LINEAR PHASE LOW-PASS IIR FRACTIONAL ORDER DIGITAL DIFFERENTIATOR

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DECLARATION

I hereby declare that the work, which is being presented in the dissertation, entitled **"Linear Phase Low-Pass IIR Fractional Order Digital Differentiator"** in partial fulfilment of the requirements for the award of degree of Master of Engineering in Electronics and Communication Engineering submitted at the Department of Electronics and Communication Engineering of Thapar University, Patiala, is an authentic record of my own work carried out under the guidance of **Dr. Sanjay Kumar (Assistant Professor)**, Department of Electronics and Communication Engineering and refers other research's work which are duly listed in reference section.

The matter presented in this dissertation has not been submitted in any other University/Institute for the award of degree.

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It is certified that the above statement made by the student is correct to the best of my knowledge and belief.

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ABSTRACT

The design of digital filters at low frequency range has become increasingly important as it can be used to design all types of filters. It is found in many applications from low frequency biomedical equipment to high frequency radars. Use of Integer order calculus for the purpose of design often results in narrow bandwidth for the low-pass. With the development of fractional order calculus in recent years the response becomes more ideal. The major objective of this work is to understand the different design strategy for digital differentiators and compare their response for various orders and to explore new design techniques for designing fractional order IIR differentiator.

A stable minimum phase, low-pass IIR digital differentiators is developed by inverting the transfer functions of a class of numerical integrators, stabilizing the resulting transfer functions and compensating their magnitudes. The class of digital integrators is first obtained by interpolating the various numerical integrators. The designed digital differentiator is modelled to find the correct response by passing some test signal. The low order and high accuracy of the filters make them attractive for real time applications.

A method for optimizing low-pass infinite impulse response (IIR) digital differentiators is presented in this dissertation. The wide band differentiator is cascaded with low-pass IIR filter resulting in linear phase low-pass IIR digital differentiator. Further the frequency response of IIR differentiator is improved by altering the gain and denominator coefficients of that differentiator. The genetic algorithm approach is used for optimizing the least square error that is defined by the fitness function. The low order optimized IIR differentiator in this paper is almost completely approximate the higher order finite impulse response (FIR) differentiator.

A novel approach for designing fractional order low-pass infinite impulse response (IIR) digital differentiators is also introduced. First, the numerical differentiators are obtained by inverting the weighted transfer function results from interpolation of various numerical integration rules. The half-order numerical differentiator is expressed in terms of higher order by using continuous fraction expansion. This discretized half order differentiator is cascaded with appropriate low-pass IIR filter resulting in linear phase low-pass IIR fractional order digital differentiator. The simulation studies have shown that, the

fractional order IIR differentiator gives better results as compared integer order IIR differentiators.

Many methods have been developed to design all types of differentiators but there is still scope of improvement in terms of parameters optimization. The design problem of differentiators is a challenging task. Therefore, there is strong motivation to make design process easy and efficient.

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ABBREVIATIONS

Abbreviations	Description
ADC	Analog to Digital Converter
CFE	Continued Fraction Expansion
DAC	Digital to Analog Converter
DD	Digital Differentiators
DSP	Digital Signal Processing
ECG	Electrocardiogram
FIR	Finite Impulse Response
FrODD	Fractional Order Digital Differentiator
GA	Genetic Algorithm
GL	Grunwald and Letnikov
IIR	Infinite Impulse Response
LP	Low-Pass
LPD	Linear Phase Differentiator
MSE	Mean Square Error
NR	Non-Recursive
PSE	Power Series Expansion
RE	Relative Error
RL	Riemann and Liouville
SA	Simulated Annealing
WLS	Weighted Least Square

CHAPTER
1

Introduction

1.1 Filters

In signal processing, the filters are the electronic circuits which perform the function especially to remove the unwanted frequency components from the signal and enrich the wanted signal. A circuit which is implemented to perform the frequency selection is termed as filter. The electronic filters are classified as [1]:

- Passive or Active
- Analog or Digital
- Infinite impulse response or Finite impulse response

1.1.1 Passive Filters or Active Filters

Passive Filters: The implementations of these filters are based on combinations of resistors (R), inductors (L) and capacitors (C). These types are conjointly known as passive filters, because the passive components usually do not depend upon an external power supply and they do not comprise any active components such as transistors. Inductors block high-frequency signals and pass low-frequency signals, while capacitors do the inverse operation. In low-pass filter, the signal is passes through an inductor while capacitor provides a path to the ground. The less attenuation is presented at low-frequency signals than high-frequency signals. A filter in which the signal passes through a capacitor and an inductor provides a path to ground, and provides less attenuation to high-frequency signals than low-frequency signals and is consequently a high-pass filter. Resistors in the circuit have no frequency-selective properties, but are added with inductors and capacitors to determine the time-constants of the circuit.

Active Filters: The implementation of active filters is based on combination of passive and active (amplifying) components, and requires an outside power source. Operational amplifiers are commonly used in active filter designs. These filters have high quality factor, and can achieve resonance without the use of inductors. However, the upper frequency limit of these filters is limited by the bandwidth of the amplifiers.

1.1.2 Analog Filter or Digital Filter

Analog Filter: An analog filter has an analog signal at both its input and its output. Both input and output are functions of a continuous variable time and can have an infinite number of values. An analog filter uses analog electronic circuits made up from components such as resistors, capacitors, inductors and certain op-amps to produce the required filtering effect. Unlike digital, analog filters works on analog signals or the so called actual signals. The transfer function of an analog filter is expressed in Laplace domain. To be stable and causal, the transfer function $H(s)$ of the filter must be a rational function of s, whose coefficients are real. For stability and causality, the poles of transfer function should lie on the left half of s-plane.

Digital Filter: A digital filter is a system that performs mathematical operations on a sampled, discrete-time signal to diminish or enrich certain aspects of that signal. This is in contrast to the other major type of electronic filter, which is analog filter, operating on continuous-time analog signals. An analog signal might be processed by a digital filter by first being digitized and represented as a sequence of numbers, then employed mathematically, and then reconstructed as a new analog signal.

In an analog filter, the input signal is "directly" employed by the circuit. A digital filter system mainly consists of an analog-to-digital converter to sample the analog input signal, followed by a microprocessor and various peripheral components such as memory to store data and filter coefficients etc. A digital-to-analog converter is required at the output stage. It operates on the digital samples of the signals. While implementing the digital filters in hardware or software , the digital logic components likes adders, subtractors, delays, etc. are required. In this filter, the filter coefficients are designed to meet the desired or expected frequency response. Mathematically the transfer function $H(z)$ of the digital filter is required to be a rational function and expressed in z-transform domain. In order to be stable and causal digital filter, the poles of the transfer function should lie inside the unit circle in z-plane.

1.1.3 Infinite Impulse Response (IIR) Filters or Finite Impulse Response (FIR) Filters

IIR Filters: The impulse response of an IIR filters is non-zero over an infinite length of time. This is in contrast to fixed-duration impulse responses of the FIR filters. IIR filters can be realized as either analog or digital filters. In digital IIR filters, the output feedback is directly apparent in the equations defining the output. Note that in different FIR filters, in designing IIR filters it is essential to carefully consider the "time zero" case in which the outputs of the filter have not yet been clearly defined.

Design of digital IIR filters is heavily reliant on that of their analog counterparts because there are plenty of resources, works and upfront design methods concerning analog feedback filter design while there are hardly any for digital IIR filters. Usually, for implementing a digital IIR filter, an analog filter (e.g. Chebyshev filter, Butterworth filter, Elliptic filter) is first designed and then is converted to a digital filter by applying discretization techniques such as Bilinear transform or Impulse invariance. The IIR filters provide better magnitude response with low order due to feedback. But the main disadvantages of IIR filter is that it provides non-linear phase at the output. IIR filters are recursive and used as an alternate.

FIR Filters: In signal processing, a FIR filter is a filter whose impulse response (or response to any finite length input) is of finite duration, because it becomes zero in finite duration of time. This is in contrast to the IIR filters, which may have internal feedback and may continue to respond indefinitely.

FIR filters can be of any form, discrete-time or continuous-time, and digital or analog. A FIR filter has a number of useful properties which sometimes make it preferable to an IIR filter. FIR filters not required any feedback. This makes FIR filter implementation simpler. The linear phase FIR filters are designed by making their coefficient sequence symmetric. This property is sometimes desired for phase-sensitive applications, for example data communications, mastering and, crossover filters.

The main disadvantage of FIR filters is that considerably more computation power in a general purpose processor is required compared to an IIR filter with similar sharpness or selectivity, especially when low cut-off frequency (relative to the sample rate) are desired.

However several digital signal processors provide specific dedicated hardware features to make FIR filters approximately as efficient as IIR for many applications.

1.2 Differentiators

A Differentiator is an electronic circuit that is designed such that the output of the circuit is approximately directly proportional to the rate of change of the input.

A passive differentiator circuit is made of only passive components like resistors, capacitors, and inductors. An active differentiator comprises some form of amplifier. The differentiator circuit is basically a high pass filter.

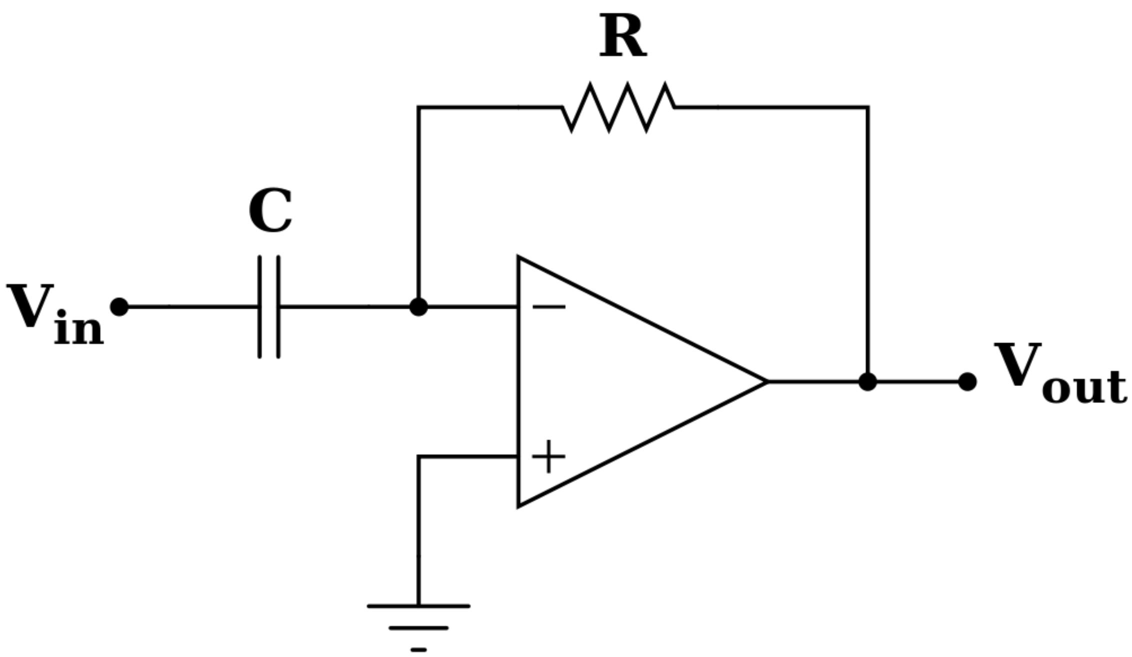


Figure 1.1: An active differentiator [2].

1.3 Operation of a Basic Differentiator

An active differentiator is shown in Figure 1.1. Any input signals are applied through the capacitor C. Capacitive reactance $X_c = 1/\omega C$ has the significant role in the analysis of the operation of a differentiator. The term capacitive reactance is directly proportional to the rate of change of input voltage applied to the capacitor. At low frequency, the capacitive reactance is very high and reactance is low for high frequency. Consequently, at low frequencies and for slow changes in input voltage, the gain of differentiator is very low, while at higher frequencies and for fast changes in input voltage, the gain of differentiator is high, producing larger output voltages.

The input voltage V_{in} is applied across capacitor C as shown in Figure 1.1. The input voltage V_{in} is calculated as follows [2]:

$$V_{in} = V_{series} = I.Z_{series} = \sqrt{R^2 + \left(\frac{1}{\omega C}\right)^2} \quad (1.1)$$

here $R << 1/\omega C$, so $V_{in} \cong 1/\omega C$, and for low frequency $\omega << 1/\omega C$, the input voltage is equivalent as $V_{in} \cong V_C$. The output voltage V_{out} across resistor R is calculated as follows:

$$V_{out} = V_R = iR = R\frac{dq}{dt} = RC\frac{dV_C}{dt} \quad (1.2)$$

$$V_{out} \cong RC\frac{dV_{in}}{dt} \quad (1.3)$$

If a constant DC voltage is applied as input to the differentiator, then the output voltage is zero. If the input voltage changes from zero to negative, the output voltage is positive. If the applied voltage across differentiator changes from zero to positive, then output voltage is negative. If a square wave is applied as input through a differentiator, then a triangular waveform is obtained at the output.

This simple differentiator becomes unstable at high frequencies and starts to oscillate. This differentiator gain at high is reduced by adding a small value capacitor across feedback resistor R_f or a resistor in series with the capacitor.

1.4 Digital Differentiators

In signal processing, differentiation is a significant tool for the applications which require the determination or estimation of the time derivatives of a given signal. It gives a measure of instantaneous rate of change or slope. Typically, for example, in radar and sonar, the velocity and acceleration are computed from the position measurements using differentiation. The transfer function of an ideal differentiator is given as follows [3]:

$$H(\omega) = j\omega \quad (1.4)$$

Depending upon the frequency region, the differentiators are classified as

a) Full band differentiator $0 < \omega < \frac{\pi}{T}$

b) Wide band differentiator $0 < \omega < \frac{0.7\pi}{T}$

c) Narrow band differentiator for small value of ω

where T is the sampling period.

1.5 Digital Differentiator Applications

- In radar and sonar, the velocity and acceleration are computed from the position measurement using differentiation. The velocity is estimated by first order differentiation and acceleration by second order [4].
- In biomedical investigation, it is often necessary to obtain the first and higher order derivatives of the biomedical data, especially at low frequency range. For example in QRS complex detection in ECG analysis [5].
- The derivatives at high frequencies are useful for solving the problems of image restoration and image texture enhancement (to detect various features, like an edge, for example, of an object in the picture) [6].
- The use of derivatives for various signals in control engineering (in auto-follow, servomechanism, robotics, artificial eye etc.) is also well known [7].
- Fractional dimension is used to measure some real-world data such as coastline, clouds, dust in the air, and network of neurons in the body [8]. The fractional dimension has been applied usually in pattern recognition and classification [9]. Fractional Order Differentiators are used to exploit such real world issues. Fractional Order Differentiators are also employed in bar code readers [10].

CHAPTER 2

Literature Survey

2.1 Survey

Kumar *et al*. [3] proposed an alternative technique to achieve the same performance with much lower order. The order of the minimax relative error digital differentiators becomes very large for extremely low relative error in the low-frequency range. Various types of digital differentiators (DD) have been designed using the minimax criterion, which amounts to minimization of the Relative Error (RE) in the frequency range of interest. The ideal DD gives negligible output at frequencies close to dc. In many practical applications such as the Doppler radar or sonar, however, accurate measurement of differentiated signals, at low frequencies, becomes necessary. This paper proposes optimal differentiators which are maximally accurate (i.e. they have minimal RE) at low frequencies. The proposed DDs have been derived from the maximally flat non-recursive (NR), low-pass (LP) digital filters. Optimal, maximally accurate digital differentiators have been derived for low-frequency range. If the frequency of differentiation is low, the better is the performance of the proposed differentiators, therefore making them suitable for many typical real-time applications.

Krishna *et al*. [6] presents the theory of operation, applications and implementation of digital differentiators. For real time applications, the differentiator must have their order small as much possible. This paper presents the different procedures for the designing of FIR and IIR type digital differentiators. The IIR Type digital differentiators are obtained by inversion and magnitude stabilization of the existing digital integrators. In some applications like controls, wave shaping, oscillators and communications require a constant 90^0 phase for differentiators. Here in this paper, an endeavoring challenge is made to study about the variation of phase angle of digital differentiators with the application of fractional delay. It has been perceived that the digital differentiators have shown superior performance compared to the well-known gradient method. With the recognized efficiency of the differentiators in the various applications, these are implemented in hardware using Verilog. The low order of these IIR digital differentiators is more favorable for real-time application. The first-order IIR differentiator from Al-Alaoui is ideal in terms of the frequency response. The IIR differentiators can be adaptively used when systems experience high dynamics. The digital differentiators of IIR type have been proved to be much more efficient in detecting edges of an image, QRS detection etc.

