can be written in matrix form as:

$$\begin{aligned} y^{(k+1)} &= -D_A^{-1}\left(L_A + U_A\right) y^{(k)} - D_A^{-1} M x^{(k)} + D_A^{-1} g \\ z^{(k+1)} &= -D_A^{-1}\left(L_A + U_A\right) z^{(k)} - D_A^{-1} N x^{(k)} + D_A^{-1} h \end{aligned} \quad \text{respectively.}$$

Sufficient condition:

The Gauss-Jacobi iterative method for solving fully fuzzy linear system of equations $\tilde{A} \otimes \tilde{x} = \tilde{b}$ converges if and only if the classical Gauss-Jacobi iterative method converges for solving the crisp linear system of equations $Ax = b$ derived from the corresponding fully fuzzy linear system of equations.

If the matrix A in the crisp linear system of equations $Ax = b$ is strictly diagonally dominant i.e., $|a_{ii}| > \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}|,\ i = 1, 2, 3, \ldots n$ then the iterations obtained in classical Gauss-Jacobi iterative method converges for any initial approximation $X^{(0)}$.

Example 5.1 Solve the following system of equations using Gauss-Jacobi method

$$\begin{aligned} (5,1,1) \otimes (x_1, y_1, z_1) \ \oplus\ (6,1,2) \otimes (x_2, y_2, z_2) &= (50,10,17) \\ (7,1,0) \otimes (x_1, y_1, z_1) \ \oplus\ (4,0,1) \otimes (x_2, y_2, z_2) &= (48,5,7) \end{aligned}$$

from the above system we found that

$$A = \begin{bmatrix} 5 & 6 \\ 7 & 4 \end{bmatrix} \quad M = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \quad N = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \quad \text{and}$$

$$b = \begin{bmatrix} 50 \\ 48 \end{bmatrix} \quad g = \begin{bmatrix} 10 \\ 5 \end{bmatrix} \quad h = \begin{bmatrix} 17 \\ 7 \end{bmatrix}$$

To solve the above problem by Gauss-Jacobi method, first of all we obtain the following equations by the method explained in chapter 3.

$$\begin{bmatrix} 5 & 6 \\ 7 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 50 \\ 48 \end{bmatrix} \tag{5.11}$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 5 & 6 \\ 7 & 4 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 10 \\ 5 \end{bmatrix} \tag{5.12}$$

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 5 & 6 \\ 7 & 4 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 17 \\ 7 \end{bmatrix} \tag{5.13}$$

(5.11) can further be written as:

$$\begin{aligned} 5x_1 + 6x_2 &= 50 \\ 7x_1 + 4x_2 &= 48 \end{aligned}$$

Since $|5| \ngtr |6|$ and $|4| \ngtr |7|$, therefore the above system of equations is not diagonally dominant. So writing the above system in diagonally dominant form as:

$$\begin{aligned} 7x_1 + 4x_2 &= 48 \\ 5x_1 + 6x_2 &= 50 \end{aligned} \tag{5.14}$$

Now, the above system of linear equations is in diagonally dominant form as $|7| > |4|$ and $|6| > |5|$. Now, to find the solution by Gauss-Jacobi method first of all (5.14) can also be written as

$$x_1 = \frac{1}{7}(48 - 4x_2)$$
$$x_2 = \frac{1}{6}(50 - 5x_1)$$

Thus, the Gauss-Jacobi's method when applied to the above system, it gives

$$\begin{aligned} x_1^{(k+1)} &= \frac{1}{7}\left(48 - 4x_2^{(k)}\right) \\ x_2^{(k+1)} &= \frac{1}{6}\left(50 - 5x_1^{(k)}\right) \end{aligned}, \qquad k = 0,1,2,\ldots$$

Now, starting with initial approximation vector $x^{(0)} = (0,0)$, we get

$$x_1^{(1)} = \frac{1}{7}\left(48 - 4x_2^{(0)}\right) = \frac{48}{7} = 6.8571$$
$$x_2^{(1)} = \frac{1}{6}\left(50 - 5x_1^{(0)}\right) = \frac{50}{6} = 8.3333$$

i.e. $\qquad x^{(1)} = (6.8571, \quad 8.3333)$

hence continuing with this we obtain

$$\begin{aligned}
x^{(2)} &= (2.0953, \quad 2.6190) & x^{(3)} &= (5.3606, \quad 6.5873) \\
x^{(4)} &= (3.0929, \quad 3.8622) & x^{(5)} &= (4.6478, \quad 5.7559) \\
x^{(6)} &= (3.5681, \quad 4.4602) & x^{(7)} &= (4.3084, \quad 5.3599) \\
x^{(8)} &= (3.7943, \quad 4.743) & x^{(9)} &= (4.4168, \quad 5.1714) \\
x^{(10)} &= (3.9021, \quad 4.6527) & x^{(11)} &= (4.1984, \quad 4.4561) \\
x^{(12)} &= (4.3108, \quad 4.8347) & x^{(13)} &= (4.0944, \quad 4.741) \\
x^{(14)} &= (4.3825, \quad 4.9213) & x^{(15)} &= (4.0449, \quad 4.6812) \\
x^{(16)} &= (4.1821, \quad 4.4323) & x^{(17)} &= (4.3244, \quad 4.1556) \\
x^{(18)} &= (4.4825, \quad 4.7296) & x^{(19)} &= (4.1545, \quad 4.5979)
\end{aligned}$$

