SOME NEW INTUITIONISTIC FUZZY ENTROPY MEASURES AND ITS APPLICATIONS TO DECISION-MAKING PROCESS

A Thesis

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CANDIDATE'S DECLARATION

I hereby certify that the work which is being presented in the thesis entitled **"Some New Intuitionistic Fuzzy Entropy Measures and Its Applications To Decision-Making Process"** in partial fulfillment of the requirement for the award of degree of Master of Science, School of Mathematics (SOM), Thapar Institute of Engineering & Technology (TIET) (Deemed to be University), Patiala is an authentic record of my own carried out under the supervision of Dr. Harish Garg, Assistant Professor, SOM, TIET Patiala.

The matter presented in this thesis has not been submitted by me for the award of any other degree of this or any other institute.

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Abstract

Multi-attribute decision-making (MADM) problems are an important part of decision theory in which we choose the best one from the set of finite alternatives based on the collective information. Traditionally, it has been assumed that the information regarding accessing the alternatives is taken in the form of real numbers. However, uncertainty and fuzziness are big issues in real-world problems nowadays and can be found everywhere as in our discussion or the way we process information. To deal with such a situation, the theory of fuzzy sets or extended fuzzy sets such as an intuitionistic fuzzy set (IFS) or interval-valued IFS (IVIFS) are the most successful ones, which characterize the attribute values in terms of membership degrees. Among these methodologies, information measures play a significant role in processing the imperfect and uncertain information. From the existing literature, it can be worth noticed that similarity, distance, entropy and inclusion measures are important tools for measuring the uncertainty associated with FS and IFS. Out of these various measures, Entropy measure is basically known as the measure of information of a revolutionary discovery named as 'Information Theory' in communication system, originated from the fundamental paper "*The mathematical Theory of Communication*" in 1948 by Claude E. Shannon.

The objective of this work is to addresses some new entropy measures to quantify the degree of fuzziness of a set in the IFS environment. In the present thesis, different kinds of the entropy measures are addressed which makes the decision more flexible and reliable corresponding to different values of the parameters. Further, based on the proposed measures, different kinds of decision-making approaches are presented in details. In the presented approaches, the characteristics of the attribute weights are taken as either partially known or completely unknown.

The present thesis is organized into five chapters which are briefly summarized as follows:

A brief account of the related work of various authors in the evaluation of MADM problems under IFS environment is presented in the first chapter. In **Chapter 2**, the basic and preliminaries related to the IFSs and the entropy measures are given.

Chapter 3 presents a new generalized parametric entropy measures under the IFS environment. The present measures is based on the three parameters, α, β and γ and it is shown that some of the existing measures are taken as a special case of it. Further, the behavior of these parameters are investigated in details. Further, we develop two approaches to deal with MADM problem with the illustration on a real-life decision making. A sensitivity analysis has also been done by taking different values of the parameters.

In **Chapter 4**, a new entropy measure based on R-norm under the intuitionistic fuzzy set environment. Further, based on the proposed measures, two MADM approaches have presented to solve the decision-making problems based on either the information of the attribute weights is completely known or partially known. Further, the applicability of the proposed measures are explained through an example.

In **Chapter 5**, we present a novel (R, S)-norm based information measure called the entropy to measure the degree of fuzziness of the IFSs. The validity of the proposed measure is tested on the linguistic variable to demonstrate it. Then, we utilized it to propose two decision-making approaches to solve the MADM problem by considering the attribute weights as either partially known or completely unknown. Finally, a practical example is provided to illustrate the decision-making process.

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Chapter 1

Introduction

Multi-attribute decision-making (MADM) problems are an important part of decision theory in which there main task is to give the ranking of alternatives based on multiple attributes. Traditionally, to solve the MADM problems decision makers give preference to alternatives based on different attributes in terms of crisp numbers. But due to lack of data for finding the outcome of decisions, crisp values are not appropriate. In order to overcome this the theory of fuzzy sets (FSs) [49] or extended fuzzy sets such as an intuitionistic fuzzy set (IFS) [4] or interval-valued IFS (IVIFS) [3] are the most successful ones, which characterize the attribute values in terms of membership degrees. Numerous attempt has been made by the researchers to solve the MADM problem by using different approaches such as aggregation operator [1, 2, 14, 15, 45, 47, 48], information measures [12, 18, 23, 27, 29, 30, 35, 38, 41, 42, 44, 51], possibility degree measures [22]. Wang and Wang [43] characterized the preference of the decision-makers in terms of interval-numbers, and then, an MADM was presented corresponding to it with completely unknown weight vectors. Jamkhaneh and Garg [26] presented some new operations for the generalized IFSs and applied them to solve decision-making problems.

Entropy measure is basically known as the measure of information of a revolutionary discovery named as 'Information Theory' in communication system, originated from the fundamental paper "*The mathematical Theory of Communication*" in 1948 by Claude E. Shannon [37]. This paper provides a mathematical set up for quantitatively defining the concepts of information. Entropy measure is remarkable contribution by Shanon in various fields of "Science and Technology". That's why Shanon is tagged as a father of information theory. Shanon's entropy is an expected (average) value of the information provided. However, classical information measures deal with information that is precise in nature. In order to overcome this, Deluca and Termini [11] proposed a set of axioms for fuzzy entropy. Later on, Szmidt and Kacprzyk [39] extended the axioms of Deluca and Termini [11] to the IFS environment. Vlachos and Sergiadis [42] extended their measure to the IFS environment. Zhang and Jiang [51] presented a measure of intuitionistic fuzzy entropy based on a generalization of measure of Deluca and Termini [11]. Renyi [33] proposed a parametric entropy measure by including a parameter α. Verma and Sharma [41] proposed an exponential order entropy in the IFS environment. Garg et al. [17] presented a generalized intuitionistic fuzzy entropy measure of order α and degree β to solve decision-making problems. Wei et al. [44] presented an entropy measure based on the trigonometric functions. Garg et al. [16] presented an entropy-based method for solving decision-making problems. Joshi and Kumar [27] presented an (R, S)-norm fuzzy information measures to solve decision-making problems. Garg and Kumar [20, 21] presented some similarity and distance measures of IFSs by using the set pair analysis theory. Selvachandran, Garg and Quek [36] presented entropy measures for the complex vague soft set. Meanwhile, decision-making methods based on some measures (such as distance, similarity degree, correlation coefficient and entropy) were proposed to deal with fuzzy IF and interval-valued IF MADM problems [8, 12, 18, 22, 28, 32].

In the above work, emphasis was given by the researchers to the attribute weights during ranking of the alternatives. It is quite obvious that the final ranking order of the alternatives highly depends on the attribute weights, because the variation of weight values may result in a different final ranking order of alternatives [9, 14, 20, 27, 32]. Now, based on the characteristics of the attribute weights, the decision-making problem can be classified into three types: (a) the decision-making situation where the attribute weights are completely known; (b) the decision-making situation where the attribute weights are completely unknown; (c) the decision-making situation where the attribute weights are partially known. Thus, based on these types, the attribute weights in MADM can be classified as subjective and objective attribute weights based on the information acquisition approach. If the decision-maker gives weights to the attributes, then such information is called subjective. The classical approaches to determine the subjective attribute weights are the analytic hierarchy process (AHP) method [34] and the Delphi method [25]. On the other hand, the objective attribute weights are determined by the decision-making matrix, and one of the most important approaches is the Shannon entropy method [37], which expresses the relative intensities of the attributes' importance to signify the average intrinsic information transmitted to the decision-maker. In the literature, several authors [2, 23, 32] have addressed the MADM problem with subjective weight information. However, some researchers formulated a nonlinear programming model to determine the attribute weights. For instance, Chen and Li [9] presented an approach to assess the attribute weights by utilizing IF entropy in the IFS environment. Garg [13] presented a generalized intuitionistic fuzzy entropy measure to determine the completely unknown attribute weight to solve the decision-making problems. Although some researchers put some efforts into determining the unknown attribute weights [19, 31, 46] under different environments, still it remains an open problem.

1.1 Objective of the Thesis

By keeping the advantages and motivated by the characteristics of the IFSs to describe the uncertainties in the data, this thesis addresses some new entropy measures to quantify the degree of fuzziness of a set in the IFS environment. In the present thesis, different kinds of the entropy measures are addressed which makes the decision more flexible and reliable corresponding to different values of the parameters. Further, based on the proposed measures, different kinds of decision-making approaches are presented in details. In the presented approaches, the characteristics of the attribute weights are taken as either partially known or completely unknown.

1.2 Structure of the Thesis

The present thesis is organized into five chapters including the present one that contains mainly the literature review. The rest of the chapters are described below:

In **Chapter 2**, the basic and preliminaries related to the IFSs and the entropy measures are given.

Chapter 3 presents a new generalized parametric entropy measures under the IFS environment. The present measures is based on the three parameters, α, β and γ and it is shown that some of the existing measures are taken as a special case of it. Further, the behavior of these parameters are investigated in details. Further, we develop two approaches to deal with MADM problem with the illustration on a real-life decision making. A sensitivity analysis has also been done by taking different values of the parameters.

In **Chapter 4**, a new entropy measure based on R-norm under the intuitionistic fuzzy set environment. Further, based on the proposed measures, two MADM approaches have presented to solve the decision-making problems based on either the information of the attribute weights is completely known or partially known. Further, the applicability of the proposed measures are explained through an example.

In **Chapter 5**, we present a novel (R, S)-norm based information measure called the entropy to measure the degree of fuzziness of the IFSs. The validity of the proposed measure is tested on the linguistic variable to demonstrate it. Then, we utilized it to propose two decision-making approaches to solve the MADM problem by considering the attribute weights as either partially known or completely unknown. Finally, a practical example is provided to illustrate the decision-making process.

Chapter 2

Preliminaries

This chapter presents some of the fundamental definitions and mathematical theory for Intuitionistic fuzzy set theory. The focus is on defining the IFSs and the entropy measures.

2.1 Basic definition

Here, we present some basic definition of IFSs as below.

Definition 2.1.1. *[4] An IFS A defined in X is an ordered pair given by:*

$$A = \{\langle x, \zeta_A(x), \vartheta_A(x)\rangle \mid x \in X\} \tag{2.1}$$

where $\zeta_A, \vartheta_A : X \longrightarrow [0,1]$ represent, respectively, the membership and non-membership degrees of the element x such that $\zeta_A, \vartheta_A \in [0,1]$ and $\zeta_A + \vartheta_A \leq 1$ for all x. For convenience, this pair is denoted by $A = \langle \zeta_A, \vartheta_A \rangle$ and called an intuitionistic fuzzy number (IFN) [47, 48].

Definition 2.1.2. *[47, 48] Let the family of all intuitionistic fuzzy sets of universal set X be denoted by IFS(X). Let A, B ∈ IFS(X) be such that then some operations can be defined as follows:*

(i) $A \subseteq B$ *if* $\zeta_A(x) \leq \zeta_B(x)$ *and* $\vartheta_A(x) \geq \vartheta_B(x)$, *for all* $x \in X$;

(ii) $A \supseteq B$ *if* $\zeta_A(x) \geq \zeta_B(x)$ *and* $\vartheta_A(x) \leq \vartheta_B(x)$, *for all* $x \in X$;

(iii) $A = B$ *iff* $\zeta_A(x) = \zeta_B(x)$ *and* $\vartheta_A(x) = \vartheta_B(x)$, *for all* $x \in X$;

(iv) $A \cup B = \{\langle x, \max(\zeta_A(x), \zeta_B(x)), \min(\vartheta_A(x), \vartheta_B(x))\rangle : x \in X\}$;

(v) $A \cap B = \{\langle x, \min(\zeta_A(x), \zeta_B(x)), \max(\vartheta_A(x), \vartheta_B(x))\rangle : x \in X\}$;

(vi) $A^c = \{\langle x, \vartheta_A(x), \zeta_A(x)\rangle : x \in X\}$.

2.2 Entropy

Definition 2.2.1. ***Entropy measure:*** *Let* $\Delta_n = \{P_j = (x_i, p_i) : p_i \geq 0, \sum_{i=1}^{n} p_i = 1\}$ *such that* $i, j = 1, 2, \ldots, n$ *represents a set of* n *probability distributions, where* x_i *denotes the random variables and* p_i *denotes their respective probabilities, then the entropy for any probability distribution is defined as*

$$H(P) = -\sum_{i=1}^{n} p_i \log p_i \tag{2.2}$$

Here, log *is to the base* 2 *and* $n \geq 2$*.*

In 1961, Renyi [33] proposed a parametric entropy measure by including a parameter α and defined as

$$H_\alpha(P) = \frac{1}{1-\alpha} \log \left(\sum_{i=1}^{n} p_i^\alpha \right), \alpha \neq 1, \alpha > 0 \tag{2.3}$$

In 1975, Sharma and Mittal [38] presented an entropy measure of order$-\alpha$ with type$-\beta$ and is given as

$$H_\alpha^\beta(P) = \frac{1}{2^{(1-\beta)} - 1} \left\{ \left(\sum_{i=1}^{n} p_i^\alpha \right)^{\frac{\beta-1}{\alpha-1}} - 1 \right\}, \alpha, \beta \neq 1, \alpha, \beta > 0 \tag{2.4}$$

Definition 2.2.2. *[39] An entropy E: IFS*$(X) \longrightarrow R^+$ *on IFS(X) is a real-valued functional satisfying the following four axioms for* $A, B \in$ *IFS*(X)

(P1) $E(A) = 0$ *if and only if* A *is a crisp set, i.e., either* $\zeta_A(x) = 1, \vartheta_A(x) = 0$ *or* $\zeta_A(x) = 0, \vartheta_A(x) = 1$ *for all* $x \in X$*.*

(P2) $E(A) = 1$ *if and only if* $\zeta_A(x) = \vartheta_A(x)$ *for all* $x \in X$*.*

(P3) $E(A) = E(A^c)$*.*

(P4) If $A \subseteq B$*, that is, if* $\zeta_A(x) \leq \zeta_B(x)$ *and* $\vartheta_A(x) \geq \vartheta_B(x)$ *for any* $x \in X$*, then* $E(A) \leq E(B)$*.*

In the below, we briefly review some of the existing entropy measures under the IFS environment.

Vlachos and Sergiadis [42] proposed the measure of intuitionistic fuzzy entropy in the IFS environment as follows:

$$E(A) = -\frac{1}{n \ln 2} \sum_{i=1}^{n} \left[\begin{array}{l} \zeta_A(x_i) \ln \zeta_A(x_i) + \vartheta_A(x_i) \ln \vartheta_A(x_i) \\ \quad - (1 - \pi_A(x_i)) \ln(1 - \pi_A(x_i)) - \pi_A(x_i) \ln 2 \end{array} \right] \tag{2.5}$$

Zhang and Jiang [51] presented a measure of intuitionistic fuzzy entropy based on a generalization of measure of Deluca and Termini [11] as:

$$E(A) = -\frac{1}{n}\sum_{i=1}^{n}\left[\begin{array}{c}\left(\frac{\zeta_A(x_i)+1-\vartheta_A(x_i)}{2}\right)\log\left(\frac{\zeta_A(x_i)+1-\vartheta_A(x_i)}{2}\right)+ \\ \left(\frac{\vartheta_A(x_i)+1-\zeta_A(x_i)}{2}\right)\log\left(\frac{\vartheta_A(x_i)+1-\zeta_A(x_i)}{2}\right)\end{array}\right] \tag{2.6}$$

Verma and Sharma [41] proposed an exponential order entropy in the IFS environment as:

$$E(A) = \frac{1}{n(\sqrt{e}-1)}\sum_{i=1}^{n}\left[\begin{array}{l}\frac{\zeta_A(x_i)+1-\vartheta_A(x_i)}{2}e^{1-\frac{\zeta_A(x_i)+1-\vartheta_A(x_i)}{2}} \\ +\frac{\vartheta_A(x_i)+1-\zeta_A(x_i)}{2}e^{1-\frac{\vartheta_A(x_i)+1-\zeta_A(x_i)}{2}} - 1\end{array}\right] \tag{2.7}$$

Garg et al. [17] generalized entropy measure $E_\alpha^\beta(A)$ of order α and degree β as:

$$E_\alpha^\beta(A) = \frac{2-\beta}{n(2-\beta-\alpha)}\sum_{i=1}^{n}\log\left[\begin{array}{l}\left(\zeta_A^{\frac{\alpha}{2-\beta}}(x_i)+\vartheta_A^{\frac{\alpha}{2-\beta}}(x_i)\right)(\zeta_A(x_i)+\vartheta_A(x_i))^{1-\frac{\alpha}{2-\beta}} \\ +2^{1-\frac{\alpha}{2-\beta}}(1-\zeta_A(x_i)-\vartheta_A(x_i))\end{array}\right] \tag{2.8}$$

where log is to the base two, $\alpha > 0$, $\beta \in [0,1]$, $\alpha+\beta \neq 2$.

Chapter 3

Generalized Parametric Entropy Measure under Intuitionistic Fuzzy Set Environment and Its Applications

In this chapter, we presented a new generalized parametric entropy measures under the IFS environment. The present measures is based on the three parameters, α, β and γ and it is shown that some of the existing measures are taken as a special case of it. Further, the behavior of these parameters are investigated in details. Further, we develop two approaches to deal with MADM problem with the illustration on a real-life decision making. A sensitivity analysis has also been done by taking different values of the parameters.

3.1 Generalized Parametric Entropy measures

In this section, we have proposed new generalized information measures for a intuitionistic fuzzy distribution

$$H^{\alpha}_{\beta,\gamma} = \frac{1}{1-\alpha}\sum_{i=1}^{n}\log\left[\frac{\zeta^{\alpha+\beta+\gamma-1}(x_i)+\vartheta^{\alpha+\beta+\gamma-1}(x_i)+\pi^{\alpha+\beta+\gamma-1}(x_i)}{\zeta^{\beta+\gamma}(x_i)+\vartheta^{\beta+\gamma}(x_i)+\pi^{\beta+\gamma}(x_i)}\right] \tag{3.1}$$

where $\alpha \neq 1$, $\beta \geq 0$, $\gamma \geq 0$, $\alpha+\beta+\gamma > 0$, $\beta+\gamma-1 \geq 0$, $\pi(x_i) = 1-\zeta(x_i)-\vartheta(x_i)$.

If $\alpha \to 1$, the measure defined in Eq.(3.1) becomes

$$H_{\beta,\gamma}(A) = -\sum_{i=1}^{n}\frac{\zeta_A^{\beta+\gamma}(x_i)\log\zeta_A(x_i)+\vartheta_A^{\beta+\gamma}(x_i)\log\vartheta_A(x_i)+\pi_A^{\beta+\gamma}(x_i)\log\pi_A(x_i)}{\zeta_A^{\beta+\gamma}(x_i)+\vartheta_A^{\beta+\gamma}(x_i)+\pi_A^{\beta+\gamma}(x_i)} \tag{3.2}$$

if $\beta = 0$ and $\gamma = 1$, then the Eq.(3.2) reduces to

$$H(A) = -\sum_{i=1}^{n} [\zeta_A(x_i) \log \zeta_A(x_i) + \vartheta_A(x_i) \log \vartheta_A(x_i) + \pi_A(x_i) \log \pi_A(x_i)] \tag{3.3}$$

Theorem 3.1.1. *The R-norm intuitionistic fuzzy information measure* $H_R^{\alpha,\beta}(A)$ *defined in Eq.*(3.1) *for IFS is a valid measure i.e., it satisfies the following properties:*

(P1) $H^{\alpha}_{\beta,\gamma}(A) = 0$ *if and only if* A *is a crisp set, i.e.* $\zeta_A(x_i) = 1, \vartheta_A(x_i) = 0$ *or* $\zeta_A(x_i) = 0, \vartheta_A(x_i) = 1$ *for all* $x_i \in X$.

(P2) $H(A) = 1$ *if and only if* $\zeta_A(x_i) = \vartheta_A(x_i) = \pi_A(x_i) = \frac{1}{3}$ $\forall x_i \in X$.

(P3) $H^{\alpha}_{\beta,\gamma}(A) \leq H^{\alpha}_{\beta,\gamma}(B)$ *if* A *is crisper than* B, *i.e., for* $\max\{\zeta_B(x_i), \vartheta_B(x_i)\} \leq \frac{1}{3}$ *if* $\zeta_A(x_i) \leq \zeta_B(x_i)$ *&* $\vartheta_A(x_i) \leq \vartheta_B(x_i)$, *and for* $\min\{\zeta_B(x_i), \vartheta_B(x_i)\} \leq \frac{1}{3}$ *if* $\zeta_A(x_i) \geq \zeta_B(x_i)$ *&* $\vartheta_A(x_i) \geq \vartheta_B(x_i)$, *for all* $x_i \in X$.

(P4) $H^{\alpha}_{\beta,\gamma}(A) = H^{\alpha}_{\beta,\gamma}(A^c)$ *for all* $A \in$ *IFS*(X).

