A STUDY OF LINEAR AND LINEAR FRACTIONAL EXTREME POINT PROGRAMMING PROBLEMS

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INDIA

DEDICATED

TO

GOD, MY PARENTS

AND

MY SUPERVISORS

CERTIFICATE

I hereby certify that the work which is being presented in the thesis entitled "**A Study of Linear and Linear Extreme Point Linear Fractional Programming Problems**" in partial fulfillment of the requirements for the award of degree of Master of Science, School of Mathematics and Computer Applications (SMCA), Thapar University, Patiala is an authentic record of my own work carried out under the supervision of **Mr. Vikas Sharma** and **Dr. Geeta Kumari**.

The matter presented in this thesis has not been submitted for the award of any other degree of this or any other university.

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ABSTRACT

The mathematical programming techniques are useful in finding the optimum value of a function of several variables under a set of constraints. Among these techniques extreme point mathematical programming is entirely a new technique. Recently a lot of work has been done on extreme point programming problems in which an objective function is optimized over a convex polyhedron with an additional requirement that optimal solution must also be an extreme point of another convex polyhedron.

The chapter-wise summary of the thesis is as follows:

Chapter 1 is introductory in nature. This chapter includes basic concepts used to find the extreme point solution of extreme point linear and linear fractional programming problem and its extension to bounded variables .

In Chapter 2 , an extreme point linear programming problem is studied and procedure used to solve extreme point linear programming problems has been discussed.

In Chapter 3, a procedure to solve an extreme point linear fractional programming problem has been studied in which the concept of ranking of extreme point solutions has been used to find the best optimal extreme point solution. To illustrate the method used, a numerical example is solved.

In Chapter 4, an extreme point linear fractional programming problem is extended to bounded variables and the procedure to solve linear fractional programming problem with bounded variables has been discussed. To illustrate the method, numerical solved in chapter 3 is solved by assuming bounds on some variables.

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Chapter 1

Introduction

This introductory chapter is divided into two sections. Section 1.1 deals with the linear extreme point programming problem in general and Section 1.2 deals with the linear fractional extreme point programming problem. A brief survey of the related work by various researchers in these areas have also been discussed.

In design, construction and maintenance of any system or project one has to take many technological and managerial decisions at several stages. The ultimate goal of all such decisions is to either minimize the effort required or maximize the desired benefit. Since the effort required or the benefit desired in any practical situation can be expressed as a function of certain decision variables, optimization can be defined as the process of finding the condition that give the maximum or minimum value of the function. Optimization theory has been developed in connection with selecting the best of many possible decisions in complex real life environment. There is no single method available for solving all optimization problems efficiently. Hence a number of optimization methods have been developed for solving different types of optimization problems. The optimum seeking methods are also known as mathematical programming techniques. These techniques are useful in finding the optimum value of a function of several variables under a set of constraints.

1.1 Extreme point programming problem

Extreme point programming problem is entirely new application of linear programming problems. A lot of work has been done recently on extreme point programming problem in which an objective function is optimized over a convex polyhedron with an additional requirement that the optimal solution must also be an extreme point of another convex polyhedron. Many practical problems - single machine scheduling problem kirby et al. [5], fixed charge problem murty[6] etc. can be well handled by techniques involved for solving an extreme point mathematical programming model. Apart from the above problems, zero-one integer programming problem can be converted into extreme point mathematical programming problem by replacing the requirement that each of the variables should be either zero or one. In addition to this a large variety of transportation and assignment problem comes under the area of application of extreme point mathematical programming model.

Extreme point linear programming problems (EPLPP) in its most general form was first studied and solved by kirby et al.[2]. Analytically, it can be stated as :

$(\mathbf{P_{1.1}})$ Max $Z = CX$

Subject to

AX = b

$X \geq 0$

and X is an extreme point of

DX = d

$X \geq 0.$

To solve $(P_{1.1})$, kirby et al. [2] considered the problem $(P_{1.2})$ which is defined as :

Max $Z = CX$

Subject to

$AX = b$

DX = d

$X \geq 0.$

The extreme point solutions of $(P_{1.2})$ are determined in a schematic way till we reach a stage where extreme point of $(P_{1.2})$ is also an extreme point of $DX = d,\ X \geq 0$, i.e., we get an optimal solution or we get an indication of no solution of $(P_{1.1})$. The technique used to find the extreme points of $(P_{1.1})$ is as : the first best extreme point solution of $(P_{1.1})$ is investigated to see whether it is an extreme point of DX = d, $X \geq 0$ and if not its second best extreme point solution is tested. In case this also is not an extreme point of DX = d, $X \geq 0$, optimal extreme point solution of $(P_{1.3})$ is obtained, where $(P_{1.3})$ is defined as:

$(\mathbf{P_{1.3}})$ Max $Z = CX$

Subject to $FX = f$, where $F = \binom{A}{D}$ and $f = \binom{b}{d}$

$X \geq 0.$

Values greater than equal to v_2(where v_2 is the second best otimal value of objective function) of the objective function at extreme point solutions adjacent to these optimal solutions of $(P_{1.3})$ are found and the one, say v which is nearest to the value v_2 is picked up and an additional constraint $CX \leq v$, is introduced in $(P_{1.3})$ to give rise to the problem

$(\mathbf{P_{1.4}})$ Max $Z = CX$

Subject to $Fx = f$

$CX \leq v$

$X \geq 0.$

By following the above technique we avoid some of the non optimal extreme point points of $(P_{1.3})$, i.e., extreme point solutions of $(P_{1.3})$ which are infeasible with respect to $AX = b$. The remaining extreme point solutions of $(P_{1.1})$ are determined by finding the extreme point solutions of $(P_{1.4})$ by above technique till an optimal solution, if any of $(P_{1.1})$ is reached.

1.2 Extreme point linear fractional programming problem

Extreme point linear fractional programming problem is presented in chapter 3. In Extreme point fractional programming problem, an objective function is to be optimized which is the ratio of two linear function. This objective function which is neither convex nor concave, is optimized subject to linear inequalities with the additional constraint that the optimal solution should also be an extreme point of another convex polyhedron. In general extreme point linear fractional programming problem can be stated as:

$$(\mathbf{P_{1.5}}) \qquad \text{Max } Z = \frac{CX + \alpha}{DX + \beta}$$

$$\text{Subject to} \quad AX = b$$

$$X \geq 0$$

$$\text{and X is an extreme point of}$$

$$RX = t$$

$$X \geq 0 \tag{1.1}$$

To solve $(P_{1.5})$, problem $(P_{1.6})$ is used which is stated as :

$$(\mathbf{P_{1.6}}) \qquad \text{Max } Z = \frac{CX + \alpha}{DX + \beta}$$

$$\text{Subject to} \quad AX = b$$

$$RX = t$$

$$X \geq 0.$$

First find extreme point solutions of $(P_{1.6})$ by the algorithm developed (discussed in detail in chapter-3) till either a stage is reached when its extreme point solution is also an extreme point of $RX = t,\ X \geq 0$, i.e., an optimal solution of $(P_{1.5})$ is obtained or an indication of no solution of $(P_{1.5})$ is obtained.

The algorithm used to find extreme points of $(P_{1.6})$ involves ranking of value of objective function in decreasing order to find best extreme point solution of $(P_{1.6})$ which is an extreme point solution of $RX = t,\ X \geq 0$.

The method to solve the of linear fractional programming problems is developed by swarup [14]. This approach is developed on lines similar to solving a LPP by simplex method, which can be briefly explained as follows :

In general linear fractional programming problem can be stated as:

$$(\mathbf{P_{1.7}}) \qquad \text{Max } Z = \frac{CX + \alpha}{DX + \beta}$$

$$\text{Subject to} \quad AX = b$$

$$X \geq 0 \tag{1.2}$$

In this approach, the following assumptions are made :

(i) Any m columns of A are linearly independent.

(ii) The denominator of the objective function is positive for all feasible solutions of $(P_{1.7})$.

Based on these assumptions it is established that optimal solution of $(P_{1.7})$ is a basic feasible solution, i.e., the optimum occurs at an extreme point of feasible region $AX = b,\ X \geq 0$. Therefore, the procedure starts with the initial basic feasible solution, moves over the set of extreme points of the feasible region in such a way that in the absence of degeneracy the value of objective function at each iteration is improved. Since the number of extreme points is finite and no extreme point is repeated the procedure converges in finite number of steps.

The algorithm developed to solve $(P_{1.7})$ is as follows:

Let X_B be initial basic feasible solution to the set of constraints of $(P_{1.7})$. Let B be the corresponding basis, $B = (b_1, b_2, ..., b_m)$. Therefore, $X_B = B^{-1}b,\ X_B \geq 0$. Let C_B and D_B be the m component row vectors having their components as the coefficient associated with the basic variables in numerator and denominator of the objective function respectively.

Corresponding to the solution X_B, let

$$Z^{(1)} = C_B X_B + \alpha$$

$$Z^{(2)} = D_B X_B + \beta$$

Therefore, the value of the objective function of $(P_{1.7})$ corresponding to solution X_B is given by $\dfrac{Z^{(1)}}{Z^{(2)}}$

It is required to determine a nonbasic variable which when inserted in the basis B, according to the procedure of simplex method, should give an improved value of objective function. Let a_j be the column of A not in B, then there exists y_{ij} such that

$a_j = BY_j$ or $Y_j = B^{-1}a_j$. Also let $Z_j^{(1)} = C_B Y_j$, $Z_j^{(2)} = D_B Y_j$. Thus $Z_j^{(1)}, Z_j^{(2)}$ and Y_j are known for every column a_j of A not in B. Suppose column b_r of B is replaced by a_j of A not in B by means of simplex method of LPP to obtain a new basic feasible solution $\hat{X_B}$ where

$$\hat{X_{B_i}} = X_{B_i} - X_{B_r}\frac{y_{ij}}{y_{rj}}, \quad i \neq r,$$

$$\hat{X_{B_r}} = \frac{X_{B_r}}{y_{rj}} = \theta.$$

Let the new value of the objective function be

$$\hat{Z} = \frac{Z^{(1)}}{Z^{(2)}},$$

where $\hat{Z}^1 = Z^{(1)} - \theta(Z_j^{(1)} - C_j)$ and $\hat{Z}^2 = Z^{(2)} - \theta(Z_j^{(2)} - D_j)$.

