

STUDY OF RAYLEIGH WAVE PROPAGATION IN ELASTIC AND MICROPOLAR SOLIDS

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Dedicated to.....

ALMIGHTY GOD

AND

MY FAMILY

CERTIFICATE

I hereby certify that the thesis entitled " *Study of Rayleigh wave propagation in elastic and inhomogeneous solids* ", which is being submitted for the award of degree of Master of Science, *Department of Mathematics and Computer Applications, Thapar University Patiala*, is a bonafied record of the work carried out by me under the supervision of **Dr. Satish Kumar Sharma**. The contents of this thesis have not been submitted to any other Institute or University for the award of degree or diploma.

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ABSTRACT

Rayleigh surface waves have been well recognized in the study of earthquakes, seismology, geophysics and geodynamics. These types of surface waves propagate in half space. These waves have been of long standing interest in seismology (Richter (1958)) and having counterparts in hydrodynamics and electromagnetics. Failure of classical theory led to the development of micropolar theories (Eringen(1966a)), in which force stress as well as a couple stress is needed to describe the motion of material particle, as opposed to classical theory, in which force stress is used to describe the motion. In the present work I have studied Rayleigh surface waves in elastic and micropolar solids. The dispersion equations in closed form are derived for Rayleigh wave propagation. It also includes the discussions of applications and recent research on micropolar elasticity. The thesis has been divided into three chapters.

Chapter-1, contains the brief historical introduction to the classical theory of elasticity and generalized Hooke's law. Classical theory of elasticity deals with those material, in which the material is considered as a continuum. The theory of elastic waves and its types are also presented in this chapter. Two types of surface waves are discussed; Rayleigh waves and Love waves. The simplest medium in which Rayleigh waves can propagate is a homogeneous isotropic half space, while the simplest medium in which Love waves can propagate consists of a homogeneous isotropic layer on a homogeneous isotropic half space.

In Chapter-2, Rayleigh waves have been studied in homogeneous, isotropic elastic half space. Generally, in Rayleigh waves, both longitudinal and transverse motion may be found in solids. Thus in this chapter, formulation of the problem is done and the dispersion equations are solved by assuming the solutions and boundary conditions. The variation of wave number and phase velocity are also shown graphically. Here I have reviewed the chapter -5 (waves in infinite media) of Text book by Graff (1991), for understanding the basic concepts and the analytical procedures involved in the analysis.

In Chapter-3, Rayleigh surface waves have been studied in micropolar half space and dispersion equations are derived. Then the variation of wave number and phase velocity are also shown

graphically .The effect of micropolarity on Rayleigh waves is also observed graphically. The dispersion relation of elastic case is also obtained as a particular case. In this chapter, I have reviewed the paper,

- Sharma, J. N. , Kumar, S. and Sharma, Y. D.(2008) ,Propagation of Rayleigh Surface Waves in Microstretch Thermoelastic Continua Under Inviscid Fluid Loadings, Journal of Thermal Stresses, 31: 1, 18-39 (in the absence of microstretch, thermal effects and fluid loading).

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LIST OF SYMBOLS USED:

λ, μ = elastic constants

α, β, γ, K = micropolar constants

ρ = density

j = microinertia

σ_{ij} = stress tensor

ε_{ij} = strain tensor

m_{ij} = couple stress tensor

$\vec{\phi}$ = micro-rotation vector in the solid

\vec{u} = displacement vector

ε = permutation symbol

q, ψ = potential function

CHAPTER 1

INTRODUCTION

1.1 THEORY OF ELASTICITY:

Solid mechanics is the branch of mechanics that concerns the behavior of solid matter under external actions (e.g., external forces, temperature changes, applied displacements, etc.). It is part of a broader study which is known as continuum mechanics. In continuum mechanics, we study the system which have properties defined at all points in space. Solid bodies can be divided into two categories; elastic body and plastic body. A body is called elastic if it returns to its original shape upon the removal of applied forces. All bodies exhibit elastic behavior under sufficiently small loads. The mathematical analysis of elastic behavior of a solid body is called Theory of elasticity. A body that does not return to its original size or shape, upon the removal of deforming forces is called plastic body.

The first name to be linked with the history of theory of elasticity is Galileo Galilei (1638), called the father of science. He was a physicist, astronomer, mathematician and a philosopher. Galileo Galilei, in sixteenth and seventeenth centuries was the first to work on strength and fracture of beams and especially cantilever beams. The best known constitutive equation for a solid is the Hooke's law of linear elasticity, which was discovered by Robert Hooke (1678). The relation between applied forces and extension of body credited to Robert Hooke. Hooke's law of elasticity is an approximation that states that the extension of a body is in direct proportion with the load applied to it and is written as $F = kx$, where F is the applied tension force, x is the resulting elongation, and k is a constant whose value depends upon the nature and dimensions of the body. This law furnishes the foundation for mathematical theory of elasticity. Solid bodies obey Hooke's law up to a limit, called the elastic limit, which refers to the maximum deformation a body will undergo without being permanently deformed. That is, when the elastic limit is exceeded then it will become a plastic body. Brittle materials like glass or cast iron break at or

near their elastic limits. Generally, at low temperatures, short time and small enough loads, most materials are elastic. As these quantities increase, plastic behavior is observed unless the material is brittle and fractures. Further, E. Mariotte (1686) in France published similar discoveries as given by Hooke's in 1680. Then a Swiss mathematician and mechanic James Bernoulli (1727) observed, that the proper way of describing deformation was to give force per unit area, or stress, as a function of the elongation per unit length, or strain, of a body under tension. Swiss mathematician and mechanic Leonhard Euler (1707–1783) proposed a linear relation between stress and strain in the form of $\sigma = E\varepsilon$, where the coefficient E is now generally called Young's modulus after English naturalist Thomas Young who developed a related idea in (1807). Further, Louis Navier (1821) had succeeded in formulating the general equations of 3-dimensional theory of elasticity. Simultaneously, Navier had developed an elasticity theory based on a simple corpuscular, or particle, model of matter in which particles interacted with their neighbors by central-force attractions between particle pairs. Augustin Cauchy, being a mathematician and engineer, Cauchy's initial interest was in the theory of elasticity arose from study of Navier's paper. Unlike Navier's, who considered intermolecular forces, Cauchy considered the notion of pressure on plane. He thus introduced the first time the notion of stress. Cauchy (1822) formalized the stress concept in the context of a general three-dimensional theory, showed its properties as consisting of a 3 by 3 symmetric array of numbers that transform as a tensor. He derived the equations of motion for a continuum in terms of the components of stress, and gave the specific development of the theory of linear elastic response for isotropic solids. As part of this work, Cauchy also introduced the equations which express the six components of strain, three longitudinal and three shears, in terms of derivatives of displacements for the case when all those derivatives are much smaller than unity.

1.2 GENERALIZED HOOKE'S LAW:

Hooke's law may also be stated in terms of Stress –Strain relationship where stress is a quantity that is proportional to the force causing a deformation and strain is a measure of the degree of deformation. Further stress can be resolved into two components; normal and shearing stress. Normal stress is the deforming force acting per unit area normal to the surface of the

body and Tangential stress is due to the deforming force acting per unit area tangential to the surface .In 3 dimensional axes of the cubic body, stress can be resolved into three parts, one normal stress and a shearing stress which itself can be resolved into two components parallel to direction of the two coordinates. Thus there are nine quantities in all acting on the face of cube. Thus in total there are nine stress components needed to describe stress state at any point of the cubic body. These are $(\sigma_{xx}, \sigma_{xy}, \sigma_{xz}, \sigma_{yx}, \sigma_{yy}, \sigma_{yz}, \sigma_{zx}, \sigma_{zy}, \sigma_{zz})$, where the first suffix refers to normal to the plane on which shear stress acts and second suffix refers to the direction of the shear on this plane. Similarly, strain can also be resolved into two components (longitudinal and shearing strain).The change in length per unit original length is the longitudinal strain and shear strain occurs when change takes place in the shape of body. In total there are nine strain components. These are defined by, $(\epsilon_{xx}, \epsilon_{xy}, \epsilon_{xz}, \epsilon_{yx}, \epsilon_{yy}, \epsilon_{yz}, \epsilon_{zx}, \epsilon_{zy}, \epsilon_{zz})$. Then the modified form of Hooke's law came into play which states that stress components are a function of strain components. This is also known as generalized form of Hooke's law. According to generalized Hooke's law, when working with a three-dimensional stress state, a 4th order (c_{ijkl}) tensor containing 81 elastic coefficients must be defined to link the stress tensor (σ_{ij}) and the strain tensor (ϵ_{ij}) i.e $\sigma_{ij} = c_{ijkl} \epsilon_{ij}$. This is the Cauchy's formulation of generalized Hooke's law. If stress and strain tensors are symmetrical i.e. then the set will consist of only six equations with, 36 Elastic constants. Then the stress and strain tensors will also have only six components. Now the numbers of constants are reduced to 36 to 21 as $c_{ij} = c_{ji}$. Then by considering different types of symmetry, elastic constants are reduced to 2 in isotropic case, as opposed to the 21 elastic constants in the general anisotropic case. The two elastic constants are usually expressed as $(\lambda$ and $\mu)$. These are called Lamé's constants in isotropic case. The constitutive equation for classical isotropic elasticity (Sokolnikoff(1956)), is as follows, in which there are the two independent elastic constants, $(\lambda$ and $\mu)$, the Lamé constants.

$$\sigma_{ij} = \lambda u_{r,r} \delta_{ij} + \mu(u_{i,j} + u_{j,i})$$

where $(i, j, r = 1, 2, 3)$, δ_{ij} is the kronecker delta, and σ_{ij} is the stress tensor.