Proof. For IFS A, we have

(P1) Sharpness: If $H^{\alpha}_{\beta,\gamma}(A) = 0$, then

$$\log\left[\frac{\zeta^{\alpha+\beta+\gamma-1}(x_i) + \vartheta^{\alpha+\beta+\gamma-1}(x_i) + \pi^{\alpha+\beta+\gamma-1}(x_i)}{\zeta^{\beta+\gamma}(x_i) + \vartheta^{\beta+\gamma}(x_i) + \pi^{\beta+\gamma}(x_i)}\right] = 0$$

Since, $\alpha \neq 1, \beta \geq 0, \gamma \geq 0, \alpha + \beta + \gamma > 0, \beta + \gamma - 1 \geq 0$, then this is possible only in the following cases:

(i) Either $\zeta_A(x_i) = 1$ i.e., $\vartheta_A(x_i) = \pi_A(x_i) = 0$ or

(ii) $\vartheta_A(x_i) = 1$ i.e., $\zeta_A(x_i) = \pi_A(x_i) = 0$ or

(iii) $\pi_A(x_i) = 1$ i.e., $\zeta_A(x_i) = \vartheta_A(x_i) = 0$

In all the above cases, $H^{\alpha}_{\beta,\gamma}(A) = 0$ implies that A is a crisp set. Conversely, if A is a crisp set then either $\zeta_A(x_i) = 1$ and $\vartheta_A(x_i) = \pi_A(x_i) = 0$ or $\vartheta_A(x_i) = 1$ and $\zeta_A(x_i) = \pi_A(x_i) = 0$ or $\pi_A(x_i) = 1$ and $\zeta_A(x_i) = \vartheta_A(x_i) = 0$. This implies that

$$\log\left[\frac{\zeta^{\alpha+\beta+\gamma-1}(x_i) + \vartheta^{\alpha+\beta+\gamma-1}(x_i) + \pi^{\alpha+\beta+\gamma-1}(x_i)}{\zeta^{\beta+\gamma}(x_i) + \vartheta^{\beta+\gamma}(x_i) + \pi^{\beta+\gamma}(x_i)}\right] = 0$$

for $\alpha \neq 1, \beta \geq 0, \gamma \geq 0, \alpha+\beta+\gamma > 0, \beta+\gamma-1 \geq 0$, which gives $H^{\alpha}_{\beta,\gamma}(A) = 0$. Hence $H^{\alpha}_{\beta,\gamma}(A) = 0$ iff A is a crisp set.

(P2) Maximality:

Since $\zeta_A(x_i)+\vartheta_A(x_i)+\pi_A(x_i) = 1$, then to obtain the maximum value of intuitionistic fuzzy entropy $H^{\alpha}_{\beta,\gamma}(A)$, we write the $\phi(\zeta_A, \vartheta_A, \pi_A) = \zeta_A(x_i) + \vartheta_A(x_i) + \pi_A(x_i) - 1$ and taking the Lagrange's multiplier λ, we consider

$$\Phi(\zeta_A, \vartheta_A, \pi_A) = H^{\alpha}_{\beta,\gamma}(\zeta_A, \vartheta_A, \pi_A) + \lambda\phi(\zeta_A, \vartheta_A, \pi_A). \tag{3.4}$$

To find the maximum value of $H^{\alpha}_{\beta,\gamma}(A)$, we differentiate Eq.(3.4) partially with respect to $\zeta_A, \vartheta_A, \pi_A$ and λ and equating them to zero, we get $\zeta_A(x_i) = \vartheta_A(x_i) = \pi_A(x_i) = \frac{1}{3}$. All the first order partial derivatives vanish iff $\zeta_A(x_i) = \vartheta_A(x_i) = \pi_A(x_i) = \frac{1}{3}$. Therefore, the stationary point of $H^{\alpha}_{\beta,\gamma}(A)$ is $\zeta_A(x_i) = \vartheta_A(x_i) = \pi_A(x_i) = \frac{1}{3}$. Next to prove $H^{\alpha}_{\beta,\gamma}(A)$ is a concave function of $A \in \text{IFS}(X)$, we calculate its Hessian at the stationary point. The Hessian of $H^{\alpha}_{\beta,\gamma}(A)$ is given by

$$\hat{H} = \begin{bmatrix} -1.8 \times 10^{-9} & -9.12 \times 10^{-10} & -9.12 \times 10^{-10} \\ -9.12 \times 10^{-10} & -1.8 \times 10^{-9} & -9.12 \times 10^{-10} \\ -9.12 \times 10^{-10} & -9.12 \times 10^{-10} & -1.8 \times 10^{-9} \end{bmatrix}$$

Thus, $\bar{H}$ is a negative semi definite matrix and hence $H^{\alpha}_{\beta,\gamma}(A)$ is a concave function having its maximum value at the point $\zeta_A(x_i) = \vartheta_A(x_i) = \pi_A(x_i) = \frac{1}{3}$.

(P3) Resolution: Since $H^{\alpha}_{\beta,\gamma}(A)$ is a concave function of $A \in \text{IFS}(X)$, then if $\max\{\zeta_A(x_i), \vartheta_A(x_i)\} \leq \frac{1}{3}$, then $\zeta_A(x_i) \leq \zeta_B(x_i)$ and $\vartheta_A(x_i) \leq \vartheta_B(x_i)$ which implies $\pi_A(x_i) \geq \pi_B(x_i) \geq \frac{1}{3}$.
Similarly,if $\min\{\zeta_A(x_i), \vartheta_A(x_i)\} \geq \frac{1}{3}$,then $\zeta_A(x_i) \leq \zeta_B(x_i)$ and $\vartheta_A(x_i) \leq \vartheta_B(x_i)$.

(P4) Symmetry: From the definition, $H^{\alpha}_{\beta,\gamma}(A) = H^{\alpha}_{\beta,\gamma}(A^c)$

Hence, $H^{\alpha}_{\beta,\gamma}(A)$ satisfies all the properties of intuitionistic fuzzy entropy measure and therefore, $H^{\alpha}_{\beta,\gamma}(A)$ is a valid information measure. □

We can write the measure defined in Eq.(3.1) as

$$H^{\alpha}_{\beta,\gamma}(A) = \lambda \sum_{i=1}^{n} \log \left[\frac{\zeta^{\rho}(x_i) + \vartheta^{\rho}(x_i) + \pi^{\rho}(x_i)}{\zeta^{\sigma}(x_i) + \vartheta^{\sigma}(x_i) + \pi^{\sigma}(x_i)} \right] \tag{3.5}$$

where $\lambda = \frac{1}{1-\alpha}$, $\rho = \alpha + \beta + \gamma - 1$ and $\sigma = \beta + \gamma$.

Definition 3.1.1. *Consider two IFSs Q and R defined over $X = \{x_1, x_2, \ldots, x_n\}$. Take the disjoint partition of X as*

$$\begin{aligned} X_1 &= \{x_i \in X : Q \subseteq R\}, \\ &= \{x_i \in X | \zeta_Q(x_i) \leq \zeta_R(x_i); \vartheta_Q(x_i) \geq \vartheta_R(x_i)\} \end{aligned}$$

and

$$\begin{aligned} X_2 &= \{x_i \in X : Q \supseteq R\}. \\ &= \{x_i \in X | \zeta_Q(x_i) \geq \zeta_R(x_i); \vartheta_Q(x_i) \leq \vartheta_R(x_i)\} \end{aligned}$$

Next, we define the joint and conditional entropies between Q and R as follows:

(a) ***Joint entropy:***

$$
\begin{aligned}
& H^{\alpha}_{\beta,\gamma}(Q\cup R) \\
= \ & \lambda\sum_{i=1}^{n}\log\left[\frac{\zeta^{\rho}_{Q\cup R}(x_i)+\vartheta^{\rho}_{Q\cup R}(x_i)+\pi^{\rho}_{Q\cup R}(x_i)}{\zeta^{\sigma}_{Q\cup R}(x_i)+\vartheta^{\sigma}_{Q\cup R}(x_i)+\pi^{\sigma}_{Q\cup R}(x_i)}\right] \\
= \ & \lambda\sum_{x_i\in X_1}\log\left[\frac{\zeta^{\rho}_{R}(x_i)+\vartheta^{\rho}_{R}(x_i)+\pi^{\rho}_{R}(x_i)}{\zeta^{\sigma}_{R}(x_i)+\vartheta^{\sigma}_{R}(x_i)+\pi^{\sigma}_{R}(x_i)}\right]+\lambda\sum_{x_i\in X_2}\log\left[\frac{\zeta^{\rho}_{Q}(x_i)+\vartheta^{\rho}_{Q}(x_i)+\pi^{\rho}_{Q}(x_i)}{\zeta^{\sigma}_{Q}(x_i)+\vartheta^{\sigma}_{Q}(x_i)+\pi^{\sigma}_{Q}(x_i)}\right]
\end{aligned}
$$

(b) ***Conditional entropy:***

$$
\begin{aligned}
& H^{\alpha}_{\beta,\gamma}(Q|R) \\
= \ & \lambda\sum_{x_i\in X_2}\left\{\log\left[\frac{\zeta^{\rho}_{Q}(x_i)+\vartheta^{\rho}_{Q}(x_i)+\pi^{\rho}_{Q}(x_i)}{\zeta^{\sigma}_{Q}(x_i)+\vartheta^{\sigma}_{Q}(x_i)+\pi^{\sigma}_{Q}(x_i)}\right]-\log\left[\frac{\zeta^{\rho}_{R}(x_i)+\vartheta^{\rho}_{R}(x_i)+\pi^{\rho}_{R}(x_i)}{\zeta^{\sigma}_{R}(x_i)+\vartheta^{\sigma}_{R}(x_i)+\pi^{\sigma}_{R}(x_i)}\right]\right\}
\end{aligned}
$$

and

$$
\begin{aligned}
& H^{\alpha}_{\beta,\gamma}(R|Q) \\
= \ & \lambda\sum_{x_i\in X_1}\left\{\log\left[\frac{\zeta^{\rho}_{R}(x_i)+\vartheta^{\rho}_{R}(x_i)+\pi^{\rho}_{R}(x_i)}{\zeta^{\sigma}_{R}(x_i)+\vartheta^{\sigma}_{R}(x_i)+\pi^{\sigma}_{R}(x_i)}\right]-\log\left[\frac{\zeta^{\rho}_{Q}(x_i)+\vartheta^{\rho}_{Q}(x_i)+\pi^{\rho}_{Q}(x_i)}{\zeta^{\sigma}_{Q}(x_i)+\vartheta^{\sigma}_{Q}(x_i)+\pi^{\sigma}_{Q}(x_i)}\right]\right\}
\end{aligned}
$$

Theorem 3.1.2. *Let Q and R be two IFSs defined on universal set $X=\{x_1,x_2,\ldots,x_n\}$, where, $Q=\{\langle x_i,\zeta_Q(x_i),\vartheta_Q(x_i)\rangle|x_i\in X\}$ and $R=\{\langle x_i,\zeta_R(x_i),\vartheta_R(x_i)\rangle|x_i\in X\}$, such that either $Q\subseteq R$ or $Q\supseteq R$ $\forall x_i\in X$, then:*

$$H^{\alpha}_{\beta,\gamma}(Q\cup R)+H^{\alpha}_{\beta,\gamma}(Q\cap R)=H^{\alpha}_{\beta,\gamma}(Q)+H^{\alpha}_{\beta,\gamma}(R)$$

Proof. Let X_1 and X_2 be the two disjoint sets of X, where,

$$X_1=\{x_i\in X: Q\subseteq R\}, X_2=\{x_i\in X: Q\supseteq R\}$$

i.e., for $x_i\in X_1$, we have $\zeta_Q(x_i)\le\zeta_R(x_i),\vartheta_Q(x_i)\ge\vartheta_R(x_i)$ and $x_i\in X_2$, implies that $\zeta_Q(x_i)\ge\zeta_R(x_i),\vartheta_Q(x_i)\le\vartheta_R(x_i)$. Therefore,

$$
\begin{aligned}
& H^{\alpha}_{\beta,\gamma}(Q\cup R)+H^{\alpha}_{\beta,\gamma}(Q\cap R) \\
= \ & \lambda\sum_{x_i\in X_1}\log\left[\frac{\zeta^{\rho}_{R}(x_i)+\vartheta^{\rho}_{R}(x_i)+\pi^{\rho}_{R}(x_i)}{\zeta^{\sigma}_{R}(x_i)+\vartheta^{\sigma}_{R}(x_i)+\pi^{\sigma}_{R}(x_i)}\right]+\lambda\sum_{x_i\in X_2}\log\left[\frac{\zeta^{\rho}_{Q}(x_i)+\vartheta^{\rho}_{Q}(x_i)+\pi^{\rho}_{Q}(x_i)}{\zeta^{\sigma}_{Q}(x_i)+\vartheta^{\sigma}_{Q}(x_i)+\pi^{\sigma}_{Q}(x_i)}\right] \\
+ \ & \lambda\sum_{x_i\in X_1}\log\left[\frac{\zeta^{\rho}_{Q}(x_i)+\vartheta^{\rho}_{Q}(x_i)+\pi^{\rho}_{Q}(x_i)}{\zeta^{\sigma}_{Q}(x_i)+\vartheta^{\sigma}_{Q}(x_i)+\pi^{\sigma}_{Q}(x_i)}\right]+\lambda\sum_{x_i\in X_2}\log\left[\frac{\zeta^{\rho}_{R}(x_i)+\vartheta^{\rho}_{R}(x_i)+\pi^{\rho}_{R}(x_i)}{\zeta^{\sigma}_{R}(x_i)+\vartheta^{\sigma}_{R}(x_i)+\pi^{\sigma}_{R}(x_i)}\right] \\
= \ & \lambda\sum_{i=1}^{n}\left\{\log\left[\frac{\zeta^{\rho}_{Q}(x_i)+\vartheta^{\rho}_{Q}(x_i)+\pi^{\rho}_{Q}(x_i)}{\zeta^{\sigma}_{Q}(x_i)+\vartheta^{\sigma}_{Q}(x_i)+\pi^{\sigma}_{Q}(x_i)}\right]+\log\left[\frac{\zeta^{\rho}_{R}(x_i)+\vartheta^{\rho}_{R}(x_i)+\pi^{\rho}_{R}(x_i)}{\zeta^{\sigma}_{R}(x_i)+\vartheta^{\sigma}_{R}(x_i)+\pi^{\sigma}_{R}(x_i)}\right]\right\} \\
= \ & H^{\alpha}_{\beta,\gamma}(Q)+H^{\alpha}_{\beta,\gamma}(R)
\end{aligned}
$$

□

Theorem 3.1.3. *Let Q and R be two IFSs defined over the set X such that either $Q \subseteq R$ or $Q \subseteq R$ then the following statements hold:*

(i) $H^{\alpha}_{\beta,\gamma}(Q \cup R) = H^{\alpha}_{\beta,\gamma}(Q) + H^{\alpha}_{\beta,\gamma}(R|Q)$;

(ii) $H^{\alpha}_{\beta,\gamma}(Q \cup R) = H^{\alpha}_{\beta,\gamma}(R) + H^{\alpha}_{\beta,\gamma}(Q|R)$;

(iii) $H^{\alpha}_{\beta,\gamma}(Q \cup R) = H^{\alpha}_{\beta,\gamma}(Q) + H^{\alpha}_{\beta,\gamma}(R|Q) = H^{\alpha}_{\beta,\gamma}(R) + H^{\alpha}_{\beta,\gamma}(Q|R)$.

Proof. For two IFSs Q and R, we have

(i) Consider

$$
\begin{aligned}
& H^{\alpha}_{\beta,\gamma}(Q \cup R) - H^{\alpha}_{\beta,\gamma}(Q) - H^{\alpha}_{\beta,\gamma}(R|Q) \\
= \; & \lambda \sum_{x_i \in X_1} \log\left[\frac{\zeta^{\rho}_R(x_i)+\vartheta^{\rho}_R(x_i)+\pi^{\rho}_R(x_i)}{\zeta^{\sigma}_R(x_i)+\vartheta^{\sigma}_R(x_i)+\pi^{\sigma}_R(x_i)}\right] + \lambda \sum_{x_i \in X_2} \log\left[\frac{\zeta^{\rho}_Q(x_i)+\vartheta^{\rho}_Q(x_i)+\pi^{\rho}_Q(x_i)}{\zeta^{\sigma}_Q(x_i)+\vartheta^{\sigma}_Q(x_i)+\pi^{\sigma}_Q(x_i)}\right] \\
- \; & \lambda \left\{ \sum_{x_i \in X_1} \log\left[\frac{\zeta^{\rho}_Q(x_i)+\vartheta^{\rho}_Q(x_i)+\pi^{\rho}_Q(x_i)}{\zeta^{\sigma}_Q(x_i)+\vartheta^{\sigma}_Q(x_i)+\pi^{\sigma}_Q(x_i)}\right] + \sum_{x_i \in X_2} \log\left[\frac{\zeta^{\rho}_Q(x_i)+\vartheta^{\rho}_Q(x_i)+\pi^{\rho}_Q(x_i)}{\zeta^{\sigma}_Q(x_i)+\vartheta^{\sigma}_Q(x_i)+\pi^{\sigma}_Q(x_i)}\right] \right\} \\
- \; & \lambda \sum_{x_i \in X_1} \left\{ \log\left[\frac{\zeta^{\rho}_R(x_i)+\vartheta^{\rho}_R(x_i)+\pi^{\rho}_R(x_i)}{\zeta^{\sigma}_R(x_i)+\vartheta^{\sigma}_R(x_i)+\pi^{\sigma}_R(x_i)}\right] - \log\left[\frac{\zeta^{\rho}_Q(x_i)+\vartheta^{\rho}_Q(x_i)+\pi^{\rho}_Q(x_i)}{\zeta^{\sigma}_Q(x_i)+\vartheta^{\sigma}_Q(x_i)+\pi^{\sigma}_Q(x_i)}\right] \right\} \\
= 0 &
\end{aligned}
$$

(ii) Consider

$$
\begin{aligned}
& H^{\alpha}_{\beta,\gamma}(Q \cup R) - H^{\alpha}_{\beta,\gamma}(R) - H^{\alpha}_{\beta,\gamma}(Q|R) \\
= \; & \lambda \sum_{x_i \in X_1} \log\left[\frac{\zeta^{\rho}_R(x_i)+\vartheta^{\rho}_R(x_i)+\pi^{\rho}_R(x_i)}{\zeta^{\sigma}_R(x_i)+\vartheta^{\sigma}_R(x_i)+\pi^{\sigma}_R(x_i)}\right] + \lambda \sum_{x_i \in X_2} \log\left[\frac{\zeta^{\rho}_Q(x_i)+\vartheta^{\rho}_Q(x_i)+\pi^{\rho}_Q(x_i)}{\zeta^{\sigma}_Q(x_i)+\vartheta^{\sigma}_Q(x_i)+\pi^{\sigma}_Q(x_i)}\right] \\
- \; & \lambda \left\{ \sum_{x_i \in X_1} \log\left[\frac{\zeta^{\rho}_R(x_i)+\vartheta^{\rho}_R(x_i)+\pi^{\rho}_R(x_i)}{\zeta^{\sigma}_R(x_i)+\vartheta^{\sigma}_R(x_i)+\pi^{\sigma}_R(x_i)}\right] + \sum_{x_i \in X_2} \log\left[\frac{\zeta^{\rho}_R(x_i)+\vartheta^{\rho}_R(x_i)+\pi^{\rho}_R(x_i)}{\zeta^{\sigma}_R(x_i)+\vartheta^{\sigma}_R(x_i)+\pi^{\sigma}_R(x_i)}\right] \right\} \\
- \; & \lambda \sum_{x_i \in X_2} \left\{ \log\left[\frac{\zeta^{\rho}_Q(x_i)+\vartheta^{\rho}_Q(x_i)+\pi^{\rho}_Q(x_i)}{\zeta^{\sigma}_Q(x_i)+\vartheta^{\sigma}_Q(x_i)+\pi^{\sigma}_Q(x_i)}\right] - \log\left[\frac{\zeta^{\rho}_R(x_i)+\vartheta^{\rho}_R(x_i)+\pi^{\rho}_R(x_i)}{\zeta^{\sigma}_R(x_i)+\vartheta^{\sigma}_R(x_i)+\pi^{\sigma}_R(x_i)}\right] \right\} \\
= 0 &
\end{aligned}
$$

(iii) This can be obtained from the parts (i) and (ii).

□

3.2 Monotone Behavior of measure of Intuitionistic fuzzy entropy by α and β

Let $A_1 = \{\langle 0.2, 0.5\rangle, \langle 0.3, 0.4\rangle, \langle 0.4, 0.5\rangle, \langle 0.3, 0.6\rangle, \langle 0.2, 0.5\rangle\}$; $A_2 = \{\langle 0.2, 0.4\rangle, \langle 0.3, 0.5\rangle, \langle 0.3, 0.3\rangle, \langle 0.3, 0.4\rangle, \langle 0.3, 0.5\rangle\}$ and $A_3 = \{\langle 0.2, 0.3\rangle, \langle 0.3, 0.4\rangle, \langle 0.4, 0.4\rangle, \langle 0.4, 0.5\rangle,$

$\langle 0.2, 0.6\rangle\}$ be three collections of standard intuitionistic fuzzy sets. Without loss of generality, we assume $\alpha = 0.5$, $\gamma = 3$ and β varies from 0 to 20. Based on these information and by using Eq. (3.1), the values of the entropy measures are calculated and summarized in Table 3.1. From this table, we can conclude that the measure of an intuitionistic fuzzy entropy given by Eq. (3.1) is monotonically decreasing function of β. However, on the other hand, the monotonic behavior of the $H^{\alpha}_{\beta,\gamma}(A)$ with respect to the parameter α is described in Table 3.2 for given value of $\beta = 0.1$ and $\gamma = 3$.