Clearly the value of objective function improves if $\hat{Z} > Z$ which imples

$$\theta[Z^{(1)}(Z^{(2)} - D_j) - Z^{(2)}(Z^{(1)} - C_j)] > 0.$$

The new basic solution will be feasible if there exist atleast one $y_{ij} > 0$, $i = 1, 2, ..., n$. thus any column a_j of A not in B if entered in the basis B gives an improved value of objective function if

(i) There is at least one $y_{ij} > 0$, $i = 1, 2, ..., n$ with $X_{Bi} > 0$

(ii) For the column $a_j, \Delta_j > 0$ and $\Delta_j = Max(\Delta_k, \Delta_k > 0)$ so that the value of objective function improves rapidly.

For every column in the basis $\Delta = 0$. The procedure will terminate, i.e, optimality will be acheived when $\Delta_j \leq 0$ for all j.

Extreme point linear fractional programming problem with bounded variables is presented in chapter 4. In this chapter, we discuss problems in which an objective function is to be optimized which is the ratio of two linear function subject to linear inequalities such that some (or all) variables have upper and lower limits, i.e., there values lies in some bounds with the additional constraint that the optimal solution should also be an extreme point of another convex polyhedron. In general extreme point linear fractional programming problem can be stated as:

($\mathbf{P_{1.8}}$) Max Z=$\dfrac{CX+\alpha}{DX+\beta}$

Subject to

AX = b

$L \leq X \leq U$, where L and U are lower and upper bounds respectively in which value of X lies and X is an extreme point of

RX = t

$L \leq X \leq U$

To solve $(P_{1.8})$, problem $(P_{1.9})$ is stated as:

($\mathbf{P_{1.9}}$) Max Z = $\dfrac{CX+\alpha}{DX+\beta}$

Subject to

AX = b

RX = t

$L \leq X \leq U.$

The technique used to find the solution of $(P_{1.9})$ has been discussed in chapter 4 where extreme points of the convex polyhedron $RX = t,\ L \leq X \leq U$ are investigated in a systematic order till feasibility in $AX = b$ is achieved. The above problem is solved by ranking the extreme points of convex polyhedron $RX = t,\ L \leq X \leq U$ till an extreme point is reached which satisfies the feasibility in $AX = b$

It may be noted that as all the techniques for extreme point programming problem and extreme point linear fractional programming problems involves the movement from one extreme point to another extreme point of a convex polyhedron, they are bound to converge in a finite number of steps as number of such extreme points is always finite and no extreme point is repeated since the vale of the objective function is decreased at each stage.

Chapter 2

Extreme Point Linear Programming Problem

In this chapter procedure to find optimal extreme point solution to an extreme point linear programming problem has been discussed. Mathematically, an extreme point linear programming problem can be stated as :

($\mathbf{P_{2.1}}$) Maximize $v = cx$

subject to

$$\text{Ax} = \text{b},$$

where x is the extreme point of

$$Dx = d, \quad x \geq 0,$$

where c is $1 \times n$, x is $n \times 1$, A is $m \times n$, b is $m \times 1$, D is $p \times n$, d is $p \times 1$ and 0 is $n \times 1$.

($\mathbf{P_{2.2}}$) Maximize $z = cx$

subject to

$$Fx = f, \quad x \geq 0,$$

where F is the $(m+p) \times n$ matrix $\binom{A}{D}$ and f is the $(m+p) \times 1$ vector $\binom{b}{d}$.

In section 2.2, we have discussed the theory of the method proposed by Kirby et al. kirby [5] to solve problem $(P_{2.1})$, which was first posed by Charnes. In sections 2.3 and 2.4, an algorithm for the above method has been summarized and a numerical example has been solved to illustrate the algorithm discussed.

2.1 Theoretical Development

To solve the $(P_{2.1})$, we consider the linear programming problem defined as $(P_{2.2})$. It may be noted that $(P_{2.1})$ is always bounded, since any solution of $(P_{2.1})$ is an extreme point of $Dx = d$, $x \geq 0$, and this set of all the extreme points is finite. However, $(P_{2.2})$ may be either bounded or unbounded. In case $(P_{2.2})$ is unbounded, it can always be converted into a bounded problem by adding a constraint $cx \leq M$, where M is an arbitrary large finite positive number determined in such a way that none of extreme point of original problem are excluded from $(P_{2.2})$. So $(P_{2.2})$ is always considered to be bounded.

Notations :

$J = \{d_j | d_j \neq 0$, where d_j is the j^{th} column of $D\}$

$J(X) = \{d_j \in J | x_j \neq 0$, where $x = (x_1, x_2, x_3, ..., x_n)\}$

$S_1 = \{x|\ Ax = b$ and x is the extreme point of $Dx = d,\ x \geq 0\}$

$S_2 = \{x|\ x$ is an extreme point of $Fx = f,\ x \geq 0\}$

$S_3 = S_2 \setminus S_1$

$X_1 = Y_1 = \{x_{11}, x_{12}, ..., x_{1k_1}\}$, is the set of all optimal extreme point solutions x_{1r}, $r = 1, ..., k_1$ of $(P_{2.2})$

$v_1 = cx_{11}$, is the optimal value of the objective function of $(P_{2.1})$

The method of solving $(P_{2.1})$ involve working with $(P_{2.2})$. There exist a number of important relations between the two problems which can be illustrated by the following theorem :

Theorem2.1 $S_1 \subseteq S_2$, i.e., every extreme point of $Dx = d$,$X \geq 0$ satisfying feasibility in $Ax = b$ is also an extreme point of $Fx = f,\ x \geq 0$.

Proof. If $S_1 = \phi$, then proof is complete. If $S_1 \neq \phi$, let $x \in S_1$. Then x is the extreme point of $Dx = d,\ x \geq 0$ and satisfies $Ax = b$, i.e., x satisfies $Fx = f$, Thus x is an element of the convex set $S_4 = \{x|\ Fx = f, x \geq 0\}$. If we prove $x \in S_2$, it will be equivalent to prove that x is extreme point of S_4. Suppose x is not an extreme point of S_4. This implies there exist $t, w \in S_4,\ t \neq w$ and $\lambda,\ 0 < \lambda < 1$, such that $x = \lambda\, t + (1-\lambda)\, w$. As $t, w \in S_4$, therefore $Ft = f$ and $Fw = f$, $t, w \geq 0$. $Ft = f$, gives $\binom{A}{D} t = \binom{b}{d}$, which further implies $Dt = d$ and $Dw = d$ which contradicts the fact that x is an extreme point of $Dx = d,\ x \geq 0$. Thus x is an extreme point of S_4. Hence $x \in S_2$ and so $S_1 \subseteq S_2$.

This theorem assures that an optimal solution to the $(P_{2.1})$ is an extreme point of $(P_{2.2})$. It also enables us to recognize the existence of three basic relations between the problems $(P_{2.1})$ and $(P_{2.2})$ as follows :

$(R1)$: $(P_{2.1})$ and $(P_{2.2})$ both have solutions.

$(R2)$: $(P_{2.2})$ has no solution and hence $(P_{2.1})$ does not have a solution.

$(R3)$: $(P_{2.2})$ has a solution but $(P_{2.1})$ has no solution.

We first apply simplex method to find solution of $(P_{2.2})$. If there is no solution to this problem, then by $(R2)$, $(P_{2.1})$ does not have solution and our procedure terminates. Hence $(P_{2.1})$ has its importance only when $(P_{2.2})$ has a solution, i.e., $S_2 \neq \phi$. We now discuss $(R1)$, i.e., the solution exist for both the problems $(P_{2.1})$ and $(P_{2.1})$. Later we consider $(R3)$ where $(P_{2.1})$ is inconsistent. The second property of an optimal solution of $(P_{2.1})$ is given by the following theorem :

Theorem2.2. If $x^* \in S_1$, then an optimal solution x of $(P_{2.1})$ is an element of S_1 which is at minimum orthogonal distance from the hyperplane $cx = cx^* = v_1$.

Proof. $(P_{2.1})$ can be restated as: Maximize cx subject to $x \in S_1$ which is equivalent to Minimize $-cx$, subject to $x \in S_1$, which is further equivalent to minimize $\dfrac{v_1 - cx}{||c||}$, subject to $x \in S_1$.

Let x be the optimal solution of $(P_{2.1})$, then $x \in S_1$. Since $S_1 \subseteq S_2$, therefore $x \in S_2$. Also x^* is the optimal solution of the $(P_{2.2})$, therefore $cx^* \geq cx$ for all

$x \in S_2$. Therefore $v_1 \geq cx$ or $v_1 - cx \geq 0$ or $\dfrac{v_1 - cx}{||c||} \geq 0$.

Hence $\dfrac{v_1 - cx}{||c||}$ represents the orthogonal distance of x from hyperplane $cx = cx^* = v_1$.

This completes the proof.

Since the $(P_{2.1})$ as restated above seems to minimize this orthogonal distance subject to $x \in S_1$, thus x, the optimal extreme point solution of $(P_{2.1})$ is at a minimum orthogonal distance from hyperplane $cx = v_1$. Hence the problem requires to find an $x \in S_1$ which is at minimum orthogonal distance from the hyperplane $cx = v_1$. To achieve the required result, the concept of 2^{nd} best, 3^{rd} best etc. extreme point solution are introduced.