1.3 FAILURE OF CLASSICAL THEORY:

Classical theory of elasticity deals with those materials, in which the material is considered as a continuum in mathematical sense. The theoretical results obtained for certain materials like steel, aluminium, concrete etc. are found to exhibit results fairly coinciding with those of experimentally observed. However, in some materials, e.g, fibrous, polymers, asphalts, remarkable discrepancies are observed between the experimental results and those obtained using classical elasticity.

The discrepancy between classical theory of elasticity and their experiments is observed in case of dynamical problems such as elastic vibrations characterized by high frequencies and short wavelength i.e. ultrasonic waves. These discrepancies are mainly because of the dominance of atomic structures of the material neglected in classical elasticity. Thus the influence of microstructure becomes more important in the case of vibrations of granular and multimolecular bodies, where new types of waves appear, not encountered in classical theory of elasticity. That is why classical theory fails in case of granular body vibrations. Also classical theory of elasticity cannot be applied in the case of bodies with definite internal structure.

1.4 MICROCONTINUUM THEORIES:

Voigt (1887) was the first who tried to correct the shortcomings of classical elasticity by taking into account the assumption that interaction between two parts through an area element inside the body is transmitted not only by a force vector but also by a moment vector giving rise to 'couple stress theory'. This assumption leads to the description of stress field by means of two asymmetric tensors: the force stress tensor and the couple stress tensor. This assumption led to the fact that not only the force stresses but also couple stresses acting on the faces of an elementary parallelepiped are asymmetric in nature. The complete theory of asymmetric elasticity was developed by Cosserat and Cosserat (1909), which was nonlinear in the beginning. They assumed that the body consists of interconnected particles in the form of small rigid bodies. During the deformation each particle is displaced by $u(x,t)$ and rotated by the functions of the

position x and time t . This is how; the concept of rotation of a point was introduced in the continuum. Thus an elastic continuum has been described such that its points possess the orientation (polar media) and for which we can speak of the rotation of a point. The theory of the brothers E. and F. Cosserat remained unnoticed and was not duly appreciated during their lifetime. This was so because the presentation was very general (the theory was non-linear, including large deformations. They attempted to construct the unified field theory, containing mechanics, optics and electrodynamics and combined by a general principle of the least action. The research in the field of the general theories of continuous media conducted in the last fifteen years, drew the attention of the scientists to Cosserat's work and several Cosserat-type theories were developed independently, e.g., Gunther (1958), Grioli (1960), Rajagopal (1960), Aero and Kuvshinskii (1960), Mindlin and Tiersten (1962), Toupin(1962), Eringen (1962), Koiter (1964), Palmov(1964), Nowacki (1974), among several others. Ofcourse, these theories were similar to Cosserat's theory, but were called by name, e.g.,Toupin's theory was called 'Cosserat theory with constrained motion', Koiter's theory was called 'Couple stress theory', Eringen's theory was called 'Indeterminate couple stress theory', Nowacki's theory was called 'Cosserat pseudo-continuum theory' etc. In Nowacki's theory, the micro-rotation vector $\vec{\phi}$ is fully described by the displacement vector \vec{u} through the formula $\vec{\phi} = \frac{1}{2} \nabla \times \vec{u}$. Later, the general Cosserat continuum theory acquired the name of 'micropolar continuum theory' following Eringen (1966a), in which the micro-rotation vector is taken independent of displacement vector. Eringen and Suhubi (1964) developed a non-linear theory for 'micro-elasticity', in which intrinsic motions of the microelements were taken into account. This theory is basically the generalization of 'Indeterminate couple stress theory' and 'Cosserat theory'. A further generalization of the continuum with microstructure leads to micromorphic continuum (Eringen (1964b)).

Micromorphic continuum treats a material body as a continuous collection of a large number of deformable particles, with each particle possessing finite size and inner structure. On the other hand, classical continuum mechanics considers a material body as a continuous collection of material points, each with infinitesimal size and no inner structure. The purpose of going beyond classical continuum mechanics is to take into account the microstructure of the material body while still keeping the advantages of continuum theory intact. Using assumptions such as infinitesimal deformation and slow motion, micromorphic theory can be reduced to Mindlin's

microstructure theory (1964). In micromorphic theory 14 independent elastic constants are required to describe the motion of micromorphic solid. In case of microstretch solids the elastic constants are reduced to 9. When the material particle is considered as rigid, i.e., neglecting the internal motion within the microstructure, micromorphic theory becomes micropolar theory and hence in 3d there are 6 independent elastic constants required to describe micropolar solid. The six elastic constants required to describe the micropolarity are $(\alpha, \beta, \gamma, \lambda, \mu, K)$.

Eliminating the distinction of macromotion of the particle and the micromotion of its inner structure, Eringen micropolar theory becomes couple stress theory (Mindlin and Tiersten, (1962); Toupin,(1962)). Moreover, when the particle reduces to a mass point, all theories reduce to classical or ordinary continuum mechanics. The theory developed by other researchers in that time is found to be in close contact with the theory of 'microelasticity'. The connection between various theories of microcontinua has been nicely presented by Eringen (1999). Eringen (1966a, 1990) developed the theories of 'micropolar continua' and 'microstretch continua', which are special cases of the theory of 'micromorphic continua' earlier developed by Eringen and his coworker(1964). Thus, the Eringen's '3M' theories (Micromorphic, Microstretch, Micropolar) are the generalization of the classical theory of elasticity.

1.5 APPLICATIONS OF MICROPOLAR ELASTICITY:

Eringen's theory of polar elasticity(1966a) keeps importance because of its applications in many physical substances, e.g., material particles having rigid directors, chopped fiber composites, platelet composites, aluminium epoxy, liquid crystal with side chains, a large class of substances like liquid crystals with rigid molecules, rigid suspensions, animal blood with rigid cells, foams, porous materials, bones, magnetic fluids, clouds with dust, concrete with sand and muddy fluids are examples of micropolar materials; polymers with flexible molecules, animal lungs, bubbly fluids, polluted air, springy suspension, mixtures with breathing elements, porous media, lattices with base, fish colonies that live in ground are examples of microstretch materials; animal blood with deformable cells and turbulent fluids with flexible vortices are best examples of micro-morphic continua. Also micropolar theory is more appropriate for geological materials like rocks, soil since this theory takes into account the intrinsic rotation and predicts the

behavior of material with inner structure. For engineering problem, it can model composites with rigid chopped fibers, elastic solids with rigid inclusion and other industrial materials such as liquid crystal.

CLASSICAL THEORY	MICROPOLAR THEORY
1. There are two elastic constants which are known as Lamé's constants (λ and μ).	1. There are six elastic constants, which are ($\alpha, \beta, \gamma, \lambda, \mu, K$).
2. The motion of a material particle is fully translational.	2. The motion of a material particle is translation as well as rotation.
3. Translation is assumed by force stress only.	3. It incorporates force stress as well as couple stress.
4. Steel, Aluminium, Concrete can be modeled by classical theory	4. A large class of substances like foams, bones, liquid crystals with rigid molecules can be modeled by micropolar elasticity.

1.6 RESEARCH ON MICROPOLAR THEORY:

The mechanical behavior of linear elastic materials with microstructure, such as polymers, polycrystals or granular materials, cannot be described adequately by the classical theory of linear elasticity, which is associated with the concepts of homogeneity and locality of stress. When the material exhibits a non-homogeneous behavior, microstructural effects are important and the state of stress has to be defined in a non-local manner. These microstructural effects can be successfully modeled in a macroscopic manner by employing micropolar and couple stress theories. The theory of micropolar elasticity introduced and developed by Eringen (1968) has aroused much interest in recent years due to its possible utility in investigating deformation properties of solids, for which the purely elastic theory is inadequate. According to Eringen (1999) the range of possible materials to be modelled by micropolar theory is very wide. Hence, various authors have done research in micropolar theory. Yang and Lakes (1981) studied the

couple stress in compact bone. In this study they considered bone to be a linear elastic material. Yang and Lakes (1982) also developed the experimental study of micropolar and couple stress elasticity in bone modeling. Rod Lakes and Robert Benedict (1982) considered the Non-centrosymmetry in micropolar elasticity. In this research, consequences of noncentrosymmetry in a micropolar or Cosserat elastic solid are considered. A solid which is isotropic with respect to coordinate rotations but not with respect to inversions is called non-centrosymmetric or chiral. Anderson and Lakes (1994) studied the size effects due to Cosserat elasticity and surface damage in closed-cell polymethacrylimide foam. Willi Pabst (2005) gave a review of micropolar theories for solids as well as for liquids.