$$x^{(20)} = (4.2297,\quad 4.1753) \qquad x^{(21)} = (4.4712,\quad 4.1216)$$
$$x^{(22)} = (4.5019,\quad 4.6073) \qquad x^{(23)} = (4.2244,\quad 4.5817)$$
$$x^{(24)} = (4.2390,\quad 4.8130) \qquad x^{(25)} = (4.1068,\quad 4.8008)$$
$$x^{(26)} = (4.1138,\quad 4.9110) \qquad x^{(27)} = (4.0508,\quad 4.9651)$$
$$x^{(28)} = (4.0542,\quad 4.9576) \qquad x^{(29)} = (4.0242,\quad 4.9548)$$
$$x^{(30)} = (4.0258,\quad 4.9798) \qquad x^{(31)} = (4.0115,\quad 4.9785)$$
$$x^{(32)} = (4.0122,\quad 4.9904) \qquad x^{(33)} = (4.0054,\quad 4.9898)$$
$$x^{(34)} = (4.0056,\quad 4.9955) \qquad x^{(35)} = (4.0058,\quad 4.9955)$$

Since we have already found the exact solution of the above system in chapter 3, Example 3.1 and is found to be (4,5). It seems that the sequence $x^{(k)}, k = 0,1,2,...$ generated by the Jacobi method will converge to the exact solution. Hence up to two decimal places we obtain

$$x = (x_1, x_2) = (4.00, 5.00)$$

Now, putting the value of (x_1, x_2) in the equations (5.12) and (5.13) we obtain

$$\begin{bmatrix} 5 & 6 \\ 7 & 4 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 5 & 6 \\ 7 & 4 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

i.e. $\begin{aligned} 5y_1 + 6y_2 &= 1 \\ 7y_1 + 4y_2 &= 1 \end{aligned}$ and $\begin{aligned} 5z_1 + 6z_2 &= 3 \\ 7z_1 + 4z_2 &= 2 \end{aligned}$

Since the above equations are not in the form of diagonally dominant form. So converting them to diagonally dominant form as:

$$\begin{aligned} 7y_1 + 4y_2 &= 1 \\ 5y_1 + 6y_2 &= 1 \end{aligned} \tag{5.15}$$

Now, solving the above equations by the same procedure that is used to solve the system (5.11), we obtain:

$$y_1 = \frac{1}{7}(1 - 4y_2)$$
$$y_2 = \frac{1}{6}(1 - 5y_1)$$

Taking the initial approximation as $y^{(0)} = (0,0)$ and continuing with Gauss-Jacobi method we obtain

$$\begin{array}{ll} y^{(1)} = (0.1428, \quad 0.1666) & y^{(2)} = (0.0476, \quad 0.0476) \\ y^{(3)} = (0.1156, \quad 0.1270) & y^{(4)} = (0.0702, \quad 0.0703) \\ y^{(5)} = (0.1026, 0.1081) & y^{(6)} = (.0810, \quad 0.0811) \\ y^{(7)} = (0.0965, \quad 0.0991) & y^{(8)} = (0.0862, 0.0862) \end{array}$$

at 8^{th} iteration we obtain $y^{(8)} = (.0862,.0862)$ which is very close to exact solution $\left(\frac{1}{11},\frac{1}{11}\right)$. Hence the value of the optimal solution up to two decimal place is:

$$y = (y_1, y_2) = (.09,.09)$$

Similarly solving

$$\begin{aligned} 5z_1 + 6z_2 &= 3 \\ 7z_1 + 4z_2 &= 2 \end{aligned} \tag{5.16}$$

Solving (5.16) we find that the value of z converges at 14^{th} iteration as follows:

$$z^{(1)} = (0.2857, \quad 0.5) \qquad z^{(2)} = (0, \quad 0.2619)$$
$$z^{(3)} = (0.1360, \quad 0.5) \qquad z^{(4)} = (0, \quad 0.3866)$$
$$z^{(5)} = (0.0648, \quad 0.5) \qquad z^{(6)} = (0, \quad 0.446)$$
$$z^{(7)} = (0.0308, \quad 0.5) \qquad z^{(8)} = (0, \quad 0.4743)$$
$$z^{(9)} = (0.0146, \quad 0.5) \qquad z^{(10)} = (0, \quad 0.4878)$$
$$z^{(11)} = (0.0069, \quad 0.5)$$

Thus the value of z up to two decimal points is

$$z = (z_1, z_2) = (0, 0.5)$$

Hence the solution of given fully fuzzy linear system of equations given in example (5.1) is as follows:

$$\tilde{x} = \begin{bmatrix} (4,.09,0) \\ (5,.09,0.5) \end{bmatrix}$$

which is the required solution of the given fully fuzzy linear system of equations.