Second Best Extreme Point Solution: The second best solution of $(P_{2.2})$, is meant by an element $s^* \in (S_2 \backslash X_1)$ such that $cs^* \geq cs$, for all $s \in (S_2 \backslash X_1)$, where X_1 is the set of optimal extreme point solutions of $(P_{2.2})$. Let $X_2 = (x_{21}, x_{22}, ..., x_{2k_2})$ be the set of all second best extreme point solution of $(P_{2.2})$, similarly the third best extreme point solution to problem $(P_{2.2})$ is an element $s^* \in (S_2 \backslash (X_1 \bigcup X_2)$ such that $cs^* \geq cs$, for all $s \in (S_2 \backslash (X_1 \bigcup X_2)$, The set of third best extreme point of $(P_{2.2})$ is denoted by $X_3 = (x_{31}, x_{32}, ..., x_{3k_3})$. In general the k^{th} best extreme point solutions of $(P_{2.2})$, is defined as an element $s^* \in S_2 \backslash \bigcup_{i=1}^{n} X_i$ such that $cs^* \geq cs$, for all $s \in S_2 \backslash \bigcup_{i=1}^{n} X_i$.

Lemma 2.1 If a linear programming problem has second best extreme point solution then it is adjacent to some optimal extreme point solution.

Proof. Consider the simplex table corresponding to an element of the second best extreme point solution of $(P_{2.2})$. Since it is not optimal there must exist at least one column say j^{th}, for which $Z_j - C_j < 0$. If the corresponding column A_j entered the basis leaving a column for which θ_j =min$\{(x_{Bi}/y_{ij})|y_{ij} > 0\}$, then this single iteration will lead to an optimal solution of LPP. As only a change of one basis vector in the second best leads to optimal solution, which follows the result that second best is adjacent to some element of extreme point solution.

Lemma 2.2 If a linear programming problem has k^{th} best extreme point solution then it is adjacent to some element of the set containing optimal, 2^{nd} best ,...,$(k-1)^{st}$ best extreme point solution.

Proof. Consider the simplex table corresponding to k^{th} best extreme point solution of $(P_{2.2})$, where $k > 1$. Since it is not the optimal solution, there must exist at least one column say j^{th}, for which $Z_j - C_j < 0$. If the corresponding column A_j entered the basis leaving a column for which $\theta_j = \min\{(x_{Bi}/y_{ij})|y_{ij} > 0\}$ then the simplex table so obtained will generate an element of $\bigcup_{i=1}^{k-1} X_i$. As simplex method moves from one extreme point solution to another along an edge, therefore the k^{th} best extreme point solution to an LPP is adjacent to an element of $\bigcup_{i=1}^{k-1} X_i$.

Lemma 2.3 If $x^* \in X_1$ is the optimal solution of $(P_{2.2})$, then the second best extreme point of solution of $(P_{2.2})$ is an element of $S_2 \backslash X_1$ which is at minimum orthogonal distance from the hyperplane $cx = cx^* = v_1$

Proof. Since x^* is an optimal solution of $(P_{2.2})$, therefore $cx \geq cx^*$ for all $x \in S_2$. Let $y^* \in (S_2 \backslash X_1)$ be the second best extreme point solution of $(P_{2.2})$, i.e., $cy^* \geq cy$ for all $y \in (S_2 \backslash X_1)$

or $-cy^* \leq -cy$ for all $y \in S_2 \backslash X_1$

or $cx^* - cy^* \leq cx^* - cy$ for all $y \in S_2 \backslash X_1$

or $\dfrac{cx^* - cy^*}{||c||} \leq \dfrac{cx^* - cy}{||c||}$ for all $y \in S_2 \backslash X_1$

or $\dfrac{v_1 - cy^*}{||c||} \leq \dfrac{v_1 - cy}{||c||}$ for all $y \in S_2 \backslash X_1$

Thus the orthogonal distance of y^* from the hyperplane $cx = v_1$ is less than or equal to the distance of any other point $y \in S_2 \backslash X_1$ from the same hyperplane . Hence y^* is at a minimum orthogonal distance.

2.2 Procedure

Let the rank of D be p and the rank of F be $m + p$, respectively. If $X \in S_2$ then it can have at most $m + p$ non zero components. Also any extreme point of $(P_{2.2})$ has at most p non zero components. Thus if $X \in S_1 \bigcap S_2$ then the set J(x) can have at most p non zero components. Thus if x is any element of S_2 and $|J(x)| > p$ then x is not an element of S_1; if $X \in S_2$ and $|J(x)| \leq p$ then $X \in S_1$ if and only if elements of $J(X)$ are linearly independent. Thus extreme point of $(P_{2.2})$ is also an extreme point of $(P_{2.1})$ if $|J(x)| \leq p$ and elements of J(x) are linearly independent.

Suppose now $X_1 \bigcap S_1 = \phi$, then by Theorem 2.1 an optimal solution of $(P_{2.1})$ will be an element of $S_1 \bigcap (S_2 \setminus X_1)$, i.e., an optimal solution of $(P_{2.1})$ must be an element of $S_1 \bigcap S_2$. Then find the set $X_2 = (x_{21}, x_{22}, ..., x_{2k_2})$ of all second best extreme point solution of $(P_{2.2})$ by the same procedure followed above. If $X_2 \bigcap S_1 \neq \phi$, then any point x^*, where $x^* \in X_2 \bigcap S_1$ is an optimal solution of $(P_{2.1})$. This follows from the fact that that any optimal solution of $(P_{2.1})$ must be an element of S_2 which is at minimal distance from the hyperplane $cx = v_1$ (by Theorem 2.1 and 2.2) and also by Lemma 2.3 that any point in X_2 is at minimum distance from the hyperplane $cx = v_1$. We now explain the procedure to find the second best extreme point solution of $(P_{2.2})$ given by [1].

ProcedureT : First obtain all the elements of X_1. Let B be the basis corresponding to an element of X_1, X_B be the vector of the basic variables and C_B be the row vector with components as the coefficient associated with the basic variables in the objective function. Thus

$$X_B = B^{-1}\ f, \quad Y_j = B^{-1}\ f \text{ and } Z_j = C_B\ Y_j.$$

For each element of X_1 determine, $H(B) = (j|\ Z_j - C_j > 0)$. Then for each optimal tableau evaluate, θ_j= $\min((x_{Bi}/y_{ij})|y_{ij} > 0)$, $j\epsilon\ H(B)$ and $\gamma_B = min_{j\ \in\ H(B)}(\theta_j(Z_j -$

$C_j)|\theta_j > 0)$ and then finally calculate δ =min $(\gamma_B|$ B is the basis of an element of X_1). This process gives the optimal table to be used and the column to enter and leave the basis to determine second best extreme point solution of $(P_{2.2})$. If the various minima obtained in the following process are unique, the second best extreme point solution is unique, otherwise it is to be generated. Let $v_2 = cx_{21}$ be the value of the objective function for the second best extreme point solution. Determine $X_2 \bigcap S_1$. In case $X_2 \bigcap S_1 \neq \phi$ then every element of $X_2 \bigcap S_1$ is an optimal solution of $(P_{2.1})$ and if $X_2 \bigcap S_1 = \phi$ then optimal solution of $(P_{2.1})$ is an element of $S_1 \bigcap (S_2 \backslash (X_1 \bigcup X_2))$. Now find the set of third best extreme point solution of $(P_{2.2})$ which is done by finding the second best solution of linear programming problem with an additional constraint $cx \leq v_2$.

This gives a new problem which is as follows:

$(\mathbf{P_{2.3}})$ Max $Z = cx$,

subject to $Fx = f$, $cx \leq v_2$, $x \geq 0$.

Solve the problem $(P_{2.3})$. Let Y_2 be the set of optimal extreme point of $(P_{2.3})$. Clearly $X_2 \subseteq Y_2$ (by Theorem 2.1). The second best extreme point solution of $(P_{2.3})$ is an element s^* of the set V, where

$V = (x|x$ is an extreme point of$Fx = f,\ cx \leq v_2,\ x \geq 0)$ such that $cs^* \geq cx$ for all $x \in S_2 \backslash (X_1 \bigcup X_2)$. The second best solution of $(P_{2.3})$ is the third best extreme point solution of $(P_{2.2})$, i.e, $X_2 \equiv X_3$ where $X_3 = (x_{31}, x_{32}, ..., x_{3k_3})$ be the third best extreme point solution of $(P_{2.2})$ and v_3 be the objective function corresponding to any element of X_3. Determine $X_3 \bigcap S_1$. If $X_3 \bigcap S_1 \neq \phi$ then every element of $X_3 \bigcap S_1$ is an optimal solution of $(P_{2.1})$, otherwise the next best solution of $(P_{2.2})$ can be found by introducing the constraint $cx \leq v_3$ in $(P_{2.2})$ and determine the second best extreme point of Max $Z = cx$, subject to $Fx = f$, $cx \leq v_3$, $x \geq 0$, which will give us the set X_4 of fourth best extreme point solution of $(P_{2.2})$. This process is continued till some stage say k^{th}, the set of k^{th} best extreme point solutions of $(P_{2.2})$ is such that $X_k \bigcap S_1 \neq \phi$, where X_k is the second best extreme point solution of the problem : Max $Z = cx$, subject to $Fx = f$, $cx \leq v_{k-1}$, $x \geq 0$, where v_{k-1} is the value of objective function at an element of $(k-1)^{st}$ best extreme point solution. The above developed method enables us to establish the geometric property of the optimal solution of $(P_{2.1})$, which can be explained in succeeding theorem.

Theorem2.3 If $\hat{x}$ is an optimal solution of $(P_{2.1})$ and $X_1 \bigcap S_1 = \phi$, then $\hat{x}$ is adjacent to some element S_3. Moreover, for all points adjacent to some elements of S_2, $\hat{x}$ is at minimal orthogonal distance from the hyperplane $cx = v_1$.