Selesnick [10] proposed an approach which describes the design of type III and type IV linear-phase finite-impulse response (FIR) low-pass digital differentiators according to the maximally flat criterion. The paper introduces a two-term recursive formula that enables the simple stable computation of the impulse response coefficients. The similar recursive formulation is valid for both Type III and Type IV solutions. The solutions cannot be obtained from a low-pass filter as in the case of a full-band differentiator. The algorithms for automatic sum generalization are used in this paper to obtain a simple two-term recurrence relation for computing the coefficients of the impulse response. There are several possible extensions to the problem described in this paper. For example, the extension of the recursive formulas to the case where the maximally flat approximation to the ideal differentiator is performed not at $\omega = 0$ but at another frequency ω_0. This type of solution is relevant when the signal is centered on a known frequency. Another remaining question is the existence of low-complexity structures for maximally flat differentiators. Those structures are multiplier less and have a regular structure. Another extension of the approach described in this paper is the design of second (and higher) order differentiators where the desired frequency response is $-\omega^2$.

Al-Alaoui [12] introduced a novel approach to designing recursive stable digital differentiators. A four-step procedure for designing the differentiator is presented. The designing procedure consists of obtaining an integrator and then modifying its transfer function appropriately to obtain a stable differentiator. An example describing a second-order recursive digital differentiator is presented. The low order of the differentiator makes it suitable for real-time applications. The resulting differentiator approximates an ideal differentiator in the pass-band region with an accuracy and range comparable to those obtained by higher order filters. In addition, it has an almost linear phase in the pass-band region.

Al-Alaoui [13] presented an approach for designing the digital differentiator from digital integrator. In this paper the first order integrator and differentiator are described and thus eminently suitable for real time applications. The obtained differentiation has almost linear phase. First, the digital integrator is obtained by interpolating two popular numerical integration techniques, the rectangular (Euler) and the trapezoidal rules. First non-minimum phase digital integrator is designed and then convert that to minimum phase digital integrator by stabilizing the poles and magnitude is compensated. The resulting integrator outperforms both the rectangular and the trapezoidal integrator in range and accuracy. The new digital differentiator is then obtained by inverting the transfer function of the digital integrator. The effective range of the resulting digital differentiator is up to 0.78 of the Nyquist frequency.

Al-Alaoui [14] introduced a technique to implement the second order differentiator from the Simpson integrator. The proposed differentiator is a stable second-order recursive differentiator suitable for applications that require fast differentiation methods. The proposed differentiator has its accuracy and the range of the magnitude response is similar as that of the Simpson integrator. The low order of the differentiator makes it suitable for real-time applications. This second-order differentiator approximates an ideal differentiator in the pass band region with accuracy comparable to that obtained by higher order filters. The proposed differentiator has an almost linear phase at low frequencies. The Simpson integrator approximates the ideal integrator for low frequencies while it amplifies the higher frequencies. Thus, the differentiator obtained by inverting the transfer function of the Simpson's integrator would yield a filter that has high accuracy differentiation capabilities at low frequencies.

Al-Alaoui [15] proposed the well-known class of first-order digital differentiators and integrators can be simply derived from a classical continuous-time approximate differentiator by using the bilinear transformation. The achieved results represent a simple and alternative method of deriving the Al-Alaoui operator (Al-Alaoui’s operator, differentiator and integrator transform). In this paper, the example of first order integrator is taken, that is obtained by interpolating the rectangular and trapezoidal integrator. As suggested by a reviewer of this letter, it might be possible to use the presented method to derive higher order digital differentiators and integrators, which might be an interesting topic for future research.

Al-Alaoui [16] developed a novel class of stable, minimum phase, second-order, low-pass digital differentiators. It is achieved by inverting the transfer functions of a class of

second-order integrators, stabilized the resulting transfer functions, and compensating their magnitudes. The transfer functions of second-order integrators are obtained by interpolating the traditional Simpson and trapezoidal integrators. The resulting interpolated integrators have a perfect -90 degree phase over the Nyquist interval and could better approximate the ideal magnitude response than either of the two traditional integrators. The resulting integrators and differentiators extend the frequency range of operation beyond that possible by using either of the two traditional integrators.
The low order and high accuracy of the resulting differentiators make them attractive for real time applications. The basic concept came from observing that the ideal integrator response lies between the responses of the traditional trapezoidal and Simpson integrators. Hence it looks realistic that interpolating the above two rules could yield integrators that better approximate the ideal integrator. Additionally there is one free parameter that can be adjusted by imposing an appropriate constraint. The low-pass differentiators have the appropriate magnitude response of zero at zero frequency and at the Nyquist frequency. The phase response of the resulting differentiators is almost linear over the frequency ranges of interest.

Ngo [17] presented a general theory of the Newton–Cotes digital integrators and differentiators, which is derived by applying the z-transform technique to the closed-form Newton–Cotes integration formula. Based on this established fundamental theory, a new wideband third-order trapezoidal digital integrator is found to be a class of trapezoidal digital integrators. Based on the designed wideband third-order trapezoidal integrator, a new digital differentiator having wideband frequency range is designed, which approximates the ideal differentiator reasonably well over the whole Nyquist frequency range and compares favorably with existing differentiators. The high accuracy of these low order novel wideband integrator and the new wideband differentiator have been proven attractive for real-time applications.

Al-Alaoui [18] introduced two novel approaches to designing approximately linear phase infinite-impulse-response (IIR) digital filters in the pass band region. Low-pass filtering and differentiation can be implemented as a single low-pass differentiator filter or by using a low-pass filter and a differentiating filter in cascade. Differentiation is used to extract information about rapid transients in the signal. The noise frequencies higher than the cutoff frequencies presented in the signal are rejected by low-pass filter. Digital filters with exact linear phase may only be obtained by designing symmetric coefficients FIR filters. IIR filters may be designed to meet the magnitude requirements with much smaller orders than their FIR counterparts, at the expense of obtaining nonlinear phases. Typically, for the same magnitude response specifications, the order of the resulting IIR filter is one sixth the order of corresponding FIR filter.

In this paper two methods are proposed to obtain novel low-pass IIR differentiators with linear phases in the pass band regions. The first method employs cascading a wide-band differentiator with appropriate low-pass filters so that a linear phase in the pass band region is obtained. The second method is a constrained optimization method where the numerator of the transfer function of the IIR filter corresponds to a linear phase filter while the coefficients of the denominator are allowed to vary to obtain an optimum solution. The approaches utilize the linear phase properties of the FIR filters and the steeper magnitude roll-off properties of the IIR filters to obtain IIR low-pass digital differentiators. In this paper it shown that the proposed low-pass differentiator have shorter transition regions, and thus better ability to suppress high frequency noise, for much lower order filters, than the corresponding FIR filters, In addition, the new low-pass differentiators exhibit almost linear phases in their corresponding pass band. The new low-pass differentiators presented in this paper compare favorably with the state-of-the-art low-pass FIR digital differentiators.

Tahmasbi *et al*. [21] introduced a novel approach is proposed for approximating Parks-McClellan low-pass differentiators using optimized low-order IIR filters. Indeed, a suitable IIR differentiator is designed for approximating Parks- McClellan Low pass differentiator using improved Al-Alaoui's method, the denominator polynomial coefficients of resulting transfer function are altered by genetic algorithm to optimize the frequency response. An appropriate fitness function is defined to optimize both magnitude and phase responses; moreover, proper weighting coefficients and GA parameters are testified for several cut-off frequencies. The fitness function for Genetic algorithm is defined total weighted least square error. For instance, in this paper, a third-order low-pass Elliptic filter having 0.1dB ripple in the pass-band with 40dB stop-band attenuation is cascaded with Al-Alaoui operator, the fourth-order low-pass IIR differentiator is developed. The resulting transfer function of IIR differentiator is optimized using GA. The simulation studies carried out in this paper have shown that the final optimized IIR LPDs yield a frequency response which is almost equal to order-30

Parks-McClellan LPDs, and also yield almost linear phase in the pass-band; furthermore, these LPDs yield steeper roll-off properties and smaller magnitude error than Al-Alaoui's one; the percentage error of magnitude response in the pass band is less than 0.5%.

Chen *et al*. [25] presented a paper for fractional-order differentiator s^r where r is a real number, describing its discretization step for its digital implementation. Two discretization schemes are presented in this paper. In first scheme, Tustin operator is directly discretized recursively. The second scheme employs direct discretization method using the Al-Alaoui operator via continued fraction expansion (CFE). This brief firstly focuses on the direct discretization method using the well-known Tustin operator which is a straightforward scheme to discretize the fractional-order derivative. The major contribution of this brief is to introduce a recursive formula for discretization with different order of approximation which simplifies the programming efforts. Moreover, the discretized transfer function is stable and minimum phase. However, the Tustin operator based discretization scheme exhibits large errors in high frequency range. A new scheme consists of both Euler and Tustin operators is proposed which yields the Al-Alaoui operator.
Using the continued fraction expansion of the Al-Alaoui operator, this paper contributes a new direct discretization scheme with a very good magnitude fit to that of the original continuous fractional differentiator. The approximate discretization is minimum phase and stable. Comprehensive discretization procedures and tiny MATLAB scripts are given.

Chen *et al*. [26] presented a new infinite impulse response (IIR)-type digital fractional order differentiator (DFOD) by using a new family of first-order digital differentiators expressed in the second-order IIR filter form. The low-pass integer first-order digital differentiators are obtained by the stable inversion of the weighted sum of Simpson integration rule and the trapezoidal integration rule. The corresponding fractional-order digital differentiator via CFE (Continued Fraction Expansion) truncation is also presented. The distinguishing point of the proposed DFOD lies in an additional tuning knob to compromise the high-frequency approximation accuracy.

Chen *et al*. [27] presented an expository review of continued fraction expansion (CFE) based discretization schemes for fractional order differentiators defined in continuous time domain. The schemes revised here is restricted only to infinite impulse response (IIR) type generating functions of first and second orders, despite the fact, high-order IIR type generating functions are possible. For the first-order IIR differentiator, the widely used Tustin operator and Al-Alaoui operator are considered. For the second order IIR differentiator, the generating function is achieved by the stable inversion of the weighted sum of Simpson integration formula and the trapezoidal integration formula, which comprises several previous discretization schemes as special cases. Numerical models and sample codes are encompassed for illustrations.

Clearly, although only first- and second-order IIR-type generating functions are reviewed in this paper, using high-order IIR-type generating functions for discretizing fractional order differentiators is totally possible. The question is if it is worthwhile to consider the high-order case. Another question is, among all possible second-order IIR-type generating functions, what the best generating function is to given a balanced frequency- and time-domain approximation of fractional order differentiators.

Ferdi *et al*. [28] introduced three techniques for transformation from *s*-to-*z* domain, power series expansion (PSE) and signal modelling are combined to develop a new procedure for efficiently computing the fractional order derivatives and integrals of discrete-time signals. First a mapping function chosen in between the *s*-plane and the *z*-plane, and then a PSE method for this mapping function raised to fractional order is performed to get the desired infinite impulse response of the ideal digital fractional operator. Finally, the impulse response that is required is being modelled as the impulse response of a linear invariant system whose rational transfer function is determined using deterministic signal modelling techniques. Three non-iterative techniques, namely Pad´e, Prony and Shanks' methods have been considered in this paper. Using Al-Alaoui's operator as *s*-to-*z* transform, computation examples demonstrates that both Prony and Shanks' method can achieve more accurate fractional differentiation and integration than Pad´e method which is equivalent to continued fraction expansion technique.

A novel procedure for the computation of fractional order derivatives and integrals of discrete-time signals has been proposed. The proposed approach allows optimal determination of the model coefficients since optimal estimation algorithms may be used, and hence improves approximation accuracy as it has been demonstrated through computation examples. It remains as topic for our future research work to consider iterative methods of signal modelling to determine the approximate rational function coefficients.

Lau *et al*. [30] presented a new method for the design of approximately linear phase IIR filters is proposed. This technique is based on first designing an approximately linear phase asymmetrical FIR filter which satisfies the magnitude specifications and then obtaining the low order IIR filter. The design of an asymmetrical FIR filter begins with the design of a symmetrical FIR filter satisfying the given specifications. Using the symmetrical filter, the asymmetrical filter is obtained with the shift, truncate, and zero pad technique. The shifting process involves shifting the impulse response coefficients of the filter to the left so that the filter becomes anti-causal. The truncation process involves truncating the anti-causal part of the filter. The zero pad process involves padding the filter to the right by zeros so that the order of the filter is restored to the original order. This filter is then reduced using model reduction techniques to obtain a low order IIR filter which meets the original magnitude response specifications while maintaining an approximately linear phase characteristic in the passband. This technique gives better results than the equalization approach in terms of the filter order. Numerical studies indicate that the order of the IIR filters obtained using asymmetrical FIR filters (as proposed in the paper) is considerably lower than those obtained using symmetrical FIR filters.

Auger [31] refocuses on three classes of s-to-z transforms. For each individual class, a closed-form expression is proposed, and for a specific element, having an imaginary part of its frequency response very close to the ideal value, is presented. All these three s-to-z transforms are then related to all-pass infinite-impulse-response fractional delay filters, permitting alternative way to select their degree of freedom. In this brief, a new expression of the Al-Alaoui differentiators has first been proposed. Being more simple and parameterized by a physically significant coefficient, this expression should encourage broader use of these s-to-z transforms. These differentiators can also be related to second-order fractional delay filters. Similar results have been presented for the first and second order s-to-z transforms recently proposed by Pei and Hsu. Closed-form expressions are given, and particular elements having a frequency response with a nearly ideal imaginary part are evidenced. These elements are deduced from complicated expansions that can easily be performed by a computer algebra system. A relationship with one fractional delay filter is possible for the first-order s-to-z transforms only. These differentiators provide an interesting way to design a discrete-time transfer function from a continuous-time one.

Upadhyay *et al*. [32] modelled a new design of a recursive wideband digital differentiator which is obtained by optimizing the pole-zero locations of existing recursive wideband digital differentiators. Further, a new integrator is then obtained by inverting the transfer function of the proposed digital differentiator design with suitable alterations. The attractiveness of these designs is that they are only of second-order recursive systems and have not more than 0.48% relative error in magnitude responses almost over the full Nyquist band, therefore these are suitable for real-time circuit applications. Simulation results carried out in this paper shows that the proposed designs have not more than 0.48% relative error in magnitude responses over the full Nyquist band except near $t, \omega = \pi$ and have either improved or comparable phase responses with existing designs of higher-order recursive systems. Thus, attractive real time signal processing applications can be achieved using the proposed differentiator and integrator designs, which are of second-order recursive systems.

Leulmi *et al*. [33] proposed an improvement of the rational approximation of continuous fractional order differentiators and integrators of type s^{α} obtained using direct discretization approach of power series expansion-signal modeling technique by using a new second order s-to-z transform obtained by inversion and stabilizing the digital integrator designed from conventional Simpson integrator and fractional delay filter. The resulting new IIR type digital fractional order differentiators have much better frequency characteristic as compared to the Al-Alaoui operator based approximations.

2.2 Gaps in the study

Based on the literature survey following gaps has been identified during study:

- Digital differentiators can leads to approximation errors in high frequency regions, and these differentiators will not give high accuracy with low order.
- In some cases the obtained transfer functions have poles that lie on the unit circle of the z-plane which obviates the application of the stabilizing method which consists of reflecting the pole that lies outside the unit circle at radius r to inside the unit circle at a radius of 1/r and compensating the magnitude response by multiplying the resulting transfer function by 1/r.
- In constrained optimization approach for designing IIR digital differentiator, there is always trade-off between magnitude response and phase response.

- The fundamental limitation for realizing linear phase IIR filters approach is variation in group delay. That will leads to higher processing delay and cause phase and harmonics distortion.

2.3 Objective of the dissertation

The main objectives of this dissertation are mentioned as follows:

- Mathematical analysis of IIR differentiators will be carried out using various numerical integration rules.
- Optimizing the frequency response of integer order low-pass IIR differentiator by using constrained optimization approach.
- A new mathematical approach for the analysis and design of IIR fractional order digital differentiator will be carried out.

2.4 Organization of the dissertation

This dissertation consists of total six chapters which are organized as follows:

Chapter 1: Introduction, it consists of introduction to various types of filters, digital differentiators and their applications.

Chapter 2: Literature Review, study of research papers of related fields in sequence has been discussed.

Chapter 3: Numerical Differentiator, in this chapter, the design methodologies of different types of numerical differentiators are carried out. In the later section, a novel approach for designing IIR differentiator from numerical integrator is introduced.

Chapter 4: Low-Pass IIR Integer Order Digital Differentiator, in this chapter, the fourth order low-pass IIR differentiator is designed. The frequency response of integer order low-pass IIR differentiator is optimized by using constrained optimization approach.

Chapter 5: Linear Phase Low-Pass IIR Fractional Order Digital Differentiator, in this chapter, a novel approach for designing IIR fractional order differentiator is introduced. The comparison of integer order versus fractional order differentiator is also discussed.

Chapter 6: Conclusions and Future Scope of Work, in this chapter, the entire work that has been carried out in this dissertation is concluded, on the basis of observations, the future scope of work has been discussed.

CHAPTER
3
Differentiators based on Numerical Analysis Techniques

3.1 Introduction

In numerical analysis, a comprehensive family of algorithms for calculating the numerical value of a definite integral are constituted in numerical integration, and by extension, the term is occasionally used to describe the numerical solution of differential equations. Such algorithms use the concept of area approximation under the curve for calculating the integration of particular function.