5.1.2 Gauss-Seidel method

The Gauss-Seidel method is simple modification of the method of simultaneous displacements.

The system of equation (5.1) can be written as:

$$\sum_{j \le i} a_{ij} x_j = b_i - \sum_{j>i}^{n} a_{ij} x_j, \qquad 1 \le i \le n \tag{5.17}$$

$$\sum_{j \le i} a_{ij} y_j = g_i - \left(\sum_{j>i}^{n} a_{ij} y_j + \sum_{j=1}^{n} m_{ij} x_j \right), \quad 1 \le i \le n \tag{5.18}$$

$$\sum_{j\le i} a_{ij} z_j = h_i - \left(\sum_{j>i}^{n} a_{ij} z_j + \sum_{j=1}^{n} n_{ij} x_j \right), \quad 1 \le i \le n \tag{5.19}$$

So, for this the Gauss-Seidel method is defined as:

$$\begin{cases} x_i^{(k+1)} = \dfrac{1}{a_{ii}} \left(b_i - \displaystyle\sum_{j<i}^{n} a_{ij} x_j^{(k+1)} - \sum_{j>i}^{n} a_{ij} x_j^{(k)} \right) \\ y_i^{(k+1)} = \dfrac{1}{a_{ii}} \left(g_i - \displaystyle\sum_{j<i}^{n} a_{ij} y_j^{(k+1)} - \sum_{j>i}^{n} a_{ij} y_j^{(k)} - \sum_{j=1}^{n} m_{ij} x_j^{(k)} \right), \quad 1 \le i \le n, \quad k \ge 0 \\ z_i^{(k+1)} = \dfrac{1}{a_{ii}} \left(h_i - \displaystyle\sum_{j<i}^{n} a_{ij} z_j^{(k+1)} - \sum_{j>i}^{n} a_{ij} z_j^{(k)} - \sum_{j=1}^{n} n_{ij} x_j^{(k)} \right) \end{cases}$$

System (5.17) can also be written as:

$$\begin{aligned} & a_{11} x_1^{(k+1)} = -a_{12} x_2^{(k)} - a_{13} x_3^{(k)} - a_{14} x_4^{(k)} - \ldots - a_{1n} x_n^{(k)} + b_1 \\ & a_{21} x_1^{(k+1)} + a_{22} x_2^{(k)} = -a_{23} x_3^{(k+1)} - a_{24} x_4^{(k)} - \ldots - a_{2n} x_n^{(k)} + b_2 \\ & a_{31} x_1^{(k+1)} + a_{32} x_2^{(k+1)} + a_{33} x_3^{(k+1)} = -a_{34} x_4^{(k)} - \ldots - a_{3n} x_n^{(k)} + b_3 \\ & \vdots \\ & a_{n1} x_1^{(k+1)} + a_{n2} x_2^{(k+1)} + a_{n3} x_3^{(k+1)} + a_{n4} x_4^{(k+1)} + \ldots + a_{n-1n-1} x_{n-1}^{(k+1)} + a_{nn} x_n^{(k+1)} = b_n \end{aligned}$$

In matrix form of the system can be written as

$$(D_A + L_A) x^{(k+1)} = b - U_A x^{(k)}$$

where D_A, L_A, U_A are diagonal, lower triangular and upper triangular matrices respectively.

Similarly (5.18) and (5.19) can also be written in matrix form as follows:

$$\begin{aligned} (D_A + L_A) y^{(k+1)} &= g - U_A y^{(k)} - M x^{(k)} \\ (D_A + L_A) z^{(k+1)} &= h - U_A z^{(k)} - M x^{(k)} \end{aligned}$$

Thus the Gauss-Seidel iterative method for solving fully fuzzy linear system of equations is as follows:

$$
\begin{aligned}
x^{(k+1)} &= -\left(D_A + L_A\right)^{-1} U_A x^{(k)} + \left(D_A + L_A\right)^{-1} b \\
y^{(k+1)} &= -\left(D_A + L_A\right)^{-1} U_A y^{(k)} - \left(D_A + L_A\right)^{-1} M x^{(k)} + \left(D_A + L_A\right)^{-1} g \\
z^{(k+1)} &= -\left(D_A + L_A\right)^{-1} U_A z^{(k)} - \left(D_A + L_A\right)^{-1} N x^{(k)} + \left(D_A + L_A\right)^{-1} h
\end{aligned}
$$

In this method also, if A is strictly diagonally dominant then the iteration always converges. Gauss-Seidel method will generally converge if the Jacobi method converges and will converge at a faster speed. Generally, for a symmetric matrix A, the rate of convergence of the Gauss-Seidel method is twice the convergence rate of the Jacobi's method. This result is true even if A is not symmetric.