Table 3.1: Monotone Behavior of $H^{\alpha}_{\beta,\gamma}$ with respect to β

β	0.2	2	6	8	10
$H^{0.5}_{\beta,3}(A_1)$	4.0835	3.7907	3.5881	3.5538	3.5344
$H^{0.5}_{\beta,3}(A_2)$	4.7145	4.4459	4.2282	4.1876	4.1647
$H^{0.5}_{\beta,3}(A_3)$	4.2061	3.9614	3.8045	3.7746	3.7568
β	12	14	16	18	20
$H^{0.5}_{\beta,3}(A_1)$	3.5232	3.5165	3.5126	3.5102	3.5088
$H^{0.5}_{\beta,3}(A_2)$	4.1518	4.1445	4.1404	4.1381	4.1368
$H^{0.5}_{\beta,3}(A_3)$	3.7460	3.7396	3.7357	3.7333	3.7319

Table 3.2: Monotone Behavior of $H^{\alpha}_{\beta,\gamma}$ with respect to α

α	0.2	2	6	8	10
$H^{\alpha}_{0.1,3}(A_1)$	4.1524	3.9497	3.7515	3.7029	3.6690
$H^{\alpha}_{0.1,3}(A_2)$	4.7708	4.5966	4.4016	4.3498	4.3128
$H^{\alpha}_{0.1,3}(A_3)$	4.2700	4.0894	3.9329	3.8946	3.8672
α	12	14	16	18	20
$H^{\alpha}_{0.1,3}(A_1)$	3.6441	3.6253	3.6107	3.5991	3.5897
$H^{\alpha}_{0.1,3}(A_2)$	4.2854	4.2646	4.2485	4.2357	4.2254
$H^{\alpha}_{0.1,3}(A_3)$	3.8467	3.8310	3.8187	3.8089	3.8009

3.3 MADM problem based on parametric measures of intuitionistic fuzzy entropy

In this section, we present an approach for solving MADM problem under IFS environment by using the proposed entropy measure.

3.3.1 Approach I: When the attribute weight is completely unknown

In this section, we present an MADM approach by considering the attribute weight is completely unknown. For it, consider a set of $'n'$ different alternatives, denoted by $A_1, A_2, \ldots, A_n$, which are evaluated by a decision maker under the $'m'$ different attributes $Q_1, Q_2, \ldots, Q_m$. Further, assume that a decision maker has given their preferences in terms of IFNs $\alpha_{ij} = \langle \zeta_{ij}, \vartheta_{ij} \rangle$ where ζ_{ij} denotes the degree of the alternative A_i satisfies under the attribute Q_j, while ϑ_{ij} denotes the degree of an alternative A_i which is dissatisfactory under Q_j such that $\zeta_{ij}, \vartheta_{ij} \in [0, 1]$ and $\zeta_{ij} + \vartheta_{ij} \leq 1$ for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$. Therefore, the collective information of all the alternatives are summarized in the form of the decision matrix G as follows:

$$G = \begin{array}{c} \\ A_1 \\ A_2 \\ \vdots \\ A_n \end{array} \begin{array}{c} \begin{array}{cccc} Q_1 & Q_2 & \cdots & Q_m \end{array} \\ \begin{pmatrix} \langle \zeta_{11}, \vartheta_{11} \rangle & \langle \zeta_{12}, \vartheta_{12} \rangle & \cdots & \langle \zeta_{1m}, \vartheta_{1m} \rangle \\ \langle \zeta_{21}, \vartheta_{21} \rangle & \langle \zeta_{22}, \vartheta_{22} \rangle & \cdots & \langle \zeta_{2m}, \vartheta_{2m} \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle \zeta_{n1}, \vartheta_{n1} \rangle & \langle \zeta_{n2}, \vartheta_{n2} \rangle & \cdots & \langle \zeta_{nm}, \vartheta_{nm} \rangle \end{pmatrix} \end{array}$$

Then, the following steps of the proposed approach are summarized to find the best alternative(s).

Step 1: If there is two different types of attributes namely cost and benefit then change the rating values of cost type into benefit type by using the following normalized formula:

$$r_{ij} = \begin{cases} \langle \zeta_{ij}, \vartheta_{ij} \rangle; & \text{if the benefit type attribute} \\ \langle \vartheta_{ij}, \zeta_{ij} \rangle; & \text{if the cost type attribute} \end{cases}$$

Step 2: By using normalized data $P = (r_{ij}) = (\zeta_{ij}, \vartheta_{ij})_{n\times m}$, the entropy measure $H^{\alpha}_{\beta,\gamma}$ as given in Eq. (3.6) is computed as:

$$H^{\alpha}_{\beta,\gamma}(Q_j) = \frac{1}{1-\alpha}\sum_{i=1}^{n}\log\left[\frac{\zeta^{\alpha+\beta+\gamma-1}(x_i)+\vartheta^{\alpha+\beta+\gamma-1}(x_i)+\pi^{\alpha+\beta+\gamma-1}(x_i)}{\zeta^{\beta+\gamma}(x_i)+\vartheta^{\beta+\gamma}(x_i)+\pi^{\beta+\gamma}(x_i)}\right] \tag{3.6}$$

where $\alpha \neq 1$, $\beta \geq 0$, $\gamma \geq 0$, $\alpha+\beta+\gamma > 0$, $\beta+\gamma-1 \geq 0$, $\pi(x_i) = 1-\zeta(x_i)-\vartheta(x_i)$

Step 3: Based on the entropy values, the weight associated with attribute Q_j is computed as follows:

$$w_j = \frac{1 - H^{\alpha}_{\beta,\gamma}(Q_j)}{m - \sum_{j=1}^{m} H^{\alpha}_{\beta,\gamma}(Q_j)} \tag{3.7}$$

Step 4: Compute the score function of each alternative by using equation as below

$$D^{\alpha}_{\beta,\gamma}(A_i) = \sum_{j=1}^{m}\zeta_{ij}\times w_j - \sum_{j=1}^{m}\vartheta_{ij}\times w_j \tag{3.8}$$

Step 5: Rank all the alternatives $A_i(i = 1, 2, \ldots, n)$ according to the highest value of $D^{\alpha}_{\beta,\gamma}(A_i)$ and hence choose the best alternative(s).

Example 3.3.1. *Let us take an example of society with 100 residents. Suppose they want to elect the president of their society. Let there be four possible alternatives A_1, A_2, A_3 and A_4. Evaluation attributes are Q_1(Honest), Q_2(Helpful) and Q_3(Easily accessible). The membership degree ζ_{ij} and non membership degree ϑ_{ij} for the alternative A_i satisfying the attribute Q_i respectively given in the following matrix:*

$$G = \begin{pmatrix} \langle 0.45, 0.35\rangle & \langle 0.50, 0.30\rangle & \langle 0.40, 0.45\rangle \\ \langle 0.65, 0.25\rangle & \langle 0.65, 0.25\rangle & \langle 0.55, 0.15\rangle \\ \langle 0.45, 0.35\rangle & \langle 0.55, 0.35\rangle & \langle 0.55, 0.20\rangle \\ \langle 0.75, 0.15\rangle & \langle 0.65, 0.20\rangle & \langle 0.35, 0.15\rangle \end{pmatrix}$$

The following steps of the proposed Approach I are executed as below

Step 1: Since all the attributes are of the same type, so no need to normalize the data.

Step 2: Without loss of generality, we take $\alpha = 0.5, \beta = 2, \gamma = 3$, and hence obtain the value of $H^{\alpha}_{\beta,\gamma}(Q_j), j = 1, 2, 3$ by using Eq. (3.6) as $H^{0.5}_{2,3}(Q_1) = 2.4681$, $H^{0.5}_{2,3}(Q_2) = 2.2685$, $H^{0.5}_{2,3}(Q_3) = 2.8536$.

Step 3: Based on these entropy values and by using Eq.(3.7), we get the weight as $w = (0.3198, 0.2764, 0.4038)^T$.

Step 4: The final score values using Eq.(3.8) are $D^{0.5}_{2,3}(A_1) = 0.0671$, $D^{0.5}_{2,3}(A_2) = 0.4000$, $D^{0.5}_{2,3}(A_3) = 0.2286$, $D^{0.5}_{2,3}(A_4) = 0.3970$.

Step 5: Based on the above result, we get the following ordering of ranks of the alternatives $A_i (i = 1, 2, 3, 4)$.

$$A_2 \succ A_4 \succ A_3 \succ A_1,$$

and thus, the best alternative is A_2.

However, in order to analyze the influence of the parameters α, β and γ on the final ranking order of the alternatives, this approach is executed by varying the values of α from 2 to 8 and β from 1 to 20 and γ from 0.1 to 10. The overall score values of each alternative long with the ranking order are summarized in Table 3.3. From this analysis, we conclude that the decision-maker can plan to choose the values of α, β and γ, hence, their respective alternatives according to this goal. Therefore, the proposed measures give various choices to the decision-maker to reach the target.

Table 3.3: Effect of α, β and γ on the measure $H^{\alpha}_{\beta,\gamma}$ by using Approach I

α	β	γ	D_1	D_2	D_3	D_4	Ranking
2	1	0.1	0.0744	0.4000	0.2253	0.4026	$A_4 \succ A_2 \succ A_3 \succ A_1$
	5	0.9	0.0666	0.4000	0.2287	0.3968	$A_2 \succ A_4 \succ A_3 \succ A_1$
	10	5	0.0664	0.4000	0.2295	0.3956	$A_2 \succ A_4 \succ A_3 \succ A_1$
	20	10	0.0667	0.4000	0.2293	0.3958	$A_2 \succ A_4 \succ A_3 \succ A_1$
4	1	0.1	0.0711	0.4000	0.2272	0.3994	$A_2 \succ A_4 \succ A_3 \succ A_1$
	5	0.9	0.0664	0.4000	0.2288	0.3966	$A_2 \succ A_4 \succ A_3 \succ A_1$
	10	5	0.0665	0.4000	0.2295	0.3956	$A_2 \succ A_4 \succ A_3 \succ A_1$
	20	10	0.0667	0.4000	0.2293	0.3958	$A_2 \succ A_4 \succ A_3 \succ A_1$
6	1	0.1	0.0697	0.4000	0.2277	0.3986	$A_2 \succ A_4 \succ A_3 \succ A_1$
	5	0.9	0.0664	0.4000	0.2289	0.3964	$A_2 \succ A_4 \succ A_3 \succ A_1$
	10	5	0.0665	0.4000	0.2295	0.3956	$A_2 \succ A_4 \succ A_3 \succ A_1$
	20	10	0.0668	0.4000	0.2293	0.3959	$A_2 \succ A_4 \succ A_3 \succ A_1$
8	1	0.1	0.0689	0.4000	0.2280	0.3981	$A_2 \succ A_4 \succ A_3 \succ A_1$
	5	0.9	0.0664	0.4000	0.2291	0.3963	$A_2 \succ A_4 \succ A_3 \succ A_1$
	10	5	0.0665	0.4000	0.2295	0.3956	$A_2 \succ A_4 \succ A_3 \succ A_1$
	20	10	0.0668	0.4000	0.2293	0.3959	$A_2 \succ A_4 \succ A_3 \succ A_1$

3.3.2 Approach II: When the attribute weight is partially known

In this section, we present an MADM approach under the condition that the attribute weights is partially known. The description of the MADM problems mentioned in Approach 3.3.1. In order to represent this incomplete information about the weights, the following relationship has been defined for $i \neq j$:

(i) A weak ranking: $w_i \geq w_j$;

(ii) A strict ranking: $w_i - w_j \geq \sigma_i; (\sigma_i \geq 0)$;

(iii) An ranking with multiplies: $w_i \geq \sigma_i w_j, (0 \leq \sigma_i \leq 1)$;

(iv) An interval form: $\lambda_i \leq w_i \leq \lambda_i + \delta_i, (0 \leq \lambda_i \leq \lambda_i + \delta_i \leq 1)$;

(v) A ranking of difference: $w_i - w_j \geq w_k - w_l, (j \neq k \neq l)$.

and denoted by Δ.

Then the proposed approach for finding the best alternative(s) are summarized into the various steps as follows:

Step 1: Similar to Approach I

Step 2: Similar to Approach I

Step 3: Compute the entropy of the alternatives $A_i(i = 1, 2, \ldots, n)$ for the attribute $Q_j(j = 1, 2, \ldots, m)$ by

$$H^{\alpha}_{\beta,\gamma}(A_i) = \lambda \sum_{j=1}^{m} \left\{ \sum_{i=1}^{n} \log \left[\frac{\zeta^{\rho}_{ij} + \vartheta^{\rho}_{ij} + \pi^{\rho}_{ij}}{\zeta^{\sigma}_{ij} + \vartheta^{\sigma}_{ij} + \pi^{\sigma}_{ij}} \right] \right\} \tag{3.9}$$

Now, by considering the information about the attribute weights $w = (w_1, w_2, \ldots, w_m)^T$ be partially known. For this, we formulate a mathematical model to determine the weight vector as follows:

$$\begin{aligned}\min H &= \sum_{i=1}^{n} H(A_i) = \sum_{i=1}^{n}\left\{\sum_{j=1}^{m} w_j H^{\alpha}_{\beta,\gamma}(A_i)\right\}\\ &= \lambda\sum_{j=1}^{m} w_j\left\{\sum_{i=1}^{n}\log\left[\frac{\zeta_{ij}^{\rho}+\vartheta_{ij}^{\rho}+\pi_{ij}^{\rho}}{\zeta_{ij}^{\sigma}+\vartheta_{ij}^{\sigma}+\pi_{ij}^{\sigma}}\right]\right\}\\ \text{s.t.}\quad & \sum_{i=1}^{m} w_j = 1; w_j \geq 0,\\ & w \in \Delta\end{aligned}$$

After solving this method we get optimal weight vector $w = (w_1, w_2, \ldots, w_m)^T$.

Step 4: Compute the score values of the alternative by

$$D_R^{\alpha,\beta}(A_i) = \sum_{j=1}^{m}\zeta_{ij}\times w_j - \sum_{j=1}^{m}\vartheta_{ij}\times w_j \tag{3.10}$$

Step 5: Rank all the alternatives $A_i(i = 1, 2, \ldots, n)$ according to the highest value of $D_R^{\alpha,\beta}(A_i)$ and hence choose the best alternative(s).

The above mentioned approach is demonstrated with a numerical example as follows:

Example 3.3.2. *Consider an MADM problem whose description is given in Example 4.3.1. Here we assume that the information about the attribute weight is partially known and is given by decision maker as* $\Delta = \{0.25 \leq w_1 \leq 0.3, 0.10 \leq w_2 \leq 0.2, 0.20 \leq w_3 \leq 0.5, w_3 \geq w_1, \sum_{i=1}^{3} w_j = 1\}$. *Now, apply the following steps on the problem*

Step 1: Since all the alternatives are of the same type, so no need to normalize the data.

Step 2: In this present study, we take $\alpha = 0.5$, $\beta = 2$ *and* $\gamma = 3$ *hence compute* $H^{\alpha}_{\beta,\gamma}(Q_j)$ *by using Eq.*(3.9) *and get* $H^{0.5}_{2,3}(Q_1) = 2.4681$, $H^{0.5}_{2,3}(Q_2) = 2.2685$, $H^{0.5}_{2,3}(Q_3) = 2.8536$.

Step 3: Solve the optimization model

$$\begin{aligned}\min(H) &= 2.4681w_1 + 2.2685w_2 + 2.8536w_3\\ \textit{subject to}\quad & 0.25 \leq w_1 \leq 0.3,\\ & 0.10 \leq w_2 \leq 0.2,\\ & 0.20 \leq w_3 \leq 0.5,\\ & w_3 \geq w_1,\\ \textit{and}\quad & w_1 + w_2 + w_3 = 1.\end{aligned}$$

and get $w = (0.30, 0.20, 0.50)^T$.

Step 4: *Using Eq.* (3.10), *we get* $D^{0.5}_{2,3}(A_1) = 0.0450$, $D^{0.5}_{2,3}(A_2) = 0.4000$, $D^{0.5}_{2,3}(A_3) = 0.2450$, $D^{0.5}_{2,3}(A_4) = 0.3700$.

Step 5: *The ranking order of the alternatives is* $A_2 \succ A_4 \succ A_3 \succ A_1$ *and hence the best alternative is* A_2.

3.4 Conclusion

In this chapter, we have presented a new information measures named as entropy $H^{\alpha}_{\beta,\gamma}$ under the IFS environment based on the parameters α, β and γ. The desirable relationship and the properties of the proposed measures are investigated in details. Also, it has been shown that the proposed measure $H^{\alpha}_{\beta,\gamma}$ is monotonically decreases with the parameters β and α. In addition to these, a decision making approach for solving MADM problem is presented by considering the attributes weights are either completely unknown or partially known. These two presented approaches are illustrated with a numerical examples. Finally, an impact of the parameters α, β and γ on to the ranking order of the process are investigated in details. Based on these different parameters, a decision maker can choice a desirable one according to his need to reach the desirable goal.

Chapter 4

R-norm Intuitionistic Fuzzy Entropy Measure and Its Applications to Decision-Making process

This chapter present a new R-norm Intuitionistic Fuzzy Information Measure under the IFS environment. Further, based on the proposed measures, an MADM approach has been presented to solve the decision-making problems. Further, the applicability of the proposed measures are explained through an example.

4.1 R-norm Intuitionistic Fuzzy Information Measure

In this section, we define a new R-norm information measure, denoted by $H_R^{\alpha,\beta}(A)$, in the IFS environment. For it, let IFS(X) be the collection of all IFSs.

Definition 4.1.1. *For all collection of IFSs $A = \{\langle x, \zeta_A(x), \vartheta_A(x)\rangle | x \in X\}$, an R-norm intuitioninstic fuzzy information measure $H_R^{\alpha,\beta}(A)$: IFS(X) $\to \Re$ is defined as follows:*

$$H_R^{\alpha,\beta}(A) = \frac{R}{R-2\alpha+\beta} \sum_{i=1}^{n} \left(1 - \left\{ \zeta_A^{\frac{R}{2\alpha-\beta}}(x_i) + \vartheta_A^{\frac{R}{2\alpha-\beta}}(x_i) + (1 - \zeta_A - \vartheta_A)^{\frac{R}{2\alpha-\beta}}(x_i) \right\}^{\frac{2\alpha-\beta}{R}} \right) \quad (4.1)$$

where $\alpha \geq 1,\ 0 < \beta \leq 1,\ R(> 0) \neq 1,\ R + \beta \neq 2\alpha.$

Theorem 4.1.1. *The R-norm intuitionistic fuzzy information measure $H_R^{\alpha,\beta}(A)$ defined in Eq.* (4.1) *for IFS is a valid measure.*

Proof. In order to prove the measure defined in Eq. (4.1) is valid measure, we need to show that it satisfies the following properties:

(P1) $H_R^{\alpha,\beta}(A) = 0$ if and only if A is a crisp set, i.e. $\zeta_A(x_i) = 1, \vartheta_A(x_i) = 0$ or $\zeta_A(x_i) = 0, \vartheta_A(x_i) = 1$ for all $x_i \in X$.

(P2) $H(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}) = \max_{A \in \mathrm{IFS}(X)} H(A)$

(P3) $H_R^{\alpha,\beta}(A) \leq H_R^{\alpha,\beta}(B)$ if A is crisper than B, i.e., for $\max\{\zeta_B(x_i), \vartheta_B(x_i)\} \leq \frac{1}{3}$ if $\zeta_A(x_i) \leq \zeta_B(x_i)$ & $\vartheta_A(x_i) \leq \vartheta_B(x_i)$, and for $\min\{\zeta_B(x_i), \vartheta_B(x_i)\} \leq \frac{1}{3}$ if $\zeta_A(x_i) \geq \zeta_B(x_i)$ & $\vartheta_A(x_i) \geq \vartheta_B(x_i)$, for all $x_i \in X$.

(P4) $H_R^{\alpha,\beta}(A) = H_R^{\alpha,\beta}(A^c)$ for all $A \in \mathrm{IFS}(X)$.

For representing the expression given in Eq. (4.1) in a more simplified way, we can define $\lambda = \frac{R}{R-2\alpha+\beta}$ and $\rho = \frac{R}{2\alpha-\beta}$. Then, expression (4.1) can be represented as follows:

$$H_R^{\alpha,\beta}(A) = \lambda \sum_{i=1}^{n} \left(1 - \left\{ \zeta_A^\rho(x_i) + \vartheta_A^\rho(x_i) + (1 - \zeta_A - \vartheta_A)^\rho(x_i) \right\}^{\frac{1}{\rho}} \right) \tag{4.2}$$

(P1) Sharpness: To show that $H_R^{\alpha,\beta}(A) = 0$ if and only if A is a crisp set, i.e., either $\zeta_A(x_i) = 1$, $\vartheta_A(x_i) = 0$ or $\zeta_A(x_i) = 0$, $\vartheta_A(x_i) = 1$ for all $x_i \in X$.