There are several algorithms which are used for calculating the integration by numerical methods. The algorithm which better approximate the area under the curve for specific function gives the less error for integration calculations. The numerical differentiators are then obtained by inverting the transfer function of resulting integrators.

3.2 First Order Approximation Scheme

The commonly used integration methods for first order approximation are forward rectangular rule, backward rule and trapezoidal rule [11].

3.2.1 The Forward Rectangular Rule

The forward rectangular rule (forward Euler method) employs a first order approximation of the function being integrated. This approximation is made by discretizing a function, $f(x,t)$ into n points and then calculating the area associated with the $n-1$ successive polygons which results [11]. In Figure 3.1 the integral of this first order system is approximated by the summation of successive polygons such as ABCD located between time points n and $n+1$.

If y_n is the integral of the function at time sample n, the integral y_{n+1} at time sample $n+1$ is equal to the integral at the previous time sample n and summed the area of newly created polygon ABCD. The polygonal area is equal to the T (time step) multiplied by the height of function $f(x,t)$ at that time sample n.

$$y_{n+1} = y_n + Tf(x_n, t_n) \tag{3.1}$$

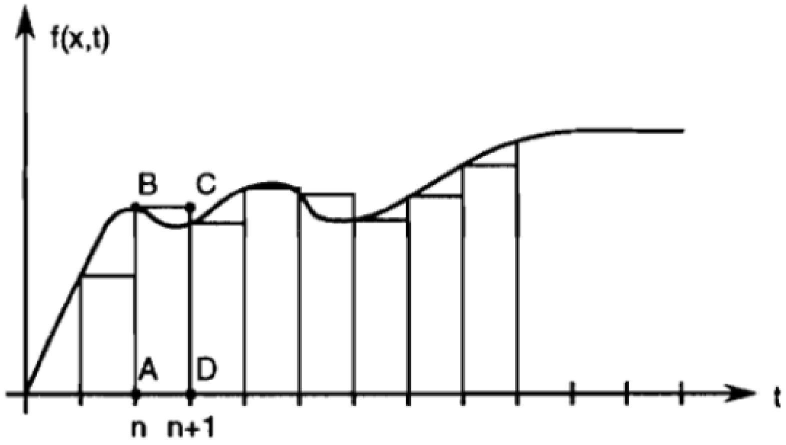


Figure 3.1: Approximation of x(t) by forward rectangular rule [11].

Equation (3.1) can be written as

$$y[(n+1)T] = y(nT) + x(nT)T \tag{3.2}$$

Applying z-transform on both sides of equation (3.2), we get

$$z\,y(z) = y(z) + Tx(z) \tag{3.3}$$

$$y(z)(z-1) = Tx(z) \tag{3.4}$$

$$H_{r\_For}(z) = \frac{y(z)}{x(z)} = \frac{T}{z-1} \tag{3.5}$$

The transfer function in equation (3.5) represents the numerical integrator resulting from the approximation of forward rectangular rule.

3.2.2 The Backward Rectangular Rule

The backward rectangular rule (backward Euler method) also employs a first order approximation of the function that being integrated. Due to the approximation of the integral based on the past value of $f(x_n, t_n)$ the introduction of instabilities into the solution is quite possible. A modification of the forward rectangular rule is to base the area of the polygon on the present value of $f(x,t)$ at time sample $n+1$. In Figure 3.2 the integral of this first order system is approximated by the summation of successive polygons such as ABCD located between time points n and $n+1$.

Again, if y_n is the integral of the function at time sample n, the integral y_{n+1} at time sample $n+1$ is equal to the integral at the previous time sample n and summed the area of newly created polygon ABCD. The polygonal area is equal to the T (time step) multiplied by the height of function $f(x, t)$ at that time sample $n + 1$.

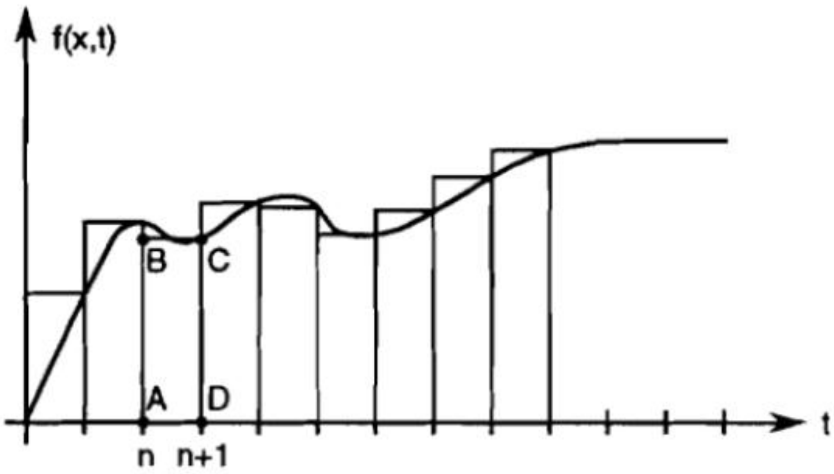


Figure 3.2: Approximation of x(t) by backward rectangular rule [11].

$$y_{n+1} = y_n + Tf(x_{n+1}, t_{n+1}) \quad (3.6)$$

Equation (3.6) can be written as

$$y[(n + 1)T] = y(nT) + x[(n + 1)T]T \quad (3.7)$$

Applying z-transform on both sides of equation (3.7), we get

$$z\, y(z) = y(z) + z\, Tx(z) \quad (3.8)$$

$$y(z)(z - 1) = z\, Tx(z) \quad (3.9)$$

$$H_{r\_Back}(z) = \frac{y(z)}{x(z)} = \frac{z\, T}{z-1} \quad (3.10)$$

The transfer function in equation (3.10) represents the numerical integrator resulting from the approximation of backward rectangular rule.

3.2.3 The Trapezoidal Rule

The Trapezoidal rule too employs a first order approximation of the function that being integrated. As demonstrated in figure 3.1 and figure 3.2, the forward rectangular (Euler) approximation is greater than the actual integral, while the backward rectangular approximation is certainly less than the actual integral. Therefore, it would make sense to take the average of two and use this more accurate approximation [11]. By this precise way, the trapezoidal rule is constructed. The trapezoidal rule is most widely used integration scheme in digital signal processing.

Trapezoidal rule of integration involves the summation of a successive number of trapezoidal regions which approximate the integral of function. As shown in the Figure 3.3 the approximate area under the curve from time n to $n+1$ is equal to the area of trapezoidal ABCD.

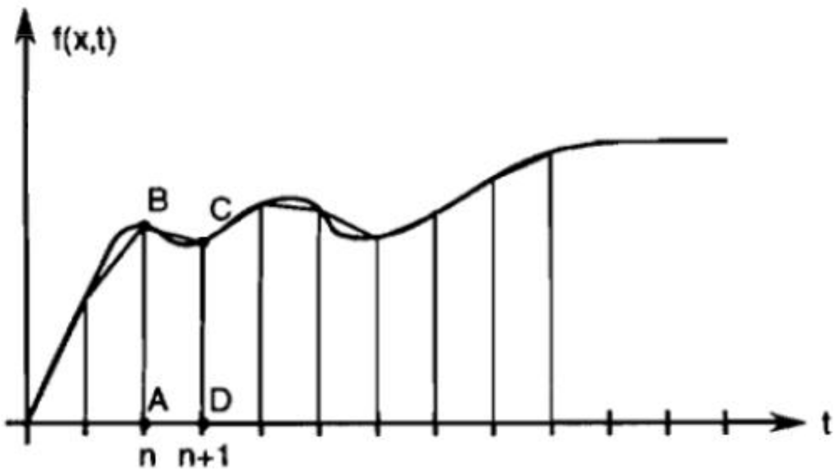


Figure 3.3: Approximation of x(t) by trapezoidal rule [11].

If y_n is the integral of the function at time sample n, the integral y_{n+1} at time sample $n+1$ is equal to the integral at the previous time sample n and summed the area of newly created trapezoidal ABCD between the time sample n and $n+1$. [Note: Trapezoidal area = $\frac{1}{2}(H_1 + H_2) * B$]

$$y_{n+1} = y_n + \frac{T}{2}[f(x_n, t_n) + f(x_{n+1}, t_{n+1})] \tag{3.11}$$

Equation (3.11) can be written as

$$y[(n+1)T] = y(nT) + \frac{T}{2}[x(nT) + x\big((n+1)T\big)] \tag{3.12}$$

Applying z-transform on both sides of equation (3.12), we get

$$z\, y(z) = y(z) + \frac{T}{2}[x(z) + \ z\, x(z)] \tag{3.13}$$

$$y(z)(z-1) = \frac{T}{2}[x(z)][z+1] \tag{3.14}$$

$$H_{Trap}(z) = \frac{y(z)}{x(z)} = \frac{T}{2}\left(\frac{z+1}{z-1}\right) \tag{3.15}$$

The transfer function in equation (3.15) represents the first order numerical integrator resulting from the approximation of trapezoidal rule.

3.3 Second Order Approximation Scheme

The second order approximations of numerical integration are based on fitting parabolic curves to the function which is to be integrated. Commonly, a function is approximated by a parabola passing through the known points, the integral of that function is then evaluated from the summation of areas under the parabolic curve. The key reason to studying second order system is that, in general, due to storage element within the most transients circuits the characteristics transients solution is a somewhat curved function. It is expected that since the desired output is a smooth curve, a better approximation would be made using a higher order curve fitting routine [11].

3.3.1 The Simpson 1/3 Rule

The Simpson 1/3 rule employs a second order approximation of the function that being integrated. Simpson's rule is an approximation of an integral obtained by passing a parabolic curve through three points [11].

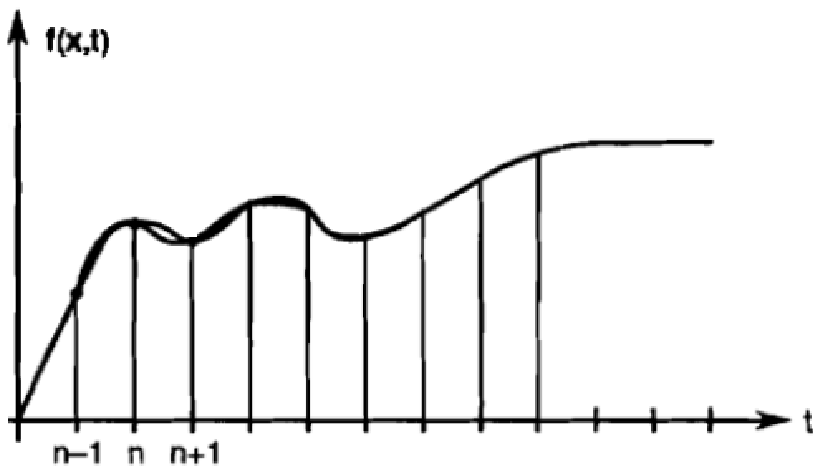


Figure 3.4: Approximation of x(t) by Simpson 1/3 rule [11].

As shown in Figure 3.4, If y_{n-1} is the integral of the function at time sample $n-1$ and y_n is the integral of the function at time sample n, the integral y_{n+1} at time sample $n+1$ is equal to the integral at the time sample $n-1$ and summed the area of newly created parabolic curve passing through time sample $n-1$ to $n+1$.

$$y_{n+1} = y_{n-1} + \frac{T}{3}[f(x_{n-1}, t_{n-1}) + 4f(x_n, t_n) + f(x_{n+1}, t_{n+1})] \tag{3.16}$$

Equation (3.16) can be written as

$$y[(n+1)T] = \mathrm{y}[(\mathrm{n}-1)T] + \frac{T}{3}[x((n-1)T) + 4x(nT) + x((n+1)T)] \qquad 3.17)$$

Applying z-transform on both sides of equation (3.17), we get

$$z\,y(z) = \frac{y(z)}{z} + \frac{T}{3}\left[\frac{x(z)}{z} + 4x(z) + z\,x(z)\right] \qquad (3.18)$$

$$y(z)\left[z - \frac{1}{z}\right] = \frac{T}{3}\left[x(z)\left(\frac{1}{z} + 4 + z\right)\right] \qquad (3.19)$$

$$y(z)\left[\frac{z^2-1}{z}\right] = \frac{T}{3}\left[x(z)\left(\frac{1+4z+z^2}{z}\right)\right] \qquad (3.20)$$

$$H_{S1/3}(z) = \frac{y(z)}{x(z)} = \frac{T}{3}\left[\frac{1+4z+z^2}{z^2-1}\right] \qquad (3.21)$$

The transfer function in equation (3.21) represents the second order numerical integrator resulting from the approximation of Simpson 1/3 rule.

3.4 Steps for Designing Differentiators from Numerical Integrators

Analog differentiators are often obtained by inverting the transfer functions of analog integrators, or vice versa. Extending this concept to non-minimum phase-digital integrators yields unstable differentiators. The stabilization approach developed below will be employed. The approach, as applied in this work, consists of the following four steps [12]:

1. Obtain the discrete-time transfer function, using the z-transform, of a digital integrator that has the desired range and accuracy. This would approximate the analog transfer function $1/s$, using the Laplace transform, of an integrator.
2. Invert the discrete-time transfer function of the above integrator.
3. Stabilize the resulting transfer function by reflecting its poles that lie outside the unit circle at radius r to inside the unit circle at a radius of $1/r$. This corresponds to adding a zero at r and a pole at $1/r$, which is equivalent to multiplying the original transfer function by an all-pass filter. For the case of a pole of order n at infinity, introduce a pole of order n at zero. In this latter case, when a zero is added at infinity, a pole is also obtained at zero [12].
4. Compensate the resulting change in the magnitude of the discrete-time transfer function, resulting from step (3), by multiplying the resulting transfer function by

$1/r$ for each reflected pole. No magnitude compensation is needed for the case of a pole of order n at infinity since the added pole of order n at the origin of the z-plane will have a magnitude of one when it is evaluated on the unit circle of the z-plane.

A minimum phase filter's zeros may lie anywhere inside the unit circle of the z-plane. Zeros are permitted to lie on the unit circle provided they are simple (i.e., they are of order one).

In our design approach we are considering linear order low power IIR digital differentiator design and fractional order IIR differentiator design through inverse integration rule and later on comparing their magnitude and phase response.

3.5 Numerical Differentiators

A class of digital integrators is first derived from the numerical integration rules [13], [14]. A class of digital differentiators are afterwards obtained by inverting the transfer function of the digital integrators and stabilizing the resulting transfer function together with magnitude compensation if necessary [12]. The transfer function of digital integrators is obtained in a straightforward manner from numerical integration rules by the simple application of the z-transform to the difference equations defined by the various numerical integration rules.

3.5.1 First Order Differentiator

This is a first-order, wide-band differentiator, obtained by inverting the transfer function of first order integrator resulting from interpolation of the trapezoidal and the rectangular integration rules. This first order differentiator is popularly known as Al-Alaoui operator. The transfer function of this differentiator is given as follows [13], [14].

$$H(z) = 0.75\, H_r(z) + 0.25\, H_t(z) \tag{3.22}$$

where $H_r(z)$ and $H_t(z)$ is the transfer function of rectangular integrator and trapezoidal integrator as mentioned in equation (3.5) and (3.15).

By putting $H_r(z)$ and $H_t(z)$ in equation (3.22), resulting in non-minimum phase integrator.

$$H_{Int\_NonMin}\,(z) = \frac{T(z+7)}{8(z-1)} \tag{3.23}$$

Compensating the magnitude of the transfer function, minimum phase integrator is achieved.

$$H_{Int\_Min}(z) \; = \frac{7T(z+\frac{1}{7})}{8(z-1)} \tag{3.24}$$

The numerical differentiator is obtained by inverting the transfer function of resulting interpolated integrator [17].

$$H(z) \; = \frac{8(z-1)}{7T(z+(\frac{1}{7}))} \tag{3.25}$$

3.5.2 Second Order Differentiator

Second-order integrator is obtained by interpolating the Simpson 1/3 integration rule. The transfer function of Simpson 1/3 integrator [14], [16].

$$H(z) \; = \frac{3(z^2-1)}{T(z^2+4z+1)} \tag{3.26}$$

Stabilize the transfer function mentioned in equation (3.26) by reflecting the poles at $z = -3.73$ by $z = -1/3.73$ and compensate the magnitude by multiplying transfer function by $1/3.73$.

The transfer function of second order differentiator is

$$H(z) \; = \frac{3(z^2-1)}{T(3.73)(z^2+0.53z+0.071)} \tag{3.27}$$

3.5.3 Ngo Third Order Differentiator

The transfer function of this differentiator is given as follows [17]

$$H(z) \; = \frac{1.18\,z^2(z-1)}{T(z+0.4226)\left(z-0.2167e^{j0.9427}\right)\left(z-0.2167e^{-j0.9427}\right)} \tag{3.28}$$

The frequency response of the resulting numerical differentiators explained in this chapter is calculated by replacing with $z = e^{j\omega}$ in their transfer function and varying the value of ω for normalized frequency from 0 to 1.