Example 5.2 Solve the system of equations considered in Example 5.1 using Gauss – Seidel method.

Solution Solving (5.14)

i.e.

$$
\begin{aligned}
x_1 &= \frac{1}{7}\left(48 - 4x_2\right) \\
x_2 &= \frac{1}{6}\left(50 - 5x_1\right)
\end{aligned}
$$

The Gauss-Seidel iterative formula for this system can be written as:

$$
\begin{aligned}
x_1^{(k+1)} &= \frac{1}{7}\left(48 - 4x_2^{(k)}\right) \\
x_2^{(k+1)} &= \frac{1}{6}\left(50 - 5x_1^{(k+1)}\right)
\end{aligned}
$$

Taking the $x^{(0)} = \left(x_1^{(0)}, x_2^{(0)}\right) = (0,0)$ we get

$$x_1^{(1)} = \frac{1}{7}\left(48 - 4x_2^{(0)}\right) = \frac{48}{7} = 6.8571$$
$$x_2^{(1)} = \frac{1}{6}\left(50 - 5x_1^{(1)}\right) = \frac{15.7145}{6} = 2.6191$$

hence continuing with this, we get

$$x^{(2)} = (5.3605, \quad 3.8662) \qquad x^{(3)} = (4.6478, \quad 4.4601)$$
$$x^{(4)} = (4.3085, \quad 4.7429) \qquad x^{(5)} = (4.1469, \quad 4.8775)$$
$$x^{(6)} = (4.07, \quad 4.9416) \qquad x^{(7)} = (4.0333, \quad 4.9722)$$
$$x^{(8)} = (4.0158, \quad 4.9868) \qquad x^{(9)} = (4.0075, \quad 4.9937)$$
$$x^{(10)} = (4.0036, \quad 4.997)$$

Since we have already found the exact solution of the above system in chapter 3, Example 3.1 and is found to be (4,5). It seems that the sequence $x^{(k)}, k = 0,1,2,...$ generated by the Gauss-Seidel method will converge to the exact solution. So, by the above results it is clear that the value of x up to two decimal points is $x = (x_1, x_2) = (4,5)$ solving (5.15) we obtain

$$y^{(1)} = (0.1426, \quad 0.0478) \qquad y^{(2)} = (0.1153, \quad 0.0706)$$
$$y^{(3)} = (0.1022, \quad 0.08151) \qquad y^{(4)} = (0.0960, \quad 0.0867)$$
$$y^{(5)} = (0.0931, \quad 0.0891)$$

Hence the value of y up to two decimal places can be written as

$$y = (y_1, y_2) = (0.09, 0.09)$$

and solving (5.16), using the same method as used for solving the system (5.14) we obtain

$$
\begin{aligned}
z^{(1)} &= (0.2860, \quad 0.2620) & z^{(2)} &= (0.1362, \quad 0.3868) \\
z^{(3)} &= (0.0649, \quad 0.4463) & z^{(4)} &= (0.0309, \quad 0.4746) \\
z^{(5)} &= (0.0148, \quad 0.4880) & z^{(6)} &= (0.0071, \quad 0.4944) \\
z^{(7)} &= (0.0035, \quad 0.4974) & &
\end{aligned}
$$

hence from the above results, we find that the value of z up to two decimal points is found to be:

$$z = (z_1, z_2) = (0, 0.5)$$

hence the solution of the given fully fuzzy linear system of equation up to two decimal places is found to be

$$\tilde{x} = \begin{bmatrix} (4, 0.09, 0) \\ (5, 0.09, 0.5) \end{bmatrix}$$

5.2 Conclusion

In this chapter, Gauss-Jacobi and Gauss-Seidel methods are used to find the solution of fully fuzzy linear system of equations. To compare the methods same numerical example is solved by using both the methods and it is shown that in number of iterations in Gauss-Jacobi method are more than numbers of iterations in Gauss-Seidel method. So, on the basis of these results it can be concluded that it is better to use Gauss-Seidel method for solving fully fuzzy linear system of equations.

Chapter 6

LINEAR PROGRAMMING METHOD FOR SOLVING FULLY FUZZY LINEAR SYSTEM OF EQUATIONS

The methods, presented in previous chapters, can be applied only for finding the solution of simultaneous fully fuzzy linear system of equations i.e the fully fuzzy linear system of equations in which numbers of fuzzy variables equals to number of equations. These methods can't be applied for solving non-simultaneous fully fuzzy linear system of equations. To overcome this shortcoming in this chapter a method is presented which can be applied for finding the solution of simultaneous as well as non simultaneous fully fuzzy linear system of equations. To illustrate the method numerical examples are solved.