1.7 ELASTIC WAVES:

A wave may be defined as a periodic disturbance in a medium that carries energy from one point to another. All waves require a source and a medium of propagation. A wave is characterized by its wavelength, frequency, and amplitude. Elastic waves are mechanical waves (i.e. a material medium is necessary for the transmission) propagating in elastic medium as an effect of forces associated with volume deformation (shear) of medium elements. External bodies causing these deformations are called wave sources. Elastic waves propagation involves exciting the movement of medium particles increasingly distance from the waves. The main factor differentiating elastic waves from any other ordered motion of medium particles is that for small disturbances, elastic wave propagation doesn't result in matter transport. Depending on restrictions imposed on elastic medium, wave propagation may vary in character.

TYPES OF ELASTIC WAVES:

- (a) Waves in infinite media
- (b) Waves in semi-infinite media

1.7.1 WAVES IN INFINITE MEDIA:

In this type of waves, we will consider the waves in which the boundary interactions are not possible, namely those involving in infinite media. The basic elasticity equation is presented below in the absence of body forces. It will be shown that two basic types of waves-longitudinal and transverse waves can propagate in an infinite medium.

Wave Equations in Terms of Potentials: Consider the equation of motion for a homogeneous isotropic medium without body forces:

$$(\lambda + \mu)\nabla(\nabla\cdot\bar{u}) + \mu\nabla^2\bar{u} = \rho\frac{\partial^2\bar{u}}{\partial t^2} \quad (1.1)$$

where $\bar{u} = (u_1, u_2, u_3)$ is the displacement vector, λ and μ are Lamé's constants of classical elasticity and ρ is the density .

For two-dimensional problems, we take

$$\bar{u} = (u, 0, w)$$

$$\vec{\psi} = (0, -\psi, 0)$$

This vector can then be decomposed in terms of scalar and vector potentials,

$$\bar{u} = \nabla q + \nabla \times \psi \quad (1.2)$$

Where, q is the scalar potential and ψ is the vector potential.

Now by inserting eq. (1.2) in equation of motion (1.1) and interchanging the order of some operations, we obtain

$$(\lambda + 2\mu)\nabla^2\mathbf{q} = \rho\ddot{\mathbf{q}}$$

$$\mu\nabla^2\psi = \rho\ddot{\psi}$$

$$\nabla^2\mathbf{q} = \frac{\rho}{(\lambda + 2\mu)}\ddot{\mathbf{q}}$$

where,

$$c_1 = \sqrt{\frac{(\lambda + 2\mu)}{\rho}}$$

$$\mu\nabla^2\psi = \rho\ddot{\psi}$$

$$\nabla^2\psi = \frac{\rho}{\mu}\ddot{\psi}$$

where,

$$c_2 = \sqrt{\frac{\mu}{\rho}}$$

Therefore general wave equations is given by,

$$\nabla^2\mathbf{q} = \frac{1}{c_1^2}\ddot{\mathbf{q}}$$

$$\nabla^2\psi = \frac{1}{c_2^2}\ddot{\psi}$$

$$c_1 = \sqrt{\frac{(\lambda + 2\mu)}{\rho}}, c_2 = \sqrt{\frac{\mu}{\rho}}$$

We have thus found that waves may propagate in the interior of elastic solid at two different velocities where, c_1 is the wave velocity of longitudinal and c_2 is the velocity of shear waves.

Longitudinal Waves (compressional/primary/P -waves):

The waves that travel parallel to the direction of motion are called longitudinal waves. In longitudinal waves, the particles in a medium oscillate back and forth about their equilibrium positions but it is the disturbance which travels, not the individual particles in the medium. It is made up of compressions and rarefactions in the medium that they are traveling in. Longitudinal waves are also called P-waves. The P-waves have arisen in seismology, where they are also called Push waves. P-waves are produced by earthquakes and recorded by seismographs. The name P-wave is often said to stand either for primary wave, as it has the highest velocity and is therefore the first to be recorded. In isotropic and homogeneous solids, the mode of propagation of a P-wave is always longitudinal. Thus, the particles in the solid have vibrations along or parallel to the travel direction of the wave energy. Examples: sound waves, waves which travel through a spring are longitudinal and P waves are for earthquakes.

Transverse waves (Shear waves/secondary/ S- waves)

The waves that travel perpendicular to the direction of motion are called Transverse waves. In a transverse wave the particle displacement is perpendicular to the direction of wave propagation. The particles do not move along with the wave; they simply oscillate up and down about their individual equilibrium positions as the wave passes by. Transverse waves are also called S-waves. S-waves have arisen in seismology, in the similar way as P-wave and are also called shake waves. The S-wave moves as a shear or transverse wave, so motion is perpendicular to the direction of wave propagation. S waves are slower than P waves, and speeds are typically around 10% of that of P waves in any given material. Its name, S for secondary, comes from the fact that it is the second direct arrival on an earthquake seismogram, after the compressional primary wave, or P-wave, because S-waves travel slower in rock. Examples: Light, S waves for earthquakes, Ocean waves, waves in a guitar string.

Difference between longitudinal and transverse waves:

In longitudinal waves the disturbances travel backwards and forwards in the same direction as the wave is travelling while in transverse waves the disturbances are at right angles to the direction of the wave.

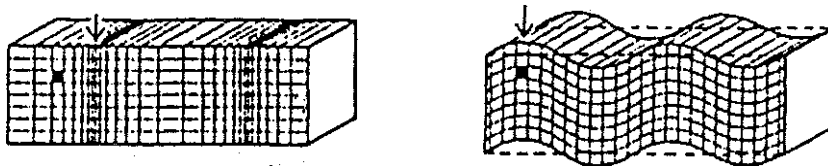


Fig.1.1. Deformations of the medium when body waves propagate from left to right: longitudinal wave (on the left), and transverse wave (on the right) (geo.mff.cuni.cz/vyuka/Novotny-Seismic SurfaceWaves)

1.7.2 WAVES IN SEMI-INFINITE MEDIA:

The second type of waves, which have studied in semi-infinite media, are known as surface waves. These waves usually form the principal phase of seismograms. They can be the most destructive type of seismic wave. It is of course the inclusion of a boundary that distinguishes the problem from the last type. Waves in semi-infinite solids, either homogeneous or inhomogeneous, have been of long standing interest in seismology. Problems in ultrasonics, delay lines, soil dynamics, blast and impact have also led to analysis of waves in half space. There are two types of surface elastic waves: Rayleigh Surface waves and Love waves.

Rayleigh Surface waves:

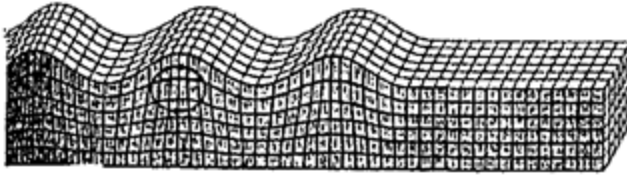
Lord Rayleigh (1887) showed that there is a wave type that could propagate along surfaces, such that the motion associated with the wave decayed exponentially with distance into material from the surface. This type of surface wave is called Rayleigh wave. In Rayleigh surface waves both

longitudinal and transverse motion may be found in solids as Rayleigh surface waves. The particles in a solid, through which a Rayleigh surface wave passes, move in elliptical paths, with the major axis of the ellipse perpendicular to the surface of the solid. Rayleigh waves, also called ground roll, are surface waves that travel as ripples with motions that are similar to those of waves on the surface of water. These waves are elliptically polarised in the plane which is determined by the normal to the surface and by the direction of propagation (Fig.- 1.2(a)). The simplest medium in which Rayleigh waves can propagate is a homogeneous isotropic half-space. The velocity of Rayleigh waves in this medium, c , is slightly less than the transverse wave velocity, and is independent of frequency. Thus, Rayleigh waves in this simple model of the medium are non-dispersive. In Rayleigh surface waves, the particle motion is a retrograde vertical ellipse (anticlockwise for a wave travelling to the right) near the surface of a homogeneous half-space.

Love waves:

Love waves (L-waves) are surface waves that cause circular shearing of the ground. They are named after A.E.H. Love (1911), a British mathematician who created a mathematical model of the waves. They usually travel slightly faster than Rayleigh waves, about 90% of the S wave velocity, and have the largest amplitude. Love waves are characterized by particle oscillations involving alternate transverse movements. The particle motion in these waves is transverse and parallel to the surface (Fig. 1.2(b)). As opposed to Rayleigh waves, Love waves cannot propagate in a homogeneous half-space. The simplest model in which Love waves can propagate consists of a homogeneous isotropic layer on a homogeneous isotropic half-space. Love waves can propagate only if the S -wave velocity generally increases with the distance from the surface of the medium.