Proof. If $\hat{x}$ is the optimal solution of $(P_{2.1})$, then $\hat{x} \in S_1$. By Theorem 2.2, $\hat{x}$ is at the minimum orthogonal distance from the hyperplane $cx = v_1$. Also $\hat{x} \in S_1$ and $S_1 \subseteq S_2$ therefore, $\hat{x} \in S_2$. Let $\hat{x}$ be N^{th} best extreme point solution of $(P_{2.2})$. Therefore $X_i \bigcap S_1 = \phi$ for $1 \leq i \leq N-1$ and hence $\bigcup_{i=1}^{N-1} X_i \subseteq S_3$, by Lemma 2.2, it is adjacent to some element of S_3. It is to be noticed in the above procedure that all the extreme points of S_2 are not required to be examined, the procedure begins with the best extreme point of $(P_{2.2})$ and proceeds to study, the second best extreme point, third best,...,Nth best extreme points and terminates as soon as an extreme point of $(P_{2.1})$ is determined. However in case (R-3), i.e $(P_{2.2})$ has a solution but $(P_{2.1})$ has no solution, it is necessary to test all the extreme points. Since $(P_{2.1})$ has no solution, therefore $S_1 = \phi$ and $X_i \bigcap S_1 = \phi$ for all i, which shows that procedure will continue infinitely. But this is impossible as S_2 is finite and $v_i > v_{i+1}$ for all i and so after finite number of steps say N it will be impossible to find out second best extreme point solution to the problem; Max $Z = cx$, subject to $Fx = f$, $cx \leq v_N$, $x \geq 0$ which is indicated by the fact that there is no optimal simplex table with a j such that $\theta_j > 0$ and $Z_j - c_j > 0$.

Statement of the Algorithm : The results proved in the previous sections will be used in the following algorithm :

Step 1. Apply the simplex method to $(P_{2.2})$. If there is no solution, then terminate else go to Step 2.

Step 2. Find the set X_i for i=1 and go to Step 3.

Step 3. Determine whether or not $X_i \bigcap S_1 \neq \phi$ by testing the linear independence of elements of $J(x)$ for each $x \in X_i$. If $X_i \bigcap S \neq \phi$, then any element of $X_i \bigcap S_1$ is the optimal solution of $(P_{2.1})$ and the procedure terminates. Otherwise let $v_i = cx_{i1}$ and go to Step 4.

Step 4. Find the set Y_i and go to Step 5. (If $(P_{2.2})$ has bounded solution then $Y_1 = X_1$).

Step 5. Using Procedure T, find the set X_{i+1} of all second best extreme point solution of $P_{2.i}$. If $X_{i+1} = \phi$, then the $(P_{2.1})$ has no solution and the procedure terminated. If $X_{i+1} \neq \phi$, then go to Step 3.

2.3 Example

Maximize $x_1 + 20x_2$ subject to

$$x_1 + x_2 \leq 11,$$

$$3x_1 + 5x_2 \leq 45,$$

(x_1, x_2) is the extreme point of

$$-5x_1 + x_2 \leq 1,$$

$$2x_1 + x_2 \leq 22,$$

$$x_1 \geq 0, \quad x_2 \geq 0.$$

We first apply simplex method to the following problem :

Maximize $\qquad x_1 + 20x_2$, subject to

$$x_1 + x_2 + x_3 = 11,$$

$$3x_1 + 5x_2 + x_4 = 45,$$

$$-5x_1 + x_2 + x_5 = 1,$$

$$2x_1 + x_2 + x_6 = 22,$$

$$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0.$$

The optimal table of above LPP is :

Tableau 2.1

	c_j	1	20	0	0	0	0	
	$z_j - c_j$	0	0	0	101/28	55/28	0	Solution
c_B	B	f_1	f_2	f_3	f_4	f_5	f_6	1150/7
0	f_3	0	0	1	$-3/14$	1/14	0	10/7
1	f_1	1	0	0	1/28	$-5/28$	0	10/7
20	f_2	0	1	0	5/28	3/28	0	57/7
0	f_6	0	0	0	$-1/4$	1/4	1	11

The columns f_3, f_4, f_5, f_6 represent the columns of slack variables, $v_1 = 1150/7$ and

$X_1 = x_{11} = (10/7, 57/7, 10/7, 0, 0, 11)$. Since Tableau 2.1 indicates no alternate optima, hence X_1 contain the single element $\mathbf{x}_{11}$. Here $p = 2$ and $J = d_1, d_2, d_5, d_6$. Since the first, second and sixth component of x_{11} are nonzero, then $J(x_{11})$ has three elements and so $X_1 \cap \ S_1 = \phi$. We then find the set X_2.

$H(B_1) \ = 4, 5$, where B_1 is the basis in Tableau 2.1.

$$\theta_4 = min(10/7 \div 1/28,\ 57/7 \div 5/28) = 40$$

$\theta_5 = min(10/7 \div 1/14,\ 57/7 \div 3/28) = 20$

$\gamma B_1 = min(40\ \times\ 101/28, 20\ \times\ 55/28 = \delta_,$. So f_5 enter the basis and f_3 leaves yielding the following tableau:

Tableau 2.2

	c_j	1	20	0	0	0	0	
	$z_j - c_j$	0	0	$-55/2$	$19/2$	0	0	Solution
c_B	B	f_1	f_2	f_3	f_4	f_5	f_6	125
0	f_5	0	0	14	-3	1	0	20
1	f_1	1	0	$5/2$	$-1/2$	0	0	5
20	f_2	0	1	$-3/2$	$1/2$	0	0	6
0	f_6	0	0	$-7/2$	$1/2$	0	1	6

$v_2 = 125$ and X_2 has single element $x_{21} = (5, 6, 0, 0, 20, 6)$, $J(x_{21}) = (d_1, d_2, d_5, d_6)$. So $J(x_{21})$ contains four elements and hence $p = 2, X_2 \bigcap\ S_1 = \phi$. We now consider the problem

$Maximize\ \ x_1\ +\ 20x_2$ subject to

$x_1 + x_2 + x_3 = 11,$

$3x_1 + 5x_2 + x_4 = 45,$

$-5x_1 + x_2 + x_5 = 1,$

$2x_1 + x_2 + x_6 = 22,$

$x_1 + 20x_2 \leq\ 125,$

$x_1, x_2, x_3, x_4, x_5, x_6 \geq\ 0.$

The above problem has two optimal extreme point solutions which are given in Tableaux 2.3 and 2.4 respectively and one second best extreme point solution is given in *Tableau* 2.5.

Tableau 2.3

	c_j	1	20	0	0	0	0	0	
	$z_j - c_j$	0	0	0	0	0	0	1	Solution
c_B	B	f_1	f_2	f_3	f_4	f_5	f_6	f_7	125
0	f_5	0	0	101/19	0	1	0	$-6/19$	20
0	f_4	0	0	$-55/19$	1	0	0	$-2/19$	0
20	f_2	0	1	$-1/19$	0	0	0	1/19	6
0	f_6	0	0	$-39/19$	0	0	1	1/19	6
1	f_1	1	0	20/19	0	0	0	$-1/19$	5

Tableau 2.4

	c_j	1	20	0	0	0	0	0	
	$z_j - c_j$	0	0	0	0	0	0	1	Solution
c_B	B	f_1	f_2	f_3	f_4	f_5	f_6	f_7	125
0	f_3	0	0	1	0	19/101	0	$-6/101$	380/101
0	f_4	0	0	0	1	55/101	0	$-28/101$	1100/101
20	f_2	0	1	0	0	1/101	0	5/101	626/101
0	f_6	0	0	0	0	39/101	1	$-7/101$	1386/101
1	f_1	1	0	0	0	$-20/101$	0	1/101	105/101

Tableau 2.5

	c_j	1	20	0	0	0	0	0	
	$z_j - c_j$	-101	0	0	0	20	0	0	Solution
c_B	B	f_1	f_2	f_3	f_4	f_5	f_6	f_7	20
0	f_3	6	0	1	0	-1	0	0	10
0	f_4	28	0	0	1	-5	0	0	40
20	f_2	-5	1	0	0	1	0	0	1
0	f_6	7	0	0	0	-1	1	0	21
0	f_7	101	0	0	0	-20	0	1	105

$v_3 = 20$ and $X_3 = \mathbf{x}_{31} = (0, 1, 10, 40, 0, 21)$. $J(\mathbf{x}_{31}) = (\mathbf{d}_2, \mathbf{d}_6)$ and since $\mathbf{d}_2$ and $\mathbf{d}_6$ are linearly independent we conclude that $\mathbf{x}_{31} \epsilon\ S_1$ and so $x_1 = 0,\ x_2 = 1$ is the optimal solution to given problem. The optimal value of objective function is $\mathbf{v}_3 = 20$.

Conclusion : In this chapter methd to solve extreme point linear programming problem is explained based upon which a numerical problem is solved whose optimal value is 20.

Chapter 3

Extreme Point Linear Fractional Programming Problem

In this chapter procedure to find an optimal extreme point solution to extreme point linear fractional programming problem has been discussed. Mathematically, it can be stated as :

$(\mathbf{P_{3.1}})$

$$\text{Maximize } Z = \frac{CX + \alpha}{DX + \beta}$$

subject to

$$AX = b,$$

where x is the extreme point of

$$RX = t, \; x \geq 0,$$

where $C, \; D$ are $1 \times n$, X is $n \times 1$, A is $m \times n$, R is $p \times \; n$, b is $m \times 1$, t is $p \times 1$, 0 is $n \times 1$ and $\alpha, \; \beta$ are scalars.