6.1 Algorithm for solving linear system of equations using linear programming method

To solve linear system of equations $\tilde{A} \otimes \tilde{x} = \tilde{b},$ we have already discussed an approach given in chapter 3. According to which the positive vectors x, y and z can be found by solving following linear system of equations synchronously:

$$\begin{cases} Ax = b, \\ Ay + Mx = g, \\ Az + Nx = h \end{cases} \tag{6.1}$$

A constraint $x - y \geq 0$ is employed to create a positive vector solution.

In this section, we employ a method based on linear programming for obtaining solution to above problem i.e. we use Two Phase or Big-M method to solve $\tilde{A} \otimes \tilde{x} = \tilde{b}.$ The steps of proposed method are as follows:

Step 1: Linear system of equations defined in (6.1) is considered as set of constraints

Step 2: Add an artificial vector to each constraint

Step 3: Add $x - y \geq 0$ as a constraint

Step 4: Define an arbitrary objective function which minimizes the sum of artificial vectors

Step 5: Solve the following linear programming problem

$$\begin{array}{ll} \text{Minimize} & x' + y' + z' \\ \text{subject to} & Ax + x' = b, \\ & Mx + Ay + y' = g, \\ & Nx + Az + z' = h, \\ & x - y \geq 0, \\ & x', y', z', x, y, z \geq 0 \end{array} \tag{6.2}$$

Let $(x'^{*}, y'^{*}, z'^{*}, x^{*}, y^{*}, z^{*})$ be the optimal solution of above linear programming problem. Optimal value of this linear programming is non-negative. If it is zero, then we have $x'^{*} = y'^{*} = z'^{*} = 0.$ Moreover we have $x^{*}, y^{*}, z^{*} \geq 0.$ From $x^{*} = 0,$ the first constraint of (6.2) imply that $Ax^{*} = b.$ Similarly for other constraints we have $Ay = g - Mx$ and $Az = h - Nx.$ In addition, the final constraint guarantees positivity property of fuzzy vector $\tilde{x}.$

If we have optimal value of (6.1) to be positive i.e. if there exists some artificial variables in optimal base, then there does not exist any positive fuzzy vector solution which satisfies $\tilde{A} \otimes \tilde{x} = \tilde{b}.$ Algorithm given in section 6.1 is defined for Two-Phase method. To employ Big-M method, algorithm is slightly different. To illustrate the linear programming approach by employing Two-Phase method and Big-M method, numerical examples are solved.

6.2 Numerical examples

In this section, to illustrate the algorithm a set of simultaneous fully fuzzy linear system of equations is solved by Two-Phase and Big-M method and it is shown that by applying both the methods, same results are obtained. Also the presented algorithm is used to a set of non-simultaneous fully fuzzy linear system of equations.

6.2.1 Two-Phase method

In this section, Two-Phase method is applied to solve a set of fully fuzzy linear system of equations.

Example 6.1

Consider the following FFLS

$$\begin{cases} (5,1,1)\otimes(x_1,y_1,z_1)\oplus(6,1,2)\otimes(x_2,y_2,z_2)=(50,10,17), \\ (7,1,0)\otimes(x_1,y_1,z_1)\oplus(4,0,1)\otimes(x_2,y_2,z_2)=(48,5,7). \end{cases}$$

Thus we have,

$$A=\begin{bmatrix} 5 & 6 \\ 7 & 4 \end{bmatrix} \qquad M=\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \qquad N=\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

$$b=\begin{bmatrix} 50 \\ 48 \end{bmatrix} \qquad g=\begin{bmatrix} 10 \\ 5 \end{bmatrix} \qquad h=\begin{bmatrix} 17 \\ 7 \end{bmatrix}$$

$$x=\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad y=\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \qquad z=\begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

Using (6.1), we have

$$\begin{bmatrix} 5 & 6 \\ 7 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 50 \\ 48 \end{bmatrix}$$

$$\begin{bmatrix} 5 & 6 \\ 7 & 4 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 10 \\ 5 \end{bmatrix}$$

$$\begin{bmatrix} 5 & 6 \\ 7 & 4 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 17 \\ 7 \end{bmatrix}$$

Applying the algorithm, we get the following linear programming problem:

$$\begin{array}{ll} \text{Minimize} & z = x_1' + x_2' + y_1' + y_2' + z_1' + z_2' \\ \text{subject to} & 5x_1 + 6x_2 + x_1' = 50, \\ & 7x_1 + 4x_2 + x_2' = 48, \\ & 5y_1 + 6y_2 + x_1 + x_2 + y_1' = 10, \\ & 7y_1 + 4y_2 + x_1 + y_2' = 5, \\ & 5z_1 + 6z_2 + x_1 + 2x_2 + z_1' = 17, \\ & 7z_1 + 4z_2 + x_2 + z_2' = 7, \\ & x_1 - y_1 \geq 0, \\ & x_2 - y_2 \geq 0, \\ & x_1', x_2', y_1', y_2', z_1', z_2', x_1, x_2, y_1, y_2, z_1, z_2 \geq 0. \end{array}$$

where $x' = \begin{bmatrix} x_1' \\ x_2' \end{bmatrix}$, $y' = \begin{bmatrix} y_1' \\ y_2' \end{bmatrix}$ and $z' = \begin{bmatrix} z_1' \\ z_2' \end{bmatrix}$ are artificial vectors.

