SOME METHODS FOR SOLVING MULTI-ATTRIBUTE DECISION MAKING PROBLEMS UNDER FUZZY ENVIRONMENT AND ITS EXTENSIONS

Thesis submitted in partial fulfillment of the requirements for the award of degree of

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by

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CERTIFICATE

This is to certify that the thesis entitled, "**Some methods for solving multi-attribute decision making problems under fuzzy environment and its extensions**" submitted by **Arshdeep Kaur** in the fulfillment of the requirement for the award of the degree of Doctor of Philosophy in the School of Mathematics, Thapar Institute of Engineering & Technology, Patiala, is a record of candidate's own work carried out by her under our supervision and guidance.

The matter presented in this thesis has not been submitted in part or full for the award of any degree in any other University or Institute.

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DECLARATION

It is certified that the thesis is entirely my own. The ideas and references cited herein have been duly acknowledged.

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Dedicated

To

My Parents

&

God

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Arshdeep Kaur.

Patiala

(ARSHDEEP KAUR)

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List of abbreviations

CLPP	Crisp linear programming problem
CLPPs	Crisp linear programming problems
FLPPs	Fuzzy linear programming problems
IF	Intuitionistic fuzzy
IFCAAO	Intuitionistic fuzzy coupla arithmetic averaging operator
IFDM	Intuitionistic fuzzy decision matrix
IFMADM	Intuitionistic fuzzy multi-attribute decision making
IFMADMPr	Intuitionistic fuzzy multi-attribute decision making problem
IFMADMPrs	Intuitionistic fuzzy multi-attribute decision making problems
IFS	Intuitionistic fuzzy set
IVIF	Interval-valued intuitionistic fuzzy
IVIFDM	Interval-valued intuitionistic fuzzy decision matrix
IVIFMADM	Interval-valued intuitionistic fuzzy multi-attribute decision making
IVIFMADMPr	Interval-valued intuitionistic fuzzy multi-attribute decision making problem
IVIFMADMPrs	Interval-valued intuitionistic fuzzy multi-attribute decision making problems
IVIFMAGDM	Interval-valued intuitionistic fuzzy multi-attribute group decision making
IVIFNIS	Interval-valued intuitionistic fuzzy negative ideal solution
IVIFPIS	Interval-valued intuitionistic fuzzy positive ideal solution
IVIFS	Interval-valued intuitionistic fuzzy set
IVPF	Interval-valued Pythagorean fuzzy
IVPFMADM	Interval-valued Pythagorean fuzzy multi-attribute decision making

IVPFMADMPrs	Interval-valued Pythagorean fuzzy multi-attribute decision making problems
IVPFS	Interval-valued Pythagorean fuzzy set
LPP	Linear programming problem
LPP’s	Linear programming problems
MADM	Multi-attribute decision making
MADMPrs	Multi-attribute decision making problems
MADMPr	Multi-attribute decision making problem
MAGDM	Multi-attribute group decision making
NLPP	Non-linear programming problem
PFS	Pythagorean fuzzy set
PIFS	Picture fuzzy set
PIFSs	Picture fuzzy sets
TFN	Triangular fuzzy number
TFNs	Triangular fuzzy numbers
TOPSIS	Technique for order preference by similarity to ideal solution
OAWV	Optimal attribute weight vector

ABSTRACT

Several methods have been proposed in the literature to solve IFMADMPrs having partially known attribute weights [50, 63, 81, 146, 222-225, 241], IVIFMADMPrs having partially known attribute weights [62, 108, 213, 258], IVPFMADMPrs having partially known attribute weights [84], IVIFMADMPrs having IVIF attribute weights [39-45, 47-49, 52, 55, 127-129, 132, 137, 209-211]. However, after a deep study, it is observed that all these existing methods are inappropriate. Also, it is observed that it is inappropriate to apply the existing method [98] for solving IVIFMAGDM problems having completely unknown attribute weights. Furthermore, it is observed that the existing definition of an IVIFS [9] and the existing definition of an IVPFS [164] are inappropriate.

Keeping all above in mind, the aim of this thesis is

(i) To point out the inappropriateness of the existing methods [50, 63, 81, 146, 222-225, 241] for solving IFMADMPrs having partially known attribute weights as well as to propose an appropriate method for solving IFMADMPrs having partially known attribute weights.

(ii) To point out the inappropriateness of the existing methods [62, 108, 213, 258] for solving IVIFMADMPrs having partially known attribute weights as well as to propose an appropriate method for solving IVIFMADMPrs having partially known attribute weights.

(iii) To point out the inappropriateness of the existing methods [84] for solving IVPFMADMPrs having partially known attribute weights as well as to propose an appropriate method for solving IVPFMADMPrs having partially known attribute weights.

(iv) To point out the inappropriateness of the existing methods [39-45, 47-49, 52, 55, 127-129, 132, 137, 209-211] for solving IVIFMADMPrs having IVIF weights as well as to propose an appropriate method for solving IVIFMADMPrs having IVIF weights.

(v) To point out the inappropriateness of the existing method [98] for solving IVIFMAGDM problems having completely unknown weights. Also, to make the researchers aware that it is inappropriate to propose a simplex method for solving an IVIF LPP without transforming it into a CLPP, inspired by Khan et al.'s simplex method [122], for resolving the inappropriateness of the existing method [98] due to the following facts:

> "*Khan et al.'s claim [123] "No mathematical incorrect assumptions have been considered in the existing simplex method [122] for solving a fuzzy LPP without transforming it into a CLPP" against Bhardwaj and Kumar's claim [16] is not valid. In fact, several mathematical incorrect assumptions, considered by Khan et al. [122], are not pointed out by Bhardwaj and Kumar [16] e.g., Bhardwaj and Kumar have not pointed out that some of the elements of the existing optimal simplex table [122] are not TFNs*."

Furthermore, to make the researchers aware that it is not possible to resolve the inappropriateness of the existing methods [98]."

(vi) To propose the appropriate definition of an IVIFS.

(vii) To propose the appropriate definition of an IVPFS.

List of published/communicated papers

1. **Arshdeep Kaur**, Amit Kumar, Commentary on "On intuitionistic fuzzy copula aggregation operators in multiple-attribute decision making", Cognitive Computation 12(4) (2020) 891-895. **(Impact Factor: 4.287)**
2. **Arshdeep Kaur**, Amit Kumar, Commentary on "A geometric approach for ranking interval-valued intuitionistic fuzzy numbers with an application to group decision-making", Computers & Industrial Engineering 135 (2019) 314-316. **(Impact Factor: 3.518)**
3. **Arshdeep Kaur**, Amit Kumar, S.S. Appadoo, A note on "Approaches to interval intuitionistic trapezoidal fuzzy multiple attribute decision making with incomplete weight information", International Journal of Fuzzy Systems 21 (2019) 1010-1011. (**Impact Factor: 3.085)**
4. **Arshdeep Kaur**, Amit Kumar, S.S. Appadoo, Commentary on "A reply to note on the paper "A simplified novel technique for solving fully fuzzy linear programming problems"", Journal of Intelligent & Fuzzy Systems 36 (2019) 5685-5691. (**Impact Factor: 1.637**)
5. **Arshdeep Kaur**, A Note on "Multiattribute decision making based on interval-valued intuitionistic values and linear programming methodology", International Journal of Mathematical Archive 9 (2018) 78-79.
6. **Arshdeep Kaur**, Amit Kumar, A note on "An interval type-2 fuzzy LINMAP method with approximate ideal solutions for multiple criteria decision analysis", Mathematical Sciences International Research Journal 7 (2019) 29-33.
7. **Arshdeep Kaur**, Akansha Mishra, Amit Kumar, Miraj Ali Khan, A note on "A new improved score function of an interval-valued Pythagorean fuzzy set based on TOPSIS method" (Communicated in Journal of Intelligent & Fuzzy Systems)

8. **Arshdeep Kaur**, Amit Kumar, Mehar method for solving IFMADM problems having partially known attribute weights (Communicated in Knowledge-Based Systems).
9. **Arshdeep Kaur**, Amit Kumar, Ajit method for solving IFMADM problems having partially known attribute weights (Communicated in Soft Computing).
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12. **Arshdeep Kaur**, Amit Kumar, Jujhar method for solving IVIFMADM problems having IVIF attribute weights (Communicated in Engineering Applications of Artificial Intelligence).
13. **Arshdeep Kaur**, Amit Kumar, Ajith Abraham, Appropriate definition of an IVIFS (Communicated in Neuro computing).
14. **Arshdeep Kaur**, Amit Kumar, Ajith Abraham, Appropriate definition of an IVPFS (Communicated in Applied Soft Computing).
15. **Arshdeep Kaur**, Amit Kumar, Commentary on “A novel picture fuzzy linguistic aggregation operator and its application to group decision making” (Communicated in Cognitive Computation).
16. **Arshdeep Kaur**, Amit Kumar, KALI ranking method for comparing interval-valued intuitionistic fuzzy sets and its application in decision making (Communicated in Cybernetics and Systems).

List of papers presented in Conferences

1. **Arshdeep Kaur**, Amit Kumar, The inclusion-based LINMAP method for multiple criteria decision analysis within an interval-valued Atanassov's intuitionistic fuzzy environment: Suggested Modification, International Symposium on OPERATIONS Research and Game Theory: Modeling and Computation organized by Indian Statistical Institute, Delhi, India during January 9-11, 2018.
2. **Arshdeep Kaur**, Amit Kumar, A note on "An interval type-2 fuzzy LINMAP method with approximate ideal solutions for multiple criteria decision analysis", International Conference on Advances in Mathematics, Engineering & Technology-2018 organized by Carmel College for Women, Nuvem, Goa in collaboration with International Multidisciplinary Research Foundation Institute for Education and Research during December 28-29, 2018.
3. **Arshdeep Kaur**, Amit Kumar, Vaishnavi method to find the optimal weight vector for the criteria of MCDM problems under picture fuzzy environment, 9th International Congress on Industrial and Applied Mathematics organized by International Council for Industrial and Applied Mathematics during July, 15-19, Valencia, Spain.
4. Amit Kumar, **Arshdeep Kaur**, Reasons for not using entropy and knowledge measure in decision making problems under intuitionistic fuzzy environment and interval-valued intuitionistic fuzzy environment, 9th International Congress on Industrial and Applied Mathematics organized by International Council for Industrial and Applied Mathematics during July, 15-19, Valencia, Spain.

Chapter 1

Introduction

In several daily life problems, there is need to follow a process to select the best alternative from all the available alternatives having same conflicting attributes or to rank all the available alternatives having same conflicting attributes. This process is called MADM and such problems are called MADMPrs [103] e.g., to rank all the students of a class or to select the best student from all the students of a class is a MADMPr. In this MADMPr students represents the alternatives and the courses assigned to all the students represents the conflicting attributes.

A MADMPr having m-alternatives $A_i, i = 1,2,\dots,m$ and n-attributes $C_j, j = 1,2,\dots,n$ can be represented by a matrix D as follows:

$$D = \begin{array}{c|ccc|} & C_1 & \dots & C_n \\ \hline A_1 & d_{11} & \cdots & d_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ A_m & d_{m1} & \cdots & d_{mn} \end{array}$$

where, d_{ij} represents the performance score/rating value of the i^{th}- alternative over the j^{th}-attribute.

In general, it is assumed that if the j^{th}-attribute C_j is a quantitative attribute then $d_{ij}, i = 1,2,\dots,m$ can be represented by a numerical value.

However, this assumption is not always valid in real-life e.g., the mileage of a car is a quantitative attribute. But, it is not always possible to represent it as a real number e.g., mileage of a car may be approximately 90 km/l.

Similarly, in general, it is assumed that if the j^{th}- attribute C_j is a qualitative attribute then $d_{ij}, i = 1,2,\dots,m, j = 1,2,\dots,n$ can be represented by a linguistic term like good, bad, very good etc. Also, it is assumed that a real number can be assigned corresponding to each linguistic term. However, this assumption is not always valid in real-life. e.g., it is illogical to assign a real-number corresponding to the linguistic term excellent students in Mathematics (students who have secured either 90 marks or more than 90 marks out of 100 marks) as a student who has secured 100 marks is surely better than the student who has secured 90 marks.

One way to handle such situations is to represent the data as a fuzzy set [263]. The MADMPrs in which atleast one element $d_{ij}, i = 1,2,\ldots,m; j = 1,2,\ldots,n$ of the decision matrix D is represented as a fuzzy set are called MADMPrs under fuzzy environment [46].

"Let X be a universal set. A fuzzy set $\tilde{A}$ *over X is defined as* $\tilde{A} = \{\langle x, \mu_A(x)\rangle | \, x \in X\}$, *where* $\mu_{\tilde{A}}: X \rightarrow [0,1]$ *and* $\mu_{\tilde{A}}(x)$ *indicates the degree of membership of x in* $\tilde{A}$."

The fuzzy set $\tilde{A}$ was proposed by considering the assumption that if the degree of membership of an element "x" in a set $\tilde{A}$ is $\mu_{\tilde{A}}(x)$ then, the degree of non-membership of the element "x" in the same set $\tilde{A}$ will be $1 - \mu_{\tilde{A}}(x)$. However, this assumption is not realistic as there may also exist a degree of hesitation $h_{\tilde{A}}(x)$ with degree of membership $\mu_{\tilde{A}}(x)$ and the degree of non-membership $\nu_{\tilde{A}}(x)$. Motivated by the same, in 1986, Atanassov [7] introduced the concept of an IFS.

"Let X be a universal set. An IFS $\tilde{A}$ *over X is defined as* $\tilde{A} = \{\langle x, \mu_{\tilde{A}}(x), \nu_{\tilde{A}}(x) \rangle | \, x \in X\}$, *where* $\mu_{\tilde{A}}(x)$ *and* $\nu_{\tilde{A}}(x)$ *represents the degree of membership and the degree of non-membership of the element* $x \in X$ *to the set* $\tilde{A}$, *respectively and satisfies the condition* $\mu_{\tilde{A}}(x) + \nu_{\tilde{A}}(x) \leq 1$.*The IFS* $\tilde{A} = \{\langle x, \mu_{\tilde{A}}(x), \nu_{\tilde{A}}(x) \rangle | \, x \in X\}$ *is also represented as* $\langle \mu_{\tilde{A}}(x), \nu_{\tilde{A}}(x) \rangle$. *"*

Yager [250] pointed out that an intuitionistic fuzzy set is constructed by considering the assumption $0 \leq \mu_{\tilde{A}}(x) + \nu_{\tilde{A}}(x) \leq 1$. However, there exist problems in which the condition $\mu_{\tilde{A}}(x) + \nu_{\tilde{A}}(x) \leq 1$ will not be satisfied but the condition $\left(\mu_{\tilde{A}}(x)\right)^2 + \left(\nu_{\tilde{A}}(x)\right)^2 \leq 1$ will be satisfied. Motivated by the same, Yager [250] proposed the concept of a PFS.

"Let X be a universal set. A PFS $\tilde{A}$ *over X is defined as* $\tilde{A} = \{x, \mu_{\tilde{A}}(x), \nu_{\tilde{A}}(x) | \, x \in X\}$, *where* $\mu_{\tilde{A}}(x), \nu_{\tilde{A}}(x) \in [0,1]$ *represents the degree of membership and the degree of non-*

membership of the element $x \in X$ *to the set* $\tilde{A}$*, respectively and satisfies the condition* $\left(\mu_{\tilde{A}}(x)\right)^2 + \left(\nu_{\tilde{A}}(x)\right)^2 \leq 1$*. Also,* $\pi_{\tilde{A}}(x) = \sqrt{1 - \left(\mu_{\tilde{A}}(x)\right)^2 - \left(\nu_{\tilde{A}}(x)\right)^2}$ *is called the degree of indeterminacy of* x*. The PFS* $\tilde{A} = \{\langle x, \mu_{\tilde{A}}(x), \nu_{\tilde{A}}(x) \rangle | \, x \in X\}$ *is also represented as* $\langle \mu_{\tilde{A}}(x), \nu_{\tilde{A}}(x) \rangle$*."*

It is obvious that in an IFS and a PFS, the degree of membership and the degree of non-membership of a decision-maker is represented as a single real number. However, there may exist uncertainty in the mind of decision-maker regarding these values. In such situations, it is better to represent the data as an IVIFS [9] and an IVPFS [164] instead of IFS and PFS respectively.

"Let X *be a universal set. An IVIFS* $\tilde{A}$ *over* X *is defined as* $\tilde{A} = \left\{\langle x, \left[\mu_{\tilde{A}}^L(x), \mu_{\tilde{A}}^U(x)\right], \left[\nu_{\tilde{A}}^L(x), \nu_{\tilde{A}}^U(x)\right]\rangle \mid x \in X\right\}$*, where,* $0 \leq \mu_{\tilde{A}}^L(x) \leq \mu_{\tilde{A}}^U(x) \leq 1$*,* $0 \leq \nu_{\tilde{A}}^L(x) \leq \nu_{\tilde{A}}^U(x) \leq 1$ *and* $0 \leq \mu_{\tilde{A}}^U(x) + \nu_{\tilde{A}}^U(x) \leq 1$*. The interval of degree of hesitation* $\left[h_{\tilde{A}}^L(x), h_{\tilde{A}}^U(x)\right]$ *can be obtained by using the expression* $\left[h_{\tilde{A}}^L(x), h_{\tilde{A}}^U(x)\right] = [1,1] - \left(\left[\mu_{\tilde{A}}^L(x), \mu_{\tilde{A}}^U(x)\right] + \left[\nu_{\tilde{A}}^L(x), \nu_{\tilde{A}}^U(x)\right]\right)$*. The IVIFS* $\tilde{A} = \left\{\langle x, \left[\mu_{\tilde{A}}^L(x), \mu_{\tilde{A}}^U(x)\right], \left[\nu_{\tilde{A}}^L(x), \nu_{\tilde{A}}^U(x)\right]\rangle \mid x \in X\right\}$ *is also represented as* $\tilde{A} = \left(\left[\mu_{\tilde{A}}^L(x), \mu_{\tilde{A}}^U(x)\right], \left[\nu_{\tilde{A}}^L(x), \nu_{\tilde{A}}^U(x)\right]\right)$*."*

"Let X *be a universal set. An IVPFS* $\tilde{A}$ *over* X *is defined as* $\tilde{A} = \left\{\langle x, \left[\mu_{\tilde{A}}^L(x), \mu_{\tilde{A}}^U(x)\right], \left[\nu_{\tilde{A}}^L(x), \nu_{\tilde{A}}^U(x)\right]\rangle \mid x \in X\right\}$*, where,* $0 \leq \mu_{\tilde{A}}^L(x) \leq \mu_{\tilde{A}}^U(x) \leq 1$*,* $0 \leq \nu_{\tilde{A}}^L(x) \leq \nu_{\tilde{A}}^U(x) \leq 1$ *and* $0 \leq \left(\mu_{\tilde{A}}^U(x)\right)^2 + \left(\nu_{\tilde{A}}^U(x)\right)^2 \leq 1$*. The interval of degree of hesitation* $\left[h_{\tilde{A}}^L(x), h_{\tilde{A}}^U(x)\right]$ *can be obtained by using the expression* $\left[h_{\tilde{A}}^L(x), h_{\tilde{A}}^U(x)\right] = \left[\sqrt{1 - \left(\mu_{\tilde{A}}^U(x)\right)^2 - \left(\nu_{\tilde{A}}^U(x)\right)^2}, \sqrt{1 - \left(\mu_{\tilde{A}}^L(x)\right)^2 - \left(\nu_{\tilde{A}}^L(x)\right)^2}\right]$*. The IVPFS* $\tilde{A} = \left\{\langle x, \left[\mu_{\tilde{A}}^L(x), \mu_{\tilde{A}}^U(x)\right], \left[\nu_{\tilde{A}}^L(x), \nu_{\tilde{A}}^U(x)\right]\rangle \mid x \in X\right\}$ *is also represented as* $\tilde{A} = \left(\left[\mu_{\tilde{A}}^L(x), \mu_{\tilde{A}}^U(x)\right], \left[\nu_{\tilde{A}}^L(x), \nu_{\tilde{A}}^U(x)\right]\right)$*."*

1.1 Classification of MADMPrs under fuzzy environment and its extensions

On the basis of the representation of $d_{ij}, i = 1,2, \ldots, m; j = 1,2, \ldots, n$, MADMPrs can be classified into different categories. Some of these categories are as follows:

(i) **IFMADMPrs:** This category contains all such MADMPrs in which atleast one element d_{ij} of the decision matrix D is represented by an IFS.

(ii) **IVIFMADMPrs:** This category contains all such MADMPrs in which atleast one element d_{ij} of the decision matrix D is represented by an IVIFS.

(iii) **IVPFMADMPrs:** This category contains all such MADMPrs in which atleast one element d_{ij} of the decision matrix D is represented by an IVPFS.

On the basis of the nature of the attribute weight, each of the above mentioned MADMPr can be further sub categorized as follows:

(i) **IF / IVIF/ IVPF MADMPrs having completely known attribute weights:** This category contains all such IF/ IVIF/ IVPF problems in which weight of each attribute is known.

(ii) **IF/ IVIF/ IVPF MADMPrs having partially known attribute weights:** This category contains all such IF/ IVIF/ IVPF problems in which some properties about attribute weights are known. Also, there is need to find the attribute weights by considering the known properties.

(iii) **IF/ IVIF/ IVPF MADMPrs having IF/ IVIF/ IVPF attribute weights:** This category contains all such IF/ IVIF/ IVPF problems in which atleast one attribute weight is represented by an IFS/ IVIFS/ IVPFS.

1.2 Aim of the thesis

Several methods have been proposed in the literature to solve IFMADMPrs having partially known attribute weights [50, 63, 81, 146, 222-225, 241], IVIFMADMPrs having partially known attribute weights [62, 108, 213, 258], IVPFMADMPrs having partially known attribute weights [84], IVIFMADMPrs having IVIF attribute weights [39-45, 47-49, 52, 55, 127-129, 132, 137, 209-211]. However, after a deep study, it is observed that all these existing methods are inappropriate. Also, it is observed that it is inappropriate to apply the existing method [98] for solving IVIFMAGDM problems having completely unknown attribute weights. Furthermore, it is observed that the existing definition of an IVIFS [9] and the existing definition of an IVPFS [164] are inappropriate.

Keeping all above in mind, the aim of this thesis is

(i) To point out the inappropriateness of the existing methods [50, 63, 81, 146, 222-225, 241] for solving IFMADMPrs having partially known attribute weights as well as to propose an appropriate method for solving IFMADMPrs having partially known attribute weights.

(ii) To point out the inappropriateness of the existing methods [62, 108, 213, 258] for solving IVIFMADMPrs having partially known attribute weights as well as to propose an appropriate method for solving IVIFMADMPrs having partially known attribute weights.

(iii) To point out the inappropriateness of the existing methods [84] for solving IVPFMADMPrs having partially known attribute weights as well as to propose an appropriate method for solving IVPFMADMPrs having partially known attribute weights.

(iv) To point out the inappropriateness of the existing methods [39-45, 47-49, 52, 55, 127-129, 132, 137, 209-211] for solving IVIFMADMPrs having IVIF weights as well as to propose an appropriate method for solving IVIFMADMPrs having IVIF weights.

(v) To point out the inappropriateness of the existing method [98] for solving IVIFMAGDM problems having completely unknown weights. Also, to make the researchers aware that it is inappropriate to propose a simplex method for solving an IVIF LPP without transforming it into a CLPP, inspired by Khan et al.'s simplex method [122], for resolving the inappropriateness of the existing method [98] due to the following facts:

> "*Khan et al.'s claim [123] "No mathematical incorrect assumptions have been considered in the existing simplex method [122] for solving a fuzzy LPP without transforming it into a CLPP" against Bhardwaj and Kumar's claim [16] is not valid. In fact, several mathematical incorrect assumptions, considered by Khan et al. [122], are not pointed out by Bhardwaj and Kumar [16] e.g., Bhardwaj and Kumar have not pointed out that some of the elements of the existing optimal simplex table [122] are not TFNs*."

Furthermore, to make the researchers aware that it is not possible to resolve the inappropriateness of the existing methods [98]."

(vi) To propose the appropriate definition of an IVIFS.

(vii) To propose the appropriate definition of an IVPFS.

1.3 Chapter wise summary of the thesis

The chapter wise summary of the thesis is as follows:

Chapter 2

Mehar method for solving IFMADMPrs having partially known attribute weights

Xu [241] claimed that there does not exist any method for solving IFMADMPrs having partially known attribute weights. To fill this gap, Xu proposed a method to solve IFMADMPrs having partially known attribute weights. Wei [223] also proposed a method for solving IFMADMPrs having partially known attribute weights. It is pertinent to mention that several researchers [50, 63, 81, 146, 222, 224, 225] have used the existing methods [223, 241] to solve IFMADMPrs having partially known attribute weights. However, after a deep study it is observed that it is inappropriate to use the existing methods [223, 241]. The aim of this chapter is to make the researchers aware about the inappropriateness of these methods as well as to propose a new method (named as Mehar method) for solving IFMADMPrs having partially known attribute weights.

Chapter 3

Ajit method for solving IFMADMPrs having partially known attribute weights

In the Mehar method (proposed in Chapter 2) as well as in the existing methods [223, 241], a CLPP has been solved to obtain the OAWV. However, the constraint $\sum_{j=1}^{n} w_j = 1$ of the considered CLPP will not be satisfied if either the condition $\sum_{j=1}^{n} w_j^l > 1$ or the condition $\sum_{j=1}^{n} w_j^u < 1$ will be satisfied, where, w_j^l and w_j^u are the known lower bound and the upper bound of the j^{th}- attribute weight i.e., no feasible solution of the considered CLPP will be obtained if either the condition $\sum_{j=1}^{n} w_j^l > 1$ or the condition $\sum_{j=1}^{n} w_j^u < 1$ will be satisfied. Therefore, the proposed Mehar method and the existing methods [223, 241] can be used only if neither the condition $\sum_{j=1}^{n} w_j^l > 1$ nor the condition $\sum_{j=1}^{n} w_j^u < 1$ will be satisfied. Keeping the same in mind, in this chapter, a new method (named as Ajit Method) has been proposed to solve such IFMADMPrs in which either the condition $\sum_{j=1}^{n} w_j^l > 1$ or the condition $\sum_{j=1}^{n} w_j^u < 1$ will be satisfied.

Chapter 4

Jujhar method for solving IVIFMADMPrs having partially known attribute weights

Wang et al. [213] claimed that there does not exist any method for solving IVIFMADMPrs having partially known attribute weights. To fill this gap, Wang et al. proposed a method to solve the IVIFMADMPrs having partially known attribute weights. Motivated by the work of Wang et al., several researchers [62, 108, 258] have proposed different methods for solving IVIFMADMPrs having partially known attribute weights. However, after a deep study it is observed that it is inappropriate to use the existing methods [62, 108, 213, 258]. The aim of this chapter is to make the researchers aware about the inappropriateness of these methods as well as to propose a new method (named as Jujhar method) to solve the IVIFMADMPrs having partially known attribute weights.

Chapter 5

Jorawar method for solving IVPFMADMPrs having partially known attribute weights

To the best of my knowledge, only the existing method [84] has been proposed to solve IVPFMADMPrs having partially known attribute weights. However after a deep study, it is observed that it is inappropriate to use this method. The aim of this chapter is to make the researchers aware about the inappropriateness of this method as well as to propose a new method (named as Jorawar method) to solve IVPFMADMPr having partially known attribute weights.

Chapter 6

Jujhar method for solving IVIFMADMPrs having IVIF attribute weights

Li [128] pointed out that there does not exist any method to solve IVIFMADMPrs having IVIF weights. To fill this gap, Li proposed a method to solve IVIFMADMPrs having IVIF weights. Motivated by Li, several other researchers [39-45, 47-49, 52, 55, 127, 129, 132, 137, 209-211] have proposed different methods to solve the IVIFMADMPrs having IVIF weights. However, after a deep study it is observed that it is inappropriate to use the existing methods [39-45, 47-49, 52, 55, 127-129, 132, 137, 209-211]. The aim of this chapter is to make the researchers aware about the inappropriateness of these methods as well as to propose a new method (named as Jujhar method) to solve IVIFMADMPrs having IVIF attribute weights.

Chapter 7

Discussion on "Evolving a linear programming technique for MAGDM problems with IVIF information"

Hajiagha et al. [98] proposed a linear programming technique for solving IVIFMAGDM problems. In this technique, firstly an IVIFDM $\widetilde{D}=\left(\left[a_{ij1},a_{ij2}\right],\left[a_{ij3},a_{ij4}\right]\right)_{m\times n}$ is obtained by aggregating the IVIFDMs $\widetilde{D}^k=\left(\left[a_{ij1}^k,a_{ij2}^k\right],\left[a_{ij3}^k,a_{ij4}^k\right]\right)_{m\times n}, k=1,2,\ldots,r,$ where, $\left(\left[a_{ij1}^k,a_{ij2}^k\right],\left[a_{ij3}^k,a_{ij4}^k\right]\right)_{m\times n}$ represents the rating value of the i^{th}- alternative over the j^{th}-attribute provided by the k^{th}-decision-maker. Then, the IVIF LPPs $(Ps); s=1,2,\ldots,m$ are constructed.

$$Max\left(\sum_{j=1}^{n}\left(\left[a_{sj1},a_{sj2}\right],\left[a_{sj3},a_{sj4}\right]\right)\times w_j\right)$$

Subject to

$$\begin{aligned}&\textstyle\sum_{j=1}^{n}\left(\left[a_{ij1},a_{ij2}\right],\left[a_{ij3},a_{ij4}\right]\right)\times w_j\leq\tilde{1}; i=1,2,\ldots,m,\\&\textstyle\sum_{j=1}^{n}w_j=1,\\&w_j\geq 0; j=1,2,\ldots,n.\end{aligned}\qquad (Ps)$$

where, $\tilde{1}=([0.90,0.95],[0.01,0.05])$ and w_j represents the normalized weight of the j^{th} attribute.

After that each IVIF LPP $(Ps), s=1,2,\ldots,m$ is transformed into its equivalent CLPP $(P's), s=1,2,\ldots,m.$

$$Min\left(\sum_{j=1}^{n}y_j\,Ln\left(\left(1-a_{sj1}\right)\left(1-a_{sj2}\right)a_{sj3}a_{sj4}\right)\right)$$

Subject to

$$\begin{aligned}&\textstyle\sum_{j=1}^{n}y_j\,Ln\left(1-a_{ij1}\right)\geq t\,Ln(1-0.90),\\&\textstyle\sum_{j=1}^{n}y_j\,Ln\left(1-a_{ij2}\right)\geq t\,Ln(1-0.95),\\&\textstyle\sum_{j=1}^{n}y_j\,Ln\left(a_{ij3}\right)\geq t\,Ln(0.01),\\&\textstyle\sum_{j=1}^{n}y_j\,Ln\left(a_{ij4}\right)\geq t\,Ln(0.05),\\&\textstyle\sum_{j=1}^{n}y_j=1,\\&t,y_j\geq 0; j=1,2,\ldots,n.\end{aligned}\qquad (P's)$$

Finally, the optimal value $\sum_{j=1}^{n} y_j \, Ln\left(\left(1-a_{sj1}\right)\left(1-a_{sj2}\right)a_{sj3}a_{sj4}\right); s=1,2,\dots,m$ of each CLPP $(P's)$, $s=1,2,\dots,m$ is used to rank the alternatives.

The aim of this chapter is

(i) To point out the inappropriateness of the existing method [98] for solving IVIFMAGDM problems having completely unknown weights.

(ii) To make the researchers aware that it is inappropriate to propose a simplex method for solving the IVIF LPP (Ps) without transforming it into the CLPP $(P's)$, inspired by Khan et al.'s simplex method [122], for resolving the inappropriateness of the existing method [98] due to the following facts:

"*Khan et al.'s claim [123] "No mathematical incorrect assumptions have been considered in the existing simplex method [122] for solving a fuzzy LPP without transforming it into a CLPP" against Bhardwaj and Kumar's claim [16] is not valid. In fact, several mathematical incorrect assumptions, considered by Khan et al. [122], are not pointed out by Bhardwaj and Kumar [16] e.g., Bhardwaj and Kumar have not pointed out that some of the elements of the existing optimal simplex table [122] are not TFNs*."

Furthermore, to make the researchers aware that it is not possible to resolve the inappropriateness of the existing methods [98]."

Chapter 8

Appropriate definition of an IVIFS

The concept of an IVIFS, proposed by Atanassov and Gargov [9], has been used by several researchers in their research work. However, after a deep study, it is observed that the existing definition of an IVIFS is not appropriate. The aim of this chapter is to make the researchers aware about the inappropriateness of the existing definition as well as to propose the appropriate definition of an IVIFS.

Chapter 9

Appropriate definition of an IVPFS

The concept of an IVPFS, proposed by Peng and Yang [164], has been used by several researchers in their work. However, after a deep study, it is observed that the existing definition of an IVPFS is not appropriate. The aim of this chapter is to make the researchers aware about the inappropriateness of the existing definition as well as to propose the appropriate definition of an IVPFS.

Chapter 10

Future scope

In this chapter, some open research problems are discussed.

Chapter 2

Mehar method for solving IFMADMPrs having partially known attribute weights[1]

Xu [241] claimed that there does not exist any method for solving IFMADMPrs having partially known attribute weights. To fill this gap, Xu proposed a method to solve IFMADMPrs having partially known attribute weights. Wei [223] also proposed a method for solving IFMADMPrs having partially known attribute weights. It is pertinent to mention that several researchers [50, 63, 81, 146, 222, 224, 225] have used the existing methods [223, 241] to solve IFMADMPrs having partially known attribute weights. However, after a deep study it is observed that it is inappropriate to use the existing methods [223, 241]. The aim of this chapter is to make the researchers aware about the inappropriateness of these methods as well as to propose a new method (named as Mehar method) for solving IFMADMPrs having partially known attribute weights.

2.1 A brief review of Xu's method

Xu [241] claimed that there does not exist any method for solving such IFMADMPrs having partially known attribute weights in which the rating value of the i^{th}-alternative over the j^{th}-attribute is represented by an IFS $\tilde{\alpha}_{ij} = \langle \mu_{ij}, \nu_{ij} \rangle$ as well as the weight of the j^{th}-attribute $w_j \in H$, where $H = \{w_j : w_j \geq w_i, w_j - w_i \geq \beta_j (> 0), w_j \geq \beta_j w_i; 0 \leq \beta_j \leq 1, 0 \leq \delta_j \leq w_j \leq \delta_j + \varepsilon_j \leq 1;\ w_j - w_i \geq w_k - w_l \text{ for } i \neq j \neq k \neq l\}$.

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[1] Some contents of this chapter have been published in "International Journal of Fuzzy Systems 21 (2019) 1010-1011" and remaining contents have been communicated in "Knowledge-Based Systems" for the possible publication.

Xu [241] proposed the following method to solve IFMADMPrs having partially known attribute weights:

Step 1: Check that all the attributes are of same type or not i.e., check that all the attributes are benefit type attribute (the larger value of the rating value indicates a greater preference) or cost type attribute (the smaller value of the rating value indicates a greater preference).

Case (i): If all the attributes are of same type then go to Step 2.

Case (ii): If some attributes are cost type attributes and the remaining attributes are benefit type attributes. Then, convert the j^{th} cost type attribute into the benefit type attribute by replacing all the elements $\tilde{\alpha}_{ij} = \langle \mu_{ij}, \nu_{ij} \rangle$ of the j^{th} column of the IFDM $\widetilde{D} = \left(\tilde{\alpha}_{ij}\right)_{m\times n}$ with $\tilde{\alpha}_{ij} = \langle \nu_{ij}, \mu_{ij} \rangle$ and go to Step 2.

Step 2: Using the expression (2.1), find the distance of each $\tilde{\alpha}_{ij} = \langle \mu_{ij}, \nu_{ij} \rangle$ from the largest IFS $\tilde{\alpha}^{+} = \langle 1,0 \rangle$.

$$d_{ij}\left(\tilde{\alpha}_{ij}, \tilde{\alpha}^{+}\right) = \frac{1}{2}\left(\left|\mu_{ij} - 1\right| + \left|\nu_{ij} - 0\right|\right), i = 1,2,\dots,m; j = 1,2,\dots,n. \tag{2.1}$$

Step 3: Using the expression (2.2), find the value of $c_j \ \forall \ j = 1,2,\dots,n$.

$$c_j = \sum_{i=1}^{m} d_{ij}\left(\tilde{\alpha}_{ij}, \tilde{\alpha}^{+}\right), j = 1,2,\dots,n \tag{2.2}$$

Step 4: Find the OAWV, $(w_1, w_2, \dots, w_n)$ by solving the CLPP (P2.1).

$Min\left[\sum_{j=1}^{n} c_j \, w_j\right]$

Subject to

$w_j \in H, w_j \geq 0, j = 1,2,\dots,n,$

$$\sum_{j=1}^{n} w_j = 1. \tag{P2.1}$$

Step 5: Using the expression (2.3), find the value of $P_i \ \forall \ i = 1,2,\dots,m$.

$$P_i = \sum_{j=1}^{n} \frac{w_j}{2} d_{ij}\left(\tilde{\alpha}_{ij}, \tilde{\alpha}^{+}\right), i = 1,2,\dots,m \tag{2.3}$$

and check that $P_p > P_q$ or $P_p < P_q$ or $P_p = P_q$, where $P_p = \sum_{j=1}^{n} \frac{w_j}{2} d_{pj}\left(\tilde{\alpha}_{pj}, \tilde{\alpha}^{+}\right)$ and

$P_q = \sum_{j=1}^{n} \frac{w_j}{2} d_{qj}\left(\tilde{\alpha}_{qj}, \tilde{\alpha}^{+}\right).$

Case (i): If $P_p < P_q$ then the ranking of the alternatives is $A_p > A_q$.

Case (ii): If $P_p > P_q$ then the ranking of the alternatives is $A_p < A_q$.

Case (iii): If $P_p = P_q$ then the ranking of the alternatives is $A_p = A_q$.

2.2 Inappropriateness of Xu's method

It is inappropriate to apply Xu's method [241] due to the following reasons:

(1) On applying Xu's method [241] more than one preference orders of the alternatives are obtained, which is inappropriate. To validate this claim a real-life IFMADMPr having partially known attribute weights, considered by Xu [241] to illustrate his proposed method, is solved with some modifications.

Xu [241] solved the following real-life IFMADMPr having partially known attribute weights to illustrate his proposed method.

There is need to rank the faculty members $A_i, i = 1,2,\ldots,6$ on the basis of the following three benefit attributes.

(i) G_1: Teaching

(ii) G_2: Research

(iii) G_3: Service

The $(i,j)^{th}$ element of the Table 2.1, represented by an IFS $\tilde{\alpha}_{ij} = \langle \mu_{ij}, \nu_{ij} \rangle$, represents the rating value of the i^{th}-faculty member over the j^{th}-attribute.

Table 2.1: Rating Values

Attributes→ ↓ Alternatives	G_1	G_2	G_3
A_1	$\langle 0.4,0.3 \rangle$	$\langle 0.6,0.1 \rangle$	$\langle 0.5,0.4 \rangle$

A_2	$\langle 0.5,0.2\rangle$	$\langle 0.3,0.4\rangle$	$\langle 0.8,0.1\rangle$
A_3	$\langle 0.7,0.2\rangle$	$\langle 0.3,0.7\rangle$	$\langle 0.6,0.2\rangle$
A_4	$\langle 0.4,0.3\rangle$	$\langle 0.6,0.2\rangle$	$\langle 0.7,0.1\rangle$
A_5	$\langle 0.6,0.2\rangle$	$\langle 0.5,0.1\rangle$	$\langle 0.4,0.6\rangle$
A_6	$\langle 0.6,0.3\rangle$	$\langle 0.7,0.2\rangle$	$\langle 0.5,0.4\rangle$

Also, $H = \{0.50 \le w_1 \le 0.55, 0.30 \le w_2 \le 0.35, w_3 \ge 0.50w_2, w_1 - w_3 \ge 0.10\}$. If it is assumed that

(i) The attribute weight w_1 satisfies the condition $0.10 \le w_1 \le 0.55$ instead of the existing condition $0.50 \le w_1 \le 0.55$.

Then, using Xu's method [241], the ranking of the alternatives of the modified IFMADMPr having partially known attribute weights can be obtained as follows:

Step 1: Since, all the attributes are of benefit type. So, according to Step 1 of Xu's method [241], discussed in Section 2.1, there is no need to apply Step 1.

Step 2: According to Step 2 of Xu's method [241], discussed in Section 2.1, there is need to calculate $d_{ij}(\tilde{\alpha}_{ij}, \tilde{\alpha}^+)\ \forall\ i = 1,2,\dots,6; j = 1,2,3$. These values are shown in Table 2.2

Table 2.2: Values of $d_{ij}(\tilde{\alpha}_{ij}, \tilde{\alpha}^+)$

$d_{11}(\langle 0.4,0.3\rangle, \langle 1,0\rangle) =$ $\frac{1}{2}(0.6 + 0.3) = 0.45$	$d_{12}(\langle 0.6,0.1\rangle, \langle 1,0\rangle) =$ $\frac{1}{2}(0.4 + 0.1) = 0.25$	$d_{13}(\langle 0.5,0.4\rangle, \langle 1,0\rangle) =$ $\frac{1}{2}(0.5 + 0.4) = 0.45$
$d_{21}(\langle 0.5,0.2\rangle, \langle 1,0\rangle) =$ $\frac{1}{2}(0.5 + 0.2) = 0.35$	$d_{22}(\langle 0.3,0.4\rangle, \langle 1,0\rangle) =$ $\frac{1}{2}(0.7 + 0.4) = 0.55$	$d_{23}(\langle 0.8,0.1\rangle, \langle 1,0\rangle) =$ $\frac{1}{2}(0.2 + 0.1) = 0.15$
$d_{31}(\langle 0.7,0.2\rangle, \langle 1,0\rangle) =$ $\frac{1}{2}(0.3 + 0.2) = 0.25$	$d_{32}(\langle 0.3,0.7\rangle, \langle 1,0\rangle) =$ $\frac{1}{2}(0.7 + 0.7) = 0.70$	$d_{33}(\langle 0.6,0.2\rangle, \langle 1,0\rangle) =$ $\frac{1}{2}(0.4 + 0.2) = 0.30$

$d_{41}(\langle 0.4,0.3\rangle,\langle 1,0\rangle)=$ $\frac{1}{2}(0.6+0.3)=0.45$	$d_{42}(\langle 0.6,0.2\rangle,\langle 1,0\rangle)=$ $\frac{1}{2}(0.4+0.2)=0.30$	$d_{43}(\langle 0.7,0.1\rangle,\langle 1,0\rangle)=$ $\frac{1}{2}(0.3+0.1)=0.20$
$d_{51}(\langle 0.6,0.2\rangle,\langle 1,0\rangle)=$ $\frac{1}{2}(0.4+0.2)=0.30$	$d_{52}(\langle 0.5,0.1\rangle,\langle 1,0\rangle)=$ $\frac{1}{2}(0.5+0.1)=0.30$	$d_{53}(\langle 0.4,0.6\rangle,\langle 1,0\rangle)=$ $\frac{1}{2}(0.6+0.6)=0.60$
$d_{61}(\langle 0.6,0.3\rangle,\langle 1,0\rangle)=$ $\frac{1}{2}(0.4+0.3)=0.35$	$d_{62}(\langle 0.7,0.2\rangle,\langle 1,0\rangle)=$ $\frac{1}{2}(0.3+0.2)=0.25$	$d_{63}(\langle 0.5,0.4\rangle,\langle 1,0\rangle)=$ $\frac{1}{2}(0.5+0.4)=0.45$

Step 3: Using Step 3 of Xu's method [241], discussed in Section 2.1,

$c_1 = \sum_{i=1}^{6} d_{i1}(\tilde{\alpha}_{i1},\tilde{\alpha}^{+}) = 0.45+0.35+0.25+0.45+0.30+0.35 \ = 2.15,$

$c_2 = \sum_{i=1}^{6} d_{i2}(\tilde{\alpha}_{i2},\tilde{\alpha}^{+}) = 0.25+0.55+0.70+0.30+0.30+0.25 \ = 2.35,$

$c_3 = \sum_{i=1}^{6} d_{i3}(\tilde{\alpha}_{i3},\tilde{\alpha}^{+}) = 0.45+0.15+0.30+0.20+0.60+0.45 \ = 2.15.$

Step 4: Using Step 4 of Xu's method [241], discussed in Section 2.1, there is need to solve the CLPP (P2.2).

$Min\ (2.15w_1 + 2.35w_2 + 2.15w_3)$

Subject to

$$\begin{cases} 0.10 \le w_1 \le 0.55, \\ 0.30 \le w_2 \le 0.35, \\ w_3 \ge 0.5w_2, \\ w_1 + w_2 + w_3 = 1, \\ w_1 \ge 0, w_2 \ge 0, w_3 \ge 0. \end{cases} \tag{P2.2}$$

It can be easily verified that on solving the CLPP (P2.2) infinite number of OAWV are obtained e.g., $(w_1,w_2,w_3) = (0.10,0.30,0.60)$ and $(w_1,w_2,w_3) = (0.50,0.30,0.20)$ both are the OAWV.

Step 5: On considering the OAWV, $(w_1,w_2,w_3) = (0.10,0.30,0.60)$ and using Step 5 of Xu's method [241], discussed in Section 2.1,

$P_1 = (0.10)(0.45) + (0.30)(0.25) + (0.60)(0.45) = 0.39,$

$P_2 = (0.10)(0.35) + (0.30)(0.55) + (0.60)(0.15) = 0.29$,

$P_3 = (0.10)(0.25) + (0.30)(0.70) + (0.60)(0.30) = 0.415$,

$P_4 = (0.10)(0.45) + (0.30)(0.30) + (0.60)(0.20) = 0.255$,

$P_5 = (0.10)(0.30) + (0.30)(0.30) + (0.60)(0.60) = 0.48$,

$P_6 = (0.10)(0.35) + (0.30)(0.25) + (0.60)(0.45) = 0.38$.

Since, $P_4 < P_2 < P_6 < P_1 < P_3 < P_5$. So, according to Step 5 of Xu's method [241], discussed in Section 2.1, the ranking of the alternatives is $A_4 > A_2 > A_6 > A_1 > A_3 > A_5$.

While, on considering the OAWV, $(w_1, w_2, w_3) = (0.50, 0.30, 0.20)$ and using Step 4 of Xu's method [241], discussed in Section 2.1,

$P_1 = (0.50)(0.45) + (0.30)(0.25) + (0.20)(0.45) = 0.39$,

$P_2 = (0.50)(0.35) + (0.30)(0.55) + (0.20)(0.15) = 0.37$,

$P_3 = (0.50)(0.25) + (0.30)(0.70) + (0.20)(0.30) = 0.395$,

$P_4 = (0.50)(0.45) + (0.30)(0.30) + (0.20)(0.20) = 0.355$,

$P_5 = (0.50)(0.30) + (0.30)(0.30) + (0.20)(0.60) = 0.36$,

$P_6 = (0.50)(0.35) + (0.30)(0.25) + (0.20)(0.45) = 0.34$.

Since, $P_6 < P_4 < P_5 < P_2 < P_1 < P_3$. So, according to Step 5 of Xu's method [241], discussed in Section 2.1, the ranking of the alternatives is $A_6 > A_4 > A_5 > A_2 > A_1 > A_3$.

It is obvious that, on applying Xu's method [241], two different ranking of the alternatives are obtained for the same IFMADMPr having partially known attribute weights, which is inappropriate. Hence, it is inappropriate to use Xu's method [241] to solve IFMADMPrs having partially known attribute weights.

(2) The ranking of the alternatives, obtained by Xu's method [241], is not appropriate. The following example clearly validates this claim.

If it is assumed that the $(i, j)^{th}$ element, represented by an IFS, of Table 2.3 represents the rating value of the i^{th}-alternative over the j^{th}-benefit attribute. Also, if it is assumed that

$H = \{0.1 \leq w_1 \leq 0.4, 0.3 \leq w_2 \leq 0.7\}$.

Table 2.3: Rating Values

Attributes→ ↓ Alternatives	G_1	G_2
A_1	$\langle 0.4,0.3 \rangle$	$\langle 0.1,0.2 \rangle$
A_2	$\langle 0.1,0 \rangle$	$\langle 0.1,0.2 \rangle$

Then, it is obvious that the ranking of the alternatives A_1 and A_2 can never be $A_1 = A_2$ as the rating values of both the alternatives corresponding to the attribute G_2 are equal. Whereas, the rating values of both the alternatives corresponding to first attribute G_1 are not equal. While, the following clearly indicates that on applying Xu's method [241], the obtained ranking of the alternatives is $A_1 = A_2$, which is inappropriate.

Using Xu's method [241] the ranking of the alternatives A_1 and A_2 can be obtained as follows:

Step 1: Since, all the attributes are of benefit type. So, according to Step 1 of Xu's method [241], discussed in Section 2.1, there is no need to apply Step 1.

Step 2: According to Step 2 of Xu's method [241], discussed in Section 2.1, there is need to calculate the values of $d_{ij}(\tilde{\alpha}_{ij}, \tilde{\alpha}^+)$ $\forall\, i = 1,2; j = 1,2$. These values are shown in Table 2.4.

Table 2.4: Values of $d_{ij}(\tilde{\alpha}_{ij}, \tilde{\alpha}^+)$

$d_{11}(\langle 0.4,0.3 \rangle, \langle 1,0 \rangle) = \frac{1}{2}(0.6 + 0.3) = 0.45$	$d_{12}(\langle 0.1,0.2 \rangle, \langle 1,0 \rangle) = \frac{1}{2}(0.9 + 0.2) = 0.55$
$d_{21}(\langle 0.1,0 \rangle, \langle 1,0 \rangle) = \frac{1}{2}(0.9 + 0) = 0.45$	$d_{22}(\langle 0.1,0.2 \rangle, \langle 1,0 \rangle) = \frac{1}{2}(0.9 + 0.2) = 0.55$

Step 3: Using Step 3 of Xu's method [241], discussed in Section 2.1,

$c_1 = \sum_{i=1}^{2} d_{i1}(\tilde{\alpha}_{i1}, \tilde{\alpha}^{+}) = 0.45 + 0.45 = 0.9,$

$c_2 = \sum_{i=1}^{2} d_{i2}(\tilde{\alpha}_{i2}, \tilde{\alpha}^{+}) = 0.55 + 0.55 = 1.1.$

Step 4: Using Step 4 of Xu's method [241], discussed in Section 2.1, there is need to solve the CLPP (P2.3).

$Min\ (0.9w_1 + 1.1w_2)$

Subject to

$$\begin{cases} 0.1 \le w_1 \le 0.4, \\ 0.3 \le w_2 \le 0.7, \\ w_1 + w_2 = 1, \\ w_1 \ge 0, w_2 \ge 0. \end{cases} \qquad \text{(P2.3)}$$

It can be easily verified that on solving the CLPP (P2.3), the obtained OAWV is $(w_1, w_2) = (0.4, 0.6)$.

Step 5: On considering the OAWV, $(w_1, w_2) = (0.4, 0.6)$ and using Step 5 of Xu's method [241], discussed in Section 2.1,

$P_1 = (0.40)(0.45) + (0.60)(0.55) = 1.4171,$

$P_2 = (0.40)(0.45) + (0.60)(0.55) = 1.4171.$

Since, $P_1 = P_2$. So, according to Step 5, of Xu's method [241], discussed in Section 2.1, the ranking of the alternatives is $A_1 = A_2$.

It is obvious that, on applying Xu's method [241], the relation $A_1 = A_2$ is obtained, which is inappropriate.

2.3 Reasons for the inappropriateness of Xu's method

In Section 2.2, it is shown that

(i) On applying Xu's method [241] more than one preference order for the alternatives are obtained.

(ii) The ranking of the alternatives, obtained by Xu's method [241], is inappropriate.

These problems are occurring due to following reasons:

(i) If there will exist two distinct attributes G_p and G_q such that $c_p = c_q$ i.e.,

$\sum_{i=1}^{m} d_{ip}(\tilde{\alpha}_{ip}, \tilde{\alpha}^{+}) = \sum_{i=1}^{m} d_{iq}(\tilde{\alpha}_{iq}, \tilde{\alpha}^{+})$. Then, the coefficient of the attribute weight w_p and w_q will be equal in the CLPP (P2.1). Therefore, all such values of w_p and w_q, corresponding to which $w_p + w_q$ will be optimal, will represent the optimal values of w_p and w_q. Due to the same reason, on solving the CLPP (P2.2), corresponding to the IFMADMPr having partially known attribute weights considered in first point of Section 2.2, the number of obtained OAWV are more than one.

(ii) If there will exist two distinct alternatives A_p and A_q such that $d_{pj}(\tilde{\alpha}_{pj}, \tilde{\alpha}^{+}) = d_{qj}(\tilde{\alpha}_{qj}, \tilde{\alpha}^{+})$. Then, the value of $P_p = \sum_{j=1}^{n} \frac{w_j}{2} d_{pj}(\tilde{\alpha}_{pj}, \tilde{\alpha}^{+})$ will be equal to $P_q = \sum_{j=1}^{n} \frac{w_j}{2} d_{qj}(\tilde{\alpha}_{qj}, \tilde{\alpha}^{+})$. Hence, the relation $A_p = A_q$ will be obtained. Due to the same reason, on solving the IFMADMPr having partially known attribute weights, considered in second point of Section 2.2, the obtained relation is $A_1 = A_2$.

2.4 A brief review of Wei's method

Wei [223] proposed the following method to solve IFMADMPrs having partially known attribute weights.

Step 1: Check that all the attributes are of same type or not i.e., check that all the attributes are benefit type attribute or cost type attribute.

Case (i): If all the attributes are of same type then go to Step 2.

Case (ii): If some attributes are cost type attributes and the remaining are benefit type attributes then convert the j^{th} cost type attribute into the benefit type attribute by replacing all the elements $\tilde{\alpha}_{ij} = \langle \mu_{ij}, \nu_{ij} \rangle$ of the j^{th} column of the IFDM $\widetilde{D} = (\tilde{\alpha}_{ij})_{m \times n}$ with $\tilde{\alpha}_{ij} = \langle \nu_{ij}, \mu_{ij} \rangle$ and go to Step 2.

Step 2: Using the expression (2.4), find $d_{ijk}(\tilde{\alpha}_{ij}, \tilde{\alpha}_{kj})\ \forall\ i = 1,2,\dots,m; j = 1,2,\dots,n; k = 1,2,\dots,m$. $d_{ijk}(\tilde{\alpha}_{ij}, \tilde{\alpha}_{kj}) = \frac{1}{2}(|\mu_{ij} - \mu_{kj}| + |\nu_{ij} - \nu_{kj}|), i = 1,2,\dots m;\ j = 1,2,\dots,n; k = 1,2,\dots,m.$ (2.4)

Step 3: Using expression (2.5), find the values of $c_j \; \forall \; j = 1,2,\ldots,n$.

$$c_j = \sum_{i=1}^{m}\sum_{k=1}^{m} d_{ijk}\left(\tilde{\alpha}_{ij},\tilde{\alpha}_{kj}\right), j = 1,2,\ldots,n. \quad (2.5)$$

Step 4: Find the OAWV, $(w_1, w_2, \ldots, w_n)$ by solving the CLPP (P2.4).

$max\left[\sum_{j=1}^{n} c_j \, w_j\right]$

Subject to

$w_j \in H, w_j \geq 0, j = 1,2,\ldots,n,$

$\sum_{j=1}^{n} w_j = 1.$ (P2.4)

Step 5: Using the expression (2.6), find the value of $P_i \; \forall \; i = 1,2,\ldots,m$.

$$P_i = 1 - \prod_{j=1}^{n}\left(1-\mu_{ij}\right)^{w_j} - \prod_{j=1}^{n}\left(\nu_{ij}\right)^{w_j}, i = 1,2,\ldots,m. \quad (2.6)$$

and check that $P_p > P_q$ or $P_p < P_q$ or $P_p = P_q$, where, $P_p = 1 - \prod_{j=1}^{n}\left(1-\mu_{pj}\right)^{w_j} - \prod_{j=1}^{n}\left(\nu_{pj}\right)^{w_j}$ and $P_q = 1 - \prod_{j=1}^{n}\left(1-\mu_{qj}\right)^{w_j} - \prod_{j=1}^{n}\left(\nu_{qj}\right)^{w_j}$.

Case (i): If $P_p < P_q$ then the ranking of the alternatives is $A_p < A_q$.

Case (ii): If $P_p > P_q$ then the ranking of the alternatives is $A_p > A_q$.

Case (iii): If $P_p = P_q$ then go to Step 6.

Step 6: Using the expression (2.7), find the value of $Q_i \; \forall \; i = 1,2,\ldots,m$.

$$Q_i = 1 - \prod_{j=1}^{n}\left(1-\mu_{ij}\right)^{w_j} + \prod_{j=1}^{n}\left(\nu_{ij}\right)^{w_j}, i = 1,2,\ldots,m \quad (2.7)$$

and check that $Q_p > Q_q$ or $Q_p < Q_q$ or $Q_p = Q_q$, where, $Q_p = 1 - \prod_{j=1}^{n}\left(1-\mu_{pj}\right)^{w_j} + \prod_{j=1}^{n}\left(\nu_{pj}\right)^{w_j}$ and $Q_q = 1 - \prod_{j=1}^{n}\left(1-\mu_{qj}\right)^{w_j} + \prod_{j=1}^{n}\left(\nu_{qj}\right)^{w_j}$.

Case (i): If $Q_p < Q_q$ then the ranking of the alternatives is $A_p < A_q$.

Case (ii): If $Q_p > Q_q$ then the ranking of the alternatives is $A_p > A_q$.

Case (iii): If $Q_p = Q_q$ then the ranking of the alternative is $A_p = A_q$.

2.5 Inappropriateness of Wei's method

It is inappropriate to use Wei's method [223] due to following reasons:

(1) On applying Wei's method [223] more than one preference orders of the alternatives are obtained, which is inappropriate. To validate this claim the real-life IFMADMPr having partially known attribute weights, considered by Wei [223] to illustrate his proposed method [223], is solved with some modifications.

Wei [223] solved the following real-life IFMADMPr having partially known attribute weights to illustrate his proposed method.

There is need to rank five companies $A_i, i = 1,2,\dots,5$ on the basis of the following four benefit attributes.

(i) G_1: risk analysis

(ii) G_2: growth analysis

(iii) G_3: social-political impact analysis

(iv) G_4: environmental impact analysis

The $(i,j)^{th}$ element of Table 2.5, represented by an IFS $\tilde{\alpha}_{ij} = \langle \mu_{ij}, \nu_{ij} \rangle$, represents the rating values of the i^{th}-company over the j^{th}-attribute.

Table 2.5: Rating values

Attributes→ ↓ Alternatives	G_1	G_2	G_3	G_4
A_1	$\langle 0.5,0.4 \rangle$	$\langle 0.6,0.3 \rangle$	$\langle 0.3,0.6 \rangle$	$\langle 0.2,0.7 \rangle$
A_2	$\langle 0.7,0.3 \rangle$	$\langle 0.7,0.2 \rangle$	$\langle 0.7,0.2 \rangle$	$\langle 0.4,0.5 \rangle$
A_3	$\langle 0.6,0.4 \rangle$	$\langle 0.5,0.4 \rangle$	$\langle 0.5,0.3 \rangle$	$\langle 0.6,0.3 \rangle$
A_4	$\langle 0.8,0.1 \rangle$	$\langle 0.6,0.3 \rangle$	$\langle 0.3,0.4 \rangle$	$\langle 0.2,0.6 \rangle$
A_5	$\langle 0.6,0.2 \rangle$	$\langle 0.4,0.3 \rangle$	$\langle 0.7,0.1 \rangle$	$\langle 0.5,0.3 \rangle$

Also, $H = \{0.15 \le w_1 \le 0.20, 0.16 \le w_2 \le 0.18, 0.30 \le w_3 \le 0.35, 0.30 \le w_4 \le 0.45\}$.

If

(i) Table 2.5 is replaced with Table 2.6

Table 2.6: Rating Values

Attributes→ ↓ Alternatives	G_1	G_2	G_3	G_4
A_1	$\langle 0.6,0.3 \rangle$	$\langle 0.5,0.4 \rangle$	$\langle 0.4,0.3 \rangle$	$\langle 0.6,0.2 \rangle$
A_2	$\langle 0.7,0.2 \rangle$	$\langle 0.7,0.3 \rangle$	$\langle 0.6,0.3 \rangle$	$\langle 0.6,0.4 \rangle$
A_3	$\langle 0.5,0.4 \rangle$	$\langle 0.6,0.4 \rangle$	$\langle 0.7,0.2 \rangle$	$\langle 0.5,0.4 \rangle$
A_4	$\langle 0.6,0.3 \rangle$	$\langle 0.8,0.1 \rangle$	$\langle 0.5,0.4 \rangle$	$\langle 0.7,0.3 \rangle$
A_5	$\langle 0.4,0.3 \rangle$	$\langle 0.6,0.2 \rangle$	$\langle 0.6,0.3 \rangle$	$\langle 0.8,0.1 \rangle$

(ii) $H = \{0.15 \le w_1 \le 0.20, 0.16 \le w_2 \le 0.18, 0.30 \le w_3 \ge 0.35, 0.30 \le w_4 \le 0.45\}$ is replaced with $H = \{0.15 \le w_1 \le 0.20, 0.10 \le w_2 \le 0.95, 0.20 \le w_3 \ge 0.35, 0.20 \le w_4 \le 0.45\}$.

Then, using Wei's method [223], the ranking of the alternatives of the modified IFMADMPr having partially known attribute weights can be obtained as follows:

Step 1: Since, all the attributes are benefit type. So, according to Step 1 of Wei's method [223], discussed in Section 2.4, there is no need to apply Step 1.

Step 2: According to Step 2 of Wei's method [223], discussed in Section 2.4, there is need to calculate $d_{ijk}(\tilde{\alpha}_{ij}, \tilde{\alpha}_{kj})\ \forall\ i = 1,2,3,4,5; j = 1,2,3,4; k = 1,2,3,4,5$. These values are shown in Table 2.7.

Table 2.7: Values of $d_{ijk}(\tilde{\alpha}_{ij},\tilde{\alpha}_{kj})$

$d_{111}(\langle 0.6,0.3\rangle,\langle 0.6,0.3\rangle)$ $=\frac{1}{2}(0+0)=0$	$d_{211}(\langle 0.5,0.4\rangle,\langle 0.5,0.4\rangle)$ $=\frac{1}{2}(0+0)=0$	$d_{311}(\langle 0.4,0.3\rangle,\langle 0.4,0.3\rangle)$ $=\frac{1}{2}(0+0)=0$	$d_{411}(\langle 0.6,0.2\rangle,\langle 0.6,0.2\rangle)$ $=\frac{1}{2}(0+0)=0$
$d_{112}(\langle 0.6,0.3\rangle,\langle 0.7,0.2\rangle)$ $=\frac{1}{2}(0.1+0.1)=0.1$	$d_{212}(\langle 0.5,0.4\rangle,\langle 0.7,0.3\rangle)$ $=\frac{1}{2}(0.2+0.1)=0.15$	$d_{312}(\langle 0.4,0.3\rangle,\langle 0.6,0.3\rangle)$ $=\frac{1}{2}(0.2+0)=0.1$	$d_{412}(\langle 0.6,0.2\rangle,\langle 0.6,0.4\rangle)$ $=\frac{1}{2}(0+0.2)=0.1$
$d_{113}(\langle 0.6,0.3\rangle,\langle 0.5,0.4\rangle)$ $=\frac{1}{2}(0.1+0.1)=0.1$	$d_{213}(\langle 0.5,0.4\rangle,\langle 0.6,0.4\rangle)$ $=\frac{1}{2}(0.1+0)=0.05$	$d_{313}(\langle 0.4,0.3\rangle,\langle 0.7,0.2\rangle)$ $=\frac{1}{2}(0.3+0.1)=0.2$	$d_{413}(\langle 0.6,0.2\rangle,\langle 0.5,0.4\rangle)$ $=\frac{1}{2}(0.1+0.2)=0.15$
$d_{114}(\langle 0.6,0.3\rangle,\langle 0.6,0.3\rangle)$ $=\frac{1}{2}(0+0)=0$	$d_{214}(\langle 0.5,0.4\rangle,\langle 0.8,0.1\rangle)$ $=\frac{1}{2}(0.3+0.3)=0.3$	$d_{314}(\langle 0.4,0.3\rangle,\langle 0.5,0.4\rangle)$ $=\frac{1}{2}(0.1+0.1)=0.1$	$d_{414}(\langle 0.6,0.2\rangle,\langle 0.7,0.3\rangle)$ $=\frac{1}{2}(0.1+0.1)=0.1$
$d_{115}(\langle 0.6,0.3\rangle,\langle 0.4,0.3\rangle)$ $=\frac{1}{2}(0.2+0)=0.1$	$d_{215}(\langle 0.5,0.4\rangle,\langle 0.6,0.2\rangle)$ $=\frac{1}{2}(0.1+0.2)=0.15$	$d_{315}(\langle 0.4,0.3\rangle,\langle 0.6,0.3\rangle)$ $=\frac{1}{2}(0.2+0)=0.1$	$d_{415}(\langle 0.6,0.2\rangle,\langle 0.8,0.1\rangle)$ $=\frac{1}{2}(0.2+0.1)=0.15$
$d_{121}(\langle 0.7,0.2\rangle,\langle 0.6,0.3\rangle)$ $=\frac{1}{2}(0.1+0.1)=0.1$	$d_{221}(\langle 0.7,0.3\rangle,\langle 0.5,0.4\rangle)$ $=\frac{1}{2}(0.2+0.1)=0.15$	$d_{321}(\langle 0.6,0.3\rangle,\langle 0.4,0.3\rangle)$ $=\frac{1}{2}(0.2+0)=0.1$	$d_{421}(\langle 0.6,0.4\rangle,\langle 0.6,0.2\rangle)$ $=\frac{1}{2}(0+0.2)=0.1$
$d_{122}(\langle 0.7,0.2\rangle,\langle 0.7,0.2\rangle)$ $=\frac{1}{2}(0+0)=0$	$d_{222}(\langle 0.7,0.3\rangle,\langle 0.7,0.3\rangle)$ $=\frac{1}{2}(0+0)=0$	$d_{322}(\langle 0.6,0.3\rangle,\langle 0.6,0.3\rangle)$ $=\frac{1}{2}(0+0)=0$	$d_{422}(\langle 0.6,0.4\rangle,\langle 0.6,0.4\rangle)$ $=\frac{1}{2}(0+0)=0$
$d_{123}(\langle 0.7,0.2\rangle,\langle 0.5,0.4\rangle)$ $=\frac{1}{2}(0.2+0.2)=0.2$	$d_{223}(\langle 0.7,0.3\rangle,\langle 0.6,0.4\rangle)$ $=\frac{1}{2}(0.1+0.1)=0.1$	$d_{323}(\langle 0.6,0.3\rangle,\langle 0.7,0.2\rangle)$ $=\frac{1}{2}(0.1+0.1)=0.1$	$d_{423}(\langle 0.6,0.4\rangle,\langle 0.5,0.4\rangle)$ $=\frac{1}{2}(0.1+0)=0.05$
$d_{124}(\langle 0.7,0.2\rangle,\langle 0.6,0.3\rangle)$ $=\frac{1}{2}(0.1+0.1)=0.1$	$d_{224}(\langle 0.7,0.3\rangle,\langle 0.8,0.1\rangle)$ $=\frac{1}{2}(0.1+0.2)=0.15$	$d_{324}(\langle 0.6,0.3\rangle,\langle 0.5,0.4\rangle)$ $=\frac{1}{2}(0.1+0.1)=0.1$	$d_{424}(\langle 0.6,0.4\rangle,\langle 0.7,0.3\rangle)$ $=\frac{1}{2}(0.1+0.1)=0.1$
$d_{125}(\langle 0.7,0.2\rangle,\langle 0.4,0.3\rangle)$ $=\frac{1}{2}(0.3+0.1)=0.2$	$d_{225}(\langle 0.7,0.3\rangle,\langle 0.6,0.2\rangle)$ $=\frac{1}{2}(0.1+0.1)=0.1$	$d_{325}(\langle 0.6,0.3\rangle,\langle 0.6,0.3\rangle)$ $=\frac{1}{2}(0+0)=0$	$d_{425}(\langle 0.6,0.4\rangle,\langle 0.8,0.1\rangle)$ $=\frac{1}{2}(0.2+0.3)=0.25$
$d_{131}(\langle 0.5,0.4\rangle,\langle 0.6,0.3\rangle)$	$d_{231}(\langle 0.6,0.4\rangle,\langle 0.5,0.4\rangle)$	$d_{331}(\langle 0.7,0.2\rangle,\langle 0.4,0.3\rangle)$	$d_{431}(\langle 0.5,0.4\rangle,\langle 0.6,0.2\rangle)$

$=\frac{1}{2}(0.1+0.1)=0.1$	$=\frac{1}{2}(0.1+0)=0.05$	$=\frac{1}{2}(0.3+0.1)=0.2$	$=\frac{1}{2}(0.1+0.2)=0.15$
$d_{132}(\langle 0.5,0.4\rangle,\langle 0.7,0.2\rangle)$ $=\frac{1}{2}(0.2+0.2)=0.2$	$d_{232}(\langle 0.6,0.4\rangle,\langle 0.7,0.3\rangle)$ $=\frac{1}{2}(0.1+0.1)=0.1$	$d_{332}(\langle 0.7,0.2\rangle,\langle 0.6,0.3\rangle)$ $=\frac{1}{2}(0.1+0.1)=0.1$	$d_{432}(\langle 0.5,0.4\rangle,\langle 0.6,0.4\rangle)$ $=\frac{1}{2}(0.1+0)=0.05$
$d_{133}(\langle 0.5,0.4\rangle,\langle 0.5,0.4\rangle)$ $=\frac{1}{2}(0+0)=0$	$d_{233}(\langle 0.6,0.4\rangle,\langle 0.6,0.4\rangle)$ $=\frac{1}{2}(0+0)=0$	$d_{333}(\langle 0.7,0.2\rangle,\langle 0.7,0.2\rangle)$ $=\frac{1}{2}(0+0)=0$	$d_{433}(\langle 0.5,0.4\rangle,\langle 0.5,0.4\rangle)$ $=\frac{1}{2}(0+0)=0$
$d_{134}(\langle 0.5,0.4\rangle,\langle 0.6,0.3\rangle)$ $=\frac{1}{2}(0.1+0.1)=0.1$	$d_{234}(\langle 0.6,0.4\rangle,\langle 0.8,0.1\rangle)$ $=\frac{1}{2}(0.2+0.3)=0.25$	$d_{334}(\langle 0.7,0.2\rangle,\langle 0.5,0.4\rangle)$ $=\frac{1}{2}(0.2+0.2)=0.2$	$d_{434}(\langle 0.5,0.4\rangle,\langle 0.7,0.3\rangle)$ $=\frac{1}{2}(0.2+0.1)=0.15$
$d_{135}(\langle 0.5,0.4\rangle,\langle 0.4,0.3\rangle)$ $=\frac{1}{2}(0.1+0.1)=0.1$	$d_{235}(\langle 0.6,0.4\rangle,\langle 0.6,0.2\rangle)$ $=\frac{1}{2}(0+0.2)=0.1$	$d_{335}(\langle 0.7,0.2\rangle,\langle 0.6,0.3\rangle)$ $=\frac{1}{2}(0.1+0.1)=0.1$	$d_{435}(\langle 0.5,0.4\rangle,\langle 0.8,0.1\rangle)$ $=\frac{1}{2}(0.3+0.3)=0.3$
$d_{141}(\langle 0.6,0.3\rangle,\langle 0.6,0.3\rangle)$ $=\frac{1}{2}(0+0)=0$	$d_{241}(\langle 0.8,0.1\rangle,\langle 0.5,0.4\rangle)$ $=\frac{1}{2}(0.3+0.3)=0.3$	$d_{341}(\langle 0.5,0.4\rangle,\langle 0.4,0.3\rangle)$ $=\frac{1}{2}(0.1+0.1)=0.1$	$d_{441}(\langle 0.7,0.3\rangle,\langle 0.6,0.2\rangle)$ $=\frac{1}{2}(0.1+0.1)=0.1$
$d_{142}(\langle 0.6,0.3\rangle,\langle 0.7,0.2\rangle)$ $=\frac{1}{2}(0.1+0.1)=0.1$	$d_{242}(\langle 0.8,0.1\rangle,\langle 0.7,0.3\rangle)$ $=\frac{1}{2}(0.1+0.2)=0.15$	$d_{342}(\langle 0.5,0.4\rangle,\langle 0.6,0.3\rangle)$ $=\frac{1}{2}(0.1+0.1)=0.1$	$d_{442}(\langle 0.7,0.3\rangle,\langle 0.6,0.4\rangle)$ $=\frac{1}{2}(0.1+0.1)=0.1$
$d_{143}(\langle 0.6,0.3\rangle,\langle 0.5,0.4\rangle)$ $=\frac{1}{2}(0.1+0.1)=0.1$	$d_{243}(\langle 0.8,0.1\rangle,\langle 0.6,0.4\rangle)$ $=\frac{1}{2}(0.2+0.3)=0.25$	$d_{343}(\langle 0.5,0.4\rangle,\langle 0.7,0.2\rangle)$ $=\frac{1}{2}(0.2+0.2)=0.2$	$d_{443}(\langle 0.7,0.3\rangle,\langle 0.5,0.4\rangle)$ $=\frac{1}{2}(0.2+0.1)=0.15$
$d_{144}(\langle 0.6,0.3\rangle,\langle 0.6,0.3\rangle)$ $=\frac{1}{2}(0+0)=0$	$d_{244}(\langle 0.8,0.1\rangle,\langle 0.8,0.1\rangle)$ $=\frac{1}{2}(0+0)=0$	$d_{344}(\langle 0.5,0.4\rangle,\langle 0.5,0.4\rangle)$ $=\frac{1}{2}(0+0)=0$	$d_{444}(\langle 0.7,0.3\rangle,\langle 0.7,0.3\rangle)$ $=\frac{1}{2}(0+0)=0$
$d_{145}(\langle 0.6,0.3\rangle,\langle 0.4,0.3\rangle)$ $=\frac{1}{2}(0.2+0)=0.1$	$d_{245}(\langle 0.8,0.1\rangle,\langle 0.6,0.2\rangle)$ $=\frac{1}{2}(0.2+0.1)=0.15$	$d_{345}(\langle 0.5,0.4\rangle,\langle 0.6,0.3\rangle)$ $=\frac{1}{2}(0.1+0.1)=0.1$	$d_{445}(\langle 0.7,0.3\rangle,\langle 0.8,0.1\rangle)$ $=\frac{1}{2}(0.1+0.2)=0.15$
$d_{151}(\langle 0.4,0.3\rangle,\langle 0.6,0.3\rangle)$ $=\frac{1}{2}(0.2+0)=0.1$	$d_{251}(\langle 0.6,0.2\rangle,\langle 0.5,0.4\rangle)$ $=\frac{1}{2}(0.1+0.2)=0.15$	$d_{351}(\langle 0.6,0.3\rangle,\langle 0.4,0.3\rangle)$ $=\frac{1}{2}(0.2+0)=0.1$	$d_{451}(\langle 0.8,0.1\rangle,\langle 0.6,0.2\rangle)$ $=\frac{1}{2}(0.2+0.1)=0.15$
$d_{152}(\langle 0.4,0.3\rangle,\langle 0.7,0.2\rangle)$	$d_{252}(\langle 0.6,0.2\rangle,\langle 0.7,0.3\rangle)$	$d_{352}(\langle 0.6,0.3\rangle,\langle 0.6,0.3\rangle)$	$d_{452}(\langle 0.8,0.1\rangle,\langle 0.6,0.4\rangle)$

$=\frac{1}{2}(0.3+0.1)=0.2$	$=\frac{1}{2}(0.1+0.1)=0.1$	$=\frac{1}{2}(0+0)=0$	$=\frac{1}{2}(0.2+0.3)=0.25$
$d_{153}(\langle 0.4,0.3\rangle,\langle 0.5,0.4\rangle)$ $=\frac{1}{2}(0.1+0.1)=0.1$	$d_{253}(\langle 0.6,0.2\rangle,\langle 0.6,0.4\rangle)$ $=\frac{1}{2}(0+0.2)=0.1$	$d_{353}(\langle 0.6,0.3\rangle,\langle 0.7,0.2\rangle)$ $=\frac{1}{2}(0.1+0.1)=0.1$	$d_{453}(\langle 0.8,0.1\rangle,\langle 0.5,0.4\rangle)$ $=\frac{1}{2}(0.3+0.3)=0.3$
$d_{154}(\langle 0.4,0.3\rangle,\langle 0.6,0.3\rangle)$ $=\frac{1}{2}(0.2+0)=0.1$	$d_{254}(\langle 0.6,0.2\rangle,\langle 0.8,0.1\rangle)$ $=\frac{1}{2}(0.2+0.1)=0.15$	$d_{354}(\langle 0.6,0.3\rangle,\langle 0.5,0.4\rangle)$ $=\frac{1}{2}(0.1+0.1)=0.1$	$d_{454}(\langle 0.8,0.1\rangle,\langle 0.7,0.3\rangle)$ $=\frac{1}{2}(0.1+0.2)=0.15$
$d_{155}(\langle 0.4,0.3\rangle,\langle 0.4,0.3\rangle)$ $=\frac{1}{2}(0+0)=0$	$d_{255}(\langle 0.6,0.2\rangle,\langle 0.6,0.2\rangle)$ $=\frac{1}{2}(0+0)=0$	$d_{355}(\langle 0.6,0.3\rangle,\langle 0.6,0.3\rangle)$ $=\frac{1}{2}(0+0)=0$	$d_{455}(\langle 0.8,0.1\rangle,\langle 0.8,0.1\rangle)$ $=\frac{1}{2}(0+0)=0$

Step 3: Using Step 3 of Wei's method [223], discussed in Section 2.4,

$c_1=\sum_{i=1}^{5}\sum_{k=1}^{5} d_{i1k}(\tilde{\alpha}_{i1},\tilde{\alpha}_{k1})=0+0.1+0.1+0+0.1+0.1+0+0.2+0.1+0.2+0.1+0.2+0+0.1+0.1+0+0.1+0.1+0+0.1+0.1+0.2+0.1+0.1+0=2.2$,

$c_2=\sum_{i=1}^{5}\sum_{k=1}^{5} d_{i2k}(\tilde{\alpha}_{i2},\tilde{\alpha}_{k2})=0+0.15+0.05+0.3+0.15+0.15+0+0.1+0.15+0.1+0.05+0.1+0+0.25+0.1+0.3+0.15+0.25+0+0.15+0.15+0.1+0.1+0.15+0=3$,

$c_3=\sum_{i=1}^{5}\sum_{k=1}^{5} d_{i3k}(\tilde{\alpha}_{i3},\tilde{\alpha}_{k3})=0+0.1+0.2+0.1+0.1+0.1+0+0.1+0.1+0+0.2+0.1+0+0.2+0.1+0.1+0.1+0.2+0+0.1+0.1+0+0.1+0.1+0=2.2$,

$c_4=\sum_{i=1}^{5}\sum_{k=1}^{5} d_{i4k}(\tilde{\alpha}_{i4},\tilde{\alpha}_{k4})=0+0.1+0.15+0.1+0.15+0.1+0+0.05+0.1+0.25+0.15+0.05+0+0.15+0.3+0.1+0.1+0.15+0+0.15+0.15+0.25+0.30+0.15+0=3$.