3.1 Theoretical Development

To solve the $(P_{3.1})$, we state problem $(P_{3.2})$ which is defined as :

$(\mathbf{P_{3.2}})$ Maximize $Z = \dfrac{CX + \alpha}{DX + \beta}$

subject to

$$FX = f,$$

$$X \geq 0,$$

where $F = \binom{A}{R}$ and $f = \binom{b}{t}$.

It may be noted that $(P_{3.1})$ is always bounded as any solution is to be extreme point of $RX = t,\ X \geq\ 0$ and extreme points of $RX = t,\ X \geq\ 0$ are finite. However $(P_{3.2})$ may be either bounded or unbounded. In case $(P_{3.2})$, is unbounded it can always be converted into a bounded problem by adding a constraint $I_n Z \leq M$, where M is an arbitrary large finite positive number determined in such a way that none of extreme point of original problem are excluded from $(P_{3.2})$. So $(P_{3.2})$ is always considered to be bounded.

Notations :

$J = \{r_j | r_j \neq 0$, where r_j is the j^{th} column of $R\}$

$J(X) = \{r_j \in J | x_j \neq 0$, where $X = (x_1, x_2, ..., x_n)$ is the basic feasible solution of $(P_{3.2})\}$

$|J(X)|=$ number of elements of $J(X)$

$S_1 = \{X |\ AX = b$ and X is the extreme point of $RX = t,\ X \geq 0\}$

$S_2 = \{X |\ X$ is an extreme point of $Fx = f,\ X \geq 0\}$

$S = \{X|\ \ FX = f,\ X \geq 0\}$

X_i= set of all i^{th} best extreme points solutions of $(P_{3.2}) =\ (X_{i1}, X_{i2}, ..., X_{is_i})$

B_i= Set of all the bases corresponding to the elements of X_i.

E_i= Set of all the bases adjacent to each element of B_i and yielding value of the objective function

less than u_i.

(Two bases are said to be adjacent to each other if either of them is obtained from the other by changing only one of its vectors).[hadley[1], kirby et al.[2] and dantzig et al.[3]]

H_i= Set of all the bases adjacent to $1^{st},\ 2^{nd}, ...,\ i^{th}$ best extreme points of $(P_{3.2})$ leaving $1^{st},\ 2^{nd}, ...,\ i^{th}$ best extreme points of $(P_{3.2})$

$=\bigcup_{j=1}^{i} E_j - \bigcup_{j=2}^{i} B_j$.

Note that $H_i \subseteq\ \bigcup_{j=1}^{i} E_j$.

Clearly $H_1 = E_1$ and $\bigcup_{j\geq 1} B_j \subseteq\ H_{i-1}$.

Remarks :

1. The set X_i of i^{th} best extreme points of $(P_{3.2})$ is adjacent to some elements of $\bigcup_{j=1}^{i-1} X_j$ kirby et al.[2].

2. $X_i \in S_2 \backslash \bigcup_{j=1}^{i-1} X_j$ and is such that $u_i \geq \dfrac{CX+\alpha}{DX+\beta}$ where X is any extreme point belonging to $S_2 \backslash \bigcup_{j=1}^{i-1} X_j$.

3. Every point of S_1 is a point of S, converse may or may not be true, i.e., $S_1 \subseteq S$.

Theorem3.1 $S_1 \subseteq S_2$, i.e., every extreme point of $RX = t,\ X \geq 0$ satisfying feasibility in $AX = b$ is also an extreme point of $FX = f,\ X \geq 0$.

Proof. Follows on the lines of Theorem 2.1.

Theorem3.2 If $\hat{X}$ is an optimal solution of $(P_{3.1})$ and $X_1 \bigcap S_1 = \phi$, then $\hat{X}$ is adjacent to some element of $S_3 = S_2 \setminus S_1$.

Proof. By Theorem 3.1, $\hat{X} \in S_2$. Let $\hat{X}$ belong to the set X_N of the N^{th} best extreme points of $(P_{3.2})$ and $\hat{X}$ is the k^{th} member of X_N i.e., it is same as X_{N_k}. Therefore, $X_i \bigcap S_1 = \phi$ for $i = 1, 2, ..., N-1$. As S_3 is the set of those extreme points of $FX = f,\ X \geq 0$ which are not extreme points of $RX = t,\ X \geq 0$, it follows that $\bigcup_{i-1}^{N-1} X_i \subseteq S_3$. As X_{N_k} is adjacent to some element of $\bigcup_{i=1}^{N-1} X_i$, it follows that X_{N_k} is adjacent to an element of S_3.

The above theorem shows that the method described above does not require that all the extreme points of S_2 should be examined. Examine only elements of $\bigcup_{i=1}^{N-1} X_i$ starting from X_i until an N^{th} best extreme point solution is found which is feasible for $(P_{3.1)})$. Thus points of S_3 are examined one by one till we reach a point in S_1

where the procedure terminated.

In case, $(P_{3.2})$ has a solution but $(P_{3.1})$ has no solution, i.e., $S_2 \bigcap S_1 = \phi$, it is necessary to examine all the extreme points of $(P_{3.2})$. As for each i, $X_i \bigcap S_1 = \phi$, it seems that the procedure will never stop. But this will not occur, since S_2 has finite number of elements. Also $u_i > u_{i+1}$. So after finite number of steps, say N, it will be impossible to get $(N+1)^{th}$ best extreme point of $(P_{3.2})$. Thus if a stage is reached when no further best extreme point solution is possible, then $(P_{3.1})$ has no solution.

3.2 Procedure

We begin by solving $(P_{3.2})$. This yields the set X_1 and $u_1$swarup [14]. If $X_1 \neq \phi$ and $X_1 \bigcap \ S_1 \neq \phi$, then any $X \in \ X_1 \bigcap \ S_1$ will be the required optimal solution of $(P_{3.1})$. But if $X_1 \bigcap \ S_1 = \phi$, i.e., no element of X_1 is an extreme point of $RX = t, \ X \geq 0$, then the optimal solution of $(P_{3.1})$ must be an element of $S_1 \bigcap \ (S_2 \backslash X_1)$. Also by Theorem 3.1 an optimal solution of $(P_{3.1})$ must be an element of $S_1 \bigcap \ S_2$. Then proceed to find the set X_2 for which we find B_1 and E_1. Bases of E_1 which yields the greatest value say u_2 of the objective function, generates the set X_2. There will be a unique second best extreme point solution of $(P_{3.2})$ (i.e., X_2 will have a single element) if there is only one element of E_1 which yields u_2. If $X_2 = \phi$, then $(P_{3.1})$ has no solution. If $X_2 \neq \phi$ and $X_2 \bigcap \ S_1 \neq \phi$ then any element $X \in \ X_2 \bigcap \ S_1$ will be an optimal solution of $(P_{3.1})$. But if $X_2 \bigcap \ S_1 = \phi$, proceed to find X_3. To determine X_3, first find B_2, E_2 and H_2. Bases of H_2 which yields the greatest value say u_3 of objective function, generates X_3. If $X_3 = \phi$, then $(P_{3.1})$ has no solution. If $X_3 \neq \phi$ and $X_3 \bigcap S_1 \neq \phi$ then any element $X \in X_3 \bigcap S_1$ will be an optimal solution of $(P_{3.1})$. And if $X_3 \neq \phi$ and $X_3 \bigcap S_1 = \phi$, the process is repeated by finding X_4, X_5 and so on till for some i we get $X_i \neq 0$ and $X_i \bigcap S_1 \neq \phi$ in which case any $X \in X_i \bigcap S_1$ will be the optimal solution of $(P_{3.1})$.

Statement of the Algorithm : The results proved in the previous sections will be used in the following algorithm:

Step 1. Solve $(P_{3.2})$. If $(P_{3.2})$ has an unbounded solution, then go to Step 5. If $(P_{3.2})$ has bounded solution, then find X_r, B_r (starting from $r = 1$) and go to step 2.

Step 2. Determine whether or not $X_r \bigcap S_1 \neq \phi$. If $X_r \bigcap S_1 \neq \phi$, terminate the process. In this case any element of $X \in X_r \bigcap S_1$ is the optimal solution of $(P_{3.1})$ yielding value U_r. If $X_r \bigcap S_1 = \phi$, go to Step 3.

Step 3. Find the set E_r and H_r (starting from $r = 1$). If $H_r = \phi$ then $(P_{3.1})$ has no solution and procedure is terminated. If $H_r \neq \phi$, then go to step 4.

Step 4. Find U_{r+1} and determine the sets B_{r+1} and X_{r+1} and go to step 2.

Step 5. Rewrite $(P_{3.2})$ to include the constraint $1X \leq M$, where M is a sufficiently large positive number and 1 is a sum vector, and return to step 1.

3.3 Example

$$\text{Maximize } \frac{2x_1 + x_2}{4x_1 + x_2 + 1}$$

$$\text{subject to } -2x_1 + x_2 \leq \ 1,$$

$$2x_1 + 5x_2 \leq \ 23,$$

$$2x_1 + x_2 \leq \ 15$$

and (x_1, x_2) is the extreme point of

$$-3x_1 + 2x_2 \leq \ 4,$$

$$x_1 + 4x_2 \leq \ 22,$$

$$5x_1 + 4x_2 \leq \ 46,$$

$$x_1 - 2x_2 \leq \ 5,$$

$$x_1, \ x_2 \geq 0$$

By adding slack variables the above problem can be rewritten as :

$$\text{Maximize } \frac{2x_1 + x_2}{4x_1 + x_2 + 1},$$

subject to

$$-2x_1 + x_2 + x_3 = 1,$$

$$2x_1 + 5x_2 + x_4 = 23,$$

$$2x_1 + x_2 + x_5 = 15,$$

$$-3x_1 + 2x_2 + x_6 = 4,$$

$$x_1 + 4x_2 + x_7 = 22,$$

$$5x_1 + 4x_2 + x_8 = 46,$$

$$x_1 - 2x_2 + x_9 = 5,$$

$$x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9 \geq 0$$

Its optimal solutions is :

Step 1. X_1=$[X_{11} = (\frac{3}{2}, 0, 0, 8, \frac{1}{2}, \frac{9}{2}, \frac{45}{2}, \frac{23}{2})]$, $u_1 = \frac{7}{11}$ and $B_1 = [B_{11} = (f_2, f_1, f_5, f_6, f_7, f_8, f_9)]$

where $f_1, f_2, ..., f_9$ are columns of F.