Firstly, we assume that $H_R^{\alpha,\beta}(A) = 0$ for $\alpha \geq 1, 0 < \beta \leq 1, R(> 0) \neq 1, R + \beta \neq 2\alpha$.. Therefore, from Eq. (4.2), we have:

$$\lambda \sum_{i=1}^{n} \left(1 - \left\{ \zeta_A^\rho(x_i) + \vartheta_A^\rho(x_i) + (1 - \zeta_A - \vartheta_A)^\rho(x_i) \right\}^{\frac{1}{\rho}} \right) = 0$$

As we have $\zeta_A^\rho(x_i) + \vartheta_A^\rho(x_i) + (1 - \zeta_A(x_i) - \vartheta_A(x_i))^\rho \leq 1$ for all $\zeta(x_i)$ and $\vartheta_A(x_i)$ and equality holds if $\zeta(x_i) = 1$, $\vartheta(x_i) = 0$ or $\zeta(x_i) = 0$, $\vartheta(x_i) = 1$. Hence, $H_R^{\alpha,\beta}(A) = 0$ if $\zeta_A(x_i) = 0$, $\vartheta_A(x_i) = 1$ or $\zeta_A(x_i) = 1$, $\vartheta_A(x_i) = 0$ i.e. A is crisp set.

Conversely, we assume that set $A = \langle x_i, \zeta_A(x_i), \vartheta_A(x_i) \rangle$ is a crisp set i.e, either $\zeta_A(x_i) = 1$, $\vartheta_A(x_i) = 0$ or $\zeta_A(x_i) = 0$, $\vartheta_A(x_i) = 1$. Now, for $\alpha \geq 1, 0 < \beta \leq 1, R(> 0) \neq 1, R + \beta \neq 2\alpha$. we can obtain that:

$$1 - \left\{ \zeta_A^\rho(x_i) + \vartheta_A^\rho(x_i) + (1 - \zeta_A - \vartheta_A)^\rho(x_i) \right\}^{\frac{1}{\rho}} = 0$$

for all $i = 1, 2, \ldots, n$, which implies $H_R^{\alpha,\beta}(A) = 0$.

Hence, $H_R^{\alpha,\beta}(A) = 0$ iff A is a crisp set.

(P2) Maximality: To find the maxima of the function $H_R^{\alpha,\beta}(A)$, differentiate Eq. (4.2) partially with respect to $\zeta_A(x_i)$ and $\vartheta_A(x_i)$, we get,

$$\begin{aligned} \frac{\partial H_R^{\alpha,\beta}(A)}{\partial(\zeta_A(x_i))} &= -\lambda \sum_{i=1}^{n} \left(\left\{ \zeta^\rho(x_i) + \vartheta^\rho(x_i) + (1 - \zeta(x_i) - \vartheta(x_i))^\rho \right\}^{\frac{1-\rho}{\rho}} \times \right. \\ &\qquad \left. \times \left\{ \zeta^{\rho-1}(x_i) - (1 - \zeta(x_i) - \vartheta(x_i))^{\rho-1} \right\} \right) \end{aligned} \tag{4.3}$$

and

$$\frac{\partial H_R^{\alpha,\beta}(A)}{\partial(\vartheta_A(x_i))} = -\lambda\sum_{i=1}^{n}\left(\left\{\zeta^{\rho}(x_i)+\vartheta^{\rho}(x_i)+(1-\zeta(x_i)-\vartheta(x_i))^{\rho}\right\}^{\frac{1-\rho}{\rho}}\times\right.$$
$$\left.\times\left\{\vartheta^{\rho-1}(x_i)-(1-\zeta(x_i)-\vartheta(x_i))^{\rho-1}\right\}\right) \tag{4.4}$$

Let $0\le\zeta_A(x_i)<\frac{1}{3}$, $0\le\vartheta_A(x_i)<\frac{1}{3}$. Then two cases arise:

Case 1. When $R>2\alpha-\beta$. In this case, we have $\lambda>0,\rho>1$ and $(\zeta^{\rho-1}(x_i)-(1-\zeta(x_i)-\vartheta(x_i))^{\rho-1})<0$, $(\vartheta^{\rho-1}(x_i)-(1-\zeta(x_i)-\vartheta(x_i))^{\rho-1})<0$, which implies $\frac{\partial H_R^{\alpha,\beta}(A)}{\partial(\zeta_A(x_i))}>0$.

Case 2. When $R<2\alpha-\beta$. In this case, we have $\lambda<0,\rho<1$ and $(\zeta^{\rho-1}(x_i)-(1-\zeta(x_i)-\vartheta(x_i))^{\rho-1})>0$, $(\vartheta^{\rho-1}(x_i)-(1-\zeta(x_i)-\vartheta(x_i))^{\rho-1})<0$, which implies $\frac{\partial H_R^{\alpha,\beta}(A)}{\partial(\zeta_A(x_i))}>0$.

Hence, $H_R^{\alpha,\beta}(A)$ is an increasing function of $\zeta_A(x_i)$ and $\vartheta_A(x_i)$ in the region $0\le\zeta_A(x_i)\le\frac{1}{3}$, $0\le\vartheta_A(x_i)\le\frac{1}{3}$. Similarly, we get $H_R^{\alpha,\beta}(A)$ is a decreasing function of $\zeta_A(x_i)$ and $\vartheta_A(x_i)$ in the region $0\le\zeta_A(x_i)\le\frac{1}{3},\frac{2}{3}\le\vartheta_A(x_i)\le 1$.

(P3) Resolution: Since $H_R^{\alpha,\beta}(A)$ is a concave function on IFS A, therefore if $\max\{\zeta_A(x),\vartheta_A(x)\}\le\frac{1}{3}$, then $\zeta_A(x_i)\le\zeta_B(x_i)$ and $\vartheta_A(x_i)\le\vartheta_B(x_i)$, which implies that

$$\zeta_A(x_i)\le\zeta_B(x_i)\le\frac{1}{3};\vartheta_A(x_i)\le\vartheta_B(x_i)\le\frac{1}{3};\pi_A(x_i)\le\pi_B(x_i)\ge\frac{1}{3}.$$

Thus, we observe that $\langle\zeta_B(x_i),\vartheta_B(x_i),\pi_B(x_i)\rangle$ is more around $\langle\frac{1}{3},\frac{1}{3},\frac{1}{3}\rangle$ than $\langle\zeta_A(x_i),\vartheta_A(x_i),\pi_A(x_i)\rangle$. Hence, $H_R^{\alpha,\beta}(A)\le H_R^{\alpha,\beta}(B)$.

Similarly, if $\min\{\zeta_A(x),\vartheta_A(x)\}\ge\frac{1}{3}$, then we get $H_R^{\alpha,\beta}(A)\le H_R^{\alpha,\beta}(B)$.

(P4) Symmetry: By the Definition of $H_R^{\alpha,\beta}(A)$, we can easily obtain that $H_R^{\alpha,\beta}(A^c)=H_R^{\alpha,\beta}(A)$

Hence $H_R^{\alpha,\beta}(A)$ satisfies all the properties of the intuitionistic measure therefore, it is a valid measure of R-norm intuitionistic fuzzy information measure. □

Remark 4.1.1. *From the proposed measure it is observed that some of existing measure can be obtained from it by assigning particular cases to R, α and β.*

1. *When $\pi_A(x_i)=0$ then the proposed measure reduces to*

$$H_R^{\alpha,\beta}(A)=\frac{R}{R-2\alpha+\beta}\sum_{i=1}^{n}\left(1-\left\{\zeta_A^{\frac{R}{2\alpha-\beta}}(x_i)+\vartheta_A^{\frac{R}{2\alpha-\beta}}(x_i)\right\}^{\frac{2\alpha-\beta}{R}}\right) \tag{4.5}$$

2. *When $2\alpha - \beta = 1$ then the proposed measure reduces to*

$$H_R^{\alpha,\beta}(A) = \frac{R}{R-1}\sum_{i=1}^{n}\left(1 - \left\{\zeta_A^R(x_i) + \vartheta_A^R(x_i) + (1-\zeta_A-\vartheta_A)^R(x_i)\right\}^{\frac{1}{R}}\right) \quad (4.6)$$

Definition 4.1.2. *Consider two IFSs A and B defined over $X = \{x_1, x_2, \ldots, x_n\}$. Take the disjoint partition of X as*

$$\begin{aligned} X_1 &= \{x_i \in X : A \subseteq B\}, \\ &= \{x_i \in X | \zeta_A(x_i) \leq \zeta_B(x_i); \vartheta_A(x_i) \geq \vartheta_B(x_i)\} \end{aligned}$$

and

$$\begin{aligned} X_2 &= \{x_i \in X : A \supseteq B\}. \\ &= \{x_i \in X | \zeta_A(x_i) \geq \zeta_B(x_i); \vartheta_A(x_i) \leq \vartheta_B(x_i)\} \end{aligned}$$

Next, we define the joint and conditional entropies between A and B as follows:

(a) ***Joint entropy:***

$$\begin{aligned} & H_R^{\alpha,\beta}(A \cup B) \\ = \ & \lambda \sum_{i=1}^{n}\left(1 - \left\{\zeta_{A\cup B}^{\rho}(x_i) + \vartheta_{A\cup B}^{\rho}(x_i) + (1-\zeta_{A\cup B}-\vartheta_{A\cup B})^{\rho}(x_i)\right\}^{\frac{1}{\rho}}\right) \\ = \ & \lambda \sum_{x_i\in X_1}\left(1 - \left\{\zeta_B^{\rho}(x_i) + \vartheta_B^{\rho}(x_i) + (1-\zeta_B-\vartheta_B)^{\rho}(x_i)\right\}^{\frac{1}{\rho}}\right) \\ + \ & \lambda \sum_{x_i\in X_2}\left(1 - \left\{\zeta_A^{\rho}(x_i) + \vartheta_A^{\rho}(x_i) + (1-\zeta_A-\vartheta_A)^{\rho}(x_i)\right\}^{\frac{1}{\rho}}\right) \end{aligned}$$

(b) ***Conditional entropy:***

$$\begin{aligned} & H_R^{\alpha,\beta}(A|B) \\ = \ & \lambda \sum_{x_i\in X_2}\left\{\left(1 - \left\{\zeta_A^{\rho}(x_i) + \vartheta_A^{\rho}(x_i) + (1-\zeta_A-\vartheta_A)^{\rho}(x_i)\right\}^{\frac{1}{\rho}}\right)\right. \\ - \ & \left.\left(1 - \left\{\zeta_B^{\rho}(x_i) + \vartheta_B^{\rho}(x_i) + (1-\zeta_B-\vartheta_B)^{\rho}(x_i)\right\}^{\frac{1}{\rho}}\right)\right\} \end{aligned}$$

and

$$\begin{aligned} & H_R^{\alpha,\beta}(B|A) \\ = \ & \lambda \sum_{x_i\in X_1}\left\{\left(1 - \left\{\zeta_B^{\rho}(x_i) + \vartheta_B^{\rho}(x_i) + (1-\zeta_B-\vartheta_B)^{\rho}(x_i)\right\}^{\frac{1}{\rho}}\right)\right. \\ - \ & \left.\left(1 - \left\{\zeta_A^{\rho}(x_i) + \vartheta_A^{\rho}(x_i) + (1-\zeta_A-\vartheta_A)^{\rho}(x_i)\right\}^{\frac{1}{\rho}}\right)\right\} \end{aligned}$$

Theorem 4.1.2. *Let A and B be two IFSs defined on universal set $X = \{x_1, x_2, \ldots, x_n\}$, where, $A = \{\langle x_i, \zeta_A(x_i), \vartheta_A(x_i)\rangle | x_i \in X\}$ and $B = \{\langle x_i, \zeta_B(x_i), \vartheta_B(x_i)\rangle | x_i \in X\}$, such that either $A \subseteq B$ or $A \supseteq B$ $\forall x_i \in X$, then we have*

$$H_R^{\alpha,\beta}(A \cup B) + H_R^{\alpha,\beta}(A \cap B) = H_R^{\alpha,\beta}(A) + H_R^{\alpha,\beta}(B)$$

Proof. Let X_1 and X_2 be two disjoint sets of X, where

$$X_1 = \{x_i \in X : A \subseteq B\}, X_2 = \{x_i \in X : A \supseteq B\}$$

i.e., for $x_i \in X_1$, we have $\zeta_A(x_i) \leq \zeta_B(x_i), \vartheta_A(x_i) \geq \vartheta_B(x_i)$ and $x_i \in X_2$, implies $\zeta_A(x_i) \geq \zeta_B(x_i), \vartheta_A(x_i) \leq \vartheta_B(x_i)$. Therefore,

$$\begin{aligned}
& H_R^{\alpha,\beta}(A \cup B) + H_R^{\alpha,\beta}(A \cap B) \\
= {} & \lambda \sum_{x_i \in X_1} \left(1 - \left\{\zeta_B^\rho(x_i) + \vartheta_B^\rho(x_i) + (1 - \zeta_B - \vartheta_B)^\rho(x_i)\right\}^{\frac{1}{\rho}}\right) \\
+ {} & \lambda \sum_{x_i \in X_2} \left(1 - \left\{\zeta_A^\rho(x_i) + \vartheta_A^\rho(x_i) + (1 - \zeta_A - \vartheta_A)^\rho(x_i)\right\}^{\frac{1}{\rho}}\right) \\
+ {} & \lambda \sum_{x_i \in X_1} \left(1 - \left\{\zeta_A^\rho(x_i) + \vartheta_A^\rho(x_i) + (1 - \zeta_A - \vartheta_A)^\rho(x_i)\right\}^{\frac{1}{\rho}}\right) \\
+ {} & \lambda \sum_{x_i \in X_2} \left(1 - \left\{\zeta_B^\rho(x_i) + \vartheta_B^\rho(x_i) + (1 - \zeta_B - \vartheta_B)^\rho(x_i)\right\}^{\frac{1}{\rho}}\right) \\
= {} & \lambda \sum_{i=1}^{n} \left(1 - \left\{\zeta_A^\rho(x_i) + \vartheta_A^\rho(x_i) + (1 - \zeta_A - \vartheta_A)^\rho(x_i)\right\}^{\frac{1}{\rho}}\right) \\
+ {} & \lambda \sum_{i=1}^{n} \left(1 - \left\{\zeta_B^\rho(x_i) + \vartheta_B^\rho(x_i) + (1 - \zeta_B - \vartheta_B)^\rho(x_i)\right\}^{\frac{1}{\rho}}\right) \\
= {} & H_R^{\alpha,\beta}(A) + H_R^{\alpha,\beta}(B)
\end{aligned}$$

□

Theorem 4.1.3. *Let A and B be two IFSs satisfying either $A \subseteq B$ or $B \subseteq A$ then the following statements hold:*

(i) $H_R^{\alpha,\beta}(A \cup B) = H_R^{\alpha,\beta}(A) + H_R^{\alpha,\beta}(B|A)$;

(ii) $H_R^{\alpha,\beta}(A \cup B) = H_R^{\alpha,\beta}(B) + H_R^{\alpha,\beta}(A|B)$;

(iii) $H_R^{\alpha,\beta}(A \cup B) = H_R^{\alpha,\beta}(A) + H_R^{\alpha,\beta}(B|A) = H_R^{\alpha,\beta}(B) + H_R^{\alpha,\beta}(A|B)$.

Proof. For two IFSs A and B, we have

(i) Consider

$$
\begin{aligned}
& H_R^{\alpha,\beta}(A\cup B) - H_R^{\alpha,\beta}(A) - H_R^{\alpha,\beta}(B|A) \\
= \; & \lambda \sum_{i=1}^{n} \left(1 - \left\{ \zeta_{A\cup B}^{\rho}(x_i) + \vartheta_{A\cup B}^{\rho}(x_i) + (1-\zeta_{A\cup B} - \vartheta_{A\cup B})^{\rho}(x_i) \right\}^{\frac{1}{\rho}} \right) \\
- \; & \lambda \sum_{i=1}^{n} \left(1 - \left\{ \zeta_{A}^{\rho}(x_i) + \vartheta_{A}^{\rho}(x_i) + (1-\zeta_{A} - \vartheta_{A})^{\rho}(x_i) \right\}^{\frac{1}{\rho}} \right) \\
- \; & \lambda \sum_{x_i \in X_1} \Bigg[\left(1 - \left\{ \zeta_{B}^{\rho}(x_i) + \vartheta_{B}^{\rho}(x_i) + (1-\zeta_{B} - \vartheta_{B})^{\rho}(x_i) \right\}^{\frac{1}{\rho}} \right) \\
& \qquad - \left(1 - \left\{ \zeta_{A}^{\rho}(x_i) + \vartheta_{A}^{\rho}(x_i) + (1-\zeta_{A} - \vartheta_{A})^{\rho}(x_i) \right\}^{\frac{1}{\rho}} \right) \Bigg] \\
= \; & \lambda \sum_{x_i \in X_1} \left(1 - \left\{ \zeta_{B}^{\rho}(x_i) + \vartheta_{B}^{\rho}(x_i) + (1-\zeta_{B} - \vartheta_{B})^{\rho}(x_i) \right\}^{\frac{1}{\rho}} \right) \\
+ \; & \lambda \sum_{x_i \in X_2} \left(1 - \left\{ \zeta_{A}^{\rho}(x_i) + \vartheta_{A}^{\rho}(x_i) + (1-\zeta_{A} - \vartheta_{A})^{\rho}(x_i) \right\}^{\frac{1}{\rho}} \right) \\
- \; & \lambda \sum_{i=1}^{n} \left(1 - \left\{ \zeta_{A}^{\rho}(x_i) + \vartheta_{A}^{\rho}(x_i) + (1-\zeta_{A} - \vartheta_{A})^{\rho}(x_i) \right\}^{\frac{1}{\rho}} \right) \\
- \; & \lambda \sum_{x_i \in X_1} \Bigg[\left(1 - \left\{ \zeta_{B}^{\rho}(x_i) + \vartheta_{B}^{\rho}(x_i) + (1-\zeta_{B} - \vartheta_{B})^{\rho}(x_i) \right\}^{\frac{1}{\rho}} \right) \\
& \qquad - \left(1 - \left\{ \zeta_{A}^{\rho}(x_i) + \vartheta_{A}^{\rho}(x_i) + (1-\zeta_{A} - \vartheta_{A})^{\rho}(x_i) \right\}^{\frac{1}{\rho}} \right) \Bigg] \\
= \; & 0
\end{aligned}
$$

(ii) Consider

$$
\begin{aligned}
& H_R^{\alpha,\beta}(A\cup B) - H_R^{\alpha,\beta}(B) - H_R^{\alpha,\beta}(A|B) \\
= \; & \lambda \sum_{i=1}^{n} \left(1 - \left\{ \zeta_{A\cup B}^{\rho}(x_i) + \vartheta_{A\cup B}^{\rho}(x_i) + (1-\zeta_{A\cup B} - \vartheta_{A\cup B})^{\rho}(x_i) \right\}^{\frac{1}{\rho}} \right) \\
- \; & \lambda \sum_{i=1}^{n} \left(1 - \left\{ \zeta_{B}^{\rho}(x_i) + \vartheta_{B}^{\rho}(x_i) + (1-\zeta_{B} - \vartheta_{B})^{\rho}(x_i) \right\}^{\frac{1}{\rho}} \right) \\
- \; & \lambda \sum_{x_i \in X_2} \Bigg[\left(1 - \left\{ \zeta_{A}^{\rho}(x_i) + \vartheta_{A}^{\rho}(x_i) + (1-\zeta_{A} - \vartheta_{A})^{\rho}(x_i) \right\}^{\frac{1}{\rho}} \right) \\
& \qquad - \left(1 - \left\{ \zeta_{B}^{\rho}(x_i) + \vartheta_{B}^{\rho}(x_i) + (1-\zeta_{B} - \vartheta_{B})^{\rho}(x_i) \right\}^{\frac{1}{\rho}} \right) \Bigg]
\end{aligned}
$$

$$
\begin{aligned}
&= \lambda \sum_{x_i \in X_1} \left(1 - \left\{\zeta_B^\rho(x_i) + \vartheta_B^\rho(x_i) + (1-\zeta_B-\vartheta_B)^\rho(x_i)\right\}^{\frac{1}{\rho}}\right) \\
&+ \lambda \sum_{x_i \in X_2} \left(1 - \left\{\zeta_A^\rho(x_i) + \vartheta_A^\rho(x_i) + (1-\zeta_A-\vartheta_A)^\rho(x_i)\right\}^{\frac{1}{\rho}}\right) \\
&- \lambda \sum_{i=1}^{n} \left(1 - \left\{\zeta_B^\rho(x_i) + \vartheta_B^\rho(x_i) + (1-\zeta_B-\vartheta_B)^\rho(x_i)\right\}^{\frac{1}{\rho}}\right) \\
&- \lambda \sum_{x_i \in X_2} \left[\left(1 - \left\{\zeta_A^\rho(x_i) + \vartheta_A^\rho(x_i) + (1-\zeta_A-\vartheta_A)^\rho(x_i)\right\}^{\frac{1}{\rho}}\right)\right. \\
&\qquad \left. - \left(1 - \left\{\zeta_B^\rho(x_i) + \vartheta_B^\rho(x_i) + (1-\zeta_B-\vartheta_B)^\rho(x_i)\right\}^{\frac{1}{\rho}}\right)\right] \\
&= 0
\end{aligned}
$$

(iii) This can be obtained from the parts (i) and (ii)

□

Theorem 4.1.4. *For two IFSs A and B such that either $A \subseteq B$ or $B \subseteq A$, the following statements hold:*

(i) $H_R^{\alpha,\beta}(A) - H_R^{\alpha,\beta}(A \cap B) = H_R^{\alpha,\beta}(A|B)$;

(ii) $H_R^{\alpha,\beta}(B) - H_R^{\alpha,\beta}(A \cap B) = H_R^{\alpha,\beta}(B|A)$.