Step 4: Using Step 4 of Wei's method [223], there is need to solve the CLPP (P2.5).

$Max(2.2w_1+3w_2+2.2w_3+3w_4)$

Subject to

$$\begin{cases} 0.15 \leq w_1 \leq 0.20, \\ 0.10 \leq w_2 \leq 0.95, \\ 0.20 \leq w_3 \leq 0.35, \\ 0.20 \leq w_4 \leq 0.45, \\ w_1 + w_2 + w_3 + w_4 = 1, \\ w_1 \geq 0, w_2 \geq 0, w_3 \geq 0, w_4 \geq 0. \end{cases} \tag{P2.5}$$

It can be easily verified that on solving the CLPP (P2.5) infinite number of OAWV are obtained e.g. $(w_1, w_2, w_3, w_4) = (0.15,0.20,0.20,0.45)$ and $(w_1, w_2, w_3, w_4) = (0.15,0.40,,0.20,0.25)$ both are the OAWV.

Step 5: On considering the OAWV, $(w_1, w_2, w_3, w_4) = (0.15,0.20,0.20,0.45)$ and using Step 5 of Wei's method [223], discussed in Section 2.4,

$P_1 = 1 - (1 - 0.6)^{0.15}(1 - 0.5)^{0.20}(1 - 0.4)^{0.20}(1 - 0.6)^{0.45} - (0.3)^{0.15}(0.4)^{0.20}(0.3)^{0.20}$
$(0.2)^{0.45} = 0.2818,$

$P_2 = 1 - (1 - 0.7)^{0.15}(1 - 0.7)^{0.20}(1 - 0.6)^{0.20}(1 - 0.6)^{0.45} - (0.2)^{0.15}(0.3)^{0.20}(0.3)^{0.20}$
$(0.4)^{0.45} = 0.3171,$

$P_3 = 1 - (1 - 0.5)^{0.15}(1 - 0.6)^{0.20}(1 - 0.7)^{0.20}(1 - 0.5)^{0.45} - (0.4)^{0.15}(0.4)^{0.20}(0.2)^{0.20}$
$(0.4)^{0.45} = 0.2201,$

$P_4 = 1 - (1 - 0.6)^{0.15}(1 - 0.8)^{0.20}(1 - 0.5)^{0.20}(1 - 0.7)^{0.45} - (0.3)^{0.15}(0.1)^{0.20}(0.4)^{0.20}$
$(0.3)^{0.45} = 0.4252,$

$P_5 = 1 - (1 - 0.4)^{0.15}(1 - 0.6)^{0.20}(1 - 0.6)^{0.20}(1 - 0.8)^{0.45} - (0.3)^{0.15}(0.2)^{0.20}(0.3)^{0.20}$
$(0.1)^{0.45} = 0.5202.$

Since, $P_5 > P_4 > P_2 > P_1 > P_3$. So, according to Step 5 of Wei's method [223], the ranking of the alternatives is $A_5 > A_4 > A_2 > A_1 > A_3$.

While, on considering the OAWV, $(w_1, w_2, w_3, w_4) = (0.15,0.40,0.20,0.25)$ and using Step 5 of Wei's method [223], discussed in Section 2.4,

$P_1 = 1 - (1 - 0.6)^{0.15}(1 - 0.5)^{0.40}(1 - 0.4)^{0.20}(1 - 0.6)^{0.25} - (0.3)^{0.15}(0.4)^{0.40}(0.3)^{0.20}$
$(0.2)^{0.25} = 0.2217,$

$P_2 = 1 - (1-0.7)^{0.15}(1-0.7)^{0.40}(1-0.6)^{0.20}(1-0.6)^{0.25} - (0.2)^{0.15}(0.3)^{0.40}(0.3)^{0.20}$

$(0.4)^{0.25} = 0.3553,$

$P_3 = 1 - (1-0.5)^{0.15}(1-0.6)^{0.40}(1-0.7)^{0.20}(1-0.5)^{0.25} - (0.4)^{0.15}(0.4)^{0.40}(0.2)^{0.20}$

$(0.4)^{0.25} = 0.2390,$

$P_4 = 1 - (1-0.6)^{0.15}(1-0.8)^{0.40}(1-0.5)^{0.20}(1-0.7)^{0.25} - (0.3)^{0.15}(0.1)^{0.40}(0.4)^{0.20}$

$(0.3)^{0.25} = 0.5004,$

$P_5 = 1 - (1-0.4)^{0.15}(1-0.6)^{0.40}(1-0.6)^{0.20}(1-0.8)^{0.25} - (0.3)^{0.15}(0.2)^{0.40}(0.3)^{0.20}$

$(0.1)^{0.25} = 0.4488.$

Since, $P_4 > P_5 > P_2 > P_3 > P_1$. So, according to Step 5 of Wei's method [223], the ranking of the alternatives is $A_4 > A_5 > A_2 > A_3 > A_1$.

It is obvious that on applying Wei's method [223] two different ranking of the alternatives are obtained for the same IFMADMPr having partially known attribute weights, which is inappropriate. Hence, it is inappropriate to use Wei's method [223] to solve IFMADMPrs having partially known attribute weights.

(2) The ranking of the alternatives, obtained by Wei's method [223], is inappropriate. The following examples clearly validate this claim.

If it is assumed that the $(i,j)^{th}$ element, represented by an IFS, of Table 2.8 represents the rating value of the i^{th}-alternative over the j^{th} benefit attribute. Also, if it is assumed that $H = \{0.1 \le w_1 \le 0.6, 0.2 \le w_2 \le 0.7\}$.

Table 2.8: Rating values

Attributes→ ↓ Alternatives	G_1	G_2
A_1	$\langle 0.4,0.3 \rangle$	$\langle 1,0 \rangle$

A_2	$\langle 0.3,0.4\rangle$	$\langle 1,0\rangle$

Then, it is obvious that the ranking of alternatives A_1 and A_2 can never be $A_1 = A_2$ as the rating values of both the alternatives corresponding to the attribute G_2 are equal. Whereas, the rating values of both the alternatives corresponding to first attribute G_1 are compliment to each other. While, the following clearly indicates that on applying Wei's method [223], the relation $A_1 = A_2$ is obtained, which is inappropriate.

Using Wei's method [223], the ranking of the alternatives A_1 and A_2 can be obtained as follows:

Step 1: Since, all the attributes are of benefit type. So, according to Step 1 of Wei' method [223], discussed in Section 2.4, there is no need to apply Step 1.

Step 2: According to Step 2 of Wei's method [223], discussed in Section 2.4, there is need to calculate $d_{ijk}\left(\tilde{\alpha}_{ij},\tilde{\alpha}_{kj}\right)\ \forall\ i = 1,2; j = 1,2; k = 1,2$. These values are shown in Table 2.8.

Table 2.8: Values of $d_{ijk}\left(\tilde{\alpha}_{ij},\tilde{\alpha}_{kj}\right)$

$d_{111}(\langle 0.4,0.3\rangle,\langle 0.4,0.3\rangle) = \frac{1}{2}(0+0) = 0$	$d_{211}(\langle 1,0\rangle,\langle 1,0\rangle) = \frac{1}{2}(0+0) = 0$
$d_{112}(\langle 0.4,0.3\rangle,\langle 0.3,0.4\rangle) = \frac{1}{2}(0.1+0.1) = 0.1$	$d_{212}(\langle 1,0\rangle,\langle 1,0\rangle) = \frac{1}{2}(0+0) = 0$
$d_{121}(\langle 0.3,0.4\rangle,\langle 0.4,0.3\rangle) = \frac{1}{2}(0.1+0.1) = 0.1$	$d_{221}(\langle 1,0\rangle,\langle 1,0\rangle) = \frac{1}{2}(0+0) = 0$
$d_{122}(\langle 0.3,0.4\rangle,\langle 0.3,0.4\rangle) = \frac{1}{2}(0+0) = 0$	$d_{222}(\langle 1,0\rangle,\langle 1,0\rangle) = \frac{1}{2}(0+0) = 0$

Step 3: Using Step 3 of Wei's method [223], discussed in Section 2.4,

$c_1 = \sum_{i=1}^{5}\sum_{k=1}^{5} d_{i1k}(\tilde{\alpha}_{i1},\tilde{\alpha}_{k1}) = 0 + 0.1 + 0.1 + 0 = 0.2,$

$c_2 = \sum_{i=1}^{5}\sum_{k=1}^{5} d_{ijk}\left(\tilde{\alpha}_{ij},\tilde{\alpha}_{kj}\right) = 0 + 0 + 0 + 0 = 0.$

Step 4: Using Step 4 of Wei's method [223], there is need to solve the CLPP (P2.6).

$Max(0.2w_1 + 0w_2)$

Subject to

$$\begin{cases} 0.1 \leq w_1 \leq 0.6, \\ 0.2 \leq w_2 \leq 0.7, \\ w_1 + w_2 = 1, \\ w_1 \geq 0, w_2 \geq 0. \end{cases} \tag{P2.6}$$

It can be easily verified that on solving the CLPP (P2.6), the obtained OAWV is $(w_1, w_2) = (0.6,0.4)$

Step 5: On considering the OAWV, $(w_1, w_2) = (0.6,0.4)$ and using Step 5 of Wei's method [223], discussed in Section 2.4,

$P_1 = 1 - (1 - 0.4)^{0.6}(1 - 1)^{0.4} - (0.3)^{0.6}(0)^{0.4} = 1,$

$P_2 = 1 - (1 - 0.3)^{0.6}(1 - 1)^{0.4} - (0.4)^{0.6}(0)^{0.4} = 1.$

Since, $P_1 = P_2$. So, according to Step 5 of Wei's method [223], there is need to apply Step 6.

Step 6: Using Step 6 of Wei's method [223], discussed in Section 2.4,

$Q_1 = 1 - (1 - 0.4)^{0.6}(1 - 1)^{0.4} + (0.3)^{0.6}(0)^{0.4} = 1,$

$Q_2 = 1 - (1 - 0.3)^{0.6}(1 - 1)^{0.4} + (0.4)^{0.6}(0)^{0.4} = 1.$

Since, $Q_1 = Q_2$. So, according to Step 6 of Wei's method [223], the ranking of the alternatives is $A_1 = A_2$.

It is obvious that on applying Wei's method [223], the relation $A_1 = A_2$ is obtained, which is inappropriate.

2.6 Reasons for the inappropriateness of Wei's method

In Section 2.5, it is shown that

(i) On applying Wei's method [223] more than one preference order for the alternatives are obtained.

(ii) The ranking of the alternatives, obtained by Wei's method [223], is inappropriate.

These problems are occurring due to following reasons:

(i) If there will exist two distinct attributes G_p and G_q such that $c_p = c_q$ i.e., $\sum_{i=1}^{m}\sum_{k=1}^{m} d_{ipk}\left(\tilde{\alpha}_{ip}, \tilde{\alpha}_{kp}\right) = \sum_{i=1}^{m}\sum_{k=1}^{m} d_{iqk}\left(\tilde{\alpha}_{iq}, \tilde{\alpha}_{kq}\right)$. Then, the coefficient of the attribute weight w_p and w_q will be equal in the CLPP (P2.4). Therefore, all such values of w_p and w_q for which $w_p + w_q$ will be optimal, will represent the optimal values of w_p and w_q. Due to the same reason, on solving the CLPP (P2.5), corresponding to the IFMADMPr having partially known attribute weights considered in first point of Section 2.5, the number of obtained OAWV are more than one.

(ii) It is pertinent to mention that if there exist $\langle \mu_{ij}, \nu_{ij} \rangle$ such that $\mu_{ij} = 1$ and $\nu_{ij} = 0$ then the value of $P_i = 1 - \prod_{j=1}^{n}\left(1 - \mu_{ij}\right)^{w_j} - \prod_{j=1}^{n} {\nu_{ij}}^{w_j}$ as well as $Q_i = 1 - \prod_{j=1}^{n}\left(1 - \mu_{ij}\right)^{w_j} + \prod_{j=1}^{n} {\nu_{ij}}^{w_j}$ will always be 1 for all values of w_j. Due to the same reason, on solving the IFMADMPr having partially known attribute weights, considered in second point of Section 2.5, the obtained relation is $A_1 = A_2$.

2.7 Inappropriateness of some other existing methods

It is inappropriate to use the existing methods [50, 63, 111, 146, 222-225, 236, 241] due to the following reasons:

(i) It can be easily verified that in all the existing methods [50, 63, 111, 146, 222-225, 236, 241], the CLPP (P2.1) or the CLPP (P2.4) has been used to obtain the OAWV with different expressions for c_j. However, as discussed in Section 2.2 and Section 2.5, it is inappropriate to use the CLPP (P2.1) and the CLPP (P2.4). Therefore, it is inappropriate to use the existing methods [50, 63, 111, 146, 222-225, 236, 241].

(ii) It can be easily verified that in the existing methods [50, 63, 111, 146, 222-225, 236, 241], either the expression (2.3) or the expression (2.6) has been used to evaluate P_i and hence for the ranking of the alternatives. However, as discussed in Section 2.2 and Section 2.5, it is inappropriate to use the expression (2.3) and the expression (2.6). Therefore, it is inappropriate to use the existing methods [50, 63, 111, 146, 222-225, 236, 241].

2.8 Inappropriateness of the existing NLPP

It is pertinent to mention that Wei [223] has used the NLPP (P2.7) to evaluate the OAWV of IFMADMPrs having completely unknown attribute weights. The NLPP (P2.7) has been obtained by replacing the linear constraint $\sum_{j=1}^{n} w_j = 1$ with the non-linear constraint $\sum_{j=1}^{n} w_j^2 = 1$ in the CLPP (P2.4).

$max\left[\sum_{j=1}^{n} c_j w_j\right]$

Subject to

$0 \leq w_j \leq 1, \sum_{j=1}^{n} w_j^2 = 1.$ (P2.7)

Also, it is pertinent to mention that Wu and Chen [236] have used the NLPP (P2.7) to evaluate the OAWV of the linguistic MADMPrs having partially known attribute weights.

Inspired by the same, one may use the NLPP (P2.7) to obtain the OAWV of IFMADMPrs having partially known attribute weights.

However, the following example clearly indicates that there is no physical meaning of the non-linear constraint $\sum_{j=1}^{n} w_j^2 = 1$. Whereas, there is a physical meaning of the linear constraint $\sum_{j=1}^{n} w_j = 1$. Therefore, it is inappropriate to use the existing NLPP (P2.7) to obtain the OAWV.

Let us consider a person has \$30,000. This person is interested to invest this amount to purchase the shares of three different companies in such a manner that a total return after one year is maximum. If it is assumed that corresponding to one unit invested amount the return after one year, corresponding to the shares of first, second and third company are 2,3 and 4 times of the total invested amount. Then, the person is interested to know the optimal amount which should be invested in first, second and third company shares.

Let us assume that the person should invest $\$a_1, \$a_2$ and $\$a_3$ to purchase first, second and third company shares. Then, the value of a_1, a_2 and a_3 can be obtained by solving the

CLPP (P2.8) or its equivalent CLPP (P2.9).

$$max(2a_1 + 3a_2 + 4a_3)$$

Subject to

$$\begin{cases} \quad 0 \le a_1 \le 30{,}000, \\ \quad 0 \le a_2 \le 30{,}000, \\ \quad 0 \le a_3 \le 30{,}000, \\ a_1 + a_2 + a_3 = 30{,}000, \\ a_1 \ge 0, a_2 \ge 0, a_3 \ge 0. \end{cases} \tag{P2.8}$$

$$max\left(30{,}000\ \left(2\left(\frac{a_1}{30{,}000}\right) + 3\left(\frac{a_2}{30{,}000}\right) + 4\left(\frac{a_3}{30{,}000}\right)\right)\right)$$

Subject to

$$\begin{cases} \qquad 0 \le \left(\frac{a_1}{30{,}000}\right) \le 1, \\ \qquad 0 \le \left(\frac{a_2}{30{,}000}\right) \le 1, \\ \qquad 0 \le \left(\frac{a_3}{30{,}000}\right) \le 1, \\ \left(\frac{a_1}{30{,}000}\right) + \left(\frac{a_2}{30{,}000}\right) + \left(\frac{a_3}{30{,}000}\right) = 1, \\ \left(\frac{a_1}{30{,}000}\right) \ge 0, \left(\frac{a_2}{30{,}000}\right) \ge 0, \left(\frac{a_3}{30{,}000}\right) \ge 0. \end{cases} \tag{P2.9}$$

Assuming, $\frac{a_1}{30{,}000} = w_1$, $\frac{a_2}{30{,}000} = w_2$ and $\frac{a_3}{30{,}000} = w_3$, the CLPP (P2.9) can be transformed into its equivalent CLPP (P2.10).

$$max\left(30{,}000\ (2w_1 + 3w_2 + 4w_3)\right)$$

Subject to

$$\begin{cases} \quad 0 \le w_1 \le 1, \\ \quad 0 \le w_2 \le 1, \\ \quad 0 \le w_3 \le 1, \\ w_1 + w_2 + w_3 = 1, \\ w_1 \ge 0, w_2 \ge 0, w_3 \ge 0. \end{cases} \tag{P2.10}$$

Now, if it is assumed that in the considered problem the total amount is B and the number of companies are n and the return of one unit invested amount corresponding to the first, second, …, n^{th}- companies are $c_1, c_2, \dots, c_n$ respectively. Then the CLPP (P2.10) will be a special case of the CLPP (P2.11).

$$max\left(B\left(\sum_{j=1}^{n} c_j w_j\right)\right)$$

Subject to

$$\begin{cases} \sum_{j=1}^{n} w_j = 1 \\ 0 \leq w_j \leq 1, j = 1,2,\dots,n. \end{cases} \tag{P2.11}$$

On the basis of the above discussion it can be easily concluded that the linear constraint $\sum_{j=1}^{n} w_j = 1$ has a physical meaning and hence it is inappropriate to replace this linear constraint with the non-linear constraint $\sum_{j=1}^{n} w_j^2 = 1$ i.e, it is inappropriate to use the NLPP (P2.7) to evaluate the OAWV.

2.9 Modified LPP

In Section 2.3 and Section 2.5, it was pointed out that if there will exist two distinct attributes G_p and G_q such that $c_p = c_q$. Then, on solving the CLPP (P2.1) and the CLPP (P2.4) infinite number of OAWV will be obtained. Due to which more than one preference orders for the alternatives will be obtained, which is inappropriate.

If in the CLPP (P2.1) and the CLPP (P2.4) the constraint $w_p = w_q$ is added in the case of $c_p = c_q$. Then, on solving the CLPP (P2.1) and the CLPP (P2.4), always a unique OAWV will be obtained.

2.10 Appropriate intuitionistic fuzzy aggregation operator

It is pertinent to mention that Xu [241] has used the expression (2.6), based upon the intuitionistic fuzzy aggregation operator $\sum_{j=1}^{n} w_j$ ×defuzzified value of $\langle \mu_{ij}, \nu_{ij} \rangle$, to find the ranking of the alternatives.

Also, Wei [223] has used the expression (2.7), based upon the intuitionistic fuzzy aggregation operator $\langle 1 - \prod_{j=1}^{n}\left(1 - \mu_{ij}\right)^{w_j}, \prod_{j=1}^{n} {\nu_{ij}}^{w_j} \rangle$, to find the ranking of the alternatives.

However, as discussed in Section 2.2 and Section 2.5, it is inappropriate to use the existing expressions (2.6) and (2.7) to find the ranking of the alternatives.

If the existing expression (2.8) and (2.9), based upon the existing intuitionistic fuzzy aggregation operator [14] $\left\langle\frac{\sum_{j=1}^{n} w_j\mu_{ij}}{\sum_{j=1}^{n} w_j}, \frac{\sum_{j=1}^{n} w_j\nu_{ij}}{\sum_{j=1}^{n} w_j}\right\rangle$, will be used to find the ranking of the alternatives. Then, the problems, occurring due to using the existing intuitionistic fuzzy aggregation operators [223, 241], can be resolved. Hence, it is appropriate to use the expression (2.8) to evaluate the value of P_i and the expression (2.9) to evaluate the value of Q_i instead of using the expression (2.6) and the expression (2.7) respectively.

$$P_i = \frac{\sum_{j=1}^{n} w_j\mu_{ij}}{\sum_{j=1}^{n} w_j} - \frac{\sum_{j=1}^{n} w_j\nu_{ij}}{\sum_{j=1}^{n} w_j} \tag{2.8}$$

$$Q_i = \frac{\sum_{j=1}^{n} w_j\mu_{ij}}{\sum_{j=1}^{n} w_j} + \frac{\sum_{j=1}^{n} w_j\nu_{ij}}{\sum_{j=1}^{n} w_j} \tag{2.9}$$

2.11 Proposed Mehar method

In this section, to resolve the inappropriateness of Xu’s method [241] and Wei’s method [223], a new method (named as Mehar method), based upon the CLPP (P2.12) as well as the existing expressions (2.8) and (2.9), has been proposed to solve the IFMADMPrs having partially known attribute weights.

The steps of the proposed Mehar method are as follows:

Step 1: Check that all the attributes are of same type or not i.e., check that all the attributes are benefit type attribute or cost type attribute.

Case (i): If all the attributes are of same type then go to Step 2.

Case (ii): If some attributes are cost type attributes and the remaining are benefit type attributes. Then, convert the j^{th} cost type attribute into the benefit type attribute by replacing all the elements $\tilde{\alpha}_{ij} = \langle\mu_{ij}, \nu_{ij}\rangle$ of the j^{th} column of the IFDM $\widetilde{D} = \left(\tilde{\alpha}_{ij}\right)_{m\times n}$ with $\tilde{\alpha}_{ij} = \langle\nu_{ij}, \mu_{ij}\rangle$ and go to Step 2.

Step 2: Using the expression (2.4), find $d_{ijk}\left(\tilde{\alpha}_{ij}, \tilde{\alpha}_{kj}\right)\ \forall\ i = 1,2,\dots,m; k = 1,2,\dots,m; j =$

$1,2,..,n$.

Step 3: Using expression (2.5), find the values of $c_j \ \forall \ j = 1,2,..,n$.

Step 4: Find the OAWV, $(w_1, w_2, \dots, w_n)$ by solving the CLPP (P2.12).

$max\left[\sum_{j=1}^{n} c_j \, w_j\right]$

Subject to

$$\begin{cases} w_j \in H, w_j \geq 0, j = 1,2,..,n, \sum_{j=1}^{n} w_j = 1 \\ \qquad\qquad w_p = w_q \ \forall \ c_p = c_q \ . \end{cases} \tag{P2.12}$$

Step 5: Using the expression (2.8), find the value of $P_i \ \forall \ i = 1,2,\dots,m$.

and check that $P_p > P_q$ or $P_p < P_q$ or $P_p = P_q$.

Case (i): If $P_p < P_q$ then the ranking of the alternatives A_p and A_q is $A_p < A_q$.

Case (ii): If $P_p > P_q$ then the ranking of the alternatives A_p and A_q is $A_p > A_q$.

Case (iii): If $P_p = P_q$ then go to Step 6.

Step 6: Using the expression (2.9), find the value of $Q_i \ \forall \ i = 1,2,\dots,m$.

and check that $Q_p > Q_q$ or $Q_p < Q_q$ or $Q_p = Q_q$.

Case (i): If $Q_p < Q_q$ then the ranking of the alternatives A_p and A_q is $A_p < A_q$.

Case (ii): If $Q_p > Q_q$ then the ranking of the alternatives A_p and A_q is $A_p > A_q$.

Case (iii): If $Q_p = Q_q$ then the ranking of the alternatives A_p and A_q is $A_p = A_q$.

2.12 Exact results of the considered IFMADMPrs having partially known attribute weights

In Section 2.2, two IFMADMPrs having partially known attribute weights were solved by Xu's method [241] and shown that the obtained results are inappropriate. Also, in Section 2.5, two IFMADMPrs were solved by Wei's method [223] and shown that the obtained results are inappropriate. In this section, the exact results of all these IFMADMPrs having partially known attribute weights are obtained by the proposed Mehar method.

2.12.1 Exact results of the first IFMADMPr having partially known attribute weights

Using the proposed Mehar method, the exact result of the IFMADMPr having partially known attribute weights, considered in the first point of Section 2.2, can be obtained as follows:

Step 1: Since, all the attributes are benefit type. So, according to Step 1 of the Mehar method, proposed in Section 2.11, there is no need to apply Step 1.

Step 2: According to Step 2 of the Mehar method, proposed in Section 2.11, there is need to calculate $d_{ijk}\left(\tilde{\alpha}_{ij},\tilde{\alpha}_{kj}\right)\ \forall\ i=1,2,\ldots,6; j=1,2,3$. These values are shown in Table 2.9.

Table 2.9: Values of $\boldsymbol{d_{ij}\left(\tilde{\alpha}_{ij},\tilde{\alpha}_{kj}\right)}$

$d_{111}(\langle 0.4,0.3\rangle,\langle 0.4,0.3\rangle) = \frac{1}{2}(0+0) = 0$	$d_{211}(\langle 0.6,0.1\rangle,\langle 0.6,0.1\rangle) = \frac{1}{2}(0+0) = 0$	$d_{311}(\langle 0.5,0.4\rangle,\langle 0.5,0.4\rangle) = \frac{1}{2}(0+0) = 0$
$d_{112}(\langle 0.4,0.3\rangle,\langle 0.5,0.2\rangle) = \frac{1}{2}(0.1+0.1) = 0.1$	$d_{212}(\langle 0.6,0.1\rangle,\langle 0.3,0.4\rangle) = \frac{1}{2}(0.3+0.3) = 0.3$	$d_{312}(\langle 0.5,0.4\rangle,\langle 0.8,0.1\rangle) = \frac{1}{2}(0.3+0.3) = 0.3$
$d_{113}(\langle 0.4,0.3\rangle,\langle 0.7,0.2\rangle) = \frac{1}{2}(0.3+0.1) = 0.2$	$d_{213}(\langle 0.6,0.1\rangle,\langle 0.3,0.7\rangle) = \frac{1}{2}(0.3+0.6) = 0.45$	$d_{313}(\langle 0.5,0.4\rangle,\langle 0.6,0.2\rangle) = \frac{1}{2}(0.1+0.2) = 0.15$
$d_{114}(\langle 0.4,0.3\rangle,\langle 0.4,0.3\rangle) = \frac{1}{2}(0+0) = 0$	$d_{214}(\langle 0.6,0.1\rangle,\langle 0.6,0.2\rangle) = \frac{1}{2}(0+0.1) = 0.05$	$d_{314}(\langle 0.5,0.4\rangle,\langle 0.7,0.1\rangle) = \frac{1}{2}(0.2+0.3) = 0.25$
$d_{115}(\langle 0.4,0.3\rangle,\langle 0.6,0.2\rangle) = \frac{1}{2}(0.2+0.1) = 0.15$	$d_{215}(\langle 0.6,0.1\rangle,\langle 0.5,0.1\rangle) = \frac{1}{2}(0.1+0) = 0.05$	$d_{315}(\langle 0.5,0.4\rangle,\langle 0.4,0.6\rangle) = \frac{1}{2}(0.1+0.2) = 0.15$
$d_{116}(\langle 0.4,0.3\rangle,\langle 0.6,0.3\rangle) = \frac{1}{2}(0.2+0) = 0.1$	$d_{216}(\langle 0.6,0.1\rangle,\langle 0.7,0.2\rangle) = \frac{1}{2}(0.1+0.1) = 0.1$	$d_{316}(\langle 0.5,0.4\rangle,\langle 0.5,0.4\rangle) = \frac{1}{2}(0+0) = 0$
$d_{121}(\langle 0.5,0.2\rangle,\langle 0.4,0.3\rangle) = \frac{1}{2}(0.1+0.1) = 0.1$	$d_{221}(\langle 0.3,0.4\rangle,\langle 0.6,0.1\rangle) = \frac{1}{2}(0.3+0.3) = 0.3$	$d_{321}(\langle 0.8,0.1\rangle,\langle 0.5,0.4\rangle) = \frac{1}{2}(0.3+0.3) = 0.3$

$d_{122}(\langle 0.5,0.2\rangle,\langle 0.5,0.2\rangle)=$ $\frac{1}{2}(0+0)=0$	$d_{222}(\langle 0.3,0.4\rangle,\langle 0.3,0.4\rangle)=$ $\frac{1}{2}(0+0)=0$	$d_{322}(\langle 0.8,0.1\rangle,\langle 0.8,0.1\rangle)=$ $\frac{1}{2}(0+0)=0$
$d_{123}(\langle 0.5,0.2\rangle,\langle 0.7,0.2\rangle)=$ $\frac{1}{2}(0.2+0)=0.1$	$d_{223}(\langle 0.3,0.4\rangle,\langle 0.3,0.7\rangle)=$ $\frac{1}{2}(0+0.3)=0.15$	$d_{323}(\langle 0.8,0.1\rangle,\langle 0.6,0.2\rangle)=$ $\frac{1}{2}(0.2+0.1)=0.15$
$d_{124}(\langle 0.5,0.2\rangle,\langle 0.4,0.3\rangle)=$ $\frac{1}{2}(0.1+0.1)=0.1$	$d_{224}(\langle 0.3,0.4\rangle,\langle 0.6,0.2\rangle)=$ $\frac{1}{2}(0.3+0.2)=0.25$	$d_{324}(\langle 0.8,0.1\rangle,\langle 0.7,0.1\rangle)=$ $\frac{1}{2}(0.1+0)=0.05$
$d_{125}(\langle 0.5,0.2\rangle,\langle 0.6,0.2\rangle)=$ $\frac{1}{2}(0.1+0)=0.05$	$d_{225}(\langle 0.3,0.4\rangle,\langle 0.5,0.1\rangle)=$ $\frac{1}{2}(0.2+0.3)=0.25$	$d_{325}(\langle 0.8,0.1\rangle,\langle 0.4,0.6\rangle)=$ $\frac{1}{2}(0.4+0.5)=0.45$
$d_{126}(\langle 0.5,0.2\rangle,\langle 0.6,0.3\rangle)=$ $\frac{1}{2}(0.1+0.1)=0.1$	$d_{226}(\langle 0.3,0.4\rangle,\langle 0.7,0.2\rangle)=$ $\frac{1}{2}(0.4+0.2)=0.3$	$d_{326}(\langle 0.8,0.1\rangle,\langle 0.5,0.4\rangle)=$ $\frac{1}{2}(0.3+0.3)=0.3$
$d_{131}(\langle 0.7,0.2\rangle,\langle 0.4,0.3\rangle)=$ $\frac{1}{2}(0.3+0.1)=0.2$	$d_{231}(\langle 0.3,0.7\rangle,\langle 0.6,0.1\rangle)=$ $\frac{1}{2}(0.3+0.6)=0.45$	$d_{331}(\langle 0.6,0.2\rangle,\langle 0.5,0.4\rangle)=$ $\frac{1}{2}(0.1+0.2)=0.15$
$d_{132}(\langle 0.7,0.2\rangle,\langle 0.5,0.2\rangle)=$ $\frac{1}{2}(0.2+0)=0.1$	$d_{232}(\langle 0.3,0.7\rangle,\langle 0.3,0.4\rangle)=$ $\frac{1}{2}(0+0.3)=0.15$	$d_{332}(\langle 0.6,0.2\rangle,\langle 0.8,0.1\rangle)=$ $\frac{1}{2}(0.2+0.1)=0.15$
$d_{133}(\langle 0.7,0.2\rangle,\langle 0.7,0.2\rangle)=$ $\frac{1}{2}(0+0)=0$	$d_{233}(\langle 0.3,0.7\rangle,\langle 0.3,0.7\rangle)=$ $\frac{1}{2}(0+0)=0$	$d_{333}(\langle 0.6,0.2\rangle,\langle 0.6,0.2\rangle)=$ $\frac{1}{2}(0+0)=0$
$d_{134}(\langle 0.7,0.2\rangle,\langle 0.4,0.3\rangle)=$ $\frac{1}{2}(0.3+0.1)=0.2$	$d_{234}(\langle 0.3,0.7\rangle,\langle 0.6,0.2\rangle)=$ $\frac{1}{2}(0.3+0.5)=0.4$	$d_{334}(\langle 0.6,0.2\rangle,\langle 0.7,0.1\rangle)=$ $\frac{1}{2}(0.1+0.1)=0.1$
$d_{135}(\langle 0.7,0.2\rangle,\langle 0.6,0.2\rangle)=$ $\frac{1}{2}(0.1+0)=0.05$	$d_{235}(\langle 0.3,0.7\rangle,\langle 0.5,0.1\rangle)=$ $\frac{1}{2}(0.2+0.6)=0.4$	$d_{335}(\langle 0.6,0.2\rangle,\langle 0.4,0.6\rangle)=$ $\frac{1}{2}(0.2+0.4)=0.3$
$d_{136}(\langle 0.7,0.2\rangle,\langle 0.6,0.3\rangle)=$ $\frac{1}{2}(0.1+0.1)=0.1$	$d_{236}(\langle 0.3,0.7\rangle,\langle 0.7,0.2\rangle)=$ $\frac{1}{2}(0.4+0.5)=0.45$	$d_{336}(\langle 0.6,0.2\rangle,\langle 0.5,0.4\rangle)=$ $\frac{1}{2}(0.1+0.2)=0.15$

$d_{141}(\langle 0.4,0.3\rangle,\langle 0.4,0.3\rangle)=\frac{1}{2}(0+0)=0$	$d_{241}(\langle 0.6,0.2\rangle,\langle 0.6,0.1\rangle)=\frac{1}{2}(0+0.1)=0.05$	$d_{341}(\langle 0.7,0.1\rangle,\langle 0.5,0.4\rangle)=\frac{1}{2}(0.2+0.3)=0.25$
$d_{142}(\langle 0.4,0.3\rangle,\langle 0.5,0.2\rangle)=\frac{1}{2}(0.1+0.1)=0.1$	$d_{242}(\langle 0.6,0.2\rangle,\langle 0.3,0.4\rangle)=\frac{1}{2}(0.3+0.2)=0.25$	$d_{342}(\langle 0.7,0.1\rangle,\langle 0.8,0.1\rangle)=\frac{1}{2}(0.1+0)=0.05$
$d_{143}(\langle 0.4,0.3\rangle,\langle 0.7,0.2\rangle)=\frac{1}{2}(0.3+0.1)=0.2$	$d_{243}(\langle 0.6,0.2\rangle,\langle 0.3,0.7\rangle)=\frac{1}{2}(0.3+0.5)=0.4$	$d_{343}(\langle 0.7,0.1\rangle,\langle 0.6,0.2\rangle)=\frac{1}{2}(0.1+0.1)=0.1$
$d_{144}(\langle 0.4,0.3\rangle,\langle 0.4,0.3\rangle)=\frac{1}{2}(0+0)=0$	$d_{244}(\langle 0.6,0.2\rangle,\langle 0.6,0.2\rangle)=\frac{1}{2}(0+0)=0$	$d_{344}(\langle 0.7,0.1\rangle,\langle 0.7,0.1\rangle)=\frac{1}{2}(0+0)=0$
$d_{145}(\langle 0.4,0.3\rangle,\langle 0.6,0.2\rangle)=\frac{1}{2}(0.2+0.1)=0.15$	$d_{245}(\langle 0.6,0.2\rangle,\langle 0.5,0.1\rangle)=\frac{1}{2}(0.1+0.1)=0.1$	$d_{345}(\langle 0.7,0.1\rangle,\langle 0.4,0.6\rangle)=\frac{1}{2}(0.3+0.5)=0.4$
$d_{146}(\langle 0.4,0.3\rangle,\langle 0.6,0.3\rangle)=\frac{1}{2}(0.2+0)=0.1$	$d_{246}(\langle 0.6,0.2\rangle,\langle 0.7,0.2\rangle)=\frac{1}{2}(0.1+0)=0.05$	$d_{346}(\langle 0.7,0.1\rangle,\langle 0.5,0.4\rangle)=\frac{1}{2}(0.2+0.3)=0.25$
$d_{151}(\langle 0.6,0.2\rangle,\langle 0.4,0.3\rangle)=\frac{1}{2}(0.2+0.1)=0.15$	$d_{251}(\langle 0.5,0.1\rangle,\langle 0.6,0.1\rangle)=\frac{1}{2}(0.1+0)=0.05$	$d_{351}(\langle 0.4,0.6\rangle,\langle 0.5,0.4\rangle)=\frac{1}{2}(0.1+0.2)=0.15$
$d_{152}(\langle 0.6,0.2\rangle,\langle 0.5,0.2\rangle)=\frac{1}{2}(0.1+0)=0.05$	$d_{252}(\langle 0.5,0.1\rangle,\langle 0.3,0.4\rangle)=\frac{1}{2}(0.2+0.3)=0.25$	$d_{352}(\langle 0.4,0.6\rangle,\langle 0.8,0.1\rangle)=\frac{1}{2}(0.4+0.5)=0.45$
$d_{153}(\langle 0.6,0.2\rangle,\langle 0.7,0.2\rangle)=\frac{1}{2}(0.1+0)=0.05$	$d_{253}(\langle 0.5,0.1\rangle,\langle 0.3,0.7\rangle)=\frac{1}{2}(0.2+0.6)=0.4$	$d_{353}(\langle 0.4,0.6\rangle,\langle 0.6,0.2\rangle)=\frac{1}{2}(0.2+0.4)=0.3$
$d_{154}(\langle 0.6,0.2\rangle,\langle 0.4,0.3\rangle)=\frac{1}{2}(0.2+0.1)=0.15$	$d_{254}(\langle 0.5,0.1\rangle,\langle 0.6,0.2\rangle)=\frac{1}{2}(0.1+0.1)=0.1$	$d_{354}(\langle 0.4,0.6\rangle,\langle 0.7,0.1\rangle)=\frac{1}{2}(0.3+0.5)=0.4$
$d_{155}(\langle 0.6,0.2\rangle,\langle 0.6,0.2\rangle)=\frac{1}{2}(0+0)=0$	$d_{255}(\langle 0.5,0.1\rangle,\langle 0.5,0.1\rangle)=\frac{1}{2}(0+0)=0$	$d_{355}(\langle 0.4,0.6\rangle,\langle 0.4,0.6\rangle)=\frac{1}{2}(0+0)=0$

$d_{156}(\langle 0.6,0.2\rangle,\langle 0.6,0.3\rangle)=\frac{1}{2}(0+0.1)=0.05$	$d_{256}(\langle 0.5,0.1\rangle,\langle 0.7,0.2\rangle)=\frac{1}{2}(0.2+0.1)=0.15$	$d_{356}(\langle 0.4,0.6\rangle,\langle 0.5,0.4\rangle)=\frac{1}{2}(0.1+0.2)=0.15$
$d_{161}(\langle 0.6,0.3\rangle,\langle 0.4,0.3\rangle)=\frac{1}{2}(0.2+0)=0.1$	$d_{261}(\langle 0.7,0.2\rangle,\langle 0.6,0.1\rangle)=\frac{1}{2}(0.1+0.1)=0.1$	$d_{361}(\langle 0.5,0.4\rangle,\langle 0.5,0.4\rangle)=\frac{1}{2}(0+0)=0$
$d_{162}(\langle 0.6,0.3\rangle,\langle 0.5,0.2\rangle)=\frac{1}{2}(0.1+0.1)=0.1$	$d_{262}(\langle 0.7,0.2\rangle,\langle 0.3,0.4\rangle)=\frac{1}{2}(0.4+0.2)=0.3$	$d_{362}(\langle 0.5,0.4\rangle,\langle 0.8,0.1\rangle)=\frac{1}{2}(0.3+0.3)=0.3$
$d_{163}(\langle 0.6,0.3\rangle,\langle 0.7,0.2\rangle)=\frac{1}{2}(0.1+0.1)=0.1$	$d_{263}(\langle 0.7,0.2\rangle,\langle 0.3,0.7\rangle)=\frac{1}{2}(0.4+0.5)=0.45$	$d_{363}(\langle 0.5,0.4\rangle,\langle 0.6,0.2\rangle)=\frac{1}{2}(0.1+0.2)=0.15$
$d_{164}(\langle 0.6,0.3\rangle,\langle 0.4,0.3\rangle)=\frac{1}{2}(0.2+0)=0.1$	$d_{264}(\langle 0.7,0.2\rangle,\langle 0.6,0.2\rangle)=\frac{1}{2}(0.1+0)=0.05$	$d_{364}(\langle 0.5,0.4\rangle,\langle 0.7,0.1\rangle)=\frac{1}{2}(0.2+0.3)=0.25$
$d_{165}(\langle 0.6,0.3\rangle,\langle 0.6,0.2\rangle)=\frac{1}{2}(0+0.1)=0.05$	$d_{265}(\langle 0.7,0.2\rangle,\langle 0.5,0.1\rangle)=\frac{1}{2}(0.2+0.1)=0.15$	$d_{365}(\langle 0.5,0.4\rangle,\langle 0.4,0.6\rangle)=\frac{1}{2}(0.1+0.2)=0.15$
$d_{166}(\langle 0.6,0.3\rangle,\langle 0.6,0.3\rangle)=\frac{1}{2}(0+0)=0$	$d_{266}(\langle 0.7,0.2\rangle,\langle 0.7,0.2\rangle)=\frac{1}{2}(0+0)=0$	$d_{366}(\langle 0.5,0.4\rangle,\langle 0.5,0.4\rangle)=\frac{1}{2}(0+0)=0$

Step 3: Using Step 3 of the Mehar method, proposed in Section 2.11,

$c_1=\sum_{i=1}^{6}\sum_{k=1}^{6} d_{i1k}(\tilde{\alpha}_{i1},\tilde{\alpha}_{k1})=0+0.1+0.2+0+0.15+0.1+0.1+0+0.1+0.1+0.05+0.1+0.2+0.1+0+0.2+0.05+0.1+0+0.1+0.2+0+0.15+0.1+0.15+0.05+0.05+0.15+0+0.05+0.1+0.1+0.1+0.1+0.05+0=3.1$,

$c_2=\sum_{i=1}^{6}\sum_{k=1}^{6} d_{i2k}(\tilde{\alpha}_{i2},\tilde{\alpha}_{k2})=0+0.3+0.45+0.05+0.05+0.1+0.3+0+0.15+0.25+0.25+0.3+0.45+0.15+0+0.4+0.4+0.45+0.05+0.25+0.4+0+0.1+0.05+0.05+0.25+0.4+0.1+0+0.15+0.1+0.30+0.45+0.05+0.15+0=6.9$,

$c_3=\sum_{i=1}^{6}\sum_{k=1}^{6} d_{i3k}(\tilde{\alpha}_{i3},\tilde{\alpha}_{k3})=0+0.3+0.15+0.25+0.15+0+0.3+0+0.15+$

$0.05 + 0.45 + 0.3 + 0.15 + 0.15 + 0 + 0.1 + 0.3 + 0.15 + 0.25 + 0.05 + 0.1 + 0 + 0.4 + 0.25 + 0.15 + 0.45 + 0.3 + 0.4 + 0 + 0.15 + 0 + 0.3 + 0.15 + 0.25 + 0.15 + 0 = 6.30.$

Step 4: Using Step 4 of the Mehar method, proposed in Section 2.11, there is need to solve the CLPP (P2.13).

$Max(3.1w_1 + 6.9w_2 + 6.3w_3)$

Subject to

$$\begin{cases} 0.1 \leq w_1 \leq 0.55, \\ 0.3 \leq w_2 \leq 0.35, \\ w_3 \geq 0.5w_2, \\ w_1 + w_2 + w_3 = 1, \\ w_1 \geq 0, w_2 \geq 0, w_3 \geq 0. \end{cases} \tag{P2.13}$$

It can be easily verified that on solving the CLPP (P2.13), the obtained OAWV is $(w_1, w_2, w_3) = (0.10,0.35,0.55)$.

Step 5: On considering the OAWV, $(w_1, w_2, w_3) = (0.10,0.35,0.55)$ and using Step 5 of Mehar method, proposed in Section 2.11,

$P_1 = \big((0.10)(0.4) + (0.35)(0.6) + (0.55)(0.5)\big) - \big((0.10)(0.3) + (0.35)(0.1) + (0.55)(0.4)\big) = 0.24,$

$P_2 = \big((0.10)(0.5) + (0.35)(0.3) + (0.55)(0.8)\big) - \big((0.10)(0.2) + (0.35)(0.4) + (0.55)(0.1)\big) = 0.38,$

$P_3 = \big((0.10)(0.7) + (0.35)(0.3) + (0.55)(0.6)\big) - \big((0.10)(0.2) + (0.35)(0.7) + (0.55)(0.2)\big) = 0.13,$

$P_4 = \big((0.10)(0.4) + (0.35)(0.6) + (0.55)(0.7)\big) - \big((0.10)(0.3) + (0.35)(0.2) + (0.55)(0.1)\big) = 0.48,$

$P_5 = \big((0.10)(0.6) + (0.35)(0.5) + (0.55)(0.4)\big) - \big((0.10)(0.2) + (0.35)(0.1) + (0.55)(0.6)\big) = 0.07,$

$P_6 = \big((0.10)(0.6) + (0.35)(0.7) + (0.55)(0.5)\big) - \big((0.10)(0.3) + (0.35)(0.2) +$

$(0.55)(0.4) = 0.26$.

Since, $P_4 > P_2 > P_6 > P_1 > P_3 > P_5$. So, according to Step 5 of the Mehar method, proposed in Section 2.11, the ranking of the alternatives is $A_4 > A_2 > A_6 > A_1 > A_3 > A_5$.

2.12.2 Exact results of second IFMADMPr having partially known attribute weights

Using the proposed Mehar method, the exact result of IFMADMPr having partially known attribute weight, considered in the second point of Section 2.2, can be obtained as follows:

Step 1: Since, all the attributes are of benefit type. So, according to Step 1 of the Mehar method, proposed in Section 2.11, there is no need to apply Step 1.

Step 2: According to Step 2 of the proposed Mehar method, discussed in Section 2.12, there is need to calculate $d_{ijk}(\tilde{\alpha}_{ij}, \tilde{\alpha}_{kj})\ \forall\ i = 1,2; j = 1,2$. These values are shown in Table 2.10.

Table 2.10: Values of $d_{ijk}(\tilde{\alpha}_{ij}, \tilde{\alpha}_{kj})$

$d_{111}(\langle 0.4,0.3\rangle, \langle 0.4,0.3\rangle) = \frac{1}{2}(0+0) = 0$	$d_{211}(\langle 0.1,0.2\rangle, \langle 0.1,0.2\rangle) = \frac{1}{2}(0+0) = 0$
$d_{112}(\langle 0.4,0.3\rangle, \langle 0.1,0\rangle) = \frac{1}{2}(0.3+0.3) = 0.3$	$d_{212}(\langle 0.1,0.2\rangle, \langle 0.1,0.2\rangle) = \frac{1}{2}(0+0) = 0$
$d_{121}(\langle 0.1,0\rangle, \langle 0.4,0.3\rangle) = \frac{1}{2}(0.3+0.3) = 0.3$	$d_{221}(\langle 0.1,0.2\rangle, \langle 0.1,0.2\rangle) = \frac{1}{2}(0+0) = 0$
$d_{122}(\langle 0.1,0\rangle, \langle 0.1,0\rangle) = \frac{1}{2}(0+0) = 0$	$d_{222}(\langle 0.1,0.2\rangle, \langle 0.1,0.2\rangle) = \frac{1}{2}(0+0) = 0$

Step 3: Using Step 3 of the Mehar method, proposed in Section 2.11,

$c_1 = \sum_{i=1}^{2} \sum_{k=1}^{2} d_{i1k}(\tilde{\alpha}_{i1}, \tilde{\alpha}_{k1}) = 0 + 0.3 + 0.3 + 0 = 0.6$,

$c_2 = \sum_{i=1}^{2} \sum_{k=1}^{2} d_{i2k}(\tilde{\alpha}_{i2}, \tilde{\alpha}_{k2}) = 0 + 0 + 0 + 0 = 0$.

Step 4: Using Step 4 of Mehar method, proposed in Section 2.11, there is need to solve the CLPP (P2.14).

$Max(0.6w_1)$

Subject to

$$\begin{cases} 0.1 \leq w_1 \leq 0.4, \\ 0.3 \leq w_2 \leq 0.7, \\ w_1 + w_2 = 1, \\ w_1 \geq 0, w_2 \geq 0. \end{cases} \tag{P2.14}$$

It can be easily verified that on solving the CLPP (P2.14) the obtained OAWV is $(w_1, w_2) = (0.4, 0.6)$

Step 5: On considering the OAWV, $(w_1, w_2) = (0.4, 0.6)$ and using Step 4 of the Mehar method, proposed in Section 2.11,

$P_1 = \big((0.4)(0.4) + (0.6)(0.1)\big) - \big((0.4)(0.3) + (0.6)(0.2)\big) = -0.02,$

$P_2 = \big((0.4)(0.1) + (0.6)(0.1)\big) - \big((0.4)(0) + (0.6)(0.2)\big) = -0.02.$

Since, $P_1 = P_2$. So, there is need to go to Step 6 of the proposed Mehar method.

Step 6: Using Step 6 of the Mehar method, proposed in Section 2.11,

$Q_1 = \big((0.4)(0.4) + (0.6)(0.1)\big) + \big((0.4)(0.3) + (0.6)(0.2)\big) = 0.46,$

$Q_2 = \big((0.4)(0.1) + (0.6)(0.1)\big) + \big((0.4)(0) + (0.6)(0.2)\big) = 0.22.$

Since, $Q_1 > Q_2$. Therefore, according to Step 6 of the proposed Mehar method, the ranking of the alternatives is $A_1 > A_2$.

2.12.3 Exact results of third IFMADMPr having partially known attribute weights

Using the proposed Mehar method, the exact result of IFMADMPr having partially known attribute weight, considered in the first point of Section 2.5, can be obtained as follows:

Since, Step 1 to Step 3 of the proposed Mehar method and Wei's method [223] are same. Also, as Step 1 to Step 3 of Wei's method [223] for the considered IFMADMPr having partially known attribute weights is discussed in the first point of Section 2.5. So, to avoid any repetition, the calculations have been started from Step 4.

Step 4: Using Step 4 of the Mehar method, proposed in Section 2.11, there is need to solve the CLPP (P2.15).

$Max(2.2w_1 + 3w_2 + 2.2w_3 + 3w_4)$

Subject to

$$\begin{cases} 0.15 \leq w_1 \leq 0.20, \\ 0.1 \leq w_2 \leq 0.95, \\ 0.20 \leq w_3 \leq 0.35, \\ 0.20 \leq w_4 \leq 0.45, \\ w_1 + w_2 + w_3 + w_4 = 1, \\ w_1 = w_3, \\ w_2 = w_4, \\ w_1 \geq 0, w_2 \geq 0, w_3 \geq 0, w_4 \geq 0. \end{cases} \tag{P2.15}$$

It can be easily verified that on solving the CLPP (P2.15) the obtained OAWV is $(w_1, w_2, w_3, w_4) = (0.2, 0.3, 0.2, 0.3)$.

Step 5: On considering the OAWV, $(w_1, w_2, w_3, w_4) = (0.2, 0.3, 0.2, 0.3)$ and using Step 5 of the Mehar method, proposed in Section 2.11,

$P_1 = \big((0.2)(0.6) + (0.3)(0.5) + (0.2)(0.4) + (0.3)(0.6)\big) - \big((0.2)(0.3) + (0.3)(0.4) + (0.2)(0.3) + (0.3)(0.2)\big) = 0.23,$

$P_2 = \big((0.2)(0.7) + (0.3)(0.7) + (0.2)(0.6) + (0.3)(0.6)\big) - \big((0.2)(0.3) + (0.3)(0.4) + (0.3)(0.3) + (0.4)(0.2)\big) = 0.34,$

$P_3 = \big((0.2)(0.5) + (0.3)(0.6) + (0.2)(0.7) + (0.3)(0.5)\big) - \big((0.2)(0.4) + (0.3)(0.4) + (0.2)(0.2) + (0.3)(0.4)\big) = 0.21,$

$P_4 = \big((0.2)(0.6) + (0.3)(0.8) + (0.2)(0.5) + (0.3)(0.7)\big) - \big((0.2)(0.3) + (0.3)(0.1) + (0.2)(0.4) + (0.3)(0.3)\big) = 0.41,$

$P_5 = \big((0.2)(0.4) + (0.3)(0.6) + (0.2)(0.6) + (0.3)(0.8)\big) - \big((0.2)(0.3) + (0.3)(0.2) + (0.2)(0.3) + (0.3)(0.1)\big) = 0.41.$

Since $P_4 = P_5 > P_2 > P_1 > P_3$. As, $P_4 = P_5$, so there is need to go to Step 6 of the proposed Mehar method.

Step 6: Using Step 6 of the Mehar method, proposed in Section 2.11,

$Q_4 = \big((0.2)(0.6) + (0.3)(0.8) + (0.2)(0.5) + (0.3)(0.7)\big) + \big((0.2)(0.3) + (0.3)(0.1) +$

$(0.2)(0.4) + (0.3)(0.3)\big) = 0.93,$

$Q_5 = \big((0.2)(0.4) + (0.3)(0.6) + (0.2)(0.6) + (0.3)(0.8)\big) + \big((0.2)(0.3) + (0.3)(0.2) +$

$(0.2)(0.3) + (0.3)(0.1)\big) = 0.83.$

Since, $Q_4 > Q_5$ and $P_4 = P_5 > P_2 > P_1 > P_3$. Therefore, according to Step 6 of the Mehar method, proposed in Section 2.11, the ranking of the alternatives is $A_4 > A_5 > A_2 > A_1 > A_3$.

2.12.4 Exact results of fourth IFMADMPr having partially known attribute weights

Using the proposed Mehar method, the exact result of IFMADMPr having partially known attribute weights, considered in the second point of Section 2.5, can be obtained as follows:

Since, Step 1 to Step 3 of the proposed Mehar method and Wei's method [223] are same. Also, as Step 1 to Step 3 of Wei's method [223], for the considered IFMADMPr having partially known attribute weights is discussed in the second point of Section 2.5. So, to avoid any repetition, the calculations have been started from Step 4.

Step 4: Using Step 4 of Mehar method, proposed in Section 2.11, there is need to solve the CLPP (P2.16).

$Max(0.2w_1 + 0.w_2)$

Subject to

$$\begin{cases} 0.1 \le w_1 \le 0.6, \\ 0.2 \le w_2 \le 0.7, \\ w_1 + w_2 = 1, \\ w_1 \ge 0, w_2 \ge 0. \end{cases} \tag{P2.16}$$

It can be easily verified that on solving the CLPP (P2.16) the obtained OAWV is $(w_1, w_2) = (0.6, 0.4)$.

Step 5: On considering the OAWV, $(w_1, w_2) = (0.6, 0.4)$ and using Step 5 of the Mehar method, proposed in Section 2.11,

$P_1 = \big((0.6)(0.4) + (0.4)(1)\big) - \big((0.6)(0.3) + (0.4)(0)\big) = 0.46,$

$P_2 = \big((0.6)(0.3) + (0.4)(1)\big) - \big((0.6)(0.4) + (0.4)(0)\big) = 0.34.$

Since, $P_1 > P_2$. Therefore, according to Step 5 of the Mehar method, proposed in Section 2.11, the ranking of the alternatives is $A_1 > A_2$.

2.13 Conclusions

The inappropriateness of the existing methods [50, 63, 11, 146, 222-225, 236, 241] are pointed out. Also, a new method (named as Mehar method) is proposed to solve the IFMADMPrs having partially known attribute weights. Furthermore, some IFMADMPrs having partially known attribute weights are solved to illustrate the proposed Mehar method.

Chapter 3

Ajit method for solving IFMADMPrs having partially known attribute weights[2]

In the Mehar method (proposed in Chapter 2) as well as in the existing methods [223, 241], a CLPP has been solved to obtain the OAWV. However, the constraint $\sum_{j=1}^{n} w_j = 1$ of the considered CLPP will not be satisfied if either the condition $\sum_{j=1}^{n} w_j^l > 1$ or the condition $\sum_{j=1}^{n} w_j^u < 1$ will be satisfied, where, w_j^l and w_j^u are the known lower bound and the upper bound of the j^{th}- attribute weight i.e., no feasible solution of the considered CLPP will be obtained if either the condition $\sum_{j=1}^{n} w_j^l > 1$ or the condition $\sum_{j=1}^{n} w_j^u < 1$ will be satisfied. Therefore, the proposed Mehar method and the existing methods [223, 241] can be used only if neither the condition $\sum_{j=1}^{n} w_j^l > 1$ nor the condition $\sum_{j=1}^{n} w_j^u < 1$ will be satisfied. Keeping the same in mind, in this chapter, a new method (named as Ajit Method) has been proposed to solve such IFMADMPrs in which either the condition $\sum_{j=1}^{n} w_j^l > 1$ or the condition $\sum_{j=1}^{n} w_j^u < 1$ will be satisfied.

3.1 Limitations of the proposed Mehar method

It is pertinent to mention that to apply the Mehar method, proposed in Section 2.11 of Chapter 2, there is need to solve the CLPP (P2.12). However, a feasible solution of the CLPP (P2.12) will exist only if neither the condition $\sum_{j=1}^{n} w_j^l > 1$ nor the condition $\sum_{j=1}^{n} w_j^u < 1$ will be satisfied.

The following examples are considered to validate this statement.

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[2] The contents of this chapter have been communicated in “Soft Computing” for the possible publication.

(1) Let us consider a person has \$30,000. This person is interested to invest this amount to purchase the shares of three different companies in such a manner that a total return after one year is maximum. If it is assumed that corresponding to one unit invested amount the return after one year, corresponding to the shares of first, second and third company are 2, 3 and 4 times of the total invested amount. Also, if it is assumed that there are the following restrictions

(i) At most \$ 5,000 can be used to purchase the shares of the first company.

(ii) At most \$ 10,000 can be used to purchase the shares of the second company.

(iii) At most \$ 8,000 can be used to purchase the shares of the third company.

Then, the person is interested to know the optimal amount which should be invested in first, second and third company shares.

Let us assume that the person should invest $\$a_1, \$a_2$ and $\$a_3$ to purchase the shares of the first, the second and the third company respectively. Then, the values of a_1, a_2 and a_3 can be obtained by solving the CLPP (P3.1) or its equivalent CLPP (P3.2).

$$max(2a_1 + 3a_2 + 4a_3)$$

Subject to

$$\begin{cases} 0 \le a_1 \le 5{,}000, \\ 0 \le a_2 \le 10{,}000, \\ 0 \le a_3 \le 8{,}000, \\ a_1 + a_2 + a_3 = 30{,}000, \\ a_1 \ge 0, a_2 \ge 0, a_3 \ge 0. \end{cases} \tag{P3.1}$$

$$max\ 30{,}000(2w_1 + 3w_2 + 4w_3)$$

Subject to

$$\begin{cases} 0 \le w_1 \le \frac{5{,}000}{30{,}000}, \\ 0 \le w_2 \le \frac{10{,}000}{30{,}000}, \\ 0 \le w_3 \le \frac{8{,}000}{30{,}000}, \\ w_1 + w_2 + w_3 = 1, \\ w_1 \ge 0, w_2 \ge 0, w_3 \ge 0. \end{cases} \tag{P3.2}$$

However, as the total maximum invested amount \$ 5,000+\$ 10,000+\$ 8,000 i.e., \$ 23,000 is less than the total available amount i.e., \$ 30,000. So, on solving the CLPP (P3.1) or its equivalent CLPP (P3.2), no feasible solution will be obtained.

(2) Let us consider a person has \$30,000. This person is interested to invest this amount to purchase the shares of three different companies in such a manner that a total return after one year is maximum. If it is assumed that corresponding to one unit invested amount the return after one year, corresponding to the shares of first, second and third company are 2, 3 and 4 times the total invested amount. Also, if it is assumed that there are the following restrictions

(i) Atleast \$ 8,000 should be used to purchase shares of the first company.
(ii) Atleast \$ 15,000 should be used to purchase shares of the second company.
(iii) Atleast \$ 12,000 should be used to purchase shares of the third company.

Then, the person is interested to know the optimal amount which should be invested in first, second and third company shares.

Let us assume that the person should invest $\$a_1, \$a_2$ and $\$a_3$ to purchase the shares of the first, the second and the third company respectively. Then, the values of a_1, a_2 and a_3 can be obtained by solving the CLPP (P3.3) or its equivalent CLPP (P3.4)

$max(2a_1 + 3a_2 + 4a_3)$

Subject to

$$\begin{cases} 8{,}000 \le a_1 \le 30{,}000, \\ 15{,}000 \le a_2 \le 30{,}000, \\ 12{,}000 \le a_3 \le 30{,}000, \\ a_1 + a_2 + a_3 = 30{,}000, \\ a_1 \ge 0, a_2 \ge 0, a_3 \ge 0. \end{cases} \tag{P3.3}$$

$max\ 30{,}000(2w_1 + 3w_2 + 4w_3)$

Subject to

$$\begin{cases} \frac{8{,}000}{30{,}000} \leq w_1 \leq 1, \\ \frac{15{,}000}{30{,}000} \leq w_2 \leq 1, \\ \frac{12{,}000}{30{,}000} \leq w_3 \leq 1, \\ w_1 + w_2 + w_3 = 1, \\ w_1 \geq 0, w_2 \geq 0, w_3 \geq 0. \end{cases} \qquad \text{(P3.4)}$$

However, as the total minimum invested amount \$ 8,000+\$ 15,000+\$ 12,000 i.e., \$ 35,000 is greater than the total available amount i.e., \$ 30,000. So, on solving the CLPP (P3.3) or its equivalent CLPP (P3.4), no feasible solution will be obtained.

3.2 Proposed CLPPs

It is obvious from Section 3.1 that the Mehar method, proposed in Section 2.11 of Chapter 2, cannot be used to solve such IFMADMPrs having partially known attribute weights in which either the condition $\sum_{j=1}^{n} w_j^l > 1$ or the condition $\sum_{j=1}^{n} w_j^u < 1$ will be satisfied. To solve such IFMADMPrs, there is need to propose two different CLPPs to evaluate the OAWV. Keeping the same in mind, in this section, a CLPP, corresponding to the condition $\sum_{j=1}^{n} w_j^u < 1$ and a CLPP corresponding to the condition $\sum_{j=1}^{n} w_j^l > 1$, are proposed.

3.2.1. Proposed CLPP corresponding to the first condition

It is obvious from the example, discussed in first point of Section 3.1 that if the total maximum investment is less than the total available amount. Then, there is need to replace the condition “The sum of total investment equal to the total available amount” with the condition “The total investment will be less than or equal to the total available amount” i.e., if the condition $\sum_{j=1}^{n} w_j^u < 1$ will be satisfied then the proposed CLPP (P3.5) should be used to evaluate the OAWV.

$max\left[\sum_{j=1}^{n} c_j w_j\right]$

Subject to

$$\begin{cases} w_j \in H, w_j \geq 0, j = 1,2,..,n, \\ \sum_{j=1}^{n} w_j \leq 1 \\ w_p = w_q \; \forall \; c_p = c_q \,. \end{cases} \qquad \text{(P3.5)}$$

3.2.2. Proposed CLPP corresponding to the second condition

It is obvious from the example, discussed in second point of Section 3.1 that if the total minimum investment is greater than the total available amount. Then, there will not exist any feasible solution for the considered problem.

It is pertinent to mention that to find the feasible solution of such type of problems, there is need to increase the total available amount. If someone is interested to know the minimum extra amount to get the feasible solution of such a problem. Then, it can be obtained by the proposed CLPP (P3.6)

$max\left[\sum_{j=1}^{n} c_j\, w_j - \sum_{j=1}^{n} \varepsilon_j\right]$

Subject to

$$\begin{cases} w_j \in H, w_j \geq 0, j = 1,2,..,n, \\ \sum_{j=1}^{n} w_j = 1, \\ \sum_{j=1}^{n} w_j^l = \text{Total available amount} + \sum_{j=1}^{n} \varepsilon_j \\ w_p = w_q \;\forall\; c_p = c_q\,. \end{cases} \tag{P3.6}$$

where,

(i) $\sum_{j=1}^{n} \varepsilon_j$ represents the total extra amount.

(ii) $\left(\text{Total available amount} + \sum_{j=1}^{n} \varepsilon_j\right)$ represents the total invested amount.

(iii) ε_j represents the percentage of total extra amount invested in j^{th}- attribute.

(iv) w_j represents the percentage of total invested amount in the j^{th}- attribute.

3.3 Results of the considered IFMADMPrs

In Section 3.2, two different IFMADMPrs were considered and pointed out that the solution of the considered problems cannot be obtained by the modified CLPP (P2.12). In this section, the solution of these IFMADMPrs are obtained with the help of the proposed CLPPs.

3.3.1. Result of the first IFMADMPr

It can be easily verified from the first point of Section 3.1 that in the considered IFMADMPr, the condition "Total maximum invested amount is less than the total available amount" is satisfying. So, according to Section 3.2.1, in the CLPP (P3.5), the constraint $a_1 + a_2 + a_3 = 30{,}000$ should be replaced with $a_1 + a_2 + a_3 \leq 30{,}000$ i.e., to find the solution of the considered IFMADMPr, there is need to solve the CLPP (P3.7) or its equivalent CLPP (P3.8) instead of the CLPP (P3.2).