$J(X_{11}) = \ r_1, r_2, r_6, r_7, r_8, r_9$

$|J(X_{11})|= 6 > 4$, Here $p = 4$.

$\therefore\ X_1 \bigcap\ S_1 = \phi$

The optimal table for X_{11} is found to be :

Tableau 3.1

c_j			2	1	0	0	0	0	0	0	0	
d_j			4	1	0	0	0	0	0	0	0	
$c_j - z_j$			0	0	$2/3$	$-1/3$	0	0	0	0	0	
$d_j - z_j$			0	0	$3/2$	$-1/2$	0	0	0	0	0	
Δ_j			0	0	$-19/6$	$-1/6$	0	0	0	0	0	Solution
D_B	C_B	B	f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8	f_9	$7/11$
1	1	f_2	0	1	$1/6$	$1/6$	o	0	0	0	0	4
4	2	f_1	1	0	$-5/12$	$1/12$	0	0	0	0	0	$3/2$
0	0	f_5	0	0	$2/3$	$-1/3$	1	0	0	0	0	8
0	0	f_6	0	0	$-19/12$	$-1/12$	0	1	0	0	0	$1/2$
0	0	f_7	0	0	$-1/4$	$-3/4$	0	0	1	0	0	$9/2$
0	0	f_8	0	0	$17/12$	$-13/12$	0	0	0	1	0	$45/2$
0	0	f_9	0	0	$3/4$	$1/4$	0	0	0	0	1	$23/2$

where $z^1 = C_{B11}X_{B11}$ and $z^2 = D_{B11}X_{B11} + 1$, C_{B11} and D_{B11} are the vectors having their

components as the coefficients associated with the basic variables in the numerator and the

denominator of the objective function $z_j^1 = C_{B11}V_j$, $z_j^2 = D_{B11}V_j$ and $V_j = B_{11}^{-1}f_j$. The optimal objective function is given by $Z = \frac{Z^1}{Z^2}$, where $\Delta_j = Z^2(c_j - z_j^1) - Z^1(d_j - z_j^2)$.

Step 2. $E_1 = [E_{11} = (f_2, f_1, f_3, f_6, f_7, f_8, f_9)]$, $E_{12} = (f_2, f_4, f_5, f_6, f_7, f_8, f_9)$, $\frac{15}{29}$ is the value of objective function yielded by $(f_2, f_1, f_3, f_6, f_7, f_8, f_9)$ and $\frac{1}{2}$ is the value of objective function yielded by $(f_2, f_4, f_5, f_6, f_7, f_8, f_9)$,

$u_2 = \frac{15}{29}$,

$B_2 = [B_{21} = (f_2, f_1, f_3, f_6, f_7, f_8, f_9)]$, $X_2 = [X_{21} = (\frac{13}{2}, 2, 12, 0, 0, \frac{39}{2}, \frac{15}{2}, \frac{11}{12}, \frac{5}{2})]$,

$J(X_{21}) = r_1, r_2, r_6, r_7, r_8, r_9$.

Here, $|J(X_{21})| = 6 > 4$.

$\therefore X_2 \bigcap S_1 = \phi$. Hence X_{21} is not an extreme point of $RX = t, \quad X \geq 0$.

Step 3. $E_2 = [E_{21} = (f_2, f_1, f_3, f_6, f_7, f_8, f_4)]$

$H_2 = [(E_1 \bigcup E_2) \backslash B_2] = [(f_2, f_4, f_5, f_6, f_7, f_8, f_9), (f_2, f_1, f_3, f_6, f_7, f_8, f_4)]$, $\dfrac{1}{2}$ is the value of objective function yielded by $(f_2, f_4, f_5, f_6, f_7, f_8, f_9)$ and $\dfrac{1}{2}$ is the value of objective function yielded by $(f_2, f_1, f_3, f_6, f_7, f_8, f_4)$

$B_3 = [B_{31} = (f_2, f_4, f_5, f_6, f_7, f_8, f_9)]$, $B_{32} = (f_2, f_1, f_3, f_6, f_7, f_8, f_9)$.

$X_3 = X_{31} = (0, 1, 0, 18, 14, 2, 18, 42, 7)$, $X_{32} = (7, 1, 14, 4, 0, 23, 11, 7, 0)$,

$J(X_{31}) = r_2, r_6, r_7, r_8, r_9$.

Here, $|J(X_{31})| = 5 > 4$

$\therefore X_{31}$ is not an extreme point of $RX = t, \ X \geq 0$.

$J(X_{32}) = r_1, r_2, r_6, r_7, r_8$,

$|J(X_{32})| = 5 > 4$

$\therefore X_{32}$ is not an extreme point of $RX = t, \ X \geq 0$ and hence $X_3 \bigcap S_1 = \phi$.

Step 4. $E_3 = [E_{31} = (f_3, f_4, f_5, f_6, f_7, f_8, f_9)]$, $E_{32} = (f_5, f_1, f_3, f_6, f_7, f_8, f_4)]$

$H_3 = [(E_1 \bigcup E_2 \bigcup E_3) \backslash (B_2 \bigcup B_3)] = [(f_3, f_4, f_5, f_6, f_7, f_8, f_9), (f_5, f_1, f_3, f_6, f_7, f_8, f_4)]$

0 is the value of objective function yielded by$(f_3, f_4, f_5, f_6, f_7, f_8, f_9)$ and $\dfrac{10}{21}$ is the

value of objective function yielded by $(f_5, f_1, f_3, f_6, f_7, f_8, f_4)$.

$u_4 = \frac{10}{21}$

$B_4 = \ [B_{41} = \ (f_5, f_1, f_3, f_6, f_7, f_8, f_4)]$

$X_4 = \ X_{41} = \ (5, 0, 11, 13, 5, 19, 17, 21, 0)$

$J(X_{41}) = \ r_1, r_6, r_7, r_8.$

$|J(X_{31})| = 4 = \ p$, Also r_1, r_6, r_7, r_8 are linearly independent.

$\therefore X_4 \bigcap \ S_1 \neq \ \phi$. Hence X_{41} is an extreme point of $RX = \ t, \quad X \geq 0$.

$\therefore$ optimal solution of $(P_{3.1})$ is $x_1 = 5, \quad x_2 = 0$ and optimal value is $\frac{10}{21}$.

Conclusion : In this chapter method to solve extreme point linear fractional programming problem is explained in which ranking of extreme point solution has been done to get best extreme point solution. Based upon the method a numerical is solved whose optimal value is $\frac{10}{21}$.

Chapter 4

Extreme Point Linear Fractional Programming Problem With Bounded Variables

This chapter is devoted to procedure of finding the optimal extreme point solution to extreme point linear fractional programming problem with bounded variables. In general, extreme point linear fractional programming problem with bounded variables can be stated as

$(\mathbf{P_{4.1}})$ $$\text{Maximize } Z = \frac{CX+\alpha}{DX+\beta}$$

subject to

$$AX = b,$$

where x is the extreme point of

$$RX = t,\ l \leq X \leq u,$$

where $C,\ D$ are $1 \times n$, X is $n \times 1,\ A$ is $m \times n$, R is $p \times\ n$, b is $m \times 1$, t is $p \times 1$, and 0 is $n \times 1$ and $\alpha,\ \beta$ are scalars.

4.1 Theoretical Development

To solve the ($P_{4.1}$), we state the problem, $P_{4.2}$ which is defined as :

($\mathbf{P_{4.2}}$) $$\text{Maximize } Z = \frac{CX + \alpha}{DX + \beta}$$

subject to $$FX = f,$$

$$l \leq X \leq u,$$

where $F = \begin{pmatrix} A \\ R \end{pmatrix}$ and $f = \begin{pmatrix} b \\ t \end{pmatrix}$.

It may be noted that ($P_{4.1}$) is always bounded as any solution is to be extreme point of $RX = t,\ l \leq X \leq u$ and extreme points of $RX = t,\ l \leq X \leq u$ are finite. However ($P_{4.2}$) may be either bounded or unbounded. In case ($P_{4.2}$), is unbounded it can always be converted into a bounded problem by adding a constraint $I_n Z \leq M$, where M is an arbitrary large finite positive number determined in such a way that none of extreme point of original problem are excluded from ($P_{4.2}$). So ($P_{4.2}$) is always considered to be bounded.

Notations :

X_i= set of all i^{th} best extreme points solutions of $(P_{4.2}) = [X_{i1}, X_{i2}, ..., X_{is_i}]$.

$x_j = j^{th}$ component of X_i.

a_j = activity vector of x_j.

B = basis corresponding to X_i.

$I = \{i \mid a_i \in B\}$

$N_1 = \{j \mid a_j \notin B,\ x_j = l_j\}$

$N_2 = \{j \mid a_j \notin B,\ x_j = u_j\}$

$A_{N_1} = \{a_j \in A \mid j \in N_1\}$

$A_{N_2} = \{a_j \in A \mid j \in N_2\}$

$X = (X_B, X_{N_1}, X_{N_2})$

$J = \{r_j | r_j \neq 0$, where r_j is the j^{th} column of $R\}$.

$J(X) = \{r_j \in J | l_j \leq x_j \leq u_j$, where $X = (x_1, x_2, x_3, ..., x_n)$ is the basic feasible solution of $(P_{4.2})$.

$|J(X)|=$ number of elements of $J(X)$.

$S_1 = \{X |\ AX = b$ and X is the extreme point of $RX = t,\ l \leq X \leq u\}$.