(a) Rayleigh waves



(b) Love waves



Fig. 1.2 The particle motion for surface waves: (a) Rayleigh waves and (b) Love waves. (geo.mff.cuni.cz/vyuka/Novotny-SeismicSurfaceWaves)

1.8 APPLICATIONS OF RAYLEIGH SURFACE WAVES:

Rayleigh waves are widely used for materials characterization, to discover the mechanical and structural properties of the object. Rayleigh waves in the ultrasonic frequency range are used in non-destructive testing applications (Victorov (1967)) to find cracks and other imperfections in materials. Rayleigh waves are also useful to detect surface flaws (Reinhardt and Dally (1970)). A wide variety of surface wave phenomenon and applications have been reported by Richard (1970). They are used at different length scales because they are easily generated and detected on the free surface of solid objects. Since they are confined in the vicinity of the free surface within a depth (\sim the wavelength) linked to the frequency of the wave, different frequencies can be used for characterization at different length scales. Low frequency Rayleigh waves generated during earthquakes are used in seismology to characterize the Earth's interior. In intermediate ranges, Rayleigh waves are used in geophysics and geotechnical engineering for the characterization of oil deposits. Rayleigh ground waves are important also for environmental noise and vibration control since they make a major contribution to traffic-induced ground vibrations and the associated structure-borne noise in buildings.

CHAPTER 2

RAYLEIGH WAVES IN A HOMOGENEOUS ISOTROPIC

ELASTIC HALF SPACE:

Rayleigh waves are the surface seismic waves that cause vertical shifting of the earth during an earthquake and therefore are of particular importance in seismology. These waves are responsible for destruction of the buildings and loss of human lives. Scattering of seismic waves due to irregularities in the surface leads to large amplification and variation in ground motion during earthquakes.

Keeping in view, the role played by Rayleigh waves, various authors have discussed the propagation of Rayleigh waves in elastic medium. Rayleigh (1887) studied the waves propagating along plane surface of an elastic solid. Love found that the ratio of the horizontal and vertical components of these waves was already close to the observed values. Love(1911) and De and Sengupta (1974) study the wave propagation in elastic medium and it has been shown that velocity of Rayleigh waves increases by a significant amount, when the wavelength is large due to influence of gravity. Reinhardt and Dally (1970) studied some characteristics of Rayleigh wave interaction with surface flaws. In this section we have studied the Rayleigh waves in a homogeneous and isotropic half-space with free boundary conditions and the solutions of the secular Rayleigh wave equation giving the velocity of the waves in medium.

2.1 FORMULATION OF THE PROBLEM:

Consider a homogeneous, isotropic and elastic half-space. Introduce a Cartesian coordinate system whose (x, y) -plane coincides with the surface of the medium, and the z -axis is positive downwards (into the medium, Fig. 2.1). We consider plane harmonic wave propagating in half space $z \geq 0$. We take origin of coordinate system (x, y, z) on top surface of solid half space. The z -axis is taken pointing vertically downward into the semi space which is represented by $z \geq 0$. The x -axis is taken along the direction of wave propagation so that all particles on a line parallel

to y-axis are equally displaced and hence all the field quantities are independent of y-coordinate. The surface of half-space is assumed to be free of stresses and couples.

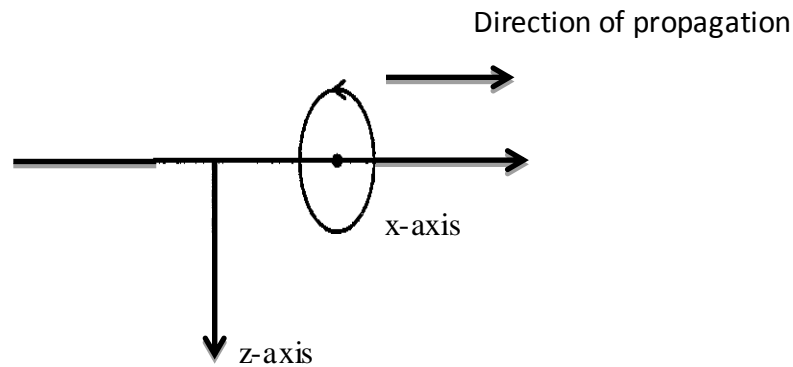


Fig. 2.1 (geo.mff.cuni.cz/vyuka/Novotny-SeismicSurfaceWaves)

The basic governing equations of elasticity in the absence of body forces and heat sources are:

$$(\lambda + \mu)\nabla(\nabla\cdot\bar{u}) + \mu\nabla^2\bar{u} = \rho\frac{\partial^2\bar{u}}{\partial t^2} \quad (2.1)$$

The constitutive relation is given by

$$\sigma_{ij} = \lambda u_{r,r}\delta_{ij} + \mu(u_{i,j} + u_{j,i}) \quad (2.2)$$

where $\bar{u} = (u_1, u_2, u_3)$ is the displacement vector, $(i, j, r=1,2,3)$, λ and μ are Lamé's constants of classical elasticity, ρ is the density, σ_{ij} are respectively the stress tensor and δ_{ij} is the Kronecker's delta.

For two-dimensional problems, we take

$$\bar{u} = (u, 0, w)$$

So that equation in non-dimensional form can be rewritten as,

$$(1 - \delta^2) \left(\frac{\partial^2 \mathbf{u}}{\partial x^2} + \frac{\partial^2 \mathbf{w}}{\partial x \partial z} \right) + \delta^2 \nabla^2 \mathbf{u} = \frac{\partial^2 \mathbf{u}}{\partial t^2} \quad (2.3)$$

$$(1 - \delta^2) \left(\frac{\partial^2 \mathbf{u}}{\partial x \partial z} + \frac{\partial^2 \mathbf{w}}{\partial z^2} \right) + \delta^2 \nabla^2 \mathbf{w} = \frac{\partial^2 \mathbf{w}}{\partial t^2} \quad (2.4)$$

Here, we have defined the quantities

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} \quad (2.5)$$

$$x' = \frac{\omega^*}{c_1} x, z' = \frac{\omega^*}{c_1} z, t' = \omega^* t,$$

$$u' = \frac{\rho \omega^* c_1}{\nu T_0} u, w' = \frac{\rho \omega^* c_1}{\nu T_0} w, T' = \frac{T}{T_0}, \delta^2 = \frac{c_2^2}{c_1^2}, \sigma'_{ij} = \frac{1}{\nu T_0} \sigma_{ij},$$

$$c_1^2 = \frac{\lambda + 2\mu}{\rho}, c_2^2 = \frac{\mu}{\rho}$$

Here primes have been suppressed for convenience. Here ω^* is characteristic frequency of medium, c_1 and c_2 are longitudinal and shear velocities.

In order to solve above equations, we introduce potential functions q and ψ in the solid through the relations

$$u = \frac{\partial q}{\partial x} + \frac{\partial \psi}{\partial z} \quad (2.6)$$

$$w = \frac{\partial q}{\partial z} - \frac{\partial \psi}{\partial x} \quad (2.7)$$

Substituting (2.6) and (2.7) in equations (2.3) and (2.4) we obtained,

$$(\nabla^2 - \frac{\partial^2}{\partial t^2})q = 0 \quad (2.8)$$

$$(\nabla^2 - \frac{1}{\delta^2} \frac{\partial^2}{\partial t^2})\psi = 0 \quad (2.9)$$

Upon using equations (2.6) and (2.7) in constitutive relations (2.2), we obtain

$$\sigma_{zz} = \nabla^2 q - 2\delta^2 (q_{,xx} + \psi_{,xz}) \quad (2.10)$$

$$\sigma_{zx} = \delta^2 (2q_{,xz} - \psi_{,xx} + \psi_{,xz}) \quad (2.11)$$

BOUNDARY CONDITIONS:

The surface of the solid is assumed to be free of stresses, couples so that $\sigma_{zz} = 0, \sigma_{zx} = 0$

The boundary conditions is to be satisfied at the surface ($z=0$) of the solid half space.

2.1.1 FORMAL SOLUTION OF THE PROBLEM:

We assume the solution of the form

$$q = q(z)e^{i\xi(x-ct)} \quad (2.12)$$

$$\psi = \psi(z)e^{i\xi(x-ct)} \quad (2.13)$$

where $c = \frac{\omega}{\xi}$ is the non-dimensional phase velocity, ω is the circular frequency, ξ is wave number. Upon using solutions (2.12) and (2.13) in equations (2.8) and (2.9), we get

$$\left(\nabla^2 - \frac{\partial^2}{\partial t^2}\right)q = 0 \quad \text{from eq.(2.8)}$$

$$\nabla^2 q = \frac{\partial^2 q}{\partial t^2}$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right)q = \frac{\partial^2 q}{\partial t^2}$$

$$(-\xi^2 + D^2)q(z) = -\xi^2 c^2 q(z)$$

$$(D^2 - \xi^2 + \xi^2 c^2)q(z) = 0$$

$$D^2 - \alpha^{*2} = 0$$

$$D = \pm \alpha^*$$

$$\text{where, } \alpha^{*2} = \xi^2(1-c^2)$$

$$q(z) = A_1 e^{\alpha^* z} + B_1 e^{-\alpha^* z}$$

$$q = (A_1 e^{\alpha^* z} + B_1 e^{-\alpha^* z}) e^{i\xi(x-ct)} \quad (2.14)$$

$$\left(\nabla^2 - \frac{1}{\delta^2} \frac{\partial^2}{\partial t^2} \right) \psi = 0 \quad \text{(from (2.9)eq.)}$$