$max(2a_1 + 3a_2 + 4a_3)$

Subject to

$$\begin{cases} 0 \leq a_1 \leq 5{,}000, \\ 0 \leq a_2 \leq 10{,}000, \\ 0 \leq a_3 \leq 8{,}000, \\ a_1 + a_2 + a_3 \leq 30{,}000, \\ a_1 \geq 0, a_2 \geq 0, a_3 \geq 0. \end{cases} \tag{P3.7}$$

$max\ 30{,}000(2w_1 + 3w_2 + 4w_3)$

Subject to

$$\begin{cases} 0 \leq w_1 \leq \frac{5{,}000}{30{,}000}, \\ 0 \leq w_2 \leq \frac{10{,}000}{30{,}000}, \\ 0 \leq w_3 \leq \frac{8{,}000}{30{,}000}, \\ w_1 + w_2 + w_3 \leq 1, \\ w_1 \geq 0, w_2 \geq 0, w_3 \geq 0. \end{cases} \tag{P3.8}$$

where, $w_1 = \frac{a_1}{30{,}000}, w_2 = \frac{a_2}{30{,}000}$ and $w_3 = \frac{a_3}{30{,}000}$.

On solving the CLPP (P3.8), the obtained non-normalized OAWV is $(w_1, w_2, w_3) = (0.1666, 0.3333, 0.2666)$ and the normalized OAWV is $(w_1, w_2, w_3) = \left(\frac{0.1666}{0.1666+0.3333+0.2666}, \frac{0.3333}{0.1666+0.3333+0.2666}, \frac{0.2666}{0.1666+0.3333+0.2666}\right) = (0.2174, 0.4348, 0.3478)$.

3.3.2. Result of the second IFMADMPr

It can be easily verified from the second point of Section 3.1 that in the considered IFMADMPr, the condition “Total minimum invested amount is greater than the total available amount” is satisfying. So, according to Section 3.1, no feasible solution will exist for this IFMADMPr. However, if someone is interested to know that the minimum extra amount required to get the solution of this problem. Then, in the CLPP (P3.4), there is need to increase the invested amount a_j by the quantity $a_j + \varepsilon_j$ in such a manner that $\sum_{j=1}^{n}\left(a_j + \varepsilon_j\right) = \sum_{j=1}^{n} a_j^l$ i.e., to find the solution of the considered problem, there is need to solve the CLPP (P3.9) or its equivalent CLPP (P3.10) instead of the CLPP (P3.4).

$max(2(a_1 + \varepsilon_1) + 3(a_2 + \varepsilon_2) + 4(a_3 + \varepsilon_3) - \varepsilon_1 - \varepsilon_2 - \varepsilon_3)$

Subject to

$$\begin{cases} 8{,}000 \leq (a_1 + \varepsilon_1) \leq 35{,}000, \\ 15{,}000 \leq (a_2 + \varepsilon_2) \leq 35{,}000, \\ 12{,}000 \leq (a_3 + \varepsilon_3) \leq 35{,}000, \\ a_1 + a_2 + a_3 = 30{,}000, \\ (a_1 + \varepsilon_1) + (a_2 + \varepsilon_2) + (a_3 + \varepsilon_3) = 35{,}000 \\ a_1 \geq 0, a_2 \geq 0, a_3 \geq 0. \end{cases} \tag{P3.9}$$

$$max\ \left(35{,}000\left(2\left(\frac{a_1+\varepsilon_1}{35{,}000}\right) + 3\left(\frac{a_2+\varepsilon_2}{35{,}000}\right) + 4\left(\frac{a_3+\varepsilon_3}{35{,}000}\right)\right) - \varepsilon_1 - \varepsilon_2 - \varepsilon_3\right)$$

Subject to

$$\begin{cases} \frac{8{,}000}{35{,}000} \leq \left(\frac{a_1+\varepsilon_1}{35{,}000}\right) \leq 1, \\ \frac{15{,}000}{35{,}000} \leq \left(\frac{a_2+\varepsilon_2}{35{,}000}\right) \leq 1, \\ \frac{12{,}000}{35{,}000} \leq \left(\frac{a_3+\varepsilon_3}{35{,}000}\right) \leq 1, \\ \frac{a_1}{35{,}000} + \frac{a_2}{35{,}000} + \frac{a_3}{35{,}000} = \frac{30{,}000}{35{,}000}, \\ \left(\frac{a_1+\varepsilon_1}{35{,}000}\right) + \left(\frac{a_2+\varepsilon_2}{35{,}000}\right) + \left(\frac{a_3+\varepsilon_3}{35{,}000}\right) = 1 \\ a_1 \geq 0, a_2 \geq 0, a_3 \geq 0. \end{cases} \tag{P3.10}$$

Assuming $\frac{a_1+\varepsilon_1}{35{,}000} = w_1$, $\frac{a_2+\varepsilon_2}{35{,}000} = w_2$ and $\frac{a_3+\varepsilon_3}{35{,}000} = w_3$ i.e., $a_1 = 35{,}000w_1 - \varepsilon_1, a_2 = 35{,}000w_2 - \varepsilon_2$ and $a_3 = 35{,}000w_3 - \varepsilon_3$, the CLPP (P3.10) can be transformed into its equivalent CLPP (P3.11).

$max\ (35{,}000(2w_1 + 3w_2 + 4w_3) - \varepsilon_1 - \varepsilon_2 - \varepsilon_3)$

Subject to

$$\begin{cases} \frac{8,000}{35,000} \leq w_1 \leq 1, \\ \frac{15,000}{35,000} \leq w_2 \leq 1, \\ \frac{12,000}{35,000} \leq w_3 \leq 1, \\ 35,000 = 30,000 + \varepsilon_1 + \varepsilon_2 + \varepsilon_3, \\ w_1 + w_2 + w_3 = 1 \\ w_1 \geq 0, w_2 \geq 0, w_3 \geq 0, \varepsilon_1 \geq 0, \varepsilon_2 \geq 0, \varepsilon_3 \geq 0. \end{cases} \quad (P3.11)$$

On solving the CLPP (P3.11), the obtained normalized OAWV is $(w_1, w_2, w_3) = (0.2286, 0.4286, 0.3428)$ and $(\varepsilon_1, \varepsilon_2, \varepsilon_3) = (0, 2500, 2500)$.

3.4 Proposed Ajit method

In this section, to overcome the limitations of the proposed Mehar method, a new method (named as Ajit method), based upon the proposed CLPPs, has been proposed to solve such IFMADMPrs having partially known attribute weights.

The steps of the proposed Ajit method are as follows:

Step 1: Check that all the attributes are of same type or not i.e., check that all the attributes are benefit type attribute or cost type attribute.

Case (i): If all the attributes are of same type then go to Step 2.

Case (ii): If some attributes are cost type attributes and the remaining are benefit type attributes. Then, convert the j^{th} cost type attribute into the benefit type attribute by replacing all the elements $\tilde{\alpha}_{ij} = \langle \mu_{ij}, \nu_{ij} \rangle$ of the j^{th} column of the IFDM $\widetilde{D} = \left(\tilde{\alpha}_{ij}\right)_{m \times n}$ with $\tilde{\alpha}_{ij} = \langle \nu_{ij}, \mu_{ij} \rangle$ and go to Step 2.

Step 2: Using the expression (3.1), find $d_{ijk}\left(\tilde{\alpha}_{ij}, \tilde{\alpha}_{kj}\right) \; \forall \; i = 1,2,\ldots,m; k = 1,2,\ldots,m;$ $j = 1,2,\ldots,n.$

$$d_{ijk}\left(\tilde{\alpha}_{ij}, \tilde{\alpha}_{kj}\right) = \frac{1}{2}\left(\left|\mu_{ij} - \mu_{kj}\right| + \left|\nu_{ij} - \nu_{kj}\right|\right), i = 1,2,\ldots m;\; j = 1,2,\ldots,n; k = 1,2,\ldots,m. \quad (3.1)$$

Step 3: Using the expression (3.2), find the values of $c_j \; \forall \; j = 1,2,..,n.$

$$c_j = \sum_{i=1}^{m} \sum_{k=1}^{m} d_{ijk}\left(\tilde{\alpha}_{ij}, \tilde{\alpha}_{kj}\right), j = 1,2,\ldots,n. \quad (3.2)$$

Step 4: Check that the condition $\sum_{j=1}^{n} w_j^u < 1$ or the condition $\sum_{j=1}^{n} w_j^l > 1$ is satisfying or not.

Case (i): If the condition $\sum_{j=1}^{n} w_j^u < 1$ is satisfying then find the OAWV, $(w_1, w_2, \dots, w_n)$ by solving the CLPP (P3.5).

Case (ii): If the condition $\sum_{j=1}^{n} w_j^l > 1$ is satisfying then find the OAWV, $(w_1, w_2, \dots, w_n)$ by solving the CLPP (P3.6).

Case (iii): If neither the condition $\sum_{j=1}^{n} w_j^u < 1$ nor the condition $\sum_{j=1}^{n} w_j^l > 1$ is satisfying then find the OAWV, $(w_1, w_2, \dots, w_n)$ by solving the CLPP (P2.12) of Chapter 2.

Step 5: Use Step 5 and Step 6 of the Mehar method, proposed in Section 2.11 of Chapter 2, to find the ranking of the alternatives.

3.5 Illustrative examples

In this section, two IFMADMPrs having partially known attribute weights, are solved to illustrate the proposed Ajit method.

3.5.1. First numerical example

If in the IFMADMPr having partially known attribute weights, considered in the Section 2.12.1 of Chapter 2,

$H = \{0.10 \leq w_1 \leq 0.55, 0.30 \leq w_2 \leq 0.35, w_3 \geq 0.5\}$ is replaced with $H = \{0.10 \leq w_1 \leq 0.20, 0.30 \leq w_2 \leq 0.40, 0.10 \leq w_3 \leq 0.25\}$. Then, the modified IFMADMPr having partially known attribute weights cannot be solved by the proposed Mehar method as the condition $\sum_{j=1}^{n} w_j^u < 1$ is satisfying.

Using the proposed Ajit method, the ranking of the alternatives of the modified IFMADMPr having partially known attribute weights can be obtained as follows:

Since, Step 1 to Step 3 of the proposed Ajit method and Wei's method [223], discussed in Section 2.4 of Chapter 2, are same. Also, as Step 1 to Step 3 of Wei's method [223] for the considered IFMADMPr having partially known attribute weights is discussed in the first point of Section 2.5 of Chapter 2. So, to avoid any repetition, the calculations have been started from Step 4.

Step 4: Since the condition $\sum_{j=1}^{n} w_j^u < 1$ is satisfying. So according to Case (i) of Step 4 of the proposed Ajit method, there is need to solve the CLPP (P3.12).

$max\ (3.1w_1 + 6.9w_2 + 6.3w_3)$

Subject to

$$\begin{cases} 0.10 \le w_1 \le 0.20, \\ 0.30 \le w_2 \le 0.40, \\ 0.10 \le w_3 \le 0.25, \\ w_1 + w_2 + w_3 \le 1, \\ w_1 \ge 0, w_2 \ge 0, w_3 \ge 0. \end{cases} \qquad \text{(P3.12)}$$

On solving the CLPP (P3.12) the obtained non-normalized OAWV is $(w_1, w_2, w_3) = (0.20, 0.40, 0.25)$ and the normalized OAWV is $(w_1, w_2, w_3) = \left(\frac{0.20}{0.20+0.40+0.25}, \frac{0.40}{0.20+0.40+0.25}, \frac{0.25}{0.20+0.40+0.25}\right) = (0.2353, 0.4705, 0.2942)$.

Step 5: On considering the normalized OAWV, $(w_1, w_2, w_3) = (0.2353, 0.4705, 0.2942)$ and using Step 5 of the proposed Ajit method,

$P_1 = \big((0.2353)(0.4) + (0.4705)(0.6) + (0.2942)(0.5)\big) - \big((0.2353)(0.3) + (0.4705)(0.1) + (0.2942)(0.4)\big) = 0.2882,$

$P_2 = \big((0.2353)(0.5) + (0.4705)(0.3) + (0.2942)(0.8)\big) - \big((0.2353)(0.2) + (0.4705)(0.4) + (0.2942)(0.1)\big) = 0.2294,$

$P_3 = \big((0.2353)(0.7) + (0.4705)(0.3) + (0.2942)(0.6)\big) - \big((0.2353)(0.2) + (0.4705)(0.7) + (0.2942)(0.2)\big) = 0.0471,$

$P_4 = \big((0.2353)(0.4) + (0.4705)(0.6) + (0.2942)(0.7)\big) - \big((0.2353)(0.3) + (0.4705)(0.2) + (0.2942)(0.1)\big) = 0.3882,$

$P_5 = \big((0.2353)(0.6) + (0.4705)(0.5) + (0.2942)(0.4)\big) - \big((0.2353)(0.2) + (0.4705)(0.1) + (0.2942)(0.6)\big) = 0.2234,$

$P_6 = \big((0.2353)(0.6) + (0.4705)(0.7) + (0.2942)(0.5)\big) - \big((0.2353)(0.3) + (0.4705)(0.2) + (0.2942)(0.4)\big) = 0.3352.$

Since $P_4 > P_6 > P_1 > P_2 > P_5 > P_3$. Therefore, according to Step 5 of the proposed Ajit method, the ranking of the alternatives is $A_4 > A_6 > A_1 > A_2 > A_5 > A_3$.

3.5.2. Second numerical example

If in the IFMADMPr having partially known attribute weights, considered in the Section 2.12.1 of Chapter 2, $H = \{0.10 \le w_1 \le 0.55, 0.30 \le w_2 \le 0.35, w_3 \ge 0.5\}$ is replaced with $H = \{0.20 \le w_1 \le 0.50, 0.60 \le w_2 \le 0.70, 0.80 \le w_3 \le 0.90\}$. Then, the modified IFMADMPr having partially known attribute weights cannot be solved by the proposed Mehar method as the condition $\sum_{j=1}^{n} w_j^l > 1$ is satisfying.

If it is assumed that the total available amount is \$100 and the minimum required amount is \$160. Then, using the proposed Ajit method the ranking of the alternatives of the modified IFMADMPr having partially known attribute weights can be obtained as follows:

Since, Step 1 to Step 3 of the proposed Ajit method and Wei's method [223], discussed in Section 2.4 of Chapter 2, are same. Also, as Step 1 to Step 3 of Wei's method [223] for the considered IFMADMPr having partially known attribute weights is discussed in the first point of Section 2.5 of Chapter 2. So, to avoid any repetition, the calculations have been started from Step 4.

Step 4: Since the condition $\sum_{j=1}^{n} w_j^l > 1$ is satisfying. So according to Case 1(b) of Step 4 of the proposed Ajit method, there is need to solve the CLPP (P3.13).

$max\ \big((3.1w_1 + 6.9w_2 + 6.3w_3) - \varepsilon_1 - \varepsilon_2 - \varepsilon_3\big)$

Subject to

$$\begin{cases} \frac{20}{160} \le w_1 \le \frac{50}{160}, \\ \frac{60}{160} \le w_2 \le \frac{70}{160}, \\ \frac{80}{160} \le w_3 \le \frac{90}{160}, \\ w_1 + w_2 + w_3 = 1, \\ 160 = 100 + \varepsilon_1 + \varepsilon_2 + \varepsilon_3, \\ w_1 \ge 0, w_2 \ge 0, w_3 \ge 0, \varepsilon_1 \ge 0, \varepsilon_2 \ge 0, \varepsilon_3 \ge 0. \end{cases} \tag{P3.13}$$

On solving the CLPP (P3.13), the obtained normalized OAWV is $(w_1, w_2, w_3) = (0.1250, 0.3750, 0.5000)$.

Step 5: On considering the OAWV $(w_1, w_2, w_3) = (0.1250, 0.3750, 0.5000)$ and using Step 5 of the proposed Ajit method,

$P_1 = \big((0.1250)(0.4) + (0.3750)(0.6) + (0.5000)(0.5)\big) - \big((0.1250)(0.3) + (0.3750)(0.1) + (0.5000)(0.4)\big) = 0.2500$,

$P_2 = \big((0.1250)(0.5) + (0.3750)(0.3) + (0.5000)(0.8)\big) - \big((0.1250)(0.2) + (0.3750)(0.4) + (0.5000)(0.1)\big) = 0.3500$,

$P_3 = \big((0.1250)(0.7) + (0.3750)(0.3) + (0.5000)(0.6)\big) - \big((0.1250)(0.2) + (0.3750)(0.7) + (0.5000)(0.2)\big) = 0.1125$,

$P_4 = \big((0.1250)(0.4) + (0.3750)(0.6) + (0.5000)(0.7)\big) - \big((0.1250)(0.3) + (0.3750)(0.2) + (0.5000)(0.1)\big) = 0.4625$,

$P_5 = \big((0.1250)(0.6) + (0.3750)(0.5) + (0.5000)(0.4)\big) - \big((0.1250)(0.2) + (0.3750)(0.1) + (0.5000)(0.6)\big) = 0.1000$,

$P_6 = \big((0.1250)(0.6) + (0.3750)(0.7) + (0.5000)(0.5)\big) - \big((0.1250)(0.3) + (0.3750)(0.2) + (0.5000)(0.4)\big) = 0.2750$.

Since, $P_4 > P_2 > P_6 > P_1 > P_3 > P_5$. Therefore, according to Step 5 of the proposed Ajit method, the ranking of the alternatives is $A_4 > A_2 > A_6 > A_1 > A_3 > A_5$.

3.6 Conclusions

The limitations of the existing methods [223] as well as the proposed Mehar method have been pointed out. Also, to overcome the limitations, a new method (named as Ajit method) has been proposed to solve such IFMADMPrs having partially known attribute weights. Furthermore, some IFMADMPrs having partially known attribute weights have been solved to illustrate the proposed Ajit method.

Chapter 4

Jujhar method for solving IVIFMADMPrs having partially known attribute weights[3]

Wang et al. [213] claimed that there does not exist any method for solving IVIFMADMPrs having partially known attribute weights. To fill this gap, Wang et al. proposed a method to solve the IVIFMADMPrs having partially known attribute weights. Motivated by the work of Wang et al., several researchers [62, 108, 258] have proposed different methods for solving IVIFMADMPrs having partially known attribute weights. However, after a deep study it is observed that it is inappropriate to use the existing methods [62, 108, 213, 258]. The aim of this chapter is to make the researchers aware about the inappropriateness of these methods as well as to propose a new method (named as Jujhar method) to solve the IVIFMADMPrs having partially known attribute weights.

4.1 A brief review of Wang et al.'s method

Wang et al. [213] claimed that there does not exist any method for solving such IVIFMADMPrs having partially known attribute weights in which the rating value of the i^{th}-alternative over the j^{th}-attribute is represented by an IVIFS $\tilde{\alpha}_{ij} = ([a_{ij1}, a_{ij2}], [a_{ij3}, a_{ij4}])$ as well as the weight of the j^{th}-attribute $w_j \in H$, where $H = \{w_j \colon w_j \geq w_i, w_j - w_i \geq \beta_j (> 0), w_j \geq \beta_j w_i; 0 \leq \beta_j \leq 1, 0 \leq \delta_j \leq w_j \leq \delta_j + \varepsilon_j \leq 1;\ w_j - w_i \geq w_k - w_l$ for $i \neq j \neq k \neq l\}$.

Wang et al. [213] proposed the following method to solve IVIFMADMPrs having partially

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[3] The contents of this chapter have been communicated in "Engineering Applications of Artificial Intelligence" for the possible publication.

known attribute weights:

Step 1: Check that all the attributes are of same type or not i.e., check that all the attributes are benefit type attribute or cost type attribute.

Case (i): If all the attributes are of same type then go to Step 2.

Case (ii): If some attributes are cost type attributes and the remaining are benefit type attributes then convert the j^{th} cost type attribute into the benefit type attribute by replacing all the elements $\tilde{\alpha}_{ij} = \left([a_{ij1}, a_{ij2}], [a_{ij3}, a_{ij4}]\right)$ of the j^{th} column of the IVIFDM $\widetilde{D} = \left(\tilde{\alpha}_{ij}\right)_{m\times n}$ with $\tilde{\alpha}_{ij} = \left([a_{ij3}, a_{ij4}], [a_{ij1}, a_{ij2}]\right)$ and go to Step 2.

Step 2: Using the expression (4.1), transform each IVIF element $\tilde{\alpha}_{ij} = \left([a_{ij1}, a_{ij2}], [a_{ij3}, a_{ij4}]\right)$ of the IVIFDM $\widetilde{D} = \left(\tilde{\alpha}_{ij}\right)_{m\times n} = \left([a_{ij1}, a_{ij2}], [a_{ij3}, a_{ij4}]\right)$ into the crisp element d_{ij}

$$d_{ij} = \left|2 - a_{ij1} - a_{ij2} - a_{ij3} - a_{ij4}\right|. \tag{4.1}$$

Step 3: Using the expression (4.2), find the value of $c_j\ \forall\ j = 1,2,\ldots,n$.

$$c_j = \sum_{i=1}^{m} d_{ij}\,, j = 1,2,\ldots,n \tag{4.2}$$

Step 4: Find the OAWV, $(w_1, w_2, \ldots, w_n)$ by solving the CLPP (P4.1).

$$max\left[\frac{\sum_{j=1}^{n} c_j w_j}{n}\right]$$

Subject to

$$w_j \in H, w_j \geq 0, j = 1,2,\ldots,n,$$

$$\sum_{j=1}^{n} w_j = 1. \tag{P4.1}$$

Step 5: Using the expression (4.3), find the value of $\tilde{P}_i\ \forall\ i = 1,2,\ldots,m$.

$$\tilde{P}_i = \left(\left[\sum_{j=1}^{n} w_j a_{ij1}\,, \sum_{j=1}^{n} w_j a_{ij2}\right], \left[\sum_{j=1}^{n} w_j a_{ij3}\,, \sum_{j=1}^{n} w_j a_{ij4}\right]\right), i = 1,2,\ldots,m \tag{4.3}$$

Step 6: Check that $S(\tilde{P}_p) > S(\tilde{P}_q)$ or $S(\tilde{P}_p) < S(\tilde{P}_q)$ or $S(\tilde{P}_p) = S(\tilde{P}_q)$, where $S(\tilde{P}_p) = \frac{a_{pj1}+a_{pj2}-a_{pj3}-a_{pj4}}{2}$ and $S(\tilde{P}_q) = \frac{a_{qj1}+a_{qj2}-a_{qj3}-a_{qj4}}{2}$.

Case (i): If $S(\tilde{P}_p) > S(\tilde{P}_q)$ then the ranking of the alternatives is $A_p > A_q$.

Case (ii): If $S(\tilde{P}_p) < S(\tilde{P}_q)$ then the ranking of the alternatives is $A_p < A_q$.

Case (iii): If $S(\tilde{P}_p) = S(\tilde{P}_q)$ then go to Step 7.

Step 7: Check that $H(\tilde{P}_p) > H(\tilde{P}_q)$ or $H(\tilde{P}_p) < H(\tilde{P}_q)$ or $H(\tilde{P}_p) = H(\tilde{P}_q)$, where $H(\tilde{P}_p) = \frac{a_{pj1}+a_{pj2}+a_{pj3}+a_{pj4}}{2}$ and $H(\tilde{P}_q) = \frac{a_{qj1}+a_{qj2}+a_{qj3}+a_{qj4}}{2}$.

Case (i): If $H(\tilde{P}_p) > H(\tilde{P}_q)$ then the ranking of the alternatives is $A_p > A_q$.

Case (ii): If $H(\tilde{P}_p) < H(\tilde{P}_q)$ then the ranking of the alternatives is $A_p < A_q$.

Case (iii): If $H(\tilde{P}_p) = H(\tilde{P}_q)$ then go to Step 8.

Step 8: Check that $T(\tilde{P}_p) > T(\tilde{P}_q)$ or $T(\tilde{P}_p) < T(\tilde{P}_q)$ or $T(\tilde{P}_p) = T(\tilde{P}_q)$, where $T(\tilde{P}_p) = a_{pj2} + a_{pj3} - a_{pj1} - a_{pj4}$ and $T(\tilde{P}_q) = a_{qj2} + a_{qj3} - a_{qj1} - a_{qj4}$.

Case (i): If $T(\tilde{P}_p) > T(\tilde{P}_q)$ then the ranking of the alternatives is $A_p < A_q$.

Case (ii): If $T(\tilde{P}_p) < T(\tilde{P}_q)$ then the ranking of the alternatives is $A_p > A_q$.

Case (iii): If $T(\tilde{P}_p) = T(\tilde{P}_q)$ then go to Step 9.

Step 9: Check that $G(\tilde{P}_p) > G(\tilde{P}_q)$ or $G(\tilde{P}_p) < G(\tilde{P}_q)$ or $G(\tilde{P}_p) = G(\tilde{P}_q)$, where $G(\tilde{P}_p) = a_{pj2} + a_{pj4} - a_{pj1} - a_{pj3}$ and $G(\tilde{P}_q) = a_{qj2} + a_{qj4} - a_{qj1} - a_{qj3}$.

Case (i): If $G(\tilde{P}_p) > G(\tilde{P}_j)$ then the ranking of the alternatives is $A_p < A_q$.

Case (ii): If $G(\tilde{P}_p) < G(\tilde{P}_j)$ then the ranking of the alternatives is $A_p > A_q$.

Case (iii): If $G(\tilde{P}_p) = G(\tilde{P}_j)$ then the ranking of the alternatives is $A_p = A_q$.

4.2 Inappropriateness of Wang et al.'s method

It is inappropriate to apply Wang et al.'s method [213] due to the following reasons:

(1) On applying Wang et al.'s method [213] more than one preference orders of the alternatives are obtained, which is inappropriate. To validate this claim a real-life IVIFMADMPr having partially known attribute weights, considered by Wang et al. [213] to illustrate his

proposed method [213], is solved with some modifications.

Wang et al. [213] solved the following real-life IVIFMADMPr having partially known attribute weights to illustrate his proposed method.

There is need to rank the faculty members $A_i, i = 1,2,\dots,4$ on the basis of the following four benefit attributes.

(i) G_1: Teaching

(ii) G_2: Research

(iii) G_3: Ability

(iv) G_4: Service

The $(i,j)^{th}$ element of Table 4.1, represented by an IVIFS $\tilde{\alpha}_{ij} = \left([a_{ij1}, a_{ij2}], [a_{ij3}, a_{ij4}]\right)$, represents the rating value of the i^{th} faculty member over the j^{th} attribute.

Table 4.1: Rating Values

Attributes→ ↓ Alternatives	G_1	G_2	G_3	G_4
A_1	$\begin{pmatrix}[0.42,0.48],\\ [0.40,0.50]\end{pmatrix}$	$\begin{pmatrix}[0.60,0.70],\\ [0.05,0.25]\end{pmatrix}$	$\begin{pmatrix}[0.40,0.50],\\ [0.20,0.50]\end{pmatrix}$	$\begin{pmatrix}[0.55,0.75],\\ [0.15,0.25]\end{pmatrix}$
A_2	$\begin{pmatrix}[0.40,0.50],\\ [0.40,0.50]\end{pmatrix}$	$\begin{pmatrix}[0.50,0.80],\\ [0.10,0.20]\end{pmatrix}$	$\begin{pmatrix}[0.30,0.60],\\ [0.30,0.40]\end{pmatrix}$	$\begin{pmatrix}[0.60,0.70],\\ [0.10,0.30]\end{pmatrix}$
A_3	$\begin{pmatrix}[0.30,0.50],\\ [0.40,0.50]\end{pmatrix}$	$\begin{pmatrix}[0.10,0.30],\\ [0.20,0.40]\end{pmatrix}$	$\begin{pmatrix}[0.70,0.80],\\ [0.10,0.20]\end{pmatrix}$	$\begin{pmatrix}[0.50,0.70],\\ [0.10,0.20]\end{pmatrix}$
A_4	$\begin{pmatrix}[0.20,0.40],\\ [0.40,0.50]\end{pmatrix}$	$\begin{pmatrix}[0.60,0.70],\\ [0.20,0.30]\end{pmatrix}$	$\begin{pmatrix}[0.50,0.60],\\ [0.20,0.30]\end{pmatrix}$	$\begin{pmatrix}[0.70,0.80],\\ [0.10,0.20]\end{pmatrix}$

Also, $H = \{0.15 \le w_1 \le 0.30, 0.15 \le w_2 \le 0.25, 0.25 \le w_3 \le 0.40, 0.30 \le w_4 \le 0.45, 2.5w_1 \le w_3\}$.

If

(i) Table 4.1 is replaced with Table 4.2

Table 4.2: Rating Values

Attributes→ ↓ Alternatives	G_1	G_2	G_3	G_4
A_1	$\begin{pmatrix}[0.10,0.20],\\ [0.30,0.40]\end{pmatrix}$	$\begin{pmatrix}[0.20,0.30],\\ [0.20,0.30]\end{pmatrix}$	$\begin{pmatrix}[0.10,0.40],\\ [0.20,0.30]\end{pmatrix}$	$\begin{pmatrix}[0.10,0.50],\\ [0.10,0.30]\end{pmatrix}$
A_2	$\begin{pmatrix}[0.30,0.40],\\ [0.10,0.20]\end{pmatrix}$	$\begin{pmatrix}[0.20,0.40],\\ [0.10,0.30]\end{pmatrix}$	$\begin{pmatrix}[0.20,0.30],\\ [0.10,0.40]\end{pmatrix}$	$\begin{pmatrix}[0.10,0.50],\\ [0.20,0.20]\end{pmatrix}$
A_3	$\begin{pmatrix}[0.30,0.40],\\ [0.10,0.20]\end{pmatrix}$	$\begin{pmatrix}[0.20,0.20],\\ [0.10,0.50]\end{pmatrix}$	$\begin{pmatrix}[0.10,0.20],\\ [0.30,0.40]\end{pmatrix}$	$\begin{pmatrix}[0.10,0.40],\\ [0.20,0.30]\end{pmatrix}$
A_4	$\begin{pmatrix}[0.10,0.30],\\ [0.10,0.50]\end{pmatrix}$	$\begin{pmatrix}[0.10,0.20],\\ [0.30,0.40]\end{pmatrix}$	$\begin{pmatrix}[0.50,0.50],\\ [0.10,0.40]\end{pmatrix}$	$\begin{pmatrix}[0.20,0.30],\\ [0.20,0.30]\end{pmatrix}$

(ii) The existing condition $2.5w_1 \leq w_3$ is removed.

Then, using Wang et al.'s method [213], the ranking of the alternatives of the modified IVIFMADMPr having partially known attribute weights can be obtained as follows:

Step 1: Since, all the attributes are of benefit type. So, according to Step 1 of Wang et al.'s method [213], discussed in Section 4.1, there is no need to apply Step 1.

Step 2: According to Step 2 of Wang et al.'s method [213], discussed in Section 4.1, there is need to calculate d_{ij} $\forall\ i = 1,2,3,4; j = 1,2,3,4$. These values are shown in Table 4.2.

Table 4.2: Values of d_{ij}

$d_{11} = 1$	$d_{12} = 1$	$d_{13} = 1$	$d_{14} = 1$
$d_{21} = 1$	$d_{22} = 1$	$d_{23} = 1$	$d_{24} = 1$
$d_{31} = 1$	$d_{32} = 1$	$d_{33} = 1$	$d_{34} = 1$

$d_{41} = 1$	$d_{42} = 1$	$d_{43} = 1$	$d_{44} = 1$

Step 3: Using Step 3 of Wang et al.'s method [213], discussed in Section 4.1,

$c_1 = \sum_{i=1}^{4} d_{i1} = 1 + 1 + 1 + 1 \ = 4,$

$c_2 = \sum_{i=1}^{4} d_{i2} = 1 + 1 + 1 + 1 \ = 4,$

$c_3 = \sum_{i=1}^{4} d_{i3} = 1 + 1 + 1 + 1 \ = 4,$

$c_4 = \sum_{i=1}^{4} d_{i4} = 1 + 1 + 1 + 1 \ = 4.$

Step 4: Using Step 4 of Wang et al.'s method [213], discussed in Section 4.1, there is need to solve the CLPP (P4.2).

$Max\left(\frac{4w_1+4w_2+4w_3+4w_4}{4}\right)$

Subject to

$$\begin{cases} 0.15 \le w_1 \le 0.30, \\ 0.15 \le w_2 \le 0.25, \\ 0.25 \le w_3 \le 0.40, \\ 0.30 \le w_4 \le 0.45, \\ w_1 + w_2 + w_3 + w_4 = 1, \\ w_1 \ge 0, w_2 \ge 0, w_3 \ge 0, w_4 \ge 0. \end{cases} \qquad \text{(P4.2)}$$

It can be easily verified that on solving the CLPP (P4.2) infinite number of OAWV are obtained e.g., $(w_1, w_2, w_3, w_4) = (0.15,0.15,0.40,0.30)$ and $(w_1, w_2, w_3, w_4) = (0.25,0.20,0.25,0.30)$ both are the OAWV.

Step 5: On considering the OAWV, $(w_1, w_2, w_3, w_4) = (0.15,0.15,0.40,0.30)$ and using Step 5 of Wang et al.'s method [213], discussed in Section 4.1,

$\tilde{P}_1 = ([(0.15)(0.10) + (0.15)(0.20) + (0.40)(0.10) + (0.30)(0.10), (0.15)(0.20) +$

$(0.15)(0.30) + (0.40)(0.40) + (0.30)(0.50)], [(0.15)(0.30) + (0.15)(0.20) + (0.40)$

$(0.20) + (0.30)(0.10), (0.15)(0.40) + (0.15)(0.30) + (0.40)(0.30) + (0.30)(0.30)]$

$= ([0.115,0.385], [0.185,0.315]),$

$\tilde{P}_2 = ([(0.15)(0.30) + (0.15)(0.20) + (0.40)(0.20) + (0.30)(0.10), (0.15)(0.40) +$

$(0.15)(0.40) + (0.40)(0.30) + (0.30)(0.50)], [(0.15)(0.10) + (0.15)(0.10) + (0.40)$
$(0.10) + (0.30)(0.20), (0.15)(0.20) + (0.15)(0.30) + (0.40)(0.40) + (0.30)(0.20)]$
$= ([0.185, 0.390], [0.130, 0.280]),$

$\tilde{P}_3 = ([(0.15)(0.30) + (0.15)(0.20) + (0.40)(0.10) + (0.30)(0.10), (0.15)(0.40) +$
$(0.15)(0.20) + (0.40)(0.20) + (0.30)(0.40)], [(0.15)(0.10) + (0.15)(0.10) + (0.40)$
$(0.30) + (0.30)(0.20), (0.15)(0.20) + (0.15)(0.50) + (0.40)(0.30) + (0.30)(0.30)]$
$= ([0.145, 0.290], [0.210, 0.315]),$

$\tilde{P}_4 = ([(0.15)(0.10) + (0.15)(0.10) + (0.40)(0.50) + (0.30)(0.20), (0.15)(0.30) +$
$(0.15)(0.20) + (0.40)(0.50) + (0.30)(0.30)], [(0.15)(0.10) + (0.15)(0.30) + (0.40)$
$(0.10) + (0.30)(0.20), (0.15)(0.50) + (0.15)(0.40) + (0.40)(0.40) + (0.30)(0.30)]$
$= ([0.290, 0.365], [0.160, 0.385]).$

While, on considering the OAWV, $(w_1, w_2, w_3, w_4) = (0.25, 0.20, 0.25, 0.30)$ and using Step 5 of Wang et al.'s method [213], discussed in Section 4.1,

$\tilde{P}_1 = ([(0.25)(0.10) + (0.20)(0.20) + (0.25)(0.10) + (0.30)(0.10), (0.25)(0.20) +$
$(0.20)(0.30) + (0.25)(0.40) + (0.30)(0.50)], [(0.25)(0.30) + (0.20)(0.20) + (0.25)$
$(0.20) + (0.30)(0.10), (0.25)(0.40) + (0.20)(0.30) + (0.25)(0.30) + (0.30)(0.30)]$
$= ([0.120, 0.360], [0.195, 0.325]),$

$\tilde{P}_2 = ([(0.25)(0.30) + (0.20)(0.20) + (0.25)(0.20) + (0.30)(0.10), (0.25)(0.40) +$
$(0.20)(0.40) + (0.25)(0.30) + (0.30)(0.50)], [(0.25)(0.10) + (0.20)(0.10) + (0.25)$
$(0.10) + (0.30)(0.20), (0.25)(0.20) + (0.20)(0.30) + (0.25)(0.40) + (0.30)(0.20)]$
$= ([0.195, 0.405], [0.130, 0.270]),$

$\tilde{P}_3 = ([(0.25)(0.30) + (0.20)(0.20) + (0.25)(0.10) + (0.30)(0.10), (0.25)(0.40) +$
$(0.20)(0.20) + (0.25)(0.20) + (0.30)(0.40)], [(0.25)(0.10) + (0.20)(0.10) + (0.25)$
$(0.30) + (0.30)(0.20), (0.25)(0.20) + (0.20)(0.50) + (0.25)(0.30) + (0.30)(0.30)]$
$= ([0.170, 0.310], [0.180, 0.340]),$

$\tilde{P}_4 = ([(0.25)(0.10) + (0.20)(0.10) + (0.25)(0.50) + (0.30)(0.20), (0.25)(0.30) + (0.20)(0.20) + (0.25)(0.50) + (0.30)(0.30)], [(0.25)(0.10) + (0.20)(0.30) + (0.25)(0.10) + (0.30)(0.20), (0.25)(0.50) + (0.20)(0.40) + (0.25)(0.40) + (0.30)(0.30)]$
$= ([0.230, 0.330], [0.170, 0.395])$.

Step 6: Using Step 6, of Wang et al.'s method [213], discussed in Section 4.1,

(i) Corresponding to the OAWV, $(w_1, w_2, w_3, w_4) = (0.15, 0.15, 0.40, 0.30)$, $S(\tilde{P}_1) = 0, S(\tilde{P}_2) = 0.0825, S(\tilde{P}_3) = -0.045, S(\tilde{P}_4) = 0.055$. Since, $S(\tilde{P}_2) > S(\tilde{P}_4) > S(\tilde{P}_1) > S(\tilde{P}_3)$. So, according to Step 6 of Wang et al.'s method [213], discussed in Section 4.1, $A_2 > A_4 > A_1 > A_3$.

(ii) Corresponding to the OAWV, $(w_1, w_2, w_3, w_4) = (0.25, 0.20, 0.25, 0.30)$, $S(\tilde{P}_1) = -0.02, S(\tilde{P}_2) = 0.10, S(\tilde{P}_3) = -0.02, S(\tilde{P}_4) = -0.0025$. Since, $S(\tilde{P}_1) = S(\tilde{P}_3)$. Therefore, according to Case (iii) of Step 6 of Wang et al.'s method [213], discussed in Section 4.1, there is need to apply Step 7.

Step 7: Using Step 7 of Wang et al.'s method [213], discussed in Section 4.1,
$H(\tilde{P}_1) = 0.50$ and $H(\tilde{P}_3) = 0.50$. Since, $H(\tilde{P}_1) = H(\tilde{P}_3)$. Therefore, according to Case (iii) of Step 7 of Wang et al.'s method [213], discussed in Section 4.1, there is need to apply Step 8.

Step 8: Using Step 8 of Wang et al.'s method [213], discussed in Section 4.1,
$T(\tilde{P}_1) = 0.02$ and $T(\tilde{P}_3) = 0.02$. Since, $T(\tilde{P}_1) = T(\tilde{P}_3)$. Therefore, according to Case (iii) of Step 8 of Wang et al.'s method [213], discussed in Section 4.1, there is need to apply Step 9.

Step 9: Using Step 9 of Wang et al.'s method [213], discussed in Section 4.1,
$G(\tilde{P}_1) = 0.185$ and $G(\tilde{P}_3) = 0.15$. Since, $G(\tilde{P}_1) > G(\tilde{P}_3)$ and $S(\tilde{P}_2) > S(\tilde{P}_4)$. Therefore, according to Wang et al.'s method [213], discussed in Section 4.1, $A_3 > A_1 > A_2 > A_4$.

It is obvious that on applying Wang et al.'s method [213] two different ranking of the alternatives are obtained for the same IVIFMADMPr having partially known attribute weights, which is inappropriate. Hence, it is inappropriate to use Wang et al.'s method [213] to solve IVIFMADMPrs having partially known attribute weights.

(2) Wang et al.'s method [213] fails to find the ranking of the alternatives. The following validates this claim.

(i) If the set $H = \{0.15 \leq w_1 \leq 0.30, 0.15 \leq w_2 \leq 0.25, 0.25 \leq w_3 \leq 0.40, 0.30 \leq w_4 \leq 0.45\}$ is replaced with $H = \{0.10 \leq w_1 \leq 0.20, 0.30 \leq w_2 \leq 0.40, 0.10 \leq w_3 \leq 0.25, 0 \leq w_4 \leq 0.05\}$. Then, no feasible solution of the CLPP (P4.2) will be obtained as the condition $\sum_{j=1}^{n} w_j^u < 1$ is satisfying i.e., the ranking of the alternatives of the modified problem cannot be obtained by Wang et al.'s method [213].

(ii) If the set $H = \{0.15 \leq w_1 \leq 0.30, 0.15 \leq w_2 \leq 0.25, 0.25 \leq w_3 \leq 0.40, 0.30 \leq w_4 \leq 0.45\}$ is replaced with $H = \{0.30 \leq w_1 \leq 0.50, 0.40 \leq w_2 \leq 0.70, 0.50 \leq w_3 \leq 0.70, 0.20 \leq w_4 \leq 0.60\}$. Then, no feasible solution of the CLPP (P4.2) will be obtained as the condition $\sum_{j=1}^{n} w_j^l > 1$ is satisfying i.e., the ranking of the alternatives of the modified problem cannot be obtained by Wang et al.'s method [213].

4.3 Reasons for the inappropriateness of Wang et al.'s method

In Section 4.2, it is shown that

(i) On applying Wang et al.'s method [213], more than one preference order for the alternatives are obtained.

(ii) Wang et al.'s method [213] fails to find the ranking of the alternatives if either the condition $\sum_{j=1}^{n} w_j^u < 1$ or the condition $\sum_{j=1}^{n} w_j^l > 1$ will be satisfied.

These problems are occurring due to following reasons:

(i) If there will exist two distinct attributes G_p and G_q such that $c_p = c_q$ i.e., $\sum_{i=1}^{m} d_{ip} = \sum_{i=1}^{m} d_{iq}$ then the coefficient of the attribute weight w_p and w_q will be equal in the CLPP (P4.2). Therefore, all such values of w_p and w_q, corresponding to which $w_p + w_q$ will be optimal, will represent the optimal values of w_p and w_q. Due to the same reason, on solving the CLPP (P4.2) corresponding to the IVIFMADMPr having partially known attribute weights, considered in the first point of Section 4.2, the number of obtained OAWV are more than one.

(ii) It is pertinent to mention that to apply Wang et al.'s method [213], there is need to solve the CLPP (P4.1). However, a feasible solution of the CLPP (P4.1) will exist only if neither the condition $\sum_{j=1}^{n} w_j^u < 1$ nor the condition $\sum_{j=1}^{n} w_j^l > 1$ will be satisfied. Due to the same reason, on solving the CLPP (P4.2) corresponding to the IVIFMADMPr having partially known attribute weights, considered in the second point of Section 4.2, infeasible solution is obtained .

4.4 A brief review of Ye's method

Ye [258] proposed the following method to solve IVIFMADMPrs having partially known attribute weights.

Step 1: Check that all the attributes are of same type or not i.e., check that all the attributes are benefit type attribute or cost type attribute.

Case (i): If all the attributes are of same type then go to Step 2.

Case (ii): If some attributes are cost type attributes and the remaining are benefit type attributes then convert the j^{th} cost type attribute into the benefit type attribute by replacing all the elements $\tilde{\alpha}_{ij} = ([a_{ij1}, a_{ij2}], [a_{ij3}, a_{ij4}])$ of the j^{th} column of the IVIFDM $\widetilde{D} = (\tilde{\alpha}_{ij})_{m\times n}$ with $\tilde{\alpha}_{ij} = ([a_{ij3}, a_{ij4}], [a_{ij1}, a_{ij2}])$ and go to Step 2.

Step 2: Using the expression (4.4), transform each IVIF element $\tilde{\alpha}_{ij} = ([a_{ij1}, a_{ij2}], [a_{ij3}, a_{ij4}])$ of the IVIFDM $\widetilde{D} = (\tilde{\alpha}_{ij})_{m\times n} = ([a_{ij1}, a_{ij2}], [a_{ij3}, a_{ij4}])$ into the crisp element d_{ij}

$$d_{ij} = \left(\frac{(a_{ij1}-a_{ij3})(2-a_{ij1}-a_{ij3})+(a_{ij2}-a_{ij4})(2-a_{ij2}-a_{ij4})}{2}\right) \tag{4.4}$$

Step 3: Using the expression (4.5), find the value of $c_j \; \forall \; j = 1,2,\dots,n$.

$$c_j = \sum_{i=1}^{m} d_{ij} \,, j = 1,2,\dots,n \tag{4.5}$$

Step 4: Find the OAWV, $(w_1, w_2, \dots, w_n)$ by solving the CLPP (P4.3).

$max\left[\sum_{j=1}^{n} c_j \, w_j\right]$

Subject to

$w_j \in H, w_j \geq 0, j = 1,2,\dots,n,$

$$\sum_{j=1}^{n} w_j = 1. \tag{P4.3}$$

Step 5: Using the expression (4.6), find the value of $P_i \; \forall \; i = 1,2,\dots,m$.

$$P_i = \sum_{j=1}^{n} w_j \, d_{ij}, i = 1,2,\dots,m \tag{4.6}$$

and check that $P_p > P_q$ or $P_p < P_q$ or $P_p = P_q$, where $P_p = \sum_{j=1}^{n} w_j \, d_{pj}$ and $P_q = \sum_{j=1}^{n} w_j \, d_{qj}$.

Case (i): If $P_p > P_q$ then the ranking of the alternatives is $A_p > A_q$.

Case (ii): If $P_p < P_q$ then the ranking of the alternatives is $A_p < A_q$.

Case (iii): If $P_p = P_q$ then the ranking of the alternatives is $A_p = A_q$.

4.5 Inappropriateness of Ye's method

It is inappropriate to apply Ye's method [258] due to the following reasons:

(1) On applying Ye's method [258] more than one preference orders of the alternatives are obtained, which is inappropriate. To validate this claim a real-life IVIFMADMPr having partially known attribute weights, considered by Ye [258] to illustrate his proposed method, is solved with some modifications.

Ye [258] solved the following real-life IVIFMADMPr having partially known attribute weights to illustrate his proposed method.

There is a panel of four alternatives for the investment company to invest money in best option $A_i, i = 1,2,3,4$. The investment company must take a decision according to the following three attribute.

(i) G_1: Risk analysis

(ii) G_2: Growth

(iii) G_3: Environmental impact

The $(i,j)^{th}$ element of the Table 4.3, represented by an IVIFS $\tilde{\alpha}_{ij} = ([a_{ij1}, a_{ij2}], [a_{ij3}, a_{ij4}])$, represents the rating value of the i^{th} company over the j^{th} attribute.

Table 4.3: Rating Values

Attributes→ ↓ Alternatives	G_1	G_2	G_3
A_1	$([0.40,0.50],[0.30,0.40])$	$([0.40,0.60],[0.20,0.40])$	$([0.10,0.30],[0.50,0.60])$
A_2	$([0.60,0.70],[0.20,0.30])$	$([0.60,0.70],[0.20,0.30])$	$([0.40,0.70],[0.10,0.20])$
A_3	$([0.30,0.60],[0.30,0.40])$	$([0.50,0.60],[0.30,0.40])$	$([0.50,0.60],[0.10,0.30])$
A_4	$([0.70,0.80],[0.10,0.20])$	$([0.60,0.70],[0.10,0.30])$	$([0.30,0.40],[0.10,0.20])$

Also, $H = \{0.20 \le w_1 \le 0.30, 0.20 \le w_2 \le 0.30, 0.30 \le w_3 \le 0.50\}$.

If

(i) Table 4.3 is replaced with Table 4.4

Table 4.4: Rating Values

Attributes→ ↓ Alternatives	G_1	G_2	G_3
A_1	$([0.40,0.50],[0.30,0.40])$	$([0.40,0.60],[0.20,0.40])$	$([0.30,0.60],[0.30,0.40])$
A_2	$([0.60,0.70],[0.20,0.30])$	$([0.60,0.70],[0.20,0.30])$	$([0.40,0.50],[0.30,0.40])$

A_3	$([0.30,0.60],[0.30,0.40])$	$([0.50,0.60],[0.30,0.40])$	$([0.70,0.80],[0.10,0.20])$
A_4	$([0.70,0.80],[0.10,0.20])$	$([0.60,0.70],[0.10,0.30])$	$([0.60,0.70],[0.20,0.30])$

(ii) It is assumed that the attribute weight w_1 satisfies the condition $0.20 \leq w_1 \leq 0.50$ instead of the existing condition $0.20 \leq w_1 \leq 0.30$.

Then, using Ye's method [258], the ranking of the alternatives of the modified IVIFMADMPr having partially known attribute weights can be obtained as follows:

Step 1: Since, all the attributes are of benefit type. So, according to Step 1 of Ye's method [258], discussed in Section 4.4, there is no need to apply Step 1.

Step 2: According to Step 2 of Ye's method [258], discussed in Section 4.4, there is need to calculate $d_{ij}\ \forall\ i = 1,2,\ldots,4; j = 1,2,3$. These values are shown in Table 4.5.

Table 4.5: Values of d_{ij}

$d_{11} = 0.12$	$d_{12} = 0.24$	$d_{13} = 0.10$
$d_{21} = 0.44$	$d_{22} = 0.44$	$d_{23} = 0.12$
$d_{31} = 0.10$	$d_{32} = 0.22$	$d_{33} = 0.66$
$d_{41} = 0.66$	$d_{42} = 0.525$	$d_{43} = 0.44$

Step 3: Using Step 3 of Ye's method [258], discussed in Section 4.4,

$c_1 = \sum_{i=1}^{4} d_{i1} = 0.12 + 0.44 + 0.10 + 0.66 = 1.32,$

$c_2 = \sum_{i=1}^{4} d_{i2} = 0.24 + 0.44 + 0.22 + 0.525 = 1.425,$

$c_3 = \sum_{i=1}^{4} d_{i3} = 0.10 + 0.12 + 0.66 + 0.44 = 1.32.$

Step 4: Using Step 4 of Ye's method [258], discussed in Section 4.4, there is need to solve the CLPP (P4.4).

$Max(1.32w_1 + 1.425w_2 + 1.32w_3)$

Subject to

$$\begin{cases} 0.20 \leq w_1 \leq 0.50, \\ 0.20 \leq w_2 \leq 0.30, \\ 0.30 \leq w_3 \leq 0.50, \\ w_1 + w_2 + w_3 = 1, \\ w_1 \geq 0, w_2 \geq 0, w_3 \geq 0. \end{cases} \tag{P4.4}$$

It can be easily verified that on solving the CLPP (P4.4) infinite number of OAWV are obtained e.g., $(w_1, w_2, w_3) = (0.20, 0.30, 0.50)$ and $(w_1, w_2, w_3) = (0.40, 0.30, 0.30)$ both are the OAWV.

Step 5: On considering the OAWV, $(w_1, w_2, w_3) = (0.20, 0.30, 0.50)$ and using Step 5 of Ye's method [258], discussed in Section 4.4,

$P_1 = (0.20)(0.12) + (0.30)(0.24) + (0.50)(0.10) = 0.1460$,

$P_2 = (0.20)(0.44) + (0.30)(0.44) + (0.50)(0.12) = 0.2800$,

$P_3 = (0.20)(0.10) + (0.30)(0.22) + (0.50)(0.66) = 0.4160$,

$P_4 = (0.20)(0.66) + (0.30)(0.525) + (0.50)(0.44) = 0.5095$.

Since, $P_4 > P_3 > P_2 > P_1$. So, according to Step 5 of Ye's method [258], discussed in Section 4.4, the ranking of the alternatives is $A_4 > A_3 > A_2 > A_1$.

While, on considering the OAWV, $(w_1, w_2, w_3) = (0.40, 0.30, 0.30)$ and using Step 5 of Ye's method [258], discussed in Section 4.4,

$P_1 = (0.40)(0.12) + (0.30)(0.24) + (0.30)(0.10) = 0.1500$,

$P_2 = (0.40)(0.44) + (0.30)(0.44) + (0.30)(0.12) = 0.3400$,

$P_3 = (0.40)(0.10) + (0.30)(0.22) + (0.30)(0.66) = 0.3040$,

$P_4 = (0.40)(0.66) + (0.30)(0.525) + (0.40)(0.44) = 0.5535$.

Since, $P_4 > P_2 > P_3 > P_1$. So, according to Step 5 of Ye's method [258], discussed in Section 4.4, the ranking of the alternatives is $A_4 > A_2 > A_3 > A_1$.

It is obvious that on applying Ye's method [258] two different ranking of the alternatives are obtained for the same IVIFMADMPr having partially known attribute weights, which is mathematically incorrect. Hence, it is inappropriate to use Ye's method [258] to solve IVIFMADMPrs having partially known attribute weights.

(2) Ye's method [258] fails to find the ranking of the alternatives. The following clearly validates this claim.

(i) If the set $H = \{0.20 \leq w_1 \leq 0.50, 0.20 \leq w_2 \leq 0.30, 0.30 \leq w_3 \leq 0.50\}$ is replaced with $H = \{0.10 \leq w_1 \leq 0.20, 0.20 \leq w_2 \leq 0.30, 0.10 \leq w_3 \leq 0.30\}$. Then, no feasible solution of the CLPP (P4.4) will be obtained as the condition $\sum_{j=1}^{n} w_j^u < 1$ is satisfying i.e., the ranking of the alternatives of the modified problem cannot be obtained by Ye's method [258].

(ii) If the set $H = \{0.20 \leq w_1 \leq 0.50, 0.20 \leq w_2 \leq 0.30, 0.30 \leq w_3 \leq 0.50\}$ is replaced with $H = \{0.50 \leq w_1 \leq 0.80, 0.20 \leq w_2 \leq 0.50, 0.50 \leq w_3 \leq 0.80\}$. Then, no feasible solution of the CLPP (P4.4) will be obtained as the condition $\sum_{j=1}^{n} w_j^l > 1$ is satisfying i.e., the ranking of the alternatives of the modified problem cannot be obtained by Ye's method [258].

(3) The ranking of the alternatives, obtained by Ye's method [258], is not appropriate. The following example clearly validates this claim.

If it is assumed that the $(i,j)^{th}$ element, represented by an IVIFS, of Table 4.6 represents the rating value of the i^{th}-alternative over the j^{th}- benefit attribute. Also, if it is assumed that $H = \{0.1 \leq w_1 \leq 0.6, 0.2 \leq w_2 \leq 0.8\}$.

Table 4.6: Rating Values

Attributes→ ↓ Alternatives	G_1	G_2
A_1	$([0.10,0.20],[0.10,0.20])$	$([0.10,0.20],[0.30,0.40])$

A_2	$([0.30,0.40],[0.30,0.40])$	$([0.10,0.20],[0.30,0.40])$

Then, it is obvious that the ranking of the alternatives A_1 and A_2 can never be $A_1 = A_2$ as the rating values of both alternatives corresponding to the attribute G_2 are equal. Whereas, the rating values of both alternatives corresponding to first attribute G_1 are not equal. While, the following clearly indicates that on applying Ye's method [258], the obtained ranking of the alternatives is $A_1 = A_2$, which is inappropriate.

Using Ye's method [258] the ranking of the alternatives A_1 and A_2 can be obtained as follows:

Step 1: Since, all the attributes are of benefit type. So, according to Step 1 of Ye's method [258], discussed in Section 4.4, there is no need to apply Step 1.

Step 2: According to Step 2 of Ye's method [258], discussed in Section 4.4, there is need to calculate the values of $d_{ij}\ \forall\ i = 1,2; j = 1,2$. These values are shown in Table 4.7.

Table 4.7: Values of d_{ij}

$d_{11} = 0$	$d_{12} = -0.30$
$d_{21} = 0$	$d_{22} = -0.30$

Step 3: Using Step 3 of Ye's method [258], discussed in Section 4.4,

$c_1 = \sum_{i=1}^{2} d_{i1} = 0 + 0 = 0,$

$c_2 = \sum_{i=1}^{2} d_{i2} = -0.30 - 0.30 = -0.60.$

Step 4: Using Step 4 of Ye's method [258], discussed in Section 4.4, there is need to solve the CLPP (P4.5).

$Max(0.0w_1 - 0.60w_2)$

Subject to

$$\begin{cases} 0.1 \leq w_1 \leq 0.6, \\ 0.2 \leq w_2 \leq 0.8, \\ w_1 + w_2 = 1, \\ w_1 \geq 0, w_2 \geq 0. \end{cases} \qquad \text{(P4.5)}$$

It can be easily verified that on solving the CLPP (P4.5), the obtained OAWV is $(w_1, w_2) = (0.6, 0.4)$.

Step 5: On considering the OAWV, $(w_1, w_2) = (0.6, 0.4)$ and using Step 5 of Ye's method [258], discussed in Section 4.4,

$P_1 = (0.60)(0) + (0.40)(-0.30) = -0.12,$

$P_2 = (0.60)(0) + (0.40)(-0.30) = -0.12.$

Since, $P_1 = P_2$. So, according to Step 5, of Ye's method [258], discussed in Section 4.4, the ranking of the alternatives is $A_1 = A_2$.

It is obvious that on applying Ye's method [258], the relation $A_1 = A_2$ is obtained, which is inappropriate.

4.6 Reasons for the inappropriateness of Ye's method

In Section 4.5, it is shown that

(i) On applying Ye's method [258], more than one preference order for the alternatives are obtained.

(ii) Ye's method [258] fails to find the ranking of the alternatives if either the condition $\sum_{j=1}^{n} w_j^u < 1$ or the condition $\sum_{j=1}^{n} w_j^l > 1$ will be satisfied.

(iii) The ranking of the alternatives, obtained by Ye's method [258], is inappropriate.

These problems are occurring due to following reasons:

(i) If there will exist two distinct attributes G_p and G_q such that $c_p = c_q$ i.e., $\sum_{i=1}^{m} d_{ip} = \sum_{i=1}^{m} d_{iq}$ then the coefficient of the attribute weight w_p and w_q will be equal in the CLPP (P4.4). Therefore, all such values of w_p and w_q, corresponding to which $w_p + w_q$ will be optimal, will represent the optimal values of w_p and w_q. Due to the same reason, on solving the CLPP (P4.4) corresponding to the IVIFMADMPr having partially known attribute weights, considered in first point of Section 4.5, the number of obtained OAWV are more than one.

(ii) It is pertinent to mention that to apply Ye's method [258], there is need to solve the LPP (P4.4). However, a feasible solution of LPP (P4.4) will exist only if neither the condition $\sum_{j=1}^{n} w_j^u < 1$ nor the condition $\sum_{j=1}^{n} w_j^l > 1$ will be satisfied. Due to the same reason, on solving the CLPP (P4.4) corresponding to the IVIFMADMPr having partially known attribute weights, considered in the second point of Section 4.5, infeasible solution is obtained.

(iii) If there will exist two distinct alternatives A_p and A_q such that $c_p = c_q$ i.e., $\sum_{i=1}^{m} d_{ip} = \sum_{i=1}^{m} d_{iq}$. Then, the value of $P_p = \sum_{j=1}^{n} w_j\, d_{pj}$ will be equal to $P_q = \sum_{j=1}^{n} w_j\, d_{qj}$. Hence, the relation $A_p = A_q$ will be obtained. Due to the same reason, on solving the IVIFMADMPr having partially known attribute weights, considered in third point of Section 4.5, the obtained relation is $A_1 = A_2$.

4.7 A brief review of Das et al.'s method

Das et al. [62] proposed the following method to solve IVIFMADMPrs having partially known attribute weights.

Step 1: Check that all the attributes are of same type or not i.e., check that all the attributes are benefit type attribute or cost type attribute.

Case (i): If all the attributes are of same type then go to Step 2.

Case (ii): If some attributes are cost type attributes and the remaining are benefit type attributes. Then, convert the j^{th} cost type attribute into the benefit type attribute by replacing all the elements $\tilde{\alpha}_{ij} = \left([a_{ij1}, a_{ij2}], [a_{ij3}, a_{ij4}]\right)$ of the j^{th} column of the IVIFDM $\tilde{D} = \left(\tilde{\alpha}_{ij}\right)_{m\times n}$ with $\tilde{\alpha}_{ij} = \left([a_{ij3}, a_{ij4}], [a_{ij1}, a_{ij2}]\right)$ and go to Step 2.

Step 2: Using the expression (4.7), transform each IVIF element

$\tilde{\alpha}_{ij} = ([a_{ij1}, a_{ij2}], [a_{ij3}, a_{ij4}])$ of the IVIFDM $\widetilde{D} = (\tilde{\alpha}_{ij})_{m\times n} = ([a_{ij1}, a_{ij2}], [a_{ij3}, a_{ij4}])$ into the crisp element d_{ij}

$$d_{ij} = \left(1 - 0.5\left(E(\tilde{\alpha}_{ij}) + \Pi_\lambda(\tilde{\alpha}_{ij})\right)\right)_{m\times n}, i = 1,2,\dots,m; j = 1,2,\dots,n. \tag{4.7}$$

where,

(i) $\Pi_\lambda(\tilde{\alpha}_{ij}) = \frac{1}{n}\sum_{i=1}^{n}\left(\lambda(1 - a_{ij2} - a_{ij4}) + (1-\lambda)(1 - a_{ij1} - a_{ij3})\right), 0 \leq \lambda \leq 1.$

(ii) $E(\tilde{\alpha}_{ij}) = \frac{1}{n}\sum_{i=1}^{n}\frac{\tilde{a}_i}{\tilde{b}_i}.$

(iii) $\tilde{a}_i = minimum\{d(\tilde{\alpha}_{ij}, P_1), d(\tilde{\alpha}_{ij}, P_2)\}.$

(iv) $\tilde{b}_i = maximum\{d(\tilde{\alpha}_{ij}, P_1), d(\tilde{\alpha}_{ij}, P_2)\}.$

(v) $d(\tilde{\alpha}_{ij}, P_1) = \frac{1}{4n}\sum_{i=1}^{n}(|a_{ij1} - 1| + |a_{ij2} - 1| + |a_{ij3} - 0| + |a_{ij4} - 0| + |1 - a_{ij2} - a_{ij4} - 0| + |1 - a_{ij1} - a_{ij3} - 0|).$

(vi) $d(\tilde{\alpha}_{ij}, P_2) = \frac{1}{4n}\sum_{i=1}^{n}(|a_{ij1} - 0| + |a_{ij2} - 0| + |a_{ij3} - 1| + |a_{ij4} - 1| + |1 - a_{ij2} - a_{ij4} - 0| + |1 - a_{ij1} - a_{ij3} - 0|).$

Step 3: Using the expression (4.8), find the value of $c_j \ \forall \ j = 1,2,\dots,n$.

$$c_j = \sum_{i=1}^{m} d_{ij}, j = 1,2,\dots,n \tag{4.8}$$

Step 4: Find the OAWV, $(w_1, w_2, \dots, w_n)$ by solving the CLPP (P4.6).

$max\left[\sum_{j=1}^{n} c_j w_j\right]$

Subject to

$w_j \in H, w_j \geq 0, j = 1,2,\dots,n,$

$$\sum_{j=1}^{n} w_j = 1. \tag{P4.6}$$

Step 5: Using the expression (4.9), find the value of $\tilde{P}_i \ \forall \ i = 1,2,\dots,m$.

$$\tilde{P}_i = \left(\left[1 - \prod_{j=1}^{n}(1 - a_{ij1})^{w_j}, 1 - \prod_{j=1}^{n}(1 - a_{ij2})^{w_j}\right], \left[\prod_{j=1}^{n}(a_{ij3})^{w_j}, \prod_{j=1}^{n}(a_{ij4})^{w_j}\right]\right), i = 1,2,\dots,m. \tag{4.9}$$

Step 6: Check that $S(\tilde{P}_p) > S(\tilde{P}_q)$ or $S(\tilde{P}_p) < S(\tilde{P}_q)$ or $S(\tilde{P}_p) = S(\tilde{P}_q)$, where $S(\tilde{P}_p) = \frac{\left(1-\prod_{j=1}^{n}(1-a_{pj1})^{w_j}\right)+\left(1-\prod_{j=1}^{n}(1-a_{pj2})^{w_j}\right)-\prod_{j=1}^{n}(a_{pj3})^{w_j}-\prod_{j=1}^{n}(a_{pj4})^{w_j}}{2}$ and $S(\tilde{P}_q) = \frac{\left(1-\prod_{j=1}^{n}(1-a_{qj1})^{w_j}\right)+\left(1-\prod_{j=1}^{n}(1-a_{qj2})^{w_j}\right)-\prod_{j=1}^{n}(a_{qj3})^{w_j}-\prod_{j=1}^{n}(a_{qj4})^{w_j}}{2}$.

Case (i): If $S(\tilde{P}_p) > S(\tilde{P}_q)$ then the ranking of the alternatives is $A_p > A_q$.

Case (ii): If $S(\tilde{P}_p) < S(\tilde{P}_q)$ then the ranking of the alternatives is $A_p < A_q$.

Case (iii): If $S(\tilde{P}_p) = S(\tilde{P}_q)$ then the ranking of the alternatives is $A_p = A_q$.

4.8 Inappropriateness of Das et al.'s method

It is inappropriate to apply Das et al.'s method [62] due to the following reasons:

(1) On applying Das et al.'s method [62] more than one preference orders of the alternatives are obtained, which is inappropriate. To validate this claim a real-life IVIFMADMPr having partially known attribute weights, considered by Das et al. [62] to illustrate his proposed method [62], is solved with some modifications.

Das et al. [62] solved the following real-life IVIFMADMPr having partially known attribute weights to illustrate his proposed method.

There is a panel of four alternatives for the investment company to invest money in best option $A_i, i = 1,2,3,4$ on the basis of the following four benefit attributes.

(i) G_1: Risk analysis

(ii) G_2: Growth

(iii) G_3: Environmental impact

(iv) G_4: Political impact

The $(i,j)^{th}$ element of the Table 4.8, represented by an IVIFS $\tilde{\alpha}_{ij} = ([a_{ij1}, a_{ij2}], [a_{ij3}, a_{ij4}])$, represents the rating value of the i^{th}-company over the j^{th}-attribute.

Table 4.8: Rating Values

Attributes→ ↓ Alternatives	G_1	G_2	G_3	G_4
A_1	$\begin{pmatrix}[0.10,0.20],\\ [0.10,0.20]\end{pmatrix}$	$\begin{pmatrix}[0.25,0.50],\\ [0.25,0.50]\end{pmatrix}$	$\begin{pmatrix}[0.40,0.50],\\ [0.30,0.50]\end{pmatrix}$	$\begin{pmatrix}[0.50,0.50],\\ [0.50,0.50]\end{pmatrix}$
A_2	$\begin{pmatrix}[0.50,0.60],\\ [0.20,0.30]\end{pmatrix}$	$\begin{pmatrix}[0.20,0.50],\\ [0.20,0.50]\end{pmatrix}$	$\begin{pmatrix}[0,0],\\ [0.25,0.75]\end{pmatrix}$	$\begin{pmatrix}[0.30,0.40],\\ [0.40,0.60]\end{pmatrix}$
A_3	$\begin{pmatrix}[0.25,0.50],\\ [0.25,0.50]\end{pmatrix}$	$\begin{pmatrix}[0.20,0.40],\\ [0.20,0.40]\end{pmatrix}$	$\begin{pmatrix}[0.20,0.30],\\ [0.40,0.70]\end{pmatrix}$	$\begin{pmatrix}[0.20,0.30],\\ [0.50,0.60]\end{pmatrix}$
A_4	$\begin{pmatrix}[0.20,0.30],\\ [0.60,0.70]\end{pmatrix}$	$\begin{pmatrix}[0.40,0.70],\\ [0.20,0.30]\end{pmatrix}$	$\begin{pmatrix}[0.20,0.50],\\ [0.20,0.50]\end{pmatrix}$	$\begin{pmatrix}[0.50,0.70],\\ [0.10,0.30]\end{pmatrix}$

Also, $H = \{w_1 \geq 0.15, 0.20 \leq w_2, 0.30 \leq w_3 \leq 0.35, w_4 \leq 0.60, w_2 + w_4 \leq 0.40\}$.

If

(i) Table 4.8 is replaced with Table 4.9

Table 4.9: Rating Values

Attributes→ ↓ Alternatives	G_1	G_2	G_3	G_4
A_1	$\begin{pmatrix}[0.10,0.20],\\ [0.10,0.20]\end{pmatrix}$	$\begin{pmatrix}[0.25,0.50],\\ [0.25,0.50]\end{pmatrix}$	$\begin{pmatrix}[0.40,0.50],\\ [0.30,0.50]\end{pmatrix}$	$\begin{pmatrix}[0.25,0.50],\\ [0.25,0.50]\end{pmatrix}$
A_2	$\begin{pmatrix}[0.50,0.60],\\ [0.20,0.30]\end{pmatrix}$	$\begin{pmatrix}[0.20,0.50],\\ [0.20,0.50]\end{pmatrix}$	$\begin{pmatrix}[0,0],\\ [0.25,0.75]\end{pmatrix}$	$\begin{pmatrix}[0.20,0.30],\\ [0.60,0.70]\end{pmatrix}$
A_3	$\begin{pmatrix}[0.25,0.50],\\ [0.25,0.50]\end{pmatrix}$	$\begin{pmatrix}[0.20,0.40],\\ [0.20,0.40]\end{pmatrix}$	$\begin{pmatrix}[0.20,0.30],\\ [0.40,0.70]\end{pmatrix}$	$\begin{pmatrix}[0.10,0.20],\\ [0.10,0.20]\end{pmatrix}$

A_4	$\begin{pmatrix}[0.20,0.30],\\ [0.60,0.70]\end{pmatrix}$	$\begin{pmatrix}[0.40,0.70],\\ [0.20,0.30]\end{pmatrix}$	$\begin{pmatrix}[0.20,0.50],\\ [0.20,0.50]\end{pmatrix}$	$\begin{pmatrix}[0.50,0.60],\\ [0.20,0.30]\end{pmatrix}$

(ii) $H = \{w_1 \geq 0.15, 0.20 \leq w_2, 0.30 \leq w_3 \leq 0.35, w_4 \leq 0.60, w_2 + w_4 \leq 0.40\}$ is replaced with $H = \{0.10 \leq w_1 \leq 0.50, 0.20 \leq w_2 \leq 0.50, 0.15 \leq w_3 \leq 0.60, 0.10 \leq w_4 \leq 0.80\}$.

Then, using Das et al.’s method [62], the ranking of the alternatives of the modified IVIFMADMPr having partially known attribute weights can be obtained as follows:

Step 1: Since, all the attributes are of benefit type. So, according to Step 1 of Das et al.’s method [62], discussed in Section 4.7, there is no need to apply Step 1.

Step 2: According to Step 2 of Das et al.’s method [62], discussed in Section 4.7, there is need to calculate $d_{ij}\ \forall\ i = 1,2,\dots,6; j = 1,2,3$. These values are shown in Table 4.10.