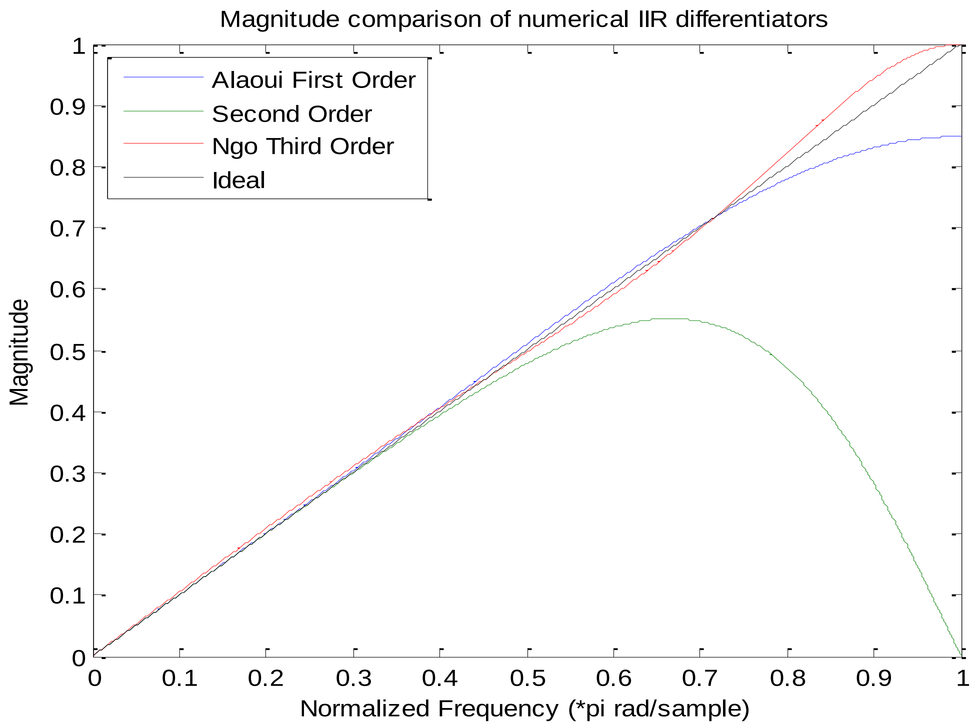


Figure 3.5: Comparison of magnitude response of various numerical differentiators.

The magnitude response of numerical differentiator explained in this section is demonstrated in Figure 3.5. The Al-Alaoui first order differentiator almost approximates the ideal differentiator up to 0.78 of full band, while the second order differentiator gives almost linear magnitude response up to 0.4 of full band of frequency range and Ngo third order differentiator almost approximate the ideal differentiator up to the full band of frequency.

The percentage magnitude error of these three respective differentiators is shown in Figure 3.6. Al-Alaoui first order differentiator gives maximum 2% error around 0.6 normalized frequency, while Ngo third order differentiator has poorer magnitude error response for full band of frequency range. Second order differentiator completely outperforms the other two differentiators for frequency less than 0.4 of full band.

Figure 3.7 shows the phase response of the three numerical differentiators mentioned in this section. The phase response of these differentiators is almost linear up to certain range of frequency. The second order differentiator has its linear phase up to 0.6 of normalized frequency in the full range, while Al-Alaoui first order differentiator and Ngo third order differentiator gives almost linear phase response for the full range of normalized frequency.

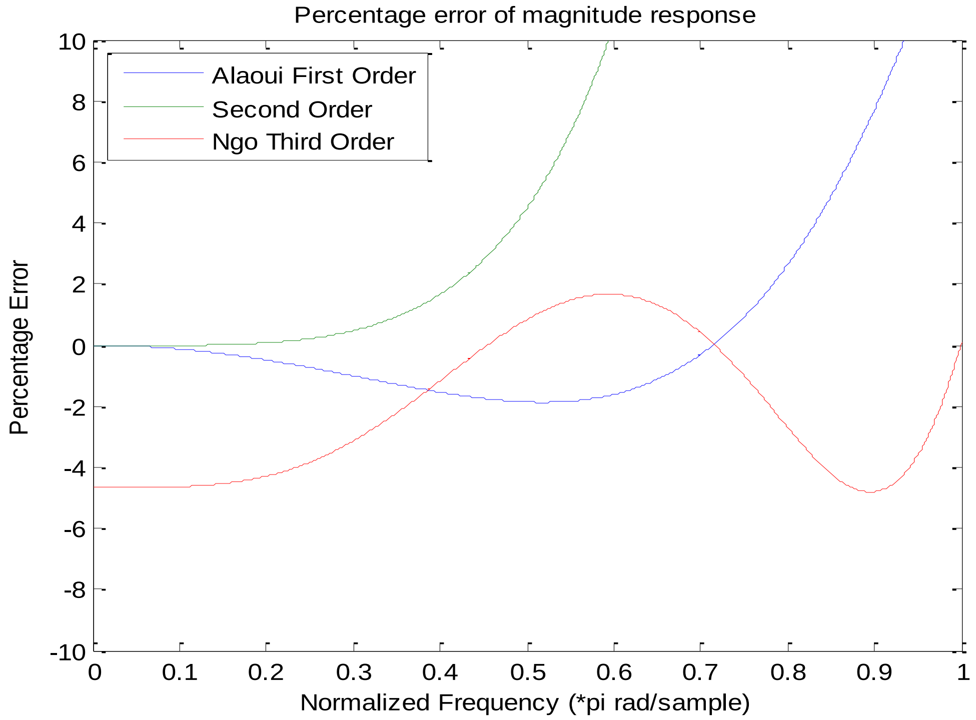


Figure 3.6: Relative percentage error of various numerical differentiators.

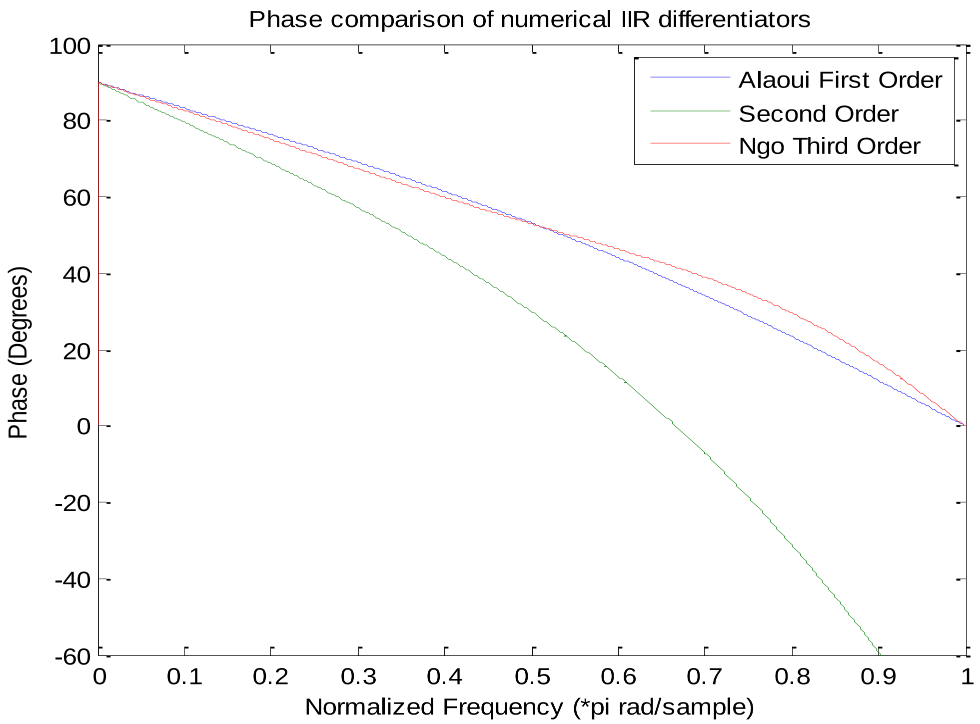


Figure 3.7: Comparison of phase response of various numerical differentiators.

The simulation results carried out in this chapter decide the selection criteria for various numerical differentiators. The Ngo third order differentiator gives poorer magnitude response. Therefore, if we are designing differentiators for very low frequencies, the second order differentiator is employed, while Al-Alaoui first order differentiator is used for designing wide band differentiators.

CHAPTER 4

Low-Pass IIR Integer Order Digital Differentiator

4.1 Introduction

The design of digital filters at low frequency range has become increasingly important as it can be used to design all types of filters [18]. The basic properties for a good digital filter are that it gives the desired magnitude response with linear phase. The linear phase filters is achieved by designing the symmetric coefficients finite impulse response (FIR) filters. A novel approach to designing approximately linear phase infinite impulse response (IIR) digital filter in the pass band region is introduced in this chapter. IIR filters are designed to meet suitable magnitude requirements with low order than FIR filters, but the main disadvantages of IIR filters is phase non-linearity. Nevertheless, it is possible to design linear phase IIR filters.

In many applications, differentiation is followed by low-pass filter. Differentiation is used to extract information about rapid transients in the signal. Low-pass filters are used to reject noise frequencies higher than the cutoff frequencies of the signal. Low-pass filtering and differentiation can be implemented as a single low-pass differentiator filter or by using a low-pass filter and a differentiating filter in cascade [18]. The resulting cascaded filters are called as low-pass IIR differentiators.

The simulation studies carried out in this chapter have shown that fourth-order IIR differentiators compare favorably with the much higher order state of the art FIR filters. Typically, for the same magnitude response specifications, the order of the resulting IIR filter is one sixth the order of corresponding FIR filter.

The proposed approach utilizes the linear phase properties of the FIR filters and the smaller transition band and the steeper roll-off properties of the IIR filters. The low order of the proposed low-pass differentiators make them suitable for real-time applications. The accuracy of the proposed low-pass IIR differentiators is comparable to that obtained by higher order FIR filters. An additional advantage is that an almost linear phase is also obtained in the pass band region.

4.2 Cascading Approach for Designing IIR Differentiators

The main objective to design linear phase filters is only achieved by designing symmetric coefficients FIR filters, the design of linear phase IIR filters are also possible. It is implemented by cascading a differentiator with low-pass filters. This key approach consists of two steps [18]:

1. Obtain a low-order IIR differentiator whose transfer function has a numerator that represents a linear phase FIR filter.
2. Cascade the above differentiator with IIR low-pass filter whose numerator also represents a linear phase IIR filter.

A. First step

Two low-pass IIR differentiator obtained by interpolating various numerical integration rules are presented.

The first on is Al-Alaoui, first-order wide-band differentiator, obtained by inverting the transfer function of first order integrator resulting from interpolation of the trapezoidal and the rectangular integration rules. The transfer function of this differentiator is given as follows [13]:

$$H(z) = \frac{8(z-1)}{7T\left(z+\frac{1}{7}\right)} \tag{4.1}$$

Second, is a second-order, low-pass differentiator, obtained by inverting the transfer function of the Simpson integrator. The resulting transfer function is then stabilized and its magnitude is compensated. The transfer function of this differentiator is given as follows [14]:

$$H(z) = \frac{3\left(z^2-1\right)}{T(3.73)(z^2+0.53z+0.071)} \tag{4.2}$$

B. Second Step

In this step an appropriate low-pass filter should be designed. This filter would be cascaded with any type of the first step differentiators. A suitable method for designing low-pass filter is applying bilinear transformation to an all pole analog low-pass prototype, or an analog filter with zeros only in the stop band [19], [20]. The resulting system should perform differentiation action up to cut-off frequency, and yield steep roll-off properties for attenuating high frequency noises.

4.3 Selecting an Appropriate Low-Pass Filter

As mentioned before in this chapter, an appropriate method to design a low-pass filter is applying bilinear transformation to either an all-pole analog low-pass prototype, or an analog filter with zeros only in the stop band. So there are several alternatives to choose the analog prototype. The best way to find a good prototype is designing linear phase differentiator (LPD) with each of them, and then comparing the results. Therefore, four types of LPDs are reported as follows [21]:

LPD I: An order-4 LPD which developed by cascading an order-3 Chebyshev-I low-pass filter with 0.1dB ripple in pass-band with Al-Alaoui operator.
LPD II: An order-4 LPD which developed by cascading an order-3 Chebyshev II low-pass filter with 40dB stop-band attenuation with Al-Alaoui operator.
LPD III: An order-4 LPD which developed by cascading an order-3 Butterworth low-pass filter with Al-Alaoui operator.
LPD IV: An order-4 LPD which developed by cascading an order-3 Elliptic low-pass filter with 0.1dB ripple in the pass-band and 40dB stop-band attenuation with Al-Alaoui operator.

The first order differentiator mentioned in equation (4.1) is cascaded with third order low-pass IIR filters at 0.42 normalized cut-off frequencies as mentioned above, results in fourth order low-pass IIR differentiators. The coefficients of resulting low-pass IIR differentiator are represented in descending power of z as in equation (4.3) below:

$$H(z) = \frac{b(1)+b(2)z^{-1}+b(3)z^{-2}+b(4)z^{-3}+b(5)z^{-4}}{a(1)+a(2)z^{-1}+a(3)z^{-2}+a(4)z^{-3}+a(5)z^{-4}} \qquad (4.3)$$

Table 4.1 shows the coefficients of the four resulting low-pass IIR differentiator at 0.42 normalized cut-off frequencies.

Figure 4.1 shows the magnitude response of various IIR LPDs. It is obvious that the LPDs made by Chebyshev-I and Elliptic filers provide steeper roll-off properties than those made by Chebyshev-II and Butterworth filters. Moreover, the best magnitude response belongs to that made by Elliptic filter.

Table 4.1: Coefficients of the fourth order low-pass IIR Differentiators with different IIR filters [21].

Coefficients	LPD I	LPD II	LPD III	LPD IV
b(1)	0.0573	0.0956	0.0935	0.0654
b(2)	0.1147	0.1732	0.1869	0.1067
b(3)	0.0000	0.0000	0.0000	0.0000
b(4)	-0.1147	-0.1732	-0.1869	-0.1067
b(5)	-0.0573	-0.0956	-0.0935	-0.0654
a(1)	1.0000	1.0000	1.0000	1.0000
a(2)	0.0133	0.6629	0.7201	0.0265
a(3)	0.4366	0.5001	0.5042	0.4638
a(4)	0.0004	0.1187	0.1166	0.0102
a(5)	-0.0092	0.0083	0.0080	-0.0083

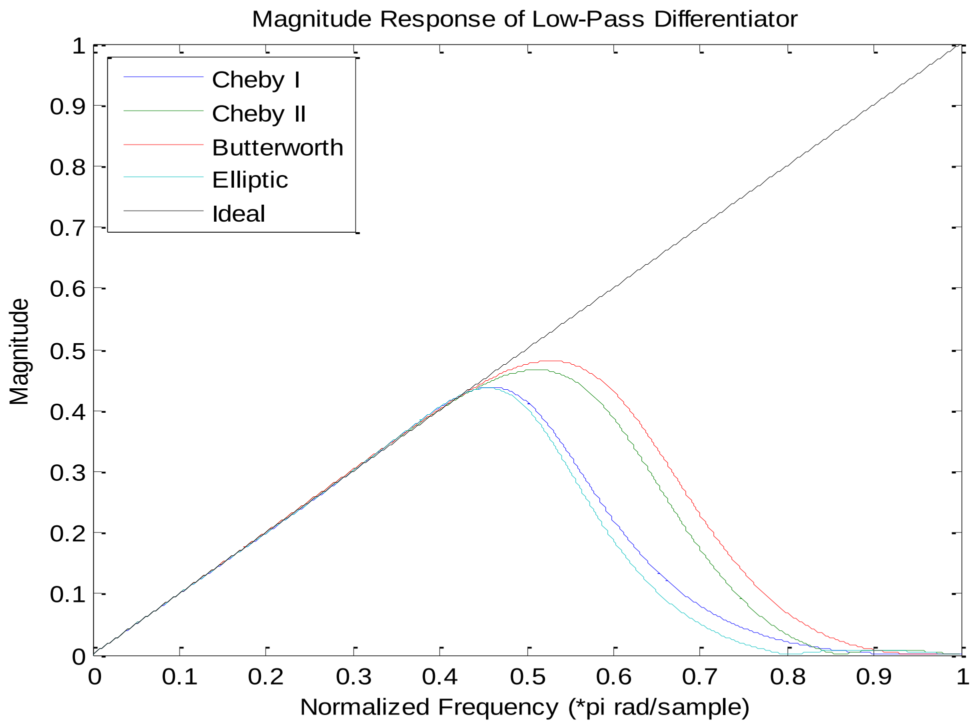


Figure 4.1: Magnitude responses of various fourth-order Low-Pass IIR differentiators.

Figure 4.2 shows the relative error of magnitude response. The figure proves that the LPDs made by Butterworth and Chebyshev-II filters provide lower percentage error than the others; there for, they are suitable for approximating maximally flat LPDs like Selesnick's [10]. For two other types, the percentage error is smaller than 1% and it can be neglected in many applications.