$$\left(D^2 - \xi^2 + \frac{1}{\delta^2} \xi^2 c^2 \right) \psi(z) = 0$$

$$D^2 - \beta^{*2} = 0$$

$$\text{where, } \beta^{*2} = \xi^2 - \frac{1}{\delta^2} \xi^2 c^2$$

$$\beta^{*2} = \xi^2 \left(1 - \frac{c^2}{\delta^2} \right)$$

$$\psi(z) = A_2 e^{\beta^* z} + B_2 e^{-\beta^* z}$$

$$\psi = (A_2 e^{\beta^* z} + B_2 e^{-\beta^* z}) e^{i\xi(x-ct)} \quad (2.15)$$

Discarding increasing terms,

$$q = B_1 e^{-\alpha^* z} e^{i\xi(x-ct)} \quad (2.16)$$

$$\psi = B_2 e^{-\beta^* z} e^{i\xi(x-ct)} \quad (2.17)$$

Now applying eq. (2.16) and (2.17) in eq. (2.10) and (2.11),

$$\sigma_{zz} = B_1[(\alpha^{*2} - \xi^2) + 2\delta^2 \xi^2] + 2i\xi\delta^2 \beta^* B_2$$

$$\sigma_{zx} = \delta^2 [B_2(\beta^{*2} + \xi^2) - 2B_1 i \xi \alpha^*]$$

Invoking boundary conditions, $\sigma_{zz} = 0, \sigma_{zx} = 0$

$$B_1(\beta^{*2} + \xi^2) + 2i\xi\beta^* B_2 = 0 \quad (2.18)$$

$$B_2(\beta^{*2} + \xi^2) - 2B_1 i \xi \alpha^* = 0 \quad (2.19)$$

This is a homogeneous system of equations for the unknown amplitudes B_1 and B_2 .

The system of eq. (2.18) and (2.19) has a non-trivial solution if the corresponding determinant is equal to zero. This yield, after multiplying the Rayleigh equation,

Now solving determinant, using eq. (2.18) and (2.19)

$$\begin{vmatrix} (\beta^{*2} + \xi^2) & 2i\xi\beta^* \\ -2i\xi\alpha^* & (\beta^{*2} + \xi^2) \end{vmatrix} = 0$$

$$(\beta^{*2} + \xi^2)^2 - 4\alpha^* \beta^* \xi^2 = 0 \quad (2.20)$$

Thus the Rayleigh equation is given as,

$$(\beta^{*2} + \xi^2)^2 - 4\alpha^* \beta^* \xi^2 = 0$$

where $\alpha^{*2} = \xi^2(1 - c^2)$,

and $\beta^{*2} = \xi^2 \left(1 - \frac{c^2}{\delta^2}\right)$

Thus the eq. (2.20) is same as obtained and discussed by Graff in (chapter 6) Propagation of waves in semi-infinite media.

2.2 NUMERICAL RESULTS AND DISCUSSIONS:

VARIATION OF WAVE NUMBER AND PHASE VELOCITY:

With the view of illustrating theoretical results derived in the preceding sections we now present some numerical results. Here we have taken aluminum-epoxy like material for numerical computations whose physical data is given by (Singh and Kumar 1998).

$$\rho = 2.19 \times 10^3 \text{ Kg/m}^3, \quad \lambda = 7.59 \times 10^{10} \text{ N/m}^2, \quad \mu = 1.89 \times 10^{10} \text{ N/m}^2$$

$$\omega^* = 4.364 \times 10^{11} / \text{sec}$$

Variation of phase velocity with respect to wave number in the case of elasticity has been studied in fig. (2.2). It is observed that the phase velocity of Rayleigh waves in half-space decreases sharply from its peak values. In the wave number range $0 \leq R \leq 1$, increases moderately in the wave number range $1 \leq R \leq 2$ following linear variations before becoming dispersion less for $R \geq 2$. The phase velocity profiles are significantly dispersive at small values of the wave number and become dispersion-less with increasing wave number.

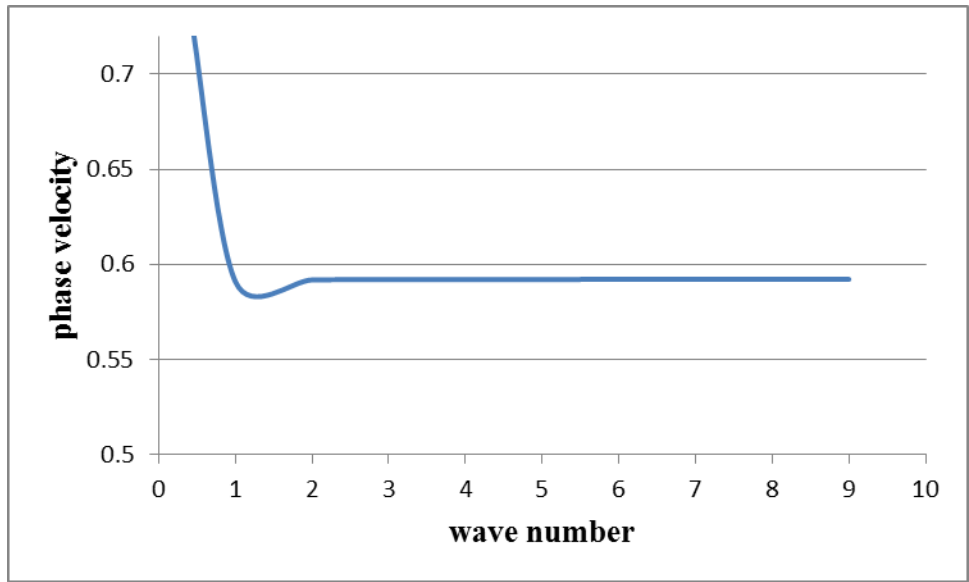


Fig. 2.2 Variation of wave number with phase velocity.

CHAPTER 3

EFFECT OF MICROPOLARITY ON RAYLEIGH SURFACE WAVES IN ELASTIC SOLIDS.

The propagation of waves in micropolar elastic half space has been discussed by Parfitt and Eringen (1969) and Ariman (1972). Parfitt and Eringen (1969) studied the reflection of plane waves from the flat boundary of a micropolar elastic half-space, while Ariman (1972) discussed the wave propagation in micropolar elastic half space. Rao and Reddy (1993) discussed Rayleigh type wave propagating on the surface of a micropolar elastic circular cylinder in an azimuthal direction. They have shown that due to the micropolar effect, there exists an extra wave, and the frequency of Rayleigh waves increases due to the micropolar effect. Kumar and Tomar (1996) studied the propagation of micropolar waves at boundary surface. Kumar & Singh (1997) discussed a problem on reflection and transmission of elastic waves at the loosely bonded interface between an elastic solid half-space and a micropolar elastic solid half-space. The study carried out in chapter-2 (Rayleigh waves in homogeneous and isotropic elastic half space) has been extended to micropolar half space. Rayleigh waves in micropolar solids have been studied by various authors under fluid loading and without fluid loading. Here I have reviewed the work of Sharma et al (2008) in the absence of fluid loading for a micropolar half space.

3.1 FORMULATION OF THE PROBLEM:

We consider a homogeneous isotropic, micropolar elastic solid space initially undisturbed. We take origin of coordinate system (x,y,z) on top surface of solid half space. The z -axis is taken pointing vertically downward into the semi space which is represented by $z \geq 0$. The x -axis is taken along the direction of wave propagation so that all particles on a line parallel to y -axis are equally displaced and hence all the field quantities are independent of y -coordinate. The surface of half –space is assumed to be free of stresses and couples. The basic governing equations of micropolar elasticity in the absence of body forces and heat sources are-

$$(\lambda + \mu)\nabla(\nabla\cdot\vec{u}) + (\mu + K)\nabla^2\vec{u} + K(\nabla\times\vec{\phi}) = \rho\frac{\partial^2\vec{u}}{\partial t^2} \quad (3.1)$$

$$(\alpha + \beta + \gamma)\nabla(\nabla\cdot\vec{\phi}) - \gamma\nabla\times(\nabla\times\vec{\phi}) + K(\nabla\times\vec{u}) - 2K\vec{\phi} = \rho j\frac{\partial^2\vec{\phi}}{\partial t^2} \quad (3.2)$$

The constitutive relations are given by

$$\sigma_{ij} = \lambda u_{r,r} \delta_{ij} + \mu(u_{i,j} + u_{j,i}) + K(u_{j,i} - \varepsilon_{ijr} \phi_r) \quad (3.3)$$

$$m_{ij} = \alpha \phi_{r,r} \delta_{ij} + \beta \phi_{j,j} + \lambda \phi_{j,i} \quad (3.4)$$

where, $\vec{u} = (u_1, u_2, u_3)$ is the displacement vector, $\vec{\phi}$ is the micro-rotation vector, λ and μ are Lamé's constants of classical elasticity, $(\alpha, \beta, \gamma, K)$ are micropolar constants of the solid, ρ is the density, j is the microinertia, σ_{ij} and m_{ij} are respectively the stress tensor and couple stress tensor and δ_{ij} is the kronecker's delta.