Table 4.10: Values of d_{ij}

$d_{11} = 0.15$	$d_{12} = 0.375$	$d_{13} = 0.467$	$d_{14} = 0.375$
$d_{21} = 0.60$	$d_{22} = 0.35$	$d_{23} = 0.50$	$d_{24} = 0.72$
$d_{31} = 0.375$	$d_{32} = 0.30$	$d_{33} = 0.483$	$d_{34} = 0.15$
$d_{41} = 0.72$	$d_{42} = 0.60$	$d_{43} = 0.35$	$d_{44} = 0.60$

Remark: For $d_{11} = \Big(1 - 0.5\big(E(\tilde{\alpha}_{11}) + \Pi_{\lambda}(\tilde{\alpha}_{11})\big)\Big)$. Here $\tilde{\alpha}_{11} = ([0.10,0.20], [0.10,0.20])$, $\lambda = 0.5$ then $\Pi_{\lambda}(\tilde{\alpha}_{11}) = \frac{1}{1}\big(0.5\ (1 - 0.20 - 0.20) + (1 - 0.5)(1 - 0.10 - 0.10)\big) = 0.70$

$d(\tilde{\alpha}_{11}, P_1) = \frac{1}{4}(|0.10 - 1| + |0.20 - 1| + |0.10 - 0| + |0.20 - 0| + |1 - 0.20 - 0.20 - 0| + |1 - 0.10 - 0.10 - 0|) = 0.85$

$d(\tilde{\alpha}_{11}, P_2) = \frac{1}{4}(|0.10 - 0| + |0.20 - 0| + |0.10 - 1| + |0.20 - 1| + |1 - 0.20 - 0.20 - 0| + |1 - 0.10 - 0.10 - 0|) = 0.85$

$\tilde{a}_1 = minimum\{d(\tilde{\alpha}_{11}, P_1), d(\tilde{\alpha}_{11}, P_2)\} = minimum\{0.85, 0.85\} = 0.85.$

$\tilde{b}_1 = maximum\{d(\tilde{\alpha}_{11}, P_1), d(\tilde{\alpha}_{11}, P_2)\} = maximum\{0.85, 0.85\} = 0.85.$

$E(\tilde{\alpha}_{11}) = \frac{1}{1}\sum_{i=1}^{1}\frac{\tilde{a}_1}{\tilde{b}_1} = 1.$

Therefore, $d_{11} = \big(1 - 0.5(1 + 0.70)\big) = 0.15.$

Step 3: Using Step 3 of Das et al.'s method [62], discussed in Section 4.7,

$c_1 = \sum_{i=1}^{4} d_{i1} = 0.15 + 0.60 + 0.375 + 0.72 = 1.845,$

$c_2 = \sum_{i=1}^{4} d_{i2} = 0.375 + 0.35 + 0.30 + 0.60 = 1.625,$

$c_3 = \sum_{i=1}^{4} d_{i3} = 0.467 + 0.50 + 0.483 + 0.35 = 1.80,$

$c_4 = \sum_{i=1}^{4} d_{i4} = 0.375 + 0.72 + 0.15 + 0.60 = 1.845.$

Step 4: Using Step 4 of Das et al.'s method [62], discussed in Section 4.7, there is need to solve the CLPP (P4.7).

$Max(1.845w_1 + 1.625w_2 + 1.80w_3 + 1.845w_4)$

Subject to

$$\begin{cases} 0.10 \le w_1 \le 0.50, \\ 0.20 \le w_2 \le 0.50, \\ 0.15 \le w_3 \le 0.60, \\ 0.1 \le w_4 \le 0.80, \\ w_1 + w_2 + w_3 + w_4 = 1, \\ w_1 \ge 0, w_2 \ge 0, w_3 \ge 0, w_4 \ge 0. \end{cases} \qquad \text{(P4.7)}$$

It can be easily verified that on solving the CLPP (P4.7) infinite number of OAWV are obtained e.g., $(w_1, w_2, w_3, w_4) = (0.10, 0.20, 0.15, 0.55)$ and $(w_1, w_2, w_3, w_4) = (0.40, 0.20, 0.15, 0.25)$ both are the OAWV.

Step 5: On considering the OAWV, $(w_1, w_2, w_3, w_4) = (0.10, 0.20, 0.15, 0.55)$ and using Step 5 of Das et al.'s method [62], discussed in Section 4.7,

$\tilde{P}_1 = ([1 - (1 - 0.10)^{0.10}(1 - 0.25)^{0.20}(1 - 0.40)^{0.15}(1 - 0.25)^{0.55}, 1 - (1 - 0.20)^{0.10}$

$(1-0.50)^{0.20}(1-0.50)^{0.15}(1-0.50)^{0.55}],[(0.10)^{0.10}(0.25)^{0.20}(0.30)^{0.15}(0.25)^{0.55},$
$(0.20)^{0.10}(0.50)^{0.20}(0.50)^{0.15}(0.50)^{0.55}])=([0.2613,0.4759],[0.2344,0.4562]),$

$\tilde{P}_2=([1-(1-0.50)^{0.10}(1-0.20)^{0.20}(1-0)^{0.15}(1-0.20)^{0.55},1-(1-0.60)^{0.10}(1-$
$0.50)^{0.20}(1-0)^{0.15}(1-0.30)^{0.55}],[(0.20)^{0.10}(0.20)^{0.20}(0.25)^{0.15}(0.60)^{0.55},(0.30)^{0.10}$
$(0.50)^{0.20}(0.75)^{0.15}(0.70)^{0.55}])=([0.2107,0.3471],[0.3784,0.6075]),$

$\tilde{P}_3=([1-(1-0.25)^{0.10}(1-0.20)^{0.20}(1-0.20)^{0.15}(1-0.10)^{0.55},1-(1-0.50)^{0.10}$
$(1-0.40)^{0.20}(1-0.30)^{0.15}(1-0.20)^{0.55}],[(0.25)^{0.10}(0.20)^{0.20}(0.40)^{0.15}(0.10)^{0.55},$
$(0.50)^{0.10}(0.40)^{0.20}(0.70)^{0.15}(0.20)^{0.55}])=([0.1519,0.2936],[0.1549,0.3038]),$

$\tilde{P}_4=([1-(1-0.20)^{0.10}(1-0.40)^{0.20}(1-0.20)^{0.15}(1-0.50)^{0.55},1-(1-0.30)^{0.10}$
$(1-0.70)^{0.20}(1-0.50)^{0.15}(1-0.60)^{0.55}],[(0.60)^{0.10}(0.20)^{0.20}(0.20)^{0.15}(0.20)^{0.55},$
$(0.70)^{0.10}(0.30)^{0.20}(0.50)^{0.15}(0.30)^{0.55}])=([0.4167,0.5870],[0.2232,0.3525]).$

While, on considering the OAWV, $(w_1,w_2,w_3,w_4)=(0.40,0.20,0.15,0.25)$ and using Step 5 of Das et al.'s method [62], discussed in Section 4.7,

$\tilde{P}_1=([1-(1-0.10)^{0.40}(1-0.25)^{0.20}(1-0.40)^{0.15}(1-0.25)^{0.25},1-(1-0.20)^{0.40}$
$(1-0.50)^{0.20}(1-0.50)^{0.15}(1-0.50)^{0.25}],[(0.10)^{0.40}(0.25)^{0.20}(0.30)^{0.15}(0.25)^{0.25},$
$(0.20)^{0.40}(0.50)^{0.20}(0.50)^{0.15}(0.50)^{0.25}])=([0.2198,0.3965],[0.1780,0.3465]),$

$\tilde{P}_2=([1-(1-0.50)^{0.40}(1-0.20)^{0.20}(1-0)^{0.15}(1-0.20)^{0.25},1-(1-0.60)^{0.40}$
$(1-0.50)^{0.20}(1-0)^{0.15}(1-0.30)^{0.25}],[(0.20)^{0.40}(0.20)^{0.20}(0.25)^{0.15}(0.60)^{0.25},$
$(0.30)^{0.40}(0.50)^{0.20}(0.75)^{0.15}(0.70)^{0.25}])=([0.3145,0.4480],[0.2721,0.4711]),$

$\tilde{P}_3=([1-(1-0.25)^{0.40}(1-0.20)^{0.20}(1-0.20)^{0.15}(1-0.10)^{0.25},1-(1-0.50)^{0.40}$
$(1-0.40)^{0.20}(1-0.30)^{0.15}(1-0.20)^{0.25}],[(0.25)^{0.40}(0.20)^{0.20}(0.40)^{0.15}(0.10)^{0.25},$
$(0.50)^{0.40}(0.40)^{0.20}(0.70)^{0.15}(0.20)^{0.25}])=([0.1970,0.3865],[0.2040,0.3999]),$

$\tilde{P}_4=([1-(1-0.20)^{0.40}(1-0.40)^{0.20}(1-0.20)^{0.15}(1-0.50)^{0.25},1-(1-0.30)^{0.40}$
$(1-0.70)^{0.20}(1-0.50)^{0.15}(1-0.60)^{0.25}],[(0.60)^{0.40}(0.20)^{0.20}(0.20)^{0.15}(0.20)^{0.25},$

$(0.70)^{0.40}(0.30)^{0.20}(0.50)^{0.15}(0.30)^{0.25}]) = ([0.3284,0.5115],[0.3103,0.4545])$.

Step 6: Using Step 6, of Das et al.'s method [62], discussed in Section 4.7,

(i) Corresponding to the first OAWV, $(w_1, w_2, w_3, w_4) = (0.10,0.20,0.15,0.55)$, $S(\tilde{P}_1) = 0.0233, S(\tilde{P}_2) = -0.2144, S(\tilde{P}_3) = -0.0066, S(\tilde{P}_4) = 0.428$. Since, $S(\tilde{P}_4) > S(\tilde{P}_1) > S(\tilde{P}_3) > S(\tilde{P}_2)$. So, according to Step 6 of Das et al.'s method [62], discussed in Section 4.7, $A_4 > A_1 > A_3 > A_2$.

(ii) Corresponding to the second OAWV, $(w_1, w_2, w_3, w_4) = (0.40,0.20,0.15,0.25)$, $S(\tilde{P}_1) = 0.0459, S(\tilde{P}_2) = 0.0096, S(\tilde{P}_3) = -0.0102, S(\tilde{P}_4) = 0.0375$. Since, $S(\tilde{P}_1) > S(\tilde{P}_4) > S(\tilde{P}_2) > S(\tilde{P}_3)$. So, according to Step 6 of Das et al.'s method [62], discussed in Section 4.7, $A_1 > A_4 > A_2 > A_3$.

It is obvious that on applying Das et al.'s method [62] two different ranking of the alternatives are obtained for the same IVIFMADMPr having partially known attribute weights, which is inappropriate. Hence, it is inappropriate to use Das et al.'s method [62] to solve IVIFMADMPrs having partially known attribute weights.

(2) Das et al.'s [62] method fails to find the ranking of the alternatives. The following clearly validates this claim.

(i) If the set $H = \{0.10 \leq w_1 \leq 0.50, 0.20 \leq w_2 \leq 0.50, 0.15 \leq w_3 \leq 0.60, 0.10 \leq w_4 \leq 0.80\}$ is replaced with $H = \{0.10 \leq w_1 \leq 0.20, 0.10 \leq w_2 \leq 0.30, 0.20 \leq w_3 \leq 0.25, 0.10 \leq w_4 \leq 0.15\}$. Then, no feasible solution of the CLPP (P4.7) will be obtained as the condition $\sum_{j=1}^{n} w_j^u < 1$ is satisfying i.e., the ranking of the alternatives of the modified problem cannot be obtained by Das et al.'s method [62].

(ii) If the set $H = \{0.10 \leq w_1 \leq 0.50, 0.20 \leq w_2 \leq 0.50, 0.15 \leq w_3 \leq 0.60, 0.10 \leq w_4 \leq 0.80\}$ is replaced with $H = \{0.30 \leq w_1 \leq 0.90, 0.40 \leq w_2 \leq 0.60, 0.70 \leq w_3 \leq 0.80, 0.30 \leq w_4 \leq 0.90\}$. Then, no feasible solution of the CLPP (P4.7) will be obtained as the condition $\sum_{j=1}^{n} w_j^l > 1$ is satisfying i.e., the ranking of the alternatives of the modified problem cannot be obtained by Das et al.'s method [62].

(3) The ranking of the alternatives, obtained by Das et al.'s method [62], is not appropriate. The following example clearly validates this claim.

If it is assumed that the $(i,j)^{th}$ element, represented by an IVIFS, of Table 4.11 represents the rating value of the i^{th}-alternative over the j^{th}-benefit attribute. Also, if it is assumed that $H = \{0.1 \le w_1 \le 0.7, 0.2 \le w_2 \le 0.6\}$.

Table 4.11: Rating Values

Attributes→ ↓ Alternatives	G_1	G_2
A_1	$([0.20,0.40],[0.20,0.40])$	$([1,1],[0,0])$
A_2	$([0.15,0.25],[0.15,0.25])$	$([1,1],[0,0])$

Then, it is obvious that the ranking of the alternatives A_1 and A_2 can never be $A_1 = A_2$ as the rating values of both the alternatives corresponding to the attribute G_2 are equal. Whereas, the rating values of both the alternatives corresponding to first attribute G_1 are not equal. While, the following clearly indicates that on applying Das et al.'s method [62], the obtained ranking of the alternatives is $A_1 = A_2$, which is inappropriate.

Using Das et al.'s method [62] the ranking of the alternatives A_1 and A_2 can be obtained as follows:

Step 1: Since, all the attributes are of benefit type. So, according to Step 1 of Das et al.'s method [62], discussed in Section 4.7, there is no need to apply Step 1.

Step 2: According to Step 2 of Das et al.'s method [62], discussed in Section 4.7, there is need to calculate the values of $d_{ij}\ \forall\ i = 1,2; j = 1,2$. These values are shown in Table 4.12.

Table 4.12: Values of d_{ij}

$d_{11} = 0.30$	$d_{12} = 0.50$
$d_{21} = 0.30$	$d_{22} = 0.50$

Step 3: Using Step 3 of Das et al.'s method [62], discussed in Section 4.7,

$c_1 = \sum_{i=1}^{2} d_{i1} = 0.30 + 0.30 = 0.60,$

$c_2 = \sum_{i=1}^{2} d_{i2} = 0.50 + 0.50 = 1.$

Step 4: Using Step 4 of Das et al.'s method [62], discussed in Section 4.7, there is need to solve the CLPP (P4.8).

$Max(0.60w_1 + 1w_2)$

Subject to

$$\begin{cases} 0.1 \le w_1 \le 0.7, \\ 0.2 \le w_2 \le 0.6, \\ w_1 + w_2 = 1, \\ w_1 \ge 0, w_2 \ge 0. \end{cases} \tag{P4.8}$$

It can be easily verified that on solving the CLPP (P4.8), the obtained OAWV is $(w_1, w_2) = (0.4, 0.6)$.

Step 5: On considering the OAWV $(w_1, w_2) = (0.40, 0.60)$ and using Step 5 of Das et al.'s method [62], discussed in Section 4.7,

$$\tilde{P}_1 = ([1-(1-0.20)^{0.40}(1-1)^{0.60}, 1-(1-0.40)^{0.40}(1-1)^{0.60}], [(0.20)^{0.40}(0)^{0.60}, (0.40)^{0.40}(0)^{0.60}] = ([1,1],[0,0]),$$

$$\tilde{P}_2 = ([1-(1-0.15)^{0.40}(1-1)^{0.60}, 1-(1-0.25)^{0.40}(1-1)^{0.60}], [(0.15)^{0.40}(0)^{0.60}, (0.25)^{0.40}(0)^{0.60}] = ([1,1],[0,0]).$$

Step 6: Using Step 6, of Das et al.'s method [62], discussed in Section 4.7,

$S(\tilde{P}_1) = 1, S(\tilde{P}_2) = 1.$

Since, $S(\tilde{P}_1) = S(\tilde{P}_2)$. So, according to Step 6, of Das et al.'s method [62], discussed in Section 4.7, the ranking of the alternatives is $A_1 = A_2$.

It is obvious that on applying Das et al.'s method [62], the relation $A_1 = A_2$ is obtained, which is inappropriate.

4.9 Reasons for the inappropriateness of Das et al.'s method

In Section 4.8, it is shown that

(i) On applying Das et al.'s method [62], more than one preference order for the alternatives are obtained.

(ii) Das et al.'s method [62] fails to find the ranking of the alternatives if either the condition $\sum_{j=1}^{n} w_j^u < 1$ or the condition $\sum_{j=1}^{n} w_j^l > 1$ will be satisfied.

(iii) The ranking of the alternatives, obtained by Das et al.'s method [62], is inappropriate.

These problems are occurring due to following reasons:

(i) If there will exist two distinct attributes G_p and G_q such that $c_p = c_q$ i.e., $\sum_{i=1}^{m} d_{ip} = \sum_{i=1}^{m} d_{iq}$ then the coefficient of the attribute weight w_p and w_q will be equal in the CLPP (P4.7). Therefore, all such values of w_p and w_q, corresponding to which $w_p + w_q$ will be optimal, will represent the optimal values of w_p and w_q. Due to the same reason, on solving the CLPP (P4.7) corresponding to the IVIFMADMPr having partially known attribute weights, considered in first point of Section 4.8, the number of obtained OAWV are more than one.

(ii) It is pertinent to mention that to apply Das et al.'s method [62], there is need to solve the CLPP (P4.7). However, a feasible solution of the CLPP (P4.7) will exist only if neither the condition $\sum_{j=1}^{n} w_j^u < 1$ nor the condition $\sum_{j=1}^{n} w_j^l > 1$ will be satisfied.

(iii) It is pertinent to mention that if there exist $([a_{ij1}, a_{ij2}], [a_{ij3}, a_{ij4}])$ such that $[a_{ij1}, a_{ij2}] = [1,1]$ and $[a_{ij3}, a_{ij4}] = [0,0]$ then the value of $P_p =$

$\frac{\left(1-\prod_{j=1}^{n}(1-a_{pj1})^{w_j}\right)+\left(1-\prod_{j=1}^{n}(1-a_{pj2})^{w_j}\right)-\prod_{j=1}^{n}a_{pj3}{}^{w_j}-\prod_{j=1}^{n}a_{pj4}{}^{w_j}}{2}$ as well as $P_q = \frac{\left(1-\prod_{j=1}^{n}(1-a_{pj1})^{w_j}\right)+\left(1-\prod_{j=1}^{n}(1-a_{pj2})^{w_j}\right)-\prod_{j=1}^{n}a_{pj3}{}^{w_j}-\prod_{j=1}^{n}a_{pj4}{}^{w_j}}{2}$ will always be 1 for all values of w_j. Due to the same reason, on solving the IVIFMADMPr having partially known attribute weights, considered in third point of Section 4.8, the obtained relation is $A_1 = A_2$.

4.10 A brief review of Joshi and Kumar's method

Joshi and Kumar [108] proposed the following method to solve IVIFMADMPrs having partially known attribute weights.

Step 1: Check that all the attributes are of same type or not i.e., check that all the attributes are benefit type attribute or cost type attribute.

Case (i): If all the attributes are of same type then go to Step 2.

Case (ii): If some attributes are cost type attributes and the remaining are benefit type attributes then convert the j^{th} cost type attribute into the benefit type attribute by replacing all the elements $\tilde{\alpha}_{ij} = ([a_{ij1}, a_{ij2}], [a_{ij3}, a_{ij4}])$ of the j^{th} column of the IVIFDM $\widetilde{D} = (\tilde{\alpha}_{ij})_{m\times n}$ with $\tilde{\alpha}_{ij} = ([a_{ij3}, a_{ij4}], [a_{ij1}, a_{ij2}])$ and go to Step 2.

Step 2: Using expression (4.10), transform each IVIF element $\tilde{\alpha}_{ij} = ([a_{ij1}, a_{ij2}], [a_{ij3}, a_{ij4}])$ of the IVIFDM $\widetilde{D} = (\tilde{\alpha}_{ij})_{m\times n} = ([a_{ij1}, a_{ij2}], [a_{ij3}, a_{ij4}])$ into the crisp element d_{ij}

$$d_{ij} = \left(\frac{a_{ij1}(1-a_{ij3})+a_{ij2}(1-a_{ij4})}{2}\right). \tag{4.10}$$

Step 3: Using the expression (4.11), find the value of c_j ∀ $j = 1,2,\ldots,n$.

$$c_j = \sum_{i=1}^{m} d_{ij}\,, j = 1,2,\ldots,n \tag{4.11}$$

Step 4: Find the OAWV, $(w_1, w_2, \ldots, w_n)$ by solving the CLPP (P4.9).

$max\left[\sum_{j=1}^{n} c_j\, w_j\right]$

Subject to

$w_j \in H, w_j \geq 0, j = 1,2,\ldots,n,$

$\sum_{j=1}^{n} w_j = 1.$ (P4.9)

Step 5: Using the expression (4.12), find the value of $P_i \ \forall \ i = 1,2,\dots,m$.

$P_i = \sum_{j=1}^{n} w_j \, d_{ij}, i = 1,2,\dots,m$ (4.12)

and check that $P_p > P_q$ or $P_p < P_q$ or $P_p = P_q$, where, $P_p = \sum_{j=1}^{n} w_j \, d_{pj}$ and $P_q = \sum_{j=1}^{n} w_j \, d_{qj}$.

Case (i): If $P_p > P_q$ then the ranking of the alternatives is $A_p > A_q$.

Case (ii): If $P_p < P_q$ then the ranking of the alternatives is $A_p < A_q$.

Case (iii): If $P_p = P_q$ then the ranking of the alternatives is $A_p = A_q$.

4.11 Inappropriateness of Joshi and Kumar's method

It is inappropriate to apply Joshi and Kumar's method [108] due to the following reasons:

(1) On applying Joshi and Kumar's method [108] more than one preference orders of the alternatives are obtained, which is inappropriate. To validate this claim a real-life IVIFMADMPr having partially known attribute weights, considered by Joshi and Kumar [108] to illustrate his proposed method [108], is solved with some modifications.

Joshi and Kumar [108] solved the following real-life IVIFMADMPr having partially known attribute weights to illustrate his proposed method.

There is need to rank the faculty members $A_i, i = 1,2,3,4$ on the basis of the following five benefit attributes.

(i) G_1: Teaching
(ii) G_2: Communication
(iii) G_3: Create interest in course
(iv) G_4: Course coverage
(v) G_5: Gives suitable illustrations

The $(i,j)^{th}$ element of the Table 4.13, represented by an IVIFS $\tilde{\alpha}_{ij} = ([a_{ij1}, a_{ij2}], [a_{ij3}, a_{ij4}])$, represents the rating value of the i^{th} faculty member over the j^{th}

attribute.

Table 4.13: Rating Values

Attributes→ ↓ Alternatives	G_1	G_2	G_3	G_4	G_5
A_1	$\begin{pmatrix}[0.16,0.16],\\ [0.17,0.50]\end{pmatrix}$	$\begin{pmatrix}[0.24,0.76],\\ [0,0]\end{pmatrix}$	$\begin{pmatrix}[0,0],\\ [0.50,0.50]\end{pmatrix}$	$\begin{pmatrix}[0.24,0.24],\\ [0.26,0.26]\end{pmatrix}$	$\begin{pmatrix}[0,0],\\ [0.25,0.75]\end{pmatrix}$
A_2	$\begin{pmatrix}[0.16,0.16],\\ [0.17,0.50]\end{pmatrix}$	$\begin{pmatrix}[0.14,0.45],\\ [0.15,0.25]\end{pmatrix}$	$\begin{pmatrix}[0.24,0.24],\\ [0.26,0.26]\end{pmatrix}$	$\begin{pmatrix}[0.207,0.207],\\ [0.26,0.26]\end{pmatrix}$	$\begin{pmatrix}[0.16,0.16],\\ [0.17,0.50]\end{pmatrix}$
A_3	$\begin{pmatrix}[0.24,0.24],\\ [0.26,0.26]\end{pmatrix}$	$\begin{pmatrix}[0.24,0.24],\\ [0.26,0.26]\end{pmatrix}$	$\begin{pmatrix}[0.24,0.24],\\ [0.26,0.26]\end{pmatrix}$	$\begin{pmatrix}[0,0],\\ [0.50,0.50]\end{pmatrix}$	$\begin{pmatrix}[0,0],\\ [0.25,0.75]\end{pmatrix}$
A_4	$\begin{pmatrix}[0.16,0.30],\\ [0.30,0.30]\end{pmatrix}$	$\begin{pmatrix}[0.125,0.375],\\ [0.127,0.374]\end{pmatrix}$	$\begin{pmatrix}[0.12,0.376],\\ [0.128,0.376]\end{pmatrix}$	$\begin{pmatrix}[0.16,0.50],\\ [0.17,0.17]\end{pmatrix}$	$\begin{pmatrix}[0,0],\\ [0.25,0.75]\end{pmatrix}$

Also, $H = \{0 \le w_1 \le 0.10, 0.10 \le w_2 \le 0.20, 0.15 \le w_3 \le 0.25, 0.20 \le w_4 \le 0.30, 0.10 \le w_5 \le 0.20\}$.

If

(i) Table 4.13 is replaced with Table 4.14

Table 4.14: Rating Values

Attributes→ ↓ Alternatives	G_1	G_2	G_3	G_4	G_5
A_1	$\begin{pmatrix}[0.16,0.16],\\ [0.17,0.50]\end{pmatrix}$	$\begin{pmatrix}[0.24,0.76],\\ [0,0]\end{pmatrix}$	$\begin{pmatrix}[0,0],\\ [0.50,0.50]\end{pmatrix}$	$\begin{pmatrix}[0.24,0.24],\\ [0.26,0.26]\end{pmatrix}$	$\begin{pmatrix}[0.24,0.24],\\ [0.26,0.26]\end{pmatrix}$
A_2	$\begin{pmatrix}[0.16,0.16],\\ [0.17,0.50]\end{pmatrix}$	$\begin{pmatrix}[0.14,0.45],\\ [0.15,0.25]\end{pmatrix}$	$\begin{pmatrix}[0.24,0.24],\\ [0.26,0.26]\end{pmatrix}$	$\begin{pmatrix}[0.207,0.207],\\ [0.26,0.26]\end{pmatrix}$	$\begin{pmatrix}[0.16,0.30],\\ [0.30,0.30]\end{pmatrix}$
A_3	$\begin{pmatrix}[0.24,0.24],\\ [0.26,0.26]\end{pmatrix}$	$\begin{pmatrix}[0.24,0.24],\\ [0.26,0.26]\end{pmatrix}$	$\begin{pmatrix}[0.24,0.24],\\ [0.26,0.26]\end{pmatrix}$	$\begin{pmatrix}[0,0],\\ [0.50,0.50]\end{pmatrix}$	$\begin{pmatrix}[0.16,0.16],\\ [0.17,0.50]\end{pmatrix}$

A_4	$\begin{pmatrix}[0.16,0.30],\\ [0.30,0.30]\end{pmatrix}$	$\begin{pmatrix}[0.125,0.375],\\ [0.127,0.374]\end{pmatrix}$	$\begin{pmatrix}[0.12,0.376],\\ [0.128,0.376]\end{pmatrix}$	$\begin{pmatrix}[0.16,0.50],\\ [0.17,0.17]\end{pmatrix}$	$\begin{pmatrix}[0.16,0.16],\\ [0.17,0.50]\end{pmatrix}$

(ii) It is assumed that the attribute weight w_1 satisfies the condition $0 \leq w_1 \leq 0.50$ instead of the existing condition $0 \leq w_1 \leq 0.10$.

(iii) It is assumed that the attribute weight w_5 satisfies the condition $0 \leq w_5 \leq 0.60$ instead of the existing condition $0.10 \leq w_5 \leq 0.20$.

Then, using Joshi and Kumar's method [108], the ranking of the alternatives of the modified IVIFMADMPr having partially known attribute weights can be obtained as follows:

Step 1: Since, all the attributes are of benefit type. So, according to Step 1 of Joshi and Kumar's method [108], discussed in Section 4.10, there is no need to apply Step 1.

Step 2: According to Step 2 of Joshi and Kumar's method [108], discussed in Section 4.10, there is need to calculate d_{ij} $\forall\, i = 1,2,\dots,6; j = 1,2,3$. These values are shown in Table 4.15.

Table 4.15: Values of d_{ij}

$d_{11} = 0.1064$	$d_{12} = 0.5000$	$d_{13} = 0$	$d_{14} = 0.1776$	$d_{15} = 0.1776$
$d_{21} = 0.1064$	$d_{22} = 0.2282$	$d_{23} = 0.1776$	$d_{24} = 0.1469$	$d_{25} = 0.1610$
$d_{31} = 0.1776$	$d_{32} = 0.1776$	$d_{33} = 0.1776$	$d_{34} = 0$	$d_{35} = 0.1064$
$d_{41} = 0.1610$	$d_{42} = 0.1719$	$d_{43} = 0.1696$	$d_{44} = 0.1064$	$d_{45} = 0.1064$

Step 3: Using Step 3 of Joshi and Kumar's method [108], discussed in Section 4.10,

$c_1 = \sum_{i=1}^{5} d_{i1} = 0.1064 + 0.1064 + 0.1776 + 0.1610 = 0.5514,$

$c_2 = \sum_{i=1}^{5} d_{i2} = 0.5000 + 0.2282 + 0.1776 + 0.1719 = 1.0770,$

$c_3 = \sum_{i=1}^{5} d_{i3} = 0 + 0.1776 + 0.1776 + 0.1696 = 0.5248,$

$c_4 = \sum_{i=1}^{5} d_{i4} = 0.1776 + 0.1469 + 0 + 0.1064 = 0.4309,$

$c_5 = \sum_{i=1}^{5} d_{i5} = 0.1776 + 0.1610 + 0.1064 + 0.1064 = 0.5514.$

Step 4: Using Step 4 of Joshi and Kumar's method [108], discussed in Section 4.10, there is need to solve the CLPP (P4.10).

$Max(0.5514w_1 + 1.077w_2 + 0.5248w_3 + 0.4309w_4 + 0.5514w_5)$

Subject to

$$\begin{cases} 0 \le w_1 \le 0.50, \\ 0.10 \le w_2 \le 0.20, \\ 0.15 \le w_3 \le 0.25, \\ 0.20 \le w_4 \le 0.30, \\ 0 \le w_5 \le 0.60, \\ w_1 + w_2 + w_3 + w_4 + w_5 = 1, \\ w_1 \ge 0, w_2 \ge 0, w_3 \ge 0, w_4 \ge 0, w_5 \ge 0. \end{cases} \qquad \text{(P4.10)}$$

It can be easily verified that on solving the CLPP (P4.10) infinite number of OAWV are obtained e.g., $(w_1, w_2, w_3, w_4, w_5) = (0,0.20,0.15,0.20,0.45)$ and $(w_1, w_2, w_3, w_4, w_5) = (0.45,0.20,0.15,0.20,0)$ both are the OAWV.

Step 5: On considering the OAWV, $(w_1, w_2, w_3, w_4, w_5) = (0,0.20,0.15,0.20,0.45)$ and using Step 5 of Joshi and Kumar's method [108], discussed in Section 4.10,

$P_1 = (0)(0.1064) + (0.20)(0.50) + (0.15)(0) + (0.20)(0.1776) + (0.45)(0.1776)$

$= 0.2154,$

$P_2 = (0)(0.1064) + (0.20)(0.2282) + (0.15)(0.1776) + (0.20)(0.1469) + (0.45)$

$(0.1610) = 0.1741,$

$P_3 = (0)(0.1776) + (0.20)(0.1776) + (0.15)(0.1776) + (0.20)(0) + (0.45)(0.1064)$

$= 0.1100,$

$P_4 = (0)(0.1610) + (0.20)(0.1719) + (0.15)(0.1696) + (0.20)(0.1064) + (0.45)$

$(0.1064) = 0.1289.$

Since, $P_1 > P_2 > P_4 > P_3$. So, according to Step 5 of Joshi and Kumar's method [108], discussed in Section 4.10, the ranking of the alternatives is $A_1 > A_2 > A_4 > A_3$.

While, on considering the OAWV, $(w_1, w_2, w_3, w_4, w_5) = (0.45,0.20,0.15,0.20,0)$ and using Step 5 of Joshi and Kumar's method [108], discussed in Section 4.10,

$P_1 = (0.45)(0.1064) + (0.20)(0.50) + (0.15)(0) + (0.20)(0.1776) + (0)(0.1776)$
$= 0.1834,$

$P_2 = (0.45)(0.1064) + (0.20)(0.2282) + (0.15)(0.1776) + (0.20)(0.1469) + (0)$
$(0.1610) = 0.1495,$

$P_3 = (0.45)(0.1776) + (0.20)(0.1776) + (0.15)(0.1776) + (0.20)(0) + (0)(0.1064)$
$= 0.1420,$

$P_4 = (0.45)(0.1610) + (0.20)(0.1719) + (0.15)(0.1696) + (0.20)(0.1064) + (0)$
$(0.1064) = 0.1535.$

Since, $P_1 > P_4 > P_2 > P_3$. So, according to Step 5 of Joshi and Kumar's method [108], discussed in Section 4.10, the ranking of the alternatives is $A_1 > A_4 > A_2 > A_3$.

It is obvious that on applying Joshi and Kumar's method [108] two different ranking of the alternatives are obtained for the same IVIFMADMPr having partially known attribute weights, which is inappropriate. Hence, it is inappropriate to use Joshi and Kumar's method [108] to solve IVIFMADMPrs having partially known attribute weights.

(2) Joshi and Kumar's method fails to find the ranking of the alternatives. The following clearly validates this claim.

(i) If the set $H = \{0 \leq w_1 \leq 0.50, 0.10 \leq w_2 \leq 0.20, 0.15 \leq w_3 \leq 0.25, 0.20 \leq w_4 \leq 0.30, 0 \leq w_5 \leq 0.60\}$ is replaced with $H = \{0 \leq w_1 \leq 0.20, 0.15 \leq w_2 \leq 0.25, 0 \leq w_3 \leq 0.10, 0.10 \leq w_4 \leq 0.15, 0 \leq w_5 \leq 0.2\}$. Then, no feasible solution of the CLPP (P4.10) will be obtained as the condition $\sum_{j=1}^{n} w_j^u < 1$ is satisfying i.e., the ranking of the alternatives of the modified problem cannot be obtained by Joshi and Kumar's method [108].

(ii) If the set $H = \{0 \leq w_1 \leq 0.50, 0.10 \leq w_2 \leq 0.20, 0.15 \leq w_3 \leq 0.25, 0.20 \leq w_4 \leq 0.30, 0 \leq w_5 \leq 0.60\}$ is replaced with $H = \{0.20 \leq w_1 \leq 0.70, 0.50 \leq w_2 \leq 0.70, 0.60 \leq w_3 \leq 0.90, 0.10 \leq w_4 \leq 0.40, 0.20 \leq w_5 \leq 0.70\}$. Then, no feasible solution of the CLPP (P4.10) will be obtained as the condition $\sum_{j=1}^{n} w_j^l > 1$ is satisfying i.e., the ranking of the alternatives of the modified problem cannot be obtained by Joshi and Kumar's method [108].

(3) The ranking of the alternatives, obtained by Joshi and Kumar's method [108], is not appropriate. The following example clearly validates this claim.

If it is assumed that the $(i,j)^{th}$ element, represented by an IVIFS, of Table 4.16 represents the rating value of the i^{th}-alternative over the j^{th}-benefit attribute. Also, if it is assumed that $H = \{0.1 \le w_1 \le 0.6, 0.2 \le w_2 \le 0.7\}$.

Table 4.16: Rating Values

Attributes→ ↓ Alternatives	G_1	G_2
A_1	$([0.20,0.40],[0.20,0.40])$	$([0.16,0.16],[0.17,0.50])$
A_2	$([0.10,0.50],[0.10,0.50])$	$([0.16,0.16],[0.17,0.50])$

Then, it is obvious that the ranking of the alternatives A_1 and A_2 can never be $A_1 = A_2$ as the rating values of both the alternatives corresponding to the attribute G_2 are equal. Whereas, the rating values of both the alternatives corresponding to first attribute G_1 are not equal. While, the following clearly indicates that on applying Joshi and Kumar's method [108], the obtained ranking of the alternatives is $A_1 = A_2$, which is inappropriate.

Using Joshi and Kumar's method [108] the ranking of the alternatives A_1 and A_2 can be obtained as follows:

Step 1: Since, all the attributes are of benefit type. So, according to Step 1 of Joshi and Kumar's method [108], discussed in Section 4.10, there is no need to apply Step 1.

Step 2: According to Step 2 of Joshi and Kumar's method [108], discussed in Section 4.10, there is need to calculate the values of d_{ij} $\forall\ i = 1,2; j = 1,2$. These values are shown in Table 4.17.

Table 4.17: Values of d_{ij}

$d_{11} = 0.30$	$d_{12} = 0.1064$
$d_{21} = 0.30$	$d_{22} = 0.1064$

Step 3: Using Step 3 of Joshi and Kumar's method [108], discussed in Section 4.10,

$c_1 = \sum_{i=1}^{2} d_{i1} = 0.30 + 0.30 = 0.60,$

$c_2 = \sum_{i=1}^{2} d_{i2} = 0.1064 + 0.1064 = 0.2128.$

Step 4: Using Step 4 of Joshi and Kumar's method [108], discussed in Section 4.10, there is need to solve the CLPP (P4.11).

$Max(0.60w_1 + 0.2128w_2)$

Subject to

$$\begin{cases} 0.1 \le w_1 \le 0.6, \\ 0.2 \le w_2 \le 0.7, \\ w_1 + w_2 = 1, \\ w_1 \ge 0, w_2 \ge 0. \end{cases} \tag{P4.11}$$

It can be easily verified that on solving the CLPP (P4.5), the obtained OAWV is $(w_1, w_2) = (0.6, 0.4)$.

Step 5: On considering the OAWV, $(w_1, w_2) = (0.6, 0.4)$ and using Step 5 of Joshi and Kumar's method [108], discussed in Section 4.10,

$P_1 = (0.60)(0.30) + (0.40)(0.1064) = 0.2225,$

$P_2 = (0.60)(0.30) + (0.40)(0.1064) = 0.2225.$

Since, $P_1 = P_2$. So, according to Step 5, of Joshi and Kumar's method [108], discussed in Section 4.10, the ranking of the alternatives is $A_1 = A_2$.

It is obvious that on applying Joshi and Kumar's method [108], the relation $A_1 = A_2$ is obtained, which is inappropriate.

4.12 Reasons for the inappropriateness of Joshi and Kumar's method

In Section 4.11, it is shown that

(i) On applying Joshi and Kumar's method [108], more than one preference order for the alternatives are obtained.

(ii) Joshi and Kumar's method [108] fails to find the ranking of the alternatives if either the condition $\sum_{j=1}^{n} w_j^u < 1$ or the condition $\sum_{j=1}^{n} w_j^l > 1$ will be satisfied.

(iii) The ranking of the alternatives, obtained by Joshi and Kumar's method [108], is inappropriate.

These problems are occurring due to following reasons:

(i) If there will exist two distinct attributes G_p and G_q such that $c_p = c_q$ i.e., $\sum_{i=1}^{m} d_{ip} = \sum_{i=1}^{m} d_{iq}$ then the coefficient of the attribute weight w_p and w_q will be equal in the CLPP (P4.10). Therefore, all such values of w_p and w_q, corresponding to which $w_p + w_q$ will be optimal, will represent the optimal values of w_p and w_q. Due to the same reason, on solving the CLPP (P4.10) corresponding to the IVIFMADMPr having partially known attribute weights, considered in first point of Section 4.11, the number of obtained OAWV are more than one.

(ii) It is pertinent to mention that to apply Joshi and Kumar's method [108], there is need to solve the CLPP (P4.10). However, a feasible solution of CLPP (P4.4) will exist only if neither the condition $\sum_{j=1}^{n} w_j^u < 1$ nor the condition $\sum_{j=1}^{n} w_j^l > 1$ will be satisfied.

(iii) If there will exist two distinct alternatives A_p and A_q such that $c_p = c_q$ i.e., $\sum_{i=1}^{m} d_{ip} = \sum_{i=1}^{m} d_{iq}$. Then, the value of $P_p = \sum_{j=1}^{n} w_j \, d_{pj}$ will be equal to $P_q = \sum_{j=1}^{n} w_j \, d_{qj}$. Hence, the relation $A_p = A_q$ will be obtained. Due to the same reason, on solving the IVIFMADMPr having partially known attribute weights, considered in third point of Section 4.11, the obtained relation is $A_1 = A_2$.

4.13 Proposed Jujhar Method

In this section, to resolve the inappropriateness of existing methods [62, 108, 213, 258], a new method (named as Jujhar method), is proposed.

The steps of the proposed Jujhar method are as follows:

Step 1: Check that all the attributes are of same type or not i.e., check that all the attributes are benefit type attribute or cost type attribute.

Case (i): If all the attributes are of same type then go to Step 2.

Case (ii): If some attributes are cost type attributes and the remaining are benefit type attributes. Then, convert the j^{th} cost type attribute into the benefit type attribute by replacing all the elements $\tilde{\alpha}_{ij} = ([a_{ij1}, a_{ij2}], [a_{ij3}, a_{ij4}])$ of the j^{th} column of the IVIFDM $\widetilde{D} = (\tilde{\alpha}_{ij})_{m \times n}$ with $\tilde{\alpha}_{ij} = ([a_{ij3}, a_{ij4}], [a_{ij1}, a_{ij2}])$ and go to Step 2.

Step 2: Using the expression (4.12), find $d_{ijk}(\tilde{\alpha}_{ij}, \tilde{\alpha}_{kj})\ \forall\ i = 1,2, \dots, m; k = 1,2, \dots, m; j = 1,2, .., n$.

$$d_{ijk}(\tilde{\alpha}_{ij}, \tilde{\alpha}_{kj}) = \frac{1}{4}(|a_{ij1} - a_{kj1}| + |a_{ij2} - a_{kj2}| + |a_{ij3} - a_{kj3}| + |a_{ij4} - a_{kj4}|), i = 1,2, \dots m;\ j = 1,2, \dots, n; k = 1,2, \dots, m \quad (4.12)$$

Step 3: Using expression (4.13), find the values of $c_j\ \forall\ j = 1,2, .., n$.

$$c_j = \sum_{i=1}^{m} \sum_{k=1}^{m} d_{ijk}(\tilde{\alpha}_{ij}, \tilde{\alpha}_{kj}), j = 1,2, \dots, n \quad (4.13)$$

Step 4: Check that the condition $\sum_{j=1}^{n} w_j^u < 1$ or the condition $\sum_{j=1}^{n} w_j^l > 1$ is satisfying or not.

Case (i): If the condition $\sum_{j=1}^{n} w_j^u < 1$ is satisfying then find the OAWV, $(w_1, w_2, \dots, w_n)$ by solving the CLPP (P3.5), proposed in Section 3.2.1 of Chapter 3.

Case (ii): If the condition $\sum_{j=1}^{n} w_j^l > 1$ is satisfying then find the OAWV, $(w_1, w_2, \dots, w_n)$ by solving the CLPP (P3.6), proposed in Section 3.2.2 of Chapter 3.

Case (iii): If neither the condition $\sum_{j=1}^{n} w_j^u < 1$ nor the condition $\sum_{j=1}^{n} w_j^l > 1$ is satisfying then find the OAWV, $(w_1, w_2, \dots, w_n)$ by solving the CLPP (P2.12) of Chapter 2.

Step 5: Use Step 5 to Step 9 of the Wang et al.'s method [213], discussed in Section 4.1, to find the ranking of the alternatives.

4.14 Exact results of the considered IVIFMADMPrs having partially known attribute weights

In Section 4.2, Section 4.5, Section 4.8 and Section 4.11, IVIFMADMPrs were solved by the existing methods [62, 108, 213, 258] and shown that the obtained results are inappropriate. In this section, the exact results of all these IVIFMADMPrs are obtained by the proposed Jujhar method.

4.14.1 Exact results of the first IVIFMADMPr having partially known attribute weights

Using the proposed Jujhar method, the exact result of the IVIFMADMPr having partially known attribute weights, considered in the first point of Section 4.2, can be obtained as follows:

Step 1: Since, all the attributes are benefit type. So, according to Step 1 of the Jujhar method, proposed in Section 4.13, there is no need to apply Step 1.

Step 2: According to Step 2 of the Jujhar method, proposed in Section 4.13, there is need to calculate $d_{ijk}(\tilde{\alpha}_{ij}, \tilde{\alpha}_{kj})\ \forall\ i = 1,2,\dots,4; j = 1,2,\dots,4$. These values are shown in Table 4.18.

Table 4.18: Values of $d_{ijk}(\tilde{\alpha}_{ij}, \tilde{\alpha}_{kj})$

$d_{111}\left(\left(\begin{matrix}[0.10,0.20],\\ [0.30,0.40]\end{matrix}\right),\left(\begin{matrix}[0.10,0.20],\\ [0.30,0.40]\end{matrix}\right)\right)=$ $\frac{1}{4}(0+0+0+0)=0$	$d_{211}\left(\left(\begin{matrix}[0.20,0.30],\\ [0.20,0.30]\end{matrix}\right),\left(\begin{matrix}[0.20,0.30],\\ [0.20,0.30]\end{matrix}\right)\right)=$ $\frac{1}{4}(0+0+0+0)=0$
$d_{112}\left(\left(\begin{matrix}[0.10,0.20],\\ [0.30,0.40]\end{matrix}\right),\left(\begin{matrix}[0.30,0.40],\\ [0.10,0.20]\end{matrix}\right)\right)=$ $\frac{1}{4}(0.2+0.2+0.2+0.2)=0.2$	$d_{212}\left(\left(\begin{matrix}[0.20,0.30],\\ [0.20,0.30]\end{matrix}\right),\left(\begin{matrix}[0.20,0.40],\\ [0.10,0.30]\end{matrix}\right)\right)=$ $\frac{1}{4}(0+0.1+0.1+0)=0.05$

$d_{113}\left(\begin{pmatrix}[0.10,0.20],\\ [0.30,0.40]\end{pmatrix},\begin{pmatrix}[0.30,0.40],\\ [0.10,0.20]\end{pmatrix}\right) =$ $\frac{1}{4}(0.2+0.2+0.2+0.2)=0.2$	$d_{213}\left(\begin{pmatrix}[0.20,0.30],\\ [0.20,0.30]\end{pmatrix},\begin{pmatrix}[0.20,0.20],\\ [0.10,0.50]\end{pmatrix}\right) =$ $\frac{1}{4}(0+0.1+0.1+0.2)=0.1$
$d_{114}\left(\begin{pmatrix}[0.10,0.20],\\ [0.30,0.40]\end{pmatrix},\begin{pmatrix}[0.10,0.30],\\ [0.10,0.50]\end{pmatrix}\right) =$ $\frac{1}{4}(0+0.1+0.2+0.1)=0.1$	$d_{214}\left(\begin{pmatrix}[0.20,0.30],\\ [0.20,0.30]\end{pmatrix},\begin{pmatrix}[0.10,0.20],\\ [0.30,0.40]\end{pmatrix}\right) =$ $\frac{1}{4}(0.1+0.1+0.1+0.1)=0.1$
$d_{121}\left(\begin{pmatrix}[0.30,0.40],\\ [0.10,0.20]\end{pmatrix},\begin{pmatrix}[0.10,0.20],\\ [0.30,0.40]\end{pmatrix}\right) =$ $\frac{1}{4}(0.2+0.2+0.2+0.2)=0.2$	$d_{221}\left(\begin{pmatrix}[0.20,0.40],\\ [0.10,0.30]\end{pmatrix},\begin{pmatrix}[0.20,0.30],\\ [0.20,0.30]\end{pmatrix}\right) =$ $\frac{1}{4}(0+0.1+0.1+0)=0.05$
$d_{122}\left(\begin{pmatrix}[0.30,0.40],\\ [0.10,0.20]\end{pmatrix},\begin{pmatrix}[0.30,0.40],\\ [0.10,0.20]\end{pmatrix}\right) =$ $\frac{1}{4}(0+0+0+0)=0$	$d_{222}\left(\begin{pmatrix}[0.20,0.40],\\ [0.10,0.30]\end{pmatrix},\begin{pmatrix}[0.20,0.40],\\ [0.10,0.30]\end{pmatrix}\right) =$ $\frac{1}{4}(0+0+0+0)=0$
$d_{123}\left(\begin{pmatrix}[0.30,0.40],\\ [0.10,0.20]\end{pmatrix},\begin{pmatrix}[0.30,0.40],\\ [0.10,0.20]\end{pmatrix}\right) =$ $\frac{1}{4}(0+0+0+0)=0$	$d_{223}\left(\begin{pmatrix}[0.20,0.40],\\ [0.10,0.30]\end{pmatrix},\begin{pmatrix}[0.20,0.20],\\ [0.10,0.50]\end{pmatrix}\right) =$ $\frac{1}{4}(0+0.2+0+0.2)=0.1$
$d_{124}\left(\begin{pmatrix}[0.30,0.40],\\ [0.10,0.20]\end{pmatrix},\begin{pmatrix}[0.10,0.30],\\ [0.10,0.50]\end{pmatrix}\right) =$ $\frac{1}{4}(0.2+0.1+0+0.3)=0.15$	$d_{224}\left(\begin{pmatrix}[0.20,0.40],\\ [0.10,0.30]\end{pmatrix},\begin{pmatrix}[0.10,0.20],\\ [0.30,0.40]\end{pmatrix}\right) =$ $\frac{1}{4}(0.1+0.2+0.2+0.1)=0.15$
$d_{131}\left(\begin{pmatrix}[0.30,0.40],\\ [0.10,0.20]\end{pmatrix},\begin{pmatrix}[0.10,0.20],\\ [0.30,0.40]\end{pmatrix}\right) =$ $\frac{1}{4}(0.2+0.2+0.2+0.2)=0.2$	$d_{231}\left(\begin{pmatrix}[0.20,0.20],\\ [0.10,0.50]\end{pmatrix},\begin{pmatrix}[0.20,0.30],\\ [0.20,0.30]\end{pmatrix}\right) =$ $\frac{1}{4}(0+0.1+0.1+0.2)=0.1$
$d_{132}\left(\begin{pmatrix}[0.30,0.40],\\ [0.10,0.20]\end{pmatrix},\begin{pmatrix}[0.30,0.40],\\ [0.10,0.20]\end{pmatrix}\right) =$ $\frac{1}{4}(0+0+0+0)=0$	$d_{232}\left(\begin{pmatrix}[0.20,0.20],\\ [0.10,0.50]\end{pmatrix},\begin{pmatrix}[0.20,0.40],\\ [0.10,0.30]\end{pmatrix}\right) =$ $\frac{1}{4}(0+0.2+0+0.2)=0.1$

$d_{133}\left(\begin{pmatrix}[0.30,0.40],\\ {[0.10,0.20]}\end{pmatrix},\begin{pmatrix}[0.30,0.40],\\ {[0.10,0.20]}\end{pmatrix}\right)=$ $\frac{1}{4}(0+0+0+0)=0$	$d_{233}\left(\begin{pmatrix}[0.20,0.20],\\ {[0.10,0.50]}\end{pmatrix},\begin{pmatrix}[0.20,0.20],\\ {[0.10,0.50]}\end{pmatrix}\right)=$ $\frac{1}{4}(0+0+0+0)=0$
$d_{134}\left(\begin{pmatrix}[0.30,0.40],\\ {[0.10,0.20]}\end{pmatrix},\begin{pmatrix}[0.10,0.30],\\ {[0.10,0.50]}\end{pmatrix}\right)=$ $\frac{1}{4}(0.2+0.1+0+0.3)=0.15$	$d_{234}\left(\begin{pmatrix}[0.20,0.20],\\ {[0.10,0.50]}\end{pmatrix},\begin{pmatrix}[0.10,0.20],\\ {[0.30,0.40]}\end{pmatrix}\right)=$ $\frac{1}{4}(0.1+0+0.2+0.1)=0.1$
$d_{141}\left(\begin{pmatrix}[0.10,0.30],\\ {[0.10,0.50]}\end{pmatrix},\begin{pmatrix}[0.10,0.20],\\ {[0.30,0.40]}\end{pmatrix}\right)=$ $\frac{1}{4}(0+0.1+0.2+0.1)=0.1$	$d_{241}\left(\begin{pmatrix}[0.10,0.20],\\ {[0.30,0.40]}\end{pmatrix},\begin{pmatrix}[0.20,0.30],\\ {[0.20,0.30]}\end{pmatrix}\right)=$ $\frac{1}{4}(0.1+0.1+0.1+0.1)=0.1$
$d_{142}\left(\begin{pmatrix}[0.10,0.30],\\ {[0.10,0.50]}\end{pmatrix},\begin{pmatrix}[0.30,0.40],\\ {[0.10,0.20]}\end{pmatrix}\right)=$ $\frac{1}{4}(0.2+0.1+0+0.3)=0.15$	$d_{242}\left(\begin{pmatrix}[0.10,0.20],\\ {[0.30,0.40]}\end{pmatrix},\begin{pmatrix}[0.20,0.40],\\ {[0.10,0.30]}\end{pmatrix}\right)=$ $\frac{1}{4}(0.1+0.2+0.2+0.1)=0.15$
$d_{143}\left(\begin{pmatrix}[0.10,0.30],\\ {[0.10,0.50]}\end{pmatrix},\begin{pmatrix}[0.30,0.40],\\ {[0.10,0.20]}\end{pmatrix}\right)=$ $\frac{1}{4}(0.2+0.1+0+0.3)=0.15$	$d_{243}\left(\begin{pmatrix}[0.10,0.20],\\ {[0.30,0.40]}\end{pmatrix},\begin{pmatrix}[0.20,0.20],\\ {[0.10,0.50]}\end{pmatrix}\right)=$ $\frac{1}{4}(0.1+0+0.2+0.1)=0.1$
$d_{144}\left(\begin{pmatrix}[0.10,0.30],\\ {[0.10,0.50]}\end{pmatrix},\begin{pmatrix}[0.10,0.30],\\ {[0.10,0.50]}\end{pmatrix}\right)=$ $\frac{1}{4}(0+0+0+0)=0$	$d_{244}\left(\begin{pmatrix}[0.10,0.20],\\ {[0.30,0.40]}\end{pmatrix},\begin{pmatrix}[0.10,0.20],\\ {[0.30,0.40]}\end{pmatrix}\right)=$ $\frac{1}{4}(0+0+0+0)=0$
$d_{311}\left(\begin{pmatrix}[0.10,0.40],\\ {[0.20,0.30]}\end{pmatrix},\begin{pmatrix}[0.10,0.40],\\ {[0.20,0.30]}\end{pmatrix}\right)=$ $\frac{1}{4}(0+0+0+0)=0$	$d_{411}\left(\begin{pmatrix}[0.10,0.50],\\ {[0.10,0.30]}\end{pmatrix},\begin{pmatrix}[0.10,0.50],\\ {[0.10,0.30]}\end{pmatrix}\right)=$ $\frac{1}{4}(0+0+0+0)=0$
$d_{312}\left(\begin{pmatrix}[0.10,0.40],\\ {[0.20,0.30]}\end{pmatrix},\begin{pmatrix}[0.20,0.30],\\ {[0.10,0.40]}\end{pmatrix}\right)=$ $\frac{1}{4}(0.1+0.1+0.1+0.1)=0.1$	$d_{412}\left(\begin{pmatrix}[0.10,0.50],\\ {[0.10,0.30]}\end{pmatrix},\begin{pmatrix}[0.10,0.50],\\ {[0.20,0.20]}\end{pmatrix}\right)=$ $\frac{1}{4}(0+0+0.1+0.1)=0.05$

$d_{313}\left(\begin{pmatrix}[0.10,0.40],\\ [0.20,0.30]\end{pmatrix},\begin{pmatrix}[0.10,0.20],\\ [0.30,0.40]\end{pmatrix}\right) =$ $\frac{1}{4}(0+0.2+0.1+0.1)=0.1$	$d_{413}\left(\begin{pmatrix}[0.10,0.50],\\ [0.10,0.30]\end{pmatrix},\begin{pmatrix}[0.10,0.40],\\ [0.20,0.30]\end{pmatrix}\right) =$ $\frac{1}{4}(0+0.1+0.1+0)=0.05$
$d_{314}\left(\begin{pmatrix}[0.10,0.40],\\ [0.20,0.30]\end{pmatrix},\begin{pmatrix}[0.50,0.50],\\ [0.10,0.40]\end{pmatrix}\right) =$ $\frac{1}{4}(0.4+0.1+0.1+0.1)=0.175$	$d_{414}\left(\begin{pmatrix}[0.10,0.50],\\ [0.10,0.30]\end{pmatrix},\begin{pmatrix}[0.20,0.30],\\ [0.20,0.30]\end{pmatrix}\right) =$ $\frac{1}{4}(0.1+0.2+0.1+0)=0.1$
$d_{321}\left(\begin{pmatrix}[0.20,0.30],\\ [0.10,0.40]\end{pmatrix},\begin{pmatrix}[0.10,0.40],\\ [0.20,0.30]\end{pmatrix}\right) =$ $\frac{1}{4}(0.1+0.1+0.1+0.1)=0.1$	$d_{421}\left(\begin{pmatrix}[0.10,0.50],\\ [0.20,0.20]\end{pmatrix},\begin{pmatrix}[0.10,0.50],\\ [0.10,0.30]\end{pmatrix}\right) =$ $\frac{1}{4}(0+0+0.1+0.1)=0.05$
$d_{322}\left(\begin{pmatrix}[0.20,0.30],\\ [0.10,0.40]\end{pmatrix},\begin{pmatrix}[0.20,0.30],\\ [0.10,0.40]\end{pmatrix}\right) =$ $\frac{1}{4}(0+0+0+0)=0$	$d_{422}\left(\begin{pmatrix}[0.10,0.50],\\ [0.20,0.20]\end{pmatrix},\begin{pmatrix}[0.10,0.50],\\ [0.20,0.20]\end{pmatrix}\right) =$ $\frac{1}{4}(0+0+0+0)=0$
$d_{323}\left(\begin{pmatrix}[0.20,0.30],\\ [0.10,0.40]\end{pmatrix},\begin{pmatrix}[0.10,0.20],\\ [0.30,0.40]\end{pmatrix}\right) =$ $\frac{1}{4}(0.1+0.1+0.2+0)=0.1$	$d_{423}\left(\begin{pmatrix}[0.10,0.50],\\ [0.20,0.20]\end{pmatrix},\begin{pmatrix}[0.10,0.40],\\ [0.20,0.30]\end{pmatrix}\right) =$ $\frac{1}{4}(0+0.1+0+0.1)=0.05$
$d_{324}\left(\begin{pmatrix}[0.20,0.30],\\ [0.10,0.40]\end{pmatrix},\begin{pmatrix}[0.50,0.50],\\ [0.10,0.40]\end{pmatrix}\right) =$ $\frac{1}{4}(0.3+0.2+0+0)=0.125$	$d_{424}\left(\begin{pmatrix}[0.10,0.50],\\ [0.20,0.20]\end{pmatrix},\begin{pmatrix}[0.20,0.30],\\ [0.20,0.30]\end{pmatrix}\right) =$ $\frac{1}{4}(0.1+0.2+0+0.1)=0.1$
$d_{331}\left(\begin{pmatrix}[0.10,0.20],\\ [0.30,0.40]\end{pmatrix},\begin{pmatrix}[0.10,0.40],\\ [0.20,0.30]\end{pmatrix}\right) =$ $\frac{1}{4}(0+0.2+0.1+0.1)=0.1$	$d_{431}\left(\begin{pmatrix}[0.10,0.40],\\ [0.20,0.30]\end{pmatrix},\begin{pmatrix}[0.10,0.50],\\ [0.10,0.30]\end{pmatrix}\right) =$ $\frac{1}{4}(0+0.1+0.1+0)=0.05$
$d_{332}\left(\begin{pmatrix}[0.10,0.20],\\ [0.30,0.40]\end{pmatrix},\begin{pmatrix}[0.20,0.30],\\ [0.10,0.40]\end{pmatrix}\right) =$ $\frac{1}{4}(0.1+0.1+0.2+0)=0.1$	$d_{432}\left(\begin{pmatrix}[0.10,0.40],\\ [0.20,0.30]\end{pmatrix},\begin{pmatrix}[0.10,0.50],\\ [0.20,0.20]\end{pmatrix}\right) =$ $\frac{1}{4}(0+0.1+0+0.1)=0.05$

$d_{333}\left(\begin{pmatrix}[0.10,0.20],\\ [0.30,0.40]\end{pmatrix},\begin{pmatrix}[0.10,0.20],\\ [0.30,0.40]\end{pmatrix}\right) =$ $\frac{1}{4}(0+0+0+0)=0$	$d_{433}\left(\begin{pmatrix}[0.10,0.40],\\ [0.20,0.30]\end{pmatrix},\begin{pmatrix}[0.10,0.40],\\ [0.20,0.30]\end{pmatrix}\right) =$ $\frac{1}{4}(0+0+0+0)=0$
$d_{334}\left(\begin{pmatrix}[0.10,0.20],\\ [0.30,0.40]\end{pmatrix},\begin{pmatrix}[0.50,0.50],\\ [0.10,0.40]\end{pmatrix}\right) =$ $\frac{1}{4}(0.4+0.3+0.2+0)=0.225$	$d_{434}\left(\begin{pmatrix}[0.10,0.40],\\ [0.20,0.30]\end{pmatrix},\begin{pmatrix}[0.20,0.30],\\ [0.20,0.30]\end{pmatrix}\right) =$ $\frac{1}{4}(0.1+0.1+0+0)=0.05$
$d_{341}\left(\begin{pmatrix}[0.50,0.50],\\ [0.10,0.40]\end{pmatrix},\begin{pmatrix}[0.10,0.40],\\ [0.20,0.30]\end{pmatrix}\right) =$ $\frac{1}{4}(0.4+0.1+0.1+0.1)=0.175$	$d_{441}\left(\begin{pmatrix}[0.20,0.30],\\ [0.20,0.30]\end{pmatrix},\begin{pmatrix}[0.10,0.50],\\ [0.10,0.30]\end{pmatrix}\right) =$ $\frac{1}{4}(0.1+0.2+0.1+0)=0.1$
$d_{342}\left(\begin{pmatrix}[0.50,0.50],\\ [0.10,0.40]\end{pmatrix},\begin{pmatrix}[0.20,0.30],\\ [0.10,0.40]\end{pmatrix}\right) =$ $\frac{1}{4}(0.3+0.2+0+0)=0.125$	$d_{442}\left(\begin{pmatrix}[0.20,0.30],\\ [0.20,0.30]\end{pmatrix},\begin{pmatrix}[0.10,0.50],\\ [0.20,0.20]\end{pmatrix}\right) =$ $\frac{1}{4}(0.1+0.2+0+0.1)=0.1$
$d_{343}\left(\begin{pmatrix}[0.50,0.50],\\ [0.10,0.40]\end{pmatrix},\begin{pmatrix}[0.10,0.20],\\ [0.30,0.40]\end{pmatrix}\right) =$ $\frac{1}{4}(0.4+0.3+0.2+0)=0.225$	$d_{443}\left(\begin{pmatrix}[0.20,0.30],\\ [0.20,0.30]\end{pmatrix},\begin{pmatrix}[0.10,0.40],\\ [0.20,0.30]\end{pmatrix}\right) =$ $\frac{1}{4}(0.1+0.1+0+0)=0.05$
$d_{344}\left(\begin{pmatrix}[0.50,0.50],\\ [0.10,0.40]\end{pmatrix},\begin{pmatrix}[0.50,0.50],\\ [0.10,0.40]\end{pmatrix}\right) =$ $\frac{1}{4}(0+0+0+0)=0$	$d_{444}\left(\begin{pmatrix}[0.20,0.30],\\ [0.20,0.30]\end{pmatrix},\begin{pmatrix}[0.20,0.30],\\ [0.20,0.30]\end{pmatrix}\right) =$ $\frac{1}{4}(0+0+0+0)=0$

Step 3: Using Step 3 of the Jujhar method, proposed in Section 4.13,

$c_1=\sum_{i=1}^{4}\sum_{k=1}^{4}\,d_{i1k}(\tilde{\alpha}_{i1},\tilde{\alpha}_{k1})=0+0.2+0.2+0.1+0.2+0+0+0.15+0.2+0+0+0.15+0.1+0.15+0.15+0=1.6$,

$c_2=\sum_{i=1}^{4}\sum_{k=1}^{4}\,d_{i2k}(\tilde{\alpha}_{i2},\tilde{\alpha}_{k2})=0+0.05+0.1+0.1+0.05+0+0.1+0.15+0.1+0.1+0+0.1+0.1+0.15+0.1+0=1.2$,

$c_3=\sum_{i=1}^{4}\sum_{k=1}^{4}\,d_{i3k}(\tilde{\alpha}_{i3},\tilde{\alpha}_{k3})=0+0.1+0.1+0.175+0.1+0+0.1+0.125+0.1+0.1+0+0.225+0.175+0.125+0.225+0=1.65$,

$c_4 = \sum_{i=1}^{4}\sum_{k=1}^{4} d_{i4k}(\tilde{\alpha}_{i4}, \tilde{\alpha}_{k4}) = 0 + 0.05 + 0.05 + 0.1 + 0.05 + 0 + 0.05 + 0.1 + 0.05 + 0.05 + 0 + 0.05 + 0.1 + 0.1 + 0.05 + 0 = 0.8.$

Step 4: Using Step 4 of the Jujhar method, proposed in Section 4.13, there is need to solve the CLPP (P4.12).

$max(1.6w_1 + 1.2w_2 + 1.65w_3 + 0.8w_4)$

Subject to

$$\begin{cases} 0.15 \le w_1 \le 0.30, \\ 0.15 \le w_2 \le 0.25, \\ 0.25 \le w_3 \le 0.40, \\ 0.30 \le w_4 \le 0.45, \\ w_1 + w_2 + w_3 + w_4 = 1, \\ w_1 \ge 0, w_2 \ge 0, w_3 \ge 0, w_4 \ge 0. \end{cases} \tag{P4.12}$$

It can be easily verified that on solving the CLPP (P4.12), the obtained OAWV is $(w_1, w_2, w_3, w_4) = (0.15, 0.15, 0.40, 0.30)$.

Step 5: On considering the OAWV, $(w_1, w_2, w_3, w_4) = (0.15, 0.15, 0.40, 0.30)$ and using Step 5 of Jujhar method, proposed in Section 4.13,

$\tilde{P}_1 = ([(0.15)(0.10) + (0.15)(0.20) + (0.40)(0.10) + (0.30)(0.10), (0.15)(0.20) + (0.15)(0.30) + (0.40)(0.40) + (0.30)(0.50)], [(0.15)(0.30) + (0.15)(0.20) + (0.40)(0.20) + (0.30)(0.10), (0.15)(0.40) + (0.15)(0.30) + (0.40)(0.30) + (0.30)(0.30)]$
$= ([0.115, 0.385], [0.185, 0.315]),$

$\tilde{P}_2 = ([(0.15)(0.30) + (0.15)(0.20) + (0.40)(0.20) + (0.30)(0.10), (0.15)(0.40) + (0.15)(0.40) + (0.40)(0.30) + (0.30)(0.50)], [(0.15)(0.10) + (0.15)(0.10) + (0.40)(0.10) + (0.30)(0.20), (0.15)(0.20) + (0.15)(0.30) + (0.40)(0.40) + (0.30)(0.20)]$
$= ([0.185, 0.390], [0.130, 0.295]),$

$\tilde{P}_3 = ([(0.15)(0.30) + (0.15)(0.20) + (0.40)(0.10) + (0.30)(0.10), (0.15)(0.40) + (0.15)(0.20) + (0.40)(0.20) + (0.30)(0.40)], [(0.15)(0.10) + (0.15)(0.10) + (0.40)(0.30) + (0.30)(0.20), (0.15)(0.20) + (0.15)(0.50) + (0.40)(0.40) + (0.30)(0.30)]$

$= ([0.145,0.290],[0.210,0.355])$,

$\tilde{P}_4 = ([(0.15)(0.10) + (0.15)(0.10) + (0.40)(0.50) + (0.30)(0.20), (0.15)(0.30) + (0.15)(0.20) + (0.40)(0.50) + (0.30)(0.30)], [(0.15)(0.10) + (0.15)(0.30) + (0.40)(0.10) + (0.30)(0.20), (0.15)(0.50) + (0.15)(0.40) + (0.40)(0.40) + (0.30)(0.30)]$

$= ([0.290,0.365],[0.160,0.385])$.

Step 6: Using Step 6 of the Jujhar method, proposed in Section 4.13,

$S(\tilde{P}_1) = 0, S(\tilde{P}_2) = 0.075, S(\tilde{P}_3) = -0.065$ and $S(\tilde{P}_4) = 0.055$.

Since, $S(\tilde{P}_2) > S(\tilde{P}_4) > S(\tilde{P}_1) > S(\tilde{P}_3)$. So, according to Step 6 of the Jujhar method, proposed in Section 4.13, the ranking of the alternatives is $A_2 > A_4 > A_1 > A_3$.

4.14.2 Exact results of the second IVIFMADMPr having partially known attribute weights

Using the proposed Jujhar method, the exact result of the IVIFMADMPr having partially known attribute weights, considered in the second point of Section 4.2, can be obtained as follows:

Since, Step 1 to Step 3 of the proposed Jujhar method are same as shown in Section 4.14.1. So, to avoid any repetition, the calculations have been started from Step 4.

Step 4: Since, the condition $\sum_{j=1}^{n} w_j^u < 1$ is satisfying. So, using Case (i) of Step 4 of the Jujhar method, proposed in Section 4.13, there is need to solve the CLPP (P4.13).

$max(1.6w_1 + 1.2w_2 + 1.65w_3 + 0.8w_4)$

Subject to

$$\left\{\begin{array}{c} 0.10 \le w_1 \le 0.20, \\ 0.30 \le w_2 \le 0.40, \\ 0.10 \le w_3 \le 0.25, \\ 0 \le w_4 \le 0.05, \\ w_1 + w_2 + w_3 + w_4 \le 1, \\ w_1 \ge 0, w_2 \ge 0, w_3 \ge 0, w_4 \ge 0. \end{array}\right. \tag{P4.13}$$

On solving the LPP (P4.13) the obtained non-normalized OAWV is $(w_1, w_2, w_3, w_4) =$

$(0.20,0.40,0.25,0.05)$ and the normalized OAWV is $(w_1,w_2,w_3,w_4)=$ $\left(\frac{0.20}{0.20+0.40+0.25+0.05},\frac{0.40}{0.20+0.40+0.25+0.05},\frac{0.25}{0.20+0.40+0.25+0.05},\frac{0.05}{0.20+0.40+0.25+0.05}\right)$ $=(0.2223,0.4445,0.2777,0.0555)$.

Step 5: On considering the normalized OAWV $(w_1,w_2,w_3,w_4)=(0.2223,0.4445,0.2777,0.0555)$ and using Step 5 of Jujhar method, proposed in Section 4.13,

$\tilde{P}_1=([(0.2223)(0.10)+(0.4445)(0.20)+(0.2777)(0.10)+(0.0555)(0.10),(0.2223)(0.20)+(0.4445)(0.30)+(0.2777)(0.40)+(0.0555)(0.50)],[(0.2223)(0.30)+(0.4445)(0.20)+(0.2777)(0.20)+(0.0555)(0.10),(0.2223)(0.40)+(0.4445)(0.30)+(0.2777)(0.30)+(0.30)(0.0555)]=([0.1444,0.3166],[0.2166,0.3222])$,

$\tilde{P}_2=([(0.2223)(0.30)+(0.4445)(0.20)+(0.2777)(0.20)+(0.0555)(0.10),(0.2223)(0.40)+(0.4445)(0.40)+(0.2777)(0.30)+(0.0555)(0.50)],[(0.2223)(0.10)+(0.4445)(0.10)+(0.2777)(0.10)+(0.0555)(0.20),(0.2223)(0.20)+(0.4445)(0.30)+(0.2777)(0.40)+(0.0555)(0.20)]=([0.2166,0.3777],[0.1055,0.2999])$,

$\tilde{P}_3=([(0.2223)(0.30)+(0.4445)(0.20)+(0.2777)(0.10)+(0.0555)(0.10),(0.2223)(0.40)+(0.4445)(0.20)+(0.2777)(0.20)+(0.0555)(0.40)],[(0.2223)(0.10)+(0.4445)(0.10)+(0.2777)(0.30)+(0.0555)(0.20),(0.2223)(0.20)+(0.4445)(0.50)+(0.2777)(0.40)+(0.30)(0.0555)]=([0.1889,0.2555],[0.1610,0.3944])$,

$\tilde{P}_4=([(0.2223)(0.10)+(0.4445)(0.10)+(0.2777)(0.50)+(0.0555)(0.20),(0.2223)(0.30)+(0.4445)(0.20)+(0.2777)(0.50)+(0.0555)(0.30)],[(0.2223)(0.10)+(0.4445)(0.30)+(0.2777)(0.10)+(0.0555)(0.20),(0.2223)(0.50)+(0.4445)(0.40)+(0.2777)(0.40)+(0.0555)(0.30)]=([0.2166,0.3110],[0.1944,0.4166])$.