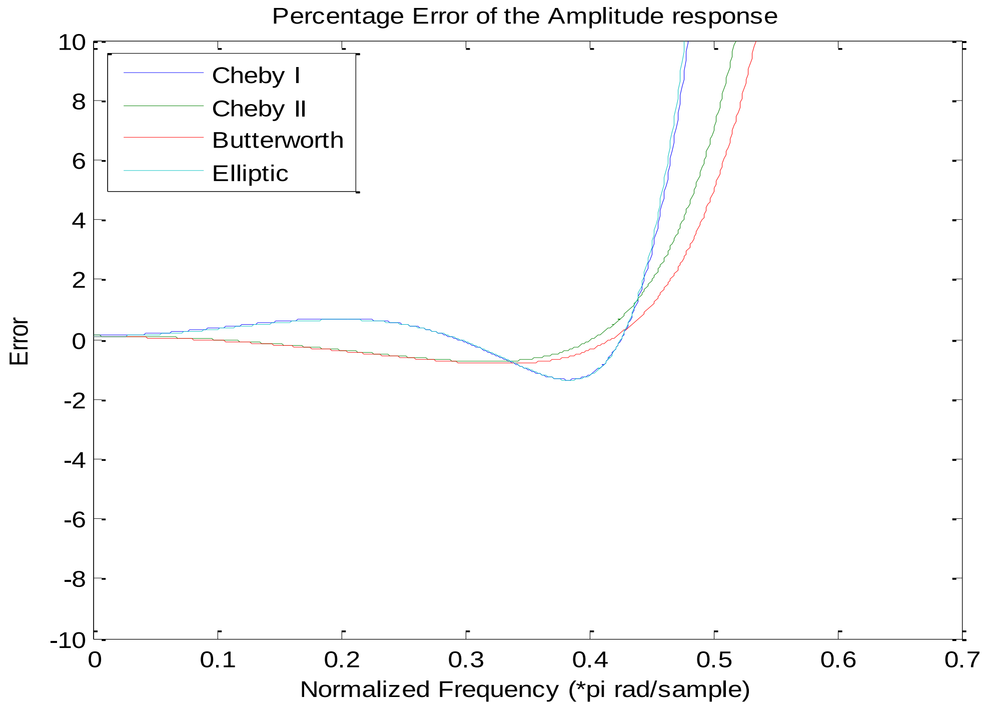


Figure 4.2: Relative percentage error of various fourth-order Low-Pass IIR differentiators.

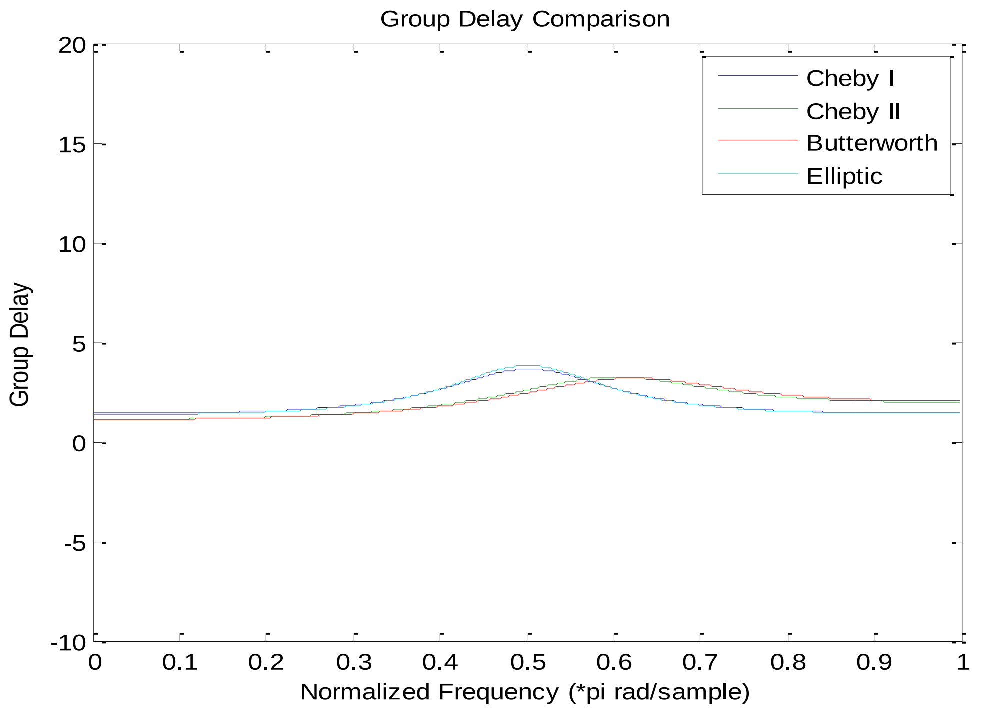


Figure 4.3: Group delay comparison between various fourth-order Low-Pass IIR differentiators.

Figure 4.3 shows the group delay. The figures mentioned above prove that the LPDs made by Butterworth and Chebyshev-II filters provide better group delays. The group delay is constant up to cut-off frequency.

According to the previous discussions, there is a trade-off for choosing analog low-pass prototype. The summary of results given as follows [21].

1. If maximally flat response is supposed for LPD, Butterworth or Chebyshev-II low-pass filters should be selected; for example, for approximating Selesnick's LPD.
2. If steep roll-off properties are important, Chebyshev-I or Elliptic low-pass filters should be selected; for instance, for approximating Parks-McClellan LPD.
3. Elliptic filter yields steeper roll-off properties than Chebyshev-I.

4.4 Low-Pass IIR Digital Differentiator

The key approach is to design an IIR filter that satisfies the desired magnitude specifications and exhibits a linear phase in the pass-band region. The linear phase IIR filters are designed by cascading the numerical differentiator whose numerator represents the linear phase filter with the appropriate low pass IIR filters [18], [19]. The resulting cascaded filter is called as linear phase low-pass IIR differentiator.

In this section, first order differentiator as mentioned in equation (4.1) is cascaded with third order low-pass Chebyshev I filter having 0.1 dB ripples in pass-band at four different cut-off frequencies. The selection of appropriate low pass IIR filter depends upon the desired characteristics as mentioned in previous section. The transfer functions of third order low-pass Chebyshev I filter at four different cut-off frequencies is given as follows [19], [20]:

$$H_C(z) = \frac{0.1061+0.3183z^{-1}+0.3183z^{-2}+0.1061z^{-3}}{1-0.5826z^{-1}+0.5504z^{-2}-0.1189z^{-3}} \quad \text{where} \quad \omega_c = 0.35 \tag{4.3}$$

$$H_C(z) = \frac{0.1576+0.4728z^{-1}+0.4728z^{-2}+0.1576z^{-3}}{1-0.1295z^{-1}+0.4551z^{-2}-0.0647z^{-3}} \quad \text{where} \quad \omega_c = 0.42 \tag{4.4}$$

$$H_C(z) = \frac{0.2465+0.7395z^{-1}+0.7395z^{-2}+0.2465z^{-3}}{1+0.4800z^{-1}+0.4845z^{-2}+0.0076z^{-3}} \quad \text{where} \quad \omega_c = 0.52 \tag{4.5}$$

$$H_C(z) = \frac{0.4533+1.3600z^{-1}+1.3600z^{-2}+0.4533z^{-3}}{1+1.4816z^{-1}+0.9597z^{-2}+0.1852z^{-3}} \quad \text{where} \quad \omega_c = 0.70 \tag{4.6}$$

The coefficients of resulting fourth order low-pass IIR differentiator are represented in descending power of z as in equation (4.7) as below [18]:

$$H(z) = \frac{b(1)+b(2)z^{-1}+b(3)z^{-2}+b(4)z^{-3}+b(5)z^{-4}}{a(1)+a(2)z^{-1}+a(3)z^{-2}+a(4)z^{-3}+a(5)z^{-4}} \quad (4.7)$$

Table 4.2: Coefficients of the fourth order low-pass IIR Differentiators at different cut-off frequencies [18].

Coefficients	ω_c= 0.35	ω_c= 0.42	ω_c= 0.52	ω_c= 0.7
b(1)	0.0386	0.0573	0.0897	0.1649
b(2)	0.0772	0.1147	0.1794	0.3298
b(3)	0.0000	0.0000	0.0000	0.0000
b(4)	-0.0772	-0.1147	-0.1794	-0.3298
b(5)	-0.0386	-0.0573	-0.0897	-0.1649
a(1)	1.0000	1.0000	1.0000	1.0000
a(2)	-0.4398	0.0133	0.6228	1.6240
a(3)	0.4672	0.4366	0.5531	0.1710
a(4)	-0.0403	0.0004	0.0768	0.3223
a(5)	-0.0170	-0.0092	0.0011	0.0265

Figure 4.4 shows the magnitude responses of the fourth-order low-pass IIR differentiators compared with the Selesnick type III FIR low-pass differentiators at the normalized cutoff frequencies 0.35, 0.42, 0.52, and 0.7 of full band, respectively.

The fourth-order low-pass IIR differentiators have almost similar magnitude response as compared with the higher order Selesnick type IV FIR low-pass differentiators and the low-pass IIR differentiator magnitudes exhibit shorter transition regions.

Figure 4.5 compares the percentage error of the low-pass IIR differentiator with a normalized cutoff frequency of 0.35 with that the FIR differentiator of Selesnick. While the FIR low-pass filters outperform the proposed differentiator, the 1% error is acceptable in many applications.

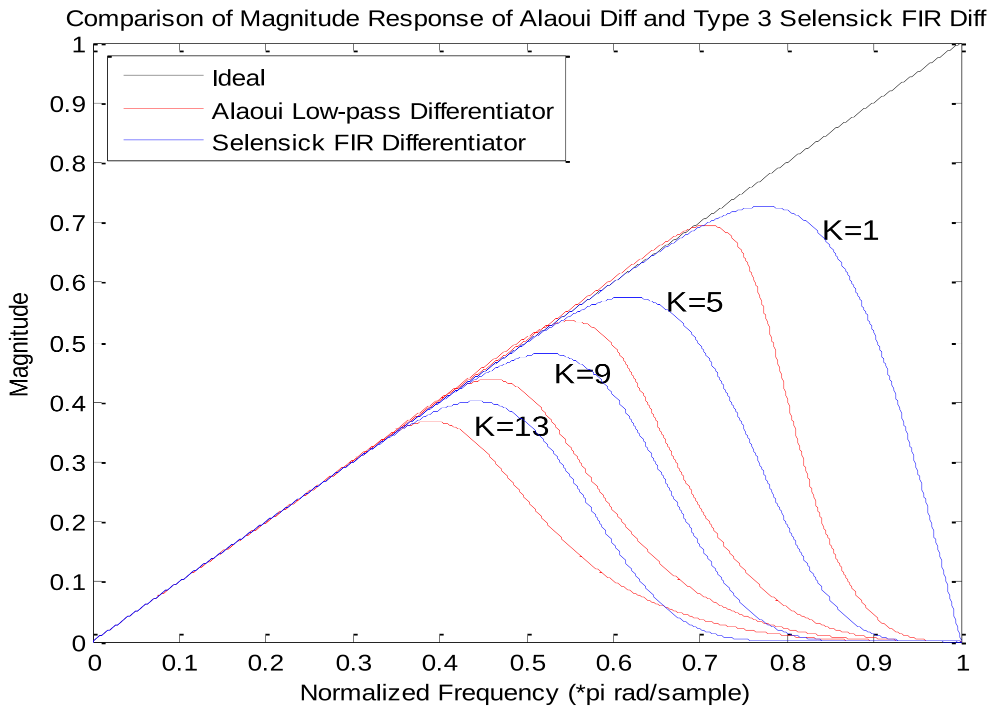


Figure 4.4: Comparison of magnitude response of Al-Alaoui fourth-order Low-Pass IIR differentiators with maximum flat Selesnick FIR differentiators.

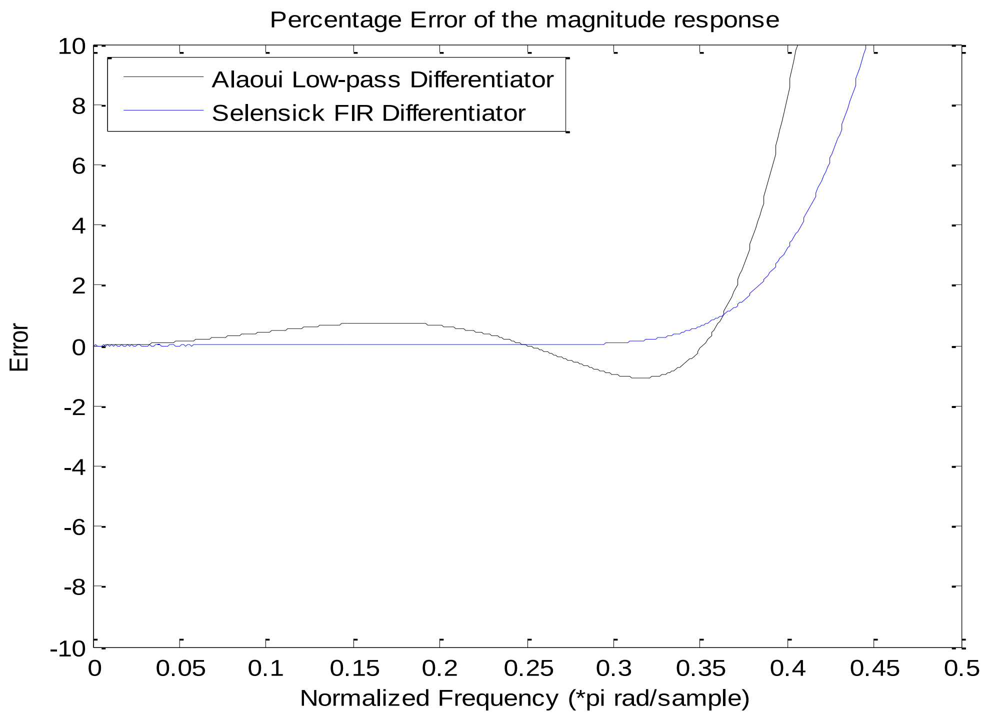


Figure 4.5: Comparison of relative percentage magnitude error between Al-Alaoui fourth order Low-Pass IIR differentiators and maximum flat Selesnick FIR differentiators.

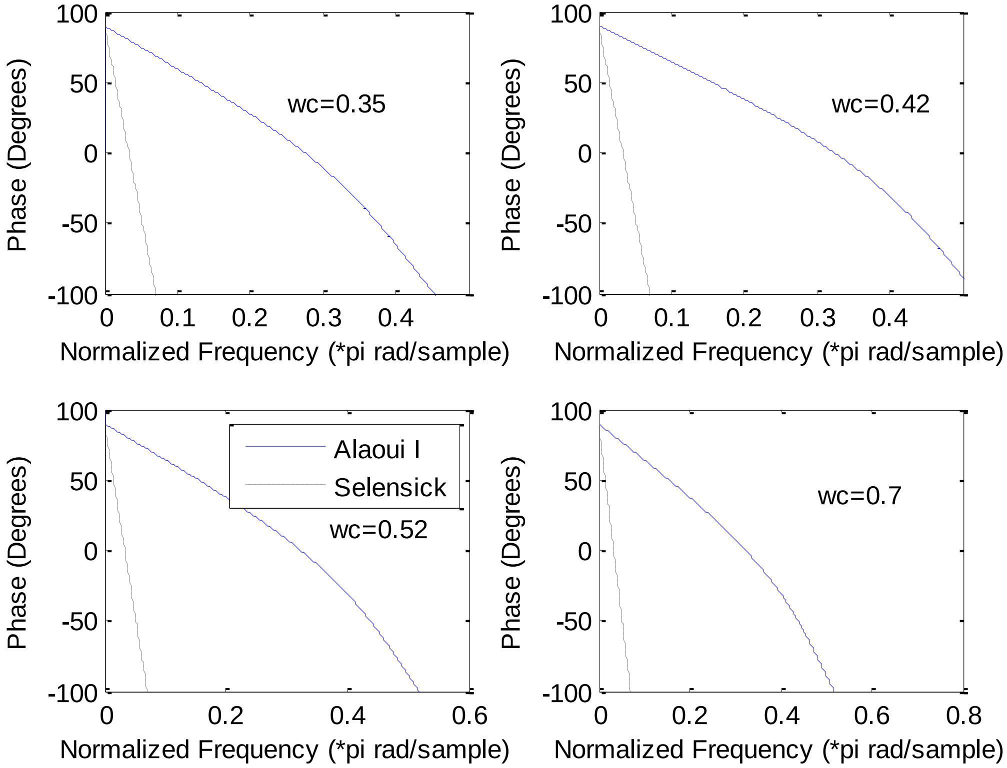


Figure 4.6: Comparison of phase response of Al-Alaoui fourth-order Low-Pass IIR differentiators with maximum flat Selesnick FIR differentiators.

Figure 4.6 shows the phase responses for four representative cutoff frequencies which clearly demonstrate that the phases of the low-pass IIR differentiators are almost linear up to the cutoff frequencies. Also shown is the much steeper phase response, due to the greater amount of delay, of the Selesnick FIR low-pass linear phase differentiators.