For two-dimensional problems, we take

$$\vec{u} = (u, 0, w)$$

$$\vec{\phi} = (0, \phi, 0)$$

So that equation in non-dimensional form can be rewritten as

$$(1 - \delta^2) \left(\frac{\partial^2 \mathbf{u}}{\partial \mathbf{x}^2} + \frac{\partial^2 \mathbf{w}}{\partial \mathbf{x} \partial \mathbf{z}} \right) + \delta^2 \nabla^2 \mathbf{u} - \mathbf{p} \frac{\partial \phi}{\partial \mathbf{z}} = \frac{\partial^2 \mathbf{u}}{\partial t^2} \quad (3.5)$$

$$(1 - \delta^2) \left(\frac{\partial^2 \mathbf{u}}{\partial \mathbf{x} \partial \mathbf{z}} + \frac{\partial^2 \mathbf{w}}{\partial \mathbf{z}^2} \right) + \delta^2 \nabla^2 \mathbf{w} + \mathbf{p} \frac{\partial \phi}{\partial \mathbf{x}} = \frac{\partial^2 \mathbf{w}}{\partial t^2} \quad (3.6)$$

$$\nabla^2 \phi + \delta^* \left(\frac{\partial \mathbf{u}}{\partial \mathbf{z}} - \frac{\partial \mathbf{w}}{\partial \mathbf{x}} \right) - 2\delta^* \phi = \frac{1}{\delta_1^2} \frac{\partial^2 \phi}{\partial t^2} \quad (3.7)$$

Here, we have defined the quantities

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}$$

$$x' = \frac{\omega^*}{c_1} x, z' = \frac{\omega^*}{c_1} z, t' = \omega^* t,$$

$$u' = \frac{\rho \omega^* c_1}{\nu T_0} u, w' = \frac{\rho \omega^* c_1}{\nu T_0} w, T' = \frac{T}{T_0}, \delta^2 = \frac{c_2^2}{c_1^2}, \delta_1^2 = \frac{c_4^2}{c_1^2}$$

$$\sigma'_{ij} = \frac{1}{\nu T_0} \sigma_{ij}, \phi' = \frac{\rho c_1^2}{\nu T_0} \phi, m'_{ij} = \frac{\omega^*}{c_1 \nu T_0} m_{ij}$$

$$c_1^2 = \frac{\lambda + 2\mu}{\rho}, c_2^2 = \frac{\mu}{\rho}, c_4^2 = \frac{\gamma}{\rho j}, p = \frac{K}{\rho c_1^2}, \delta^* = \frac{K c_1^2}{\gamma \omega^*}$$

Here primes have been suppressed for convenience. Here ω^* is characteristic frequency of medium, c_1 and c_2 are longitudinal and shear velocities.

In order to solve above equations, we introduce potential functions q and ψ in the solid through the relations

$$u = \frac{\partial q}{\partial x} + \frac{\partial \psi}{\partial z}$$

$$(3.8) \quad w = \frac{\partial q}{\partial z} - \frac{\partial \psi}{\partial x}$$

$$(3.9)$$

Substituting eq. (3.8) and (3.9) in eq. (3.5) to (3.7) we obtained,

$$\left(\nabla^2 - \frac{\partial^2}{\partial t^2} \right) q = 0 \quad (3.10)$$

$$\left(\nabla^2 \psi - \frac{1}{\delta^2} \ddot{\psi} \right) = \frac{p}{\delta^2} \phi \quad (3.11)$$

$$\nabla^2 \phi - 2\delta^* \phi - \frac{1}{\delta_1^2} \ddot{\phi} + \delta^* \nabla^2 \psi = 0 \quad (3.12)$$

Upon using eq. (3.8) and (3.9) in constitutive relations (3.3) and (3.4), we obtain

$$\sigma_{zz} = \nabla^2 q - 2 \left(\delta^2 - \frac{p}{2} \right) (q_{,xx} + \psi_{,xz}) \quad (3.13)$$

$$\sigma_{zx} = \ddot{\psi} + (2\delta^2 - p)(q_{,xz} - \psi_{,xx}) \quad (3.14)$$

$$m_{zy} = \frac{\lambda}{\rho_1^2} \frac{\partial \phi}{\partial z} \quad (3.15)$$

BOUNDARY CONDITIONS:

The surface of the solid is assumed to be free of stresses, couples and, so that

$$\sigma_{zz} = 0, \sigma_{zx} = 0, m_{zy} = 0.$$

The boundary conditions to be satisfied at the surface ($z=0$) of the solid half space.

3.1.1 FORMAL SOLUTION OF THE PROBLEM:

We assume the solution of the form

$$q = q(z) e^{i\xi(x-ct)} \quad (3.16)$$

$$\psi = \psi(z) e^{i\xi(x-ct)} \quad (3.17)$$

$$\phi = \phi(z) e^{i\xi(x-ct)} \quad (3.18)$$

where $c = \frac{\omega}{\xi}$ is the non-dimensional phase velocity, ω is the circular frequency, ξ is wave

number. Upon using solutions (3.16) to (3.18) in equations (3.10) to (3.12), we get

$$\left(\nabla^2 - \frac{\partial^2}{\partial t^2} \right) q = 0 \quad \text{from eq. (3.10)}$$

$$\nabla^2 \mathbf{q} = \frac{\partial^2 \mathbf{q}}{\partial t^2}$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} \right) \mathbf{q} = \frac{\partial^2 \mathbf{q}}{\partial t^2}$$

$$(-\xi^2 + D^2) \mathbf{q}(z) = -\xi^2 c^2 \mathbf{q}(z)$$

$$(D^2 - \xi^2 + \xi^2 c^2) \mathbf{q}(z) = 0$$

$$D^2 - m_1^2 = 0$$

$$D = \pm m_1$$

$$\text{Where, } m_1^2 = \xi^2 (1 - c^2)$$

$$\mathbf{q}(z) = A_1 e^{m_1 z} + B_1 e^{-m_1 z}$$

$$\mathbf{q} = (A_1 e^{m_1 z} + B_1 e^{-m_1 z}) e^{i\xi(x-ct)} \quad (3.19)$$

$$\nabla^2 \phi - 2\delta^* \phi - \frac{1}{\delta_1^2} \ddot{\phi} + \delta^* \nabla^2 \psi = 0 \quad \text{from eq. (3.12)}$$

$$\ddot{\phi} = -\omega^2 \phi$$

$$\left(\nabla^2 - 2\delta^* + \frac{\omega^2}{\delta_1^2} \right) \phi + \delta^* \nabla^2 \psi = 0 \quad (3.20)$$

$$\frac{\mathbf{p}}{\delta^2} \phi - \nabla^2 \psi + \frac{1}{\delta^2} \ddot{\psi} = 0 \quad \text{from eq. (3.11)}$$

$$\frac{\mathbf{p}}{\delta^2} \phi - \nabla^2 \psi - \frac{1}{\delta^2} \omega^2 \psi = 0$$

$$\frac{\mathbf{p}}{\delta^2} \phi - \left(\nabla^2 + \frac{\omega^2}{\delta^2} \right) \psi = 0 \quad (3.21)$$

Now solving eq. (3.20) and (3.21),

$$\left(\nabla^2 - 2\delta^* + \frac{\omega^2}{\delta_1^2}\right)\phi + \delta^*\nabla^2\psi = 0$$

$$\frac{\mathbf{p}}{\delta^2}\phi - \left(\nabla^2 + \frac{\omega^2}{\delta^2}\right)\psi = 0$$

$$\left[\left(\nabla^2 + \frac{\omega^2}{\delta^2}\right)\left(\nabla^2 - 2\delta^* + \frac{\omega^2}{\delta_1^2}\right) + \frac{\mathbf{p}}{\delta^2}\delta^*\nabla^2\right]\psi = 0$$

$$\left[\nabla^4 + \nabla^2\left(-2\delta^* + \frac{\omega^2}{\delta_1^2} + \frac{\mathbf{p}}{\delta^2}\delta^* + \frac{\omega^2}{\delta^2}\right) + \left(\frac{\omega^4}{\delta^2\delta_1^2} - 2\frac{\delta^*\omega^2}{\delta^2}\right)\right]\psi = 0$$

$$\left[\nabla^4 + \nabla^2\left(-2\frac{\delta^*}{\omega^2} + \frac{1}{\delta_1^2} + \frac{\mathbf{p}}{\delta^2\omega^2}\delta^* + \frac{1}{\delta^2}\right)\omega^2 + \omega^4\left(\frac{1}{\delta^2\delta_1^2} - 2\frac{\delta^*}{\delta^2\omega^2}\right)\right]\psi = 0$$

$$(\nabla^4 + \nabla^2\omega^2\mathbf{a}_1^2 + \omega^4\mathbf{a}_2^2)\psi = 0$$

$$[(\nabla^2 + \mathbf{a}_1^2\omega^2)(\nabla^2 + \mathbf{a}_2^2\omega^2)]\psi = 0$$