Step 6: Using Step 6 of the Jujhar method, proposed in Section 4.13,

$S(\tilde{P}_1)=-0.0389, S(\tilde{P}_2)=0.0944, S(\tilde{P}_3)=-0.0555$ and $S(\tilde{P}_4)=-0.0417$.

Since, $S(\tilde{P}_2)>S(\tilde{P}_1)>S(\tilde{P}_4)>S(\tilde{P}_3)$. So, according to Step 6 of the Jujhar method, proposed in Section 4.13, the ranking of the alternatives is $A_2 > A_1 > A_4 > A_3$.

4.14.3 Exact results of the third IVIFMADMPr having partially known attribute weights

Using the proposed Jujhar method, the exact result of the IVIFMADMPr having partially known attribute weights, considered in the second point of Section 4.2, can be obtained as follows:

Since, Step 1 to Step 3 of the proposed Jujhar method are same as shown in Section 4.14.1. So, to avoid any repetition, the calculations have been started from Step 4.

Step 4: Since, the condition $\sum_{j=1}^{n} w_j^l > 1$ is satisfying. So, using Case (ii) of Step 4 of the Jujhar method, proposed in Section 4.13, there is need to solve the CLPP (P4.14).

$max\ (1.6w_1 + 1.2w_2 + 1.65w_3 + 0.8w_4 - \varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4)$

Subject to

$$\begin{cases} \frac{30}{140} \le w_1 \le \frac{50}{140}, \\ \frac{40}{140} \le w_2 \le \frac{70}{140}, \\ \frac{50}{140} \le w_3 \le \frac{70}{140}, \\ \frac{20}{140} \le w_4 \le \frac{60}{140}, \\ w_1 + w_2 + w_3 + w_4 = 1, \\ 140 = 100 + \varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4, \\ w_1 \ge 0, w_2 \ge 0, w_3 \ge 0, w_4 \ge 0, \varepsilon_1 \ge 0, \varepsilon_2 \ge 0, \varepsilon_3 \ge 0, \varepsilon_4 \ge 0. \end{cases} \qquad \text{(P4.14)}$$

On solving the LPP (P4.14) the obtained normalized OAWV is $(w_1, w_2, w_3, w_4) = (0.2142, 0.2857, 0.3573, 0.1428)$.

Step 5: On considering the normalized OAWV, $(w_1, w_2, w_3, w_4) = (0.2142, 0.2857, 0.3573, 0.1428)$ and using Step 5 of Jujhar method, proposed in Section 4.13,

$\tilde{P}_1 = ([(0.2142)(0.10) + (0.2857)(0.20) + (0.3573)(0.10) + (0.1428)(0.10), (0.2142)(0.20) + (0.2857)(0.30) + (0.3573)(0.40) + (0.1428)(0.50)], [(0.2142)(0.30) + (0.2857)(0.20) + (0.3573)(0.20) + (0.1428)(0.10), (0.2142)(0.40) + (0.2857)(0.30) + (0.3573)(0.30) + (0.30)(0.1428)] = ([0.1285, 0.3428], [0.2071, 0.3214])$,

$\tilde{P}_2 = ([(0.2142)(0.30) + (0.2857)(0.20) + (0.3573)(0.20) + (0.1428)(0.10), (0.2142)$ $(0.40) + (0.2857)(0.40)+(0.3573)(0.30) + (0.1428)(0.50)], [(0.2142)(0.10) +$ $(0.2857)(0.10) + (0.3573)(0.10) + (0.1428)(0.20), (0.2142)(0.20) + (0.2857)(0.30) +$ $(0.3573)(0.40) + (0.1428)(0.20)] = ([0.2071,0.3785], [0.1142,0.3000]),$

$\tilde{P}_3 = ([(0.2142)(0.30) + (0.2857)(0.20) + (0.3573)(0.10) + (0.1428)(0.10), (0.2142)$ $(0.40) + (0.2857)(0.20)+(0.3573)(0.20) + (0.1428)(0.40)], [(0.2142)(0.10) +$ $(0.2857)(0.10) + (0.3573)(0.30) + (0.1428)(0.20), (0.2142)(0.20) + (0.2857)(0.50) +$ $(0.3573)(0.40) + (0.1428)(0.30)] = ([0.1714,0.2714], [0.1857,0.3714]),$

$\tilde{P}_4 = ([(0.2142)(0.10) + (0.2857)(0.10) + (0.3573)(0.50) + (0.1428)(0.20), (0.2142)$ $(0.30) + (0.2857)(0.20)+(0.3573)(0.50) + (0.1428)(0.30)], [(0.2142)(0.10) +$ $(0.2857)(0.30) + (0.3573)(0.10) + (0.1428)(0.20), (0.2142)(0.50) + (0.2857)(0.40) +$ $(0.3573)(0.40) + (0.1428)(0.30)] = ([0.2572,0.3429], [0.1714,0.4071]).$

Step 6: Using Step 6 of the Jujhar method, proposed in Section 4.13,

$S(\tilde{P}_1) = -0.0286, S(\tilde{P}_2) = 0.0857, S(\tilde{P}_3) = -0.0571$ and $S(\tilde{P}_4) = 0.0108$.

Since, $S(\tilde{P}_2) > S(\tilde{P}_4) > S(\tilde{P}_1) > S(\tilde{P}_3)$. So, according to Step 6 of the Jujhar method, proposed in Section 4.13, the ranking of the alternatives is $A_2 > A_4 > A_1 > A_3$.

4.14.4 Exact results of the fourth IVIFMADMPr having partially known attribute weights

Using the proposed Jujhar method, the exact result of the IVIFMADMPr having partially known attribute weights, considered in the first point of Section 4.5, can be obtained as follows:

Step 1: Since, all the attributes are benefit type. So, according to Step 1 of the Jujhar method, proposed in Section 4.13, there is no need to apply Step 1.

Step 2: According to Step 2 of the Jujhar method, proposed in Section 4.13, there is need to calculate $d_{ijk}(\tilde{\alpha}_{ij}, \tilde{\alpha}_{kj})$ $\forall$ $i = 1,2,...,4; j = 1,2,3$. These values are shown in Table 4.19.

Table 4.19: Values of $d_{ijk}(\tilde{\alpha}_{ij}, \tilde{\alpha}_{kj})$

$d_{111}\left(\begin{pmatrix}[0.40,0.50],\\ {[0.30,0.40]}\end{pmatrix},\begin{pmatrix}[0.40,0.50],\\ {[0.30,0.40]}\end{pmatrix}\right)=$ $\frac{1}{4}(0+0+0+0)=0$	$d_{211}\left(\begin{pmatrix}[0.40,0.60],\\ {[0.20,0.40]}\end{pmatrix},\begin{pmatrix}[0.40,0.60],\\ {[0.20,0.40]}\end{pmatrix}\right)=$ $\frac{1}{4}(0+0+0+0)=0$
$d_{112}\left(\begin{pmatrix}[0.40,0.50],\\ {[0.30,0.40]}\end{pmatrix},\begin{pmatrix}[0.60,0.70],\\ {[0.20,0.30]}\end{pmatrix}\right)=$ $\frac{1}{4}(0.2+0.2+0.1+0.1)=0.15$	$d_{212}\left(\begin{pmatrix}[0.40,0.60],\\ {[0.20,0.40]}\end{pmatrix},\begin{pmatrix}[0.60,0.70],\\ {[0.20,0.30]}\end{pmatrix}\right)=$ $\frac{1}{4}(0.2+0.1+0+0.1)=0.1$
$d_{113}\left(\begin{pmatrix}[0.40,0.50],\\ {[0.30,0.40]}\end{pmatrix},\begin{pmatrix}[0.30,0.60],\\ {[0.30,0.40]}\end{pmatrix}\right)=$ $\frac{1}{4}(0.1+0.1+0+0)=0.05$	$d_{213}\left(\begin{pmatrix}[0.40,0.60],\\ {[0.20,0.40]}\end{pmatrix},\begin{pmatrix}[0.50,0.60],\\ {[0.30,0.40]}\end{pmatrix}\right)=$ $\frac{1}{4}(0.1+0+0.1+0)=0.05$
$d_{114}\left(\begin{pmatrix}[0.40,0.50],\\ {[0.30,0.40]}\end{pmatrix},\begin{pmatrix}[0.70,0.80],\\ {[0.10,0.20]}\end{pmatrix}\right)=$ $\frac{1}{4}(0.3+0.3+0.2+0.2)=0.25$	$d_{214}\left(\begin{pmatrix}[0.40,0.60],\\ {[0.20,0.40]}\end{pmatrix},\begin{pmatrix}[0.60,0.70],\\ {[0.10,0.30]}\end{pmatrix}\right)=$ $\frac{1}{4}(0.2+0.1+0.1+0.1)=0.125$
$d_{121}\left(\begin{pmatrix}[0.60,0.70],\\ {[0.20,0.30]}\end{pmatrix},\begin{pmatrix}[0.40,0.50],\\ {[0.30,0.40]}\end{pmatrix}\right)=$ $\frac{1}{4}(0.2+0.2+0.1+0.1)=0.15$	$d_{221}\left(\begin{pmatrix}[0.60,0.70],\\ {[0.20,0.30]}\end{pmatrix},\begin{pmatrix}[0.40,0.60],\\ {[0.20,0.40]}\end{pmatrix}\right)=$ $\frac{1}{4}(0.2+0.1+0+0.1)=0.1$
$d_{122}\left(\begin{pmatrix}[0.60,0.70],\\ {[0.20,0.30]}\end{pmatrix},\begin{pmatrix}[0.60,0.70],\\ {[0.20,0.30]}\end{pmatrix}\right)=$ $\frac{1}{4}(0+0+0+0)=0$	$d_{222}\left(\begin{pmatrix}[0.60,0.70],\\ {[0.20,0.30]}\end{pmatrix},\begin{pmatrix}[0.60,0.70],\\ {[0.20,0.30]}\end{pmatrix}\right)=$ $\frac{1}{4}(0+0+0+0)=0$
$d_{123}\left(\begin{pmatrix}[0.60,0.70],\\ {[0.20,0.30]}\end{pmatrix},\begin{pmatrix}[0.30,0.60],\\ {[0.30,0.40]}\end{pmatrix}\right)=$ $\frac{1}{4}(0.3+0.1+0.1+0.1)=0.15$	$d_{223}\left(\begin{pmatrix}[0.60,0.70],\\ {[0.20,0.30]}\end{pmatrix},\begin{pmatrix}[0.50,0.60],\\ {[0.30,0.40]}\end{pmatrix}\right)=$ $\frac{1}{4}(0.1+0.1+0.1+0.1)=0.1$
$d_{124}\left(\begin{pmatrix}[0.60,0.70],\\ {[0.20,0.30]}\end{pmatrix},\begin{pmatrix}[0.70,0.80],\\ {[0.10,0.20]}\end{pmatrix}\right)=$ $\frac{1}{4}(0.1+0.1+0.1+0.1)=0.1$	$d_{224}\left(\begin{pmatrix}[0.60,0.70],\\ {[0.20,0.30]}\end{pmatrix},\begin{pmatrix}[0.60,0.70],\\ {[0.10,0.30]}\end{pmatrix}\right)=$ $\frac{1}{4}(0+0+0.1+0)=0.025$

$d_{131}\left(\begin{pmatrix}[0.30,0.60],\\ [0.30,0.40]\end{pmatrix},\begin{pmatrix}[0.40,0.50],\\ [0.30,0.40]\end{pmatrix}\right) =$ $\frac{1}{4}(0.1+0.1+0+0)=0.05$	$d_{231}\left(\begin{pmatrix}[0.50,0.60],\\ [0.30,0.40]\end{pmatrix},\begin{pmatrix}[0.40,0.60],\\ [0.20,0.40]\end{pmatrix}\right) =$ $\frac{1}{4}(0.1+0+0.1+0)=0.05$
$d_{132}\left(\begin{pmatrix}[0.30,0.60],\\ [0.30,0.40]\end{pmatrix},\begin{pmatrix}[0.60,0.70],\\ [0.20,0.30]\end{pmatrix}\right) =$ $\frac{1}{4}(0.3+0.1+0.1+0.1)=0.15$	$d_{232}\left(\begin{pmatrix}[0.50,0.60],\\ [0.30,0.40]\end{pmatrix},\begin{pmatrix}[0.60,0.70],\\ [0.20,0.30]\end{pmatrix}\right) =$ $\frac{1}{4}(0.1+0.1+0.1+0.1)=0.1$
$d_{133}\left(\begin{pmatrix}[0.30,0.60],\\ [0.30,0.40]\end{pmatrix},\begin{pmatrix}[0.30,0.60],\\ [0.30,0.40]\end{pmatrix}\right) =$ $\frac{1}{4}(0+0+0+0)=0$	$d_{233}\left(\begin{pmatrix}[0.50,0.60],\\ [0.30,0.40]\end{pmatrix},\begin{pmatrix}[0.50,0.60],\\ [0.30,0.40]\end{pmatrix}\right) =$ $\frac{1}{4}(0+0+0+0)=0$
$d_{134}\left(\begin{pmatrix}[0.30,0.60],\\ [0.30,0.40]\end{pmatrix},\begin{pmatrix}[0.70,0.80],\\ [0.10,0.20]\end{pmatrix}\right) =$ $\frac{1}{4}(0.4+0.2+0.2+0.2)=0.25$	$d_{234}\left(\begin{pmatrix}[0.50,0.60],\\ [0.30,0.40]\end{pmatrix},\begin{pmatrix}[0.60,0.70],\\ [0.10,0.30]\end{pmatrix}\right) =$ $\frac{1}{4}(0.1+0.1+0.2+0.1)=0.125$
$d_{141}\left(\begin{pmatrix}[0.70,0.80],\\ [0.10,0.20]\end{pmatrix},\begin{pmatrix}[0.40,0.50],\\ [0.30,0.40]\end{pmatrix}\right) =$ $\frac{1}{4}(0.3+0.3+0.2+0.2)=0.25$	$d_{241}\left(\begin{pmatrix}[0.60,0.70],\\ [0.10,0.30]\end{pmatrix},\begin{pmatrix}[0.40,0.60],\\ [0.20,0.40]\end{pmatrix}\right) =$ $\frac{1}{4}(0.2+0.1+0.1+0.1)=0.125$
$d_{142}\left(\begin{pmatrix}[0.70,0.80],\\ [0.10,0.20]\end{pmatrix},\begin{pmatrix}[0.60,0.70],\\ [0.20,0.30]\end{pmatrix}\right) =$ $\frac{1}{4}(0.1+0.1+0.1+0.1)=0.1$	$d_{242}\left(\begin{pmatrix}[0.60,0.70],\\ [0.10,0.30]\end{pmatrix},\begin{pmatrix}[0.60,0.70],\\ [0.20,0.30]\end{pmatrix}\right) =$ $\frac{1}{4}(0+0+0.1+0)=0.025$
$d_{143}\left(\begin{pmatrix}[0.70,0.80],\\ [0.10,0.20]\end{pmatrix},\begin{pmatrix}[0.30,0.60],\\ [0.30,0.40]\end{pmatrix}\right) =$ $\frac{1}{4}(0.4+0.2+0.2+0.2)=0.25$	$d_{243}\left(\begin{pmatrix}[0.60,0.70],\\ [0.10,0.30]\end{pmatrix},\begin{pmatrix}[0.50,0.60],\\ [0.30,0.40]\end{pmatrix}\right) =$ $\frac{1}{4}(0.1+0.1+0.2+0.1)=0.125$
$d_{144}\left(\begin{pmatrix}[0.70,0.80],\\ [0.10,0.20]\end{pmatrix},\begin{pmatrix}[0.70,0.80],\\ [0.10,0.20]\end{pmatrix}\right) =$ $\frac{1}{4}(0+0+0+0)=0$	$d_{244}\left(\begin{pmatrix}[0.60,0.70],\\ [0.10,0.30]\end{pmatrix},\begin{pmatrix}[0.60,0.70],\\ [0.10,0.30]\end{pmatrix}\right) =$ $\frac{1}{4}(0+0+0+0)=0$

$d_{311}\left(\begin{pmatrix}[0.30,0.60],\\ {[0.30,0.40]}\end{pmatrix},\begin{pmatrix}[0.30,0.60],\\ {[0.30,0.40]}\end{pmatrix}\right) =$ $\frac{1}{4}(0+0+0+0)=0$	$d_{331}\left(\begin{pmatrix}[0.70,0.80],\\ {[0.10,0.20]}\end{pmatrix},\begin{pmatrix}[0.30,0.60],\\ {[0.30,0.40]}\end{pmatrix}\right) =$ $\frac{1}{4}(0.4+0.2+0.2+0.2)=0.25$
$d_{312}\left(\begin{pmatrix}[0.30,0.60],\\ {[0.30,0.40]}\end{pmatrix},\begin{pmatrix}[0.40,0.50],\\ {[0.30,0.40]}\end{pmatrix}\right) =$ $\frac{1}{4}(0.1+0.1+0+0)=0.05$	$d_{332}\left(\begin{pmatrix}[0.70,0.80],\\ {[0.10,0.20]}\end{pmatrix},\begin{pmatrix}[0.40,0.50],\\ {[0.30,0.40]}\end{pmatrix}\right) =$ $\frac{1}{4}(0.3+0.3+0.2+0.2)=0.25$
$d_{313}\left(\begin{pmatrix}[0.30,0.60],\\ {[0.30,0.40]}\end{pmatrix},\begin{pmatrix}[0.70,0.80],\\ {[0.10,0.20]}\end{pmatrix}\right) =$ $\frac{1}{4}(0.4+0.2+0.2+0.2)=0.25$	$d_{333}\left(\begin{pmatrix}[0.70,0.80],\\ {[0.10,0.20]}\end{pmatrix},\begin{pmatrix}[0.70,0.80],\\ {[0.10,0.20]}\end{pmatrix}\right) =$ $\frac{1}{4}(0+0+0+0)=0$
$d_{314}\left(\begin{pmatrix}[0.30,0.60],\\ {[0.30,0.40]}\end{pmatrix},\begin{pmatrix}[0.60,0.70],\\ {[0.20,0.30]}\end{pmatrix}\right) =$ $\frac{1}{4}(0.3+0.1+0.1+0.1)=0.15$	$d_{334}\left(\begin{pmatrix}[0.70,0.80],\\ {[0.10,0.20]}\end{pmatrix},\begin{pmatrix}[0.60,0.70],\\ {[0.20,0.30]}\end{pmatrix}\right) =$ $\frac{1}{4}(0.1+0.1+0.1+0.1)=0.1$
$d_{321}\left(\begin{pmatrix}[0.40,0.50],\\ {[0.30,0.40]}\end{pmatrix},\begin{pmatrix}[0.30,0.60],\\ {[0.30,0.40]}\end{pmatrix}\right) =$ $\frac{1}{4}(0.1+0.1+0+0)=0.05$	$d_{341}\left(\begin{pmatrix}[0.60,0.70],\\ {[0.20,0.30]}\end{pmatrix},\begin{pmatrix}[0.30,0.60],\\ {[0.30,0.40]}\end{pmatrix}\right) =$ $\frac{1}{4}(0.3+0.1+0.1+0.1)=0.15$
$d_{322}\left(\begin{pmatrix}[0.40,0.50],\\ {[0.30,0.40]}\end{pmatrix},\begin{pmatrix}[0.40,0.50],\\ {[0.30,0.40]}\end{pmatrix}\right) =$ $\frac{1}{4}(0+0+0+0)=0$	$d_{342}\left(\begin{pmatrix}[0.60,0.70],\\ {[0.20,0.30]}\end{pmatrix},\begin{pmatrix}[0.40,0.50],\\ {[0.30,0.40]}\end{pmatrix}\right) =$ $\frac{1}{4}(0.2+0.2+0.1+0.1)=0.15$
$d_{323}\left(\begin{pmatrix}[0.40,0.50],\\ {[0.30,0.40]}\end{pmatrix},\begin{pmatrix}[0.70,0.80],\\ {[0.10,0.20]}\end{pmatrix}\right) =$ $\frac{1}{4}(0.3+0.3+0.2+0.2)=0.25$	$d_{343}\left(\begin{pmatrix}[0.60,0.70],\\ {[0.20,0.30]}\end{pmatrix},\begin{pmatrix}[0.70,0.80],\\ {[0.10,0.20]}\end{pmatrix}\right) =$ $\frac{1}{4}(0.1+0.1+0.1+0.1)=0.1$
$d_{324}\left(\begin{pmatrix}[0.40,0.50],\\ {[0.30,0.40]}\end{pmatrix},\begin{pmatrix}[0.60,0.70],\\ {[0.20,0.30]}\end{pmatrix}\right) =$ $\frac{1}{4}(0.2+0.2+0.1+0.1)=0.15$	$d_{344}\left(\begin{pmatrix}[0.60,0.70],\\ {[0.20,0.30]}\end{pmatrix},\begin{pmatrix}[0.60,0.70],\\ {[0.20,0.30]}\end{pmatrix}\right) =$ $\frac{1}{4}(0+0+0+0)=0$

Step 3: Using Step 3 of the Jujhar method, proposed in Section 4.13,

$c_1 = \sum_{i=1}^{4}\sum_{k=1}^{4} d_{i1k}(\tilde{\alpha}_{i1},\tilde{\alpha}_{k1}) = 0 + 0.15 + 0.05 + 0.25 + 0.15 + 0 + 0.15 + 0.1 + 0.05 + 0.15 + 0 + 0.25 + 0.25 + 0.1 + 0.25 + 0 = 1.9$,

$c_2 = \sum_{i=1}^{4}\sum_{k=1}^{4} d_{i2k}(\tilde{\alpha}_{i2},\tilde{\alpha}_{k2}) = 0 + 0.1 + 0.05 + 0.125 + 0.1 + 0 + 0.1 + 0.025 + 0.05 + 0.1 + 0 + 0.125 + 0.125 + 0.025 + 0.125 + 0 = 1.05$,

$c_3 = \sum_{i=1}^{4}\sum_{k=1}^{4} d_{i3k}(\tilde{\alpha}_{i3},\tilde{\alpha}_{k3}) = 0 + 0.05 + 0.25 + 0.15 + 0.05 + 0 + 0.25 + 0.15 + 0.25 + 0.25 + 0 + 0.1 + 0.15 + 0.15 + 0.1 + 0 = 1.9$.

Step 4: Using Step 4 of the Jujhar method, proposed in Section 4.13, there is need to solve the CLPP (P4.15).

$max(1.9w_1 + 1.05w_2 + 1.9w_3)$

Subject to

$$\begin{cases} 0.20 \le w_1 \le 0.50, \\ 0.20 \le w_2 \le 0.30, \\ 0.30 \le w_3 \le 0.50, \\ w_1 + w_2 + w_3 = 1, \\ w_1 = w_3, \\ w_1 \ge 0, w_2 \ge 0, w_3 \ge 0. \end{cases} \qquad \text{(P4.15)}$$

It can be easily verified that on solving the CLPP (P4.15), the obtained OAWV is $(w_1, w_2, w_3) = (0.40, 0.20, 0.40)$.

Step 5: On considering the OAWV, $(w_1, w_2, w_3) = (0.40, 0.20, 0.40)$ and using Step 5 of Jujhar method, proposed in Section 4.13,

$\tilde{P}_1 = ([(0.40)(0.40) + (0.20)(0.40) + (0.40)(0.30), (0.40)(0.50) + (0.20)(0.60) + (0.40)(0.60)], [(0.40)(0.30) + (0.20)(0.20) + (0.40)(0.30), (0.40)(0.40) + (0.20)(0.40) + (0.40)(0.40)] = ([0.36, 0.56], [0.28, 0.40])$,

$\tilde{P}_2 = ([(0.40)(0.60) + (0.20)(0.60) + (0.40)(0.40), (0.40)(0.70) + (0.20)(0.70) + (0.40)(0.50)], [(0.40)(0.20) + (0.20)(0.20) + (0.40)(0.30), (0.40)(0.30) + (0.20)(0.30) + (0.40)(0.40)] = ([0.52, 0.62], [0.24, 0.34])$,

$\tilde{P}_3 = ([(0.40)(0.30) + (0.20)(0.50) + (0.40)(0.70), (0.40)(0.60) + (0.20)(0.60) +$

$(0.40)(0.80)],[(0.40)(0.30)+(0.20)(0.30)+(0.40)(0.10),(0.40)(0.40)+(0.20)$ $(0.40)+(0.40)(0.20)]=([0.50,0.68],[0.22,0.32])$,

$\tilde{P}_4=([(0.40)(0.70)+(0.20)(0.60)+(0.40)(0.60),(0.40)(0.80)+(0.20)(0.70)$ $+(0.40)(0.70)],[(0.40)(0.10)+(0.20)(0.10)+(0.40)(0.20),(0.40)(0.20)+(0.20)$ $(0.30)+(0.40)(0.30)]=([0.64,0.74],[0.14,0.26])$.

Step 6: Using Step 6 of the Jujhar method, proposed in Section 4.13,

$S(\tilde{P}_1)=0.12, S(\tilde{P}_2)=0.28, S(\tilde{P}_3)=0.32$ and $S(\tilde{P}_4)=0.49$.

Since, $S(\tilde{P}_4)>S(\tilde{P}_3)>S(\tilde{P}_2)>S(\tilde{P}_1)$. So, according to Step 6 of the Jujhar method, proposed in Section 4.13, the ranking of the alternatives is $A_4>A_3>A_2>A_1$.

4.14.5 Exact results of the fifth IVIFMADMPr

Using the proposed Jujhar method, the exact result of the IVIFMADMPr having partially known attribute weights, considered in the second point of Section 4.5, can be obtained as follows:

Since, Step 1 to Step 3 of the proposed Jujhar method are same as shown in Section 4.14.4. So, to avoid any repetition, the calculations have been started from Step 4.

Step 4: Since, the condition $\sum_{j=1}^{n} w_j^u<1$ is satisfying. So, using Case (i) of Step 4 of the Jujhar method, proposed in Section 4.13, there is need to solve the CLPP (P4.16).

$max(1.9w_1+1.05w_2+1.9w_3)$

Subject to

$$\begin{cases} 0.10\le w_1\le 0.20, \\ 0.20\le w_2\le 0.30, \\ 0.10\le w_3\le 0.30, \\ w_1+w_2+w_3\le 1, \\ w_1=w_3, \\ w_1\ge 0, w_2\ge 0, w_3\ge 0. \end{cases} \quad \text{(P4.16)}$$

On solving the LPP (P4.16) the obtained non-normalized OAWV is $(w_1,w_2,w_3)=$ $(0.20, 0.30, 0.20)$ and the normalized OAWV is $(w_1,w_2,w_3)=$

$\left(\frac{0.20}{0.20+0.30+0.20}, \frac{0.30}{0.20+0.30+0.20}, \frac{0.20}{0.20+0.30+0.20}\right) = (0.2857{,}0.4286{,}0.2857)$.

Step 5: On considering the normalized OAWV, $(w_1, w_2, w_3) = (0.2857{,}0.4286{,}0.2857)$ and using Step 5 of Jujhar method, proposed in Section 4.13,

$\tilde{P}_1 = ([(0.2857)(0.40) + (0.4286)(0.40) + (0.2857)(0.30), (0.2857)(0.50) + (0.4286)(0.60)+(0.2857)(0.60)], [(0.2857)(0.30) +(0.4286)(0.20) + (0.2857)(0.30), (0.2857)(0.40) + (0.4286)(0.40) + (0.2857)(0.40)] = ([0.3714{,}0.5714], [0.2571{,}0.4000])$,

$\tilde{P}_2 = ([(0.2857)(0.60) + (0.4286)(0.60) + (0.2857)(0.40), (0.2857)(0.70) + (0.4286)(0.70)+(0.2857)(0.50)], [(0.2857)(0.20) +(0.4286)(0.20) + (0.2857)(0.30), (0.2857)(0.30) + (0.4286)(0.30) + (0.2857)(0.40)] = ([0.5428{,}0.6428], [0.2285{,}0.3285])$,

$\tilde{P}_3 = ([(0.2857)(0.30) + (0.4286)(0.50) + (0.2857)(0.70), (0.2857)(0.60) + (0.4286)(0.60)+(0.2857)(0.80)], [(0.2857)(0.30) +(0.4286)(0.30) + (0.2857)(0.10), (0.2857)(0.40) + (0.4286)(0.40) + (0.2857)(0.20)] = ([0.5000{,}0.6571], [0.2428{,}0.3428])$,

$\tilde{P}_4 = ([(0.2857)(0.70) + (0.4286)(0.60) + (0.2857)(0.60), (0.2857)(0.80) + (0.4286)(0.70)+(0.2857)(0.70)], [(0.2857)(0.10) +(0.4286)(0.10) + (0.2857)(0.20), (0.2857)(0.20) + (0.4286)(0.30) + (0.2857)(0.30)] = ([0.6285{,}0.7285], [0.1285{,}0.2714])$.

Step 6: Using Step 6 of the Jujhar method, proposed in Section 4.13,

$S(\tilde{P}_1) = 0.1428, S(\tilde{P}_2) = 0.3143, S(\tilde{P}_3) = 0.2857$ and $S(\tilde{P}_4) = 0.4785$.

Since, $S(\tilde{P}_4) > S(\tilde{P}_2) > S(\tilde{P}_3) > S(\tilde{P}_1)$. So, according to Step 6 of the Jujhar method, proposed in Section 4.13, the ranking of the alternatives is $A_4 > A_2 > A_3 > A_1$.

4.14.6 Exact results of the sixth IVIFMADMPr having partially known attribute weights

Using the proposed Jujhar method, the exact result of the IVIFMADMPr having partially known attribute weights, considered in the second point of Section 4.5, can be obtained as follows:

Since, Step 1 to Step 3 of the proposed Jujhar method are same as shown in Section 4.14.4.

So, to avoid any repetition, the calculations have been started from Step 4.

Step 4: Since, the condition $\sum_{j=1}^{n} w_j^l > 1$ is satisfying. So, using Case (ii) of Step 4 of the Jujhar method, proposed in Section 4.13, there is need to solve the CLPP (P4.17).

$max\ (1.9w_1 + 1.05w_2 + 1.9w_3 - \varepsilon_1 - \varepsilon_2 - \varepsilon_3)$

Subject to

$$\left\{\begin{array}{c} \frac{20}{120} \le w_1 \le \frac{40}{120}, \\ \frac{30}{120} \le w_2 \le \frac{50}{120}, \\ \frac{70}{120} \le w_3 \le \frac{80}{120}, \\ w_1 + w_2 + w_3 = 1, \\ w_1 = w_3, \\ 120 = 100 + \varepsilon_1 + \varepsilon_2 + \varepsilon_3, \\ w_1 \ge 0, w_2 \ge 0, w_3 \ge 0, \varepsilon_1 \ge 0, \varepsilon_2 \ge 0, \varepsilon_3 \ge 0. \end{array}\right. \tag{P4.17}$$

On solving the CLPP (P4.17), the obtained normalized OAWV is $(w_1, w_2, w_3) = (0.4167, 0.1666, 0.4167)$.

Step 5: On considering the normalized OAWV, $(w_1, w_2, w_3) = (0.4167, 0.1666, 0.4167)$ and using Step 5 of Jujhar method, proposed in Section 4.13,

$\tilde{P}_1 = ([(0.4167)(0.40) + (0.1666)(0.40) + (0.4167)(0.30), (0.4167)(0.50) + (0.1666)(0.60) + (0.4167)(0.60)], [(0.4167)(0.30) + (0.1666)(0.20) + (0.4167)(0.30), (0.4167)(0.40) + (0.1666)(0.40) + (0.4167)(0.40)] = ([0.3583, 0.5583], [0.2833, 0.4000])$,

$\tilde{P}_2 = ([(0.4167)(0.60) + (0.1666)(0.60) + (0.4167)(0.40), (0.4167)(0.70) + (0.1666)(0.70) + (0.4167)(0.50)], [(0.4167)(0.20) + (0.1666)(0.20) + (0.4167)(0.30), (0.4167)(0.30) + (0.1666)(0.30) + (0.4167)(0.40)] = ([0.5166, 0.6166], [0.2416, 0.3416])$,

$\tilde{P}_3 = ([(0.4167)(0.30) + (0.1666)(0.50) + (0.4167)(0.70), (0.4167)(0.60) + (0.1666)(0.60) + (0.4167)(0.80)], [(0.4167)(0.30) + (0.1666)(0.30) + (0.4167)(0.10), (0.4167)(0.40) + (0.1666)(0.40) + (0.4167)(0.20)] = ([0.5000, 0.6833], [0.2166, 0.3166])$,

$\tilde{P}_4 = ([(0.4167)(0.70) + (0.1666)(0.60) + (0.4167)(0.60), (0.4167)(0.80) + (0.1666)(0.70) + (0.4167)(0.70)], [(0.4167)(0.10) + (0.1666)(0.10) + (0.4167)(0.20), (0.4167)$

$(0.20) + (0.1666)(0.30) + (0.4167)(0.30)] = ([0.6416,0.7416],[0.1416,0.2583])$.

Step 6: Using Step 6 of the Jujhar method, proposed in Section 4.13,

$S(\tilde{P}_1) = 0.1166, S(\tilde{P}_2) = 0.2750, S(\tilde{P}_3) = 0.3250$ and $S(\tilde{P}_4) = 0.4916$.

Since, $S(\tilde{P}_4) > S(\tilde{P}_3) > S(\tilde{P}_2) > S(\tilde{P}_1)$. So, according to Step 6 of the Jujhar method, proposed in Section 4.13, the ranking of the alternatives is $A_4 > A_3 > A_2 > A_1$.

4.14.7 Exact results of seventh IVIFMADMPr having partially known attribute weights

Using the proposed Jujhar method, the exact result of the IVIFMADMPr having partially known attribute weights, considered in the third point of Section 4.5, can be obtained as follows:

Step 1: Since, all the attributes are benefit type. So, according to Step 1 of the Jujhar method, proposed in Section 4.13, there is no need to apply Step 1.

Step 2: According to Step 2 of the Jujhar method, proposed in Section 4.13, there is need to calculate $d_{ijk}(\tilde{\alpha}_{ij}, \tilde{\alpha}_{kj})\ \forall\ i = 1,2; j = 1,2$. These values are shown in Table 4.20.

Table 4.20: Values of $\boldsymbol{d_{ijk}(\tilde{\alpha}_{ij}, \tilde{\alpha}_{kj})}$

$d_{111}\left(\begin{pmatrix}[0.10,0.20],\\ [0.10,0.20]\end{pmatrix},\begin{pmatrix}[0.10,0.20],\\ [0.10,0.20]\end{pmatrix}\right) =$ $\frac{1}{4}(0 + 0 + 0 + 0) = 0$	$d_{211}\left(\begin{pmatrix}[0.10,0.20],\\ [0.30,0.40]\end{pmatrix},\begin{pmatrix}[0.10,0.20],\\ [0.30,0.40]\end{pmatrix}\right) =$ $\frac{1}{4}(0 + 0 + 0 + 0) = 0$
$d_{112}\left(\begin{pmatrix}[0.10,0.20],\\ [0.10,0.20]\end{pmatrix},\begin{pmatrix}[0.30,0.40],\\ [0.30,0.40]\end{pmatrix}\right) =$ $\frac{1}{4}(0.2 + 0.2 + 0.2 + 0.2) = 0.20$	$d_{212}\left(\begin{pmatrix}[0.10,0.20],\\ [0.30,0.40]\end{pmatrix},\begin{pmatrix}[0.10,0.20],\\ [0.30,0.40]\end{pmatrix}\right) =$ $\frac{1}{4}(0 + 0 + 0 + 0) = 0$
$d_{121}\left(\begin{pmatrix}[0.30,0.40],\\ [0.30,0.40]\end{pmatrix},\begin{pmatrix}[0.10,0.20],\\ [0.10,0.20]\end{pmatrix}\right) =$ $\frac{1}{4}(0.2 + 0.2 + 0.2 + 0.2) = 0.20$	$d_{213}\left(\begin{pmatrix}[0.10,0.20],\\ [0.30,0.40]\end{pmatrix},\begin{pmatrix}[0.10,0.20],\\ [0.30,0.40]\end{pmatrix}\right) =$ $\frac{1}{4}(0 + 0 + 0 + 0) = 0$

$d_{122}\left(\begin{pmatrix}[0.30,0.40],\\ [0.30,0.40]\end{pmatrix},\begin{pmatrix}[0.30,0.40],\\ [0.30,0.40]\end{pmatrix}\right)=$ $\frac{1}{4}(0+0+0+0)=0$	$d_{214}\left(\begin{pmatrix}[0.10,0.20],\\ [0.30,0.40]\end{pmatrix},\begin{pmatrix}[0.10,0.20],\\ [0.30,0.40]\end{pmatrix}\right)=$ $\frac{1}{4}(0+0+0+0)=0$

Step 3: Using Step 3 of the Jujhar method, proposed in Section 4.13,

$c_1=\sum_{i=1}^{2}\sum_{k=1}^{2} d_{i1k}(\tilde{\alpha}_{i1},\tilde{\alpha}_{k1})=0+0.2+0.2+0+0+0+0+0=0.4,$

$c_2=\sum_{i=1}^{2}\sum_{k=1}^{2} d_{i1k}(\tilde{\alpha}_{i1},\tilde{\alpha}_{k1})=0+0+0+0+0+0+0+0=0.$

Step 4: Using Step 4 of the Jujhar method, proposed in Section 4.13, there is need to solve the CLPP (P4.18).

$max(0.4w_1)$

Subject to

$$\begin{cases}0.1\le w_1\le 0.6,\\ 0.2\le w_2\le 0.8,\\ w_1+w_2=1,\\ w_1\ge 0, w_2\ge 0.\end{cases} \qquad \text{(P4.18)}$$

It can be easily verified that on solving the CLPP (P4.20) the obtained OAWV is $(w_1,w_2)=(0.6,0.4)$

Step 5: On considering the OAWV, $(w_1,w_2)=(0.6,0.4)$ and using Step 5 of the Jujhar method, proposed in Section 4.13,

$\tilde{P}_1=([(0.60)(0.10)+(0.40)(0.10),(0.60)(0.10)+(0.40)(0.30)],[(0.60)(0.10)+(0.40)(0.30),(0.60)(0.20)+(0.40)(0.40)]=([0.10,0.18],[0.18,0.28]),$

$\tilde{P}_2=([(0.60)(0.30)+(0.40)(0.10),(0.60)(0.40)+(0.40)(0.20)],[(0.60)(0.30)+(0.40)(0.30),(0.40)(0.20)+(0.40)(0.40)]=([0.22,0.32],[0.30,0.40]).$

Step 6: Using Step 6 of the Jujhar method, proposed in Section 4.13,

$S(\tilde{P}_1)=-0.09$ and $S(\tilde{P}_2)=-0.08.$

Since, $S(\tilde{P}_1) < S(\tilde{P}_2)$. Therefore, according to case (ii) of Step 6 of the proposed Jujhar method, the ranking of the alternatives is $A_2 > A_1$.

4.14.8 Exact results of eighth IVIFMADMPr having partially known attribute weights

Using the proposed Jujhar method, the exact result of the IVIFMADMPr having partially known attribute weights, considered in the first point of Section 4.8, can be obtained as follows:

Step 1: Since, all the attributes are benefit type. So, according to Step 1 of the Jujhar method, proposed in Section 4.13, there is no need to apply Step 1.

Step 2: According to Step 2 of the Jujhar method, proposed in Section 4.13, there is need to calculate $d_{ijk}(\tilde{\alpha}_{ij}, \tilde{\alpha}_{kj})\ \forall\ i = 1,2,\dots,6; j = 1,2,3$. These values are shown in Table 4.21.

Table 4.21: Values of $d_{ijk}(\tilde{\alpha}_{ij}, \tilde{\alpha}_{kj})$

$d_{111}\left(\begin{pmatrix}[0.10,0.20],\\ [0.10,0.20]\end{pmatrix},\begin{pmatrix}[0.10,0.20],\\ [0.10,0.20]\end{pmatrix}\right) =$ $\frac{1}{4}(0+0+0+0) = 0$	$d_{211}\left(\begin{pmatrix}[0.25,0.50],\\ [0.25,0.50]\end{pmatrix},\begin{pmatrix}[0.25,0.50],\\ [0.25,0.50]\end{pmatrix}\right) =$ $\frac{1}{4}(0+0+0+0) = 0$
$d_{112}\left(\begin{pmatrix}[0.10,0.20],\\ [0.10,0.20]\end{pmatrix},\begin{pmatrix}[0.50,0.60],\\ [0.20,0.30]\end{pmatrix}\right) =$ $\frac{1}{4}(0.4+0.4+0.1+0.1) = 0.25$	$d_{212}\left(\begin{pmatrix}[0.25,0.50],\\ [0.25,0.50]\end{pmatrix},\begin{pmatrix}[0.20,0.50],\\ [0.20,0.50]\end{pmatrix}\right) =$ $\frac{1}{4}(0.05+0+0.05+0) = 0.025$
$d_{113}\left(\begin{pmatrix}[0.10,0.20],\\ [0.10,0.20]\end{pmatrix},\begin{pmatrix}[0.25,0.50],\\ [0.25,0.50]\end{pmatrix}\right) =$ $\frac{1}{4}(0.15+0.3+0.15+0.3) = 0.225$	$d_{213}\left(\begin{pmatrix}[0.25,0.50],\\ [0.25,0.50]\end{pmatrix},\begin{pmatrix}[0.20,0.40],\\ [0.20,0.40]\end{pmatrix}\right) =$ $\frac{1}{4}(0.05+0.1+0.05+0.1) = 0.075$
$d_{114}\left(\begin{pmatrix}[0.10,0.20],\\ [0.10,0.20]\end{pmatrix},\begin{pmatrix}[0.20,0.30],\\ [0.60,0.70]\end{pmatrix}\right) =$ $\frac{1}{4}(0.1+0.1+0.5+0.5) = 0.3$	$d_{214}\left(\begin{pmatrix}[0.25,0.50],\\ [0.25,0.50]\end{pmatrix},\begin{pmatrix}[0.40,0.70],\\ [0.20,0.30]\end{pmatrix}\right) =$ $\frac{1}{4}(0.15+0.2+0.05+0.2) = 0.15$
$d_{121}\left(\begin{pmatrix}[0.50,0.60],\\ [0.20,0.30]\end{pmatrix},\begin{pmatrix}[0.10,0.20],\\ [0.10,0.20]\end{pmatrix}\right) =$ $\frac{1}{4}(0.4+0.4+0.1+0.1) = 0.25$	$d_{221}\left(\begin{pmatrix}[0.20,0.50],\\ [0.20,0.50]\end{pmatrix},\begin{pmatrix}[0.25,0.50],\\ [0.25,0.50]\end{pmatrix}\right) =$ $\frac{1}{4}(0.05+0+0.05+0) = 0.025$

$d_{122}\left(\begin{pmatrix}{[0.50,0.60]},\\{[0.20,0.30]}\end{pmatrix},\begin{pmatrix}{[0.50,0.60]},\\{[0.20,0.30]}\end{pmatrix}\right)=$ $\frac{1}{4}(0+0+0+0)=0$	$d_{222}\left(\begin{pmatrix}{[0.20,0.50]},\\{[0.20,0.50]}\end{pmatrix},\begin{pmatrix}{[0.20,0.50]},\\{[0.20,0.50]}\end{pmatrix}\right)=$ $\frac{1}{4}(0+0+0+0)=0$
$d_{123}\left(\begin{pmatrix}{[0.50,0.60]},\\{[0.20,0.30]}\end{pmatrix},\begin{pmatrix}{[0.25,0.50]},\\{[0.25,0.50]}\end{pmatrix}\right)=$ $\frac{1}{4}(0.25+0.1+0.05+0.2)=0.15$	$d_{223}\left(\begin{pmatrix}{[0.20,0.50]},\\{[0.20,0.50]}\end{pmatrix},\begin{pmatrix}{[0.20,0.40]},\\{[0.20,0.40]}\end{pmatrix}\right)=$ $\frac{1}{4}(0+0.1+0+0.1)=0.05$
$d_{124}\left(\begin{pmatrix}{[0.50,0.60]},\\{[0.20,0.30]}\end{pmatrix},\begin{pmatrix}{[0.20,0.30]},\\{[0.60,0.70]}\end{pmatrix}\right)=$ $\frac{1}{4}(0.3+0.3+0.4+0.4)=0.35$	$d_{224}\left(\begin{pmatrix}{[0.20,0.50]},\\{[0.20,0.50]}\end{pmatrix},\begin{pmatrix}{[0.40,0.70]},\\{[0.20,0.30]}\end{pmatrix}\right)=$ $\frac{1}{4}(0.2+0.2+0+0.2)=0.15$
$d_{131}\left(\begin{pmatrix}{[0.25,0.50]},\\{[0.25,0.50]}\end{pmatrix},\begin{pmatrix}{[0.10,0.20]},\\{[0.10,0.20]}\end{pmatrix}\right)=$ $\frac{1}{4}(0.15+0.3+0.15+0.3)=0.225$	$d_{231}\left(\begin{pmatrix}{[0.20,0.40]},\\{[0.20,0.40]}\end{pmatrix},\begin{pmatrix}{[0.25,0.50]},\\{[0.25,0.50]}\end{pmatrix}\right)=$ $\frac{1}{4}(0.05+0.1+0.05+0.1)=0.075$
$d_{132}\left(\begin{pmatrix}{[0.25,0.50]},\\{[0.25,0.50]}\end{pmatrix},\begin{pmatrix}{[0.50,0.60]},\\{[0.20,0.30]}\end{pmatrix}\right)=$ $\frac{1}{4}(0.25+0.1+0.05+0.2)=0.15$	$d_{232}\left(\begin{pmatrix}{[0.20,0.40]},\\{[0.20,0.40]}\end{pmatrix},\begin{pmatrix}{[0.20,0.50]},\\{[0.20,0.50]}\end{pmatrix}\right)=$ $\frac{1}{4}(0+0.1+0+0.1)=0.05$
$d_{133}\left(\begin{pmatrix}{[0.25,0.50]},\\{[0.25,0.50]}\end{pmatrix},\begin{pmatrix}{[0.25,0.50]},\\{[0.25,0.50]}\end{pmatrix}\right)=$ $\frac{1}{4}(0+0+0+0)=0$	$d_{233}\left(\begin{pmatrix}{[0.20,0.40]},\\{[0.20,0.40]}\end{pmatrix},\begin{pmatrix}{[0.20,0.40]},\\{[0.20,0.40]}\end{pmatrix}\right)=$ $\frac{1}{4}(0+0+0+0)=0$
$d_{134}\left(\begin{pmatrix}{[0.25,0.50]},\\{[0.25,0.50]}\end{pmatrix},\begin{pmatrix}{[0.20,0.30]},\\{[0.60,0.70]}\end{pmatrix}\right)=$ $\frac{1}{4}(0.05+0.2+0.35+0.2)=0.20$	$d_{234}\left(\begin{pmatrix}{[0.20,0.40]},\\{[0.20,0.40]}\end{pmatrix},\begin{pmatrix}{[0.40,0.70]},\\{[0.20,0.30]}\end{pmatrix}\right)=$ $\frac{1}{4}(0.2+0.3+0+0.1)=0.15$
$d_{141}\left(\begin{pmatrix}{[0.20,0.30]},\\{[0.60,0.70]}\end{pmatrix},\begin{pmatrix}{[0.10,0.20]},\\{[0.10,0.20]}\end{pmatrix}\right)=$ $\frac{1}{4}(0.1+0.1+0.5+0.5)=0.3$	$d_{241}\left(\begin{pmatrix}{[0.40,0.70]},\\{[0.20,0.30]}\end{pmatrix},\begin{pmatrix}{[0.25,0.50]},\\{[0.25,0.50]}\end{pmatrix}\right)=$ $\frac{1}{4}(0.15+0.2+0.05+0.2)=0.15$

$d_{142}\left(\begin{pmatrix}[0.20,0.30],\\ [0.60,0.70]\end{pmatrix},\begin{pmatrix}[0.50,0.60],\\ [0.20,0.30]\end{pmatrix}\right)=$ $\frac{1}{4}(0.3+0.3+0.4+0.4)=0.35$	$d_{242}\left(\begin{pmatrix}[0.40,0.70],\\ [0.20,0.30]\end{pmatrix},\begin{pmatrix}[0.20,0.50],\\ [0.20,0.50]\end{pmatrix}\right)=$ $\frac{1}{4}(0.2+0.2+0+0.2)=0.15$
$d_{143}\left(\begin{pmatrix}[0.20,0.30],\\ [0.60,0.70]\end{pmatrix},\begin{pmatrix}[0.25,0.50],\\ [0.25,0.50]\end{pmatrix}\right)=$ $\frac{1}{4}(0.05+0.2+0.35+0.2)=0.2$	$d_{243}\left(\begin{pmatrix}[0.40,0.70],\\ [0.20,0.30]\end{pmatrix},\begin{pmatrix}[0.20,0.40],\\ [0.20,0.40]\end{pmatrix}\right)=$ $\frac{1}{4}(0.2+0.3+0+0.1)=0.15$
$d_{144}\left(\begin{pmatrix}[0.20,0.30],\\ [0.60,0.70]\end{pmatrix},\begin{pmatrix}[0.20,0.30],\\ [0.60,0.70]\end{pmatrix}\right)=$ $\frac{1}{4}(0+0+0+0)=0$	$d_{244}\left(\begin{pmatrix}[0.40,0.70],\\ [0.20,0.30]\end{pmatrix},\begin{pmatrix}[0.40,0.70],\\ [0.20,0.30]\end{pmatrix}\right)=$ $\frac{1}{4}(0+0+0+0)=0$
$d_{311}\left(\begin{pmatrix}[0.40,0.50],\\ [0.30,0.50]\end{pmatrix},\begin{pmatrix}[0.40,0.50],\\ [0.30,0.50]\end{pmatrix}\right)=$ $\frac{1}{4}(0+0+0+0)=0$	$d_{411}\left(\begin{pmatrix}[0.25,0.50],\\ [0.25,0.50]\end{pmatrix},\begin{pmatrix}[0.25,0.50],\\ [0.25,0.50]\end{pmatrix}\right)=$ $\frac{1}{4}(0+0+0+0)=0$
$d_{312}\left(\begin{pmatrix}[0.40,0.50],\\ [0.30,0.50]\end{pmatrix},\begin{pmatrix}[0,0],\\ [0.25,0.75]\end{pmatrix}\right)=$ $\frac{1}{4}(0.4+0.5+0.05+0.25)=0.3$	$d_{412}\left(\begin{pmatrix}[0.25,0.50],\\ [0.25,0.50]\end{pmatrix},\begin{pmatrix}[0.20,0.30],\\ [0.60,0.70]\end{pmatrix}\right)=$ $\frac{1}{4}(0.05+0.2+0.35+0.2)=0.2$
$d_{313}\left(\begin{pmatrix}[0.40,0.50],\\ [0.30,0.50]\end{pmatrix},\begin{pmatrix}[0.20,0.30],\\ [0.40,0.70]\end{pmatrix}\right)=$ $\frac{1}{4}(0.2+0.2+0.1+0.2)=0.175$	$d_{413}\left(\begin{pmatrix}[0.25,0.50],\\ [0.25,0.50]\end{pmatrix},\begin{pmatrix}[0.10,0.20],\\ [0.10,0.20]\end{pmatrix}\right)=$ $\frac{1}{4}(0.15+0.3+0.15+0.3)=0.225$
$d_{314}\left(\begin{pmatrix}[0.40,0.50],\\ [0.30,0.50]\end{pmatrix},\begin{pmatrix}[0.20,0.50],\\ [0.20,0.50]\end{pmatrix}\right)=$ $\frac{1}{4}(0.2+0+0.1+0)=0.075$	$d_{414}\left(\begin{pmatrix}[0.25,0.50],\\ [0.25,0.50]\end{pmatrix},\begin{pmatrix}[0.50,0.60],\\ [0.20,0.30]\end{pmatrix}\right)=$ $\frac{1}{4}(0.25+0.1+0.05+0.2)=0.15$
$d_{321}\left(\begin{pmatrix}[0,0],\\ [0.25,0.75]\end{pmatrix},\begin{pmatrix}[0.40,0.50],\\ [0.30,0.50]\end{pmatrix}\right)=$ $\frac{1}{4}(0.4+0.5+0.05+0.25)=0.3$	$d_{421}\left(\begin{pmatrix}[0.20,0.30],\\ [0.60,0.70]\end{pmatrix},\begin{pmatrix}[0.25,0.50],\\ [0.25,0.50]\end{pmatrix}\right)=$ $\frac{1}{4}(0.05+0.2+0.35+0.2)=0.20$

$d_{322}\left(\begin{pmatrix}[0,0],\\ [0.25,0.75]\end{pmatrix},\begin{pmatrix}[0,0],\\ [0.25,0.75]\end{pmatrix}\right) =$ $\frac{1}{4}(0+0+0+0)=0$	$d_{422}\left(\begin{pmatrix}[0.20,0.30],\\ [0.60,0.70]\end{pmatrix},\begin{pmatrix}[0.20,0.30],\\ [0.60,0.70]\end{pmatrix}\right) =$ $\frac{1}{4}(0+0+0+0)=0$
$d_{323}\left(\begin{pmatrix}[0,0],\\ [0.25,0.75]\end{pmatrix},\begin{pmatrix}[0.20,0.30],\\ [0.40,0.70]\end{pmatrix}\right) =$ $\frac{1}{4}(0.2+0.3+0.15+0.05)=0.175$	$d_{423}\left(\begin{pmatrix}[0.20,0.30],\\ [0.60,0.70]\end{pmatrix},\begin{pmatrix}[0.10,0.20],\\ [0.10,0.20]\end{pmatrix}\right) =$ $\frac{1}{4}(0.1+0.1+0.5+0.5)=0.3$
$d_{324}\left(\begin{pmatrix}[0,0],\\ [0.25,0.75]\end{pmatrix},\begin{pmatrix}[0.20,0.50],\\ [0.20,0.50]\end{pmatrix}\right) =$ $\frac{1}{4}(0.2+0.5+0.05+0.25)=0.25$	$d_{424}\left(\begin{pmatrix}[0.20,0.30],\\ [0.60,0.70]\end{pmatrix},\begin{pmatrix}[0.50,0.60],\\ [0.20,0.30]\end{pmatrix}\right) =$ $\frac{1}{4}(0.3+0.3+0.4+0.4)=0.35$
$d_{331}\left(\begin{pmatrix}[0.20,0.30],\\ [0.40,0.70]\end{pmatrix},\begin{pmatrix}[0.40,0.50],\\ [0.30,0.50]\end{pmatrix}\right) =$ $\frac{1}{4}(0.2+0.2+0.1+0.2)=0.175$	$d_{431}\left(\begin{pmatrix}[0.10,0.20],\\ [0.10,0.20]\end{pmatrix},\begin{pmatrix}[0.25,0.50],\\ [0.25,0.50]\end{pmatrix}\right) =$ $\frac{1}{4}(0.15+0.3+0.15+0.3)=0.225$
$d_{332}\left(\begin{pmatrix}[0.20,0.30],\\ [0.40,0.70]\end{pmatrix},\begin{pmatrix}[0,0],\\ [0.25,0.75]\end{pmatrix}\right) =$ $\frac{1}{4}(0.2+0.3+0.15+0.05)=0.175$	$d_{432}\left(\begin{pmatrix}[0.10,0.20],\\ [0.10,0.20]\end{pmatrix},\begin{pmatrix}[0.20,0.30],\\ [0.60,0.70]\end{pmatrix}\right) =$ $\frac{1}{4}(0.1+0.1+0.5+0.5)=0.3$
$d_{333}\left(\begin{pmatrix}[0.20,0.30],\\ [0.40,0.70]\end{pmatrix},\begin{pmatrix}[0.20,0.30],\\ [0.40,0.70]\end{pmatrix}\right) =$ $\frac{1}{4}(0+0+0+0)=0$	$d_{433}\left(\begin{pmatrix}[0.10,0.20],\\ [0.10,0.20]\end{pmatrix},\begin{pmatrix}[0.10,0.20],\\ [0.10,0.20]\end{pmatrix}\right) =$ $\frac{1}{4}(0+0+0+0)=0$
$d_{334}\left(\begin{pmatrix}[0.20,0.30],\\ [0.40,0.70]\end{pmatrix},\begin{pmatrix}[0.20,0.50],\\ [0.20,0.50]\end{pmatrix}\right) =$ $\frac{1}{4}(0+0.2+0.2+0.2)=0.15$	$d_{434}\left(\begin{pmatrix}[0.10,0.20],\\ [0.10,0.20]\end{pmatrix},\begin{pmatrix}[0.50,0.60],\\ [0.20,0.30]\end{pmatrix}\right) =$ $\frac{1}{4}(0.4+0.4+0.1+0.1)=0.25$
$d_{341}\left(\begin{pmatrix}[0.20,0.50],\\ [0.20,0.50]\end{pmatrix},\begin{pmatrix}[0.40,0.50],\\ [0.30,0.50]\end{pmatrix}\right) =$ $\frac{1}{4}(0.2+0+0.1+0)=0.075$	$d_{441}\left(\begin{pmatrix}[0.50,0.60],\\ [0.20,0.30]\end{pmatrix},\begin{pmatrix}[0.25,0.50],\\ [0.25,0.50]\end{pmatrix}\right) =$ $\frac{1}{4}(0.25+0.1+0.05+0.2)=0.15$

$d_{342}\left(\begin{pmatrix}[0.20,0.50],\\ [0.20,0.50]\end{pmatrix},\begin{pmatrix}[0,0],\\ [0.25,0.75]\end{pmatrix}\right) =$ $\frac{1}{4}(0.2+0.5+0.05+0.25)=0.25$	$d_{442}\left(\begin{pmatrix}[0.50,0.60],\\ [0.20,0.30]\end{pmatrix},\begin{pmatrix}[0.20,0.30],\\ [0.60,0.70]\end{pmatrix}\right) =$ $\frac{1}{4}(0.3+0.3+0.4+0.4)=0.35$
$d_{343}\left(\begin{pmatrix}[0.20,0.50],\\ [0.20,0.50]\end{pmatrix},\begin{pmatrix}[0.20,0.30],\\ [0.40,0.70]\end{pmatrix}\right) =$ $\frac{1}{4}(0+0.2+0.2+0.2)=0.15$	$d_{443}\left(\begin{pmatrix}[0.50,0.60],\\ [0.20,0.30]\end{pmatrix},\begin{pmatrix}[0.10,0.20],\\ [0.10,0.20]\end{pmatrix}\right) =$ $\frac{1}{4}(0.4+0.4+0.1+0.1)=0.25$
$d_{344}\left(\begin{pmatrix}[0.20,0.50],\\ [0.20,0.50]\end{pmatrix},\begin{pmatrix}[0.20,0.50],\\ [0.20,0.50]\end{pmatrix}\right) =$ $\frac{1}{4}(0+0+0+0)=0$	$d_{444}\left(\begin{pmatrix}[0.50,0.60],\\ [0.20,0.30]\end{pmatrix},\begin{pmatrix}[0.50,0.60],\\ [0.20,0.30]\end{pmatrix}\right) =$ $\frac{1}{4}(0+0+0+0)=0$

Step 3: Using Step 3 of the Jujhar method, proposed in Section 4.13,

$c_1=\sum_{i=1}^{4}\sum_{k=1}^{4} d_{i1k}(\tilde{\alpha}_{i1},\tilde{\alpha}_{k1}) = 0+0.25+0.225+0.3+0.25+0+0.15+0.35+0.225+0.15+0+0.2+0.3+0.35+0.2+0=2.95$,

$c_2=\sum_{i=1}^{4}\sum_{k=1}^{4} d_{i2k}(\tilde{\alpha}_{i2},\tilde{\alpha}_{k2}) = 0+0.025+0.075+0.15+0.025+0+0.05+0.15+0.075+0.05+0+0.15+0.15+0.15+0.15+0=1.2$,

$c_3=\sum_{i=1}^{4}\sum_{k=1}^{4} d_{i3k}(\tilde{\alpha}_{i3},\tilde{\alpha}_{k3}) = 0+0.3+0.175+0.075+0.3+0+0.175+0.25+0.175+0.175+0+0.15+0.075+0.25+0.15+0=2.25$,

$c_4=\sum_{i=1}^{4}\sum_{k=1}^{4} d_{i4k}(\tilde{\alpha}_{i4},\tilde{\alpha}_{k4}) = 0+0.2+0.225+0.15+0.2+0+0.3+0.35+0.225+0.3+0+0.25+0.15+0.35+0.25+0=2.95$.

Step 4: Using Step 4 of the Jujhar method, proposed in Section 4.13, there is need to solve the CLPP (P4.19).

$max(2.95w_1+1.2w_2+2.25w_3+2.95w_4)$

Subject to

$$\begin{cases} 0.10 \le w_1 \le 0.50, \\ 0.20 \le w_2 \le 0.50, \\ 0.15 \le w_3 \le 0.60, \\ 0.10 \le w_4 \le 0.80, \\ w_1 + w_2 + w_3 + w_4 = 1, \\ w_1 = w_4, \\ w_1 \ge 0, w_2 \ge 0, w_3 \ge 0, w_4 \ge 0. \end{cases} \tag{P4.19}$$

It can be easily verified that on solving the CLPP (P4.19), the obtained OAWV is $(w_1, w_2, w_3, w_4) = (0.3250, 0.20, 0.15, 0.3250)$.

Step 5: On considering the OAWV, $(w_1, w_2, w_3, w_4) = (0.3250, 0.20, 0.15, 0.3250)$ and using Step 5 of Jujhar method, proposed in Section 4.13,

$\tilde{P}_1 = ([(0.3250)(0.10) + (0.20)(0.25) + (0.15)(0.40) + (0.3250)(0.25), (0.3250)(0.20) + (0.20)(0.50) + (0.15)(0.50) + (0.3250)(0.50)], [(0.3250)(0.10) + (0.20)(0.25) + (0.15)(0.30) + (0.3250)(0.25), (0.3250)(0.20) + (0.20)(0.50) + (0.15)(0.50) + (0.3250)(0.50)] = ([0.2237, 0.4025], [0.2087, 0.4025])$,

$\tilde{P}_2 = ([(0.3250)(0.50) + (0.20)(0.20) + (0.15)(0) + (0.3250)(0.20), (0.3250)(0.60) + (0.20)(0.50) + (0.15)(0) + (0.3250)(0.30)], [(0.3250)(0.20) + (0.20)(0.20) + (0.15)(0.25) + (0.3250)(0.60), (0.3250)(0.30) + (0.20)(0.50) + (0.15)(0.75) + (0.3250)(0.70)] = ([0.2675, 0.3925], [0.3375, 0.5375])$,

$\tilde{P}_3 = ([(0.3250)(0.25) + (0.20)(0.20) + (0.15)(0.20) + (0.3250)(0.10), (0.3250)(0.50) + (0.20)(0.40) + (0.15)(0.30) + (0.3250)(0.20)], [(0.3250)(0.25) + (0.20)(0.20) + (0.15)(0.40) + (0.3250)(0.10), (0.3250)(0.50) + (0.20)(0.40) + (0.15)(0.70) + (0.3250)(0.20)] = ([0.1837, 0.3525], [0.2137, 0.4125])$,

$\tilde{P}_4 = ([(0.3250)(0.20) + (0.20)(0.40) + (0.15)(0.20) + (0.3250)(0.50), (0.3250)(0.30) + (0.20)(0.70) + (0.15)(0.50) + (0.3250)(0.60)], [(0.3250)(0.60) + (0.20)(0.20) + (0.15)(0.20) + (0.3250)(0.20), (0.3250)(0.70) + (0.20)(0.30) + (0.15)(0.50) + (0.3250)(0.30)] = ([0.3375, 0.5075], [0.3300, 0.4600])$.

Step 6: Using Step 6 of the Jujhar method, proposed in Section 4.13,

$S(\tilde{P}_1) = 0.0075, S(\tilde{P}_2) = -0.1075, S(\tilde{P}_3) = -0.045$ and $S(\tilde{P}_4) = 0.0275$.

Since, $S(\tilde{P}_4) > S(\tilde{P}_1) > S(\tilde{P}_3) > S(\tilde{P}_2)$. So, according to Step 6 of the Jujhar method, proposed in Section 4.13, the ranking of the alternatives is $A_4 > A_1 > A_3 > A_2$.

4.14.9 Exact results of the ninth IVIFMADMPr having partially known attribute weights

Using the proposed Jujhar method, the exact result of the IVIFMADMPr having partially known attribute weights, considered in the second point of Section 4.8, can be obtained as follows:

Since, Step 1 to Step 3 of the proposed Jujhar method are same as shown in Section 4.14.9. So, to avoid any repetition, the calculations have been started from Step 4.

Step 4: Since, the condition $\sum_{j=1}^{n} w_j^u < 1$ is satisfying. So, using Case (i) of Step 4 of the Jujhar method, proposed in Section 4.13, there is need to solve the CLPP (P4.20).

$max(2.95w_1 + 1.2w_2 + 2.25w_3 + 2.95w_4)$

Subject to

$$\begin{cases} 0.10 \leq w_1 \leq 0.20, \\ 0.10 \leq w_2 \leq 0.30, \\ 0.20 \leq w_3 \leq 0.25, \\ 0.10 \leq w_4 \leq 0.15, \\ w_1 + w_2 + w_3 + w_4 \leq 1, \\ w_1 = w_4, \\ w_1 \geq 0, w_2 \geq 0, w_3 \geq 0, w_4 \geq 0. \end{cases} \tag{P4.20}$$

On solving the CLPP (P4.20) the obtained non-normalized OAWV is $(w_1, w_2, w_3, w_4) = (0.15, 0.30, 0.25, 0.15)$ and the normalized OAWV is $(w_1, w_2, w_3, w_4) = \left(\frac{0.15}{0.15+0.30+0.25+0.15}, \frac{0.30}{0.15+0.30+0.25+0.15}, \frac{0.25}{0.15+0.30+0.25+0.15}, \frac{0.15}{0.15+0.30+0.25+0.15}\right)$

$= (0.1765, 0.3529, 0.2941, 0.1765)$.

Step 5: On considering the normalized OAWV, $(w_1, w_2, w_3, w_4) = (0.1765, 0.3529, 0.2941, 0.1765)$ and using Step 5 of Jujhar method, proposed in Section 4.13,

$\tilde{P}_1 = ([(0.1765)(0.10) + (0.3529)(0.25) + (0.2941)(0.40) + (0.1765)(0.25), (0.1765)$

$(0.20) + (0.3529)(0.50)+(0.2941)(0.50) + (0.1765)(0.50)], [(0.1765)(0.10) +$
$(0.3529)(0.25) + (0.2941)(0.30) + (0.1765)(0.25), (0.1765)(0.20) + (0.3529)(0.50) +$
$(0.2941)(0.50) + (0.1765)(0.50)] = ([0.2676,0.4470], [0.2382,0.4470]),$

$\tilde{P}_2 = ([(0.1765)(0.50) + (0.3529)(0.20) + (0.2941)(0) + (0.1765)(0.20), (0.1765)$
$(0.60) + (0.3529)(0.50)+(0.2941)(0) + (0.1765)(0.30)], [(0.1765)(0.20) +(0.3529)$
$(0.20) + (0.2941)(0.25) + (0.1765)(0.60), (0.1765)(0.30) + (0.3529)(0.50) +$
$(0.2941)(0.75) + (0.1765)(0.70)] = ([0.1941,0.3353], [0.2853,0.5735]),$

$\tilde{P}_3 = ([(0.1765)(0.25) + (0.3529)(0.20) + (0.2941)(0.20) + (0.1765)(0.10), (0.1765)$
$(0.50) + (0.3529)(0.40)+(0.2941)(0.30) + (0.1765)(0.20)], [(0.1765)(0.25) +$
$(0.3529)(0.20) + (0.2941)(0.40) + (0.1765)(0.10), (0.1765)(0.50) + (0.3529)(0.40) +$
$(0.2941)(0.70) + (0.1765)(0.20)] = ([0.1911,0.3529], [0.2499,0.4705]),$

$\tilde{P}_4 = ([(0.1765)(0.10) + (0.3529)(0.10) + (0.2941)(0.50) + (0.1765)(0.20), (0.1765)$
$(0.30) + (0.3529)(0.20)+(0.2941)(0.50) + (0.1765)(0.30)], [(0.1765)(0.10) +$
$(0.3529)(0.30) + (0.2941)(0.10) + (0.1765)(0.20), (0.1765)(0.50) + (0.3529)(0.40) +$
$(0.2941)(0.40) + (0.1765)(0.30)] = ([0.3235,0.5529], [0.2706,0.4294]).$

Step 6: Using Step 6 of the Jujhar method, proposed in Section 4.13,
$S(\tilde{P}_1) = 0.0147, S(\tilde{P}_2) = -0.1647, S(\tilde{P}_3) = -0.0882$ and $S(\tilde{P}_4) = 0.0882$.

Since, $S(\tilde{P}_4) > S(\tilde{P}_1) > S(\tilde{P}_3) > S(\tilde{P}_2)$. So, according to Step 6 of the Jujhar method, proposed in Section 4.13, the ranking of the alternatives is $A_4 > A_1 > A_3 > A_2$.

4.14.10 Exact results of the tenth IVIFMADMPr having partially known attribute weights

Using the proposed Jujhar method, the exact result of the IVIFMADMPr having partially known attribute weights, considered in the second point of Section 4.8, can be obtained as follows:

Since, Step 1 to Step 3 of the proposed Jujhar method are same as shown in Section 4.14.8. So, to avoid any repetition, the calculations have been started from Step 4.

Step 4: Since, the condition $\sum_{j=1}^{n} w_j^l > 1$ is satisfying. So, using Case (ii) of Step 4 of the Jujhar method, proposed in Section 4.13, there is need to solve the CLPP (P4.21).

$max\ (2.95w_1 + 1.2w_2 + 2.25w_3 + 2.95w_4 - \varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4)$

Subject to

$$\begin{cases} \frac{20}{170} \le w_1 \le \frac{40}{170}, \\ \frac{50}{170} \le w_2 \le \frac{60}{170}, \\ \frac{70}{170} \le w_3 \le \frac{80}{170}, \\ \frac{30}{170} \le w_4 \le \frac{90}{170}, \\ w_1 + w_2 + w_3 + w_4 = 1, \\ 170 = 100 + \varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4, \\ w_1 = w_4, \\ w_1 \ge 0, w_2 \ge 0, w_3 \ge 0, w_4 \ge 0, \varepsilon_1 \ge 0, \varepsilon_2 \ge 0, \varepsilon_3 \ge 0, \varepsilon_4 \ge 0. \end{cases} \tag{P4.21}$$

On solving the CLPP (P4.21) the obtained normalized OAWV is $(w_1, w_2, w_3, w_4) = (0.1765, 0.2353, 0.4117, 0.1765)$.

Step 5: On considering the normalized OAWV, $(w_1, w_2, w_3, w_4) = (0.1765, 0.2353, 0.4117, 0.1765)$ and using Step 5 of Jujhar method, proposed in Section 4.13,

$\tilde{P}_1 = ([(0.1765)(0.10) + (0.2353)(0.25) + (0.4117)(0.40) + (0.1765)(0.25), (0.1765)(0.20) + (0.2353)(0.50) + (0.4117)(0.50) + (0.1765)(0.50)], [(0.1765)(0.10) + (0.2353)(0.25) + (0.4117)(0.30) + (0.1765)(0.25), (0.1765)(0.20) + (0.2353)(0.50) + (0.4117)(0.50) + (0.1765)(0.50)] = ([0.2852, 0.4470], [0.2441, 0.4470]),$

$\tilde{P}_2 = ([(0.1765)(0.50) + (0.2353)(0.20) + (0.4117)(0) + (0.1765)(0.20), (0.1765)(0.60) + (0.2353)(0.50) + (0.4117)(0) + (0.1765)(0.30)], [(0.1765)(0.20) + (0.2353)(0.20) + (0.4117)(0.25) + (0.1765)(0.60), (0.1765)(0.30) + (0.2353)(0.50) + (0.4117)(0.75) + (0.1765)(0.70)] = ([0.1706, 0.2765], [0.2911, 0.6029]),$

$\tilde{P}_3 = ([(0.1765)(0.25) + (0.2353)(0.20) + (0.4117)(0.20) + (0.1765)(0.10), (0.1765)$ $(0.50) + (0.2353)(0.40)+(0.4117)(0.30) + (0.1765)(0.20)], [(0.1765)(0.25) +$ $(0.2353)(0.20) + (0.4117)(0.40) + (0.1765)(0.10), (0.1765)(0.50) + (0.2353)(0.40) +$ $(0.4117)(0.70) + (0.1765)(0.20)] = ([0.1911,0.3411], [0.2735,0.5058]),$

$\tilde{P}_4 = ([(0.1765)(0.20) + (0.2353)(0.40) + (0.4117)(0.20) + (0.1765)(0.50), (0.1765)$ $(0.30) + (0.2353)(0.70)+(0.4117)(0.50) + (0.1765)(0.60)], [(0.1765)(0.60) +$ $(0.2353)(0.20) + (0.4117)(0.20) + (0.1765)(0.20), (0.1765)(0.70) + (0.2353)(0.30) +$ $(0.4117)(0.50) + (0.1765)(0.30)] = ([0.3001,0.5294], [0.2706,0.4529]).$

Step 6: Using Step 6 of the Jujhar method, proposed in Section 4.13,

$S(\tilde{P}_1) = 0.0205, S(\tilde{P}_2) = -0.2234, S(\tilde{P}_3) = -0.1235$ and $S(\tilde{P}_4) = 0.0530$.