$S_2 = \{X |\ X$ is an extreme point of $Fx = f,\ l \leq X \leq u\}$

$S = \{X |\ FX = f,\ l \leq X \leq u\}$

B_i= Set of all the bases corresponding to the elements of X_i.

E_i= Set of all the bases adjacent to each element of B_i and yielding value of the objective function

less than u_i.

(Two bases are said to be adjacent to each other if either of them is obtained from the other by changing only one of its vectors).[hadley[1], kirby'72 et al.[2] and dantzig et al.[3]]

H_i= Set of all the bases adjacent to $1^{st},\ 2^{nd}, ...,\ i^{th}$ best extreme points of $(P_{4.2})$ leaving $1^{st},\ 2^{nd}, ...,\ i^{th}$ best extreme points of $(P_{4.2})$

$=\bigcup_{j=1}^{i} E_j \setminus \bigcup_{j=2}^{i} B_j.$

Note that $H_i \subseteq \ \bigcup_{j=1}^{i} E_j$.

Clearly $H_1 = E_1$ and $\bigcup_{j\geq 1} B_j \subseteq \ H_{i-1}$.

The vector X_B contains those variables which appear in the basis B, similarly X_{N_1} and X_{N_2} are vectors which are formed by variables which are at their lower and upper bounds, respectively.

C_B (D_B) is the cost vector associated with basic variables in the numerator (denominator).

C_{N_1} (D_{N_1}) is the cost vector associated with nonbasic variables which are at their lower bound in the numerator (denominator).

C_{N_2} (D_{N_2}) is the cost vector associated with nonbasic variables which are at their upper bound in the numerator (denominator).

l_j is j^{th} component of L and u_j is j^{th} component of U.

For all $j = \ 1, 2, \ldots, n$, $Y_j = \{y_{ij}\} = (B)^{-1}a_j$.

Any basic feasible solution corresponding to the basis B is given by vector $X = (X_B, X_{N_1}, X_{N_2})$.

So $BX_B + A_{N_1}X_{N_1} + A_{N_2}X_{N_2} = b$.

$$X_B + \sum_{j\in N_1} Y_j x_j + \sum_{j\in N_2} Y_j x_j = (B)^{-1}b \tag{4.1}$$

For a basic feasible solution (bfs), $X_B = X_B^o,\ x_j = l_j,\ j \in N_1$ and $x_j = \ u_j,\ j \in N_2$, we get

$$X_B^o + \sum_{j\in N_1} Y_j l_j + \sum_{j\in N_2} Y_j u_j = (B)^{-1}b \tag{4.2}$$

The equations (4.1) and (4.2) imply that

$X_B + \sum_{j\in N_1} Y_j(x_j - l_j) + \sum_{j\in N_2} Y_j(x_j - u_j) = X_B$ (4.3)

The value objective function is given by $Z(X) = \dfrac{Z^1(X)}{Z^2(X)}$ where

$Z^1(X) = \ CX + \alpha.$

$$= C_B X_B + C_{N_1} X_{N_1} + C_{N_2} X_{N_2} + \alpha.$$

$$= C_B X_B + \sum_{j\in N_1} c_j l_j + \sum_{j\in N_2} c_j u_j + \alpha.$$

Then by using equation (4.2) and taking $Z_j^1 = C_B^1 Y_j$, we obtain

$$Z^1 = C_B^1[(B)^{-1}b] - \sum_{j\in N_1}(Z_j^1 - c_j)l_j - \sum_{j\in N_2}(Z_j^1 - c_j)u_j + \alpha.$$

Similarly, we obtain

$$Z^2 = D_B^1[(B)^{-1}b] - \sum_{j\in N_1}(Z_j^2 - d_j)l_j - \sum_{j\in N_2}(Z_j^2 - d_j)u_j + \beta.$$

To prove the optimality of X, we introduce the following notations.

$\triangle_j = Z^1(Z_j^2 - d_j) - Z^2(Z_j^1 - c_j),$

$J_1 = \{j \mid j \in N_1 \ \ \textit{such that} \ \triangle_j = 0\},$

$J_2 = \{j \mid j \in N_2 \ \ \textit{such that} \ \triangle_j = 0\}.$

Theorem4.1 $S_1 \subseteq S_2$, i.e., every extreme point of $RX = t,\ lX \geq 0$ satisfying feasibility in $AX = b$ is also an extreme point of $FX = f,\ X \geq 0$.

Proof. Follows on the lines of Theorem 2.1.

Theorem4.2 If $\hat{X}$ is an optimal solution of $(P_{4.1})$ and $X_1 \bigcap S_1 = \phi$, then $\hat{X}$ is adjacent to some element of $S_3 = \ S_2 \setminus S_1$.

Proof. Follows on the lines of Theorem 3.2.

4.2 Procedure

In this section first we discuss the simplex method for linear fractional programming problem ($P_{4.2}$) with bounded variables, which is an extension of simplex method for linear fractional programming problem with non-negative variables (see swarup[14]) and linear programming problem with bounded variables m.s.bazaraa et al. [**?**]

Consider the linear fractional programming problem :

$(\mathbf{P_{4.3}})\qquad \max_X F(X) = \dfrac{CX+\alpha}{DX+\beta}$

subject to $\qquad FX = f,$

$L \quad \leq X \leq U.$

Let X be an extreme point of $P_{4.2}$ then we have following observations :

$Z^1(X) = C_B(B^{-1}b) - \sum_{j\in N_1}(Z_j^1 - c_j)l_j - \sum_{j\in N_2}(Z_j^1 - c_j)u_j + \alpha$

$Z^2(X) = D_B(B^{-1}b) - \sum_{j\in N_1}(Z_j^2 - d_j)l_j - \sum_{j\in N_2}(Z_j^2 - d_j)u_j + \beta$

$where\ Z_j^1 = \sum\limits_{i\in I} c_{B_i}y_{ij},\ Z_j^2 = \sum\limits_{i\in I} d_{B_i}y_{ij}.$

Assume that at a current bfs $X^0 = (x_j^0)$, we have $x_j^0 = l_j$ for all $j \in N_1$ and $x_j^0 = u_j$ for all $j \in N_2$. The value of the objective function is given by

$Z(X^0) = \dfrac{Z^1(X^0)}{Z^2(X^0)}.$

In order to find a better bfs, we know from the theory of simplex method that one of the nonbasic vectors must enter the basis. Since nonbasic variables are either at their upper bounds or at their lower bounds, consider for instance that the nonbasic variable which undergoes change is currently at its lower bound, i.e. $x_r = l_r$. If this variable undergoes a change θ_r, then the new solution is defined as $\widehat{X} = (\hat{x}_j)$ where

$\hat{x_B} = \hat{x_B} - y_{ir}\theta_r \quad \forall\ i \in I$

$\hat{x_r} = l_r + \theta_r,$

$$\hat{x_j} = x_j^0,\ j \in (N_1 \cup N_2) \setminus \{r\} \tag{4.4}$$

$$Z(\hat{X}) = \frac{Z^1(X^0) - \theta_r(Z_r^1 - c_r)}{Z^2(X^0) - \theta_r(Z_r^2 - d_r)} \tag{4.5}$$

This new solution is a feasible extreme point provided

$$\theta_r = \min\left\{(u_r - l_r), \left(\frac{x_{B_i} - l_{B_i}}{y_{ir}}\middle| y_{ir} > 0, i \in I\right), \left(\frac{u_{B_i} - x_{B_i}}{-y_{ir}}\middle| y_{ir} < 0, i \in I\right)\right\}.$$

Then the following possibilities may arise:

(i) If $\theta_r = u_r - l_r$, then x_r attains its lower bound and remains nonbasic. Change in the values of basic variables $\hat{x}_{B_i}$ for all $i \in I$ and Z is given by equations (4.4) and (4.5), respectively.

(ii) If $\theta_r = \dfrac{x_{B_s} - l_{B_s}}{-y_{sr}}$ for some s, then x_r becomes basic and x_{B_s} departs and attains its lower bound, $\hat{x}_{B_i}$ and Z change as shown above in equations (4.4) and (4.5) respectively.

(iii) If $\theta_r = \dfrac{u_{B_s} - x_{B_s}}{y_{sr}}$ for some s, then x_r becomes basic and x_{B_s} departs and attains its upper bound, $\hat{x}_{B_i}, i \in I$ and Z are given by

$\hat{x_r} = u_r - \theta_r,$

$$Z(\hat{X}) = \frac{Z^1(X^0) + \theta_r(Z_r^1 - c_r)}{Z^2(X^0) + \theta_r(Z_r^2 - d_r)}.$$

Then

$$Z(\widehat{X}) - Z(X^0) = \frac{-\theta_r \triangle_r}{(Z^2(X^0) - \phi_r(Z_r^2 - d_r))Z^2(X^0)}.$$

From above discussion we see that the choice of the variable x_r which enters the basis is made among those nonbasic variables which give maximum improvement in the value of objective function. Preceding discussion leads to following observations :

Observation1. Given a basic feasible solution $X = (X_B, X_{N_1}, X_{N_2})$ to $(P_{4.3})$ with value of the objective function given by $Z(X)$ such that $\triangle_j \leq 0$ for all $j \in N_1$, $\triangle_j \geq 0$ for all $j \in N_2$ and $\triangle_j = 0$ for all $j \in I$, then $Z(X)$ is the maximum value of $(P_{4.3})$ and the basic feasible solution X is an optimal bfs.

Observation2. In case the objective function of $(P_{4.3})$ is linear, i.e., of the form

$$CX = \sum_{j=1}^{n} c_j x_j,$$

then $\triangle_j = -(Z_j - c_j)$, where $Z_j = \sum_{i \in I} c_{B_i} y_{ij}$. From observation 1, it follows that X is an optimal bfs of linear programming problem with bounded variables if $(Z_j - c_j) \geq 0$ for all $j \in N_1$, $(Z_j - c_j) \leq 0$ for all $j \in N_2$ and $(Z_j - c_j) = 0$ for all basic variables .