$$\nabla^2 + \mathbf{a}_1^2\omega^2 = 0 \tag{3.22}$$

$$\nabla^2 + \mathbf{a}_2^2\omega^2 = 0 \tag{3.23}$$

$$\sum\mathbf{a}_1^2 + \mathbf{a}_2^2 = -2\frac{\delta^*}{\omega^2} + \frac{1}{\delta_1^2} + \frac{\mathbf{p}}{\delta^2\omega^2}\delta^* + \frac{1}{\delta^2} + \frac{1}{\delta^2\delta_1^2} - 2\frac{\delta^*}{\delta^2\omega^2}$$

$$\sum\mathbf{a}_1^2\mathbf{a}_2^2 = \left(-2\frac{\delta^*}{\omega^2} + \frac{1}{\delta_1^2} + \frac{\mathbf{p}}{\delta^2\omega^2}\delta^* + \frac{1}{\delta^2}\right)\left(\frac{1}{\delta^2\delta_1^2} - 2\frac{\delta^*}{\delta^2\omega^2}\right)$$

$$\nabla^2\psi = \left(\frac{\partial^2}{\partial\mathbf{x}^2} + \frac{\partial^2}{\partial\mathbf{z}^2}\right)\psi(\mathbf{z}) e^{i\xi(\mathbf{x}-ct)}$$

$$\nabla^2\psi = (-\xi^2 + \mathbf{D}^2)\psi(\mathbf{z}) e^{i\xi(\mathbf{x}-ct)}$$

$$\nabla^2\psi = (-\xi^2 + \mathbf{D}^2)\psi$$

Now putting value in eq. (3.22) and (3.23)

$$(D^2 - \xi^2 + a_1^2 \omega^2) \psi = 0$$

$$D^2 - m_2^2 = 0$$

where,

$$m_2^2 = \xi^2 - a_1^2 \omega^2$$

$$m_2^2 = \xi^2 \left(1 - \frac{a_1^2 \omega^2}{\xi^2} \right)$$

$$m_2^2 = \xi^2 (1 - a_1^2 c^2)$$

similarly ,

$$D^2 - m_3^2 = 0$$

$$m_3^2 = \xi^2 (1 - a_2^2 c^2)$$

Clubing we get,

$$\psi = (A_2 e^{m_2 z} + B_2 e^{-m_2 z} + A_3 e^{m_3 z} + B_3 e^{-m_3 z}) e^{i\xi(x-ct)} \quad (3.24)$$

Now from eq. (3.21), we have

$$\phi = \frac{\delta^2}{p} \left(\nabla^2 \psi + \frac{\omega^2}{\delta^2} \psi \right) \quad (3.25)$$

$$\nabla^2 \psi = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} \right) (A_2 e^{m_2 z} + B_2 e^{-m_2 z} + A_3 e^{m_3 z} + B_3 e^{-m_3 z}) e^{i\xi(x-ct)}$$

$$\nabla^2 \psi = \left[\xi^2 (A_2 e^{m_2 z} + B_2 e^{-m_2 z} + A_3 e^{m_3 z} + B_3 e^{-m_3 z}) \right] + (m_2^2 A_2 e^{m_2 z} + m_2^2 B_2 e^{-m_2 z} + m_3^2 A_3 e^{m_3 z} + m_3^2 B_3 e^{-m_3 z}) e^{i\xi(x-ct)}$$

Now substituting in eq. (3.26),

$$\phi = \frac{\delta^2}{p} \left\{ \left[(-\xi^2 + m_2^2)A_2 e^{m_2 z} + (-\xi^2 + m_2^2)B_2 e^{-m_2 z} + (-\xi^2 + m_3^2)A_3 e^{m_3 z} + (-\xi^2 + m_3^2)B_3 e^{-m_3 z} \right] + \frac{\omega^2}{\delta^2} \right\} e^{i\xi(x-ct)}$$

$$\phi = \frac{\delta^2}{p} \left[\left(m_2^2 - \xi^2 + \frac{\omega^2}{\delta^2} \right) (A_2 e^{m_2 z} + B_2 e^{-m_2 z}) + \left(m_3^2 - \xi^2 + \frac{\omega^2}{\delta^2} \right) (A_3 e^{m_3 z} + B_3 e^{-m_3 z}) \right] e^{i\xi(x-ct)} \quad (3.26)$$

Thus eq. (3.27) becomes,

$$\phi = \frac{\delta^2}{p} \left[(m_2^2 - \beta^{*2})(A_2 e^{m_2 z} + B_2 e^{-m_2 z}) + (m_3^2 - \beta^{*2})(A_3 e^{m_3 z} + B_3 e^{-m_3 z}) \right] e^{i\xi(x-ct)} \quad (3.27)$$

These solutions constitute a pair of partial waves propagating along positive and negative z -directions with A_k and B_k amplitude ($k=1,2,3$) being unknown amplitudes. For the wave propagation in generalized elastic semispace ($z=0$), we choose the formal solution by discarding the increasing terms. So we get,

$$q = (B_1 e^{-m_1 z}) e^{i\xi(x-ct)} \quad (3.28)$$

$$\psi = (B_2 e^{-m_2 z} + B_3 e^{-m_3 z}) e^{i\xi(x-ct)} \quad (3.29)$$

$$\phi = \frac{\delta^2}{p} \left[(m_2^2 - \beta^{*2})(B_2 e^{-m_2 z}) + (m_3^2 - \beta^{*2})(B_3 e^{-m_3 z}) \right] e^{i\xi(x-ct)} \quad (3.30)$$

3.2 DISPERSION EQUATIONS:

This section is devoted to the derivation of secular equations of Rayleigh surface waves in the considered media under different situations. Upon invoking the boundary conditions, we obtain a system of algebraic equations in the amplitudes of B_k ($k=1,2,3$) of partial waves. This system will have a non-trivial solution if determinant of the coefficients of B_k ($k=1,2,3$) vanishes. This requirement leads to the secular equations governing the propagation of Rayleigh waves in the considered media.

Upon invoking the boundary conditions, $\sigma_{zz} = 0, \sigma_{zx} = 0, m_{zy} = 0,$

$$\mathbf{B}_1 \left(m_1^2 - \xi^2 + 2 \left(\delta^2 - \frac{\mathbf{P}}{2} \right) \xi^2 \right) + 2i\xi m_2 \left(\delta^2 - \frac{\mathbf{P}}{2} \right) \mathbf{B}_2 + 2i\xi m_3 \left(\delta^2 - \frac{\mathbf{P}}{2} \right) \mathbf{B}_3 = 0 \quad (3.31)$$

$$\mathbf{B}_2 \left[-\xi^2 c^2 + 2 \left(\delta^2 - \frac{\mathbf{P}}{2} \right) \xi^2 \right] - 2i\xi m_1 \left(\delta^2 - \frac{\mathbf{P}}{2} \right) \mathbf{B}_1 + \left[-\xi^2 c^2 + 2 \left(\delta^2 - \frac{\mathbf{P}}{2} \right) \xi^2 \right] \mathbf{B}_3 = 0 \quad (3.32)$$

$$-m_2(m_2^2 - \beta^{*2})\mathbf{B}_2 - m_3(m_3^2 - \beta^{*2})\mathbf{B}_3 = 0 \quad (3.33)$$

Now from eq. (3.31), (3.32) and (3.33) we have,

$$\begin{aligned} & \mathbf{B}_1 \left(m_1^2 - \xi^2 + 2 \left(\delta^2 - \frac{\mathbf{P}}{2} \right) \xi^2 \right) + 2i\xi m_2 \left(\delta^2 - \frac{\mathbf{P}}{2} \right) \mathbf{B}_2 + 2i\xi m_3 \left(\delta^2 - \frac{\mathbf{P}}{2} \right) \mathbf{B}_3 = 0 \\ & \left\{ \left[-\xi^2 c^2 + 2 \left(\delta^2 - \frac{\mathbf{P}}{2} \right) \xi^2 \right] \mathbf{B}_2 - 2i\xi m_1 \left(\delta^2 - \frac{\mathbf{P}}{2} \right) \mathbf{B}_1 + \left[-\xi^2 c^2 + 2 \left(\delta^2 - \frac{\mathbf{P}}{2} \right) \xi^2 \right] \mathbf{B}_3 \right\} = 0 \\ & -m_2(m_2^2 - \beta^{*2})\mathbf{B}_2 - m_3(m_3^2 - \beta^{*2})\mathbf{B}_3 = 0 \end{aligned}$$

$$\begin{vmatrix} 0 & (m_1^2 - \xi^2) + 2(\delta^2 - \frac{\mathbf{P}}{2})\xi^2 & 0 & 2i\xi m_2(\delta^2 - \frac{\mathbf{P}}{2}) & 0 & 2i\xi m_3(\delta^2 - \frac{\mathbf{P}}{2}) \\ 0 & -(\delta^2 - \frac{\mathbf{P}}{2})2i\xi m_1 & 0 & -\xi^2 c^2 + 2(\delta^2 - \frac{\mathbf{P}}{2})\xi^2 & 0 & -\xi^2 c^2 + 2(\delta^2 - \frac{\mathbf{P}}{2})\xi^2 \\ 0 & 0 & 0 & -m_2(m_2^2 - \beta^{*2}) & 0 & -m_3(m_3^2 - \beta^{*2}) \end{vmatrix} = 0$$

$$\begin{aligned}
& m_2(m_2^2 - \beta^{*2}) \begin{vmatrix} 0 & (m_1^2 - \xi^2) + 2(\delta^2 - \frac{p}{2})\xi^2 & 0 & 0 & 2i\xi m_3(\delta^2 - \frac{p}{2}) \\ 0 & -(\delta^2 - \frac{p}{2})2i\xi m_1 & 0 & 0 & -\xi^2 c^2 + 2(\delta^2 - \frac{p}{2})\xi^2 \end{vmatrix} \\
& + m_3(m_3^2 - \beta^{*2}) \begin{vmatrix} 0 & (m_1^2 - \xi^2) + 2(\delta^2 - \frac{p}{2})\xi^2 & 0 & 2i\xi m_2(\delta^2 - \frac{p}{2}) & 0 \\ 0 & -(\delta^2 - \frac{p}{2})2i\xi m_1 & 0 & -\xi^2 c^2 + 2(\delta^2 - \frac{p}{2})\xi^2 & 0 \end{vmatrix} = 0
\end{aligned}$$