Since, $S(\tilde{P}_4) > S(\tilde{P}_1) > S(\tilde{P}_3) > S(\tilde{P}_2)$. So, according to Step 6 of the Jujhar method, proposed in Section 4.13, the ranking of the alternatives is $A_4 > A_1 > A_3 > A_2$.

4.14.11 Exact result of eleventh IVIFMADMPr having partially known attribute weights

Using the proposed Jujhar method, the exact result of the IVIFMADMPr having partially known attribute weights, considered in the third point of Section 4.8, can be obtained as follows:

Step 1: Since, all the attributes are benefit type. So, according to Step 1 of the Jujhar method, proposed in Section 4.13, there is no need to apply Step 1.

Step 2: According to Step 2 of the Jujhar method, proposed in Section 4.13, there is need to calculate $d_{ijk}(\tilde{\alpha}_{ij}, \tilde{\alpha}_{kj})\ \forall\ i = 1,2; j = 1,2$. These values are shown in Table 4.22.

Table 4.22: Values of $d_{ijk}(\tilde{\alpha}_{ij}, \tilde{\alpha}_{kj})$

$d_{111}\left(\begin{pmatrix}[0.20,0.40],\\ [0.20,0.40]\end{pmatrix}, \begin{pmatrix}[0.20,0.40],\\ [0.20,0.40]\end{pmatrix}\right) =$ $\frac{1}{4}(0 + 0 + 0 + 0) = 0$	$d_{211}\left(\begin{pmatrix}[1,1],\\ [0,0]\end{pmatrix}, \begin{pmatrix}[1,1],\\ [0,0]\end{pmatrix}\right)$ $= \frac{1}{4}(0 + 0 + 0 + 0) = 0$

$d_{112}\left(\begin{pmatrix}[0.20,0.40],\\ [0.20,0.40]\end{pmatrix},\begin{pmatrix}[0.15,0.25],\\ [0.15,0.25]\end{pmatrix}\right)=$ $\frac{1}{4}(0.05+0.15+0.05+0.15)=0.10$	$d_{212}\left(\begin{pmatrix}[1,1],\\ [0,0]\end{pmatrix},\begin{pmatrix}[1,1],\\ [0,0]\end{pmatrix}\right)$ $=\frac{1}{4}(0+0+0+0)=0$
$d_{121}\left(\begin{pmatrix}[0.15,0.25],\\ [0.15,0.25]\end{pmatrix},\begin{pmatrix}[0.20,0.40],\\ [0.20,0.40]\end{pmatrix}\right)=$ $\frac{1}{4}(0.05+0.15+0.05+0.15)=0.10$	$d_{213}\left(\begin{pmatrix}[1,1],\\ [0,0]\end{pmatrix},\begin{pmatrix}[1,1],\\ [0,0]\end{pmatrix}\right)$ $=\frac{1}{4}(0+0+0+0)=0$
$d_{122}\left(\begin{pmatrix}[0.15,0.25],\\ [0.15,0.25]\end{pmatrix},\begin{pmatrix}[0.15,0.25],\\ [0.15,0.25]\end{pmatrix}\right)=$ $\frac{1}{4}(0+0+0+0)=0$	$d_{214}\left(\begin{pmatrix}[1,1],\\ [0,0]\end{pmatrix},\begin{pmatrix}[1,1],\\ [0,0]\end{pmatrix}\right)$ $=\frac{1}{4}(0+0+0+0)=0$

Step 3: Using Step 3 of the Jujhar method, proposed in Section 4.13,

$c_1=\sum_{i=1}^{2}\sum_{k=1}^{2} d_{i1k}(\tilde{\alpha}_{i1},\tilde{\alpha}_{k1})=0+0.1+0.1+0+0+0+0+0=0.2,$

$c_2=\sum_{i=1}^{2}\sum_{k=1}^{2} d_{i1k}(\tilde{\alpha}_{i1},\tilde{\alpha}_{k1})=0+0+0+0+0+0+0+0=0.$

Step 4: Using Step 4 of the Jujhar method, proposed in Section 4.13, there is need to solve the CLPP (P4.22).

$max(0.4w_1)$

Subject to

$$\begin{cases}0.1\le w_1\le 0.7,\\ 0.2\le w_2\le 0.6,\\ w_1+w_2=1,\\ w_1\ge 0, w_2\ge 0.\end{cases} \qquad \text{(P4.22)}$$

It can be easily verified that on solving the CLPP (P4.22) the obtained OAWV is $(w_1,w_2)=(0.7,0.3)$.

Step 5: On considering the OAWV, $(w_1,w_2)=(0.7,0.3)$ and using Step 5 of the Jujhar method, proposed in Section 4.13,

$\tilde{P}_1=([(0.70)(0.20)+(0.30)(1),(0.70)(0.40)+(0.30)(1)],[(0.70)(0.20)+(0.30)(0),(0.70)(0.40)+(0.30)(0)]=([0.44,0.58],[0.14,0.28]),$

$\tilde{P}_2 = ([(0.70)(0.15) + (0.30)(1), (0.70)(0.25) + (0.30)(1)], [(0.70)(0.15) + (0.30)(0), (0.70)(0.25) + (0.30)(0)] = ([0.405,0.475], [0.105,0.175])$.

Step 6: Using Step 6 of the Jujhar method, proposed in Section 4.13,

$S(\tilde{P}_1) = 0.30$ and $S(\tilde{P}_2) = 0.30$.

Since, $S(\tilde{P}_1) = S(\tilde{P}_2)$.So, there is need to apply Step 7 of the Jujhar method, proposed in Section 4.13.

Step 7: Using Step 7 of the Jujhar method, proposed in Section 4.13,

$H(\tilde{P}_1) = 0.72$ and $H(\tilde{P}_2) = 0.58$.

Since, $H(\tilde{P}_1) > H(\tilde{P}_2)$. Therefore, according to case (ii) of Step 7 of the proposed Jujhar method, the ranking of the alternatives is $A_1 > A_2$.

4.14.12 Exact results of twelfth IVIFMADMPr having partially known attribute weights

Using the proposed Jujhar method, the exact result of the IVIFMADMPr having partially known attribute weights, considered in the first point of Section 4.11, can be obtained as follows:

Step 1: Since, all the attributes are benefit type. So, according to Step 1 of the Jujhar method, proposed in Section 4.13, there is no need to apply Step 1.

Step 2: According to Step 2 of the Jujhar method, proposed in Section 4.13, there is need to calculate $d_{ijk}(\tilde{\alpha}_{ij}, \tilde{\alpha}_{kj}) \; \forall \; i = 1,2, \ldots, 4; j = 1,2, \ldots, 5$. These values are shown in Table 4.23.

Table 4.23: Values of $d_{ijk}(\tilde{\alpha}_{ij}, \tilde{\alpha}_{kj})$

$d_{111}\left(\begin{pmatrix}[0.16,0.16],\\ [0.17,0.50]\end{pmatrix}, \begin{pmatrix}[0.16,0.16],\\ [0.17,0.50]\end{pmatrix}\right) =$ $\frac{1}{4}(0+0+0+0) = 0$	$d_{211}\left(\begin{pmatrix}[0.24,0.76],\\ [0,0]\end{pmatrix}, \begin{pmatrix}[0.24,0.76],\\ [0,0]\end{pmatrix}\right) =$ $\frac{1}{4}(0+0+0+0) = 0$

$d_{112}\left(\begin{pmatrix}[0.16,0.16],\\ [0.17,0.50]\end{pmatrix},\begin{pmatrix}[0.16,0.16],\\ [0.17,0.50]\end{pmatrix}\right) =$ $\frac{1}{4}(0+0+0+0) = 0$	$d_{212}\left(\begin{pmatrix}[0.24,0.76],\\ [0,0]\end{pmatrix},\begin{pmatrix}[0.14,0.45],\\ [0.15,0.25]\end{pmatrix}\right) =$ $\frac{1}{4}(0.1+0.31+0.15+0.25) = 0.2025$
$d_{113}\left(\begin{pmatrix}[0.16,0.16],\\ [0.17,0.50]\end{pmatrix},\begin{pmatrix}[0.24,0.24],\\ [0.26,0.26]\end{pmatrix}\right) =$ $\frac{1}{4}(0.08+0.08+0.09+0.24) = 0.1225$	$d_{213}\left(\begin{pmatrix}[0.24,0.76],\\ [0,0]\end{pmatrix},\begin{pmatrix}[0.24,0.24],\\ [0.26,0.26]\end{pmatrix}\right) =$ $\frac{1}{4}(0+0.52+0.26+0.26) = 0.2600$
$d_{114}\left(\begin{pmatrix}[0.16,0.16],\\ [0.17,0.50]\end{pmatrix},\begin{pmatrix}[0.16,0.30],\\ [0.30,0.30]\end{pmatrix}\right) =$ $\frac{1}{4}(0+0.14+0.13+0.2) = 0.1175$	$d_{214}\left(\begin{pmatrix}[0.24,0.76],\\ [0,0]\end{pmatrix},\begin{pmatrix}[0.125,0.375],\\ [0.127,0.374]\end{pmatrix}\right) =$ $\frac{1}{4}(0.115+0.385+0.127+0.374) = 0.2502$
$d_{121}\left(\begin{pmatrix}[0.16,0.16],\\ [0.17,0.50]\end{pmatrix},\begin{pmatrix}[0.16,0.16],\\ [0.17,0.50]\end{pmatrix}\right) =$ $\frac{1}{4}(0+0+0+0) = 0$	$d_{221}\left(\begin{pmatrix}[0.14,0.45],\\ [0.15,0.25]\end{pmatrix},\begin{pmatrix}[0.24,0.76],\\ [0,0]\end{pmatrix}\right) =$ $\frac{1}{4}(0.1+0.31+0.15+0.25) = 0.2025$
$d_{122}\left(\begin{pmatrix}[0.16,0.16],\\ [0.17,0.50]\end{pmatrix},\begin{pmatrix}[0.16,0.16],\\ [0.17,0.50]\end{pmatrix}\right) =$ $\frac{1}{4}(0+0+0+0) = 0$	$d_{222}\left(\begin{pmatrix}[0.14,0.45],\\ [0.15,0.25]\end{pmatrix},\begin{pmatrix}[0.14,0.45],\\ [0.15,0.25]\end{pmatrix}\right) =$ $\frac{1}{4}(0+0+0+0) = 0$
$d_{123}\left(\begin{pmatrix}[0.16,0.16],\\ [0.17,0.50]\end{pmatrix},\begin{pmatrix}[0.24,0.24],\\ [0.26,0.26]\end{pmatrix}\right) =$ $\frac{1}{4}(0.08+0.08+0.09+0.24) = 0.1225$	$d_{223}\left(\begin{pmatrix}[0.14,0.45],\\ [0.15,0.25]\end{pmatrix},\begin{pmatrix}[0.24,0.24],\\ [0.26,0.26]\end{pmatrix}\right) =$ $\frac{1}{4}(0.1+0.21+0.11+0.01) = 0.1075$
$d_{124}\left(\begin{pmatrix}[0.16,0.16],\\ [0.17,0.50]\end{pmatrix},\begin{pmatrix}[0.16,0.30],\\ [0.30,0.30]\end{pmatrix}\right) =$ $\frac{1}{4}(0+0.14+0.13+0.2) = 0.1175$	$d_{224}\left(\begin{pmatrix}[0.14,0.45],\\ [0.15,0.25]\end{pmatrix},\begin{pmatrix}[0.125,0.375],\\ [0.127,0.374]\end{pmatrix}\right) =$ $\frac{1}{4}(0.015+0.075+0.023+0.124) = 0.2370$
$d_{131}\left(\begin{pmatrix}[0.24,0.24],\\ [0.26,0.26]\end{pmatrix},\begin{pmatrix}[0.16,0.16],\\ [0.17,0.50]\end{pmatrix}\right) =$ $\frac{1}{4}(0.08+0.08+0.09+0.24) = 0.1225$	$d_{231}\left(\begin{pmatrix}[0.24,0.24],\\ [0.26,0.26]\end{pmatrix},\begin{pmatrix}[0.24,0.76],\\ [0,0]\end{pmatrix}\right) =$ $\frac{1}{4}(0+0.52+0.26+0.26) = 0.26$

$d_{132}\left(\begin{pmatrix}[0.24,0.24],\\ [0.26,0.26]\end{pmatrix},\begin{pmatrix}[0.16,0.16],\\ [0.17,0.50]\end{pmatrix}\right)=$ $\frac{1}{4}(0.08+0.08+0.09+0.24)=0.1225$	$d_{232}\left(\begin{pmatrix}[0.24,0.24],\\ [0.26,0.26]\end{pmatrix},\begin{pmatrix}[0.14,0.45],\\ [0.15,0.25]\end{pmatrix}\right)=$ $\frac{1}{4}(0.1+0.21+0.09+0.01)=0.1025$
$d_{133}\left(\begin{pmatrix}[0.24,0.24],\\ [0.26,0.26]\end{pmatrix},\begin{pmatrix}[0.24,0.24],\\ [0.26,0.26]\end{pmatrix}\right)=$ $\frac{1}{4}(0+0+0+0)=0$	$d_{233}\left(\begin{pmatrix}[0.24,0.24],\\ [0.26,0.26]\end{pmatrix},\begin{pmatrix}[0.24,0.24],\\ [0.26,0.26]\end{pmatrix}\right)=$ $\frac{1}{4}(0+0+0+0)=0$
$d_{134}\left(\begin{pmatrix}[0.24,0.24],\\ [0.26,0.26]\end{pmatrix},\begin{pmatrix}[0.16,0.30],\\ [0.30,0.30]\end{pmatrix}\right)=$ $\frac{1}{4}(0.08+0.06+0.04+0.04)=0.055$	$d_{234}\left(\begin{pmatrix}[0.24,0.24],\\ [0.26,0.26]\end{pmatrix},\begin{pmatrix}[0.125,0.375],\\ [0.127,0.374]\end{pmatrix}\right)=$ $\frac{1}{4}(0.115+0.135+0.133+0.114)=0.1242$
$d_{141}\left(\begin{pmatrix}[0.16,0.30],\\ [0.30,0.30]\end{pmatrix},\begin{pmatrix}[0.16,0.16],\\ [0.17,0.50]\end{pmatrix}\right)=$ $\frac{1}{4}(0+0.14+0.13+0.2)=0.1175$	$d_{241}\left(\begin{pmatrix}[0.125,0.375],\\ [0.127,0.374]\end{pmatrix},\begin{pmatrix}[0.24,0.76],\\ [0,0]\end{pmatrix}\right)=$ $\frac{1}{4}(0.115+0.385+0.127+0.374)=0.2502$
$d_{142}\left(\begin{pmatrix}[0.16,0.30],\\ [0.30,0.30]\end{pmatrix},\begin{pmatrix}[0.16,0.16],\\ [0.17,0.50]\end{pmatrix}\right)=$ $\frac{1}{4}(0+0.14+0.13+0.2)=0.1175$	$d_{242}\left(\begin{pmatrix}[0.125,0.375],\\ [0.127,0.374]\end{pmatrix},\begin{pmatrix}[0.14,0.45],\\ [0.15,0.25]\end{pmatrix}\right)=$ $\frac{1}{4}(0.015+0.075+0.023+0.124)=0.0592$
$d_{143}\left(\begin{pmatrix}[0.16,0.30],\\ [0.30,0.30]\end{pmatrix},\begin{pmatrix}[0.24,0.24],\\ [0.26,0.26]\end{pmatrix}\right)=$ $\frac{1}{4}(0.08+0.06+0.04+0.04)=0.055$	$d_{243}\left(\begin{pmatrix}[0.125,0.375],\\ [0.127,0.374]\end{pmatrix},\begin{pmatrix}[0.24,0.24],\\ [0.26,0.26]\end{pmatrix}\right)=$ $\frac{1}{4}(0.115+0.135+0.133+0.114)=0.1242$
$d_{144}\left(\begin{pmatrix}[0.16,0.30],\\ [0.30,0.30]\end{pmatrix},\begin{pmatrix}[0.16,0.30],\\ [0.30,0.30]\end{pmatrix}\right)=$ $\frac{1}{4}(0+0+0+0)=0$	$d_{244}\left(\begin{pmatrix}[0.125,0.375],\\ [0.127,0.374]\end{pmatrix},\begin{pmatrix}[0.125,0.375],\\ [0.127,0.374]\end{pmatrix}\right)=$ $\frac{1}{4}(0+0+0+0)=0$
$d_{311}\left(\begin{pmatrix}[0,0],\\ [0.50,0.50]\end{pmatrix},\begin{pmatrix}[0,0],\\ [0.50,0.50]\end{pmatrix}\right)=$ $\frac{1}{4}(0+0+0+0)=0$	$d_{411}\left(\begin{pmatrix}[0.24,0.24],\\ [0.26,0.26]\end{pmatrix},\begin{pmatrix}[0.24,0.24],\\ [0.26,0.26]\end{pmatrix}\right)=$ $\frac{1}{4}(0+0+0+0)=0$

$d_{312}\left(\begin{pmatrix}[0,0],\\ [0.50,0.50]\end{pmatrix},\begin{pmatrix}[0.24,0.24],\\ [0.26,0.26]\end{pmatrix}\right)=$ $\frac{1}{4}(0.24+0.24+0.24+0.24)=0.24$	$d_{412}\left(\begin{pmatrix}[0.24,0.24],\\ [0.26,0.26]\end{pmatrix},\begin{pmatrix}[0.207,0.207],\\ [0.22,0.36]\end{pmatrix}\right)=$ $\frac{1}{4}(0.033+0.033+0.04+0.1)=0.0515$
$d_{313}\left(\begin{pmatrix}[0,0],\\ [0.50,0.50]\end{pmatrix},\begin{pmatrix}[0.24,0.24],\\ [0.26,0.26]\end{pmatrix}\right)=$ $\frac{1}{4}(0.24+0.24+0.24+0.24)=0.24$	$d_{413}\left(\begin{pmatrix}[0.24,0.24],\\ [0.26,0.26]\end{pmatrix},\begin{pmatrix}[0,0],\\ [0.50,0.50]\end{pmatrix}\right)=$ $\frac{1}{4}(0.24+0.24+0.24+0.24)=0.2400$
$d_{314}\left(\begin{pmatrix}[0,0],\\ [0.50,0.50]\end{pmatrix},\begin{pmatrix}[0.12,0.376],\\ [0.128,0.376]\end{pmatrix}\right)=$ $\frac{1}{4}(0.12+0.376+0.372+0.124)=0.2480$	$d_{414}\left(\begin{pmatrix}[0.24,0.24],\\ [0.26,0.26]\end{pmatrix},\begin{pmatrix}[0.16,0.50],\\ [0.17,0.17]\end{pmatrix}\right)=$ $\frac{1}{4}(0.08+0.26+0.09+0.09)=0.1300$
$d_{321}\left(\begin{pmatrix}[0.24,0.24],\\ [0.26,0.26]\end{pmatrix},\begin{pmatrix}[0,0],\\ [0.50,0.50]\end{pmatrix}\right)=$ $\frac{1}{4}(0.24+0.24+0.24+0.24)=0.24$	$d_{421}\left(\begin{pmatrix}[0.207,0.207],\\ [0.22,0.36]\end{pmatrix},\begin{pmatrix}[0.24,0.24],\\ [0.26,0.26]\end{pmatrix}\right)=$ $\frac{1}{4}(0.033+0.033+0.04+0.1)=0.0515$
$d_{322}\left(\begin{pmatrix}[0.24,0.24],\\ [0.26,0.26]\end{pmatrix},\begin{pmatrix}[0.24,0.24],\\ [0.26,0.26]\end{pmatrix}\right)=$ $\frac{1}{4}(0+0+0+0)=0$	$d_{422}\left(\begin{pmatrix}[0.207,0.207],\\ [0.22,0.36]\end{pmatrix},\begin{pmatrix}[0.207,0.207],\\ [0.22,0.36]\end{pmatrix}\right)=$ $\frac{1}{4}(0+0+0+0)=0$
$d_{323}\left(\begin{pmatrix}[0.24,0.24],\\ [0.26,0.26]\end{pmatrix},\begin{pmatrix}[0.24,0.24],\\ [0.26,0.26]\end{pmatrix}\right)=$ $\frac{1}{4}(0+0+0+0)=0$	$d_{423}\left(\begin{pmatrix}[0.207,0.207],\\ [0.22,0.36]\end{pmatrix},\begin{pmatrix}[0,0],\\ [0.50,0.50]\end{pmatrix}\right)=$ $\frac{1}{4}(0.207+0.207+0.28+0.14)=0.208$
$d_{324}\left(\begin{pmatrix}[0.24,0.24],\\ [0.26,0.26]\end{pmatrix},\begin{pmatrix}[0.12,0.376],\\ [0.128,0.376]\end{pmatrix}\right)=$ $\frac{1}{4}(0.12+0.136+0.132+0.116)=0.1260$	$d_{424}\left(\begin{pmatrix}[0.207,0.207],\\ [0.22,0.36]\end{pmatrix},\begin{pmatrix}[0.16,0.50],\\ [0.17,0.17]\end{pmatrix}\right)=$ $\frac{1}{4}(0.047+0.293+0.05+0.19)=0.145$
$d_{331}\left(\begin{pmatrix}[0.24,0.24],\\ [0.26,0.26]\end{pmatrix},\begin{pmatrix}[0,0],\\ [0.50,0.50]\end{pmatrix}\right)=$ $\frac{1}{4}(0.24+0.24+0.24+0.24)=0.24$	$d_{431}\left(\begin{pmatrix}[0,0],\\ [0.50,0.50]\end{pmatrix},\begin{pmatrix}[0.24,0.24],\\ [0.26,0.26]\end{pmatrix}\right)=$ $\frac{1}{4}(0.24+0.24+0.24+0.24)=0.24$

$d_{332}\left(\begin{pmatrix}[0.24,0.24],\\ [0.26,0.26]\end{pmatrix},\begin{pmatrix}[0.24,0.24],\\ [0.26,0.26]\end{pmatrix}\right) =$ $\frac{1}{4}(0+0+0+0) = 0$	$d_{432}\left(\begin{pmatrix}[0,0],\\ [0.50,0.50]\end{pmatrix},\begin{pmatrix}[0.207,0.207],\\ [0.22,0.36]\end{pmatrix}\right) =$ $\frac{1}{4}(0.207+0.207+0.28+0.14) = 0.2085$
$d_{333}\left(\begin{pmatrix}[0.24,0.24],\\ [0.26,0.26]\end{pmatrix},\begin{pmatrix}[0.24,0.24],\\ [0.26,0.26]\end{pmatrix}\right) =$ $\frac{1}{4}(0+0+0+0) = 0$	$d_{433}\left(\begin{pmatrix}[0,0],\\ [0.50,0.50]\end{pmatrix},\begin{pmatrix}[0,0],\\ [0.50,0.50]\end{pmatrix}\right)$ $= \frac{1}{4}(0+0+0+0) = 0$
$d_{334}\left(\begin{pmatrix}[0.24,0.24],\\ [0.26,0.26]\end{pmatrix},\begin{pmatrix}[0.12,0.376],\\ [0.128,0.376]\end{pmatrix}\right) =$ $\frac{1}{4}(0.12+0.136+0.132+0.116) = 0.1260$	$d_{434}\left(\begin{pmatrix}[0,0],\\ [0.50,0.50]\end{pmatrix},\begin{pmatrix}[0.16,0.50],\\ [0.17,0.17]\end{pmatrix}\right) =$ $\frac{1}{4}(0.16+0.50+0.33+0.33) = 0.33$
$d_{341}\left(\begin{pmatrix}[0.12,0.376],\\ [0.128,0.376]\end{pmatrix},\begin{pmatrix}[0,0],\\ [0.50,0.50]\end{pmatrix}\right) =$ $\frac{1}{4}(0.12+0.376+0.372+0.124) = 0.2480$	$d_{441}\left(\begin{pmatrix}[0.16,0.50],\\ [0.17,0.17]\end{pmatrix},\begin{pmatrix}[0.24,0.24],\\ [0.26,0.26]\end{pmatrix}\right) =$ $\frac{1}{4}(0.08+0.26+0.09+0.09) = 0.13$
$d_{342}\left(\begin{pmatrix}[0.12,0.376],\\ [0.128,0.376]\end{pmatrix},\begin{pmatrix}[0.24,0.24],\\ [0.26,0.26]\end{pmatrix}\right) =$ $\frac{1}{4}(0.12+0.136+0.132+0.116) = 0.1260$	$d_{442}\left(\begin{pmatrix}[0.16,0.50],\\ [0.17,0.17]\end{pmatrix},\begin{pmatrix}[0.207,0.207],\\ [0.22,0.36]\end{pmatrix}\right) =$ $\frac{1}{4}(0.047+0.293+0.05+0.19) = 0.145$
$d_{343}\left(\begin{pmatrix}[0.12,0.376],\\ [0.128,0.376]\end{pmatrix},\begin{pmatrix}[0.24,0.24],\\ [0.26,0.26]\end{pmatrix}\right) =$ $\frac{1}{4}(0.12+0.136+0.132+0.116) = 0.1260$	$d_{443}\left(\begin{pmatrix}[0.16,0.50],\\ [0.17,0.17]\end{pmatrix},\begin{pmatrix}[0,0],\\ [0.50,0.50]\end{pmatrix}\right) =$ $\frac{1}{4}(0.16+0.50+0.33+0.33) = 0.33$
$d_{344}\left(\begin{pmatrix}[0.12,0.376],\\ [0.128,0.376]\end{pmatrix},\begin{pmatrix}[0.12,0.376],\\ [0.128,0.376]\end{pmatrix}\right) =$ $\frac{1}{4}(0+0+0+0) = 0$	$d_{444}\left(\begin{pmatrix}[0.16,0.50],\\ [0.17,0.17]\end{pmatrix},\begin{pmatrix}[0.16,0.50],\\ [0.17,0.17]\end{pmatrix}\right) =$ $\frac{1}{4}(0+0+0+0) = 0$
$d_{511}\left(\begin{pmatrix}[0.24,0.24],\\ [0.26,0.26]\end{pmatrix},\begin{pmatrix}[0.24,0.24],\\ [0.26,0.26]\end{pmatrix}\right) =$ $\frac{1}{4}(0+0+0+0) = 0$	$d_{531}\left(\begin{pmatrix}[0.16,0.16],\\ [0.17,0.50]\end{pmatrix},\begin{pmatrix}[0.24,0.24],\\ [0.26,0.26]\end{pmatrix}\right) =$ $\frac{1}{4}(0.08+0.08+0.09+0.24) = 0.1225$

$d_{512}\left(\begin{pmatrix}[0.24,0.24],\\ [0.26,0.26]\end{pmatrix},\begin{pmatrix}[0.16,0.30],\\ [0.30,0.30]\end{pmatrix}\right)=$ $\frac{1}{4}(0.08+0.06+0.04+0.04)=0.055$	$d_{532}\left(\begin{pmatrix}[0.16,0.16],\\ [0.17,0.50]\end{pmatrix},\begin{pmatrix}[0.16,0.30],\\ [0.30,0.30]\end{pmatrix}\right)=$ $\frac{1}{4}(0+0.14+0.13+0.2)=0.1175$
$d_{513}\left(\begin{pmatrix}[0.24,0.24],\\ [0.26,0.26]\end{pmatrix},\begin{pmatrix}[0.16,0.16],\\ [0.17,0.50]\end{pmatrix}\right)=$ $\frac{1}{4}(0.08+0.08+0.09+0.24)=0.1225$	$d_{533}\left(\begin{pmatrix}[0.16,0.16],\\ [0.17,0.50]\end{pmatrix},\begin{pmatrix}[0.16,0.16],\\ [0.17,0.50]\end{pmatrix}\right)=$ $\frac{1}{4}(0+0+0+0)=0$
$d_{514}\left(\begin{pmatrix}[0.24,0.24],\\ [0.26,0.26]\end{pmatrix},\begin{pmatrix}[0.16,0.16],\\ [0.17,0.50]\end{pmatrix}\right)=$ $\frac{1}{4}(0.08+0.08+0.09+0.24)=0.1225$	$d_{534}\left(\begin{pmatrix}[0.16,0.16],\\ [0.17,0.50]\end{pmatrix},\begin{pmatrix}[0.16,0.16],\\ [0.17,0.50]\end{pmatrix}\right)=$ $\frac{1}{4}(0+0+0+0)=0$
$d_{521}\left(\begin{pmatrix}[0.16,0.30],\\ [0.30,0.30]\end{pmatrix},\begin{pmatrix}[0.24,0.24],\\ [0.26,0.26]\end{pmatrix}\right)=$ $\frac{1}{4}(0.08+0.06+0.04+0.04)=0.055$	$d_{541}\left(\begin{pmatrix}[0.16,0.16],\\ [0.17,0.50]\end{pmatrix},\begin{pmatrix}[0.24,0.24],\\ [0.26,0.26]\end{pmatrix}\right)=$ $\frac{1}{4}(0.08+0.08+0.09+0.24)=0.1225$
$d_{522}\left(\begin{pmatrix}[0.16,0.30],\\ [0.30,0.30]\end{pmatrix},\begin{pmatrix}[0.16,0.30],\\ [0.30,0.30]\end{pmatrix}\right)=$ $\frac{1}{4}(0+0+0+0)=0$	$d_{542}\left(\begin{pmatrix}[0.16,0.16],\\ [0.17,0.50]\end{pmatrix},\begin{pmatrix}[0.16,0.30],\\ [0.30,0.30]\end{pmatrix}\right)=$ $\frac{1}{4}(0+0.14+0.13+0.2)=0.1175$
$d_{523}\left(\begin{pmatrix}[0.16,0.30],\\ [0.30,0.30]\end{pmatrix},\begin{pmatrix}[0.16,0.16],\\ [0.17,0.50]\end{pmatrix}\right)=$ $\frac{1}{4}(0+0.14+0.13+0.2)=0.1175$	$d_{543}\left(\begin{pmatrix}[0.16,0.16],\\ [0.17,0.50]\end{pmatrix},\begin{pmatrix}[0.16,0.16],\\ [0.17,0.50]\end{pmatrix}\right)=$ $\frac{1}{4}(0+0+0+0)=0$
$d_{524}\left(\begin{pmatrix}[0.16,0.30],\\ [0.30,0.30]\end{pmatrix},\begin{pmatrix}[0.16,0.16],\\ [0.17,0.50]\end{pmatrix}\right)=$ $\frac{1}{4}(0+0.14+0.13+0.2)=0.1175$	$d_{544}\left(\begin{pmatrix}[0.16,0.16],\\ [0.17,0.50]\end{pmatrix},\begin{pmatrix}[0.16,0.16],\\ [0.17,0.50]\end{pmatrix}\right)=$ $\frac{1}{4}(0+0+0+0)=0$

Step 3: Using Step 3 of the Jujhar method, proposed in Section 4.13,

$c_1=\sum_{i=1}^{4}\sum_{k=1}^{4} d_{i1k}(\tilde{\alpha}_{i1},\tilde{\alpha}_{k1})=0+0+0.1225+0.1175+0+0+0.1225+0.1175+0.1225+0.1225+0+0.055+0.1175+0.1175+0.055+0=1.07$,

$c_2=\sum_{i=1}^{4}\sum_{k=1}^{4} d_{i2k}(\tilde{\alpha}_{i2},\tilde{\alpha}_{k2})=0+0.2025+0.2600+0.2502+0.2025+0+$

$0.1075 + 0.237 + 0.26 + 0.1025 + 0 + 0.1242 + 0.2502 + 0.0592 + 0.1242 + 0 =$

$2.7128,$

$c_3 = \sum_{i=1}^{4}\sum_{k=1}^{4} d_{i3k}(\tilde{\alpha}_{i3}, \tilde{\alpha}_{k3}) = 0 + 0.24 + 0.24 + 0.248 + 0.24 + 0 + 0 + 0.126 +$

$0.24 + 0 + 0 + 0.126 + 0.248 + 0.126 + 0.126 + 0 = 1.96,$

$c_4 = \sum_{i=1}^{4}\sum_{k=1}^{4} d_{i4k}(\tilde{\alpha}_{i4}, \tilde{\alpha}_{k4}) = 0 + 0.0515 + 0.24 + 0.13 + 0.0515 + 0 + 0.2085 +$

$0.145 + 0.24 + 0.2085 + 0 + 0.33 + 0.13 + 0.145 + 0.33 + 0 = 2.21,$

$c_5 = \sum_{i=1}^{4}\sum_{k=1}^{4} d_{i5k}(\tilde{\alpha}_{i5}, \tilde{\alpha}_{k5}) = 0 + 0.055 + 0.1225 + 0.1225 + 0.055 + 0 + 0.1175 +$

$0.1175 + 0.1225 + 0.1175 + 0 + 0 + 0.1225 + 0.1175 + 0 + 0 = 1.07.$

Step 4: Using Step 4 of the Jujhar method, proposed in Section 4.13, there is need to solve the CLPP (P4.23).

$max(1.07w_1 + 2.7128w_2 + 1.96w_3 + 2.21w_4 + 1.07w_5)$

Subject to

$$\begin{cases} 0 \le w_1 \le 0.50, \\ 0.10 \le w_2 \le 0.20, \\ 0.15 \le w_3 \le 0.25, \\ 0.20 \le w_4 \le 0.30, \\ 0.10 \le w_5 \le 0.60, \\ w_1 + w_2 + w_3 + w_4 + w_5 = 1, \\ w_1 = w_5, \\ w_1 \ge 0, w_2 \ge 0, w_3 \ge 0, w_4 \ge 0, w_5 \ge 0. \end{cases} \tag{P4.23}$$

It can be easily verified that on solving the CLPP (P4.23), the obtained OAWV is $(w_1, w_2, w_3, w_4, w_5) = (0.1250, 0.20, 0.25, 0.30, 0.1250)$.

Step 5: On considering the OAWV, $(w_1, w_2, w_3, w_4, w_5) = (0.1250, 0.20, 0.25, 0.30, 0.1250)$ and using Step 5 of Jujhar method, proposed in Section 4.13,

$\tilde{P}_1 = ([(0.1250)(0.16) + (0.20)(0.24) + (0.25)(0) + (0.30)(0.24) + (0.1250)(0.24),$

$(0.1250)(0.16) + (0.20)(0.76) + (0.25)(0)+(0.30)(0.24) + (0.1250)(0.24)], [(0.1250)$

$(0.17) + (0.20)(0) + (0.25)(0.50) + (0.30)(0.26) + (0.1250)(0.26), (0.1250)(0.50) +$

$(0.20)(0) + (0.25)(0.50) + (0.30)(0.26) + (0.1250)(0.26)]$

$= ([0.1700, 0.2740], [0.2567, 0.2980]),$

$\tilde{P}_2 = ([(0.1250)(0.16) + (0.20)(0.14) + (0.25)(0.24) + (0.30)(0.207) + (0.1250)$
$(0.16), (0.1250)(0.16) + (0.20)(0.45) + (0.25)(0.24) + (0.30)(0.207) + (0.1250)(0.30)]$
$, [(0.1250)(0.17) + (0.20)(0.15) + (0.25)(0.26) + (0.30)(0.22) + (0.1250)(0.30),$
$(0.1250)(0.50) + (0.20)(0.25) + (0.25)(0.26) + (0.30)(0.36) + (0.1250)(0.30)]$
$= ([0.1901, 0.2696], [0.2197, 0.3230]),$

$\tilde{P}_3 = ([(0.1250)(0.24) + (0.20)(0.24) + (0.25)(0.24) + (0.30)(0) + (0.1250)(0.16),$
$(0.1250)(0.24) + (0.20)(0.24) + (0.25)(0.24) + (0.30)(0) + (0.1250)(0.16)], [(0.1250)$
$(0.26) + (0.20)(0.26) + (0.25)(0.26) + (0.30)(0.50) + (0.1250)(0.17), (0.1250)$
$(0.26) + (0.20)(0.26) + (0.25)(0.26) + (0.30)(0.50) + (0.1250)(0.50)]$
$= ([0.1580, 0.1580], [0.3207, 0.3620]),$

$\tilde{P}_4 = ([(0.1250)(0.16) + (0.20)(0.125) + (0.25)(0.12) + (0.30)(0.16) + (0.1250)$
$(0.16), (0.1250)(0.30) + (0.20)(0.375) + (0.25)(0.376) + (0.30)(0.50) + (0.1250)$
$(0.16)], [(0.1250)(0.30) + (0.20)(0.127) + (0.25)(0.128) + (0.30)(0.17) + (0.1250)$
$(0.17), (0.1250)(0.30) + (0.20)(0.374) + (0.25)(0.376) + (0.30)(0.17) + (0.1250)$
$(0.50)] = ([0.1430, 0.3765], [0.1671, 0.3198]).$

Step 6: Using Step 6 of the Jujhar method, proposed in Section 4.13,
$S(\tilde{P}_1) = -0.0553, S(\tilde{P}_2) = -0.0415, S(\tilde{P}_3) = -0.1833$ and $S(\tilde{P}_4) = 0.0163$.

Since, $S(\tilde{P}_4) > S(\tilde{P}_2) > S(\tilde{P}_1) > S(\tilde{P}_3)$. So, according to Step 6 of the Jujhar method, proposed in Section 4.13, the ranking of the alternatives is $A_4 > A_2 > A_1 > A_3$.

4.14.13 Exact results of the thirteenth IVIFMADMPr having partially known attribute weights

Using the proposed Jujhar method, the exact result of the IVIFMADMPr having partially known attribute weights, considered in the second point of Section 4.11, can be obtained as follows:

Since, Step 1 to Step 3 of the proposed Jujhar method are same as shown in Section 4.14.12. So, to avoid any repetition, the calculations have been started from Step 4.

Step 4: Since, the condition $\sum_{j=1}^{n} w_j^u < 1$ is satisfying. So, using Case (i) of Step 4 of the Jujhar method, proposed in Section 4.13, there is need to solve the CLPP (P4.24).

$max(1.07w_1 + 2.7128w_2 + 1.96w_3 + 2.21w_4 + 1.07w_5)$

Subject to

$$\begin{cases} 0 \leq w_1 \leq 0.20, \\ 0.15 \leq w_2 \leq 0.25, \\ 0 \leq w_3 \leq 0.1, \\ 0.10 \leq w_4 \leq 0.15, \\ 0 \leq w_5 \leq 0.20, \\ w_1 + w_2 + w_3 + w_4 + w_5 \leq 1, \\ w_1 = w_5, \\ w_1 \geq 0, w_2 \geq 0, w_3 \geq 0, w_4 \geq 0, w_5 \geq 0. \end{cases} \tag{P4.24}$$

On solving the LPP (P4.24) the obtained non-normalized OAWV is $(w_1, w_2, w_3, w_4, w_5) = (0.20, 0.25, 0.10, 0.15, 0.20)$ and the normalized OAWV is $(w_1, w_2, w_3, w_4, w_5) = \left(\frac{0.20}{0.20+0.25+0.10+0.15+0.20}, \frac{0.25}{0.20+0.25+0.10+0.15+0.20}, \frac{0.10}{0.20+0.25+0.10+0.15+0.20}, \frac{0.15}{0.20+0.25+0.10+0.15+0.20}, \frac{0.20}{0.20+0.25+0.10+0.15+0.20}\right)$
$= (0.2222, 0.2778, 0.1112, 0\ .1666, 0.2222)$.

Step 5: On considering the normalized OAWV $(w_1, w_2, w_3, w_4, w_5) = (0.2222, 0.2778, 0.1112, 0\ .1666, 0.2222)$ and using Step 5 of Jujhar method, proposed in Section 4.13,

$\tilde{P}_1 = ([(0.2222)(0.16) + (0.2778)(0.24) + (0.1112)(0) + (0.1666)(0.24) + (0.2222)(0.24), (0.2222)(0.16) + (0.2778)(0.76) + (0.1112)(0) + (0.1666)(0.24) + (0.2222)(0.24)], [(0.2222)(0.17) + (0.2778)(0) + (0.1112)(0.50) + (0.1666)(0.26) + (0.2222)(0.26), (0.2222)(0.50) + (0.2778)(0) + (0.1112)(0.50) + (0.1666)(0.26) + (0.2222)(0.26)\big) = ([0.1955, 0.3399], [0.1944, 0.2678])$,

$\tilde{P}_2 = ([(0.2222)(0.16) + (0.2778)(0.14) + (0.1112)(0.24) + (0.1666)(0.207) +$

$(0.2222)(0.16), (0.2222)(0.16) + (0.2778)(0.45) + (0.1112)(0.24) + (0.1666)$
$(0.207) + (0.2222)(0.30)], [(0.2222)(0.17) + (0.2778)(0.15) + (0.1112)(0.26) +$
$(0.1666)(0.22) + (0.2222)(0.30), (0.2222)(0.50) + (0.2778)(0.25) + (0.1112)(0.26) +$
$(0.1666)(0.36) + (0.2222)(0.30)]) = ([0.1711, 0.2883], [0.2116, 0.3360]),$

$\tilde{P}_3 = ([(0.2222)(0.24) + (0.2778)(0.24) + (0.1112)(0.24) + (0.1666)(0) + (0.2222)$
$(0.16), (0.2222)(0.24) + (0.2778)(0.24) + (0.1112)(0.24) + (0.1666)(0) + (0.2222)$
$(0.16)], [(0.2222)(0.26) + (0.2778)(0.26) + (0.1112)(0.26) + (0.1666)(0.50) +$
$(0.2222)(0.17), (0.2222)(0.26) + (0.2778)(0.26) + (0.1112)(0.26) + (0.1666)(0.50) +$
$(0.2222)(0.50)]) = ([0.1822, 0.1822], [0.2799, 0.3533]),$

$\tilde{P}_4 = ([(0.2222)(0.16) + (0.2778)(0.125) + (0.1112)(0.12) + (0.1666)(0.16) +$
$(0.2222)(0.16), (0.2222)(0.30) + (0.2778)(0.375) + (0.1112)(0.376) + (0.1666)$
$(0.50) + (0.2222)(0.16)], [(0.2222)(0.30) + (0.2778)(0.127) + (0.1112)(0.128) +$
$(0.1666)(0.17) + (0.2222)(0.17), (0.2222)(0.30) + (0.2778)(0.374) + (0.1112)$
$(0.376) + (0.1666)(0.17) + (0.2222)(0.50)]) = ([0.1458, 0.3314], [0.1822, 0.3517])$

Step 6: Using Step 6 of the Jujhar method, proposed in Section 4.13,

$S(\tilde{P}_1) = 0.0366, S(\tilde{P}_2) = -0.0441, S(\tilde{P}_3) = -0.1344$ and $S(\tilde{P}_4) = -0.0283$.

Since, $S(\tilde{P}_1) > S(\tilde{P}_4) > S(\tilde{P}_2) > S(\tilde{P}_3)$. So, according to Step 6 of the Jujhar method, proposed in Section 4.13, the ranking of the alternatives is $A_1 > A_4 > A_2 > A_3$.

4.14.14 Exact results of the fourteenth IVIFMADMPr having partially known attribute weights

Using the proposed Jujhar method, the exact result of the IVIFMADMPr having partially known attribute weights, considered in the second point of Section 4.11, can be obtained as follows:

Since, Step 1 to Step 3 of the proposed Jujhar method are same as shown in Section 4.14.12. So, to avoid any repetition, the calculations have been started from Step 4.

Step 4: Since, the condition $\sum_{j=1}^{n} w_j^l > 1$ is satisfying. So, using Case (ii) of Step 4 of the Jujhar method, proposed in Section 4.13, there is need to solve the CLPP (P4.25).

$max\ (1.07w_1 + 2.7128w_2 + 1.96w_3 + 2.21w_4 + 1.07w_5 - \varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4 - \varepsilon_5)$

Subject to

$$\begin{cases} \frac{20}{170} \le w_1 \le \frac{70}{170}, \\ \frac{50}{170} \le w_2 \le \frac{70}{170}, \\ \frac{60}{170} \le w_3 \le \frac{90}{170}, \\ \frac{10}{170} \le w_4 \le \frac{40}{170}, \\ \frac{20}{170} \le w_5 \le \frac{70}{170}, \\ w_1 + w_2 + w_3 + w_4 + w_5 = 1, \\ 170 = 100 + \varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4 + \varepsilon_5, \\ w_1 = w_5, \\ w_1 \ge 0, w_2 \ge 0, w_3 \ge 0, w_4 \ge 0, w_5 \ge 0, \varepsilon_1 \ge 0, \varepsilon_2 \ge 0, \varepsilon_3 \ge 0, \varepsilon_4 \ge 0, \varepsilon_5 \ge 0. \end{cases} \qquad \text{(P4.25)}$$

On solving the LPP (P4.25) the obtained normalized OAWV is $(w_1, w_2, w_3, w_4, w_5) = (0.1176, 0.3531, 0.3529, 0.0588, 0.1176)$.

Step 5: On considering the normalized OAWV, $(w_1, w_2, w_3, w_4, w_5) = (0.1176, 0.3531, 0.3529, 0.0588, 0.1176)$ and using Step 5 of Jujhar method, proposed in Section 4.13,

$\tilde{P}_1 = ([(0.1176)(0.16) + (0.3531)(0.24) + (0.3529)(0) + (0.0588)(0.24) + (0.1176)(0.24), (0.1176)(0.16) + (0.3531)(0.76) + (0.3529)(0) + (0.0588)(0.24) + (0.1176)(0.24)], [(0.1176)(0.17) + (0.3531)(0) + (0.3529)(0.50) + (0.0588)(0.26) + (0.1176)(0.26), (0.1176)(0.50) + (0.3531)(0) + (0.3529)(0.50) + (0.0588)(0.26) + (0.1176)(0.26)]) = ([0.1458, 0.3295], [0.2423, 0.2811])$,

$\tilde{P}_2 = ([(0.1176)(0.16) + (0.3531)(0.14) + (0.3529)(0.24) + (0.0588)(0.207) + (0.1176)(0.16), (0.1176)(0.16) + (0.3531)(0.45) + (0.3529)(0.24) + (0.0588)(0.207) + (0.1176)(0.30)], [(0.1176)(0.17) + (0.3531)(0.15) + (0.3529)(0.26) + (0.0588)(0.22) + (0.1176)(0.30), (0.1176)(0.50) + (0.3531)(0.25) + (0.3529)(0.26) +$

$(0.0588)(0.36)+(0.1176)(0.30)])=([0.1839,0.3098],[0.2129,0.2952])$,

$\tilde{P}_3=([(0.1176)(0.24)+(0.3531)(0.24)+(0.3529)(0.24)+(0.0588)(0)+(0.1176)$ $(0.16),(0.1176)(0.24)+(0.3531)(0.24)+(0.3529)(0.24)+(0.0588)(0)+(0.1176)$ $(0.16)],[(0.1176)(0.26)+(0.3531)(0.26)+(0.3529)(0.26)+(0.0588)(0.50)+$ $(0.1176)(0.17),(0.1176)(0.26)+(0.3531)(0.26)+(0.3529)(0.26)+(0.0588)(0.50)+$ $(0.1176)(0.50)])=([0.2164,0.2164],[0.2635,0.3023])$,

$\tilde{P}_4=([(0.1176)(0.16)+(0.3531)(0.125)+(0.3529)(0.12)+(0.0588)(0.16)+$ $(0.1176)(0.16),(0.1176)(0.30)+(0.3531)(0.375)+(0.3529)(0.376)+(0.0588)$ $(0.50)+(0.1176)(0.16)],[(0.1176)(0.30)+(0.3531)(0.127)+(0.3529)(0.128)+$ $(0.0588)(0.17)+(0.1176)(0.17),(0.1176)(0.30)+(0.3531)(0.374)+(0.3529)$ $(0.376)+(0.0588)(0.17)+(0.1176)(0.50)])=([0.1335,0.3485],[0.1552,0.3688])$.

Step 6: Using Step 6 of the Jujhar method, proposed in Section 4.13,

$S(\tilde{P}_1)=-0.0240, S(\tilde{P}_2)=-0.0072, S(\tilde{P}_3)=-0.0665$ and $S(\tilde{P}_4)=-0.0210$.

Since, $S(\tilde{P}_2)>S(\tilde{P}_4)>S(\tilde{P}_1)>S(\tilde{P}_3)$. So, according to Step 6 of the Jujhar method, proposed in Section 4.13, the ranking of the alternatives is $A_2>A_4>A_1>A_3$.

4.14.15 Exact result of fifteenth IVIFMADMPr having partially known attribute weights

Using the proposed Jujhar method, the exact result of the IVIFMADMPr having partially known attribute weights, considered in the third point of Section 4.11, can be obtained as follows:

Step 1: Since, all the attributes are benefit type. So, according to Step 1 of the Jujhar method, proposed in Section 4.13, there is no need to apply Step 1.

Step 2: According to Step 2 of the Jujhar method, proposed in Section 4.13, there is need to calculate $d_{ijk}(\tilde{\alpha}_{ij},\tilde{\alpha}_{kj})\ \forall\ i=1,2; j=1,2$. These values are shown in Table 4.24.

Table 4.24: Values of $d_{ijk}(\tilde{\alpha}_{ij},\tilde{\alpha}_{kj})$

$d_{111}\left(\begin{pmatrix}[0.20,0.40],\\ [0.20,0.40]\end{pmatrix},\begin{pmatrix}[0.20,0.40],\\ [0.20,0.40]\end{pmatrix}\right)=$ $\frac{1}{4}(0+0+0+0)=0$	$d_{211}\left(\begin{pmatrix}[0.16,0.16],\\ [0.17,0.50]\end{pmatrix},\begin{pmatrix}[0.16,0.16],\\ [0.17,0.50]\end{pmatrix}\right)$ $=\frac{1}{4}(0+0+0+0)=0$
$d_{112}\left(\begin{pmatrix}[0.20,0.40],\\ [0.20,0.40]\end{pmatrix},\begin{pmatrix}[0.10,0.50],\\ [0.10,0.50]\end{pmatrix}\right)=$ $\frac{1}{4}(0.1+0.1+0.1+0.1)=0.1$	$d_{212}\left(\begin{pmatrix}[0.16,0.16],\\ [0.17,0.50]\end{pmatrix},\begin{pmatrix}[0.16,0.16],\\ [0.17,0.50]\end{pmatrix}\right)$ $=\frac{1}{4}(0+0+0+0)=0$
$d_{121}\left(\begin{pmatrix}[0.10,0.50],\\ [0.10,0.50]\end{pmatrix},\begin{pmatrix}[0.20,0.40],\\ [0.20,0.40]\end{pmatrix}\right)=$ $\frac{1}{4}(0.1+0.1+0.1+0.1)=0.1$	$d_{213}\left(\begin{pmatrix}[0.16,0.16],\\ [0.17,0.50]\end{pmatrix},\begin{pmatrix}[0.16,0.16],\\ [0.17,0.50]\end{pmatrix}\right)$ $=\frac{1}{4}(0+0+0+0)=0$
$d_{122}\left(\begin{pmatrix}[0.10,0.50],\\ [0.10,0.50]\end{pmatrix},\begin{pmatrix}[0.10,0.50],\\ [0.10,0.50]\end{pmatrix}\right)=$ $\frac{1}{4}(0+0+0+0)=0$	$d_{214}\left(\begin{pmatrix}[0.16,0.16],\\ [0.17,0.50]\end{pmatrix},\begin{pmatrix}[0.16,0.16],\\ [0.17,0.50]\end{pmatrix}\right)$ $=\frac{1}{4}(0+0+0+0)=0$

Step 3: Using Step 3 of the Jujhar method, proposed in Section 4.13,

$c_1=\sum_{i=1}^{2}\sum_{k=1}^{2} d_{i1k}(\tilde{\alpha}_{i1},\tilde{\alpha}_{k1})=0+0.1+0.1+0+0+0+0+0=0.2,$

$c_2=\sum_{i=1}^{2}\sum_{k=1}^{2} d_{i1k}(\tilde{\alpha}_{i1},\tilde{\alpha}_{k1})=0+0+0+0+0+0+0+0=0.$

Step 4: Using Step 4 of the Jujhar method, proposed in Section 4.13, there is need to solve the CLPP (P4.26).

$max(0.2w_1)$

Subject to

$$\begin{cases}0.1\le w_1\le 0.6,\\ 0.2\le w_2\le 0.7,\\ w_1+w_2=1,\\ w_1\ge 0, w_2\ge 0.\end{cases} \tag{P4.26}$$

It can be easily verified that on solving the CLPP (P4.26) the obtained OAWV is $(w_1,w_2)=(0.6,0.4)$

Step 5: On considering the OAWV, $(w_1,w_2)=(0.6,0.4)$ and using Step 5 of the Jujhar method, proposed in Section 4.13,

$\tilde{P}_1 = ([(0.60)(0.20) + (0.40)(0.16), (0.60)(0.40) + (0.40)(0.16)], [(0.60)(0.20) + (0.40)(0.17), (0.60)(0.40) + (0.40)(0.50)] = ([0.1840,0.3040], [0.1880,0.4400]),$

$\tilde{P}_2 = ([(0.60)(0.1) + (0.40)(0.16), (0.60)(0.50) + (0.40)(0.16)], [(0.60)(0.1) + (0.40)(0.17), (0.60)(0.50) + (0.40)(0.50)] = ([0.1240,0.3640], [0.1280,0.5000]).$

Step 6: Using Step 6 of the Jujhar method, proposed in Section 4.13,

$S(\tilde{P}_1) = -0.0735$ and $S(\tilde{P}_2) = -0.0700$.

Since, $S(\tilde{P}_2) > S(\tilde{P}_1)$. Therefore, according to case (i) of Step 6 of the proposed Jujhar method, the ranking of the alternatives is $A_2 > A_1$.

4.15 Conclusions

The inappropriateness of the existing methods [62, 108, 213, 258] are pointed out. Also, a new method (named as Jujhar method) is proposed to solve the IVIFMADMPrs having partially known attribute weights. Furthermore, the proposed Jujhar method has been illustrated with the help of some IVIFMADMPrs having partially known attribute weights.

Chapter 5

Jorawar method for solving IVPFMADMPrs having partially known attribute weights[4]

To the best of my knowledge, only the existing method [84] has been proposed to solve IVPFMADMPrs having partially known attribute weights. However after a deep study, it is observed that it is inappropriate to use this method. The aim of this chapter is to make the researchers aware about the inappropriateness of this method as well as to propose a new method (named as Jorawar method) to solve IVPFMADMPr having partially known attribute weights.

5.1 A brief review of Garg's method

Garg [84] proposed the following method to solve IVPFMADMPrs having partially known attribute weights.

Step 1: Check that all the attributes are of same type or not i.e., check that all the attributes are benefit type attribute or cost type attribute.

Case (i): If all the attributes are of same type then go to Step 2.

Case (ii): If some attributes are cost type attributes and the remaining are benefit type attributes. Then, convert the j^{th} cost type attribute into the benefit type attribute by replacing all the elements $\tilde{\alpha}_{ij} = ([a_{ij1}, a_{ij2}], [a_{ij3}, a_{ij4}])$ of the j^{th} column of the IVPFDM $\widetilde{D} = (\tilde{\alpha}_{ij})_{m\times n}$ with $\tilde{\alpha}_{ij} = ([a_{ij3}, a_{ij4}], [a_{ij1}, a_{ij2}])$ and go to Step 2.

Step 2: Using the expression (5.1), transform each IVPF element $\tilde{\alpha}_{ij} =$

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[4] The contents of this chapter have been communicated in "International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems" for the possible publication

$([a_{ij1}, a_{ij2}], [a_{ij3}, a_{ij4}])$ of the IVPFDM $\widetilde{D} = (\tilde{\alpha}_{ij})_{m\times n} = ([a_{ij1}, a_{ij2}], [a_{ij3}, a_{ij4}])$ into the crisp element d_{ij}.

$$d_{ij} = \left(\frac{({a_{ij1}}^2 - {a_{ij3}}^2)\left(1 + \sqrt{1 - {a_{ij1}}^2 - {a_{ij3}}^2}\right) + ({a_{ij2}}^2 - {a_{ij4}}^2)\left(1 + \sqrt{1 - {a_{ij2}}^2 - {a_{ij4}}^2}\right)}{2}\right) \tag{5.1}$$

Step 3: Using the expression (5.2), find the value of $c_j \ \forall\ j = 1,2,\dots,n$.

$$c_j = \sum_{i=1}^{m} d_{ij}\ , j = 1,2,\dots,n \tag{5.2}$$

Step 4: Find the OAWV, $(w_1, w_2, \dots, w_n)$ by solving the CLPP (P5.1).

$max\left[\sum_{j=1}^{n} c_j\, w_j\right]$

Subject to

$w_j \in H, w_j \geq 0, j = 1,2,\dots,n,$

$\sum_{j=1}^{n} w_j = 1.$ (P5.1)

Step 5: Using the expression (5.3), find the value of $P_i \ \forall\ i = 1,2,\dots,m$.

$$P_i = \sum_{j=1}^{n} w_j\, d_{ij}, i = 1,2,\dots,m \tag{5.3}$$

and check that $P_p > P_q$ or $P_p < P_q$ or $P_p = P_q$, where, $P_p = \sum_{j=1}^{n} w_j\, d_{pj}$ and $P_q = \sum_{j=1}^{n} w_j\, d_{qj}$.

Case (i): If $P_p > P_q$ then the ranking of the alternatives is $A_p > A_q$.

Case (ii): If $P_p < P_q$ then the ranking of the alternatives is $A_p < A_q$.

Case (iii): If $P_p = P_q$ then the ranking of the alternatives is $A_p = A_q$.

5.2 Inappropriateness of Garg's method

It is inappropriate to apply Garg's method [84] due to the following reasons:

(1) On applying Garg's method [84] more than one preference orders of the alternatives are obtained, which is inappropriate. To validate this claim a real-life IVPFMADMPr having partially known attribute weights, considered by Garg [84] to illustrate his proposed method, is solved with some modifications.

Garg [84] solved the following real-life IVPFMADMPr having partially known attribute weights to illustrate his proposed method.

There is a panel of four alternatives for the investment company to invest money in best option $A_i, i = 1,2,3,4$ on the basis of the following four attributes.

(i) G_1: Risk analysis

(ii) G_2: Growth

(iii) G_3: Environmental impact

(iv) G_4: Social-political impact

The $(i,j)^{th}$ element of the Table 5.1, represented by an IVPFS $\tilde{\alpha}_{ij} = \left([a_{ij1}, a_{ij2}], [a_{ij3}, a_{ij4}]\right)$, represents the rating value of the i^{th} company over the j^{th} attribute.

Table 5.1: Rating Values

Attributes→ ↓ Alternatives	G_1	G_2	G_3	G_4
A_1	$\begin{pmatrix}[0.30,0.40],\\ [0.60,0.80]\end{pmatrix}$	$\begin{pmatrix}[0.50,0.60],\\ [0.30,0.50]\end{pmatrix}$	$\begin{pmatrix}[0.30,0.70],\\ [0.50,0.60]\end{pmatrix}$	$\begin{pmatrix}[0.40,0.50],\\ [0.20,0.40]\end{pmatrix}$
A_2	$\begin{pmatrix}[0.40,0.60],\\ [0.60,0.70]\end{pmatrix}$	$\begin{pmatrix}[0.60,0.70],\\ [0.40,0.50]\end{pmatrix}$	$\begin{pmatrix}[0.40,0.70],\\ [0.30,0.50]\end{pmatrix}$	$\begin{pmatrix}[0.30,0.40],\\ [0.40,0.60]\end{pmatrix}$
A_3	$\begin{pmatrix}[0.30,0.40],\\ [0.50,0.70]\end{pmatrix}$	$\begin{pmatrix}[0.50,0.60],\\ [0.40,0.50]\end{pmatrix}$	$\begin{pmatrix}[0.50,0.60],\\ [0.70,0.80]\end{pmatrix}$	$\begin{pmatrix}[0.70,0.80],\\ [0.30,0.50]\end{pmatrix}$
A_4	$\begin{pmatrix}[0.30,0.40],\\ [0.70,0.80]\end{pmatrix}$	$\begin{pmatrix}[0.60,0.70],\\ [0.30,0.40]\end{pmatrix}$	$\begin{pmatrix}[0.50,0.60],\\ [0.40,0.50]\end{pmatrix}$	$\begin{pmatrix}[0.50,0.60],\\ [0.70,0.80]\end{pmatrix}$

Also, $H = \{0.10 \le w_1 \le 0.20, 0.20 \le w_2 \le 0.30, 0.30 \le w_3 \le 0.50, 0.20 \le w_4 \le 0.30\}$.

If

(i) Table 5.1 is replaced with Table 5.2

Table 5.2: Rating Values

Attributes→ ↓ Alternatives	G_1	G_2	G_3	G_4
A_1	$\begin{pmatrix}[0.30,0.40],\\ [0.60,0.80]\end{pmatrix}$	$\begin{pmatrix}[0.50,0.60],\\ [0.30,0.50]\end{pmatrix}$	$\begin{pmatrix}[0.30,0.70],\\ [0.50,0.60]\end{pmatrix}$	$\begin{pmatrix}[0.30,0.40],\\ [0.70,0.80]\end{pmatrix}$
A_2	$\begin{pmatrix}[0.40,0.60],\\ [0.60,0.70]\end{pmatrix}$	$\begin{pmatrix}[0.60,0.70],\\ [0.40,0.50]\end{pmatrix}$	$\begin{pmatrix}[0.40,0.70],\\ [0.30,0.50]\end{pmatrix}$	$\begin{pmatrix}[0.30,0.40],\\ [0.60,0.80]\end{pmatrix}$
A_3	$\begin{pmatrix}[0.30,0.40],\\ [0.50,0.70]\end{pmatrix}$	$\begin{pmatrix}[0.50,0.60],\\ [0.40,0.50]\end{pmatrix}$	$\begin{pmatrix}[0.50,0.60],\\ [0.70,0.80]\end{pmatrix}$	$\begin{pmatrix}[0.40,0.60],\\ [0.60,0.70]\end{pmatrix}$
A_4	$\begin{pmatrix}[0.30,0.40],\\ [0.70,0.80]\end{pmatrix}$	$\begin{pmatrix}[0.60,0.70],\\ [0.30,0.40]\end{pmatrix}$	$\begin{pmatrix}[0.50,0.60],\\ [0.40,0.50]\end{pmatrix}$	$\begin{pmatrix}[0.30,0.40],\\ [0.50,0.70]\end{pmatrix}$

(ii) It is assumed that the attribute weight w_1 satisfies the condition $0.20 \leq w_1 \leq 0.30$ instead of the existing condition $0.10 \leq w_1 \leq 0.20$.

Then, using Garg's method [84], the ranking of the alternatives of the modified IVPFMADMPr having partially known attribute weights can be obtained as follows:

Step 1: Since, the attributes G_1 and G_4 are of cost type. So, according to Step 1 of Garg's method [84], discussed in Section 5.1, Table 5.2 is transformed into Table 5.3.

Attributes→ ↓ Alternatives	G_1	G_2	G_3	G_4
A_1	$\begin{pmatrix}[0.60,0.80],\\ [0.30,0.40]\end{pmatrix}$	$\begin{pmatrix}[0.50,0.60],\\ [0.30,0.50]\end{pmatrix}$	$\begin{pmatrix}[0.30,0.70],\\ [0.50,0.60]\end{pmatrix}$	$\begin{pmatrix}[0.70,0.80],\\ [0.30,0.40]\end{pmatrix}$

A_2	$\begin{pmatrix}[0.60,0.70],\\ [0.40,0.60]\end{pmatrix}$	$\begin{pmatrix}[0.60,0.70],\\ [0.40,0.50]\end{pmatrix}$	$\begin{pmatrix}[0.40,0.70],\\ [0.30,0.50]\end{pmatrix}$	$\begin{pmatrix}[0.60,0.80],\\ [0.30,0.40]\end{pmatrix}$
A_3	$\begin{pmatrix}[0.50,0.70],\\ [0.30,0.40]\end{pmatrix}$	$\begin{pmatrix}[0.50,0.60],\\ [0.40,0.50]\end{pmatrix}$	$\begin{pmatrix}[0.50,0.60],\\ [0.70,0.80]\end{pmatrix}$	$\begin{pmatrix}[0.60,0.70],\\ [0.40,0.60]\end{pmatrix}$
A_4	$\begin{pmatrix}[0.70,0.80],\\ [0.30,0.40]\end{pmatrix}$	$\begin{pmatrix}[0.60,0.70],\\ [0.30,0.40]\end{pmatrix}$	$\begin{pmatrix}[0.50,0.60],\\ [0.40,0.50]\end{pmatrix}$	$\begin{pmatrix}[0.50,0.70],\\ [0.30,0.40]\end{pmatrix}$

Step 2: According to Step 2 of Garg's method [84], discussed in Section 5.1, there is need to calculate d_{ij} $\forall\ i = 1,2, \dots ,4; j = 1,2, \dots ,4$. These values are shown in Table 5.4.

Table 5.4: Values of d_{ij}

$d_{11} = 0.5824$	$d_{12} = 0.2343$	$d_{13} = -0.0548$	$d_{14} = 0.6769$
$d_{21} = 0.2595$	$d_{22} = 0.3505$	$d_{23} = 0.2465$	$d_{24} = 0.5824$
$d_{31} = 0.4076$	$d_{32} = 0.1689$	$d_{33} = -0.3212$	$d_{34} = 0.2595$
$d_{41} = 0.6769$	$d_{42} = 0.4977$	$d_{43} = 0.1689$	$d_{44} = 0.4076$

Step 3: Using Step 3 of Garg's method [84], discussed in Section 5.1,

$c_1 = \sum_{i=1}^{4} d_{i1} = 0.5824 + 0.2595 + 0.4076 + 0.6769 = 1.9264,$

$c_2 = \sum_{i=1}^{4} d_{i2} = 0.2343 + 0.3505 + 0.1689 + 0.4977 = 1.2514,$

$c_3 = \sum_{i=1}^{4} d_{i3} = -0.0548 + 0.2465 - 0.3212 + 0.1689 = 0.0394,$

$c_4 = \sum_{i=1}^{4} d_{i4} = 0.6769 + 0.5824 + 0.2595 + 0.4076 = 1.9264.$

Step 4: Using Step 4 of Garg's method [84], discussed in Section 5.1, there is need to solve the CLPP (P5.2).

$Max(1.9264w_1 + 1.2514w_2 + 0.0394w_3 + 1.9264w_4)$

Subject to

$$\begin{cases} 0.20 \le w_1 \le 0.30, \\ 0.20 \le w_2 \le 0.30, \\ 0.30 \le w_3 \le 0.50, \\ 0.20 \le w_4 \le 0.30, \\ w_1 + w_2 + w_3 + w_4 = 1, \\ w_1 \ge 0, w_2 \ge 0, w_3 \ge 0. \end{cases} \tag{P5.2}$$

It can be easily verified that on solving the CLPP (P5.2) infinite number of OAWV are obtained e.g., $(w_1, w_2, w_3, w_4) = (0.20, 0.20, 0.30, 0.30)$ and $(w_1, w_2, w_3, w_4) = (0.30, 0.20, 0.30, 0.20)$ both are the OAWV.

Step 5: On considering the OAWV, $(w_1, w_2, w_3, w_4) = (0.20, 0.20, 0.30, 0.30)$ and using Step 5 of Garg's method [84], discussed in Section 5.1,

$P_1 = (0.20)(0.5824) + (0.20)(0.2343) + (0.30)(-0.0548) + (0.30)(0.6769) = 0.3499$,

$P_2 = (0.20)(0.2595) + (0.20)(0.3505) + (0.30)(0.2465) + (0.30)(0.5824) = 0.3706$,

$P_3 = (0.20)(0.4076) + (0.20)(0.1689) + (0.30)(-0.3212) + (0.30)(0.2595) = 0.0967$,

$P_4 = (0.20)(0.6769) + (0.20)(0.4977) + (0.30)(0.1689) + (0.30)(0.4076) = 0.4078$.

Since, $P_4 > P_2 > P_1 > P_3$. So, according to Step 5 of Garg's method [84], discussed in Section 5.1, the ranking of the alternatives is $A_4 > A_2 > A_1 > A_3$.

While, on considering the OAWV, $(w_1, w_2, w_3, w_4) = (0.30, 0.20, 0.30, 0.20)$ and using Step 5 of Garg's method [84], discussed in Section 5.1,

$P_1 = (0.30)(0.5824) + (0.20)(0.2343) + (0.30)(-0.0548) + (0.20)(0.6769) = 0.3405$,

$P_2 = (0.30)(0.2595) + (0.20)(0.3505) + (0.30)(0.2465) + (0.20)(0.5824) = 0.3383$,

$P_3 = (0.30)(0.4076) + (0.20)(0.1689) + (0.30)(-0.3212) + (0.20)(0.2595) = 0.1116$,

$P_4 = (0.30)(0.6769) + (0.20)(0.4977) + (0.30)(0.1689) + (0.20)(0.4076) = 0.4348$.

Since, $P_4 > P_1 > P_2 > P_3$. So, according to Step 5 of Garg's method [84], discussed in Section 5.1, the ranking of the alternatives is $A_4 > A_1 > A_2 > A_3$.

It is obvious that on applying Garg's method [84] two different ranking of the alternatives are obtained for the same IVPFMADMPr having partially known attribute weights, which is mathematically incorrect. Hence, it is inappropriate to use Garg's method [84] to solve IVPFMADMPrs having partially known attribute weights.

(2) Garg's method [84] fails to find the ranking of the alternatives. The following validates this claim.

(i) If the set $H = \{0.10 \leq w_1 \leq 0.20, 0.20 \leq w_2 \leq 0.30, 0.30 \leq w_3 \leq 0.50, 0.20 \leq w_4 \leq 0.30\}$ is replaced with $H = \{0.10 \leq w_1 \leq 0.20, 0.20 \leq w_2 \leq 0.25, 0.10 \leq w_3 \leq 0.15, 0.10 \leq w_4 \leq 0.20\}$. Then, no feasible solution of the CLPP (P5.2) will be obtained as the condition $\sum_{j=1}^{n} w_j^u < 1$ is satisfying i.e., the ranking of the alternatives of the modified problem cannot be obtained by Garg's method [84].

(ii) If the set $H = \{0.10 \leq w_1 \leq 0.20, 0.20 \leq w_2 \leq 0.30, 0.30 \leq w_3 \leq 0.50, 0.20 \leq w_4 \leq 0.30\}$ is replaced with $H = \{0.30 \leq w_1 \leq 0.80, 0.40 \leq w_2 \leq 0.60, 0.50 \leq w_3 \leq 0.70, 0.30 \leq w_4 \leq 0.80\}$. Then, no feasible solution of the CLPP (P5.2) will be obtained as the condition $\sum_{j=1}^{n} w_j^l > 1$ is satisfying i.e., the ranking of the alternatives of the modified problem cannot be obtained by Garg's method [84].

(2) The ranking of the alternatives, obtained by Garg's method [84], is not appropriate. The following example clearly validates this claim.

If it is assumed that the $(i, j)^{th}$ element, represented by an IVPFS, of Table 5.5 represents the rating value of the i^{th}-alternative over the j^{th}-benefit attribute. Also, if it is assumed that $H = \{0.1 \leq w_1 \leq 0.5, 0.2 \leq w_2 \leq 0.7\}$.