Proof. For two IFSs A and B, we get

(i) Consider

$$
\begin{aligned}
& H_R^{\alpha,\beta}(A) - H_R^{\alpha,\beta}(A \cap B) \\
&= \lambda \sum_{x_i \in X_1} \left(1 - \left\{\zeta_A^\rho(x_i) + \vartheta_A^\rho(x_i) + (1-\zeta_A-\vartheta_A)^\rho(x_i)\right\}^{\frac{1}{\rho}}\right) \\
&+ \lambda \sum_{x_i \in X_2}^{n} \left(1 - \left\{\zeta_A^\rho(x_i) + \vartheta_A^\rho(x_i) + (1-\zeta_A-\vartheta_A)^\rho(x_i)\right\}^{\frac{1}{\rho}}\right) \\
&- \lambda \sum_{x_i \in X_1} \left(1 - \left\{\zeta_A^\rho(x_i) + \vartheta_A^\rho(x_i) + (1-\zeta_A-\vartheta_A)^\rho(x_i)\right\}^{\frac{1}{\rho}}\right) \\
&+ \lambda \sum_{x_i \in X_2} \left(1 - \left\{\zeta_B^\rho(x_i) + \vartheta_B^\rho(x_i) + (1-\zeta_B-\vartheta_B)^\rho(x_i)\right\}^{\frac{1}{\rho}}\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \lambda \sum_{x_i \in X_2} \Bigg[\left(1 - \left\{ \zeta_A^\rho(x_i) + \vartheta_A^\rho(x_i) + (1-\zeta_A-\vartheta_A)^\rho(x_i) \right\}^{\frac{1}{\rho}} \right) \\
&\qquad - \left(1 - \left\{ \zeta_B^\rho(x_i) + \vartheta_B^\rho(x_i) + (1-\zeta_B-\vartheta_B)^\rho(x_i) \right\}^{\frac{1}{\rho}} \right) \Bigg] \\
&= H_R^{\alpha,\beta}(A|B)
\end{aligned}
$$

(ii) Consider

$$
\begin{aligned}
& H_R^{\alpha,\beta}(B) - H_R^{\alpha,\beta}(A \cap B) \\
&= \lambda \sum_{x_i \in X_1} \left(1 - \left\{ \zeta_B^\rho(x_i) + \vartheta_B^\rho(x_i) + (1-\zeta_B-\vartheta_B)^\rho(x_i) \right\}^{\frac{1}{\rho}} \right) \\
&+ \lambda \sum_{x_i \in X_2} \left(1 - \left\{ \zeta_A^\rho(x_i) + \vartheta_A^\rho(x_i) + (1-\zeta_A-\vartheta_A)^\rho(x_i) \right\}^{\frac{1}{\rho}} \right) \\
&- \lambda \sum_{x_i \in X_1} \left(1 - \left\{ \zeta_A^\rho(x_i) + \vartheta_A^\rho(x_i) + (1-\zeta_A-\vartheta_A)^\rho(x_i) \right\}^{\frac{1}{\rho}} \right) \\
&+ \lambda \sum_{x_i \in X_2} \left(1 - \left\{ \zeta_B^\rho(x_i) + \vartheta_B^\rho(x_i) + (1-\zeta_B-\vartheta_B)^\rho(x_i) \right\}^{\frac{1}{\rho}} \right) \\
&= \lambda \sum_{x_i \in X_1} \Bigg[\left(1 - \left\{ \zeta_B^\rho(x_i) + \vartheta_B^\rho(x_i) + (1-\zeta_B-\vartheta_B)^\rho(x_i) \right\}^{\frac{1}{\rho}} \right) \\
&\qquad - \left(1 - \left\{ \zeta_A^\rho(x_i) + \vartheta_A^\rho(x_i) + (1-\zeta_A-\vartheta_A)^\rho(x_i) \right\}^{\frac{1}{\rho}} \right) \Bigg] \\
&= H_R^{\alpha,\beta}(B|A)
\end{aligned}
$$

□

4.2 Monotone Behavior of $H_R^{\alpha,\beta}$ with respect to parameters R, α and β

Consider $A_1 = \{\langle 0.2, 0.5\rangle, \langle 0.3, 0.4\rangle, \langle 0.4, 0.5\rangle, \langle 0.3, 0.6\rangle, \langle 0.2, 0.5\rangle\}$; $A_2 = \{\langle 0.2, 0.4\rangle, \langle 0.3, 0.5\rangle, \langle 0.3, 0.3\rangle, \langle 0.3, 0.4\rangle, \langle 0.3, 0.5\rangle\}$ and $A_3 = \{\langle 0.2, 0.3\rangle, \langle 0.3, 0.4\rangle, \langle 0.4, 0.4\rangle, \langle 0.4, 0.5\rangle, \langle 0.2, 0.6\rangle\}$ be three collections of standard IFSs.

In order to analyze the monotonic behavior of the proposed $H_R^{\alpha,\beta}$ with respect to the parameter β, we investigate an analysis in which we freeze the parameter R and α to be 0.6 and 1.5 respectively. However, the value of β varies from 0.1 to 0.9 with an increment of 0.1 and hence the corresponding values of $H_R^{\alpha,\beta}$ with these parametric setting are summarized in Table 4.1. From this table, we can conclude that the proposed measure given by Eq. (4.1) is monotonically decreasing function of β. Similarly, we investigate the monotonic behavior of the $H_R^{\alpha,\beta}$ with respect to the parameters α and R and their results are summarized in Tables 4.2 and 4.3 respectively. From these tables, we observe that the proposed function is monotonic increasing w.r.t. α for given value of $\beta = 0.1$ and $R = 0.6$, while the function is monotonic decreasing w.r.t. R for given value of $\beta = 0.1$ and $\alpha = 100$.

Table 4.1: Monotone Behavior of $H_R^{\alpha,\beta}$ with respect to β

β	0.1	0.2	0.3	0.4	0.5
$H_{0.6}^{1.5,\beta}(A_1)$	79.1361	68.7261	59.7727	52.0655	45.4248
$H_{0.6}^{1.5,\beta}(A_2)$	83.8468	72.7860	63.2762	55.0926	48.0441
$H_{0.6}^{1.5,\beta}(A_3)$	80.4783	69.8820	60.7694	52.9259	46.1686
β	0.6	0.7	0.8	0.9	
$H_{0.6}^{1.5,\beta}(A_1)$	39.6975	34.7531	30.4800	26.7830	
$H_{0.6}^{1.5,\beta}(A_2)$	41.9673	36.7233	32.1931	28.2755	
$H_{0.6}^{1.5,\beta}(A_3)$	40.3414	35.3114	30.9648	27.2048	

Table 4.2: Monotone Behavior of $H_R^{\alpha,\beta}$ with respect to α

α	$H_{0.6}^{\alpha,0.1}(A_1)$	$H_{0.6}^{\alpha,0.1}(A_2)$	$H_{0.6}^{\alpha,0.1}(A_3)$
1.5	79.1361	83.8468	80.64783
2	346.6634	368.6175	352.9506
5	7.1985e+06	7.7126e+06	7.3470e+06
50	2.4774e+77	2.6664e+77	2.5322e+77

4.3 Application of proposed measure in MADM process

In this section, we present a decision-making method for solving the MADM problem based on the R-norm Intuitionistic fuzzy information measure. For it, the description of the problem under IFS environment is given in Section 3.3.1 of the Chapter 3. In the

Table 4.3: Monotone Behavior of $H_R^{\alpha,\beta}$ with respect to R

R	$H_R^{100,0.1}(A_1)$	$H_R^{100,0.1}(A_2)$	$H_R^{100,0.1}(A_3)$
0.6	4.0877e+156	4.4008e+156	4.1785e+156
4	2.1234e+22	2.2827e+22	2.1695e+22
25	1.4079e+03	1.5004e+03	1.4344e+03
200	4.9880	5.2905	5.0656
2000	2.7664	3.0688	2.8460

following, we present two approaches by considering the attribute weights information is either partially known or completely unknown.

4.3.1 Approach I: When the attribute weight information is completely unknown

The steps of the proposed approach for MADM problems are summarized as follows:

Step 1: The collective information given by an expert related to the evaluation of each alternative $A_i (i = 1, 2, \dots, n)$ under the attributes $Q_j (j = 1, 2, \dots, m)$ are summarized in the form of the decision matrix G as

$$G = \begin{array}{c} \\ A_1 \\ A_2 \\ \vdots \\ A_n \end{array} \begin{array}{c} \begin{array}{cccc} Q_1 & Q_2 & \cdots & Q_m \end{array} \\ \begin{pmatrix} \langle \zeta_{11}, \vartheta_{11} \rangle & \langle \zeta_{12}, \vartheta_{12} \rangle & \cdots & \langle \zeta_{1m}, \vartheta_{1m} \rangle \\ \langle \zeta_{21}, \vartheta_{21} \rangle & \langle \zeta_{22}, \vartheta_{22} \rangle & \cdots & \langle \zeta_{2m}, \vartheta_{2m} \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle \zeta_{n1}, \vartheta_{n1} \rangle & \langle \zeta_{n2}, \vartheta_{n2} \rangle & \cdots & \langle \zeta_{nm}, \vartheta_{nm} \rangle \end{pmatrix} \end{array}$$

Step 2: Normalize the information G into $R = (r_{ij})$, if required, where r_{ij} is computed by using the following equation

$$r_{ij} = \begin{cases} \langle \zeta_{ij}, \vartheta_{ij} \rangle; \text{ if the benefit type attribute} \\ \langle \vartheta_{ij}, \zeta_{ij} \rangle; \text{ if the cost type attribute} \end{cases} \tag{4.7}$$

Step 3: Utilize the information $R = (r_{ij})$ and hence compute the entropy values of each attribute $Q_j(j = 1, 2, \ldots, m)$ by using Eq. (4.8) as

$$H_R^{\alpha,\beta}(Q_j) = \frac{R}{R-2\alpha+\beta} \sum_{i=1}^{n} \left(1 - \left\{ \zeta_{ij}^{\frac{R}{2\alpha-\beta}} + \vartheta_{ij}^{\frac{R}{2\alpha-\beta}} + (1 - \zeta_{ij} - \vartheta_{ij})^{\frac{R}{2\alpha-\beta}} \right\}^{\frac{2\alpha-\beta}{R}} \right) \quad (4.8)$$

where $\lambda = \frac{R}{R-2\alpha+\beta}$, $\rho = \frac{R}{2\alpha-\beta}$.

Step 4: The weight vector associated with attribute $Q_j(j = 1, 2, \ldots, m)$ are computed by

$$w_j = \frac{1 - H_R^{\alpha,\beta}(x_j)}{m - \sum_{j=1}^{m} H_R^{\alpha,\beta}(x_j)} \quad (4.9)$$

Step 5: Construct the overall value of the alternative $A_i(i = 1, 2, \ldots, n)$ by using a score function as follows:

$$D_R^{\alpha,\beta}(A_i) = \sum_{j=1}^{m} \zeta_{ij} \times w_j - \sum_{j=1}^{m} \vartheta_{ij} \times w_j \quad (4.10)$$

Step 6: The ranking order of the alternatives is obtained from the descending order of the values obtained from Eq. (4.10) and hence choose the best alternative(s).

The working of the the proposed approach is illustrated with a numerical example as follows:

Example 4.3.1. *Consider a customer who intends to invest a money in a market. After carefully analyzing the market scenario, they decided to invest the money out of the five different types of companies namely, Food Company* (A_1)*, Arm Company* (A_2)*, Insurance Company* (A_3)*, Steel Company* (A_4) *and Medicine Company* (A_5)*. The four different attributes* Q_1 *"Benefits in the short term",* Q_2 *"Benefits in the mid term",* Q_3 *"Benefits in the long term" and* Q_4 *"Risk of the production strategy" taken into account during the decision. Also, the information about the attribute weight is completely unknown. Then, the following steps of the proposed approach are executed which are explained as below:*

Step 1: An expert evaluate each alternative and summarized their rating values in terms of IFNs which are given as

$$G = \begin{pmatrix} \langle 0.5, 0.4 \rangle & \langle 0.5, 0.4 \rangle & \langle 0.4, 0.3 \rangle & \langle 0.2, 0.6 \rangle \\ \langle 0.7, 0.2 \rangle & \langle 0.5, 0.2 \rangle & \langle 0.2, 0.6 \rangle & \langle 0.5, 0.4 \rangle \\ \langle 0.6, 0.2 \rangle & \langle 0.2, 0.3 \rangle & \langle 0.5, 0.2 \rangle & \langle 0.1, 0.6 \rangle \\ \langle 0.8, 0.1 \rangle & \langle 0.6, 0.3 \rangle & \langle 0.3, 0.3 \rangle & \langle 0.3, 0.6 \rangle \\ \langle 0.6, 0.2 \rangle & \langle 0.4, 0.5 \rangle & \langle 0.9, 0.0 \rangle & \langle 0.4, 0.5 \rangle \end{pmatrix} \quad (4.11)$$

Step 2: *Since Q_4 is the cost type attribute, so we normalize the information by using Eq.* (4.7) *and get*

$$F = \begin{pmatrix} \langle 0.5, 0.4\rangle & \langle 0.5, 0.4\rangle & \langle 0.4, 0.3\rangle & \langle 0.6, 0.2\rangle \\ \langle 0.7, 0.2\rangle & \langle 0.5, 0.2\rangle & \langle 0.2, 0.6\rangle & \langle 0.4, 0.5\rangle \\ \langle 0.6, 0.2\rangle & \langle 0.2, 0.3\rangle & \langle 0.5, 0.2\rangle & \langle 0.6, 0.1\rangle \\ \langle 0.8, 0.1\rangle & \langle 0.6, 0.3\rangle & \langle 0.3, 0.3\rangle & \langle 0.6, 0.3\rangle \\ \langle 0.6, 0.2\rangle & \langle 0.4, 0.5\rangle & \langle 0.9, 0.0\rangle & \langle 0.5, 0.4\rangle \end{pmatrix} \tag{4.12}$$

Step 3: *Without loss of generality, we take $R = 0.6, \alpha = 1.5, \beta = 0.1$, hence compute $H_R^{\alpha,\beta}(Q_j)$, $(j = 1, 2, 3, 4)$ for each attribute by using Eq.* (4.8)*. The result corresponding to it are $H_{0.6}^{1.5,0.1}(x_1) = 70.4239$, $H_{0.6}^{1.5,0.1}(x_2) = 76.7455$, $H_{0.6}^{1.5,0.1}(x_3) = 68.4528$, $H_{0.6}^{1.5,0.1}(x_4) = 73.7463$.*

Step 4: *Use Eq.*(4.9)*, we get $w = (w_1, w_2, w_3, w_4) = (0.2433, 0.2654, 0.2364, 0.2549)$.*

Step 5: *The final score values of the alternative are computed by using Eq.* (4.10) *are $D_{0.6}^{1.5,0.1}(a_1) = 0.1765, D_{0.6}^{1.5,0.1}(a_2) = 0.0812, D_{0.6}^{1.5,0.1}(a_3) = 0.2691, D_{0.6}^{1.5,0.1}(a_4) = 0.3264, D_{0.6}^{1.5,0.1}(a_5) = 0.3090$.*

Step 6: *Based on the score values, we obtain the ranking order of the alternatives as*

$$A_4 \succ A_5 \succ A_3 \succ A_1 \succ A_2,$$

and thus A_4 is the best alternative for investing the money.

However, in order to analyze the influence of the parameters R, α and β on the final ranking order of the alternatives, this approach is executed by varying the values of R from 0.6 to 2000, α from 2 to 8 and β from 0.1 to 0.8. The overall score values of each alternative along with the ranking order are summarized in Table 4.4. From this analysis, we conclude that the decision-maker can plan to choose the values of R, α and β, hence, their respective alternatives according to this goal.

4.3.2 Approach II: When the information about the attribute weight is partially known

In this section, we assume that the attribute weight information as provided by the decision maker is partially known. Let Δ be the set of all such information. The relationship between the incomplete information is summarized in Section 3.3.2. By considering all these features, we present an approach to compute the attribute weight as well as a method to solve the MADM approach, which are summarized in the following steps.

Table 4.4: Effect of R, α and β on the measure $H_R^{\alpha,\beta}$

α	β	R	Overall score value of the alternative					Ranking results
			A_1	A_2	A_3	A_4	A_5	
2	0.1	0.6	0.1766	0.0821	0.2689	0.3269	0.3077	$A_4 \succ A_5 \succ A_3 \succ A_1 \succ A_2$
	0.3	4	0.1766	0.0723	0.2656	0.3181	0.3105	$A_4 \succ A_5 \succ A_3 \succ A_1 \succ A_2$
	0.4	25	0.1778	0.0515	0.2510	0.2956	0.3027	$A_5 \succ A_4 \succ A_3 \succ A_1 \succ A_2$
	0.6	200	0.1782	0.0482	0.2482	0.2918	0.3004	$A_5 \succ A_4 \succ A_3 \succ A_1 \succ A_2$
	0.8	2000	0.1783	0.0479	0.2479	0.2914	0.3000	$A_5 \succ A_4 \succ A_3 \succ A_1 \succ A_2$
2.5	0.1	0.6	0.1766	0.0825	0.2687	0.3271	0.3071	$A_4 \succ A_5 \succ A_3 \succ A_1 \succ A_2$
	0.3	4	0.1764	0.0742	0.2670	0.3203	0.3114	$A_4 \succ A_5 \succ A_3 \succ A_1 \succ A_2$
	0.4	25	0.1777	0.0531	0.2520	0.2973	0.3032	$A_5 \succ A_4 \succ A_3 \succ A_1 \succ A_2$
	0.6	200	0.1782	0.0483	0.2483	0.2919	0.3005	$A_5 \succ A_4 \succ A_3 \succ A_1 \succ A_2$
	0.8	2000	0.1783	0.0479	0.2479	0.2914	0.3000	$A_5 \succ A_4 \succ A_3 \succ A_1 \succ A_2$
4	0.1	0.6	0.1763	0.0750	0.2676	0.3211	0.3117	$A_4 \succ A_5 \succ A_3 \succ A_1 \succ A_2$
	0.3	4	0.1762	0.0772	0.2688	0.3233	0.3120	$A_4 \succ A_5 \succ A_3 \succ A_1 \succ A_2$
	0.4	25	0.1775	0.0582	0.2555	0.3027	0.3047	$A_5 \succ A_4 \succ A_3 \succ A_1 \succ A_2$
	0.6	200	0.1782	0.0487	0.2487	0.2924	0.3008	$A_5 \succ A_4 \succ A_3 \succ A_1 \succ A_2$
	0.8	2000	0.1783	0.0479	0.2479	0.2914	0.3001	$A_5 \succ A_4 \succ A_3 \succ A_1 \succ A_2$
6	0.1	0.6	0.1765	0.0822	0.2683	0.3267	0.3071	$A_4 \succ A_5 \succ A_3 \succ A_1 \succ A_2$
	0.3	4	0.1763	0.0791	0.2693	0.3250	0.3112	$A_4 \succ A_5 \succ A_3 \succ A_1 \succ A_2$
	0.4	25	0.1772	0.0639	0.2596	0.3090	0.3068	$A_4 \succ A_5 \succ A_3 \succ A_1 \succ A_2$
	0.6	200	0.1781	0.0491	0.2491	0.2929	0.3012	$A_5 \succ A_4 \succ A_3 \succ A_1 \succ A_2$
	0.8	2000	0.1782	0.0480	0.2480	0.2915	0.3001	$A_5 \succ A_4 \succ A_3 \succ A_1 \succ A_2$
8	0.1	0.6	0.1764	0.0820	0.2682	0.3265	0.3074	$A_4 \succ A_5 \succ A_3 \succ A_1 \succ A_2$
	0.3	4	0.1764	0.0804	0.2693	0.3259	0.3100	$A_4 \succ A_5 \succ A_3 \succ A_1 \succ A_2$
	0.4	25	0.1770	0.0679	0.2625	0.3134	0.3084	$A_4 \succ A_5 \succ A_3 \succ A_1 \succ A_2$
	0.6	200	0.1780	0.0495	0.2495	0.2934	0.3017	$A_5 \succ A_4 \succ A_3 \succ A_1 \succ A_2$
	0.8	2000	0.1782	0.0480	0.2480	0.2915	0.3002	$A_5 \succ A_4 \succ A_3 \succ A_1 \succ A_2$

Step 1: Collection the information in terms of IF decision matrix G.

Step 2: Normalize the data, if required, by using Eq. (4.7) and obtain normalized matrix R.