Solving above linear programming problem using TORA, we get the optimal solution as:

$$x_1' = x_2' = y_1' = y_2' = z_1' = z_2' = 0$$

and $\quad x_1 = 4, x_2 = 5, \quad y_1 = 0.09, y_2 = 0.09,$

$$z_1 = 0, z_2 = 0.5.$$

Thus the solution is

$$\tilde{x} = \begin{bmatrix} (4, 0.09, 0) \\ (5, 0.09, 0.5) \end{bmatrix}$$

6.2.2 Big-M method

If we employ Big-M method to solve the linear system of equations, the algorithm defined earlier remains same with the difference that now our objective function becomes

$$\text{Minimize} \qquad M_1 x' + M_2 y' + M_3 z'$$

where M_1, M_2 and M_3 be three positive big numbers. If we select M_1 bigger than M_2 and M_3, x can enter to the base faster than y and z. Since x is used in the other constraints, it may accelerate the rate of convergence of algorithm. Now we shall solve the same numerical problem with Big-M method.

Example 6.2

$$\begin{cases} (5,1,1)\otimes(x_1,y_1,z_1)\oplus(6,1,2)\otimes(x_2,y_2,z_2)=(50,10,17), \\ (7,1,0)\otimes(x_1,y_1,z_1)\oplus(4,0,1)\otimes(x_2,y_2,z_2)=(48,5,7). \end{cases}$$

Applying the algorithm, we get the following linear programming problem:

$$\begin{array}{ll} \text{Minimize} & z = M_1 x_1' + M_1 x_2' + M_2 y_1' + M_2 y_2' + M_3 z_1' + M_3 z_2' \\ \text{subject to} & 5x_1 + 6x_2 + x_1' = 50, \\ & 7x_1 + 4x_2 + x_2' = 48, \\ & 5y_1 + 6y_2 + x_1 + x_2 + y_1' = 10, \\ & 7y_1 + 4y_2 + x_1 + y_2' = 5, \\ & 5z_1 + 6z_2 + x_1 + 2x_2 + z_1' = 17, \\ & 7z_1 + 4z_2 + x_2 + z_2' = 7, \\ & x_1 - y_1 \geq 0, \\ & x_2 - y_2 \geq 0, \\ & x_1', x_2', y_1', y_2', z_1', z_2', x_1, x_2, y_1, y_2, z_1, z_2 \geq 0. \\ & M_1, M_2 \text{ and } M_3 \text{ are positive big numbers.} \end{array}$$

Solving above linear programming problem using TORA, we get the optimal solution as:

$$x_1' = x_2' = y_1' = y_2' = z_1' = z_2' = 0$$

and $x_1 = 4, x_2 = 5,$ $y_1 = 0.09, y_2 = 0.09,$

$z_1 = 0,\ z_2 = 0.5.$

Thus the solution is:

$$\tilde{x} = \begin{bmatrix} (4, 0.09, 0) \\ (5, 0.09, 0.5) \end{bmatrix}$$

Example 6.3

$$\tilde{A} = \begin{bmatrix} (2,0.2,0.2) & (1,0.4,0.3) & (3,0.3,0.4) & (4,0.2,0.1) \\ (2,0.3,0.1) & (3,0.4,0.2) & (2,0.2,0.3) & (1,0.1,0.3) \\ (1,0.3,0.2) & (3,0.4,0.2) & (2,0.2,0.3) & (1,0.1,0.3) \\ (2,0.4,0.5) & (4,0.5,0.2) & (2,0.6,1.2) & (3,0.3,0.3) \end{matrix}$$

$$\begin{matrix} (5,0.5,0.2) & (6,0.4,0.3) & (7,0.3,0.5) & (5,0.2,0,8) & (9,1,0,9) \\ (2,0.9,0.2) & (3,0.8,0.3) & (4,0.7,0.3) & (2,0.6,0.3) & (3,0.5,0.2) \\ (2,0.9,0.2) & (3,0.8,0.3) & (4,0.7,0.3) & (2,0.6,0.3) & (3,0.5,0.2) \\ (2,0.1,0.6) & (6,0.2,0.3) & (8,0.3,0.3) & (7,0.4,0.9) & (6,0.5,0.2) \end{bmatrix}$$

and

$$\tilde{b} = \begin{bmatrix} (109.9,13.9,14.1) \\ (48,12.5,7.9) \\ (84.9,7.5,8.3) \\ (103,10.5,13.5) \end{bmatrix}$$