Table 5.5: Rating Values

Attributes→ ↓ Alternatives	G_1	G_2
A_1	$([0.20,0.50], [0.20,0.50])$	$([0.60,0.80], [0.30,0.40])$

A_2	$([0.30,0.60],[0.30,0.60])$	$([0.60,0.80],[0.30,0.40])$

Then, it is obvious that the ranking of the alternatives A_1 and A_2 can never be $A_1 = A_2$ as the rating values of both the alternatives corresponding to the attribute G_2 are equal. Whereas, the rating values of both the alternatives corresponding to first attribute G_1 are not equal. While, the following clearly indicates that on applying Garg's method [84], the obtained ranking of the alternatives is $A_1 = A_2$, which is inappropriate.

Using Garg's method [84] the ranking of the alternatives A_1 and A_2 can be obtained as follows:

Step 1: Since, all the attributes are of benefit type. So, according to Step 1 of Garg's method [84], discussed in Section 5.2, there is no need to apply Step 1.

Step 2: According to Step 2 of Garg's method [84], discussed in Section 5.2, there is need to calculate the values of d_{ij} $\forall$ $i = 1,2; j = 1,2$. These values are shown in Table 5.6.

Table 5.6: Values of d_{ij}

$d_{11} = 0$	$d_{12} = 0.5824$
$d_{21} = 0$	$d_{22} = 0.5824$

Step 3: Using Step 3 of Garg's method [84], discussed in Section 5.1,

$c_1 = \sum_{i=1}^{2} d_{i1} = 0 + 0 = 0,$

$c_2 = \sum_{i=1}^{2} d_{i2} = 0.5824 + 0.5824 = 1.1648.$

Step 4: Using Step 4 of Garg's method [84], discussed in Section 5.2, there is need to solve the CLPP (P5.3).

$Max(0.0w_1 + 1.1648w_2)$

Subject to

$$\begin{cases} 0.1 \leq w_1 \leq 0.5, \\ 0.2 \leq w_2 \leq 0.7, \\ w_1 + w_2 = 1, \\ w_1 \geq 0, w_2 \geq 0. \end{cases} \tag{P5.3}$$

It can be easily verified that on solving the CLPP (P5.3), the obtained OAWV is $(w_1, w_2) = (0.3, 0.7)$.

Step 5: On considering the OAWV, $(w_1, w_2) = (0.3, 0.7)$ and using Step 5 of Garg's method [84], discussed in Section 5.1,

$P_1 = (0.30)(0) + (0.70)(0.5824) = 0.4076,$

$P_2 = (0.30)(0) + (0.70)(0.5824) = 0.4076.$

Since, $P_1 = P_2$. So, according to Step 5, of Garg's method [84], discussed in Section 5.1, the ranking of the alternatives is $A_1 = A_2$.

It is obvious that on applying Garg's method [84], the relation $A_1 = A_2$ is obtained, which is inappropriate.

5.3 Reasons for the inappropriateness of Garg's method

In Section 5.2, it is shown that

(i) On applying Garg's method [84] more than one preference order for the alternatives are obtained.

(ii) Garg's method [84] fails to find the ranking of the alternatives if either the condition $\sum_{j=1}^{n} w_j^u < 1$ or the condition $\sum_{j=1}^{n} w_j^l > 1$ will be satisfied.

(iii) The ranking of the alternatives, obtained by Garg's method [84], is inappropriate.

These problems are occurring due to following reasons:

(i) If there will exist two distinct attributes G_p and G_q such that $c_p = c_q$ i.e., $\sum_{i=1}^{m} d_{ip} = \sum_{i=1}^{m} d_{iq}$. Then, the coefficient of the attribute weight w_p and w_q will be equal in the CLPP (P5.2). Therefore, all such values of w_p and w_q, corresponding to which $w_p + w_q$ will be optimal, will represent the optimal values of w_p and w_q. Due to the same reason, on solving the CLPP (P5.2) corresponding to the IVPFMADMPr having partially known attribute weights, considered in first point of Section 5.2, the number of obtained OAWV are more than one.

(ii) It is pertinent to mention that to apply Garg's method [84], there is need to solve the LPP (P5.2). However, a feasible solution of LPP (P5.2) will exist only if neither the condition $\sum_{j=1}^{n} w_j^u < 1$ nor the condition $\sum_{j=1}^{n} w_j^l > 1$ will be satisfied.

(iii) If there will exist two distinct alternatives A_p and A_q such that $c_p = c_q$ i.e., $\sum_{i=1}^{m} d_{ip} = \sum_{i=1}^{m} d_{iq}$. Then, the value of $P_p = \sum_{j=1}^{n} w_j\, d_{pj}$ will be equal to $P_q = \sum_{j=1}^{n} w_j\, d_{qj}$. Hence, the relation $A_p = A_q$ will be obtained. Due to the same reason, on solving the IVPFMADMPr having partially known attribute weights, considered in fourth point of Section 5.2, the obtained relation is $A_1 = A_2$.

5.4 Proposed Jorawar method

In this section, to resolve the inappropriateness of existing method [84], a new method (named as Jorawar method), is proposed.

The steps of the proposed Jorawar method are as follows:

Step 1: Check that all the attributes are of same type or not i.e., check that all the attributes are benefit type attribute or cost type attribute.

Case (i): If all the attributes are of same type then go to Step 2.

Case (ii): If some attributes are cost type attributes and the remaining are benefit type attributes. Then, convert the j^{th} cost type attribute into the benefit type attribute by replacing all the elements $\tilde{\alpha}_{ij} = \left([a_{ij1}, a_{ij2}], [a_{ij3}, a_{ij4}]\right)$ of the j^{th} column of the IVPFDM $\widetilde{D} = \left(\tilde{\alpha}_{ij}\right)_{m\times n}$ with $\tilde{\alpha}_{ij} = \left([a_{ij3}, a_{ij4}], [a_{ij1}, a_{ij2}]\right)$ and go to Step 2.

Step 2: Using the expression (5.4), find $d_{ijk}\left(\tilde{\alpha}_{ij}, \tilde{\alpha}_{kj}\right)\ \forall\ i = 1,2,\dots,m; k = 1,2,\dots,m; j = 1,2,..,n$.

$d_{ijk}\left(\tilde{\alpha}_{ij},\tilde{\alpha}_{kj}\right)=\frac{1}{4}\left(\left|a_{ij1}-a_{kj1}\right|+\left|a_{ij2}-a_{kj2}\right|+\left|a_{ij3}-a_{kj3}\right|+\left|a_{ij4}-a_{kj4}\right|\right), i=$

$1,2,\ldots m;\ j=1,2,\ldots,n; k=1,2,\ldots,m$ (5.4)

Step 3: Using expression (5.5), find the values of $c_j\ \forall\, j=1,2,..,n$.

$$c_j=\sum_{i=1}^{m}\sum_{k=1}^{m} d_{ijk}\left(\tilde{\alpha}_{ij},\tilde{\alpha}_{kj}\right), j=1,2,\ldots,n \tag{5.5}$$

Step 4: Check that the condition $\sum_{j=1}^{n} w_j^u<1$ or the condition $\sum_{j=1}^{n} w_j^l>1$ is satisfying or not.

Case (i): If the condition $\sum_{j=1}^{n} w_j^u<1$ is satisfying then find the OAWV, $(w_1,w_2,\ldots,w_n)$ by solving the CLPP (P3.5), proposed in Section 3.2.1 of Chapter 3.

Case (ii): If the condition $\sum_{j=1}^{n} w_j^l>1$ is satisfying then find the OAWV, $(w_1,w_2,\ldots,w_n)$ by solving the CLPP (P3.6), proposed in Section 3.2.2 of Chapter 3.

Case (iii): If neither the condition $\sum_{j=1}^{n} w_j^u<1$ nor the condition $\sum_{j=1}^{n} w_j^l>1$ is satisfying then find the OAWV, $(w_1,w_2,\ldots,w_n)$ by solving the CLPP (P2.12) of Chapter 2.

Step 5: Using the expression (5.6), find the value of $\tilde{P}_i\ \forall\ i=1,2,\ldots,m$.

$$\tilde{P}_i=\left(\left[\sum_{j=1}^{n} w_j a_{ij1},\sum_{j=1}^{n} w_j a_{ij2}\right],\left[\sum_{j=1}^{n} w_j a_{ij3},\sum_{j=1}^{n} w_j a_{ij4}\right]\right), i=1,2,\ldots,m \tag{5.6}$$

Step 6: Check that $S(\tilde{P}_p)>S(\tilde{P}_q)$ or $S(\tilde{P}_p)<S(\tilde{P}_q)$ or $S(\tilde{P}_p)=S(\tilde{P}_q)$, where $S(\tilde{P}_p)=\frac{a_{pj1}^2+a_{pj2}^2-a_{pj3}^2-a_{pj4}^2}{2}$ and $S(\tilde{P}_q)=\frac{a_{qj1}^2+a_{qj2}^2-a_{qj3}^2-a_{qj4}^2}{2}$.

Case (i): If $S(\tilde{P}_p)>S(\tilde{P}_q)$ then the ranking of the alternatives is $A_p>A_q$.

Case (ii): If $S(\tilde{P}_p)<S(\tilde{P}_q)$ then the ranking of the alternatives is $A_p<A_q$.

Case (iii): If $S(\tilde{P}_p)=S(\tilde{P}_q)$ then go to Step 7.

Step 7: Check that $H(\tilde{P}_p)>H(\tilde{P}_q)$ or $H(\tilde{P}_p)<H(\tilde{P}_q)$ or $H(\tilde{P}_p)=S(\tilde{P}_q)$, where $H(\tilde{P}_p)=\frac{a_{pj1}^2+a_{pj2}^2+a_{pj3}^2+a_{pj4}^2}{2}$ and $H(\tilde{P}_q)=\frac{a_{qj1}^2+a_{qj2}^2+a_{qj3}^2+a_{qj4}^2}{2}$.

Case (i): If $H(\tilde{P}_p)>H(\tilde{P}_q)$ then the ranking of the alternatives is $A_p>A_q$.

Case (ii): If $H(\tilde{P}_p)<H(\tilde{P}_q)$ then the ranking of the alternatives is $A_p<A_q$.

Case (iii): If $H(\tilde{P}_p) = H(\tilde{P}_q)$ then go to Step 8.

Step 8: Check that $T(\tilde{P}_p) > T(\tilde{P}_q)$ or $T(\tilde{P}_p) < S(\tilde{P}_q)$ or $T(\tilde{P}_p) = S(\tilde{P}_q)$, where $T(\tilde{P}_p) = a_{pj2}^2 + a_{pj3}^2 - a_{pj1}^2 - a_{pj4}^2$ and $T(\tilde{P}_q) = a_{qj2}^2 + a_{qj3}^2 - a_{qj1}^2 - a_{qj4}^2$.

Case (i): If $T(\tilde{P}_p) > T(\tilde{P}_q)$ then the ranking of the alternatives is $A_p < A_q$.

Case (ii): If $T(\tilde{P}_p) < T(\tilde{P}_q)$ then the ranking of the alternatives is $A_p > A_q$.

Case (iii): If $T(\tilde{P}_p) = T(\tilde{P}_q)$ then go to Step 9.

Step 9: Check that $G(\tilde{P}_p) > G(\tilde{P}_q)$ or $G(\tilde{P}_p) < G(\tilde{P}_q)$ or $G(\tilde{P}_p) = G(\tilde{P}_q)$, where $G(\tilde{P}_p) = a_{pj2}^2 + a_{pj4}^2 - a_{pj1}^2 - a_{pj3}^2$ and $G(\tilde{P}_q) = a_{qj2}^2 + a_{qj4}^2 - a_{qj1}^2 - a_{qj3}^2$.

Case (i): If $G(\tilde{P}_p) > G(\tilde{P}_q)$ then the ranking of the alternatives is $A_p < A_q$.

Case (ii): If $G(\tilde{P}_p) < G(\tilde{P}_q)$ then the ranking of the alternatives is $A_p > A_q$.

Case (iii): If $G(\tilde{P}_p) = G(\tilde{P}_q)$ then the ranking of the alternatives is $A_p = A_q$.

5.5 Exact results of the considered IVPFMADMPrs having partially known attribute weights

In Section 5.3, IVPFMADMPrs were solved by the existing methods [84] and shown that the obtained results are inappropriate. In this section, the exact results of all these IVPFMADMPrs are obtained by the proposed Jorawar method.

5.5.1 Exact results of the first IVPFMADMPr having partially known attribute weights

Using the proposed Jorawar method, the exact result of the IVPFMADMPr having partially known attribute weights, considered in the first point of Section 5.3, can be obtained as follows:

Step 1: Since, attributes G_1 and G_4 are cost type. So, according to Step 1 of the Jorawar method, proposed in Section 5.4,

Table 5.8: Rating Values

Attributes→ ↓ Alternatives	G_1	G_2	G_3	G_4
A_1	$\begin{pmatrix}[0.60,0.80],\\ {[0.30,0.40]}\end{pmatrix}$	$\begin{pmatrix}[0.50,0.60],\\ {[0.30,0.50]}\end{pmatrix}$	$\begin{pmatrix}[0.30,0.70],\\ {[0.50,0.60]}\end{pmatrix}$	$\begin{pmatrix}[0.70,0.80],\\ {[0.30,0.40]}\end{pmatrix}$
A_2	$\begin{pmatrix}[0.60,0.70],\\ {[0.40,0.60]}\end{pmatrix}$	$\begin{pmatrix}[0.60,0.70],\\ {[0.40,0.50]}\end{pmatrix}$	$\begin{pmatrix}[0.40,0.70],\\ {[0.30,0.50]}\end{pmatrix}$	$\begin{pmatrix}[0.60,0.80],\\ {[0.30,0.40]}\end{pmatrix}$
A_3	$\begin{pmatrix}[0.50,0.70],\\ {[0.30,0.40]}\end{pmatrix}$	$\begin{pmatrix}[0.50,0.60],\\ {[0.40,0.50]}\end{pmatrix}$	$\begin{pmatrix}[0.50,0.60],\\ {[0.70,0.80]}\end{pmatrix}$	$\begin{pmatrix}[0.60,0.70],\\ {[0.40,0.60]}\end{pmatrix}$
A_4	$\begin{pmatrix}[0.70,0.80],\\ {[0.30,0.40]}\end{pmatrix}$	$\begin{pmatrix}[0.60,0.70],\\ {[0.30,0.40]}\end{pmatrix}$	$\begin{pmatrix}[0.50,0.60],\\ {[0.40,0.50]}\end{pmatrix}$	$\begin{pmatrix}[0.50,0.70],\\ {[0.30,0.40]}\end{pmatrix}$

Step 2: According to Step 2 of the Jorawar method, proposed in Section 5.4, there is need to calculate $d_{ijk}(\tilde{\alpha}_{ij},\tilde{\alpha}_{kj})\ \forall\ i=1,2,\ldots,4; j=1,2,\ldots,4$. These values are shown in Table 5.9.

Table 5.9: Values of $\boldsymbol{d_{ijk}(\tilde{\alpha}_{ij},\tilde{\alpha}_{kj})}$

$d_{111}\left(\begin{pmatrix}[0.60,0.80],\\ {[0.30,0.40]}\end{pmatrix},\begin{pmatrix}[0.60,0.80],\\ {[0.30,0.40]}\end{pmatrix}\right)=$ $\frac{1}{4}(0+0+0+0)=0$	$d_{211}\left(\begin{pmatrix}[0.50,0.60],\\ {[0.30,0.50]}\end{pmatrix},\begin{pmatrix}[0.50,0.60],\\ {[0.30,0.50]}\end{pmatrix}\right)=$ $\frac{1}{4}(0+0+0+0)=0$
$d_{112}\left(\begin{pmatrix}[0.60,0.80],\\ {[0.30,0.40]}\end{pmatrix},\begin{pmatrix}[0.60,0.70],\\ {[0.40,0.60]}\end{pmatrix}\right)=$ $\frac{1}{4}(0+0.1+0.1+0.2)=0.1$	$d_{212}\left(\begin{pmatrix}[0.50,0.60],\\ {[0.30,0.50]}\end{pmatrix},\begin{pmatrix}[0.60,0.70],\\ {[0.40,0.50]}\end{pmatrix}\right)=$ $\frac{1}{4}(0.1+0.1+0.1+0)=0.075$
$d_{113}\left(\begin{pmatrix}[0.60,0.80],\\ {[0.30,0.40]}\end{pmatrix},\begin{pmatrix}[0.50,0.70],\\ {[0.30,0.40]}\end{pmatrix}\right)=$ $\frac{1}{4}(0.1+0.1+0+0)=0.05$	$d_{213}\left(\begin{pmatrix}[0.50,0.60],\\ {[0.30,0.50]}\end{pmatrix},\begin{pmatrix}[0.50,0.60],\\ {[0.40,0.50]}\end{pmatrix}\right)=$ $\frac{1}{4}(0+0+0.1+0)=0.025$
$d_{114}\left(\begin{pmatrix}[0.60,0.80],\\ {[0.30,0.40]}\end{pmatrix},\begin{pmatrix}[0.70,0.80],\\ {[0.30,0.40]}\end{pmatrix}\right)=$ $\frac{1}{4}(0.1+0+0+0)=0.025$	$d_{214}\left(\begin{pmatrix}[0.50,0.60],\\ {[0.30,0.50]}\end{pmatrix},\begin{pmatrix}[0.60,0.70],\\ {[0.30,0.40]}\end{pmatrix}\right)=$ $\frac{1}{4}(0.1+0.1+0+0.1)=0.075$

$d_{121}\left(\begin{pmatrix}[0.60,0.70],\\ [0.40,0.60]\end{pmatrix},\begin{pmatrix}[0.60,0.80],\\ [0.30,0.40]\end{pmatrix}\right)=$ $\frac{1}{4}(0+0.1+0.1+0.2)=0.1$	$d_{221}\left(\begin{pmatrix}[0.60,0.70],\\ [0.40,0.50]\end{pmatrix},\begin{pmatrix}[0.50,0.60],\\ [0.30,0.50]\end{pmatrix}\right)=$ $\frac{1}{4}(0.1+0.1+0.1+0)=0.075$
$d_{122}\left(\begin{pmatrix}[0.60,0.70],\\ [0.40,0.60]\end{pmatrix},\begin{pmatrix}[0.60,0.70],\\ [0.40,0.60]\end{pmatrix}\right)=$ $\frac{1}{4}(0+0+0+0)=0$	$d_{222}\left(\begin{pmatrix}[0.60,0.70],\\ [0.40,0.50]\end{pmatrix},\begin{pmatrix}[0.60,0.70],\\ [0.40,0.50]\end{pmatrix}\right)=$ $\frac{1}{4}(0+0+0+0)=0$
$d_{123}\left(\begin{pmatrix}[0.60,0.70],\\ [0.40,0.60]\end{pmatrix},\begin{pmatrix}[0.50,0.70],\\ [0.30,0.40]\end{pmatrix}\right)=$ $\frac{1}{4}(0.1+0+0.1+0.2)=0.1$	$d_{223}\left(\begin{pmatrix}[0.60,0.70],\\ [0.40,0.50]\end{pmatrix},\begin{pmatrix}[0.50,0.60],\\ [0.40,0.50]\end{pmatrix}\right)=$ $\frac{1}{4}(0.1+0.1+0+0)=0.05$
$d_{124}\left(\begin{pmatrix}[0.60,0.70],\\ [0.40,0.60]\end{pmatrix},\begin{pmatrix}[0.70,0.80],\\ [0.30,0.40]\end{pmatrix}\right)=$ $\frac{1}{4}(0.1+0.1+0.1+0.2)=0.125$	$d_{224}\left(\begin{pmatrix}[0.60,0.70],\\ [0.40,0.50]\end{pmatrix},\begin{pmatrix}[0.60,0.70],\\ [0.30,0.40]\end{pmatrix}\right)=$ $\frac{1}{4}(0+0+0.1+0.1)=0.05$
$d_{131}\left(\begin{pmatrix}[0.50,0.70],\\ [0.30,0.40]\end{pmatrix},\begin{pmatrix}[0.60,0.80],\\ [0.30,0.40]\end{pmatrix}\right)=$ $\frac{1}{4}(0.1+0.1+0+0)=0.05$	$d_{231}\left(\begin{pmatrix}[0.50,0.60],\\ [0.40,0.50]\end{pmatrix},\begin{pmatrix}[0.50,0.60],\\ [0.30,0.50]\end{pmatrix}\right)=$ $\frac{1}{4}(0+0+0.1+0)=0.025$
$d_{132}\left(\begin{pmatrix}[0.50,0.70],\\ [0.30,0.40]\end{pmatrix},\begin{pmatrix}[0.60,0.70],\\ [0.40,0.60]\end{pmatrix}\right)=$ $\frac{1}{4}(0.1+0+0.1+0.2)=0.1$	$d_{232}\left(\begin{pmatrix}[0.50,0.60],\\ [0.40,0.50]\end{pmatrix},\begin{pmatrix}[0.60,0.70],\\ [0.40,0.50]\end{pmatrix}\right)=$ $\frac{1}{4}(0.1+0.1+0+0)=0.05$
$d_{133}\left(\begin{pmatrix}[0.50,0.70],\\ [0.30,0.40]\end{pmatrix},\begin{pmatrix}[0.50,0.70],\\ [0.30,0.40]\end{pmatrix}\right)=$ $\frac{1}{4}(0+0+0+0)=0$	$d_{233}\left(\begin{pmatrix}[0.50,0.60],\\ [0.40,0.50]\end{pmatrix},\begin{pmatrix}[0.50,0.60],\\ [0.40,0.50]\end{pmatrix}\right)=$ $\frac{1}{4}(0+0+0+0)=0$
$d_{134}\left(\begin{pmatrix}[0.50,0.70],\\ [0.30,0.40]\end{pmatrix},\begin{pmatrix}[0.70,0.80],\\ [0.30,0.40]\end{pmatrix}\right)=$ $\frac{1}{4}(0.2+0.1+0+0)=0.075$	$d_{234}\left(\begin{pmatrix}[0.50,0.60],\\ [0.40,0.50]\end{pmatrix},\begin{pmatrix}[0.60,0.70],\\ [0.30,0.40]\end{pmatrix}\right)=$ $\frac{1}{4}(0.1+0.1+0.1+0.1)=0.1$

$d_{141}\left(\begin{pmatrix}[0.70,0.80],\\ [0.30,0.40]\end{pmatrix},\begin{pmatrix}[0.60,0.80],\\ [0.30,0.40]\end{pmatrix}\right)=$ $\frac{1}{4}(0.1+0+0+0)=0.025$	$d_{241}\left(\begin{pmatrix}[0.60,0.70],\\ [0.30,0.40]\end{pmatrix},\begin{pmatrix}[0.50,0.60],\\ [0.30,0.50]\end{pmatrix}\right)=$ $\frac{1}{4}(0.1+0.1+0+0.1)=0.075$
$d_{142}\left(\begin{pmatrix}[0.70,0.80],\\ [0.30,0.40]\end{pmatrix},\begin{pmatrix}[0.60,0.70],\\ [0.40,0.60]\end{pmatrix}\right)=$ $\frac{1}{4}(0.1+0.1+0.1+0.2)=0.125$	$d_{242}\left(\begin{pmatrix}[0.60,0.70],\\ [0.30,0.40]\end{pmatrix},\begin{pmatrix}[0.60,0.70],\\ [0.40,0.50]\end{pmatrix}\right)=$ $\frac{1}{4}(0+0+0.1+0.1)=0.05$
$d_{143}\left(\begin{pmatrix}[0.70,0.80],\\ [0.30,0.40]\end{pmatrix},\begin{pmatrix}[0.50,0.70],\\ [0.30,0.40]\end{pmatrix}\right)=$ $\frac{1}{4}(0.2+0.1+0+0)=0.075$	$d_{243}\left(\begin{pmatrix}[0.60,0.70],\\ [0.30,0.40]\end{pmatrix},\begin{pmatrix}[0.50,0.60],\\ [0.40,0.50]\end{pmatrix}\right)=$ $\frac{1}{4}(0.1+0.1+0.1+0.1)=0.1$
$d_{144}\left(\begin{pmatrix}[0.70,0.80],\\ [0.30,0.40]\end{pmatrix},\begin{pmatrix}[0.70,0.80],\\ [0.30,0.40]\end{pmatrix}\right)=$ $\frac{1}{4}(0+0+0+0)=0$	$d_{244}\left(\begin{pmatrix}[0.60,0.70],\\ [0.30,0.40]\end{pmatrix},\begin{pmatrix}[0.60,0.70],\\ [0.30,0.40]\end{pmatrix}\right)=$ $\frac{1}{4}(0+0+0+0)=0$
$d_{311}\left(\begin{pmatrix}[0.30,0.70],\\ [0.50,0.60]\end{pmatrix},\begin{pmatrix}[0.30,0.70],\\ [0.50,0.60]\end{pmatrix}\right)=$ $\frac{1}{4}(0+0+0+0)=0$	$d_{411}\left(\begin{pmatrix}[0.70,0.80],\\ [0.30,0.40]\end{pmatrix},\begin{pmatrix}[0.70,0.80],\\ [0.30,0.40]\end{pmatrix}\right)=$ $\frac{1}{4}(0+0+0+0)=0$
$d_{312}\left(\begin{pmatrix}[0.30,0.70],\\ [0.50,0.60]\end{pmatrix},\begin{pmatrix}[0.40,0.70],\\ [0.30,0.50]\end{pmatrix}\right)=$ $\frac{1}{4}(0.1+0+0.2+0.1)=0.1$	$d_{412}\left(\begin{pmatrix}[0.70,0.80],\\ [0.30,0.40]\end{pmatrix},\begin{pmatrix}[0.60,0.80],\\ [0.30,0.40]\end{pmatrix}\right)=$ $\frac{1}{4}(0.1+0+0+0)=0.025$
$d_{313}\left(\begin{pmatrix}[0.30,0.70],\\ [0.50,0.60]\end{pmatrix},\begin{pmatrix}[0.50,0.60],\\ [0.70,0.80]\end{pmatrix}\right)=$ $\frac{1}{4}(0.2+0.1+0.2+0.2)=0.175$	$d_{413}\left(\begin{pmatrix}[0.70,0.80],\\ [0.30,0.40]\end{pmatrix},\begin{pmatrix}[0.60,0.70],\\ [0.40,0.60]\end{pmatrix}\right)=$ $\frac{1}{4}(0.1+0.1+0.1+0.2)=0.125$
$d_{314}\left(\begin{pmatrix}[0.30,0.70],\\ [0.50,0.60]\end{pmatrix},\begin{pmatrix}[0.50,0.60],\\ [0.40,0.50]\end{pmatrix}\right)=$ $\frac{1}{4}(0.2+0.1+0.1+0.1)=0.125$	$d_{414}\left(\begin{pmatrix}[0.70,0.80],\\ [0.30,0.40]\end{pmatrix},\begin{pmatrix}[0.50,0.70],\\ [0.30,0.40]\end{pmatrix}\right)=$ $\frac{1}{4}(0.2+0.1+0+0)=0.075$

$d_{321}\left(\begin{pmatrix}[0.40,0.70],\\ {[0.30,0.50]}\end{pmatrix},\begin{pmatrix}[0.30,0.70],\\ {[0.50,0.60]}\end{pmatrix}\right)=$ $\frac{1}{4}(0.1+0+0.2+0.1)=0.1$	$d_{421}\left(\begin{pmatrix}[0.60,0.80],\\ {[0.30,0.40]}\end{pmatrix},\begin{pmatrix}[0.70,0.80],\\ {[0.30,0.40]}\end{pmatrix}\right)=$ $\frac{1}{4}(0.1+0+0+0)=0.025$
$d_{322}\left(\begin{pmatrix}[0.40,0.70],\\ {[0.30,0.50]}\end{pmatrix},\begin{pmatrix}[0.40,0.70],\\ {[0.30,0.50]}\end{pmatrix}\right)=$ $\frac{1}{4}(0+0+0+0)=0$	$d_{422}\left(\begin{pmatrix}[0.60,0.80],\\ {[0.30,0.40]}\end{pmatrix},\begin{pmatrix}[0.60,0.80],\\ {[0.30,0.40]}\end{pmatrix}\right)=$ $\frac{1}{4}(0+0+0+0)=0$
$d_{323}\left(\begin{pmatrix}[0.40,0.70],\\ {[0.30,0.50]}\end{pmatrix},\begin{pmatrix}[0.50,0.60],\\ {[0.70,0.80]}\end{pmatrix}\right)=$ $\frac{1}{4}(0.1+0.1+0.4+0.3)=0.225$	$d_{423}\left(\begin{pmatrix}[0.60,0.80],\\ {[0.30,0.40]}\end{pmatrix},\begin{pmatrix}[0.60,0.70],\\ {[0.40,0.60]}\end{pmatrix}\right)=$ $\frac{1}{4}(0+0.1+0.1+0.2)=0.1$
$d_{324}\left(\begin{pmatrix}[0.40,0.70],\\ {[0.30,0.50]}\end{pmatrix},\begin{pmatrix}[0.50,0.60],\\ {[0.40,0.50]}\end{pmatrix}\right)=$ $\frac{1}{4}(0.1+0.1+0.1+0)=0.075$	$d_{424}\left(\begin{pmatrix}[0.60,0.80],\\ {[0.30,0.40]}\end{pmatrix},\begin{pmatrix}[0.50,0.70],\\ {[0.30,0.40]}\end{pmatrix}\right)=$ $\frac{1}{4}(0.1+0.1+0+0)=0.05$
$d_{331}\left(\begin{pmatrix}[0.50,0.60],\\ {[0.70,0.80]}\end{pmatrix},\begin{pmatrix}[0.30,0.70],\\ {[0.50,0.60]}\end{pmatrix}\right)=$ $\frac{1}{4}(0.2+0.1+0.2+0.2)=0.175$	$d_{431}\left(\begin{pmatrix}[0.60,0.70],\\ {[0.40,0.60]}\end{pmatrix},\begin{pmatrix}[0.70,0.80],\\ {[0.30,0.40]}\end{pmatrix}\right)=$ $\frac{1}{4}(0.1+0.1+0.1+0.2)=0.125$
$d_{332}\left(\begin{pmatrix}[0.50,0.60],\\ {[0.70,0.80]}\end{pmatrix},\begin{pmatrix}[0.40,0.70],\\ {[0.30,0.50]}\end{pmatrix}\right)=$ $\frac{1}{4}(0.1+0.1+0.4+0.3)=0.225$	$d_{432}\left(\begin{pmatrix}[0.60,0.70],\\ {[0.40,0.60]}\end{pmatrix},\begin{pmatrix}[0.60,0.80],\\ {[0.30,0.40]}\end{pmatrix}\right)=$ $\frac{1}{4}(0+0.1+0.1+0.2)=0.1$
$d_{333}\left(\begin{pmatrix}[0.50,0.60],\\ {[0.70,0.80]}\end{pmatrix},\begin{pmatrix}[0.50,0.60],\\ {[0.70,0.80]}\end{pmatrix}\right)=$ $\frac{1}{4}(0+0+0+0)=0$	$d_{433}\left(\begin{pmatrix}[0.60,0.70],\\ {[0.40,0.60]}\end{pmatrix},\begin{pmatrix}[0.60,0.70],\\ {[0.40,0.60]}\end{pmatrix}\right)=$ $\frac{1}{4}(0+0+0+0)=0$
$d_{334}\left(\begin{pmatrix}[0.50,0.60],\\ {[0.70,0.80]}\end{pmatrix},\begin{pmatrix}[0.50,0.60],\\ {[0.40,0.50]}\end{pmatrix}\right)=$ $\frac{1}{4}(0+0+0.3+0.3)=0.15$	$d_{434}\left(\begin{pmatrix}[0.60,0.70],\\ {[0.40,0.60]}\end{pmatrix},\begin{pmatrix}[0.50,0.70],\\ {[0.30,0.40]}\end{pmatrix}\right)=$ $\frac{1}{4}(0.1+0+0.1+0.2)=0.1$

$d_{341}\left(\begin{pmatrix}[0.50,0.60],\\ [0.40,0.50]\end{pmatrix},\begin{pmatrix}[0.30,0.70],\\ [0.50,0.60]\end{pmatrix}\right)=$ $\frac{1}{4}(0.2+0.1+0.1+0.1)=0.125$	$d_{441}\left(\begin{pmatrix}[0.50,0.70],\\ [0.30,0.40]\end{pmatrix},\begin{pmatrix}[0.70,0.80],\\ [0.30,0.40]\end{pmatrix}\right)=$ $\frac{1}{4}(0.2+0.1+0+0)=0.075$
$d_{342}\left(\begin{pmatrix}[0.50,0.60],\\ [0.40,0.50]\end{pmatrix},\begin{pmatrix}[0.40,0.70],\\ [0.30,0.50]\end{pmatrix}\right)=$ $\frac{1}{4}(0.1+0.1+0.1+0)=0.075$	$d_{442}\left(\begin{pmatrix}[0.50,0.70],\\ [0.30,0.40]\end{pmatrix},\begin{pmatrix}[0.60,0.80],\\ [0.30,0.40]\end{pmatrix}\right)=$ $\frac{1}{4}(0.1+0.1+0+0)=0.05$
$d_{343}\left(\begin{pmatrix}[0.50,0.60],\\ [0.40,0.50]\end{pmatrix},\begin{pmatrix}[0.50,0.60],\\ [0.70,0.80]\end{pmatrix}\right)=$ $\frac{1}{4}(0+0+0.3+0.3)=0.15$	$d_{443}\left(\begin{pmatrix}[0.50,0.70],\\ [0.30,0.40]\end{pmatrix},\begin{pmatrix}[0.60,0.70],\\ [0.40,0.60]\end{pmatrix}\right)=$ $\frac{1}{4}(0.1+0+0.1+0.2)=0.1$
$d_{344}\left(\begin{pmatrix}[0.50,0.60],\\ [0.40,0.50]\end{pmatrix},\begin{pmatrix}[0.50,0.60],\\ [0.40,0.50]\end{pmatrix}\right)=$ $\frac{1}{4}(0+0+0+0)=0$	$d_{444}\left(\begin{pmatrix}[0.50,0.70],\\ [0.30,0.40]\end{pmatrix},\begin{pmatrix}[0.50,0.70],\\ [0.30,0.40]\end{pmatrix}\right)=$ $\frac{1}{4}(0+0+0+0)=0$

Step 3: Using Step 3 of the Jorawar method, proposed in Section 5.4,

$c_1=\sum_{i=1}^{4}\sum_{k=1}^{4} d_{i1k}(\tilde{\alpha}_{i1},\tilde{\alpha}_{k1})=0+0.1+0.05+0.025+0.1+0+0.1+0.125+0.05+0.1+0+0.075+0.025+0.125+0.075+0=0.95$,

$c_2=\sum_{i=1}^{4}\sum_{k=1}^{4} d_{i2k}(\tilde{\alpha}_{i2},\tilde{\alpha}_{k2})=0+0.075+0.025+0.075+0.075+0+0.05+0.05+0.025+0.05+0+0.1+0.075+0.05+0.1+0=0.75$,

$c_3=\sum_{i=1}^{4}\sum_{k=1}^{4} d_{i3k}(\tilde{\alpha}_{i3},\tilde{\alpha}_{k3})=0+0.1+0.175+0.125+0.1+0+0.225+0.075+0.175+0.225+0+0.15+0.125+0.075+0.15+0=1.70$,

$c_4=\sum_{i=1}^{4}\sum_{k=1}^{4} d_{i4k}(\tilde{\alpha}_{i4},\tilde{\alpha}_{k4})=0+0.025+0.125+0.075+0.025+0+0.1+0.05+0.125+0.1+0+0.1+0.075+0.05+0.1+0=0.95$.

Step 4: Using Step 4 of the Jorawar method, proposed in Section 5.4, there is need to solve the CLPP (P5.4).

$max(0.95w_1+0.75w_2+1.7w_3+0.95w_4)$

Subject to

$$\begin{cases} 0.10 \le w_1 \le 0.30, \\ 0.20 \le w_2 \le 0.30, \\ 0.30 \le w_3 \le 0.50, \\ 0.20 \le w_4 \le 0.30, \\ w_1 + w_2 + w_3 + w_4 = 1, \\ w_1 = w_4, \\ w_1 \ge 0, w_2 \ge 0, w_3 \ge 0, w_4 \ge 0. \end{cases} \qquad \text{(P5.4)}$$

It can be easily verified that on solving the CLPP (P4.12), the obtained OAWV is $(w_1, w_2, w_3, w_4) = (0.20, 0.20, 0.40, 0.20)$.

Step 5: On considering the OAWV, $(w_1, w_2, w_3, w_4) = (0.20, 0.20, 0.40, 0.20)$ and using Step 5 of Jorawar method, proposed in Section 5.4,

$\tilde{P}_1 = ([(0.20)(0.30) + (0.20)(0.50) + (0.40)(0.30) + (0.20)(0.30), (0.20)(0.40) + (0.20)(0.60) + (0.40)(0.70) + (0.20)(0.40)], [(0.20)(0.60) + (0.20)(0.30) + (0.40)(0.50) + (0.20)(0.70), (0.20)(0.70) + (0.20)(0.50) + (0.40)(0.50) + (0.20)(0.80)])$

$= ([0.34, 0.56], [0.52, 0.60]),$

$\tilde{P}_2 = ([(0.20)(0.40) + (0.20)(0.60) + (0.40)(0.40) + (0.20)(0.30), (0.20)(0.60) + (0.20)(0.70) + (0.40)(0.70) + (0.20)(0.40)], [(0.20)(0.60) + (0.20)(0.40) + (0.40)(0.30) + (0.20)(0.60), (0.20)(0.70) + (0.20)(0.50) + (0.40)(0.50) + (0.20)(0.80)])$

$= ([0.42, 0.62], [0.44, 0.60]),$

$\tilde{P}_3 = ([(0.20)(0.30) + (0.20)(0.50) + (0.40)(0.50) + (0.20)(0.40), (0.20)(0.40) + (0.20)(0.60) + (0.40)(0.60) + (0.20)(0.60)], [(0.20)(0.50) + (0.20)(0.40) + (0.40)(0.70) + (0.20)(0.60), (0.20)(0.70) + (0.20)(0.50) + (0.40)(0.80) + (0.20)(0.70)])$

$= ([0.44, 0.56], [0.58, 0.70]),$

$\tilde{P}_4 = ([(0.20)(0.30) + (0.20)(0.60) + (0.40)(0.50) + (0.20)(0.30), (0.20)(0.40) + (0.20)(0.70) + (0.40)(0.60) + (0.20)(0.40)], [(0.20)(0.70) + (0.20)(0.30) + (0.40)(0.40) + (0.20)(0.50), (0.20)(0.80) + (0.20)(0.40) + (0.40)(0.50) + (0.20)(0.70)])$

$= ([0.44, 0.54], [0.46, 0.58]).$

Step 6: Using Step 6 of the Jorawar method, proposed in Section 5.4,

$S(\tilde{P}_1) = -0.1006, S(\tilde{P}_2) = 0.0036, S(\tilde{P}_3) = -0.1596$ and $S(\tilde{P}_4) = -0.0332$.

Since, $S(\tilde{P}_2) > S(\tilde{P}_4) > S(\tilde{P}_1) > S(\tilde{P}_3)$. So, according to Step 6 of the Jujhar method, proposed in Section 5.4, the ranking of the alternatives is $A_2 > A_4 > A_1 > A_3$.

5.5.2 Exact results of the second IVPFMADMPr having partially known attribute weights

Using the proposed Jorawar method, the exact result of the IVPFMADMPr having partially known attribute weights, considered in the second point of Section 5.2, can be obtained as follows:

Since, Step 1 to Step 3 of the proposed Jorawar method are same as shown in Section 5.5.1. So, to avoid any repetition, the calculations have been started from Step 4.

Step 4: Since, the condition $\sum_{j=1}^{n} w_j^u < 1$ is satisfying. So, using Case (i) of Step 4 of the Jorawar method, proposed in Section 5.4, there is need to solve the CLPP (P5.5).

$max(0.95w_1 + 0.75w_2 + 1.7w_3 + 0.95w_4)$

Subject to

$$\begin{cases} 0.10 \leq w_1 \leq 0.20, \\ 0.20 \leq w_2 \leq 0.25, \\ 0.10 \leq w_3 \leq 0.15, \\ 0.1 \leq w_4 \leq 0.20, \\ w_1 + w_2 + w_3 + w_4 \leq 1, \\ w_1 = w_4, \\ w_1 \geq 0, w_2 \geq 0, w_3 \geq 0, w_4 \geq 0. \end{cases} \tag{P5.5}$$

On solving the LPP (P5.5) the obtained non-normalized OAWV is $(w_1, w_2, w_3, w_4) = (0.20, 0.25, 0.15, 0.20)$ and the normalized OAWV is $(w_1, w_2, w_3, w_4) = \left(\frac{0.20}{0.20+0.25+0.15+0.20}, \frac{0.25}{0.20+0.25+0.15+0.20}, \frac{0.15}{0.20+0.25+0.15+0.20}, \frac{0.20}{0.20+0.25+0.15+0.20}\right)$

$= (0.25, 0.3125, 0.1875, 0.25)$.

Step 5: On considering the normalized OAWV $(w_1, w_2, w_3, w_4) = (0.25, 0.3125, 0.1875, 0.25)$ and using Step 5 of Jorawar method, proposed in Section 5.4,

$\tilde{P}_1 = ([(0.25)(0.30) + (0.3125)(0.50) + (0.1875)(0.30) + (0.25)(0.30), (0.25)(0.40) + (0.3125)(0.60) + (0.1875)(0.70) + (0.25)(0.40)], [(0.25)(0.60) + (0.3125)(0.30) + (0.1875)(0.50) + (0.25)(0.70), (0.25)(0.80) + (0.3125)(0.50) + (0.1875)(0.60) + (0.25)(0.80)]) = ([0.3625, 0.5187], [0.5125, 0.6687])$,

$\tilde{P}_2 = ([(0.25)(0.40) + (0.3125)(0.60) + (0.1875)(0.40) + (0.25)(0.30), (0.25)(0.60) + (0.3125)(0.70) + (0.1875)(0.70) + (0.25)(0.40)], [(0.25)(0.60) + (0.3125)(0.40) + (0.1875)(0.30) + (0.25)(0.60), (0.25)(0.70) + (0.3125)(0.50) + (0.1875)(0.50) + (0.25)(0.80)]) = ([0.4375, 0.6000], [0.4812, 0.6250])$,

$\tilde{P}_3 = ([(0.25)(0.30) + (0.3125)(0.50) + (0.1875)(0.50) + (0.25)(0.40), (0.25)(0.40) + (0.3125)(0.60) + (0.1875)(0.60) + (0.25)(0.60)], [(0.25)(0.50) + (0.3125)(0.40) + (0.1875)(0.70) + (0.25)(0.60), (0.25)(0.70) + (0.3125)(0.50) + (0.1875)(0.80) + (0.25)(0.70)]) = ([0.4250, 0.5500], [0.5312, 0.6562])$,

$\tilde{P}_4 = ([(0.25)(0.30) + (0.3125)(0.50) + (0.1875)(0.30) + (0.25)(0.30), (0.25)(0.40) + (0.3125)(0.60) + (0.1875)(0.70) + (0.25)(0.40)], [(0.25)(0.60) + (0.3125)(0.30) + (0.1875)(0.50) + (0.25)(0.70), (0.25)(0.80) + (0.3125)(0.50) + (0.1875)(0.60) + (0.25)(0.80)]) = ([0.4312, 0.5312], [0.4687, 0.5930])$.

Step 6: Using Step 6 of the Jorawar method, proposed in Section 5.4,

$S(\tilde{P}_1) = -0.1546, S(\tilde{P}_2) = -0.0353, S(\tilde{P}_3) = -0.1148$ and $S(\tilde{P}_4) = -0.0520$.

Since, $S(\tilde{P}_2) > S(\tilde{P}_4) > S(\tilde{P}_3) > S(\tilde{P}_1)$. So, according to Step 6 of the Jujhar method, proposed in Section 4.13, the ranking of the alternatives is $A_2 > A_4 > A_3 > A_1$.

5.5.3 Exact results of the third IVPFMADMPr having partially known attribute weights

Using the proposed Jorawar method, the exact result of the IVPFMADMPr having partially known attribute weights, considered in the second point of Section 5.3, can be obtained as follows:

Since, Step 1 to Step 3 of the proposed Jorawar method are same as shown in Section 5.5.1. So, to avoid any repetition, the calculations have been started from Step 4.

Step 4: Since, the condition $\sum_{j=1}^{n} w_j^l > 1$ is satisfying. So, using Case (ii) of Step 4 of the Jorawar method, proposed in Section 5.4, there is need to solve the CLPP (P5.6).

$max\ (0.95w_1 + 0.75w_2 + 1.7w_3 + 0.95w_4 - \varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4)$

Subject to

$$\left\{\begin{array}{c} \frac{30}{150} \le w_1 \le \frac{80}{150}, \\ \frac{40}{150} \le w_2 \le \frac{60}{150}, \\ \frac{50}{150} \le w_3 \le \frac{70}{150}, \\ \frac{30}{150} \le w_4 \le \frac{80}{150}, \\ w_1 + w_2 + w_3 + w_4 = 1, \\ w_1 = w_4, \\ 150 = 100 + \varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4, \\ w_1 \ge 0, w_2 \ge 0, w_3 \ge 0, w_4 \ge 0, \varepsilon_1 \ge 0, \varepsilon_2 \ge 0, \varepsilon_3 \ge 0, \varepsilon_4 \ge 0. \end{array}\right. \qquad \text{(P5.6)}$$

On solving the CLPP (P5.6) the obtained OAWV is $(w_1, w_2, w_3, w_4) = (0.2000, 0.2666, 0.3334, 0.2000)$.

Step 5: On considering the OAWV $(w_1, w_2, w_3, w_4) = (0.2000, 0.2666, 0.3334, 0.2000)$ and using Step 5 of Jorawar method, proposed in Section 5.4,

$\tilde{P}_1 = ([(0.2000)(0.30) + (0.2666)(0.50) + (0.3334)(0.30) + (0.2000)(0.30), (0.2000)(0.40) + (0.2666)(0.60) + (0.3334)(0.70) + (0.2000)(0.40)], [(0.2000)(0.60) + (0.2666)(0.30) + (0.3334)(0.50) + (0.2000)(0.70), (0.2000)(0.80) + (0.2666)(0.50) + (0.3334)(0.60) + (0.2000)(0.80)]) = ([0.3773, 0.5853], [0.5626, 0.7173])$,

$\tilde{P}_2 = ([(0.2000)(0.40) + (0.2666)(0.60) + (0.3334)(0.40) + (0.2000)(0.30), (0.2000)(0.60) + (0.2666)(0.70) + (0.3334)(0.70) + (0.2000)(0.40)], [(0.2000)(0.60) + (0.2666)(0.40) + (0.3334)(0.30) + (0.2000)(0.60), (0.2000)(0.70) + (0.2666)(0.50) + (0.3334)(0.50) + (0.2000)(0.80)]) = ([0.4573, 0.6520], [0.4946, 0.6640])$,

$\tilde{P}_3 = ([(0.2000)(0.30) + (0.2666)(0.50) + (0.3334)(0.50) + (0.2000)(0.40), (0.2000)$

$(0.40) + (0.2666)(0.60) + (0.3334)(0.60) + (0.2000)(0.60)], [(0.2000)(0.50) + (0.2666)(0.40) + (0.3334)(0.70) + (0.2000)(0.60), (0.2000)(0.70) + (0.2666)(0.50) + (0.3334)(0.80) + (0.2000)(0.70)]) = ([0.4720, 0.6080], [0.6080, 0.7360])$,

$\tilde{P}_4 = ([(0.2000)(0.30) + (0.2666)(0.60) + (0.3334)(0.50) + (0.2000)(0.30), (0.2000)(0.40) + (0.2666)(0.70) + (0.3334)(0.60) + (0.2000)(0.40)], [(0.2000)(0.70) + (0.2666)(0.30) + (0.3334)(0.40) + (0.2000)(0.50), (0.2000)(0.80) + (0.2666)(0.40) + (0.3334)(0.50) + (0.2000)(0.70)]) = ([0.4706, 0.5786], [0.4933, 0.6293])$.

Step 6: Using Step 6 of the Jorawar method, proposed in Section 5.4,

$S(\tilde{P}_1) = -0.1730, S(\tilde{P}_2) = -0.0256, S(\tilde{P}_3) = -0.1594$ and $S(\tilde{P}_4) = -0.0415$.

Since, $S(\tilde{P}_2) > S(\tilde{P}_4) > S(\tilde{P}_3) > S(\tilde{P}_1)$. So, according to Step 6 of the Jorawar method, proposed in Section 5.4, the ranking of the alternatives is $A_2 > A_4 > A_3 > A_1$.

5.5.4 Exact results of fourth IVPFMADMPr having partially known attribute weights

Using the proposed Jorawar method, the exact result of the IVPFMADMPr having partially known attribute weights, considered in the fourth point of Section 5.3, can be obtained as follows:

Step 1: Since, all the attributes are benefit type. So, according to Step 1 of the Jorawar method, proposed in Section 5.4, there is no need to apply Step 1.

Step 2: According to Step 2 of the Jorawar method, proposed in Section 5.4, there is need to calculate $d_{ijk}(\tilde{\alpha}_{ij}, \tilde{\alpha}_{kj})\ \forall\ i = 1,2,\dots,6; j = 1,2,3$. These values are shown in Table 5.10.

Table 5.10: Values of $d_{ijk}(\tilde{\alpha}_{ij}, \tilde{\alpha}_{kj})$

$d_{111}\left(\begin{pmatrix}[0.20,0.50],\\ [0.20,0.50]\end{pmatrix},\begin{pmatrix}[0.20,0.50],\\ [0.20,0.50]\end{pmatrix}\right) =$ $\frac{1}{4}(0+0+0+0) = 0$	$d_{211}\left(\begin{pmatrix}[0.60,0.80],\\ [0.30,0.40]\end{pmatrix},\begin{pmatrix}[0.60,0.80],\\ [0.30,0.40]\end{pmatrix}\right) =$ $\frac{1}{4}(0+0+0+0) = 0$

$d_{112}\left(\begin{pmatrix}[0.20,0.50],\\ [0.20,0.50]\end{pmatrix},\begin{pmatrix}[0.30,0.60],\\ [0.30,0.60]\end{pmatrix}\right)=$ $\frac{1}{4}(0.1+0.1+0.1+0.1)=0.10$	$d_{212}\left(\begin{pmatrix}[0.60,0.80],\\ [0.30,0.40]\end{pmatrix},\begin{pmatrix}[0.60,0.80],\\ [0.30,0.40]\end{pmatrix}\right)=$ $\frac{1}{4}(0+0+0+0)=0$
$d_{121}\left(\begin{pmatrix}[0.30,0.60],\\ [0.30,0.60]\end{pmatrix},\begin{pmatrix}[0.20,0.50],\\ [0.20,0.50]\end{pmatrix}\right)=$ $\frac{1}{4}(0.1+0.1+0.1+0.1)=0.10$	$d_{221}\left(\begin{pmatrix}[0.60,0.80],\\ [0.30,0.40]\end{pmatrix},\begin{pmatrix}[0.60,0.80],\\ [0.30,0.40]\end{pmatrix}\right)=$ $\frac{1}{4}(0+0+0+0)=0$
$d_{122}\left(\begin{pmatrix}[0.30,0.60],\\ [0.30,0.60]\end{pmatrix},\begin{pmatrix}[0.30,0.60],\\ [0.30,0.60]\end{pmatrix}\right)=$ $\frac{1}{4}(0+0+0+0)=0$	$d_{222}\left(\begin{pmatrix}[0.60,0.80],\\ [0.30,0.40]\end{pmatrix},\begin{pmatrix}[0.60,0.80],\\ [0.30,0.40]\end{pmatrix}\right)=$ $\frac{1}{4}(0+0+0+0)=0$

Step 3: Using Step 3 of the Jorawar method, proposed in Section 5.4,

$c_1=\sum_{i=1}^{2}\sum_{k=1}^{2} d_{i1k}(\tilde{\alpha}_{i1},\tilde{\alpha}_{k1})=0+0.1+0.1+0+0=0.2,$

$c_2=\sum_{i=1}^{2}\sum_{k=1}^{2} d_{i1k}(\tilde{\alpha}_{i1},\tilde{\alpha}_{k1})=0+0+0+0=0.$

Step 4: Using Step 4 of the Jorawar method, proposed in Section 5.4, there is need to solve the CLPP (P5.7).

$max(0.2w_1)$

Subject to

$$\begin{cases}0.1\leq w_1\leq 0.5,\\ 0.2\leq w_2\leq 0.7,\\ w_1+w_2=1,\\ w_1\geq 0, w_2\geq 0.\end{cases} \qquad \text{(P5.7)}$$

It can be easily verified that on solving the CLPP (P5.7) the obtained OAWV is $(w_1,w_2)=(0.5,0.5)$

Step 5: On considering the OAWV, $(w_1,w_2)=(0.5,0.5)$ and using Step 5 of the Jorawar method, proposed in Section 5.4,

$\tilde{P}_1=([(0.50)(0.20)+(0.50)(0.60),(0.50)(0.50)+(0.50)(0.80)],[(0.50)(0.20)+(0.50)(0.30),(0.50)(0.50)+(0.50)(0.40)])=([0.40,0.65],[0.25,0.45]),$

$\tilde{P}_2 = ([(0.50)(0.30) + (0.50)(0.60), (0.50)(0.60) + (0.50)(0.80)], [(0.50)(0.30) + (0.50)(0.30), (0.50)(0.60) + (0.50)0.40]) = ([0.45,0.70], [0.30,0.50])$.

Step 6: Using Step 6 of the Jorawar method, proposed in Section 5.4,

$S(\tilde{P}_1) = 0.1587$ and $S(\tilde{P}_2) = 0.1762$.

Since, $S(\tilde{P}_2) > S(\tilde{P}_1)$. Therefore, according to case (ii) of Step 7 of the proposed Jorawar method, the ranking of the alternatives is $A_2 > A_1$.

5.6 Conclusions

The inappropriateness of the existing method [84] is pointed out. Also, a new method (named as Jorawar method) is proposed to solve the IVPFMADMPrs having partially known attribute weights. Furthermore, the proposed Jorawar method has been illustrated with the help of some IVPFMADMPrs having partially known attribute weights.

Chapter 6

Jujhar method for solving IVIFMADMPrs having IVIF attribute weights[5]

Li [128] pointed out that there does not exist any method to solve IVIFMADMPrs having IVIF weights. To fill this gap, Li proposed a method to solve IVIFMADMPrs having IVIF weights. Motivated by Li, several other researchers [35-45, 47-49, 52, 55, 127, 129, 132, 137, 209-211] have proposed different methods to solve the IVIFMADMPrs having IVIF weights. However, after a deep study it is observed that it is inappropriate to use the existing methods [35-45, 47-49, 52, 55, 127-129, 132, 137, 209-211]. The aim of this chapter is to make the researchers aware about the inappropriateness of these methods as well as to propose a new method (named as Jujhar method) to solve IVIFMADMPrs having IVIF attribute weights.

6.1 A brief review about the existing work

Li [128] pointed out that there does not exist any method for solving such MADMPrs in which rating values of each alternative over each attribute as well as each attribute weight is represented by an IVIF set [9]. To fill this gap, Li [128] proposed a method to solve IVIFMADMPrs having IVIF weights. In this method, firstly, two non-linear programming problems, corresponding to each alternative, are solved to evaluate the crisp attribute weights corresponding to the known IVIF attribute weights. Then, the obtained crisp attribute weights are used to find the ranking of the alternatives. Li [129] used the same method to solve IVIFMADMPrs with crisp weights.

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[5] The contents of this chapter have been communicated in “Engineering Applications of Artificial Intelligence” for the possible publication

Li [127] also proposed a method to solve IVIFMADMPrs having IVIF weights. In this method, firstly, two LPPs, corresponding to each alternative, are solved to evaluate the crisp attribute weights corresponding to the known IVIF attribute weights. Then, the obtained crisp attribute weights are used to find the ranking of the alternatives.

Chen and Chiou [40] pointed out the flaws of Li's method [128]. Also, to resolve the flaws, Chen and Chiou [40] proposed a method to solve IVIFMADMPrs having IVIF weights. In this method, firstly, a NLPP is solved to evaluate the crisp attribute weights corresponding to the known IVIF attribute weights. Then, the obtained crisp attribute weights are used to find the ranking of the alternatives. Although, Chen and Chiou [40] have not pointed out the flaws of Li's method [128]. However, it can be easily verified that the flaws, pointed out by Chen and Chiou [40] in Li's method [127], are also occurring in Li's method [128].

Chen and Huang [45] pointed out the flaws of Li's method [128] as well as Chen and Chiou's method [40]. Also, to resolve the flaws, Chen and Huang [45] proposed a method to solve IVIFMADMPrs having IVIF weights. In this method, firstly, the CLPP (P6.1) is solved to evaluate the crisp attribute weights corresponding to the known IVIF attribute weights. Then, the obtained crisp attribute weights are used to find the ranking of the alternatives.

$$Max\left(\sum_{j=1}^{n}\left(\sum_{i=1}^{m}\left(w_j d_{ij}\right)\right)\right)$$

Subject to

$$\begin{cases} w_{j1} \le w_j \le 1 - w_{j3}, \mathrm{j} = 1,2,\dots,\mathrm{n}, \\ \sum_{j=1}^{n} w_j = 1\,, \\ 0 \le w_j \le 1, \mathrm{j} = 1,2,\dots,\mathrm{n}. \end{cases} \tag{P6.1}$$

where,

(i) $d_{ij} = \frac{a_{ij1}+a_{ij2}-a_{ij3}-a_{ij4}}{2}$ represents the crisp value corresponding to the IVIF rating value $\left([a_{ij1}, a_{ij2}], [a_{ij3}, a_{ij4}]\right)$ of the i^{th}- alternative over the j^{th}- attribute.

(ii) w_j represents the unknown crisp weight of the j^{th}- attribute corresponding to the known

IVIF weight $\tilde{w}_j = ([w_{j1}, w_{j2}], [w_{j3}, w_{j4}])$ of the j^{th}- attribute.

(iii) m represents the number of alternatives.

(iv) n represents the number of attributes.

Wang and Chen [209] pointed out the flaws of Li's method and Chen and Chiou's method [40]. Also, to resolve the flaws Wang and Chen [209] proposed a method for solving IVIFMADMPrs having IVIF weights. In this method firstly, a CLPP based upon an existing similarity measure, is used to evaluate the crisp attribute weight w_j corresponding to the IVIF weight $\tilde{w}_j = ([w_{j1}, w_{j2}], [w_{j3}, w_{j4}])$. Then, the obtained crisp attribute weights are used to find the ranking of the alternatives.

Chen and Huang [44] also pointed out the flaws of Li's method [128] as well as Chen and Chiou's method [40]. Furthermore, to resolve the flaws, Chen and Huang [44] proposed a method to solve IVIFMADMPrs having IVIF weights. In this method, firstly, the CLPP (P6.1) with the following modification is solved to evaluate the crisp attribute weights corresponding to the known IVIF attribute weights. Then, the obtained crisp attribute weights are used to find the ranking of the alternatives.

"Replace $d_{ij} = \frac{a_{ij1}+a_{ij2}-a_{ij3}-a_{ij4}}{2}$ with $d_{ij} = a_{ij1} + a_{ij2} - 1 + \frac{a_{ij3}+a_{ij4}}{2}$."

Wang and Chen [210] pointed out the flaws Chen and Huang's method [45]. Also, to resolve the flaws, Wang and Chen [210] proposed a method to solve IVIFMADMPrs having IVIF weights. In this method, firstly, the CLPP (P6.1) with the following modification is solved to evaluate the crisp attribute weights corresponding to the known IVIF attribute weights. Then, the obtained crisp attribute weights are used to find the ranking of the alternatives.

"Replace $d_{ij} = \frac{a_{ij1}+a_{ij2}-a_{ij3}-a_{ij4}}{2}$ with $d_{ij} = \frac{a_{ij1}+a_{ij2}+\sqrt{a_{ij2}a_{ij4}}(1-a_{ij1}-a_{ij3})+\sqrt{a_{ij1}a_{ij3}}(1-a_{ij2}-a_{ij4})}{2}$."

Cheng [56] proposed a method to solve IVIFMADMPrs having IVIF weights. In this method, the IVIF weights are directly used to aggregate the IVIF rating values. However, it is inappropriate to do the same as sum of the IVIF weights will never be $([1,1],[0,0])$. Therefore, it is not appropriate to use Cheng's IVIFMADM method [56].

Gupta et al. [93] proposed a method for solving IVIFMADMPr having IVIF weights. In this method, a CLPP has been solved to obtain the attribute weight w_j corresponding to the IVIF weight $\widetilde{w}_j = \left(\left[w_{j1}, w_{j2}\right], \left[w_{j3}, w_{j4}\right]\right)$. It is pertinent to mention that, in the CLPP, considered by Gupta et al. [93], the constraint $w_{j1} \leq w_j \leq 1 - w_{j3}$ has been used. However, it is inappropriate to use this constraint as if $w_{j1} = 1 - w_{j3}$ then the obtained value will be always w_{j1} and will be independent from the remaining values of w_{j2} and w_{j3}. Therefore, it is inappropriate to use Gupta et al.'s IVIFMADM method [93].

Chen and Han [41] as well as Wang and Chen [211] pointed out that it is inappropriate to use the CLPP (P1) due to the following reason:

"If there will exist two distinct attributes G_p and G_q such that $c_p = c_q$ i.e., $\sum_{i=1}^{m} d_{ip} = \sum_{i=1}^{m} d_{iq}$ then the coefficient of the attribute weight w_p and w_q will be equal in the CLPP (P6.1). Therefore, all such values of w_p and w_q, corresponding to which $w_p + w_q$ will be optimal, will represent the optimal values of w_p and w_q. Due to the same reason, on solving the CLPP (P6.1) corresponding to the IVIFMADMPr having IVIF attribute weights, the number of obtained optimal weight vectors are more than one, which is mathematically incorrect."

Also, to resolve the flaws, Chen and Han [41] as well as Wang and Chen [211] proposed different methods to solve IVIFMADMPrs having IVIF weights. In these methods, firstly, d_{ij} is transformed into u_{ij} such that $\sum_{i=1}^{m} u_{ij} \neq u$ (constant) $\forall j$. Then, the CLPP (P6.1) is solved to evaluate the crisp attribute weights corresponding to the known IVIF attribute weights. Finally, the obtained crisp attribute weights are used to find the ranking of the alternatives.

Chen and Kuo [47] pointed out that it is inappropriate to use the CLPP (P6.1) due to the following reasons:

(i) If there will exist two distinct attributes G_p and G_q such that $c_p = c_q$ i.e., $\sum_{i=1}^{m} d_{ip} = \sum_{i=1}^{m} d_{iq}$ then the coefficient of the attribute weight w_p and w_q will be equal in the CLPP (P6.1). Therefore, all such values of w_p and w_q, corresponding to which $w_p + w_q$ will be optimal, will represent the optimal values of w_p and w_q. Due to the same reason, on solving the CLPP (P6.1) corresponding to the IVIFMADMPr having IVIF attribute weights, the number of obtained OAWV are more than one, which is mathematically incorrect.

(iii) If $\sum_{j=1}^{n} w_{j1} > 1$ or $\sum_{j=1}^{n}(1 - w_{j3}) < 1$. Then, no feasible solution exist for the CLPP (P6.1) i.e., no OAWV is obtained on solving the CLPP (P6.1).

Also, to resolve the flaws, Chen and Kuo [47] proposed the crisp NLPP (P6.2) to evaluate the crisp attribute weights corresponding to the known IVIF attribute weights.

$$max \left(\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{\substack{k=1\\ i\neq k}}^{m}\left|tanh(w_j \times p_{ij}) - tanh(w_j \times p_{kj})\right| + 0.1 \times \sum_{i=1}^{m}\sum_{j=1}^{n} tanh(w_j \times p_{ij}) - \sum_{j=1}^{n}(\varepsilon_{j1} + \varepsilon_{j2})\right)$$

Subject to

$$\begin{cases} w_j + \varepsilon_{j1} \geq w_{j1}, \\ w_j - \varepsilon_{j2} \leq 1 - w_{j3}, \\ \sum_{j=1}^{n} w_j = 1, \\ 0 \leq w_j \leq 1, \mathrm{j} = 1,2,\dots,\mathrm{n}, \\ 0 \leq \varepsilon_{j1} \leq 1, \mathrm{j} = 1,2,\dots,\mathrm{n}, \\ 0 \leq \varepsilon_{j2} \leq 1, \mathrm{j} = 1,2,\dots,\mathrm{n}. \end{cases} \qquad \text{(P6.2)}$$

where,

(i) $d_{ij} = \frac{a_{ij1} - a_{ij3} + a_{ij2} - a_{ij4}}{2} + 1$ represents the crisp value corresponding to the IVIF rating value $([a_{ij1}, a_{ij2}], [a_{ij3}, a_{ij4}])$ of the i^{th}- alternative over the j^{th}- attribute.

(ii) $d_{kj} = \frac{a_{kj1} - a_{kj3} + a_{kj2} - a_{kj4}}{2} + 1$ represents the crisp value corresponding to the IVIF rating value $([a_{kj1}, a_{kj2}], [a_{kj3}, a_{kj4}])$ of the k^{th}- alternative over the j^{th}- attribute.

(iii) ε_{j1} and ε_{j2} are the unknown variables.

(iv) w_j represents the unknown crisp weight of the j^{th}- attribute corresponding to the known IVIF weight $\widetilde{w}_j = ([w_{j1}, w_{j2}], [w_{j3}, w_{j4}])$ of the j^{th}- attribute.

(v) w_{j1}, w_{j2}, w_{j3} and w_{j4} are the elements of the known IVIF weight $\widetilde{w}_j = ([w_{j1}, w_{j2}], [w_{j3}, w_{j4}])$ of the j^{th}- attribute.

(vi) m represents the number of alternatives.

(vii) n represents the number of attributes.

Chen and Han [42] pointed out the flaws of Chen and Han's [41] method. Also, to resolve the flaws, Chen and Han [42] proposed the crisp NLPP (P6.3) to evaluate the crisp attribute weights corresponding to the known IVIF weights.

$$max\left(\sum_{j=1}^{n}\sum_{i=1}^{m}\left|\frac{(d_{ij})^{w_j} - (d_{ij}+0.1)^{-w_j}}{(d_{ij})^{w_j} + (d_{ij}+0.1)^{-w_j}}\right|\right)$$

Subject to

$$\begin{cases} w_{j1} \le w_j \le 1 - w_{j3}, j = 1,2, \dots, n, \\ \quad \sum_{j=1}^{n} w_j = 1,, \mathrm{j} = 1,2, \dots, \mathrm{n}. \end{cases} \qquad \text{(P6.3)}$$

where,

(i) $d_{ij} = \frac{a_{ij1} + a_{ij2} - a_{ij3} - a_{ij4}}{2}$ represents the crisp value corresponding to IVIF rating value $([a_{ij1}, a_{ij2}], [a_{ij3}, a_{ij4}])$ of the i^{th}- alternative over the j^{th}- attribute.

(ii) w_j represents the unknown crisp weight of the j^{th}- attribute corresponding to the known IVIF weight $\widetilde{w}_j = ([w_{j1}, w_{j2}], [w_{j3}, w_{j4}])$ of the j^{th}- attribute.

(iii) w_{j1}, w_{j2}, w_{j3} and w_{j4} are the elements of the known IVIF weight $\widetilde{w}_j = ([w_{j1}, w_{j2}], [w_{j3}, w_{j4}])$ of the j^{th}- attribute.

(iv) m represents the number of alternatives.

(v) n represents the number of attributes.

Chen et al. [48] pointed out the flaws of Wang and Chen's method [211]. Also, to resolve the flaws, Chen et al. [48] proposed the crisp NLPP (P6.4) to evaluate the crisp attribute weights corresponding to the known IVIF weights.

$$max\left(\sum_{j=1}^{n} tanh\left(w_j \times E_j\right) + \sum_{i=1}^{m}\sum_{j=1}^{n} tanh\left(w_j \times d_{ij}\right) - \sum_{j=1}^{n}\left(\varepsilon_{j1} + \varepsilon_{j2}\right)\right)$$

Subject to

$$\begin{cases} w_{j1} - \varepsilon_{j1} \leq w_j \leq \left(1 - w_{j3}\right) + \varepsilon_{j2}, \\ \sum_{j=1}^{n} w_j = 1, \\ 0 \leq w_j \leq 1, \mathrm{j} = 1,2, \dots, \mathrm{n}, \\ 0 \leq \varepsilon_{j1} \leq 1, \mathrm{j} = 1,2, \dots, \mathrm{n}, \\ 0 \leq \varepsilon_{j2} \leq 1, \mathrm{j} = 1,2, \dots, \mathrm{n}. \end{cases} \qquad \text{(P6.4)}$$

where,

(i) $d_{ij} = \frac{a_{ij1} - a_{ij3} + a_{ij2} - a_{ij4}}{2} + 1$ represents the crisp value corresponding to the IVIF rating value $\left(\left[a_{ij1}, a_{ij2}\right], \left[a_{ij3}, a_{ij4}\right]\right)$ of the i^{th}- alternative over the j^{th}- attribute.

(ii) $E_j = \begin{cases} 0 \; if \; \sum_{j=1}^{n}\left(log_2 m + \sum_{i=1}^{m} k_{ij}\left(log_2 k_{ij}\right)\right) = 0 \\ \frac{log_2 m + \sum_{i=1}^{m} k_{ij}\left(log_2 k_{ij}\right)}{\sum_{j=1}^{n}\left(log_2 m + \sum_{i=1}^{m} k_{ij}\left(log_2 k_{ij}\right)\right)} \quad otherwise \end{cases},$

where $k_{ij} \begin{cases} 0 \quad if \; \sum_{i=1}^{m} d_{ij} = 0 \\ \frac{d_{ij}}{\sum_{i=1}^{m} p_{ij}} \quad otherwise \end{cases}$

(iii) ε_{j1} and ε_{j2} are the unknown variables.

(iv) w_j represents the unknown crisp weight of the j^{th}- attribute corresponding to the known IVIF weight $\widetilde{w}_j = \left(\left[w_{j1}, w_{j2}\right], \left[w_{j3}, w_{j4}\right]\right)$ of the j^{th}- attribute.

(v) w_{j1}, w_{j2}, w_{j3} and w_{j4} are the elements of the known IVIF weight $\widetilde{w}_j = \left(\left[w_{j1}, w_{j2}\right], \left[w_{j3}, w_{j4}\right]\right)$ of the j^{th}- attribute.

(vi) m represents the number of alternatives.

(vii) n represents the number of attributes.

Chen and Han [43] pointed out that it is inappropriate to use the CLPP (P6.1) due to the following reasons:

(i) If there will exist two distinct attributes G_p and G_q such that $c_p = c_q$ i.e., $\sum_{i=1}^{m} d_{ip} = \sum_{i=1}^{m} d_{iq}$ then the coefficient of the attribute weight w_p and w_q will be equal in the CLPP (P6.1). Therefore, all such values of w_p and w_q, corresponding to which $w_p + w_q$ will be optimal, will represent the optimal values of w_p and w_q. Due to the same reason, on solving the CLPP (P6.1) corresponding to the IVIFMADMPr having IVIF attribute weights, the number of obtained OAWV are more than one, which is mathematically incorrect.

(ii) If $\sum_{j=1}^{n} w_{j1} > 1$ or $\sum_{j=1}^{n}\left(1 - w_{j3}\right) < 1$. Then, no feasible solution exist for the CLPP (P6.1) i.e., no OAWV is obtained on solving the CLPP (P6.1).