Step 3: Compute the values of $H_R^{\alpha,\beta}$ for each attribute $Q_j (j = 1, 2, \ldots, m)$ by using

$$H_R^{\alpha,\beta}(Q_j) = \frac{R}{R-2\alpha+\beta} \sum_{j=1}^{m} \left\{ \sum_{i=1}^{n} \left(1 - \left\{ \zeta_{ij}^{\frac{R}{2\alpha-\beta}} + \vartheta_{ij}^{\frac{R}{2\alpha-\beta}} + (1 - \zeta_{ij} - \vartheta_{ij})^{\frac{R}{2\alpha-\beta}} \right\}^{\frac{2\alpha-\beta}{R}} \right) \right\} \tag{4.13}$$

Step 4: Formulate an optimization model based on the given information and attribute weight Δ as

$$\begin{aligned}\min H &= \sum_{i=1}^{n}\left\{\sum_{j=1}^{m} w_j H_R^{\alpha,\beta}(r_{ij})\right\}\\ &= \lambda\sum_{j=1}^{m} w_j\left\{\sum_{i=1}^{n}\left(1-\left\{\zeta_{ij}^{\rho}+\vartheta_{ij}^{\rho}+(1-\zeta_{ij}-\vartheta_{ij})^{\rho}\right\}^{\frac{1}{\rho}}\right)\right\}\\ \text{s.t.}\quad & \sum_{i=1}^{m} w_j = 1; w_j > 0\\ & w \in \Delta,\\ & \lambda = \frac{R}{R-2\alpha+\beta}, \rho = \frac{R}{2\alpha-\beta}\end{aligned}$$

After solving this method we get optimal weight vector $w = (w_1, w_2, \ldots, w_m)^T$.

Step 5: Compute the score function of each alternative by

$$D_R^{\alpha,\beta}(A_i) = \sum_{j=1}^{m}\zeta_{ij}\times w_j - \sum_{j=1}^{m}\vartheta_{ij}\times w_j \tag{4.14}$$

Step 6: Rank all the alternatives $A_i (i = 1, 2, \ldots, n)$ based on descending value of $D_R^{\alpha,\beta}(A_i)$ and hence choose the best alternative(s).

To demonstrate the above mentioned approach, a numerical example discussed as follows:

Example 4.3.2. *Consider an MADM problem which we discussed in the Example 4.3.1. Here, we assume that the information about the attribute weight is partially known and is given by decision maker as* $\Delta = \{0.12 \leq w_1 \leq 0.2,\ 0.25 \leq w_2 \leq 0.45,\ 0.05 \leq w_3 \leq 0.30,\ 0.15 \leq w_4 \leq 0.30,\ w_3 \geq w_1,\ \sum_{i=1}^{4} w_j = 1\}$. *Thus, based on this information, the steps of the proposed Approach II are executed as follows:*

Step 1: The information about the alternatives are given in Eq. (4.11).

Step 2: The normalized information is given in Eq. 4.12.

Step 3: Without loss of generality, we take $R = 0.6, \alpha = 1.5, \beta = 0.1$, *hence compute* $H_R^{\alpha,\beta}(Q_j)$ *for each attribute by using Eq.* (4.13) *and get* $H_{0.6}^{1.5,0.1}(x_1) = 70.4239$, $H_{0.6}^{1.5,0.1}(x_2) = 76.7455$, $H_{0.6}^{1.5,0.1}(x_3) = 68.4528$, $H_{0.6}^{1.5,0.1}(x_4) = 73.7463$.

Step 4: The optimization model for the considered problem is formulated as

$$\begin{aligned} \min(H) &= 70.4239w_1 + 76.7455w_2 + 68.4528w_3 + 73.7463w_4 \\ \textit{subject to} \quad & 0.12 \leq w_1 \leq 0.2, \\ & 0.25 \leq w_2 \leq 0.45, \\ & 0.05 \leq w_3 \leq 0.30, \\ & 0.15 \leq w_4 \leq 0.30, \\ & w_3 \geq w_1, \\ \textit{and} \quad & w_1 + w_2 + w_3 + w_4 = 1. \end{aligned}$$

and hence get the optimal values by using MATLAB as $w = (0.20, 0.25, 0.30, 0.25)^T$.

Step 5: The score values by using Eq. (4.14) *are computed as*

$$D_{0.6}^{1.5,0.1}(a_1) = 0.1750, D_{0.6}^{1.5,0.1}(a_2) = 0.0300, D_{0.6}^{1.5,0.1}(a_3) = 0.2700,$$
$$D_{0.6}^{1.5,0.1}(a_4) = 0.2900, D_{0.6}^{1.5,0.1}(a_5) = 0.3500.$$

Step 6: Based on these values, the ranking order of the alternatives is

$$A_5 \succ A_4 \succ A_3 \succ A_1 \succ A_2,$$

and hence the best alternative is A_5.

4.4 Conclusion

In this chapter, we present a R-norm intuitionistic fuzzy information measure under IFS environment with the parameters α and β. The validity of the proposed measure is tested by proving their several properties. Also, some of the existing measures can be deduced from the proposed measures by setting a particular values to the parameters R, α or β. The joint and the conditional R-norm entropies are also proposed in this chapter along with their desirable relations. The monotonicity behavior of proposed $H_R^{\alpha,\beta}$ measure is described where we found that the measure is monotonically decreasing with respect to the parameters β and R while increasing with α parameter. Finally, based on this measure, an attempt has been made to solve the MADM problem to get the ranking order of the alternatives. From the proposed measures, the decision maker can choose the different parametric values of α, β and R and hence choose a desired alternative(s) according to his preference.

Chapter 5

(R, S)-Norm Entropy Measure of Intuitionistic Fuzzy Sets and Its Applications in Multi-Attribute Decision-Making

In this chapter, we present a novel (R, S)-norm based information measure called the entropy to measure the degree of fuzziness of the IFSs. The validity of the proposed measure is tested on the linguistic variable to demonstrate it. Then, we utilized it to propose two decision-making approaches to solve the MADM problem by considering the attribute weights as either partially known or completely unknown. Finally, a practical example is provided to illustrate the decision-making process.

5.1 Proposed (R, S)-Norm Intuitionistic Fuzzy Information Measure

In this section, we define a new (R, S)-norm information measure, denoted by H_R^S, in the IFS environment. For it, let Ω be the collection of all IFSs.

Definition 5.1.1. *For a collection of IFSs $A = \{(x, \zeta_A(x), \vartheta_A(x)) \mid x \in X\}$, an information measure $H_R^S : \Omega^n \to \mathbf{R}$; $n \geq 2$ is defined as follows:*

$$H_R^S(A) = \begin{cases} \frac{R\times S}{n(R-S)} \sum_{i=1}^{n} \left[\begin{array}{c} \left(\zeta_A^S(x_i) + \vartheta_A^S(x_i) + \pi_A^S(x_i)\right)^{\frac{1}{S}} \\ -\left(\zeta_A^R(x_i) + \vartheta_A^R(x_i) + \pi_A^R(x_i)\right)^{\frac{1}{R}} \end{array} \right]; & \text{either } R > 1,\ 0 < S < 1 \\ & \text{or } 0 < R < 1,\ S > 1 \\ \frac{R}{n(R-1)} \sum_{i=1}^{n} \left[1 - \left(\zeta_A^R(x_i) + \vartheta_A^R(x_i) + \pi_A^R(x_i)\right)^{\frac{1}{R}}\right]; & \text{when } S = 1;\ 0 < R < 1 \\ \frac{S}{n(1-S)} \sum_{i=1}^{n} \left[\left(\zeta_A^S(x_i) + \vartheta_A^S(x_i) + \pi_A^S(x_i)\right)^{\frac{1}{S}} - 1\right]; & \text{when } R = 1;\ 0 < S < 1 \\ \frac{-1}{n} \sum_{i=1}^{n} \left[\begin{array}{c} \zeta_A(x_i)\log\zeta_A(x_i) + \vartheta_A(x_i)\log\vartheta_A(x_i) \\ +\pi_A(x_i)\log\pi_A(x_i) \end{array} \right]; & R = 1 = S. \end{cases} \tag{5.1}$$

Theorem 5.1.1. *An intuitionistic fuzzy entropy measure $H_R^S(A)$ defined in Eq.* (5.1) *for IFSs is a valid measure, i.e., it satisfies the following properties.*

(P1) $H_R^S(A) = 0$ *if and only if A is a crisp set, i.e.,* $\zeta_A(x_i) = 1, \vartheta_A(x_i) = 0$ *or* $\zeta_A(x_i) = 0, \vartheta_A(x_i) = 1$ *for all $x_i \in X$.*

(P2) $H_R^S(A) = 1$ *if and only if* $\zeta_A(x_i) = \vartheta_A(x_i)$ *for all $x_i \in X$.*

(P3) $H_R^S(A) \leq H_R^S(B)$ *if A is crisper than B, i.e., if* $\zeta_A(x_i) \leq \zeta_B(x_i)$ *&* $\vartheta_A(x_i) \leq \vartheta_B(x_i)$, *for* $\max\{\zeta_B(x_i), \vartheta_B(x_i)\} \leq \frac{1}{3}$ *and* $\zeta_A(x_i) \geq \zeta_B(x_i)$ *&* $\vartheta_A(x_i) \geq \vartheta_B(x_i)$, *for* $\min\{\zeta_B(x_i), \vartheta_B(x_i)\} \leq \frac{1}{3}$ *for all $x_i \in X$.*

(P4) $H_R^S(A) = H_R^S(A^c)$ *for all $A \in IFS(X)$.*

Proof. To prove that the measure defined by Eq. (5.1) is a valid information measure, we will have to prove that it satisfies the four properties defined in the definition of the intuitionistic fuzzy information measure.

(P1) Sharpness: In order to prove (P1), we need to show that $H_R^S(A) = 0$ if and only if A is a crisp set, i.e., either $\zeta_A(x) = 1, \vartheta_A(x) = 0$ or $\zeta_A(x) = 0, \vartheta_A(x) = 1$ for all $x \in X$.

Firstly, we assume that $H_R^S(A) = 0$ for $R, S > 0$ and $R \neq S$. Therefore, from Equation (5.1), we have:

$$\frac{R\times S}{n(R-S)} \sum_{i=1}^{n} \left(\begin{array}{c} \left(\zeta_A^S(x_i) + \vartheta_A^S(x_i) + \pi_A^S(x_i)\right)^{\frac{1}{S}} \\ -\left(\zeta_A^R(x_i) + \vartheta_A^R(x_i) + \pi_A^R(x_i)\right)^{\frac{1}{R}} \end{array} \right) = 0$$

$$\Rightarrow \quad \left(\zeta_A^S(x_i) + \vartheta_A^S(x_i) + \pi_A^S(x_i)\right)^{\frac{1}{S}} - \left(\zeta_A^R(x_i) + \vartheta_A^R(x_i) + \pi_A^R(x_i)\right)^{\frac{1}{R}} = 0 \text{ for all } i = 1, 2, \ldots, n.$$

$$i.e., \quad \left(\zeta_A^S(x_i) + \vartheta_A^S(x_i) + \pi_A^S(x_i)\right)^{\frac{1}{S}} = \left(\zeta_A^R(x_i) + \vartheta_A^R(x_i) + \pi_A^R(x_i)\right)^{\frac{1}{R}} \text{ for all } i = 1, 2, \ldots, n.$$

Since $R, S > 0$ and $R \neq S$, therefore, the above equation is satisfied only if $\zeta_A(x_i) = 0, \vartheta_A(x_i) = 1$ or $\zeta_A(x_i) = 1, \vartheta_A(x_i) = 0$ for all $i = 1, 2, \ldots, n$.

Conversely, we assume that set $A = (\zeta_A, \vartheta_A)$ is a crisp set i.e., either $\zeta_A(x_i) = 0$ or 1. Now, for $R, S > 0$ and $R \neq S$, we can obtain that:

$$\left(\zeta_A^S(x_i) + \vartheta_A^S(x_i) + \pi_A^S(x_i)\right)^{\frac{1}{S}} - \left(\zeta_A^R(x_i) + \vartheta_A^R(x_i) + \pi_A^R(x_i)\right)^{\frac{1}{R}} = 0$$

for all $i = 1, 2, \ldots, n$, which gives that $H_R^S(A) = 0$.
Hence, $H_R^S(A) = 0$ iff A is a crisp set.

(P2) Maximality: We will find maxima of the function $H_R^S(A)$; for this purpose, we will differentiate Equation (5.1) with respect to $\zeta_A(x_i)$ and $\vartheta_A(x_i)$. We get,

$$\frac{\partial H_R^S(A)}{\partial \zeta_A(x_i)} = \frac{R\times S}{n(R-S)} \sum_{i=1}^{n} \left\{ \begin{array}{c} \left(\zeta_A^S(x_i) + \vartheta_A^S(x_i) + \pi_A^S(x_i)\right)^{\frac{1-S}{S}} \left(\zeta_A^{S-1}(x_i) - \pi_A^{S-1}(x_i)\right) \\ -\left(\zeta_A^R(x_i) + \vartheta_A^R(x_i) + \pi_A^R(x_i)\right)^{\frac{1-R}{R}} \left(\zeta_A^{R-1}(x_i) - \pi_A^{R-1}(x_i)\right) \end{array} \right\} \tag{5.2}$$

and:

$$\frac{\partial H_R^S(A)}{\partial \vartheta_A(x_i)} = \frac{R\times S}{n(R-S)} \sum_{i=1}^{n} \left\{ \begin{array}{c} \left(\zeta_A^S(x_i) + \vartheta_A^S(x_i) + \pi_A^S(x_i)\right)^{\frac{1-S}{S}} \left(\vartheta_A^{S-1}(x_i) - \pi_A^{S-1}(x_i)\right) \\ -\left(\zeta_A^R(x_i) + \vartheta_A^R(x_i) + \pi_A^R(x_i)\right)^{\frac{1-R}{R}} \left(\vartheta_A^{R-1}(x_i) - \pi_A^{R-1}(x_i)\right) \end{array} \right\} \tag{5.3}$$

In order to check the convexity of the function, we calculate its second order derivatives as follows:

$$\frac{\partial^2 H_R^S(A)}{\partial^2 \zeta_A(x_i)} = \frac{R\times S}{n(R-S)} \sum_{i=1}^{n} \left\{ \begin{array}{l} (1-S)\left(\zeta_A^S(x_i) + \vartheta_A^S(x_i) + \pi_A^S(x_i)\right)^{\frac{1-2S}{S}} \left(\zeta_A^{S-1}(x_i) - \pi_A^{S-1}(x_i)\right)^2 \\ +(S-1)\left(\zeta_A^S(x_i) + \vartheta_A^S(x_i) + \pi_A^S(x_i)\right)^{\frac{1-S}{S}} \left(\zeta_A^{S-2}(x_i) + \pi_A^{S-2}(x_i)\right) \\ -(1-R)\left(\zeta_A^R(x_i) + \vartheta_A^R(x_i) + \pi_A^R(x_i)\right)^{\frac{1-2R}{R}} \left(\zeta_A^{R-1}(x_i) - \pi_A^{R-1}(x_i)\right)^2 \\ -(R-1)\left(\zeta_A^R(x_i) + \vartheta_A^R(x_i) + \pi_A^R(x_i)\right)^{\frac{1-R}{R}} \left(\zeta_A^{R-2}(x_i) + \pi_A^{R-2}(x_i)\right) \end{array} \right\};$$

$$\frac{\partial^2 H_R^S(A)}{\partial^2 \vartheta_A(x_i)} = \frac{R\times S}{n(R-S)} \sum_{i=1}^{n} \left\{ \begin{array}{l} (1-S)\left(\zeta_A^S(x_i) + \vartheta_A^S(x_i) + \pi_A^S(x_i)\right)^{\frac{1-2S}{S}} \left(\vartheta_A^{S-1}(x_i) - \pi_A^{S-1}(x_i)\right)^2 \\ +(S-1)\left(\zeta_A^S(x_i) + \vartheta_A^S(x_i) + \pi_A^S(x_i)\right)^{\frac{1-S}{S}} \left(\vartheta_A^{S-2}(x_i) + \pi_A^{S-2}(x_i)\right) \\ -(1-R)\left(\zeta_A^R(x_i) + \vartheta_A^R(x_i) + \pi_A^R(x_i)\right)^{\frac{1-2R}{R}} \left(\vartheta_A^{R-1}(x_i) - \pi_A^{R-1}(x_i)\right)^2 \\ -(R-1)\left(\zeta_A^R(x_i) + \vartheta_A^R(x_i) + \pi_A^R(x_i)\right)^{\frac{1-R}{R}} \left(\vartheta_A^{R-2}(x_i) + \pi_A^{R-2}(x_i)\right) \end{array} \right\};$$

and

$$\frac{\partial^2 H_R^S(A)}{\partial \vartheta_A(x_i)\partial \zeta_A(x_i)} = \frac{R\times S}{n(R-S)} \sum_{i=1}^{n} \left\{ \begin{array}{l} (1-S)\left(\zeta_A^S(x_i) + \vartheta_A^S(x_i) + \pi_A^S(x_i)\right)^{\frac{1-2S}{S}} \times \\ \quad \times \left(\vartheta_A^{S-1}(x_i) - \pi_A^{S-1}(x_i)\right)\left(\zeta_A^{S-1}(x_i) - \pi_A^{S-1}(x_i)\right) \\ -(1-R)\left(\zeta_A^R(x_i) + \vartheta_A^R(x_i) + \pi_A^R(x_i)\right)^{\frac{1-2R}{R}} \times \\ \quad \times \left(\vartheta_A^{R-1}(x_i) - \pi_A^{R-1}(x_i)\right)\left(\zeta_A^{R-1}(x_i) - \pi_A^{R-1}(x_i)\right) \end{array} \right\}$$

To find the maximum/minimum point, we set $\frac{\partial H_R^S(A)}{\partial \zeta_A(x_i)} = 0$ and $\frac{\partial H_R^S(A)}{\partial \vartheta_A(x_i)} = 0$, which gives that $\zeta_A(x_i) = \vartheta_A(x_i) = \pi_A(x_i) = \frac{1}{3}$ for all i and hence called the critical point of the function H_R^S.

(a) When $R < 1, S > 1$, then at the critical point $\zeta_A(x_i) = \vartheta_A(x_i) = \pi_A(x_i) = \frac{1}{3}$, we compute that:

$$\frac{\partial^2 H_R^S(A)}{\partial^2 \zeta_A(x_i)} < 0$$

$$\text{and} \qquad \frac{\partial^2 H_R^S(A)}{\partial^2 \zeta_A(x_i)} \cdot \frac{\partial^2 H_R^S(A)}{\partial^2 \vartheta_A(x_i)} - \left(\frac{\partial^2 H_R^S(A)}{\partial \vartheta_A(x_i) \partial \zeta_A(x_i)}\right)^2 > 0$$

Therefore, the Hessian matrix of $H_R^S(A)$ is negative semi-definite, and hence, $H_R^S(A)$ is a concave function. As the critical point of H_R^S is $\zeta_A = \vartheta_A = \frac{1}{3}$ and by the concavity, we get that $H_R^S(A)$ has a relative maximum value at $\zeta_A = \vartheta_A = \frac{1}{3}$.

(b) When $R > 1, S < 1$, then at the critical point, we can again easily obtain that:

$$\frac{\partial^2 H_R^S(A)}{\partial^2 \zeta_A(x_i)} < 0$$

$$\text{and} \qquad \frac{\partial^2 H_R^S(A)}{\partial^2 \zeta_A(x_i)} \cdot \frac{\partial^2 H_R^S(A)}{\partial^2 \vartheta_A(x_i)} - \left(\frac{\partial^2 H_R^S(A)}{\partial \vartheta_A(x_i) \zeta_A(x_i)}\right)^2 > 0$$

This proves that $H_R^S(A)$ is a concave function and its global maximum at $\zeta_A(x_i) = \vartheta_A(x_i) = \frac{1}{3}$.

Thus, for all $R, S > 0; R < 1, S < 1$ or $R > 1, S < 1$, the global maximum value of $H_R^S(A)$ attains at the point $\zeta_A(x_i) = \vartheta_A(x_i) = \frac{1}{3}$, i.e., $H_R^S(A)$ is maximum if and only if A is the most fuzzy set.

(P3) Resolution: Since $H_R^S(A)$ is a concave function on the IFS A, therefore, if $\max\{\zeta_A(x), \vartheta_A(x)\} \leq \frac{1}{3}$, then $\zeta_A(x_i) \leq \zeta_(x_i)$ and $\vartheta_A(x_i) \leq \vartheta_B(x_i)$, which implies that:

$$\zeta_A(x_i) \leq \zeta_B(x_i) \leq \frac{1}{3}; \vartheta_A(x_i) \leq \vartheta_B(x_i) \leq \frac{1}{3}; \pi_A(x_i) \geq \pi_B(x_i) \geq \frac{1}{3}$$

Thus, we observe that $(\zeta_B(x_i), \vartheta_B(x_i), \pi_B(x_i))$ is more around $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ than $(\zeta_A(x_i), \vartheta_A(x_i), \pi_A(x_i))$. Hence, $H_R^S(A) \leq H_R^B(B)$. Similarly, if $\min\{\zeta_A(x_i), \vartheta_A(x_i)\} \geq \frac{1}{3}$, then we get $H_R^S(A) \leq H_R^B(B)$.

(P4) Symmetry: By the definition of $H_R^S(A)$, we can easily obtain that $H_R^S(A^c) = H_R^S(A)$.