On expanding the determinant, we get

$$\begin{aligned}
& 4\xi^2 m_1 m_2 m_3 \left(\delta^2 - \frac{p}{2} \right)^2 (m_2^2 - m_3^2) + \left[-\xi^2 c^2 + 2 \left(\delta^2 - \frac{p}{2} \right) \xi^2 \right]^2 \left[m_3(m_3^2 - \beta^{*2}) - m_2(m_2^2 - \beta^{*2}) \right] = 0 \\
& 4\xi^2 m_1 m_2 m_3 \delta^4 \left(1 - \frac{p}{2\delta^2} \right)^2 (m_2^2 - m_3^2) + \delta^4 \left(\beta^{*2} + \xi^2 - \frac{p\xi^2}{\delta^2} \right)^2 \left[m_3(m_3^2 - \beta^{*2}) - m_2(m_2^2 - \beta^{*2}) \right] = 0 \quad (3.34)
\end{aligned}$$

Eq. (3.34) represents the Rayleigh wave equation in micropolar medium, which is same obtained by Sharma et al (2008).

3.2.1 SPECIAL CASE OF DISPERSION EQUATION:

ELASTIC HALF SPACE:

The equation (3.34) in the absence of micropolar effect ($p=0$) and by taking $m_2 = \beta^*$, reduces to,

$$(\beta^{*2} + \xi^2)^2 - 4\alpha^* \beta^* \xi^2 = 0$$

This is the Rayleigh wave equation for homogeneous, isotropic, elastic solid, which is solved in chapter-2.

3.3 NUMERICAL RESULTS AND DISCUSSIONS:

With the view of illustrating theoretical results derived in the preceding sections we now present some numerical results. Though theory of micropolar elastic solids was established a decade back, their material constants are rare in experimental data and hence are not sufficient for analysis. Thus a hypothetical data is used to study problems related to micropolar theory. Here we have taken aluminum-epoxy like material for numerical computations whose physical data is given by (Singh and Kumar 1998).

$$\rho = 2.19 \times 10^3 \text{ Kg/m}^3, \quad \lambda = 7.59 \times 10^{10} \text{ N/m}^2, \quad \mu = 1.89 \times 10^{10} \text{ N/m}^2$$

$$K = 0.0149 \times 10^{10} \text{ N/m}^2, \quad \gamma = 0.268 \times 10^6 \text{ N}, \quad j = 0.196 \times 10^{-4} \text{ m}^2,$$

$$\omega^* = 4.364 \times 10^{11} / \text{sec}$$

VARIATION OF WAVE NUMBER AND PHASE VELOCITY:

Variation of phase velocity for insulated conditions with respect to wave number in the case of coupled theory of elasticity has been studied in fig. (3.1). It is observed that the phase velocity of Rayleigh waves in half-space decreases sharply from its peak values. In the wave number range $0 \leq R \leq 1$, increases moderately in the wave number range $1 \leq R \leq 2$ following linear variations before becoming dispersion less for $R \geq 2$. The phase velocity profiles are significantly dispersive at small values of the wave number and become dispersionless with increasing wave number. This shows that at long wavelengths the Rayleigh waves are dispersive because of resistance due to starting of interaction between various coupled fields and degree of dispersion decrease with decreasing wavelengths. Ultimately waves become dispersionless at short wavelengths when a complete saturation between interacting fields is setup in the considered solid. This is attributed to the fact that initially the coupling between the interacting fields is weak and becomes stronger and stronger at small wavelengths resulting in the pure surface wave formation.

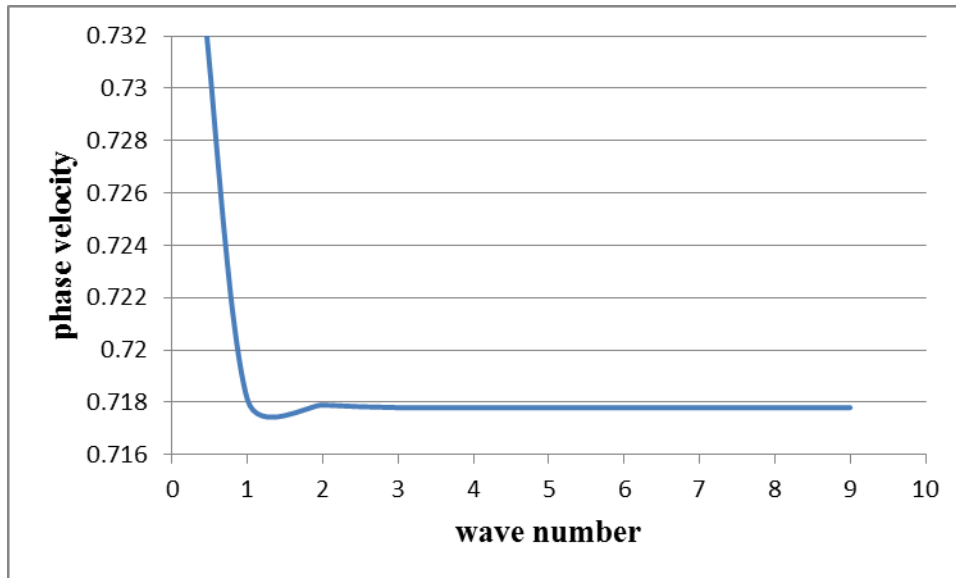


Fig. 3.1 Variation of Wave number with phase velocity.

Variation of wave-number and phase velocity for both elastic and micropolar solids is shown in fig. 3.2. It is clear from the fig. that when effects of micropolarity are considered, then the value of phase velocity corresponding to wave-number increases but it follows the same trend as in elastic solids.

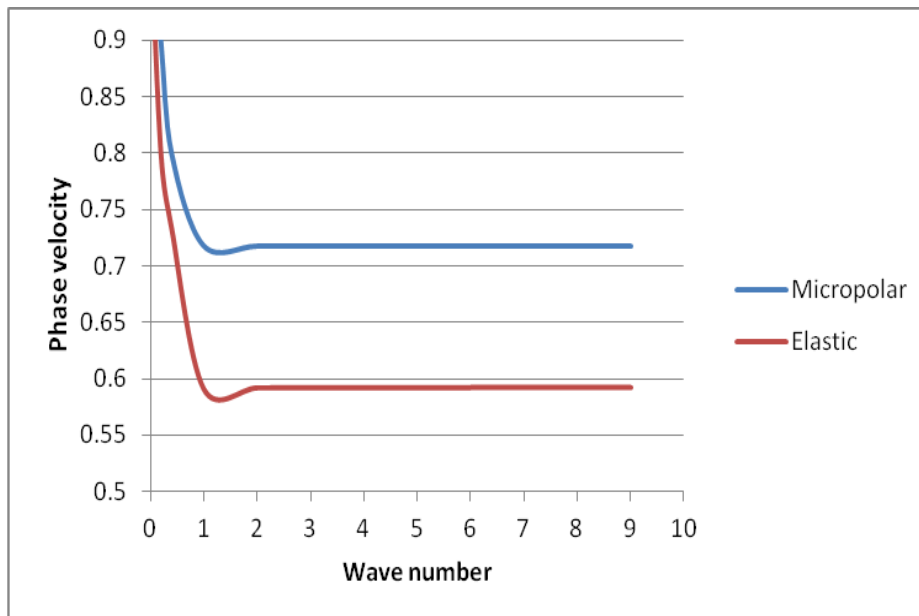


Fig. 3.2 Variation of Wave number with phase velocity.

CONCLUSIONS:

The dispersion equation for the Rayleigh waves in elastic half space and effect of micropolarity in Rayleigh waves has been concluded in the thesis. Rayleigh waves are widely used for material characterization, to discover the mechanical and structural properties of the objects. Moreover, Rayleigh waves are used in seismology, earth-quakes, bone modeling, geophysics and geotechnical engineering for the characterization of oil deposits. Seeing the applications of Rayleigh waves in geophysics and biomechanics further extension of our model may be useful in this field.

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