4.5 Constrained Optimization Approach

The low-pass IIR differentiators designed by cascading approach does not have the suitable flat magnitude response and have the large transition band as compared with the FIR differentiators [18]. These properties of IIR differentiators are further improved by altering the gain factor and denominator coefficients of designed IIR differentiators [18], [21]. If we are designing the differentiator for maximum flat property, it must be approximated with Selesnick maximum flat FIR differentiator. For the steep roll-off property, the differentiator is approximated with Parks McClellan FIR differentiator [21].

The two low-pass IIR differentiators is designed by cascading the first order differentiator in equation (4.1) with low-pass Chebyshev I and Elliptic low-pass filter are optimized in this section.

The resulting low-pass IIR differentiator in equation (4.7) is being optimized. Only gain factor and denominator coefficients of the transfer function are adjusted in such a manner that the frequency response is going to be improved.

$$H(z) = K\left(\frac{1+c(1)z^{-1}+c(2)z^{-2}+c(3)z^{-3}+c(4)z^{-4}}{a(1)+a(2)z^{-1}+a(3)z^{-2}+a(4)z^{-3}+a(5)z^{-4}}\right) \tag{4.8}$$

where $K = b(1)$, $c(n) = \frac{b(n+1)}{b(1)}, n = 1,..,4$ and $b(n)$ is nth numerator coefficients of equation (4.7).

The frequency response of the differentiator is calculated by replacing with $z = e^{j\omega}$ in equation (4.7) and varying the value of ω for normalized frequency from 0 to 1. The fourth order low-pass IIR differentiator in equation (4.7) will almost approximate the 30th order FIR differentiator as mentioned in [18]. The low order of IIR differentiator will reduce the design and computational complexity.

The frequency response of differentiator is represented as follows:

$$H(\omega) = A(\omega)e^{j\theta(\omega)} \tag{4.9}$$

where $A(\omega)$ is the magnitude response and $\theta(\omega)$ is the phase response of the differentiator. The group delay of differentiator is expressed as [18], [19]:

$$\tau_g(\omega) = -\frac{d\theta(\omega)}{d\omega} \tag{4.10}$$

As we are optimizing the frequency response of IIR differentiators, the main objective is to minimize the least square error of magnitude and phase response. The fitness function for minimizing the least square error is defined as follows [18]:

$$\varepsilon = (1-\gamma)\sum_{n=1}^{L} w_n[A(\omega_n) - A_d(\omega_n)]^2 + \gamma\sum_{n=1}^{L} v_n\left[\tau_g(\omega_n) - \tau_g(\omega_0) - \tau_d(\omega_n)\right]^2 \tag{4.11}$$

Where ε is the total weighted least square error over all frequencies:

$\omega_1, \omega_2, \ldots, \omega_L$ For $0 \leq \omega_n \leq \pi$ and $\omega_n = n\pi/L$

where $\{\gamma\}, \{w_n\}, \{v_n\}$ are weighting factors selected by the designer. L is the number of frequency components; greater number of L increases the accuracy. The magnitude error at a particular frequency is $A(\omega_n) - A_d(\omega_n)$, where $A_d(\omega_n)$ is the magnitude response and the delay error is $\tau_g(\omega_n) - \tau_g(\omega_0) - \tau_d(\omega_n)$, where $\tau_g(\omega_0)$ is the delay at some nominal center frequency in the pass-band and $\tau_d(\omega_n)$ is the desired delay response of the differentiator relative to $\tau_g(\omega_n)$.

Genetic Algorithm is used for minimizing the total least square error. For the suitable coefficients, which gives the minimum value of fitness function are used as optimized coefficients. The desired response for approximating the differentiator is chosen accordingly. The FIR Selesnick maximum flat differentiator is used where flat magnitude response is important. If steep roll-off property is important, Parks McClellan differentiator is used. We optimize low-pass differentiator at following cutoff frequency with the corresponding weighting factors.

a) ω_c = 0.5π:
   γ = 0.1
   w_n = 1 for $\omega_n = 0, \ldots., \omega_c$ and w_n= 0.65 for $\omega_n = \omega_c + \Delta\omega, \ldots., \pi$
   v_n = 1 for $\omega_n = 0, \ldots., \omega_c$ and v_n= 0 for $\omega_n = \omega_c + \Delta\omega, \ldots., \pi$
   $\Delta\omega = \frac{\pi}{1000}$
   stop when $\varepsilon < 25$

b) ω_c = 0.7π:
   γ = 0.05
   w_n = 1 for $\omega_n = 0, \ldots., \omega_c$ and w_n= 0.6 for $\omega_n = \omega_c + \Delta\omega, \ldots., \pi$
   v_n = 1 for $\omega_n = 0, \ldots., \omega_c$ and v_n= 0.7 for $\omega_n = \omega_c + \Delta\omega, \ldots., \pi$
   stop when $\varepsilon < 40$

The frequency response of proposed optimized IIR differentiators is demonstrated in Figure 4.7 to Figure 4.12. From the Figure 4.7, it is clear that the magnitude roll-off of proposed IIR differentiator (ω_c = 0.5) is lies between Al-Alaoui low-pass IIR differentiator and optimized Al-Alaoui IIR differentiator.

The percentage relative error of magnitude response of proposed optimized IIR differentiator as compared with other differentiators is shown in Figure 4.8. The previous optimized Al-Alaoui low-pass IIR differentiator has slightly better phase response than the proposed optimized IIR differentiator but the proposed optimized IIR differentiator has less percentage error as compared with others in the desired range of frequency. Figure 4.9 shows the phase response of proposed optimized IIR differentiator. So, there is trade-off between relative percentage magnitude error and phase response.

Table 4.3: Coefficients of the optimized low-pass IIR Differentiators.

Coefficients	ω_c= **0.5**	ω_c= **0.7**
K	0.0887	0.1669
a(2)	0.5984	1.6063
a(3)	0.5423	1.1849
a(4)	0.0522	0.32739
a(5)	-0.0009	0.03569

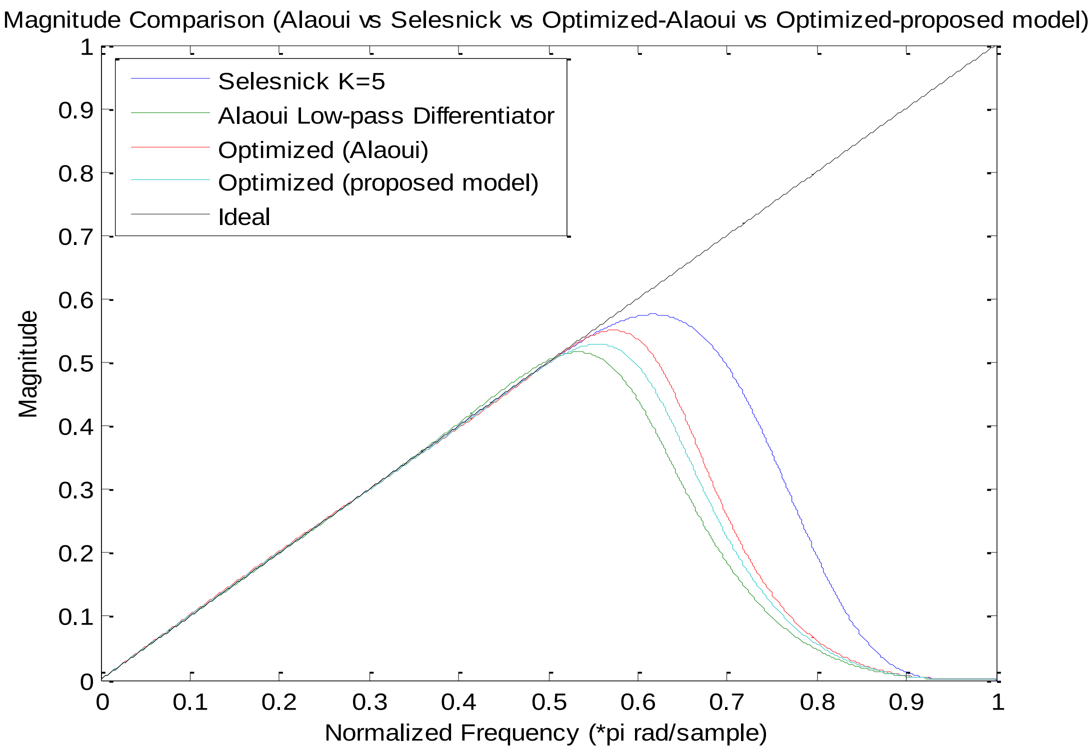


Figure 4.7: Comparison of magnitude response of Al-Alaoui low-pass differentiator, Selesnick FIR differentiator K=5, optimized Al-Alaoui and proposed optimized IIR differentiator.

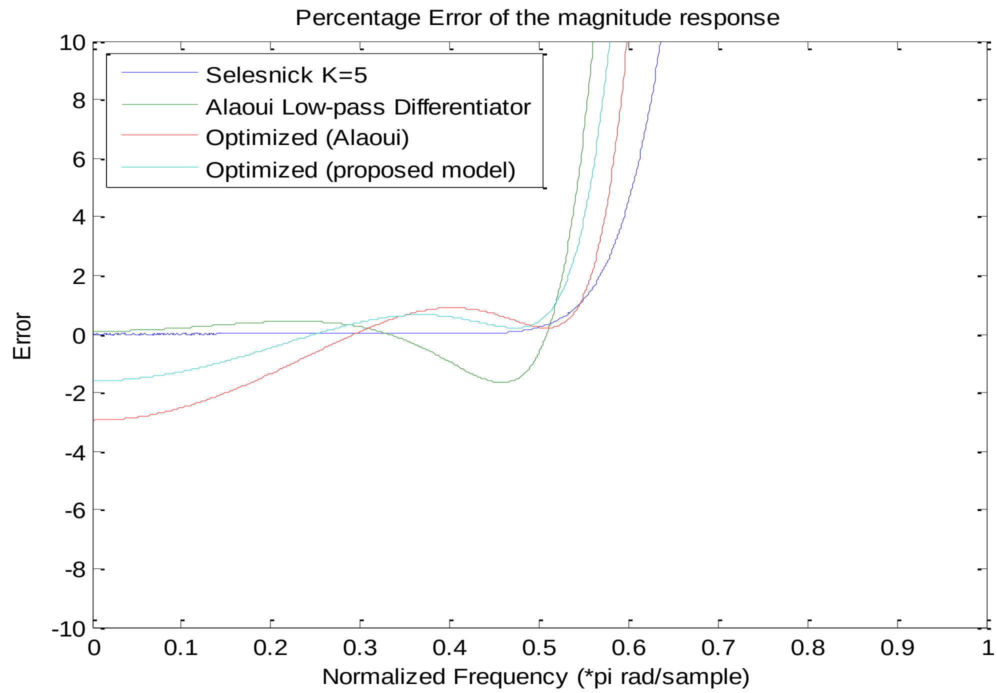


Figure 4.8: Comparison of percentage relative magnitude error of Al-Alaoui low-pass differentiator, Selesnick FIR differentiator K=5, optimized Al-Alaoui and proposed optimized IIR differentiator.

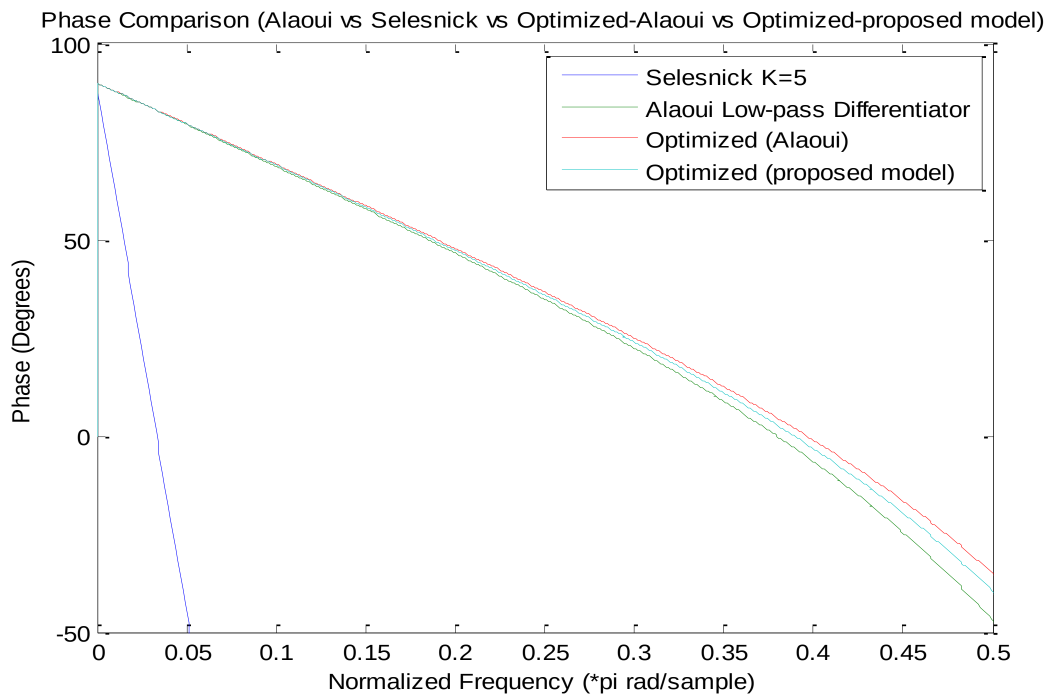
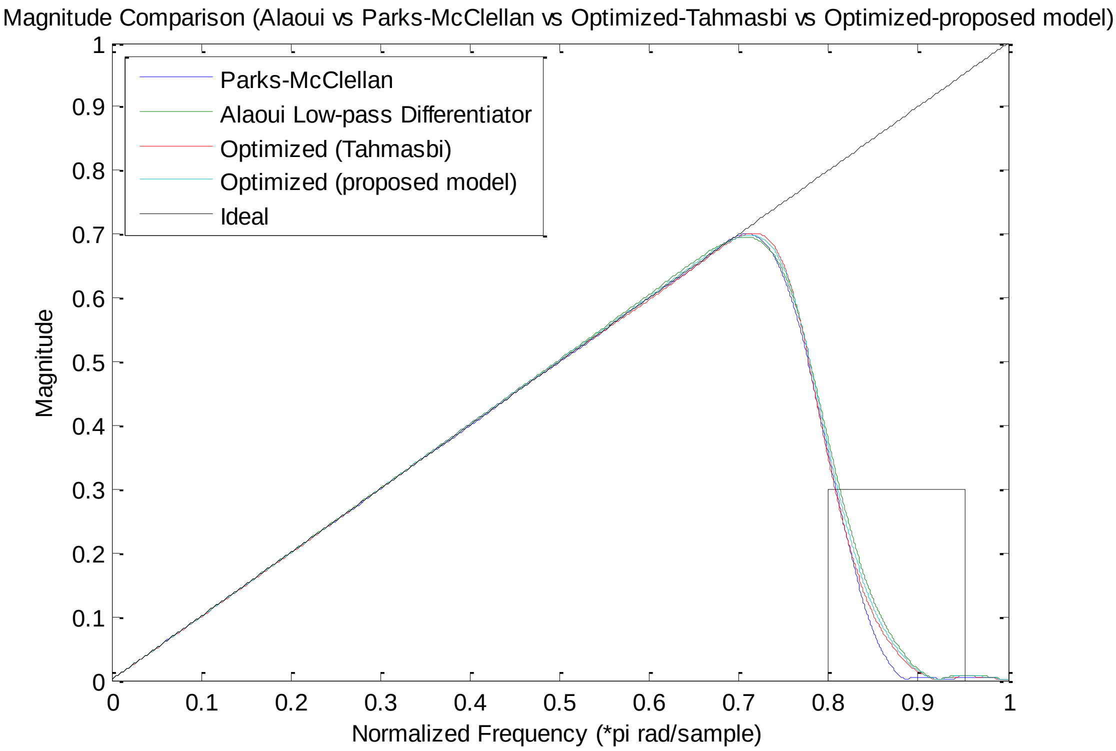
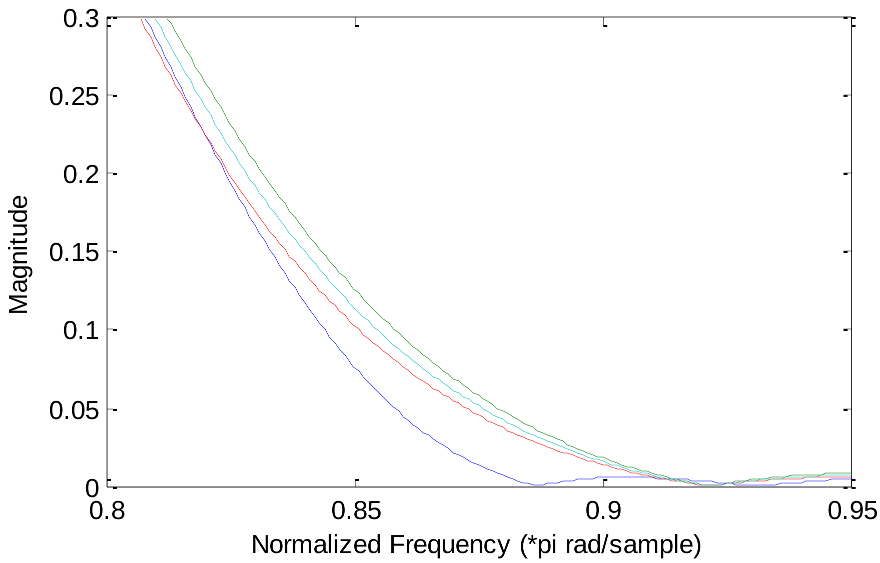


Figure 4.9: Comparison of phase response of Al-Alaoui low-pass differentiator, Selesnick FIR differentiator K=5, optimized Al-Alaoui and proposed optimized IIR differentiator.