Hence $H_R^S(A)$ satisfies all the properties of the intuitionistic fuzzy information measure and, therefore, is a valid measure of intuitionistic fuzzy entropy. □

Consider two IFSs A and B defined over $X = \{x_1, x_2, \ldots, x_n\}$. Take the disjoint partition of X as:

$$\begin{aligned} X_1 &= \{x_i \in X \mid A \subseteq B\}, \\ &= \{x_i \in X \mid \zeta_A(x) \leq \zeta_B(x); \vartheta_A(x) \geq \vartheta_B(x)\} \end{aligned}$$

and:

$$\begin{aligned} X_2 &= \{x_i \in X \mid A \supseteq B\} \\ &= \{x_i \in X \mid \zeta_A(x) \geq \zeta_B(x); \vartheta_A(x) \leq \vartheta_B(x)\} \end{aligned}$$

Next, we define the joint and conditional entropies between IFSs A and B as follows:

1. Joint entropy:

$$\begin{aligned} & H_R^S(A \cup B) \\ = & \frac{R \times S}{n(R-S)} \sum_{i=1}^{n} \left[\begin{array}{l} \left(\zeta_{A\cup B}^S(x_i) + \vartheta_{A\cup B}^S(x_i) + (1 - \zeta_{A\cup B}(x_i) - \vartheta_{A\cup B}(x_i))^S\right)^{\frac{1}{S}} \\ - \left(\zeta_{A\cup B}^R(x_i) + \vartheta_{A\cup B}^R(x_i) + (1 - \zeta_{A\cup B}(x_i) - \vartheta_{A\cup B}(x_i))^R\right)^{\frac{1}{R}} \end{array} \right] \\ = & \frac{R \times S}{n(R-S)} \sum_{x_i \in X_1} \left[\begin{array}{r} \left(\zeta_B^S(x_i) + \vartheta_B^S(x_i) + (1 - \zeta_B(x_i) - \vartheta_B(x_i))^S\right)^{\frac{1}{S}} \\ - \left(\zeta_B^R(x_i) + \vartheta_B^R(x_i) + (1 - \zeta_B(x_i) - \vartheta_B(x_i))^R\right)^{\frac{1}{R}} \end{array} \right] \\ + & \frac{R \times S}{n(R-S)} \sum_{x_i \in X_2} \left[\begin{array}{r} \left(\zeta_A^S(x_i) + \vartheta_A^S(x_i) + (1 - \zeta_A(x_i) - \vartheta_A(x_i))^S\right)^{\frac{1}{S}} \\ - \left(\zeta_A^R(x_i) + \vartheta_A^R(x_i) + (1 - \zeta_A(x_i) - \vartheta_A(x_i))^R\right)^{\frac{1}{R}} \end{array} \right] \end{aligned}$$

2. Conditional entropy:

$$\begin{aligned} & H_R^S(A|B) \\ = & \tfrac{R\times S}{n(R-S)} \sum_{x_i \in X_2} \left[\begin{array}{l} \left(\zeta_A^S(x_i) + \vartheta_A^S(x_i) + \pi_A^S(x_i)\right)^{\frac{1}{S}} - \left(\zeta_A^R(x_i) + \vartheta_A^R(x_i) + \pi_A^R(x_i)\right)^{\frac{1}{R}} \\ - \left(\zeta_B^S(x_i) + \vartheta_B^S(x_i) + \pi_B^S(x_i)\right)^{\frac{1}{S}} + \left(\zeta_B^R(x_i) + \vartheta_B^R(x_i) + \pi_B^R(x_i)\right)^{\frac{1}{R}} \end{array} \right] \end{aligned}$$

and:

$$\begin{aligned} & H_R^S(B|A) \\ = & \tfrac{R\times S}{n(R-S)} \sum_{x_i \in X_1} \left[\begin{array}{l} \left(\zeta_B^S(x_i) + \vartheta_B^S(x_i) + \pi_B^S(x_i)\right)^{\frac{1}{S}} - \left(\zeta_B^R(x_i) + \vartheta_B^R(x_i) + \pi_B^R(x_i)\right)^{\frac{1}{R}} \\ - \left(\zeta_A^S(x_i) + \vartheta_A^S(x_i) + \pi_A^S(x_i)\right)^{\frac{1}{S}} + \left(\zeta_A^R(x_i) + \vartheta_A^R(x_i) + \pi_A^R(x_i)\right)^{\frac{1}{R}} \end{array} \right] \end{aligned}$$

Theorem 5.1.2. *Let A and B be the two IFSs defined on universal set $X = \{x_1, x_2, \ldots, x_n\}$, where, $A = \{\langle x_i, \zeta_A(x_i), \vartheta_A(x_i)\rangle \mid x_i \in X\}$ and $B = \{\langle x_i, \zeta_B(x_i), \vartheta_B(x_i)\rangle \mid x_i \in X\}$, such that either $A \subseteq B$ or $A \supseteq B$ $\forall$ $x_i \in X$, then:*

$$H_R^S(A \cup B) + H_R^S(A \cap B) = H_R^S(A) + H_R^S(B)$$

Proof. Let X_1 and X_2 be the two disjoint sets of X, where,

$$X_1 = \{x \in X : A \subseteq B\}, \quad X_2 = \{x \in X : A \supseteq B\}$$

i.e., for $x_i \in X_1$, we have $\zeta_A(x_i) \leq \zeta_B(x_i), \vartheta_A(x_i) \geq \vartheta_B(x_i)$ and $x_i \in X_2$, implying that $\zeta_A(x_i) \geq \zeta_B(x_i), \vartheta_A(x_i) \leq \vartheta_B(x_i)$. Therefore,

$$\begin{aligned}
& H_R^S(A \cup B) + H_R^S(A \cap B) \\
= & \frac{R \times S}{n(R-S)} \sum_{i=1}^{n} \left[\begin{array}{l} \left(\zeta_{A\cup B}^S(x_i) + \vartheta_{A\cup B}^S(x_i) + (1 - \zeta_{A\cup B}(x_i) - \vartheta_{A\cup B}(x_i))^S \right)^{\frac{1}{S}} \\ - \left(\zeta_{A\cup B}^R(x_i) + \vartheta_{A\cup B}^R(x_i) + (1 - \zeta_{A\cup B}(x_i) - \vartheta_{A\cup B}(x_i))^R \right)^{\frac{1}{R}} \end{array} \right] \\
+ & \frac{R \times S}{n(R-S)} \sum_{i=1}^{n} \left[\begin{array}{l} \left(\zeta_{A\cap B}^S(x_i) + \vartheta_{A\cap B}^S(x_i) + (1 - \zeta_{A\cap B}(x_i) - \vartheta_{A\cap B}(x_i))^S \right)^{\frac{1}{S}} \\ - \left(\zeta_{A\cap B}^R(x_i) + \vartheta_{A\cap B}^R(x_i) + (1 - \zeta_{A\cap B}(x_i) - \vartheta_{A\cap B}(x_i))^R \right)^{\frac{1}{R}} \end{array} \right] \\
= & \frac{R \times S}{n(R-S)} \sum_{x_i \in X_1} \left[\left(\zeta_B^S(x_i) + \vartheta_B^S(x_i) + \pi_B^S(x_i) \right)^{\frac{1}{S}} - \left(\zeta_B^R(x_i) + \vartheta_B^R(x_i) + \pi_B^R(x_i) \right)^{\frac{1}{R}} \right] \\
+ & \frac{R \times S}{n(R-S)} \sum_{x_i \in X_2} \left[\left(\zeta_A^S(x_i) + \vartheta_A^S(x_i) + \pi_A^S(x_i) \right)^{\frac{1}{S}} - \left(\zeta_A^R(x_i) + \vartheta_A^R(x_i) + \pi_A^R(x_i) \right)^{\frac{1}{R}} \right] \\
+ & \frac{R \times S}{n(R-S)} \sum_{x_i \in X_1} \left[\left(\zeta_A^S(x_i) + \vartheta_A^S(x_i) + \pi_A^S(x_i) \right)^{\frac{1}{S}} - \left(\zeta_A^R(x_i) + \vartheta_A^R(x_i) + \pi_A^R(x_i) \right)^{\frac{1}{R}} \right] \\
+ & \frac{R \times S}{n(R-S)} \sum_{x_i \in X_2} \left[\left(\zeta_B^S(x_i) + \vartheta_B^S(x_i) + \pi_B^S(x_i) \right)^{\frac{1}{S}} - \left(\zeta_B(x_i)^R + \vartheta_B(x_i)^R + \pi_B^R(x_i) \right)^{\frac{1}{R}} \right] \\
= & H_R^S(A) + H_R^S(B)
\end{aligned}$$

□

Theorem 5.1.3. *The maximum and minimum values of the entropy $H_R^S(A)$ are independent of the parameters R and S.*

Proof. As from the above theorem, we conclude that the entropy is maximum if and only if A is the most IFS and minimum when A is a crisp set. Therefore, it is enough to show that the value of $H_R^S(A)$ in these conditions is independent of R and S. When A is the most IFS, i.e., $\zeta_A(x_i) = \vartheta_A(x_i)$, for all $x_i \in X$, then $H_R^S(A) = 1$, and when A is a crisp set, i.e., either $\zeta_A(x_i) = 0$, $\vartheta_A(x_i) = 1$ or $\zeta_A(x_i) = 1, \vartheta_A(x_i) = 0$ for all $x_i \in X$, then $H_R^S(A) = 0$. Hence, in both cases, $H_R^S(A)$ is independent of the parameters R and S. □

Remark 5.1.1. *From the proposed measure, it is observed that some of the existing measures can be obtained from it by assigning particular cases to R and S. For instance,*

(i) When $\pi_A(x_i) = 0$ for all $x_i \in X$, then the proposed measures reduce to the entropy measure of Joshi and Kumar [27].

(ii) When $R = S$ and $S > 0$, then the proposed measures are reduced by the measure of Taneja [40].

(iii) *When $R = 1$ and $R \neq S$, then the measure is equivalent to the R-norm entropy presented by Boekee and Van der Lubbe [6].*

(iv) *When $R = S = 1$, then the proposed measure is the well-known Shannon's entropy.*

(v) *When $S = 1$ and $R \neq S$, then the proposed measure becomes the measure of Bajaj et al. [5].*

Theorem 5.1.4. *Let A and B be two IFSs defined over the set X such that either $A \subseteq B$ or $B \subseteq A$, then the following statements hold:*

(P1) $H_R^S(A \cup B) = H_R^S(A) + H_R^S(B|A)$;

(P2) $H_R^S(A \cup B) = H_R^S(B) + H_R^S(A|B)$;

(P3) $H_R^S(A \cup B) = H_R^S(A) + H_R^S(B|A) = H_R^S(B) + H_R^S(A|B)$.

Proof. For two IFSs A and B and by using the definitions of joint, conditional and the proposed entropy measures, we get:

(P1) Consider:

$$
\begin{aligned}
& H_R^S(A \cup B) - H_R^S(A) - H_R^S(B|A) \\
= & \ \tfrac{R\times S}{n(R-S)} \sum_{i=1}^{n} \left[\begin{array}{l} \left(\zeta_{A\cup B}^S(x_i) + \vartheta_{A\cup B}^S(x_i) + (1-\zeta_{A\cup B}(x_i) - \vartheta_{A\cup B}(x_i))^S\right)^{\frac{1}{S}} \\ - \left(\zeta_{A\cup B}^R(x_i) + \vartheta_{A\cup B}^R(x_i) + (1-\zeta_{A\cup B}(x_i) - \vartheta_{A\cup B}(x_i))^R\right)^{\frac{1}{R}} \end{array} \right] \\
- & \ \tfrac{R\times S}{n(R-S)} \sum_{i=1}^{n} \left[\left(\zeta_A^S(x_i) + \vartheta_A^S(x_i) + \pi_A^S(x_i)\right)^{\frac{1}{S}} - \left(\zeta_A^R(x_i) + \vartheta_A^R(x_i) + \pi_A^R(x_i)\right)^{\frac{1}{R}} \right] \\
- & \ \tfrac{R\times S}{n(R-S)} \sum_{x_i \in X_1} \left[\begin{array}{l} \left(\zeta_B^S(x_i) {}^{+} \vartheta_B^S(x_i) + \pi_B^S(x_i)\right)^{\frac{1}{S}} - \left(\zeta_B^R(x_i) + \vartheta_B^R(x_i) + \pi_B^R(x_i)\right)^{\frac{1}{R}} \\ - \left(\zeta_A^S(x_i) + \vartheta_A^S(x_i) + \pi_A^S(x_i)\right)^{\frac{1}{S}} + \left(\zeta_A^R(x_i) + \vartheta_A^R(x_i) + \pi_A^R(x_i)\right)^{\frac{1}{R}} \end{array} \right] \\
= & \ \tfrac{R\times S}{n(R-S)} \sum_{x_i \in X_1} \left[\left(\zeta_B^S(x_i) + \vartheta_B^S(x_i) + \pi_B^S(x_i)\right)^{\frac{1}{S}} - \left(\zeta_B^R(x_i) + \vartheta_B^R(x_i) + \pi_B^R(x_i)\right)^{\frac{1}{R}} \right] \\
+ & \ \tfrac{R\times S}{n(R-S)} \sum_{x_i \in X_2} \left[\left(\zeta_A^S(x_i) + \vartheta_A^S(x_i) + \pi_A^S(x_i)\right)^{\frac{1}{S}} - \left(\zeta_A^R(x_i) + \vartheta_A^R(x_i) + \pi_A^R(x_i)\right)^{\frac{1}{R}} \right] \\
- & \ \tfrac{R\times S}{n(R-S)} \sum_{x_i \in X_1} \left[\left(\zeta_A^S(x_i) + \vartheta_A^S(x_i) + \pi_A^S(x_i)\right)^{\frac{1}{S}} - \left(\zeta_A^R(x_i) + \vartheta_A^R(x_i) + \pi_A^R(x_i)\right)^{\frac{1}{R}} \right] \\
+ & \ \tfrac{R\times S}{n(R-S)} \sum_{x_i \in X_2} \left[\left(\zeta_A^S(x_i) + \vartheta_A^S(x_i) + \pi_A^S(x_i)\right)^{\frac{1}{S}} - \left(\zeta_A^R(x_i) + \vartheta_A^R(x_i) + \pi_A^R(x_i)\right)^{\frac{1}{R}} \right] \\
- & \ \tfrac{R\times S}{n(R-S)} \sum_{x_i \in X_1} \left[\begin{array}{l} \left(\zeta_B^S(x_i) {}^{+} \vartheta_B^S(x_i) + \pi_B^S(x_i)\right)^{\frac{1}{S}} - \left(\zeta_B^R(x_i) + \vartheta_B^R(x_i) + \pi_B^R(x_i)\right)^{\frac{1}{R}} \\ - \left(\zeta_A^S(x_i) + \vartheta_A^S(x_i) + \pi_A^S(x_i)\right)^{\frac{1}{S}} + \left(\zeta_A^R(x_i) + \vartheta_A^R(x_i) + \pi_A^R(x_i)\right)^{\frac{1}{R}} \end{array} \right] \\
= & \ 0
\end{aligned}
$$

(P2) Consider:

$$\begin{aligned}
&H_R^S(A\cup B)-H_R^S(B)-H_R^S(A|B)\\
=\ &\tfrac{R\times S}{n(R-S)}\sum_{i=1}^{n}\left[\begin{array}{l}\left(\zeta_{A\cup B}^S(x_i)+\vartheta_{A\cup B}^S(x_i)+(1-\zeta_{A\cup B}(x_i)-\vartheta_{A\cup B}(x_i))^S\right)^{\frac{1}{S}}\\ -\left(\zeta_{A\cup B}^R(x_i)+\vartheta_{A\cup B}^R(x_i)+(1-\zeta_{A\cup B}(x_i)-\vartheta_{A\cup B}(x_i))^R\right)^{\frac{1}{R}}\end{array}\right]\\
-\ &\tfrac{R\times S}{n(R-S)}\sum_{i=1}^{n}\left[\left(\zeta_B^S(x_i)+\vartheta_B^S(x_i)+\pi_B^S(x_i)\right)^{\frac{1}{S}}-\left(\zeta_B^R(x_i)+\vartheta_B^R(x_i)+\pi_B^R(x_i)\right)^{\frac{1}{R}}\right]\\
-\ &\tfrac{R\times S}{n(R-S)}\sum_{x_i\in X_2}\left[\begin{array}{l}\left(\zeta_A^S(x_i)^+\vartheta_A^S(x_i)+\pi_A^S(x_i)\right)^{\frac{1}{S}}-\left(\zeta_A^R(x_i)+\vartheta_A^R(x_i)+\pi_A^R(x_i)\right)^{\frac{1}{R}}\\ -\left(\zeta_B^S(x_i)+\vartheta_B^S(x_i)+\pi_B^S(x_i)\right)^{\frac{1}{S}}+\left(\zeta_B^R(x_i)+\vartheta_B^R(x_i)+\pi_B^R(x_i)\right)^{\frac{1}{R}}\end{array}\right]\\
=\ &\tfrac{R\times S}{n(R-S)}\sum_{x_i\in X_1}\left[\left(\zeta_B^S(x_i)+\vartheta_B^S(x_i)+\pi_B^S(x_i)\right)^{\frac{1}{S}}-\left(\zeta_B^R(x_i)+\vartheta_B^R(x_i)+\pi_B^R(x_i)\right)^{\frac{1}{R}}\right]\\
+\ &\tfrac{R\times S}{n(R-S)}\sum_{x_i\in X_2}\left[\left(\zeta_A^S(x_i)+\vartheta_A^S(x_i)+\pi_A^S(x_i)\right)^{\frac{1}{S}}-\left(\zeta_A^R(x_i)+\vartheta_A^R(x_i)+\pi_A^R(x_i)\right)^{\frac{1}{R}}\right]\\
-\ &\tfrac{R\times S}{n(R-S)}\sum_{x_i\in X_1}\left[\left(\zeta_B^S(x_i)+\vartheta_B^S(x_i)+\pi_B^S(x_i)\right)^{\frac{1}{S}}-\left(\zeta_B^R(x_i)+\vartheta_B^R(x_i)+\pi_B^R(x_i)\right)^{\frac{1}{R}}\right]\\
+\ &\tfrac{R\times S}{n(R-S)}\sum_{x_i\in X_2}\left[\left(\zeta_B^S(x_i)+\vartheta_B^S(x_i)+\pi_B^S(x_i)\right)^{\frac{1}{S}}-\left(\zeta_B^R(x_i)+\vartheta_B^R(x_i)+\pi_B^R(x_i)\right)^{\frac{1}{R}}\right]\\
-\ &\tfrac{R\times S}{n(R-S)}\sum_{x_i\in X_2}\left[\begin{array}{l}\left(\zeta_A^S(x_i)+\vartheta_A^S(x_i)+\pi_A^S(x_i)\right)^{\frac{1}{S}}-\left(\zeta_A^R(x_i)+\vartheta_A^R(x_i)+\pi_A^R(x_i)\right)^{\frac{1}{R}}\\ \left(\zeta_B^S(x_i)+\vartheta_B^S(x_i)+\pi_B^S(x_i)\right)^{\frac{1}{S}}-\left(\zeta_B^R(x_i)+\vartheta_B^R(x_i)+\pi_B^R(x_i)\right)^{\frac{1}{R}}\end{array}\right]\\
=\ &0
\end{aligned}$$

(P3) This can be deduced from Parts (1) and (2).

□

Before elaborating on the comparison between the proposed entropy function and other entropy functions, we state a definition [10] for an IFS of the form $A = \langle x, \zeta_A(x_i), \vartheta_A(x_i) \mid x \in X\rangle$ defined on universal set X, which is as follows:

$$A^n = \{\langle x, [\zeta_A(x_i)]^n, 1-[1-\vartheta_A(x_i)]^n\rangle \mid x \in X\} \tag{5.4}$$

Definition 5.1.2. *The concentration of an IFS A of the universe X is denoted by $CON(A)$ and is defined by:*

$$CON(A) = \{\langle x, \zeta_{CON(A)}(x), \vartheta_{CON(A)}(x)\rangle \mid x \in X\}$$

where $\zeta_{(CON(A))}(x) = [\zeta_A(x)]^2$, $\vartheta_{CON(A)}(x)) = 1-[1-\vartheta_A(x)]^2$, i.e., the operation of the concentration of an IFS is defined by $CON(A) = A^2$.