Thus, we have

$$A = \begin{bmatrix} 2 & 1 & 3 & 4 & 5 & 6 & 7 & 5 & 9 \\ 2 & 3 & 2 & 1 & 2 & 3 & 4 & 2 & 3 \\ 1 & 1 & 2 & 3 & 1 & 9 & 5 & 4 & 6 \\ 2 & 4 & 2 & 3 & 2 & 6 & 8 & 7 & 6 \end{bmatrix} \quad M = \begin{bmatrix} 0.2 & 0.4 & 0.3 & 0.2 & 0.5 & 0.4 & 0.3 & 0.2 & 1 \\ 0.3 & 0.4 & 0.2 & 0.1 & 0.9 & 0.8 & 0.7 & 0.6 & 0.5 \\ 0.3 & 0.5 & 0.3 & 0.2 & 0.2 & 0.2 & 0.2 & 0.2 & 0.2 \\ 0.4 & 0.5 & 0.6 & 0.3 & 0.1 & 0.2 & 0.3 & 0.4 & 0.5 \end{bmatrix}$$

$$N = \begin{bmatrix} 0.2 & 0.3 & 0.4 & 0.1 & 0.2 & 0.3 & 0.5 & 0.8 & 0.9 \\ 0.4 & 0.2 & 0.3 & 0.3 & 0.2 & 0.3 & 0.3 & 0.3 & 0.2 \\ 0.2 & 0.2 & 0.1 & 0.3 & 0.5 & 0.3 & 0.6 & 0.3 & 0.2 \\ 0.5 & 0.2 & 1.2 & 0.3 & 0.6 & 0.3 & 0.3 & 0.9 & 0.2 \end{bmatrix}$$

$$b = \begin{bmatrix} 109.9 \\ 48 \\ 84.9 \\ 103 \end{bmatrix} \qquad g = \begin{bmatrix} 13.9 \\ 12.5 \\ 7.5 \\ 10.5 \end{bmatrix} \qquad h = \begin{bmatrix} 14.1 \\ 7.9 \\ 8.3 \\ 13.5 \end{bmatrix}$$

Applying the linear programming method, we get the following linear programming problem:

Minimize $\quad x_1' + x_2' + x_3' + x_4' + y_1' + y_2' + y_3' + y_4' + z_1' + z_2' + z_3' + z_4'$

subject to $\quad 2x_1 + x_2 + 3x_3 + 4x_4 + 5x_5 + 6x_6 + 7x_7 + 5x_8 + 9x_9 + x_1' = 109.9,$

$$2x_1 + 3x_2 + 2x_3 + x_4 + 2x_5 + 3x_6 + 4x_7 + 2x_8 + 3x_9 + x_2' = 48,$$

$$x_1 + x_2 + 2x_3 + 3x_4 + x_5 + 9x_6 + 5x_7 + 4x_8 + 6x_9 + x_3' = 84.9,$$

$$2x_1 + 4x_2 + 2x_3 + 3x_4 + 2x_5 + 6x_6 + 8x_7 + 7x_8 + 6x_9 + x_4' = 103,$$

$$2y_1 + y_2 + 3y_3 + 4y_4 + 5y_5 + 6y_6 + 7y_7 + 5y_8 + 9y_9 + 0.2x_1 + 0.4x_2$$

$$+0.3x_3 + 0.2x_4 + 0.5x_5 + 0.4x_6 + 0.3x_7 + 0.2x_8 + x_9 + y_1' = 13.9,$$

$$2y_1 + 3y_2 + 2y_3 + y_4 + 2y_5 + 3y_6 + 4y_7 + 2y_8 + 3y_9 + 0.3x_1 + 0.4x_2$$

$$+0.2x_3 + 0.1x_4 + 0.9x_5 + 0.8x_6 + 0.7x_7 + 0.6x_8 + 0.5x_9 + y_2' = 12.5,$$

$$y_1 + y_2 + 2y_3 + 3y_4 + y_5 + 9y_6 + 5y_7 + 4y_8 + 6y_9 + 0.3x_1 + 0.5x_2$$

$$+0.3x_3 + 0.2x_4 + 0.2x_5 + 0.2x_6 + 0.2x_7 + 0.2x_8 + 0.2x_9 + y_3' = 7.5,$$

$$2y_1 + 4y_2 + 2y_3 + 3y_4 + 2y_5 + 6y_6 + 8y_7 + 7y_8 + 6y_9 + 0.4x_1 + 0.5x_2$$