(a)



(b)

Figure 4.10: (a) Comparison of magnitude response of Al-Alaoui low-pass differentiator, Parks-McClellan FIR differentiator, optimized Tahmasbi and proposed optimized IIR differentiator.

(b) Magnified view of particular portion of magnitude response.

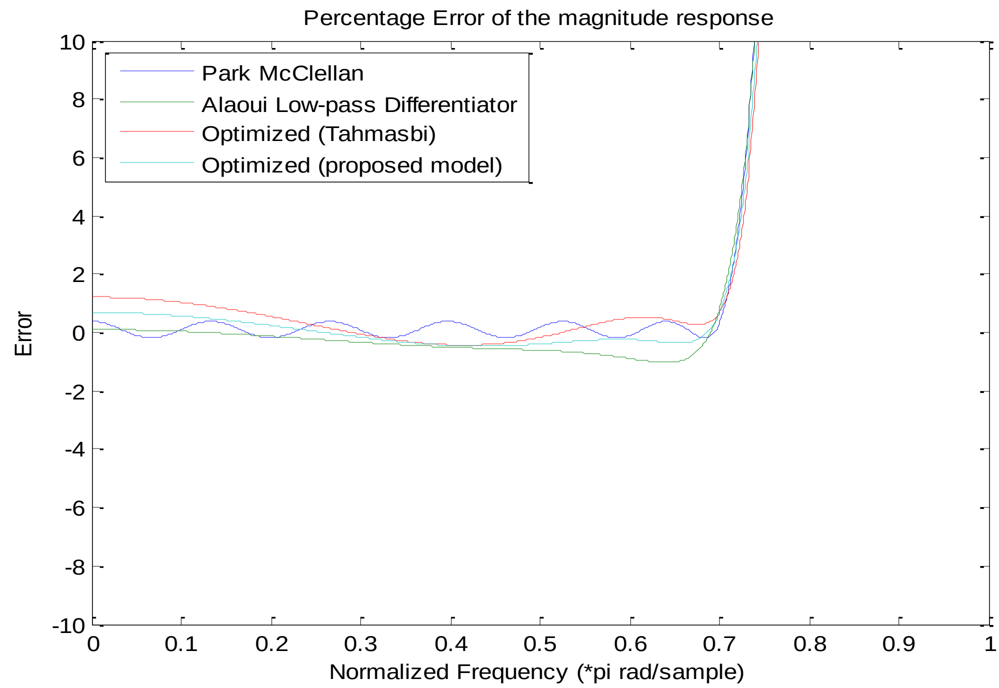


Figure 4.11: Comparison of percentage relative magnitude error of Al-Alaoui low-pass differentiator, Parks-McClellan FIR differentiator, optimized Tahmasbi and proposed optimized IIR differentiator.

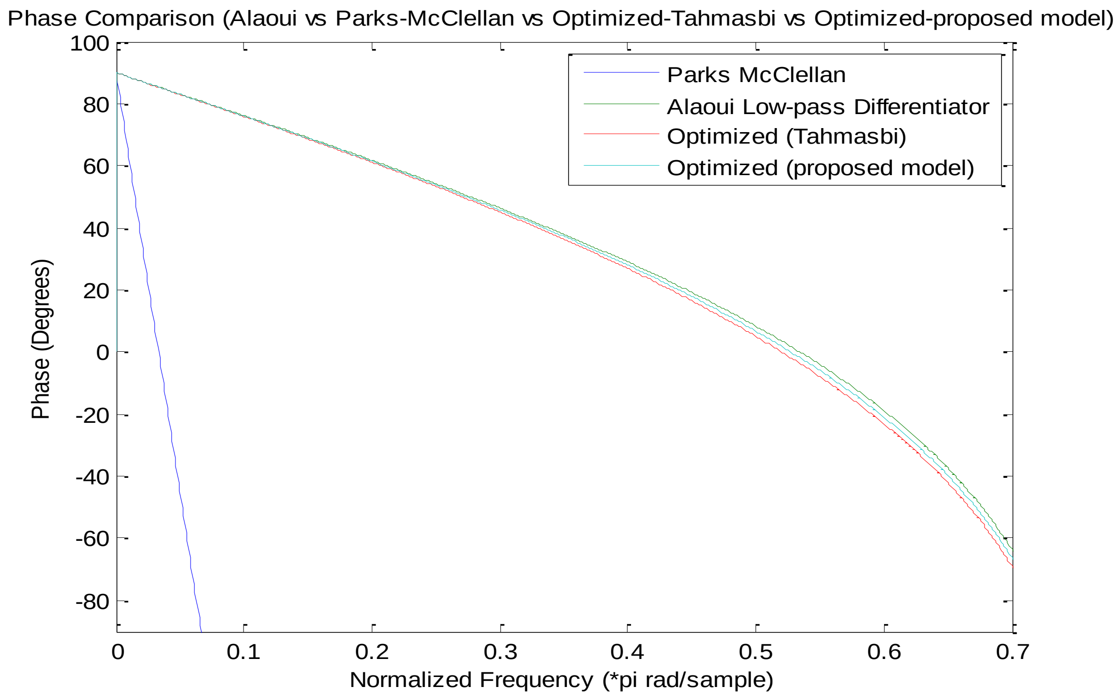


Figure 4.12: Comparison of percentage relative magnitude error of Al-Alaoui low-pass differentiator, Parks-McClellan FIR differentiator, optimized Tahmasbi and proposed optimized IIR differentiator.

The magnitude response of proposed optimized IIR differentiator (ω_c = 0.7) as compared with other differentiators is demonstrated in Figure 4.10. As shown in the magnified view of particular portion in Figure 4.10, it is clear that the magnitude roll-off of proposed IIR differentiator is lies between Al-Alaoui low-pass IIR differentiator and optimized IIR Tahmasbi differentiator. The percentage relative error of magnitude response of proposed optimized IIR differentiator as compared with other differentiators is shown in Figure 4.11. The proposed optimized IIR differentiator has less percentage error as compared with others in the desired range of frequency. Figure 4.12 shows the phase response of proposed optimized IIR differentiator with magnified view of its particular portion. The phase response of proposed optimized IIR differentiator is slightly linear as compared with optimized Tahmasbi IIR differentiator.

CHAPTER

5

Linear Phase Low-Pass IIR Fractional Order Digital Differentiator

5.1 Introduction

The theory of fractional order calculus (FOC) was refined usually in the nineteenth century despite of being 300-years old topic. Recent books bestow a good source of references on fractional calculus [22, 23]. In spite of, utilizing FOC to dynamic systems control is just a recent focus of interest. The fractional order of differentiation and integration is more suitable in control system applications [24]. Fractional Calculus is generalization of ordinary differentiation and integration to non-integer order i.e. taking real number powers of differentiation operator.

$$D^{v} f(x) = \frac{d^{v} f(x)}{dx^{v}} \tag{5.1}$$

If v is an integer then the case is called integer order differentiation. Positive real number corresponds to fractional order differentiation. The historical developments culminated in three calculi which are based on the work of Riemann-Liouville (RL), Grunwald-Letnikov (GL), and Caputo. The classical form of fractional calculus is given by the Riemann-Liouville integral. It is given as follows [22]:

$${}_{a}D_{t}^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_{a}^{t} (t-\tau)^{\alpha-1} f(\tau)\, d\tau \tag{5.2}$$

The important step in digital implementation of IIR fractional order differentiators is the discretization of the fractional-order differentiators s^{r}, where s is the transfer function of ideal differentiator in Laplace domain and r is any real number [25, 26]. In general, there are two discretization methods: direct discretization and indirect discretization. In indirect discretization methods [27], two steps are required, i.e., frequency domain fitting in continuous time domain first and then discretizing the filter *s*-domain transfer function. Other frequency-domain fitting methods can also be used but without guaranteeing the stable minimum-phase discretization. Existing direct discretization methods include the application of the direct power series expansion (PSE) of the Euler operator [27], continuous fractional expansion (CFE) of the Tustin operator [26], [28], and numerical integration based method. However, as pointed out in [25], [26], the Tustin operator based discretization scheme exhibits large errors in high frequency range. A new mixed scheme of Euler and Tustin operators is proposed in [13], [15] which yield the so-called Al-Alaoui operator. These discretization methods for s^r are in infinite impulse response (IIR) form. Recently, there are some reported methods to directly obtain the digital fractional order differentiators in finite impulse response (FIR) form. However, using an FIR filter to approximate s^r may be less efficient due to very high order of the FIR filter. In this paper, we focus on discretizing fractional differentiators in IIR forms.

The half-order numerical differentiator is expressed in higher order terms by using continuous fraction expansion (CFE). The discretize half order differentiator is cascaded with third order low-pass Chebyshev filter resulting in linear phase low-pass IIR fractional order digital differentiator (FrODD). The FrODD gives better performance to the real-time signal as shown by the simulation studies.

5.2 Numerical Differentiators and Its Discretization

A class of digital integrators is primarily derived from a class of numerical integration rules [13]. A class of digital differentiators is consequently obtained by inverting the transfer functions of the obtained integrators and stabilizing the resulting transfer functions together with magnitude compensation if necessary [14]. The trapezoidal and the Simpson rules are among the most popular methods for approximating the evaluation of the definite integrals [13], [15].

5.2.1 The First Order Differentiator: Al-Alaoui Operator

This is a first-order, obtained by interpolating the trapezoidal and the rectangular integration rules. It is also called as Al-Alaoui operator [13].

$$H(z) = aH_R(z) + (1-a)H_T(z),\ a \in [0,1] \tag{5.3}$$

Where a = 0.75 is actually a weighting factor. $H_R(z)$ and $H_T(z)$ are the z-domain transfer functions of the rectangular and the trapezoidal integrators. The transfer function of this resulting differentiator is given as follows [13]:

$$H(z) = \frac{8(z-1)}{7T\left(z+\frac{1}{7}\right)} \tag{5.4}$$

Here we presents the CFE of Al-Alaoui operator, the resulting discrete transfer function, approximating fractional order differentiator can be expressed as [26], [27]:

$$H_n^r(z) \approx \left(\frac{8}{7T}\right)^r CFE\left\{\left(\frac{1-z^{-1}}{1+0.1428z^{-1}}\right)^r\right\} \tag{5.5}$$

The discretization of the half-differentiator (r = 0.5 and n = 5) sampled at 0.001 sec is calculated numerically, and the approximated transfer function is given as follows [22]:

$$H_5(z) = \frac{2.47e04z^5-5.999e04z^4+4.941e04z^3-1.512e04z^2+956.9z+98.48}{730.7z^5-1357z^4+745.7z^3-89.48z^2-15.52z+1} \tag{5.6}$$

5.2.2 The Second Order Differentiator

The second order integrator obtained by interpolating the Simpson and trapezoidal integration rules [16]

$$H(z) = aH_S(z) + (1-a)H_T(z),\ a \in\ [0,1] \tag{5.7}$$

where a is actually a weighting factor. $H_S(z)$ and $H_T(z)$ are the z-domain transfer functions of the Simpson and the trapezoidal integrators [14], [16].

$$H_S(z) = \frac{T\left(z^2+4z+1\right)}{3(z^2-1)} \tag{5.8}$$

$$H_T(z) = \frac{T(z+1)}{2(z-1)} \tag{5.9}$$

By combining equation (5.8) and (5.9), the general weighted integrator with weighing factor a is given as follows:

$$H(z) = \frac{T(3-a)\{z^2+[2(3+a)/(3-a)]z+1\}}{6(z^2-1)} \tag{5.10}$$

By inverting the transfer function of above integrator in equation (10), the second order differentiator is obtained [16].

$$H(z) = \frac{6\left(z^2-1\right)}{T(3-a)\{z^2+[2(3+a)/(3-a)]z+1\}} \tag{5.11}$$

The discretization of the half-differentiator (r = 0.5, n = 3, 4 and a = 0.75) sampled at 0.001 sec is calculated numerically and the approximated transfer function is given as follows [26], [27]:

$$H_3(z) = \frac{968.1z^3 - 442z^2 - 820.8z + 363}{32.47z^3 - 4z^2 - 16.24z + 1} \tag{5.12}$$

$$H_4(z) = \frac{477z^4 + 968.1z^3 - 919z^2 - 820.8z + 422.7}{16z^4 + 37.8z^3 - 12z^2 - 18.9z + 1} \tag{5.13}$$

5.3 IIR Fractional Order Digital Differentiator

To obtain low-pass IIR FrODD cascade the discretized half order differentiator with an appropriate low-pass IIR filter [18]. The half-order differentiator is expressed in terms of higher order terms by using CFE [26], [27]. The choice for selecting an appropriate low-pass IIR filter is depends upon the requirements as mentioned in the chapter four. If steep roll-off characteristics are important, Chebyshev I or elliptical low-pass filter is selected.

The third order low-pass Chebyshev I filter with 0.1 dB ripple in pass-band at 0.52 normalized cut-off frequency is given as follows [19], [20]:

$$H_c(z) = \frac{0.2465 + 0.7395z^{-1} + 0.7395z^{-2} + 0.2465z^{-3}}{1 + 0.4800z^{-1} + 0.4845z^{-2} + 0.0076z^{-3}} \tag{5.14}$$

To obtain the FrODD, the cascading approach is used [18]. The discretized half-differentiator in equation (5.12), (5.13) and (5.6) are cascaded with low-pass IIR filter in equation (5.14), results in sixth, seventh and eighth order linear phase low-pass IIR FrODD. The coefficients of resulting low-pass IIR FrODD are expressed in descending power of z as in equation (5.15) below:

$$H(z) = \frac{b(1) + b(2)z^{-1} + b(3)z^{-2} + b(4)z^{-3} + b(5)z^{-4} + b(6)z^{-5} + b(7)z^{-6} + b(8)z^{-7} + b(9)z^{-8}}{a(1) + a(2)z^{-1} + a(3)z^{-2} + a(4)z^{-3} + a(5)z^{-4} + a(6)z^{-5} + a(7)z^{-6} + a(8)z^{-7} + a(9)z^{-8}} \tag{5.15}$$

Table 5.1 shows the coefficients $a(i)$ and $b(i)$ of resulting low-pass IIR FrODD at normalized cut-off frequency 0.5. The discretized differentiator transfer function in equation (5.6), (5.12) and (5.13) are calculated at sampling intervals of 0.001sec, whereas the Chebyshev filter transfer function in equation (5.14) is calculated at the sampling intervals of 1 sec. So it becomes necessary to first normalize the transfer function in equation (5.6), (5.12) and (5.13) for the sampling intervals of 1 sec before cascading them with low-pass IIR filter.

The frequency response of the resulting low-pass IIR FrODD is calculated by replacing with $z = e^{j\omega}$ in the resulting transfer function as mentioned in equation (5.15).

Table 5.1: Coefficients of the low-pass IIR FrODD.

Coefficients	Sixth Order	Seventh Order	Eighth Order
b(1)	0.1308	0.1308	0.1483
b(2)	0.3328	0.6580	0.0847
b(3)	0.1024	0.9369	0.3390
b(4)	-0.3321	-0.0538	-0.1330
b(5)	-0.2453	-1.0493	0.2632
b(6)	0.0363	-0.5788	-0.0421
b(7)	0.0449	0.1229	-0.0718
b(8)	0.0000	0.1159	0.0075
b(9)	0.0000	0.0000	0.0006
a(1)	1.0000	1.0000	1.0000
a(2)	0.3568	2.8425	-1.3771
a(3)	-0.0748	0.8685	0.6136
a(4)	-0.2614	-0.3903	-0.5248
a(5)	-0.2284	-0.8506	0.4003
a(6)	-0.0111	-0.5486	-0.0603
a(7)	0.00023	0.0213	-0.0106
a(8)	0.0000	0.00048	0.00052
a(9)	0.0000	0.00000	0.000011

The magnitude response of the proposed low-pass FrODD as compared with their counterparts integer order differentiator in [18] is demonstrated in Figure 5.1. The proposed fractional order differentiator almost completely approximates the ideal fractional differentiator up to its cut-off frequency range.

Figure 5.2 shows the percentage magnitude error relative to the ideal differentiator. It shows that relative percentage error of magnitude response of proposed FrODD have less percentage error as compared with integer order differentiator in the desired range cut-off frequency.