Definition 5.1.3. *The dilation of an IFS A of the universe X is denoted by $DIL(A)$ and is defined by:*

$$DIL(A) = \{\langle x, \zeta_{DIL(A)}(x), \vartheta_{DIL(A)}(x)\rangle \mid x \in X\}$$

where $\zeta_{DIL(A)}(x) = [\zeta_A(x)]^{1/2}$ *and* $\vartheta_{DIL(A)}(x) = 1 - [1 - \vartheta_A(x)]^{1/2}$, *i.e., the operation of the dilation of an IFS is defined by* $DIL(A) = A^{1/2}$

Example 5.1.1. *Consider a universe of the discourse* $X = \{x_1, x_2, x_3, x_4, x_5\}$, *and an IFS A "LARGE" of X may be defined by:*

$$LARGE = \{(x_1, 0.1, 0.8), (x_2, 0.3, 0.5), (x_3, 0.5, 0.4), (x_4, 0.9, 0), (x_5, 1, 0)\}$$

Using the operations as defined in Eq. (5.4), we have generated the following IFSs

$$A^{1/2}, A^2, A^3, A^4$$

, which are defined as follows:

$A^{1/2}$ may be treated as "More or less LARGE"

A^2 may be treated as "very LARGE"

A^3 may be treated as "quite very LARGE"

A^4 may be treated as "very very LARGE"

and their corresponding sets are computed as:

$$\begin{aligned}
A^{\frac{1}{2}} &= \{(x_1, 0.3162, 0.5528), (x_2, 0.5477, 0.2929), (x_3, 0.7071, 0.2254), (x_4, 0.9487, 0), (x_5, 1, 0)\} \\
A^2 &= \{(x_1, 0.01, 0.96), (x_2, 0.09, 0.75), (x_3, 0.25, 0.64), (x_4, 0.81, 0), (x_5, 1, 0)\} \\
A^3 &= \{(x_1, 0.001, 0.9920), (x_2, 0.0270, 0.8750), (x_3, 0.1250, 0.7840), (x_4, 0.7290, 0), (x_5, 1, 0)\} \\
A^4 &= \{(x_1, 0.0001, 0.9984), (x_2, 0.0081, 0.9375), (x_3, 0.0625, 0.8704), (x_4, 0.6561, 0), (x_5, 1, 0)\}
\end{aligned}$$

From the viewpoint of mathematical operations, the entropy values of the above defined IFSs, $A^{1/2}$, A, A^2, A^3 and A^4, have the following requirement:

$$E(A^{1/2}) > E(A) > E(A^2) > E(A^3) > E(A^4) \tag{5.5}$$

Based on the dataset given in the above, we compute the entropy measure for them at different values of R and S. The result corresponding to these different pairs of values is summarized in Table 5.1 along with the existing approaches' results. From these computed values, it is observed that the ranking order of the linguistic variable by the proposed entropy follows the pattern as described in Eq. (5.5) for some suitable pairs of (R, S), while the performance order pattern corresponding to [7, 39, 50] and [24] is $E(A) > E(A^{1/2}) > E(A^2) > E(A^3) > E(A^4)$, which does not satisfy the requirement given in Eq. (5.5). Hence, the proposed entropy measure is a good alternative and performs better than the existing measures. Furthermore, for different pairs of (R, S), a decision-maker may have more choices to access the alternatives from the viewpoint of structured linguistic variables.

Table 5.1: Entropy measures values corresponding to existing approaches, as well as the proposed approach.

Entropy Measure	$A^{\frac{1}{2}}$	A	A^2	A^3	A^4
$E_{\{BB\}}$ [7]	0.0818	0.1000	0.0980	0.0934	0.0934
$E_{\{SK\}}$ [39]	0.3446	0.3740	0.1970	0.1309	0.1094
$E_{\{ZL\}}$ [50]	0.4156	0.4200	0.2380	0.1546	0.1217
$E_{\{HY\}}$ [24]	0.3416	0.3440	0.2610	0.1993	0.1613
$E_{\{ZJ\}}$ [51]	0.2851	0.3050	0.1042	0.0383	0.0161
$E_{0.4}^{0.2}$ [17]	0.5995	0.5981	0.5335	0.4631	0.4039
H_R^S (proposed measure)					
$R = 0.3, S = 2$	2.3615	2.3589	1.8624	1.4312	1.1246
$R = 0.5, S = 2$	0.8723	0.8783	0.6945	0.5392	0.4323
$R = 0.7, S = 2$	0.5721	0.5769	0.4432	0.3390	0.2725
$R = 2.5, S = 0.3$	2.2882	2.2858	1.8028	1.3851	1.0890
$R = 2.5, S = 0.5$	0.8309	0.8368	0.6583	0.5104	0.4103
$R = 2.5, S = 0.7$	0.5369	0.5415	0.4113	0.3138	0.2538

5.2 MADM Problem Based on the Proposed Entropy Measure

In this section, we present a method for solving the MADM problem based on the proposed entropy measure. The description of the MADM is given in Section 3.3.1 of the Chapter 3. In the following, we present two approaches for solving MADM problem.

5.2.1 Approach I: When the Attribute Weight Is Completely Unknown

The steps of the proposed approach for MADM problems are summarized as follows:

Step 1: Collect the information and summarized in IF decision-matrix D as follows:

$$D = \begin{array}{c} \\ A_1 \\ A_2 \\ \vdots \\ A_n \end{array} \begin{array}{c} \begin{array}{cccc} Q_1 & Q_2 & \dots & Q_m \end{array} \\ \begin{pmatrix} \langle \zeta_{11}, \vartheta_{11} \rangle & \langle \zeta_{12}, \vartheta_{12} \rangle & \dots & \langle \zeta_{1m}, \vartheta_{1m} \rangle \\ \langle \zeta_{21}, \vartheta_{21} \rangle & \langle \zeta_{22}, \vartheta_{22} \rangle & \dots & \langle \zeta_{2m}, \vartheta_{2m} \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle \zeta_{n1}, \vartheta_{n1} \rangle & \langle \zeta_{n2}, \vartheta_{n2} \rangle & \dots & \langle \zeta_{nm}, \vartheta_{nm} \rangle \end{pmatrix} \end{array} \tag{5.6}$$

Step 2: Normalize the rating values, if required by using Eq. (4.7) as described in Chapter 4 and get matrix R.

Step 3: Based on the matrix R, the information entropy of attribute $Q_j (j = 1, 2, \dots, m)$ is computed as:

$$(H_R^S)_j = \frac{R \times S}{n(R - S)} \sum_{i=1}^{n} \left[\left(\zeta_{ij}^S + \vartheta_{ij}^S + \pi_{ij}^S \right)^{\frac{1}{S}} - \left(\zeta_{ij}^R + \vartheta_{ij}^R + \pi_{ij}^R \right)^{\frac{1}{R}} \right] \tag{5.7}$$

where $R, S > 0$ and $R \neq S$.

Step 4: Compute the attributes weight $\omega_j (j = 1, 2, \dots, n)$ by

$$\omega_j = \frac{1 - \kappa_j}{\sum_{j=1}^{m} (1 - \kappa_j)} = \frac{1 - \kappa_j}{m - \sum_{j=1}^{m} \kappa_j} \tag{5.8}$$

where $\kappa_j = \sum_{i=1}^{n} H_R^S(r_{ij}), j = 1, 2, \dots, m.$

Step 5: Compute the score values of each alternative by multiplying the score function of each criterion by its assigned weight as:

$$Q(A_i) = \sum_{j=1}^{m} \omega_j (\zeta_{ij} - \vartheta_{ij}); \quad i = 1, 2, \dots, n \tag{5.9}$$

Step 6: Rank all the alternatives $A_i (i = 1, 2, \dots, n)$ according to the highest value of $Q(A_i)$ and, hence, choose the best alternative.

The above-mentioned approach has been illustrated with a practical example of the decision-maker, which can be read as:

Example 5.2.1. *Consider a decision-making problem from the field of the recruitment sector. Assume that a pharmaceutical company wants to select a lab technician for a micro-bio laboratory. For this, the company has published a notification in a newspaper and considered the four attributes required for technician selection, namely academic record* (Q_1)*, personal interview evaluation* (Q_2)*, experience* (Q_3) *and technical capability* (Q_4)*. On the basis of the notification conditions, only five candidates* A_1, A_2, A_3, A_4 *and* A_5 *as alternatives are interested and selected to be presented to the panel of experts for this post. Then, the main object of the company is to choose the best candidate among them for the task. In order to describe the ambiguity and uncertainties in the data, the preferences related to each alternative are represented in the IFS environment. The preferences of each alternative are represented in the form of IFNs as follows:*

$$D = \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \end{array} \begin{array}{c} \begin{array}{cccc} Q_1 & Q_2 & Q_3 & Q_4 \end{array} \\ \begin{pmatrix} \langle 0.7, 0.2\rangle & \langle 0.5, 0.4\rangle & \langle 0.6, 0.2\rangle & \langle 0.6, 0.3\rangle \\ \langle 0.7, 0.1\rangle & \langle 0.5, 0.2\rangle & \langle 0.7, 0.2\rangle & \langle 0.4, 0.5\rangle \\ \langle 0.6, 0.3\rangle & \langle 0.5, 0.1\rangle & \langle 0.5, 0.3\rangle & \langle 0.6, 0.2\rangle \\ \langle 0.8, 0.1\rangle & \langle 0.6, 0.3\rangle & \langle 0.3, 0.7\rangle & \langle 0.6, 0.3\rangle \\ \langle 0.6, 0.3\rangle & \langle 0.4, 0.6\rangle & \langle 0.7, 0.2\rangle & \langle 0.5, 0.4\rangle \end{pmatrix} \end{array} \tag{5.10}$$

Then, the steps of the proposed approach are followed to find the best alternative(s) as below:

Step 1: *Since all the attributes are of the same type, so there is no need for the normalization process.*

Step 2: *Without loss of generality, we take* $R = 0.3$ *and* $S = 2$ *and, hence, compute the entropy measurement value for each attribute by using Eq.* (5.7)*. The results corresponding to it are* $H_R^S(Q_1) = 3.4064$, $H_R^S(Q_2) = 3.372$, $H_R^S(Q_3) = 3.2491$ *and* $H_R^S(Q_4) = 3.7564$.

Step 3: *Based on these entropy values, the weight of each criterion is calculated as* $\omega = (0.2459,\ 0.2425,\ 0.2298,\ 0.2817)^T$.

Step 4: *The overall weighted score values of the alternative corresponding to* $R = 0.3$, $S = 2$ *and* $\omega = (0.2459, 0.2425, 0.2298, 0.2817)^T$ *obtained by using Eq.* (5.9) *are* $Q(A_1) = 0.3237$, $Q(A_2) = 0.3071$, $Q(A_3) = 0.3294$, $Q(A_4) = 0.2375$ *and* $Q(A_5) = 0.1684$.

Step 5: *Since* $Q(A_3) > Q(A_1) > Q(A_2) > Q(A_4) > Q(A_5)$*, hence the ranking order of the alternatives is* $A_3 \succ A_1 \succ A_2 \succ A_4 \succ A_5$*. Thus, the best alternative is* A_3.

However, in order to analyze the influence of the parameters R *and* S *on the final ranking order of the alternatives, the steps of the proposed approach are executed by varying the values of* R *from 0.1 to 1.0 and* S *from 1.0 to 5.0. The overall score values of each alternative along with the ranking order are summarized in Table 5.2. From this analysis, we conclude that the decision-maker can plan to choose the values of* R *and* S *and, hence, their respective alternatives according to his goal. Therefore, the proposed measures give various choices to the decision-maker to reach the target.*

Table 5.2: Effect of R and S on the entropy measure H_R^S by using Approach I.

S	R	$H_R^S(A_1)$	$H_R^S(A_2)$	$H_R^S(A_3)$	$H_R^S(A_4)$	$H_R^S(A_5)$	Ranking Order
	0.1	0.3268	0.3084	0.3291	0.2429	0.1715	$A_3 \succ A_1 \succ A_2 \succ A_4 \succ A_5$
	0.3	0.3241	0.3081	0.3292	0.2374	0.1690	$A_3 \succ A_1 \succ A_2 \succ A_4 \succ A_5$
1.2	0.5	0.3165	0.2894	0.3337	0.2368	0.1570	$A_3 \succ A_1 \succ A_2 \succ A_4 \succ A_5$
	0.7	0.1688	-0.0988	0.4296	0.2506	-0.0879	$A_3 \succ A_4 \succ A_1 \succ A_5 \succ A_2$
	0.9	0.3589	0.3992	0.3065	0.2328	0.2272	$A_2 \succ A_1 \succ A_3 \succ A_4 \succ A_5$
	0.1	0.3268	0.3084	0.3291	0.2429	0.1715	$A_3 \succ A_1 \succ A_2 \succ A_4 \succ A_5$
	0.3	0.3239	0.3076	0.3293	0.2374	0.1688	$A_3 \succ A_1 \succ A_2 \succ A_4 \succ A_5$
1.5	0.5	0.3132	0.2811	0.3359	0.2371	0.1515	$A_3 \succ A_1 \succ A_2 \succ A_4 \succ A_5$
	0.7	0.4139	0.5404	0.2712	0.2272	0.3185	$A_2 \succ A_1 \succ A_5 \succ A_3 \succ A_2$
	0.9	0.3498	0.3741	0.3125	0.2334	0.2121	$A_2 \succ A_1 \succ A_3 \succ A_4 \succ A_5$
	0.1	0.3268	0.3084	0.3291	0.2429	0.1715	$A_3 \succ A_1 \succ A_2 \succ A_4 \succ A_5$
	0.3	0.3237	0.3071	0.3294	0.2375	0.1684	$A_3 \succ A_1 \succ A_2 \succ A_4 \succ A_5$
2.0	0.5	0.3072	0.2666	0.3396	0.2381	0.1415	$A_3 \succ A_1 \succ A_2 \succ A_4 \succ A_5$
	0.7	0.3660	0.4140	0.3022	0.2308	0.2393	$A_2 \succ A_1 \succ A_3 \succ A_5 \succ A_4$
	0.9	0.3461	0.3631	0.3150	0.2331	0.2062	$A_2 \succ A_1 \succ A_3 \succ A_4 \succ A_5$
	0.1	0.3268	0.3084	0.3291	0.2429	0.1715	$A_3 \succ A_1 \succ A_2 \succ A_4 \succ A_5$
	0.3	0.3235	0.3067	0.3295	0.2376	0.1681	$A_3 \succ A_1 \succ A_2 \succ A_4 \succ A_5$
2.5	0.5	0.3010	0.2517	0.3436	0.2396	0.1308	$A_3 \succ A_1 \succ A_2 \succ A_4 \succ A_5$
	0.7	0.3578	0.3920	0.3074	0.2304	0.2261	$A_2 \succ A_1 \succ A_3 \succ A_4 \succ A_5$
	0.9	0.3449	0.3591	0.3158	0.2322	0.2045	$A_2 \succ A_1 \succ A_3 \succ A_4 \succ A_5$
	0.1	0.3268	0.3084	0.3291	0.2429	0.1715	$A_3 \succ A_1 \succ A_2 \succ A_4 \succ A_5$
	0.3	0.3234	0.3064	0.3296	0.2376	0.1678	$A_3 \succ A_1 \succ A_2 \succ A_4 \succ A_5$
3.0	0.5	0.2946	0.2368	0.3476	0.2417	0.1199	$A_3 \succ A_1 \succ A_4 \succ A_2 \succ A_5$
	0.7	0.3545	0.3829	0.3095	0.2298	0.2209	$A_2 \succ A_1 \succ A_3 \succ A_4 \succ A_5$
	0.9	0.3442	0.3570	0.3161	0.2314	0.2037	$A_2 \succ A_1 \succ A_3 \succ A_4 \succ A_5$
	0.1	0.3268	0.3084	0.3291	0.2429	0.1715	$A_3 \succ A_1 \succ A_2 \succ A_4 \succ A_5$
	0.3	0.3231	0.3058	0.3298	0.2379	0.1674	$A_3 \succ A_1 \succ A_2 \succ A_4 \succ A_5$
5.0	0.5	0.2701	0.1778	0.3638	0.2520	0.0767	$A_3 \succ A_1 \succ A_4 \succ A_2 \succ A_5$
	0.7	0.3496	0.3706	0.3123	0.2277	0.2137	$A_2 \succ A_1 \succ A_3 \succ A_4 \succ A_5$
	0.9	0.3428	0.3532	0.3168	0.2293	0.2020	$A_2 \succ A_1 \succ A_3 \succ A_4 \succ A_5$

5.2.2 Approach II: When the Attribute Weight Is Partially Known

In this section, we assume that the attribute weight information as provided by the decision maker is partially known denoted by Δ whose relationship is summarized in Section 3.3.2.

The following steps of the proposed approach are summarized under this environment.

Step 1: Similar to Approach I.

Step 2: similar to Approach I.

Step 3: The overall entropy of the alternative $A_i (i = 1, 2, \ldots, n)$ for the attribute Q_j is given by:

$$\begin{aligned} H(A_i) &= \sum_{j=1}^{m} H_R^S(r_{ij}) \\ &= \frac{R \times S}{n(R-S)} \sum_{j=1}^{m} \left\{ \sum_{i=1}^{n} \left((\zeta_{ij}^S + \vartheta_{ij}^S + \pi_{ij}^S)^{\frac{1}{S}} - (\zeta_{ij}^R + \vartheta_{ij}^R + \pi_{ij}^R)^{\frac{1}{R}} \right) \right\} \end{aligned} \tag{5.11}$$

where $R, S > 0$ and $R \neq S$.

Formulate a linear programming model to determine the weight vector as follows:

$$\begin{aligned} \min H &= \sum_{i=1}^{n} \left\{ \sum_{j=1}^{m} \omega_j H_R^S(r_{ij}) \right\} \\ &= \frac{R \times S}{n(R-S)} \sum_{j=1}^{m} \omega_j \left\{ \sum_{i=1}^{n} \left((\zeta_{ij}^S + \vartheta_{ij}^S + \pi_{ij}^S)^{\frac{1}{S}} - (\zeta_{ij}^R + \vartheta_{ij}^R + \pi_{ij}^R)^{\frac{1}{R}} \right) \right\} \\ \text{s.t. } & \sum_{j=1}^{m} \omega_j = 1 \\ & \omega_j \geq 0; \omega \in \Delta \end{aligned}$$

After solving this model, we get the optimal weight vector $\omega = (\omega_1, \omega_2, \ldots, \omega_m)^T$.

Step 4: Construct the weighted sum of each alternative by multiplying the score function of each criterion by its assigned weight as:

$$Q(A_i) = \sum_{j=1}^{m} \omega_j (\zeta_{ij} - \vartheta_{ij}); \quad i = 1, 2, \ldots, n \tag{5.12}$$

Step 5: Rank all the alternative $A_i (i = 1, 2, \ldots, n)$ according to the highest value of $Q(A_i)$ and, hence, choose the best alternative.

To demonstrate the above-mentioned approach, a numerical example has been taken, which is stated as below.

Example 5.2.2. *Consider an MADM problem, which was stated and described in Example 5.2.1. Assume that the information about the attribute weight is partially known and is given by the decision-maker as* $\Delta = \{0.15 \leq \omega_1 \leq 0.45, 0.2 \leq \omega_2 \leq 0.5, 0.1 \leq \omega_3 \leq 0.3, 0.1 \leq \omega_4 \leq 0.2, \omega_1 \geq \omega_4, \sum_{j=1}^{4} \omega_j = 1\}$. *Then, based on the rating values as mentioned in Equation* (5.10)*, the following steps of the Approach II are executed as below:*

Step 1: Similar to Step 1 of Approach I.

Step 2: Similar to Step 2 of Approach I.

Step 3: Without loss of generality, we take $R = 0.3$ *and* $S = 2$ *and, hence, compute the entropy measurement value for each attribute by using Eq.* (5.11). *The results corresponding to it are* $H_R^S(G_1) = 3.4064$, $H_R^S(G_2) = 3.372$, $H_R^S(G_3) = 3.2491$ *and* $H_R^S(G_4) = 3.7564$.

Step 4: Formulate the optimization model as:

$$\begin{aligned} \min H &= 3.4064\omega_1 + 3.372\omega_2 + 3.2491\omega_3 + 3.7564\omega_4 \\ \textit{subject to} \quad & 0.15 \leq \omega_1 \leq 0.45, \\ & 0.2 \leq \omega_2 \leq 0.5, \\ & 0.1 \leq \omega_3 \leq 0.3, \\ & 0.1 \leq \omega_4 \leq 0.2, \\ & \omega_1 \geq \omega_4, \\ \textit{and} \quad & \omega_1 + \omega_2 + \omega_3 + \omega_4 = 1. \end{aligned}$$

and after solving with the help of MATLAB software, we can obtain the weight vector as $\omega = (0.15, 0.45, 0.30, 0.10)^T$.

Step 5: The overall weighted score values of the alternative corresponding to $R = 0.3$, $S = 2$ *and* $\omega = (0.15, 0.45, 0.30, 0.10)^T$ *obtained by using Eq.* (5.12) *are* $Q(A_1) = 0.2700$, $Q(A_2) = 0.3650$, $Q(A_3) = 0.3250$ *and* $Q(A_4) = 0.1500$ *and* $Q(A_5) = 0.1150$.

Step 6: Since $Q(A_2) > Q(A_3) > Q(A_1) > Q(A_4) > Q(A_5)$, *hence the ranking order of the alternatives is* $A_2 \succ A_3 \succ A_1 \succ A_4 \succ A_5$. *Thus, the best alternative is* A_2.

5.3 Conclusion

In this chapter, we addressed a novel (R, S)-norm-based information measure in the IFS environment to measure the degree of fuzziness of a set. The prominent characteristics of the measures are studied in details. From this measure, it is computed that some of the existing measures are taken as a special case of it. In addition to these and to explore the structural characteristics and functioning of the proposed measures, a decision-making method is presented to solve MADM problem under the characteristics that attribute weights are either partially known or completely unknown. Approaches are explained with a numerical example and based on the different parametric values of R and S, the decision-maker(s) may have different choices to make a decision according to his/her choice. From the studies, it is concluded that the proposed work provides a new and easy way to handle the uncertainty and vagueness in the data and, hence, provides an alternative way to solve the decision-making problem in the IFS environment.

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