$$+0.6x_3 + 0.3x_4 + 0.1x_5 + 0.2x_6 + 0.3x_7 + 0.4x_8 + 0.5x_9 + y_4' = 10.5,$$

$$2z_1 + z_2 + 3z_3 + 4z_4 + 5z_5 + 6z_6 + 7z_7 + 5z_8 + 9z_9 + 0.2x_1 + 0.4x_2$$

$$+0.3x_3 + 0.2x_4 + 0.5x_5 + 0.4x_6 + 0.3x_7 + 0.2x_8 + x_9 + z_1' = 14.1,$$

$$2z_1 + 3z_2 + 2z_3 + z_4 + 2z_5 + 3z_6 + 4z_7 + 2z_8 + 3z_9 + 0.3x_1 + 0.4x_2$$

$$+0.2x_3 + 0.1x_4 + 0.9x_5 + 0.8x_6 + 0.7x_7 + 0.6x_8 + 0.5x_9 + z_2' = 7.9,$$

$$z_1 + z_2 + 2z_3 + 3z_4 + z_5 + 9z_6 + 5z_7 + 4z_8 + 6z_9 + 0.3x_1 + 0.5x_2$$

$$+0.3x_3 + 0.2x_4 + 0.2x_5 + 0.2x_6 + 0.2x_7 + 0.2x_8 + 0.2x_9 + z_3' = 8.3,$$

$$2z_1 + 4z_2 + 2z_3 + 3z_4 + 2z_5 + 6z_6 + 8z_7 + 7z_8 + 6z_9 + 0.4x_1 + 0.5x_2$$

$$+0.6x_3 + 0.3x_4 + 0.1x_5 + 0.2x_6 + 0.3x_7 + 0.4x_8 + 0.5x_9 + z_4' = 13.5,$$

$$x_1 - y_1 \geq 0, \qquad x_2 - y_2 \geq 0,$$

$$x_3 - y_3 \geq 0, \qquad x_4 - y_4 \geq 0,$$

$$x_5 - y_5 \geq 0, \qquad x_6 - y_6 \geq 0,$$

$$x_7 - y_7 \geq 0, \qquad x_8 - y_8 \geq 0,$$

$$x_9 - y_9 \geq 0,$$

$$x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8, y_9, z_1, z_2, z_3, z_4$$

$$, z_5, z_6, z_7, z_8, z_9, x_1', x_2', x_3', x_4', y_1', y_2', y_3', y_4', z_1', z_2', z_3', z_4' \geq 0.$$

Solving above system of equations using some mathematical programming software package (e.g. TORA), we get the objective value equal to 0. Thus, above fully fuzzy linear system of equations has positive solution:

$$x_1' = 0, x_2' = 0, x_3' = 0, x_4' = 0, y_1' = 0, y_2' = 0, y_3' = 0, y_4' = 0,$$

$$z_1' = 0, z_2' = 0, z_3' = 0, z_4' = 0\,.$$

$$x_1 = 0,\ \ x_2 = 0,\ \ x_3 = 1.26,\ \ x_4 = 0,\ \ x_5 = 0,\ \ x_6 = 1.61,\ \ x_7 = 4.47,\ \ x_8 = 3.17,\ \ x_9 = 5.48$$

$$y_1 = 0,\ \ y_2 = 0,\ \ y_3 = 1.26,\ \ x_4 = 0,\ \ x_5 = 0,\ \ y_6 = 0.10,\ \ y_7 = 0,\ \ y_8 = 0.07,\ \ y_9 = 0.08$$

$$z_1 = 0,\ \ z_2 = 0.76,\ \ z_3 = 0,\ \ z_4 = 0,\ \ z_5 = 0,\ \ z_6 = 0.05,\ \ z_7 = 0.23,\ \ z_8 = 0.15,\ \ z_9 = 0$$

Thus the solution is

$$\tilde{x} = \begin{bmatrix} (x_1, y_1, z_1) \\ (x_2, y_2, z_2) \\ (x_3, y_3, z_3) \\ (x_4, y_4, z_4) \\ (x_5, y_5, z_5) \\ (x_6, y_6, z_6) \\ (x_7, y_6, z_6) \\ (x_8, y_8, z_8) \\ (x_9, y_9, z_9) \end{bmatrix} = \begin{bmatrix} (0,0,0) \\ (0,0,0) \\ (1.26,1.26,0) \\ (0,0,0) \\ (0,0,0) \\ (1.61,0.10,0.05) \\ (4.47,0,0) \\ (3.17,0.07,0.15) \\ (5.48,0.08,0) \end{bmatrix}$$

6.3 Conclusion

In this chapter, the fully fuzzy linear fuzzy systems, i.e. fuzzy linear system of equations with fuzzy coefficients involving fuzzy variables are solved using linear programming method. This method gives the positive fuzzy vector solution, if it exists. Moreover, the proposed scheme is more efficient and useful in the manner that this can be applied to solve non-squared fuzzy systems also.

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