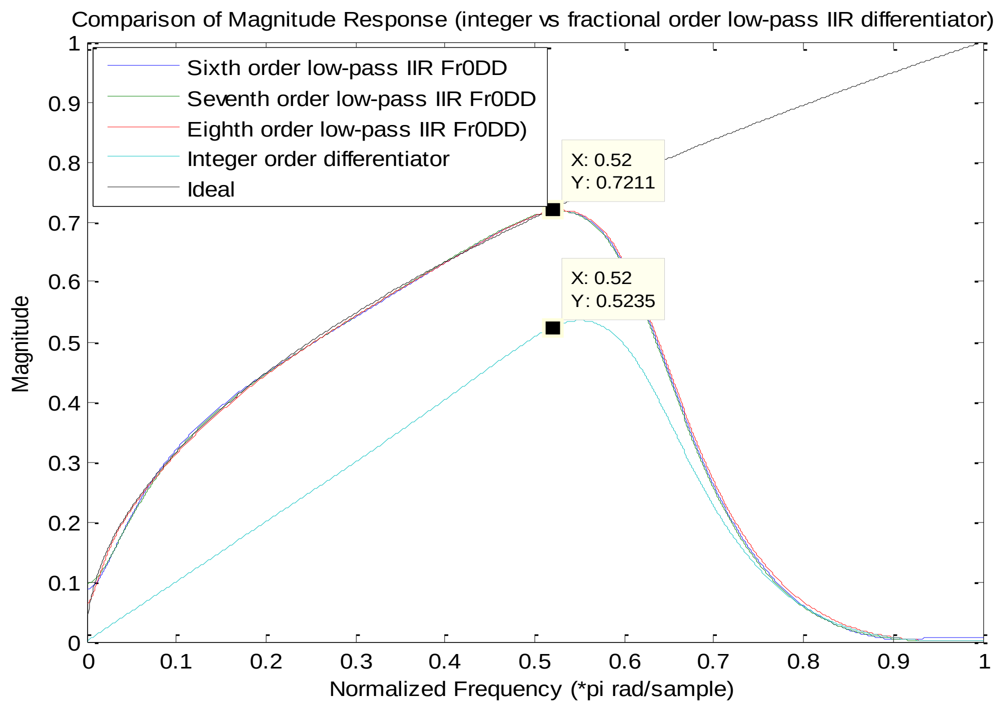


Figure 5.1: Comparison of magnitude response of proposed low-pass IIR FrODD with integer order low-pass IIR differentiator.

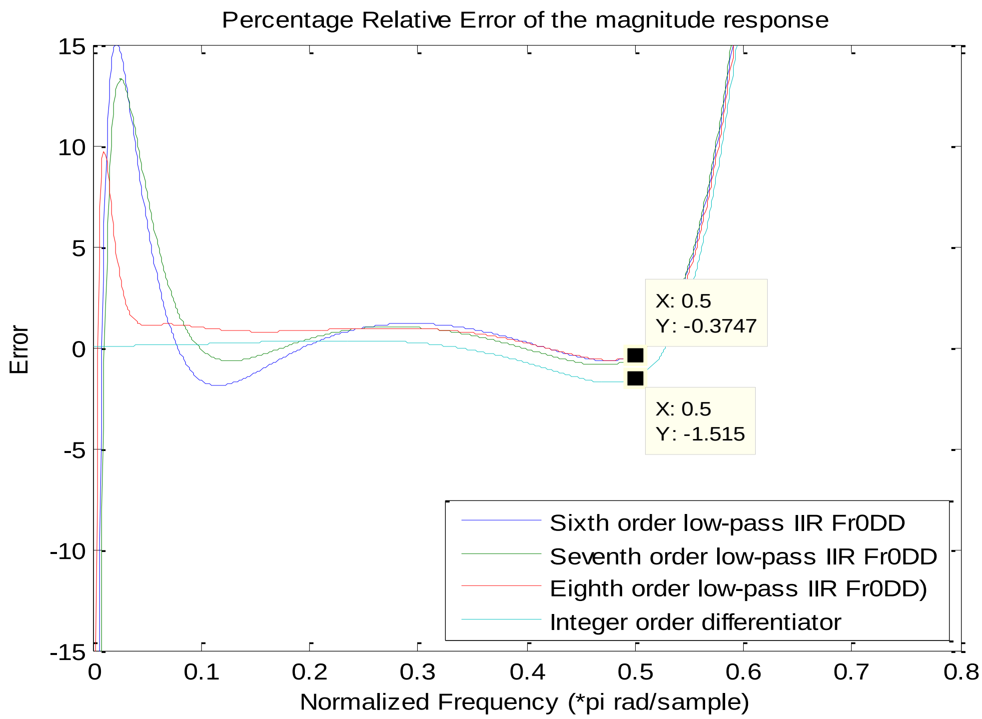


Figure 5.2: Comparison of percentage relative magnitude error of low-pass FrODD with integer order differentiator.

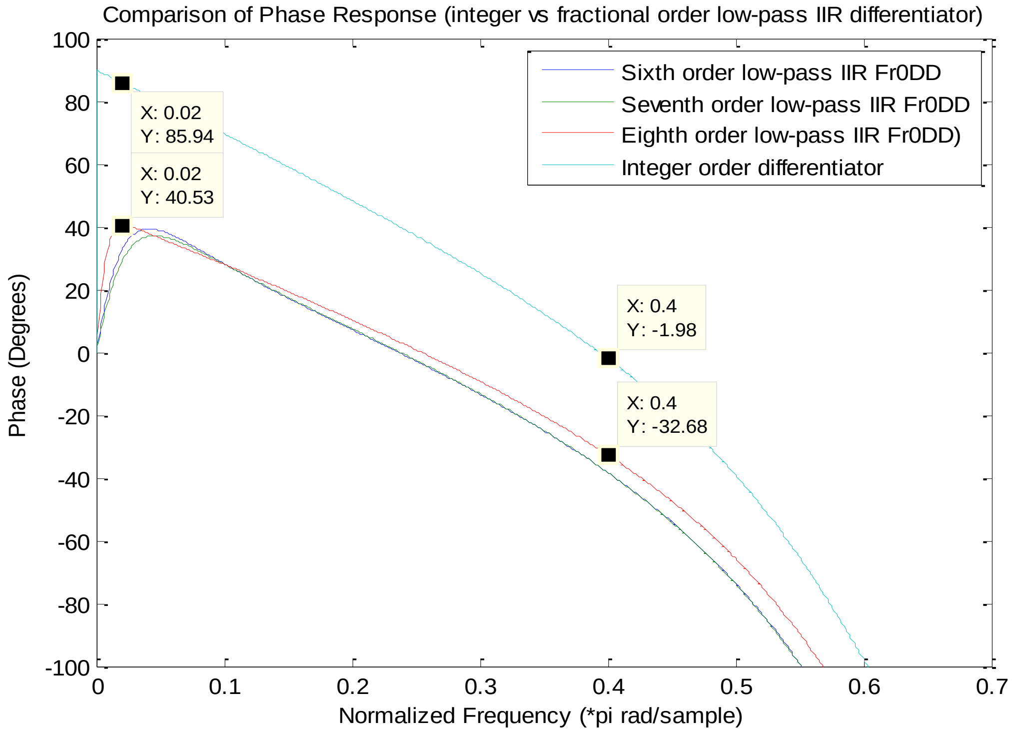


Figure 5.3: Comparison of phase response of low-pass IIR FrODD with integer order low-pass IIR differentiator.

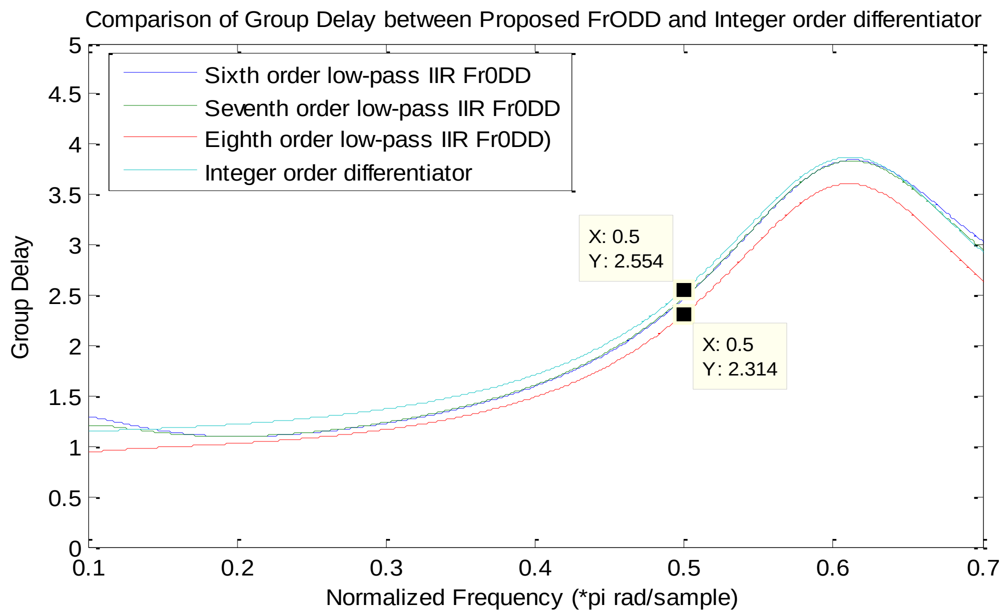


Figure 5.4: Comparison of group delay of low-pass FrODD with integer order differentiator.

Figure 5.3 shows the phase response of proposed FrODD as compared with integer order differentiator as mentioned in previous chapter. The phase response of proposed FrODD has almost linear phase response up to the cut-off frequency range. The proposed FrODD has less phase delay for low frequency. Also, Figure 5.4 shows group delay comparison between both, the proposed FrODD has less group delay as compared with integer order differentiator.

5.4 Comparative analysis between Fractional Order Differentiator and Integer Order Differentiator

When we pass any noisy signal through the differentiator, the differentiator will attenuate the noise amplitude to some extent. The performance comparison of the IIR FrODD with integer order differentiator is explained by passing some test signal through the differentiator. In this paper, a sinusoidal signal with Gaussian noise (SNR = 20dB) is passed through both differentiators.

The output behavior of both proposed FrODD and integer order differentiator to noise corrupted sinusoidal signal is shown in Figure 5.5. Both the FrODD and integer order differentiator will attenuate the noise presents in the input signal to some extent as shown in Figure 5.5, but the integer order differentiator also reduce the signal amplitude. Also proposed FrODD provides less phase delay for the sinusoidal input signal. The output SNR of any system with its mathematical analysis is illustrated below. The ratio of received signal power to the noise power can be defined as the output SNR of that system [29].

$$SNR_{output} = \frac{|H(x+n)|^2}{\sigma^2} \tag{5.16}$$

where H is the gain of the system, x is amplitude of the input signal, n is noise amplitude which is presented in input signal and σ^2 is the noise power. In this case an input sinusoidal with Gaussian noise (SNR = 20 dB) is passed through the differentiator.

$$SNR_{output} = \frac{|H|^2\{|x|^2+|n|^2\}}{\sigma^2} \tag{5.17}$$

$$SNR_{output} = |H|^2\left\{\frac{|x|^2}{\sigma^2} + \frac{\sigma^2}{\sigma^2}\right\} \tag{5.18}$$

$$SNR_{output} = |H|^2\{SNR_{input} + 1\} \tag{5.19}$$

$$SNR_{output} \propto |H|^2$$

The above expression in equation (5.19) for the output SNR explained that, the output SNR of any system is directly proportional to the gain of that transfer function.

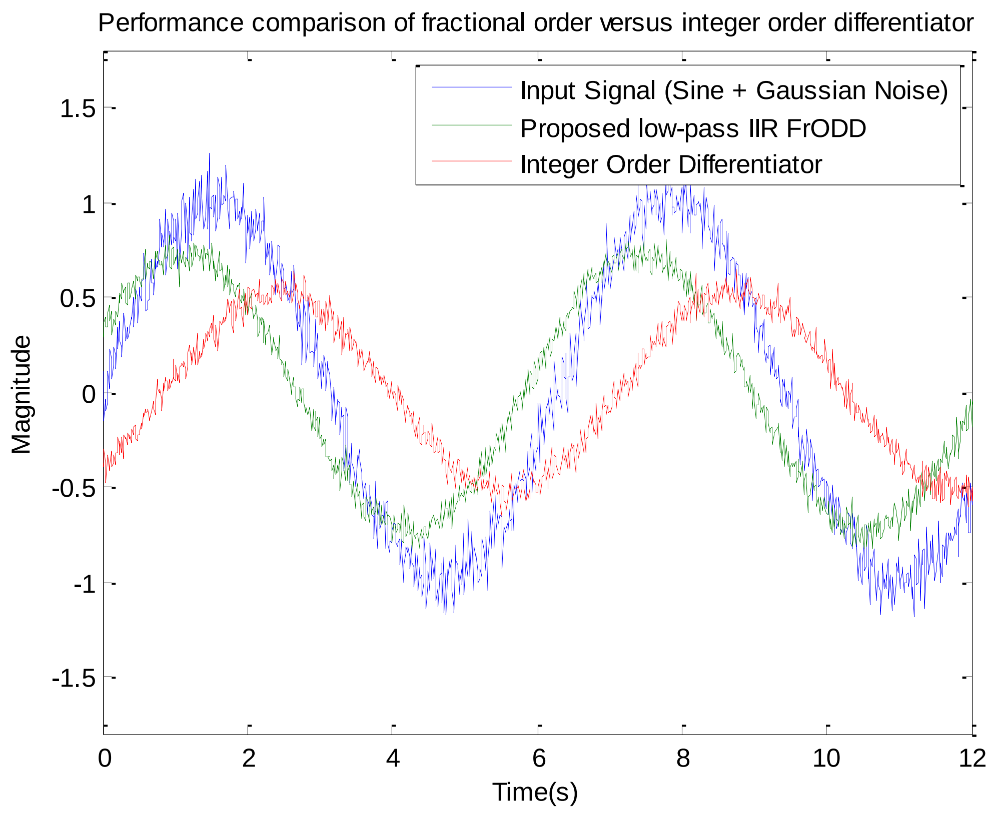


Figure 5.5: Performance comparison of FrODD with integer order differentiator for noise corrupted sinusoidal signal.

The proposed FrODD provides higher gain as compared with integer order differentiator as shown in Figure 5.1. From the above mathematical expressions and simulation results as shown in Figure 5.5, it is clear that the IIR FrODD gives appropriate results as compared with integer order IIR differentiator.

CHAPTER

6

Conclusions and Future Scope of Work

6.1 Conclusions

In this dissertation, IIR digital filters and various low-pass IIR differentiators have been discussed along with their frequency response characteristics. Various advantages and applications of digital filters are also listed in this dissertation work.

Mathematical techniques for the design of integer order order digital differentiator have been discussed. Several techniques to design digital differentiator available in literatures are also discussed.

A method for optimizing low-pass IIR digital differentiators is introduced in chapter four. The frequency response of the differentiator is improved by altering the gain factor and the denominator coefficients of the IIR differentiator. The *genetic algorithm* is used for calculating the appropriate coefficients to minimize the value of fitness function. The higher order Selesnick FIR differentiator and Parks-McClellan FIR differentiator are approximated by optimizing the coefficients of low-pass IIR differentiators. The low-order optimized differentiators almost give the same frequency response as compare with these FIR differentiators. The proposed optimized differentiators have less percentage magnitude error response in the desired cut-off frequency range. It is shown through the simulation results, that if we are optimizing the differentiator, there is always a trade-off between magnitude roll-off, phase response and percentage relative error of the magnitude response.

A novel approach for designing linear phase low-pass IIR fractional order digital differentiator (FrODD) is also introduced. The cascading approach is used for designing IIR FrODD. The proposed fractional order differentiator provides higher gain as compared with integer order IIR differentiator for the same frequency. The proposed FrODD also provide less group delay. The fractional order differentiators also have less percentage magnitude error for the desired range within cut-off frequency. The fractional order of differentiation and integration is also useful in control system applications.

6.2 Future Scope of Work

The application of higher order low pass IIR differentiators could well be used in image edge detection, complex QRS detection where the higher derivative is required to detect the peaks. It is interesting to seek other possible applications of the proposed filter design method in the future.

Further improvements in fractional order IIR digital differentiator by optimizing the frequency response based on various optimization techniques is an open research topic.

The design of differintegrators is also an area of further research. There is also chance to explore the design of low pass integrators, differintegrators and extension to fractional order cases to accomplish real world situations.

PUBLICATIONS

1. Rajeev Kumar and Sanjay Kumar, "A new optimized low-pass IIR digital differentiator," *Progress In Science and Engineering Research Journal*, vol. 2, no. 3/6, pp. 164-170, 2014.
2. Rajeev Kumar and Sanjay Kumar, "A new low-pass IIR fractional order digital differentiator," *Springer - Signal Image and Video Processing (under review),* submission date - June 2014.

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