Now by solving $(P_{4.2})$ by the method discussed above we reach the first optimal table, yielding the set X_1 and $Z(X_1)$. If $X_1 \neq \phi$ and $X_1 \bigcap S_1 \neq \phi$, then any $X \in X_1 \bigcap S_1$ will be the required optimal solution of $(P_{4.1})$.

But if $X_1 \bigcap S_1 = \phi$, i.e., no element of X_1 is an extreme point of $RX = t$, $X \geq 0$, then the optimal solution of $(P_{4.1})$ must be an element of $S_1 \bigcap (S_2 \backslash X_1)$. Also by Theorem 4.1 an optimal solution of $(P_{4.1})$ must be an element of $S_1 \bigcap S_2$. Then proceed to find the set X_2 for which we find B_1 and E_1. Bases of E_1 which yields the greatest value say $Z(X_2)$ of the objective function, generates the set X_2. There will be a unique second best extreme point solution of $(P_{4.2})$ (i.e., X_2 will have a single element) if there is only one element of E_1 which yields $Z(X_2)$. If $X_2 = \phi$, then $(P_{4.1})$ has no solution. If $X_2 \neq \phi$ and $X_2 \bigcap S_1 \neq \phi$ then any element $X \in X_2 \bigcap S_1$ will be an optimal solution of $(P_{4.1})$. But if $X_2 \bigcap S_1 = \phi$, proceed to find X_3. To determine

X_3, first find B_2, E_2 and H_2. Bases of H_2 which yields the greatest value say $Z(X_3)$ of objective function, generates X_3. If $X_3 = \phi$, then $(P_{4.1})$ has no solution. If $X_3 \neq \phi$ and $X_3 \bigcap S_1 \neq \phi$ then any element $X \in X_3 \bigcap S_1$ will be an optimal solution of $(P_{4.1})$. And if $X_3 \neq \phi$ and $X_3 \bigcap S_1 = \phi$ the process is repeated by finding X_4, X_5 and so on till for some i we get $X_i \neq 0$ and $X_i \bigcap S_1 \neq \phi$ in which case any $X \in X_i \bigcap S_1$ will be the optimal solution of $(P_{4.1})$.

Statement of the Algorithm :

Step 1. Solve $P_{4.2}$. If $(P_{4.2})$ has an unbounded solution, then go to Step 5. If $(P_{4.2})$ has bounded solution, then find X_r, B_r (starting from $r = 1$) and go to step 2.

Step 2. Determine whether or not $X_r \bigcap S_1 \neq \phi$. If $X_r \bigcap S_1 \neq \phi$, terminate the process. In this case any element of $X \in X_r \bigcap S_1$ is the optimal solution of $(P_{4.1})$ yielding value U_r. If $X_r \bigcap S_1 = \phi$, go to Step 3.

Step 3. Find the set E_r and H_r (starting from $r = 1$). If $H_r = \phi$ then $(P_{4.1})$ has no solution and procedure is terminated. If $H_r \neq \phi$, then go to step 4.

Step 4. Find U_{r+1} and determine the sets B_{r+1} and X_{r+1} and go to step 2.

Step 5. Rewrite $(P_{4.2})$ to include the constraint $1X \leq M$, where M is a sufficiently large positive number and I_n is a sum vector, and return to step 1.

4.3 Example

$$\text{Maximize } \frac{2x_1 + x_2}{4x_1 + x_2 + 1}$$

$$\text{subject to } -2x_1 + x_2 \leq 1,$$

$$2x_1 + 5x_2 \leq 23,$$

$$2x_1 + x_2 \leq 15$$

and (x_1, x_2) is the extreme point of

$$-3x_1 + 2x_2 \leq 4,$$

$$x_1 + 4x_2 \leq 22,$$

$$5x_1 + 4x_2 \leq 46,$$

$$x_1 - 2x_2 \leq 5,$$

$$\text{where, } \quad 1 \leq x_1 \leq 6, \; 0 \leq x_2 \leq 5$$

By adding slack variables the above problem can be rewritten as :

$$\text{Maximize } \frac{2x_1 + x_2}{4x_1 + x_2 + 1},$$

subject to

$$-2x_1 + x_2 + x_3 = 1,$$

$$2x_1 + 5x_2 + x_4 = 23,$$

$$2x_1 + x_2 + x_5 = 15,$$

$$-3x_1 + 2x_2 + x_6 = 4,$$

$$x_1 + 4x_2 + x_7 = 22,$$

$$5x_1 + 4x_2 + x_8 = 46,$$

$$x_1 - 2x_2 + x_9 = 5,$$

$$1 \leq x_1 \leq 6,\ 0 \leq x_2 \leq 5, x_3, x_4, x_5, x_6, x_7, x_8, x_9 \geq 0$$

Its optimal solutions is :

Step 1. X_1=$[X_{11} = (6, 1, 0, 18, 14, 2, 18, 42, 7)]$, $Z(X_1) = \dfrac{1}{2}$ and

$B_1 = [B_{11} = (f_2, f_4, f_5, f_6, f_7, f_8, f_9)]$

where $f_1, f_2, ..., f_9$ are columns of F.

$J(X_{11}) =\ r_2, r_6, r_7, r_8, r_9$

$|J(X_{11})|= 5 > 4$, Here $p = 4$.

$\therefore\ X_1 \bigcap\ S_1 = \phi$

The optimal table for X_{11} is found to be :

Tableau 4.1

		c_j	2	1	0	0	0	0	0	0	0	
		d_j	4	1	0	0	0	0	0	0	0	
		$z_j - c_j$	−4	0	1	0	0	0	0	0	0	
		$z_j - d_j$	−6	0	1	0	0	0	0	0	0	
Δ_j			26	0	13	0	0	0	0	0	0	Solution
D_B	C_B	B	f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8	f_9	1/2
1	1	f_2	−2	1	1	0	0	0	0	0	0	1
0	0	f_4	12	0	−5	1	0	0	0	0	0	18
0	0	f_5	4	0	−1	0	1	0	0	0	0	14
0	0	f_6	1	0	−2	0	0	1	0	0	0	2
0	0	f_7	9	0	−4	0	0	0	1	0	0	18
0	0	f_8	13	0	−4	0	0	0	0	1	0	42
0	0	f_9	−3	0	2	0	0	0	0	0	1	7

where $Z^1(X) = C_B(B^{-1}b) - \sum_{j \in N_1}(Z_j^1 - c_j)l_j - \sum_{j \in N_2}(Z_j^1 - c_j)u_j + \alpha$

$Z^2(X) = D_B^1[(B)^{-1}b] - \sum\limits_{j \in N_1}(Z_j^2 - d_j)l_j - \sum\limits_{j \in N_2}(Z_j^2 - d_j)u_j + \beta.$, Here $\alpha = 0$ and $\beta = 1$

C_B and D_B are the vectors having their components as the coefficients associated with the basic variables in the numerator and the denominator of the objective function.

The optimal objective function is given by $Z = \dfrac{Z^1(X)}{Z^2(X)}$.

Step 2. $E_1 = [E_{11} = (f_3, f_4, f_5, f_6, f_7, f_8, f_9)]$, $E_{12} = (f_1, f_4, f_5, f_6, f_7, f_8, f_9)$, $\frac{12}{25}$ is the value of objective function yielded by $(f_3, f_4, f_5, f_6, f_7, f_8, f_9)$ and $\dfrac{11}{23}$ is the value of objective function yielded by $(f_1, f_4, f_5, f_6, f_7, f_8, f_9)$,

$Z(X_2) = \dfrac{11}{23}$,

$B_2 = [B_{21} = (f_1, f_4, f_5, f_6, f_7, f_8, f_9)]$, $X_2 = [X_{21} = (\frac{11}{2}, 0, 0, 24, 16, \dfrac{3}{2}, \dfrac{45}{2}, \dfrac{97}{2}, \dfrac{11}{2})]$,

$J(X_{21}) = r_1, r_6, r_7, r_8, r_9$.

Here, $|J(X_{21})| = 5 > 4$.

$\therefore X_2 \bigcap S_1 = \phi$. Hence X_{21} is not an extreme point of $RX = t, \quad X \geq 0$.

Step 3. $E_2 = [E_{21} = (f_3, f_4, f_5, f_6, f_7, f_8, f_4)]$

$H_2 = [(E_1 \bigcup E_2) \backslash B_2] = [(f_3, f_4, f_5, f_6, f_7, f_8, f_9), (f_1, f_4, f_5, f_6, f_7, f_8, f_4), (f_3, f_4, f_5, f_6, f_7, f_8, f_9)]$

$\backslash (f_1, f_4, f_5, f_6, f_7, f_8, f_4) = (f_3, f_4, f_5, f_6, f_7, f_8, f_4)$

$\dfrac{12}{25}$ is the value of objective function yielded by $(f_3, f_4, f_5, f_6, f_7, f_8, f_9)$

$B_3 = [B_{31} = (f_3, f_4, f_5, f_6, f_7, f_8, f_9)]$

$J(X_{31}) = r_6, r_7, r_8, r_9$.

$|J(X_{31})| = 4 = p$

$\therefore X_{31}$ is an extreme point of $RX = t, \ X \geq 0$.

$\therefore$ optimal solution of $(P_{4.2})$ is $x_1 = 6, \quad x_2 = 0$ and optimal value is $\dfrac{12}{25}$.

Conclusion : In this chapter extreme point linear fractional programming is explained with bounded variables, i.e., some of the variables of the problem attains some bounds. The technique used to solve linear fractional programming with bounded variables is presented and based on that a numerical has been solved whose objective value is $\frac{12}{25}$.

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