Also, to resolve the flaws, Chen and Han [43] proposed the crisp NLPP (P6.5) to evaluate the crisp attribute weights corresponding to the known IVIF attribute weights.

$$max\left(\sum_{j=1}^{n}\sum_{i=1}^{m}\left|\frac{(d_{ij}+1)^{w_j}-(d_{ij}+1.1)^{-w_j}}{(d_{ij}+1)^{w_j}+(d_{ij}+1.1)^{-w_j}}\right| - \frac{1}{10}\sum_{j=1}^{n}\left(\varepsilon_{j1} + \varepsilon_{j2}\right)\right)$$

Subject to

$$\begin{cases} w_{j1} - \varepsilon_{j1} \leq w_j \leq 1 - w_{j3} + \varepsilon_{j2}, j = 1,2,\ldots,n, \\ \sum_{j=1}^{n} w_j = 1, \\ 0 \leq w_j \leq 1, \mathrm{j} = 1,2,\ldots,\mathrm{n}. \end{cases} \qquad \text{(P6.5)}$$

where,

(i) $d_{ij} = a_{ij1} + a_{ij2} - 1 + \frac{a_{ij3}+a_{ij4}}{2}$ represents the crisp value corresponding to IVIF rating value $\left(\left[a_{ij1}, a_{ij2}\right], \left[a_{ij3}, a_{ij4}\right]\right)$ of the i^{th}- alternative over the j^{th}- attribute.

(ii) ε_{j1} and ε_{j2} are the unknown variables.

(iii) w_j represents the unknown crisp weight of the j^{th}- attribute corresponding to the known IVIF weight $\widetilde{w}_j = \left(\left[w_{j1}, w_{j2}\right], \left[w_{j3}, w_{j4}\right]\right)$ of the j^{th}- attribute.

(iv) w_{j1}, w_{j2}, w_{j3} and w_{j4} are the elements of the known IVIF weight $\widetilde{w}_j = ([w_{j1}, w_{j2}], [w_{j3}, w_{j4}])$ of the j^{th}- attribute.

(v) m represents the number of alternatives.

(vi) n represents the number of attributes.

6.2 Outcomes of the existing work

On the basis of the results, discussed in Section 6.1, it can be concluded that

(i) No one has pointed out the inappropriateness of Wang and Chen's method [211].

(ii) No one has pointed out the inappropriateness of Chen and Kuo's method [47].

(iii) No one has pointed out the inappropriateness of Chen and Han's method [42].

(iv) No one has pointed out the inappropriateness of Chen et al.'s method [48].

(v) No one has pointed out the inappropriateness of Chen and Han's method [43].

(vi) To apply any of the existing methods [42, 43, 47, 48, 211], there is need to solve the crisp NLPP's for which a lot of computational efforts are required.

6.3 A brief review of Wang and Chen's method

To point out the inappropriateness of Wang and Chen's method, firstly there is need to discuss Wang and Chen's method [211]. Therefore, a brief review of Wang and Chen's method [211] is presented in this section.

Wang and Chen proposed the following method to find the ranking of the alternatives for IVIFMADMPrs having IVIF weights.

Step 1: Check that all the attributes are of same type or not i.e., check that all the attributes are benefit type attribute or cost type attribute.

Case (i): If all the attributes are of same type then go to Step 2.

Case (ii): If some attributes are cost type attributes and the remaining are benefit type attributes. Then, transform the j^{th} cost type attribute into the benefit type attribute by replacing all the elements $\tilde{\alpha}_{ij} = ([a_{ij1}, a_{ij2}], [a_{ij3}, a_{ij4}])$ of the j^{th} column of the IVIFDM $\widetilde{D} = (\tilde{\alpha}_{ij})_{m\times n}$ with

$\tilde{\alpha}_{ij} = ([a_{ij3}, a_{ij4}], [a_{ij1}, a_{ij2}])$ and go to Step 2.

Step 2: Using the expression (6.1), transform each IVIF element $\tilde{\alpha}_{ij} = ([a_{ij1}, a_{ij2}], [a_{ij3}, a_{ij4}])$ of the IVIFDM $\tilde{D} = (\tilde{\alpha}_{ij})_{m\times n} = ([a_{ij1}, a_{ij2}], [a_{ij3}, a_{ij4}])$ into the crisp element d_{ij}

$$d_{ij} = \frac{(a_{ij1}+a_{ij2})(a_{ij1}+a_{ij3})-(a_{ij3}+a_{ij4})(a_{ij2}+a_{ij4})}{2}. \tag{6.1}$$

Step 3: Find $c_j = \sum_{i=1}^{m} d_{ij}$ $\forall\, j = 1,2,\dots,n$ and check that c_j is distinct $\forall\, j = 1,2,\dots,n$ or not.

Case (i): If c_j is distinct $\forall\, j = 1,2,\dots,n$ then go to Step 6.

Case (ii): If c_j is same for two or more values of j then go to Step 4.

Step 4: If c_j is same for s-values of j, where $2 \le s \le n$. Then, find the standard deviation of all the elements for each of these s- columns and go to Step 5. (The standard deviation σ_j of all the elements d_{ij} of the j^{th} column of the crisp decision matrix $D = (d_{ij})_{m\times n}$ is defined as:

$$\sigma_j = \sqrt{\frac{1}{m}\sum_{i=1}^{m}\left(d_{ij} - \frac{\sum_{i=1}^{m} d_{ij}}{m}\right)^2}).$$

Step 5: Using the following procedure, transform the crisp decision matrix $D = (d_{ij})_{m\times n}$ into a new crisp decision matrix $D' = (d'_{ij})_{m\times n}$.

Add a real number $\delta_1 = 0.0001$ in the first element of all those columns of the crisp decision matrix $D = (d_{ij})_{m\times n}$ corresponding to which the obtained standard deviation is minimum. Add a real number $\delta_2 = 0.0002$ in the first element of all those columns of the crisp decision matrix $D = (d_{ij})_{m\times n}$ corresponding to which the obtained standard deviation is next to minimum. Repeat the same procedure for all the s- columns by increasing 0.0001 in the previous value of δ and go to Step 6.

Step 6: Find the optimal weight w_j of the j^{th}- attribute by solving the CLPP (P6.6) (If Case (i) of Step 3 is satisfied) or the CLPP (P6.7) (If Case (ii) of Step 3 is satisfied).

$$max\ \left[\sum_{i=1}^{m}\sum_{j=1}^{n} d_{ij}w_j\right]$$

$$\begin{cases} 0 \le w_{j1} \le w_j \le 1 - w_{j3} \le 1, j = 1,2,\ldots,n\ , \\ \qquad\qquad \sum_{j=1}^{n} w_j = 1. \end{cases} \qquad \text{(P6.6)}$$

where, d_{ij} is the $(i,j)^{th}$element of the crisp decision matrix $D = \left(d_{ij}\right)_{m\times n}$.

$$max\ \left[\sum_{i=1}^{m}\sum_{j=1}^{n} d'_{ij}w_j\right] \qquad \text{(P6.7)}$$

Subject to

Constraints of the CLPP (P6.6)

where, d'_{ij} is the $(i,j)^{th}$ element of the crisp decision matrix $D' = \left(d'_{ij}\right)_{m\times n}$.

Step 7: Using the expression (6.2), find the value of $\tilde{P}_i\ \forall\ i = 1,2,\ldots,m$.

$$\tilde{P}_i = \left(\left[\sum_{j=1}^{n} w_j \times a_{ij1}\,,\sum_{j=1}^{n} w_j \times a_{ij2}\right], \left[\sum_{j=1}^{n} w_j \times a_{ij3}\,,\sum_{j=1}^{n} w_j \times a_{ij4}\right]\right), i = 1,2,\ldots,m \quad (6.2)$$

Step 8: Check that $S(\tilde{P}_p) > S(\tilde{P}_q)$ or $S(\tilde{P}_p) < S(\tilde{P}_q)$ or $S(\tilde{P}_p) = S(\tilde{P}_q)$, where

$S(\tilde{P}_p) = \frac{(a_{pj1}+a_{pj2})(a_{pj1}+a_{pj3})-(a_{pj3}+a_{pj4})(a_{pj2}+a_{pj4})}{2}$ and $S(\tilde{P}_q) = \frac{(a_{qj1}+a_{qj2})(a_{qj1}+a_{qj3})-(a_{qj3}+a_{qj4})(a_{qj2}+a_{qj4})}{2}$.

Case (i): If $S(\tilde{P}_p) > S(\tilde{P}_q)$ then the ranking of the alternatives is $A_p > A_q$.

Case (ii): If $S(\tilde{P}_p) < S(\tilde{P}_q)$ then the ranking of the alternatives is $A_p < A_q$.

Case (iii): If $S(\tilde{P}_p) = S(\tilde{P}_q)$ then go to Step 9.

Step 9: Check that $H(\tilde{P}_p) > H(\tilde{P}_q)$ or $H(\tilde{P}_p) < H(\tilde{P}_q)$ or $H(\tilde{P}_p) = H(\tilde{P}_q)$, where $H(\tilde{P}_p) = \frac{(1-a_{pj1}-a_{pj2})(1-a_{pj1}-a_{pj3})+(1-a_{pj3}-a_{pj4})(1-a_{pj2}-a_{pj4})}{2}$ and $H(\tilde{P}_q) = \frac{(1-a_{qj1}-a_{qj2})(1-a_{qj1}-a_{qj3})+(1-a_{qj3}-a_{qj4})(1-a_{qj2}-a_{qj4})}{2}$.

Case (i): If $H(\tilde{P}_p) > H(\tilde{P}_q)$ then the ranking of the alternatives is $A_p > A_q$.

Case (ii): If $H(\tilde{P}_p) < H(\tilde{P}_q)$ then the ranking of the alternatives is $A_p < A_q$.

Case (iii): If $H(\tilde{P}_p) = H(\tilde{P}_q)$ then $A_p = A_q$.

6.4 Inappropriateness of Wang and Chen's method

It is inappropriate to use Wang and Chen's method [211] due to following reasons

(1) On applying Wang and Chen's method [211] more than one preference orders of the alternatives are obtained, which is inappropriate. To validate this claim an IVIFMADMPr having completely known attribute weights, considered by Wang and Chen [211] to illustrate his proposed method, is solved.

The $(i,j)^{th}$ element of Table 6.1, represented by an IVIFS $\tilde{\alpha}_{ij} = \left(\left[a_{ij1}, a_{ij2}\right], \left[a_{ij3}, a_{ij4}\right]\right)$, represents the rating value of the i^{th}-alternative over the j^{th}-attribute.

Table 6.1: Rating Values

Attributes→ ↓ Alternatives	G_1	G_2
A_1	$([0.40, 0.50], [0.30, 0.40])$	$([0.40, 0.60], [0.20, 0.40])$
A_2	$([0.40, 0.60], [0.20, 0.40])$	$([0.40, 0.50], [0.30, 0.40])$

Also, if it is assumed that $\tilde{w}_1 = ([0, 0.10], [0.20, 0.40])$ and $\tilde{w}_2 = ([0.10, 0.20], [0.10, 0.70])$ represents the IVIF weights.

Then, using Wang and Chen's method [211], the ranking of the alternatives of the considered IVIFMADMPr having IVIF attribute weights can be obtained as follows:

Step 1: Since, all the attributes are of benefit type. So, according to Step 1 of Wang and Chen's method [211], discussed in Section 6.3, there is no need to apply Step 1.

Step 2: According to Step 2 of Wang and Chen's method [211], discussed in Section 6.3, there is need to calculate d_{ij} $\forall$ $i = 1,2; j = 1,2$. These values are shown in Table 6.2.

Table 6.2: Values of d_{ij}

$d_{11} = 0$	$d_{12} = 0$

$d_{21} = 0$	$d_{22} = 0$

Step 3: Using Step 3 of Wang's method [211], discussed in Section 6.3,

$c_1 = \sum_{i=1}^{2} d_{i1} = 0 + 0 = 0$, $c_2 = \sum_{i=1}^{2} d_{i2} = 0 + 0 = 0$.

Since $\sum_{i=1}^{2} d_{i1} = \sum_{i=1}^{2} d_{i2}$, so according to case (ii) of Step 3 of Wang and Chen's method, there is need to apply Step 4 of Wang and Chen's method [211], discussed in Section 6.3.

Step 4: Using Step 4 of Wang's method [211], discussed in Section 6.3, the standard deviations σ_1 and σ_2 of the elements 0, 0 of the first column of the crisp decision matrix D and the elements 0, 0 of the second column of the crisp decision matrix D are $\sigma_1 = 0$, $\sigma_2 = 0$ respectively.

Step 5: Since $\sigma_1 = \sigma_2$, so according to Step 5 of Wang and Chen's IVIFMADM method [211], there is need to obtain a new crisp decision matrix D' by adding $\delta_1 = 0.0001$ in the first elements of both the columns of the crisp decision matrix i.e., the crisp decision matrix $D = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ will be transformed into the crisp decision matrix $D' = \begin{pmatrix} 0 + 0.0001 & 0 + 0.0001 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0.0001 & 0.0001 \\ 0 & 0 \end{pmatrix}$.

Step 6: Using Step 6 of Wang and Chen's method [211], discussed in Section 6.3, there is need to solve the CLPP (P6.8).

$$Max\left(w_1 \times (0.0001) + w_1 \times (0.0) + w_2 \times (0.0001) + w_2 \times (0.0)\right)$$

Subject to

$$\begin{cases} 0.1 \le w_1 \le 0.8, \\ 0.2 \le w_2 \le 0.9, \\ w_1 + w_2 = 1, \\ w_1 \ge 0, w_2 \ge 0. \end{cases} \qquad \text{(P6.8)}$$

It can be easily verified that on solving the CLPP (P6.8) infinite number of OAWV are obtained e.g., $(w_1, w_2) = (0.80, 0.20)$ and $(w_1, w_2) = (0.20, 0.80)$ both are the OAWV.

Step 7: On considering the OAWV, $(w_1, w_2) = (0.80, 0.20)$ and using Step 7 of Wang and

Chen's method [211], discussed in Section 6.3,

$\tilde{P}_1 = ([(0.80)(0.40) + (0.20)(0.40), (0.80)(0.50) + (0.20)(0.60)],[(0.80)(0.30) + (0.20)$
$(0.20), (0.80)(0.40) + (0.20)(0.40)]) = ([0.40,0.52], [0.28,0.40])$,

$\tilde{P}_2 = ([(0.80)(0.40) + (0.20)(0.40), (0.80)(0.60) + (0.20)(0.50)],[(0.80)(0.20) + (0.20)$
$(0.30), (0.80)(0.40) + (0.20)(0.40)]) = ([0.40,0.58], [0.22,0.40])$.

While, on considering the OAWV, $(w_1, w_2) = (0.20,0.80)$ and using Step 7 of Wang and Chen's method [211], discussed in Section 6.3,

$\tilde{P}_1 = ([(0.20)(0.40) + (0.80)(0.40), (0.20)(0.50) + (0.80)(0.60)],[(0.20)(0.30) + (0.80)$
$(0.20), (0.20)(0.40) + (0.80)(0.40)]) = ([0.40,0.58], [0.22,0.40])$,

$\tilde{P}_2 = ([(0.20)(0.40) + (0.80)(0.40), (0.20)(0.60) + (0.80)(0.50)],[(0.20)(0.20) + (0.80)$
$(0.30), (0.20)(0.40) + (0.80)(0.40)]) = ([0.40,0.52], [0.28,0.40])$.

Step 8: Using Step 8, of Wang and Chen's method [211], discussed in Section 6.3,

(i) Corresponding to the OAWV, $(w_1, w_2) = (0.80,0.20)$, $S(\tilde{P}_1) = 0, S(\tilde{P}_2) = 0$. Since, $S(\tilde{P}_1) = S(\tilde{P}_2)$. Therefore, according to Case (iii) of Step 8 of Wang and Chen's method [211], discussed in Section 6.3, there is need to apply Step 9.

(ii) Corresponding to the OAWV, $(w_1, w_2) = (0.20,0.80)$, $S(\tilde{P}_1) = 0, S(\tilde{P}_2) = 0$. Since, $S(\tilde{P}_1) = S(\tilde{P}_2)$. Therefore, according to Case (iii) of Step 9 of Wang and Chen's method [211], discussed in Section 6.3, there is need to apply Step 9.

Step 9: Using Step 9, of Wang and Chen's method [211], discussed in Section 6.3,

(i) Corresponding to the OAWV, $(w_1, w_2) = (0.80,0.20)$, $H(\tilde{P}_1) = 0.2255, H(\tilde{P}_2) = 0.2373$. Since, $H(\tilde{P}_2) > H(\tilde{P}_1)$. So, according to Step 9 of Wang and Chen's method [211], discussed in Section 6.3, $A_2 > A_1$.

(ii) Corresponding to the OAWV, $(w_1, w_2) = (0.20,0.80)$, $H(\tilde{P}_1) = 0.2373, H(\tilde{P}_2) = 0.2255$. Since, $H(\tilde{P}_1) > H(\tilde{P}_2)$. So, according to Step 9 of Wang and Chen's method

[211], discussed in Section 6.3, $A_1 > A_2$.

(2) Wang and Chen's method [211] fails to find the ranking of the alternatives. The following clearly validates this claim.

(i) If the IVIF weights $\widetilde{w}_1 = ([0, 0.10], [0.20, 0.40])$ and $\widetilde{w}_2 = ([0.10, 0.20], [0.10, 0.70])$ are replaced with $\widetilde{w}_1 = ([0.10, 0.10], [0.80, 0.90])$ and $\widetilde{w}_2 = ([0.10, 0.20], [0.50, 0.70])$. Then, no feasible solution of the considered IVIFMADMPr will be obtained as the condition $\sum_{j=1}^{n} w_j^u < 1$ will be satisfied i.e., the ranking of the alternatives of the modified problem cannot be obtained by Wang and Chen's method [211].

(ii) If the IVIF weights $\widetilde{w}_1 = ([0, 0.10], [0.20, 0.40])$ and $\widetilde{w}_2 = ([0.10, 0.20], [0.10, 0.70])$ are replaced with $\widetilde{w}_1 = ([0.60, 0.70], [0.20, 0.30])$ and $\widetilde{w}_2 = ([0.60, 0.70], [0.20, 0.30])$. Then, no feasible solution of the considered IVIFMADMPr will be obtained as the condition $\sum_{j=1}^{n} w_j^l > 1$ will be satisfied i.e., the ranking of the alternatives of the modified problem cannot be obtained by Wang and Chen's method [211].

6.5 A brief review of Chen and Kuo's method

To point out the inappropriateness of Chen and Kuo's method [47], there is need to discuss Chen and Kuo's method [47]. Therefore, a brief review of Chen and Kuo's method [47] is presented in this section.

Chen and Kuo [47] proposed a method for solving IVIFMADMPrs in which the rating value of the i^{th}- alternative over the j^{th}- attribute is represented by IVIFS $\tilde{\alpha}_{ij} = \left(\left[a_{ij1}, a_{ij2}\right], \left[a_{ij3}, a_{ij4}\right]\right)$ as well as the weights of the j^{th}- attribute $\widetilde{w}_j$ is represented by IVIFS $\widetilde{w}_j = \left(\left[w_{j1}, w_{j2}\right], \left[w_{j3}, w_{j4}\right]\right)$.

Step 1: Check that all the attributes are of same type or not i.e., check that all the attributes are benefit type attribute or cost type attribute.

Case (i): If all the attributes are of same type then go to Step 2.

Case (ii): If some attributes are cost type attributes and the remaining are benefit type attributes. Then, convert the j^{th} cost type attribute into the benefit type attribute by replacing all the elements $\tilde{\alpha}_{ij} = \left(\left[a_{ij1}, a_{ij2}\right], \left[a_{ij3}, a_{ij4}\right]\right)$ of the j^{th} column of the IVIFDM $\widetilde{D} = \left(\tilde{\alpha}_{ij}\right)_{m\times n}$ with $\tilde{\alpha}_{ij} = \left(\left[a_{ij3}, a_{ij4}\right], \left[a_{ij1}, a_{ij2}\right]\right)$ and go to Step 2.

Step 2: Using the expression (6.3), transform each IVIF element $\tilde{\alpha}_{ij} = \left(\left[a_{ij1}, a_{ij2}\right], \left[a_{ij3}, a_{ij4}\right]\right)$ of the IVIFDM $\widetilde{D} = \left(\tilde{\alpha}_{ij}\right)_{m\times n} = \left(\left[a_{ij1}, a_{ij2}\right], \left[a_{ij3}, a_{ij4}\right]\right)$ into the crisp element d_{ij}

$$d_{ij} = \left(\frac{a_{ij1} - a_{ij3} + a_{ij2} - a_{ij4}}{2} + 1\right). \tag{6.3}$$

Step 3: Using the expression (6.4), find the value of $c_j \; \forall \; j = 1,2,\ldots,n$.

$$c_j = \sum_{i=1}^{m} d_{ij}\, , j = 1,2,\ldots,n \tag{6.4}$$

Step 4: Find the OAWV, $(w_1, w_2, \ldots, w_n)$ by solving the CLPP (P6.9).

$$Maximize \left(\sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{\substack{k=1 \\ i\neq k}}^{m} \left|tanh\left(w_j \times d_{ij}\right) - tanh\left(w_j \times d_{kj}\right)\right| + 0.1 \times \sum_{i=1}^{m} \sum_{j=1}^{n} tanh\left(w_j \times d_{ij}\right) - \sum_{j=1}^{n}\left(\varepsilon_{j1} + \varepsilon_{j2}\right)\right)$$

Subject to

$$\left\{\begin{array}{l} 1 - w_{j3} + \varepsilon_{j2} \geq w_j \geq w_{j1} - \varepsilon_{j1}, \\ \sum_{j=1}^{n} w_j = 1, \\ 0 \leq w_j \leq 1, \\ 0 \leq \varepsilon_{j1} \leq 1, \\ 0 \leq \varepsilon_{j2} \leq 1, \\ 1 \leq j \leq n. \end{array}\right. \tag{P6.9}$$

Step 5: Using the expression (6.5), find the value of $\tilde{P}_i \; \forall \; i = 1,2,\ldots,m$.

$$\tilde{P}_i = \left(\left[\frac{\sum_{j=1}^{n} a_{ij1} \times w_j}{\sum_{j=1}^{n} w_j}, \frac{\sum_{j=1}^{n} a_{ij2} \times w_j}{\sum_{j=1}^{n} w_j}\right], \left[\frac{\sum_{j=1}^{n} a_{ij3} \times w_j}{\sum_{j=1}^{n} w_j}, \frac{\sum_{j=1}^{n} a_{ij4} \times w_j}{\sum_{j=1}^{n} w_j}\right]\right), i = 1,2,\ldots,m \tag{6.5}$$

Step 6: Check that $S\left(\tilde{P}_p\right) > S\left(\tilde{P}_q\right)$ or $S\left(\tilde{P}_p\right) < S\left(\tilde{P}_q\right)$ or $S\left(\tilde{P}_p\right) = S\left(\tilde{P}_q\right)$, where $S\left(\tilde{P}_p\right) =$

$$\frac{\frac{\sum_{j=1}^{n} a_{pj1}\times w_j}{\sum_{j=1}^{n} w_j}+\frac{\sum_{j=1}^{n} a_{pj2}\times w_j}{\sum_{j=1}^{n} w_j}-\frac{\sum_{j=1}^{n} a_{pj3}\times w_j}{\sum_{j=1}^{n} w_j}-\frac{\sum_{j=1}^{n} a_{pj4}\times w_j}{\sum_{j=1}^{n} w_j}}{2} \text{ and}$$

$$S(\tilde{P}_q)=\frac{\frac{\sum_{j=1}^{n} a_{qj1}\times w_j}{\sum_{j=1}^{n} w_j}+\frac{\sum_{j=1}^{n} a_{qj2}\times w_j}{\sum_{j=1}^{n} w_j}-\frac{\sum_{j=1}^{n} a_{qj3}\times w_j}{\sum_{j=1}^{n} w_j}-\frac{\sum_{j=1}^{n} a_{qj4}\times w_j}{\sum_{j=1}^{n} w_j}}{2}.$$

Case (i): If $S(\tilde{P}_p) > S(\tilde{P}_q)$ then the ranking of the alternatives is $A_p > A_q$.

Case (ii): If $S(\tilde{P}_p) < S(\tilde{P}_q)$ then the ranking of the alternatives is $A_p < A_q$.

Case (iii): If $S(\tilde{P}_p) = S(\tilde{P}_q)$ then go to Step 7.

Step 7: Check that $H(\tilde{P}_p) > H(\tilde{P}_q)$ or $H(\tilde{P}_p) < H(\tilde{P}_q)$ or $H(\tilde{P}_p) = H(\tilde{P}_q)$, where $H(\tilde{P}_p) =$

$$\frac{\frac{\sum_{j=1}^{n} a_{pj1}\times w_j}{\sum_{j=1}^{n} w_j}+\frac{\sum_{j=1}^{n} a_{pj2}\times w_j}{\sum_{j=1}^{n} w_j}+\frac{\sum_{j=1}^{n} a_{pj3}\times w_j}{\sum_{j=1}^{n} w_j}+\frac{\sum_{j=1}^{n} a_{pj4}\times w_j}{\sum_{j=1}^{n} w_j}}{2} \quad \text{and} \quad H(\tilde{P}_q)=$$

$$\frac{\frac{\sum_{j=1}^{n} a_{qj1}\times w_j}{\sum_{j=1}^{n} w_j}+\frac{\sum_{j=1}^{n} a_{qj2}\times w_j}{\sum_{j=1}^{n} w_j}+\frac{\sum_{j=1}^{n} a_{qj3}\times w_j}{\sum_{j=1}^{n} w_j}+\frac{\sum_{j=1}^{n} a_{qj4}\times w_j}{\sum_{j=1}^{n} w_j}}{2}.$$

Case (i): If $H(\tilde{P}_p) > H(\tilde{P}_q)$ then the ranking of the alternatives is $A_p > A_q$.

Case (ii): If $H(\tilde{P}_p) < H(\tilde{P}_q)$ then the ranking of the alternatives is $A_p < A_q$.

Case (iii): If $H(\tilde{P}_p) = H(\tilde{P}_q)$ then go to Step 8.

Step 8: Check that $T(\tilde{P}_p) > T(\tilde{P}_q)$ or $T(\tilde{P}_p) < T(\tilde{P}_q)$ or $T(\tilde{P}_p) = T(\tilde{P}_q)$, where

$$T(\tilde{P}_p)=\frac{\sum_{j=1}^{n} a_{pj2}\times w_j}{\sum_{j=1}^{n} w_j}+\frac{\sum_{j=1}^{n} a_{pj3}\times w_j}{\sum_{j=1}^{n} w_j}-\frac{\sum_{j=1}^{n} a_{pj1}\times w_j}{\sum_{j=1}^{n} w_j}-\frac{\sum_{j=1}^{n} a_{pj4}\times w_j}{\sum_{j=1}^{n} w_j} \quad \text{and} \quad T(\tilde{P}_q)=\frac{\sum_{j=1}^{n} a_{qj2}\times w_j}{\sum_{j=1}^{n} w_j}+$$

$$\frac{\sum_{j=1}^{n} a_{qj3}\times w_j}{\sum_{j=1}^{n} w_j}-\frac{\sum_{j=1}^{n} a_{qj1}\times w_j}{\sum_{j=1}^{n} w_j}-\frac{\sum_{j=1}^{n} a_{qj4}\times w_j}{\sum_{j=1}^{n} w_j}.$$

Case (i): If $T(\tilde{P}_p) > T(\tilde{P}_q)$ then the ranking of the alternatives is $A_p < A_q$.

Case (ii): If $T(\tilde{P}_p) < T(\tilde{P}_q)$ then the ranking of the alternatives is $A_p > A_q$.

Case (iii): If $T(\tilde{P}_p) = T(\tilde{P}_q)$ then go to Step 9.

Step 9: Check that $G(\tilde{P}_p) > G(\tilde{P}_q)$ or $G(\tilde{P}_p) < G(\tilde{P}_q)$ or $G(\tilde{P}_p) = G(\tilde{P}_q)$, where $G(\tilde{P}_p) =$

$$\frac{\sum_{j=1}^{n} a_{pj2}\times w_j}{\sum_{j=1}^{n} w_j}+\frac{\sum_{j=1}^{n} a_{pj4}\times w_j}{\sum_{j=1}^{n} w_j}-\frac{\sum_{j=1}^{n} a_{pj1}\times w_j}{\sum_{j=1}^{n} w_j}-\frac{\sum_{j=1}^{n} a_{pj3}\times w_j}{\sum_{j=1}^{n} w_j} \text{ and } G(\tilde{P}_q)=\frac{\sum_{j=1}^{n} a_{qj2}\times w_j}{\sum_{j=1}^{n} w_j}$$

$$+\frac{\sum_{j=1}^{n} a_{qj4}\times w_j}{\sum_{j=1}^{n} w_j}-\frac{\sum_{j=1}^{n} a_{qj1}\times w_j}{\sum_{j=1}^{n} w_j}-\frac{\sum_{j=1}^{n} a_{qj3}\times w_j}{\sum_{j=1}^{n} w_j}.$$

Case (i): If $G(\tilde{P}_p) > G(\tilde{P}_q)$ then the ranking of the alternatives is $A_p < A_q$.

Case (ii): If $G(\tilde{P}_p) < G(\tilde{P}_q)$ then the ranking of the alternatives is $A_p > A_q$.

Case (iii): If $G(\tilde{P}_p) = G(\tilde{P}_q)$ then the ranking of the alternatives is $A_p = A_q$.

6.6 Inappropriateness of Chen and Kuo's method

On applying Chen and Kuo's method [47] more than one preference orders of the alternatives are obtained, which is inappropriate. To validate this claim an IVIFMADMPr having completely known attribute weights, considered by Chen and Kuo [47] to illustrate their proposed method [47], is solved with some modifications.

If it is assumed that the $(i,j)^{th}$ element of Table 6.3, represented by an IVIFS $\tilde{\alpha}_{ij} = ([a_{ij1}, a_{ij2}], [a_{ij3}, a_{ij4}])$, represents the rating value of the i^{th}-alternative over the j^{th}-benefit attribute. Also, if it is assumed that $\tilde{w}_1 = ([0.15, 0.20], [0.20, 0.40])$ and $\tilde{w}_2 = ([0.15, 0.20], [0.20, 0.70])$ represents the IVIF weights.

Table 6.3: Rating Values

Attributes→ ↓ Alternatives	G_1	G_2
A_1	$([0.10,0.40],[0.20,0.30])$	$([0.01,0.04],[0.02,0.03])$
A_2	$([0.02,0.04],[0.01,0.05])$	$([0.20,0.50],[0.30,0.40])$

Then, using Chen and Kuo's method [47], the ranking of the alternatives of the modified IVIFMADMPr having IVIF attribute weights can be obtained as follows:

Step 1: Since, all the attributes are of benefit type. So, according to Step 1 of Chen and Kuo's method [47], discussed in Section 6.5, there is no need to apply Step 1.

Step 2: According to Step 2 of Chen and Kuo's method [47], discussed in Section 6.5, there is need to calculate $d_{ij}\ \forall\ i = 1,2; j = 1,2$. These values are shown in Table 6.4.

Table 6.4: Values of d_{ij}

$d_{11} = 1$	$d_{12} = 1$
$d_{21} = 1$	$d_{22} = 1$

Step 3: Using Step 3 of Chen and Kuo's method [47], discussed in Section 6.5,

$c_1 = \sum_{i=1}^{2} d_{i1} = 1 + 1 = 2,$

$c_2 = \sum_{i=1}^{2} d_{i2} = 1 + 1 = 2.$

Step 4: Using Step 4 of Chen and Kuo's method [47], discussed in Section 6.5, there is need to solve the CLPP (P6.10).

$$Maximize\ \left(0.1 \times \left(2tanh(w_1) + 2tanh(w_2)\right) - \left(\varepsilon_{j1} + \varepsilon_{j2} + \varepsilon_{j3} + \varepsilon_{j4}\right)\right)$$

Subject to

$$\begin{cases} 0.80 + \varepsilon_{j2} \geq w_1 \geq 0.15 - \varepsilon_{j1}, \\ 0.80 + \varepsilon_{j3} \geq w_2 \geq 0.15 - \varepsilon_{j3}, \\ w_1 + w_2 = 1, \\ 0 \leq w_1 \leq 1, \\ 0 \leq w_2 \leq 1, \\ 0 \leq \varepsilon_{j1} \leq 1, \\ 0 \leq \varepsilon_{j2} \leq 1, \\ 0 \leq \varepsilon_{j3} \leq 1, \\ 0 \leq \varepsilon_{j4} \leq 1. \end{cases} \quad \text{(P6.10)}$$

It can be easily verified that on solving the CLPP (P6.10) infinite number of OAWV are obtained e.g., $(w_1, w_2) = (0.20, 0.80)$ and $(w_1, w_2) = (0.80, 0.20)$ both are the OAWV.

Step 5: On considering the OAWV, $(w_1, w_2) = (0.20, 0.80)$ and using Step 5 of Chen and Kuo's method [47], discussed in Section 6.5,

$$\tilde{P}_1 = ([(0.20)(0.10) + (0.80)(0.01), (0.20)(0.20) + (0.80)(0.02)], [(0.20)(0.30) + (0.80)(0.03), (0.20)(0.40) + (0.80)(0.04)]) = ([0.028, 0.112], [0.056, 0.084]),$$

$\tilde{P}_2 = ([(0.20)(0.02) + (0.80)(0.20), (0.20)(0.04) + (0.80)(0.5)], [(0.20)(0.01) + (0.80)(0.30), (0.20)(0.05) + (0.80)(0.40)]) = ([0.164, 0.408], [0.242, 0.33])$.

While, on considering the OAWV, $(w_1, w_2) = (0.80, 0.20)$ and using Step 5 of Chen and Kuo's method [47], discussed in Section 6.5,

$\tilde{P}_1 = ([(0.80)(0.10) + (0.20)(0.01), (0.80)(0.20) + (0.20)(0.02)], [(0.80)(0.30) + (0.20)(0.03), (0.80)(0.40) + (0.20)(0.04)]) = ([0.082, 0.328], [0.164, 0.246])$,

$\tilde{P}_2 = ([(0.80)(0.02) + (0.20)(0.20), (0.80)(0.04) + (0.20)(0.5)], [(0.80)(0.01) + (0.20)(0.30), (0.80)(0.05) + (0.20)(0.40)]) = ([0.056, 0.132], [0.068, 0.120])$.

Step 6: Using Step 6, of Chen and Kuo's method [47], discussed in Section 6.5,

(i) Corresponding to the OAWV, $(w_1, w_2) = (0.20, 0.80)$, $S(\tilde{P}_1) = 0, S(\tilde{P}_2) = 0$. Since, $S(\tilde{P}_1) = S(\tilde{P}_2)$. So, according to Step 6 of Chen and Kuo's method [47], discussed in Section 6.5, there is need to apply Step 7.

(ii) Corresponding to the OAWV, $(w_1, w_2) = (0.80, 0.20)$, $S(\tilde{P}_1) = 0, S(\tilde{P}_2) = 0$. Since, $S(\tilde{P}_1) = S(\tilde{P}_2)$. So, according to Step 6 of Chen and Kuo's method [47], discussed in Section 6.5, there is need to apply Step 7.

Step 7: Using Step 7 of Chen and Kuo's method [47], discussed in Section 6.5,

(i) Corresponding to the OAWV, $(w_1, w_2) = (0.20, 0.80)$, $H(\tilde{P}_1) = 0.14, H(\tilde{P}_2) = 0.572$. Since, $H(\tilde{P}_2) > H(\tilde{P}_1)$. So, according to Step 6 of Chen and Kuo's method [47], discussed in Section 6.5, $A_2 > A_1$.

(ii) Corresponding to the OAWV, $(w_1, w_2) = (0.80, 0.20)$, $H(\tilde{P}_1) = 0.41, H(\tilde{P}_2) = 0.188$. Since, $H(\tilde{P}_1) > H(\tilde{P}_2)$. So, according to Step 6 of Chen and Kuo's method [47], discussed in Section 6.5, $A_1 > A_2$.

It is obvious that on applying Chen and Kuo's method [47] two different ranking of the alternatives are obtained for the same IVIFMADMPr having IVIF attribute weights, which is mathematically incorrect. Hence, it is inappropriate to use Chen and Kuo's method [47] to solve IVIFMADMPrs having IVIF attribute weights.

6.7 A brief review of Chen and Han's method

To point out the inappropriateness of Chen and Han's method [42], there is need to discuss Chen and Han's method [42]. Therefore, a brief review of Chen and Han's method [42] is presented in this section.

Chen and Han [42] proposed a method for solving IVIFMADMPrs in which the rating value of the i^{th}- alternative over the j^{th}- attribute is represented by IVIFS $\tilde{\alpha}_{ij} = ([a_{ij1}, a_{ij2}], [a_{ij3}, a_{ij4}])$ as well as the weights of the j^{th}- attribute $\widetilde{w}_j$ is represented by IVIFS $\widetilde{w}_j = ([w_{j1}, w_{j2}], [w_{j3}, w_{j4}])$.

Step 1: Check that all the attributes are of same type or not i.e., check that all the attributes are benefit type attribute or cost type attribute.

Case (i): If all the attributes are of same type then go to Step 2.

Case (ii): If some attributes are cost type attributes and the remaining are benefit type attributes. Then, convert the j^{th} cost type attribute into the benefit type attribute by replacing all the elements $\tilde{\alpha}_{ij} = ([a_{ij1}, a_{ij2}], [a_{ij3}, a_{ij4}])$ of the j^{th} column of the IVIFDM $\widetilde{D} = (\tilde{\alpha}_{ij})_{m\times n}$ with $\tilde{\alpha}_{ij} = ([a_{ij3}, a_{ij4}], [a_{ij1}, a_{ij2}])$ and go to Step 2.

Step 2: Using the expression (6.6), transform each IVIF element $\tilde{\alpha}_{ij} = ([a_{ij1}, a_{ij2}], [a_{ij3}, a_{ij4}])$ of the IVIFDM $\widetilde{D} = (\tilde{\alpha}_{ij})_{m\times n} = ([a_{ij1}, a_{ij2}], [a_{ij3}, a_{ij4}])$ into the crisp element d_{ij}.

$$d_{ij} = \frac{a_{ij1} + a_{ij2} - a_{ij3} - a_{ij4}}{2} \tag{6.6}$$

Step 3: Find $c_j = \sum_{i=1}^{m} d_{ij} \; \forall \, j = 1,2,\dots,n$ and check that c_j is distinct $\forall \, j = 1,2,\dots,n$ or not.

Case (i): If c_j is distinct $\forall \, j = 1,2,\dots,n$ then go to Step 6.

Case (ii): If c_j is same for two or more values of j then go to Step 4.

Step 4: If c_j is same for two or more values of j. Then, replace the $(i,j)^{th}$ element $\tilde{\alpha}_{ij} = \left(\left[a_{ij1}, a_{ij2}\right], \left[a_{ij3}, a_{ij4}\right]\right)$ of the IVIFDM, $\widetilde{D} = \left(\tilde{\alpha}_{ij}\right)_{m\times n}$ with $\tilde{\alpha}'_{ij} = \left(\left[1-\left(1-a_{ij1}\right)^{\lambda}, 1-\left(1-a_{ij2}\right)^{\lambda}\right], \left[{a_{ij3}}^{\lambda}, {a_{ij4}}^{\lambda}\right]\right)$, to obtain a new IVIFDM $\widetilde{D}' = \left(\tilde{\alpha}'_{ij}\right)_{m\times n}$ and hence a new crisp decision matrix $D' = \left(d'_{ij}\right)_{m\times n} = \left(S\left(\tilde{\alpha}'_{ij}\right)\right)_{m\times n}$ and check that for $\lambda = 2$, Case (i) of Step 3 is satisfying or not, where $d'_{ij} = c'_j$.

Case (i): If Case (i) of Step 3 is satisfied then go to Step 5.

Case (ii): If Case (i) of Step 3 is not satisfied then repeat the same procedure with $\lambda = 3$ and so on until Case (i) of Step 3 is not satisfied. When it is satisfied then go to Step 5.

Step 5: Find the OAWV, $(w_1, w_2, \dots, w_n)$ by solving the CLPP (P6.11). (If Case (i) of Step 3 is satisfied) or the CLPP (P6.12) (If Case (ii) of Step 3 is satisfied).

$max\left[\sum_{j=1}^{n} c_j w_j\right]$

Subject to

$0 \le w_{j1} \le w_j \le 1 - w_{j3} \le 1, j = 1,2,\dots,n\,,$ (P6.11)

$\sum_{j=1}^{n} w_j = 1.$

where, d_{ij} is the $(i,j)^{th}$element of the crisp decision matrix $D = \left(d_{ij}\right)_{m\times n}$.

$max\left[\sum_{i=1}^{m}\sum_{j=1}^{n} c'_j w_j\right]$ (P6.12)

Subject to

Constraints of the CLPP (P6.11)

where, d'_{ij} is the $(i,j)^{th}$element of the crisp decision matrix $D' = \left(d'_{ij}\right)_{m\times n}$.

Step 6: Using the expression (6.7), find the value of $\tilde{P}_i \; \forall \; i = 1,2,\dots,m$.

$$\tilde{P}_i = \left(\left[1-\prod_{j=1}^{n}\left(1-a_{ij1}\right)^{w_j}, 1-\prod_{j=1}^{n}\left(1-a_{ij2}\right)^{w_j}\right], \left[\prod_{j=1}^{n}\left(a_{ij3}\right)^{w_j}, \prod_{j=1}^{n}\left(a_{ij4}\right)^{w_j}\right]\right), i = 1,2,\dots,m \qquad (6.7)$$

Step 7: Check that $S\left(\tilde{P}_p\right) > S\left(\tilde{P}_q\right)$ or $S\left(\tilde{P}_p\right) < S\left(\tilde{P}_q\right)$ or $S\left(\tilde{P}_p\right) = S\left(\tilde{P}_q\right)$, where

$S(\tilde{P}_p) = \frac{1-\prod_{j=1}^{n}(1-a_{pj1})^{w_j}+1-\prod_{j=1}^{n}(1-a_{pj2})^{w_j}-\prod_{j=1}^{n}(a_{pj3})^{w_j}-\prod_{j=1}^{n}(a_{pj3})^{w_j}}{2}$ and

$S(\tilde{P}_q) = \frac{1-\prod_{j=1}^{n}(1-a_{qj1})^{w_j}+1-\prod_{j=1}^{n}(1-a_{qj2})^{w_j}-\prod_{j=1}^{n}(a_{qj3})^{w_j}-\prod_{j=1}^{n}(a_{qj3})^{w_j}}{2}$.

Case (i): If $S(\tilde{P}_p) > S(\tilde{P}_q)$ then the ranking of the alternatives is $A_p > A_q$.

Case (ii): If $S(\tilde{P}_p) < S(\tilde{P}_q)$ then the ranking of the alternatives is $A_p < A_q$.

Case (iii): If $S(\tilde{P}_p) = S(\tilde{P}_q)$ then go to Step 8.

Step 7: Check that $H(\tilde{P}_p) > H(\tilde{P}_q)$ or $H(\tilde{P}_p) < H(\tilde{P}_q)$ or $H(\tilde{P}_p) = H(\tilde{P}_q)$, where $H(\tilde{P}_p) = \frac{1-\prod_{j=1}^{n}(1-a_{pj1})^{w_j}+1-\prod_{j=1}^{n}(1-a_{pj2})^{w_j}-\prod_{j=1}^{n}(a_{pj3})^{w_j}-\prod_{j=1}^{n}(a_{pj3})^{w_j}}{2}$ and

$H(\tilde{P}_q) = \frac{1-\prod_{j=1}^{n}(1-a_{qj1})^{w_j}+1-\prod_{j=1}^{n}(1-a_{qj2})^{w_j}-\prod_{j=1}^{n}(a_{qj3})^{w_j}-\prod_{j=1}^{n}(a_{qj3})^{w_j}}{2}$.

Case (i): If $H(\tilde{P}_p) > H(\tilde{P}_q)$ then the ranking of the alternatives is $A_p > A_q$.

Case (ii): If $H(\tilde{P}_p) < H(\tilde{P}_q)$ then the ranking of the alternatives is $A_p < A_q$.

Case (iii): If $H(\tilde{P}_p) = H(\tilde{P}_q)$ then $A_p = A_q$.

6.8 Inappropriateness of Chen and Han's method

It is inappropriate to use Chen and Han's method [42] due to following reasons

(1) On applying Chen and Han's method [42] more than one preference orders of the alternatives are obtained, which is inappropriate. To validate this claim an IVIFMADMPr having completely known IVIF attribute weights, considered by Chen and Han [42] to illustrate their proposed method [42], is solved with some modifications.

If it is assumed that the $(i,j)^{th}$ element of Table 6.5, represented by an IVIFS $\tilde{\alpha}_{ij} = ([a_{ij1}, a_{ij2}], [a_{ij3}, a_{ij4}])$, represents the rating value of the i^{th}-alternative over the j^{th}-attribute.

Table 6.5: Rating Values

Attributes→ ↓ Alternatives	G_1	G_2
A_1	$([0.10,0.20],[0.30,0.40])$	$([0.30,0.40],[0.40,0.50])$
A_2	$([0.30,0.40],[0.40,0.50])$	$([0.10,0.20],[0.30,0.40])$

Also, if it is assumed that $\widetilde{w}_1 = ([0, 0.1], [0.2, 0.4])$ and $\widetilde{w}_2 = ([0.1, 0.2], [0.1, 0.7])$ represents the IVIF weights.

Then, using Chen and Han's method [42], the ranking of the alternatives of the IVIFMADMPr having completely known IVIF attribute weights can be obtained as follows:

Step 1: Since, all the attributes are of benefit type. So, according to Step 1 of Chen and Han's method [42], discussed in Section 6.7, there is no need to apply Step 1.

Step 2: According to Step 2 of Chen and Han's method [42], discussed in Section 6.7, there is need to calculate $d_{ij}\ \forall\ i = 1,2; j = 1,2$. These values are shown in Table 6.6.

Table 6.6: Values of d_{ij}

$d_{11} = -0.20$	$d_{12} = -0.10$
$d_{21} = -0.10$	$d_{22} = -0.20$

and calculate

$c_1 = \sum_{i=1}^{2} d_{i1} = -0.20 - 0.10 = -0.30,$

$c_2 = \sum_{i=1}^{2} d_{i2} = -0.10 - 0.20 = -0.30.$

Step 3: Using Step 3 of Chen and Han's method [42], discussed in Section 6.7,

Since $c_1 = c_2$, so according to Step 3 of Chen and Han's IVIFMADM method [42], replacing the elements $\tilde{\alpha}_{ij} = \left([a_{ij1}, a_{ij2}], [a_{ij3}, a_{ij4}]\right)$ of the IVIFDM $\widetilde{D} = \left(\tilde{\alpha}_{ij}\right)_{2\times 2}$

$$= \begin{pmatrix} ([a_{111}, a_{112}], [a_{113}, a_{114}]) & ([a_{121}, a_{122}], [a_{123}, a_{124}]) \\ ([a_{211}, a_{212}], [a_{213}, a_{214}]) & ([a_{221}, a_{222}], [a_{223}, a_{224}]) \end{pmatrix}$$

$= \begin{pmatrix} ([0.10,0.20], [0.30,0.40]) & ([0.30,0.40], [0.40,0.50]) \\ ([0.30,0.40], [0.40,0.50]) & ([0.10,0.20], [0.30,0.40]) \end{pmatrix}$ with $\tilde{\alpha}'_{ij} = \left(\left[1-\left(1-a_{ij1}\right)^{\lambda}, 1-\left(1-a_{ij2}\right)^{\lambda}\right], \left[{a_{ij3}}^{\lambda}, {a_{ij4}}^{\lambda}\right]\right)$, the obtained new IVIFDM is $\widetilde{D}' = \left(\tilde{\alpha}'_{ij}\right)_{2\times 2}$

$$= \begin{pmatrix} \begin{pmatrix} [1-(1-0.10)^{\lambda}, 1-(1-0.20)^{\lambda}], \\ [0.30^{\lambda}, 0.40^{\lambda}] \end{pmatrix} & \begin{pmatrix} [1-(1-0.30)^{\lambda}, 1-(1-0.40)^{\lambda}], \\ [0.40^{\lambda}, 0.50^{\lambda}] \end{pmatrix} \\ \begin{pmatrix} [1-(1-0.30)^{\lambda}, 1-(1-0.40)^{\lambda}], \\ [0.40^{\lambda}, 0.50^{\lambda}] \end{pmatrix} & \begin{pmatrix} [1-(1-0.10)^{\lambda}, 1-(1-0.20)^{\lambda}], \\ [0.30^{\lambda}, 0.40^{\lambda}] \end{pmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} \left([1-0.90^{\lambda}, 1-0.80^{\lambda}], [0.30^{\lambda}, 0.40^{\lambda}]\right) & \left([1-0.70^{\lambda}, 1-0.60^{\lambda}], [0.40^{\lambda}, 0.50^{\lambda}]\right) \\ \left([1-0.70^{\lambda}, 1-0.60^{\lambda}], [0.40^{\lambda}, 0.50^{\lambda}]\right) & \left([1-0.90^{\lambda}, 1-0.80^{\lambda}], [0.30^{\lambda}, 0.40^{\lambda}]\right) \end{pmatrix}.$$

and hence, the obtained new crisp decision matrix $D' = \left(d'_{ij}\right)_{2\times 2} =$

$$\begin{pmatrix} \frac{1-0.90^{\lambda}-0.30^{\lambda}+1-0.80^{\lambda}-0.40^{\lambda}}{2} & \frac{1-0.70^{\lambda}-0.40^{\lambda}+1-0.60^{\lambda}-0.50^{\lambda}}{2} \\ \frac{1-0.70^{\lambda}-0.40^{\lambda}+1-0.60^{\lambda}-0.50^{\lambda}}{2} & \frac{1-0.90^{\lambda}-0.30^{\lambda}+1-0.80^{\lambda}-0.40^{\lambda}}{2} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{2-0.90^{\lambda}-0.30^{\lambda}-0.80^{\lambda}-0.40^{\lambda}}{2} & \frac{2-0.70^{\lambda}-0.40^{\lambda}-0.60^{\lambda}-0.50^{\lambda}}{2} \\ \frac{2-0.70^{\lambda}-0.40^{\lambda}-0.60^{\lambda}-0.50^{\lambda}}{2} & \frac{2-0.90^{\lambda}-0.30^{\lambda}-0.80^{\lambda}-0.40^{\lambda}}{2} \end{pmatrix}.$$

It is obvious that $d'_{11} + d'_{21} = d'_{12} + d'_{22} = \frac{4-0.90^{\lambda}-0.30^{\lambda}-0.80^{\lambda}-0.40^{\lambda}-0.70^{\lambda}-0.40^{\lambda}-0.60^{\lambda}-0.50^{\lambda}}{2}$ for all values of λ.

Therefore, it is not possible to apply further steps of Chen and Han's IVIFMADM method [42] as these steps can be applied only if there exist a λ such that $\sum_{i=1}^{m} d'_{ij}$ is different for all values of j. While, for the considered IVIFMADMPr, this condition will never be satisfied. Hence, Chen and Han's IVIFMADM method [42] fails to solve the considered IVIFMADMPr.

(2) Chen and Han's method [42] fails to find the ranking of the alternatives. The following clearly validates this claim.

(i) If the set $\widetilde{w}_1 = ([0, 0.10], [0.20, 0.40])$ and $\widetilde{w}_2 = ([0.10, 0.20], [0.10, 0.70])$ is replaced with $\tilde{w}_1 = ([0, 0.10], [0.80, 0.90])$ and $\tilde{w}_2 = ([0.10, 0.20], [0.50, 0.70])$. Then, no feasible solution of above IVIFMADMPr will be obtained as the condition $\sum_{j=1}^{n} w_j^u < 1$ is satisfying i.e., the ranking of the alternatives of the modified problem cannot be obtained by Chen and Han's method [42].

(ii) If the set $\tilde{w}_1 = ([0, 0.10], [0.20, 0.40])$ and $\tilde{w}_2 = ([0.10, 0.20], [0.10, 0.70])$ is replaced with $\tilde{w}_1 = ([0.60, 0.70], [0.20, 0.30])$ and $\tilde{w}_2 = ([0.60, 0.70], [0.20, 0.30])$. Then, no feasible solution of above IVIFMADMPr will be obtained as the condition $\sum_{j=1}^{n} w_j^l > 1$ is satisfying i.e., the ranking of the alternatives of the modified problem cannot be obtained by Chen and Han's method [42].

(3) The ranking of the alternatives, obtained by Chen and Han's method [42], is not appropriate. The following example clearly validates this claim.

If it is assumed that the $(i, j)^{th}$ element, represented by an IVIFS, of Table 6.7 represents the rating value of the i^{th}-alternative over the j^{th}-attribute. Also, if it is assumed that $\tilde{w}_1 = ([0, 0.1], [0, 0.4])$ and $\tilde{w}_2 = ([0, 0.2], [0, 0.7])$ represents the IVIF weights.

Table 6.7: Rating Values

Attributes→ ↓ Alternatives	G_1	G_2
A_1	$([0.10,0.30], [0.20,0.40])$	$([0.10,0.50], [0.20,0.40])$
A_2	$([0.10, 0.30], [0.10, 0.50])$	$([0.10,0.50], [0.20,0.40])$

Then, it is obvious that the ranking of the alternatives A_1 and A_2 can never be $A_1 = A_2$ as the rating values of both the alternatives corresponding to the attribute G_2 are equal. Whereas, the rating values of both the alternatives corresponding to first attribute G_1 are not equal. While, the following clearly indicates that on applying Chen and Han's method [42], the obtained ranking of the alternatives is $A_1 = A_2$, which is inappropriate.

Using Chen and Han's method [42] the ranking of the alternatives A_1 and A_2 can be obtained as follows:

Step 1: Since, all the attributes are of benefit type. So, according to Step 1 of Chen and Han's method [42], discussed in Section 6.7, there is no need to apply Step 1.

Step 2: According to Step 2 of Chen and Han's method [42], discussed in Section 6.7, there is need to calculate the values of d_{ij} $\forall$ $i = 1,2; j = 1,2$. These values are shown in Table 6.8.

Table 6.8: Values of d_{ij}

$d_{11} = -0.10$	$d_{12} = 0$
$d_{21} = -0.10$	$d_{22} = 0$

Step 3: Using Step 3 of Chen and Han's method [42],

$c_1 = \sum_{i=1}^{2} d_{i1} = -0.10 - 0.10 = -0.20,$

$c_2 = \sum_{i=1}^{2} d_{i2} = 0 + 0 = 0.$

Step 4: Since $c_1 \neq c_2$, so according to Step 3 of Chen and Han's IVIFMADM method [42], there is no need to apply Step 4, go to Step 5.

Step 5: Using Step 5 of Chen and Han's method [42], discussed in Section 6.7, there is need to solve the CLPP (P6.13).

$$Max[(-0.10)w_1 + (-0.10)w_1 + (0.00)w_2 + (0.00)w_2]$$

$$\text{s.t.} \begin{cases} 0 \leq w_1 \leq 1, \\ 0 \leq w_2 \leq 1, \\ w_1 + w_2 = 1. \end{cases} \qquad \text{(P6.13)}$$

It can be easily verified that on solving the CLPP (P6.13), the obtained OAWV is $(w_1, w_2) = (0,1)$.

Step 5: On considering the OAWV, $(w_1, w_2) = (0,1)$ and using Step 5 of Chen and Han's method [42], discussed in Section 6.7,

$\tilde{P}_1 = ([1-(1-0.10)^0(1-0.10)^1, 1-(1-0.30)^0(1-0.50)^1], [(0.2)^0(0.20)^1, (0.40)^0$ $(0.40)^1]) = ([0.10, 0.50], [0.20, 0.40]),$

$\tilde{P}_2 = ([1-(1-0.10)^0(1-0.10)^1, 1-(1-0.30)^0(1-0.50)^1], [(0.1)^0(0.20)^1, (0.50)^0$ $(0.40)^1]) = ([0.10, 0.50], [0.20, 0.40]).$

Step 6: Using Step 6, of Chen and Han's method [42], discussed in Section 6.7, $S(\tilde{P}_1) = -0.10, S(\tilde{P}_2) = -0.10$. Since, $S(\tilde{P}_1) = S(\tilde{P}_2)$. Therefore, according to Case (iii) of Step 6 of Chen and Han's method [42], proposed in Section 6.7, there is need to apply Step 7.

Step 7: Using Step 6, of Chen and Han's method [42], discussed in Section 6.7, $H(\tilde{P}_1) = 0.50, H(\tilde{P}_2) = 0.50$.

Since, $H(\tilde{P}_1) = H(\tilde{P}_2)$. So, according to Step 7, of Chen and Han's method [42], discussed in Section 6.5, the ranking of the alternatives is $A_1 = A_2$.

It is obvious that on applying Chen and Han's method [42], the relation $A_1 = A_2$ is obtained, which is inappropriate.

6.9 A brief review of Chen et al.'s method

Chen et al. [48] proposed the following method to solve IVIFMADMPrs having IVIF weights.

Step 1: Check that all the attributes are of same type or not i.e., check that all the attributes are benefit type attribute or cost type attribute.

Case (i): If all the attributes are of same type then go to Step 2.

Case (ii): If some attributes are cost type attributes and the remaining are benefit type attributes. Then, convert the j^{th} cost type attribute into the benefit type attribute by replacing all the elements $\tilde{\alpha}_{ij} = ([a_{ij1}, a_{ij2}], [a_{ij3}, a_{ij4}])$ of the j^{th} column of the IVIFDM $\widetilde{D} = (\tilde{\alpha}_{ij})_{m\times n}$ with $\tilde{\alpha}_{ij} = ([a_{ij3}, a_{ij4}], [a_{ij1}, a_{ij2}])$ and go to Step 2.

Step 2: Using the expression (6.8), transform each IVIF element

$\tilde{\alpha}_{ij} = \left([a_{ij1}, a_{ij2}], [a_{ij3}, a_{ij4}]\right)$ of the IVIFDM $\tilde{D} = \left(\tilde{\alpha}_{ij}\right)_{m\times n} = \left([a_{ij1}, a_{ij2}], [a_{ij3}, a_{ij4}]\right)$ into the crisp element d_{ij}.

$$d_{ij} = \left(\frac{a_{ij1} - a_{ij3} + a_{ij2} - a_{ij4}}{2} + 1\right) \tag{6.8}$$

Step 3: Find the OAWV, $(w_1, w_2, \ldots, w_n)$ by solving the crisp NLPP (P6.4).

Step 4: Using the expression (6.9), find the value of $\tilde{P}_i\ \forall\ i = 1,2,\ldots,m$.

$$\tilde{P}_i = \left(\left[\sum_{j=1}^{n} w_j a_{ij1}, \sum_{j=1}^{n} w_j a_{ij2}\right], \left[\sum_{j=1}^{n} w_j a_{ij3}, \sum_{j=1}^{n} w_j a_{ij4}\right]\right), i = 1,2,\ldots,m \tag{6.9}$$

Step 6: Check that $S(\tilde{P}_p) > S(\tilde{P}_q)$ or $S(\tilde{P}_p) < S(\tilde{P}_q)$ or $S(\tilde{P}_p) = S(\tilde{P}_q)$, where $S(\tilde{P}_p) = \frac{\sum_{j=1}^{n} w_j a_{pj1} + \sum_{j=1}^{n} w_j a_{pj2} - \sum_{j=1}^{n} w_j a_{pj3} - \sum_{j=1}^{n} w_j a_{pj4}}{2}$ and

$S(\tilde{P}_q) = \frac{\sum_{j=1}^{n} w_j a_{qj1} + \sum_{j=1}^{n} w_j a_{qj2} - \sum_{j=1}^{n} w_j a_{qj3} - \sum_{j=1}^{n} w_j a_{qj4}}{2}$.

Case (i): If $S(\tilde{P}_p) > S(\tilde{P}_q)$ then the ranking of the alternatives is $A_p > A_q$.

Case (ii): If $S(\tilde{P}_p) < S(\tilde{P}_q)$ then the ranking of the alternatives is $A_p < A_q$.

Case (iii): If $S(\tilde{P}_p) = S(\tilde{P}_q)$ then go to Step 7.

Step 7: Check that $H(\tilde{P}_p) > H(\tilde{P}_q)$ or $H(\tilde{P}_p) < H(\tilde{P}_q)$ or $H(\tilde{P}_p) = H(\tilde{P}_q)$, where

$H(\tilde{P}_p) = \frac{\sum_{j=1}^{n} w_j a_{pj1} + \sum_{j=1}^{n} w_j a_{pj2} + \sum_{j=1}^{n} w_j a_{pj3} + \sum_{j=1}^{n} w_j a_{pj4}}{2}$ and

$H(\tilde{P}_q) = \frac{\sum_{j=1}^{n} w_j a_{qj1} + \sum_{j=1}^{n} w_j a_{qj2} + \sum_{j=1}^{n} w_j a_{qj3} + \sum_{j=1}^{n} w_j a_{qj4}}{2}$.

Case (i): If $H(\tilde{P}_p) > H(\tilde{P}_q)$ then the ranking of the alternatives is $A_p > A_q$.

Case (ii): If $H(\tilde{P}_p) < H(\tilde{P}_q)$ then the ranking of the alternatives is $A_q < A_q$.

Case (iii): If $H(\tilde{P}_p) = H(\tilde{P}_q)$ then go to Step 8.

Step 8: Check that $T(\tilde{P}_p) > T(\tilde{P}_q)$ or $T(\tilde{P}_p) < T(\tilde{P}_q)$ or $T(\tilde{P}_p) = T(\tilde{P}_q)$, where $T(\tilde{P}_p) = \sum_{j=1}^{n} w_j a_{pj2} + \sum_{j=1}^{n} w_j a_{pj3} - \sum_{j=1}^{n} w_j a_{pj1} - \sum_{j=1}^{n} w_j a_{pj4}$ and $T(\tilde{P}_q) = \sum_{j=1}^{n} w_j a_{qj2} + \sum_{j=1}^{n} w_j a_{qj3} - \sum_{j=1}^{n} w_j a_{qj1} - \sum_{j=1}^{n} w_j a_{qj4}$.

Case (i): If $T(\tilde{P}_p) > T(\tilde{P}_q)$ then the ranking of the alternatives is $A_p < A_q$.

Case (ii): If $T(\tilde{P}_p) < T(\tilde{P}_q)$ then the ranking of the alternatives is $A_p > A_q$.

Case (iii): If $T(\tilde{P}_p) = T(\tilde{P}_q)$ then go to Step 9.

Step 9: Check that $G(\tilde{P}_p) > G(\tilde{P}_q)$ or $G(\tilde{P}_p) < G(\tilde{P}_q)$ or $G(\tilde{P}_p) = G(\tilde{P}_q)$, where $G(\tilde{P}_p) = \sum_{j=1}^{n} w_j a_{pj2} + \sum_{j=1}^{n} w_j a_{pj4} - \sum_{j=1}^{n} w_j a_{pj1} - \sum_{j=1}^{n} w_j a_{pj3}$ and $G(\tilde{P}_q) = \sum_{j=1}^{n} w_j a_{qj2} + \sum_{j=1}^{n} w_j a_{qj4} - \sum_{j=1}^{n} w_j a_{qj1} - \sum_{j=1}^{n} w_j a_{qj3}$.

Case (i): If $G(\tilde{P}_p) > G(\tilde{P}_q)$ then the ranking of the alternatives is $A_p < A_q$.

Case (ii): If $G(\tilde{P}_p) < G(\tilde{P}_q)$ then the ranking of the alternatives is $A_p > A_q$.

Case (iii): If $G(\tilde{P}_p) = G(\tilde{P}_q)$ then the ranking of the alternatives is $A_p = A_q$.

6.10 Inappropriateness of Chen et al.'s method

In can be easily concluded from Section 3.1 of Chapter 3, that

(i) If $\sum_{j=1}^{n} w_j^l > 1$ then no feasible solution of the IVIFMADMPr will exist. However, if one is interested to find the solution then there is need to increase the total available amount i.e., there is need to replace the weight w_j of the j^{th} attribute with $w_j + \varepsilon_j$ where $\varepsilon_j > 0$.

(ii) If $\sum_{j=1}^{n} w_j^u < 1$ then no feasible solution of the IVIFMADMPr will exist. However, if one is interested to find the solution then to find the feasible solution of IVIFMADM there is need to replace the constraint $\sum_{j=1}^{n} w_j = 1$ with the constraint $\sum_{j=1}^{n} w_j \leq 1$.

(iii) If neither the condition $\sum_{j=1}^{n} w_j^l > 1$ nor the condition $\sum_{j=1}^{n} w_j^u < 1$ will be satisfied. Then, there is neither any need to replace w_j of the j^{th} attribute with $w_j + \varepsilon_j$ nor any need to replace the constraint $\sum_{j=1}^{n} w_j = 1$ with the constraint $\sum_{j=1}^{n} w_j \leq 1$.

However, it is obvious from the NLPP (P6.4) that Chen et al. [48] have not considered these conditions to propose the NLPP (P6.4). Furthermore, to find the feasible solution of the IVIFMADMPr, Chen et al. [48] have reduced the lower bound w_j^l and/or increased the upper bound w_j^u. According to Section 3.1 of Chapter 3, this modification physically indicates that the restrictions have been relaxed instead of increasing the amount. Due to all these reasons it is inappropriate to use the NLPP (P6.4) to find the crisp attribute weight w_j, corresponding to the IVIF weights $\widetilde{w}_j = ([w_{j1}, w_{j2}], [w_{j3}, w_{j4}])$. Hence, it is inappropriate to use Chen et al.'s method [48].

6.11 A brief review of Chen and Han's method

Chen and Han [43] proposed the following method for solving IVIFMADMPrs having IVIF weights.

Step 1: Check that all the attributes are of same type or not i.e., check that all the attributes are benefit type attribute or cost type attribute.

Case (i): If all the attributes are of same type then go to Step 2.

Case (ii): If some attributes are cost type attributes and the remaining are benefit type attributes. Then, convert the j^{th} cost type attribute into the benefit type attribute by replacing all the elements $\tilde{\alpha}_{ij} = ([a_{ij1}, a_{ij2}], [a_{ij3}, a_{ij4}])$ of the j^{th} column of the IVIFDM $\widetilde{D} = (\tilde{\alpha}_{ij})_{m\times n}$ with $\tilde{\alpha}_{ij} = ([a_{ij3}, a_{ij4}], [a_{ij1}, a_{ij2}])$ and go to Step 2.

Step 2: Check that either the condition $\sum_{j=1}^{n} w_j^l > 1$ or the condition $\sum_{j=1}^{n} w_j^u < 1$ **is s**atisfying.

Case (i) If $\sum_{j=1}^{n} w_j^l > 1$ then transform the IVIF weight $\widetilde{w}_j = ([w_{j1}, w_{j2}], [w_{j3}, w_{j4}])$ into the IVIF weight $\widetilde{w}_j = \left(\left[\frac{w_{j1}}{\sum_{j=1}^{n} w_j}, w_{j2}\right], [w_{j3}, w_{j4}]\right)$.

Case (ii) If $\sum_{j=1}^{n} w_j^u < 1$ then transform the IVIF weight $\widetilde{w}_j = ([w_{j1}, w_{j2}], [w_{j3}, w_{j4}])$ into the IVIF weight $\widetilde{w}_j = \left([w_{j1}, w_{j2}], \left[w_{j3}, \frac{w_{j4}}{\sum_{j=1}^{n} w_j}\right]\right)$.

Step 3: Using the expression (6.10), transform each IVIF element $\tilde{\alpha}_{ij} = ([a_{ij1}, a_{ij2}], [a_{ij3}, a_{ij4}])$ of the IVIFDM $\widetilde{D} = (\tilde{\alpha}_{ij})_{m\times n} = ([a_{ij1}, a_{ij2}], [a_{ij3}, a_{ij4}])$ into the crisp element d_{ij}

$$d_{ij} = \left(a_{ij1} + a_{ij2} - 1 + \frac{a_{ij3} + a_{ij4}}{2}\right). \tag{6.10}$$

Step 4: Find the OAWV, $(w_1, w_2, \dots, w_n)$ by solving the crisp NLPP (P6.5).

where, $\begin{cases} w_{j1} = \frac{w_{j1}}{\sum_{j=1}^{n} w_j} & if\ \sum_{j=1}^{n} w_j^{l} > 1 \\ w_{j3} = \frac{w_{j3}}{\sum_{j=1}^{n} w_j} & if\ \sum_{j=1}^{n} w_j^{u} < 1 \\ w_{j1} = w_{j1} \text{ and } w_{j3} = w_{j3} & \text{otherwise} \end{cases}$

Step 5: Using the expression (6.11), find the value of $\tilde{P}_i\ \forall\ i = 1,2,\dots,m$.

$$\tilde{P}_i = \left(\left[\prod_{j=1}^{n}\left(a_{ij1}\right)^{w_j}, \prod_{j=1}^{n}\left(a_{ij2}\right)^{w_j}\right], \left[1-\prod_{j=1}^{n}\left(1-a_{ij3}\right)^{w_j}, 1-\prod_{j=1}^{n}\left(1-a_{ij4}\right)^{w_j}\right]\right), i = 1,2,\dots,m \qquad (6.11)$$

Step 6: Check that $S(\tilde{P}_p) > S(\tilde{P}_q)$ or $S(\tilde{P}_p) < S(\tilde{P}_q)$ or $S(\tilde{P}_p) = S(\tilde{P}_q)$, where $S(\tilde{P}_p) = \frac{\prod_{j=1}^{n}(a_{pj1})^{w_j} + \prod_{j=1}^{n}(a_{pj2})^{w_j} - 1 - \prod_{j=1}^{n}(1-a_{pj3})^{w_j} - 1 - \prod_{j=1}^{n}(1-a_{pj4})^{w_j}}{2}$ and

$$S(\tilde{P}_q) = \frac{\prod_{j=1}^{n}(a_{qj1})^{w_j} + \prod_{j=1}^{n}(a_{qj2})^{w_j} - 1 - \prod_{j=1}^{n}(1-a_{qj3})^{w_j} - 1 - \prod_{j=1}^{n}(1-a_{qj4})^{w_j}}{2}.$$

Case (i): If $S(\tilde{P}_p) > S(\tilde{P}_q)$ then the ranking of the alternatives is $A_p > A_q$.

Case (ii): If $S(\tilde{P}_p) < S(\tilde{P}_q)$ then the ranking of the alternatives is $A_p < A_q$.

Case (iii): If $S(\tilde{P}_p) = S(\tilde{P}_q)$ then go to Step 7.

Step 7: Check that $H(\tilde{P}_p) > H(\tilde{P}_q)$ or $H(\tilde{P}_p) < H(\tilde{P}_q)$ or $H(\tilde{P}_p) = H(\tilde{P}_q)$, where

$$H(\tilde{P}_p) = \frac{\prod_{j=1}^{n}(a_{pj1})^{w_j} + \prod_{j=1}^{n}(a_{pj2})^{w_j} + 1 - \prod_{j=1}^{n}(1-a_{pj3})^{w_j} + 1 - \prod_{j=1}^{n}(1-a_{pj4})^{w_j}}{2} \text{ and}$$

$$H(\tilde{P}_q) = \frac{\prod_{j=1}^{n}(a_{qj1})^{w_j} + \prod_{j=1}^{n}(a_{qj2})^{w_j} + 1 - \prod_{j=1}^{n}(1-a_{qj3})^{w_j} + 1 - \prod_{j=1}^{n}(1-a_{qj4})^{w_j}}{2}.$$

Case (i): If $H(\tilde{P}_p) > H(\tilde{P}_q)$ then the ranking of the alternatives is $A_p > A_q$.

Case (ii): If $H(\tilde{P}_p) < H(\tilde{P}_q)$ then the ranking of the alternatives is $A_p < A_q$.

Case (iii): If $H(\tilde{P}_p) = H(\tilde{P}_q)$ then go to Step 8.

Step 8: Check that $T(\tilde{P}_p) > T(\tilde{P}_q)$ or $T(\tilde{P}_p) < T(\tilde{P}_q)$ or $T(\tilde{P}_p) = T(\tilde{P}_q)$, where $T(\tilde{P}_p) = \prod_{j=1}^{n}(a_{pj2})^{w_j} + 1 - \prod_{j=1}^{n}(1-a_{pj3})^{w_j} - \prod_{j=1}^{n}(a_{pj1})^{w_j} - 1 - \prod_{j=1}^{n}(1-a_{pj4})^{w_j}$ and

$$T(\tilde{P}_q) = \prod_{j=1}^{n}(a_{qj2})^{w_j} + 1 - \prod_{j=1}^{n}(1-a_{qj3})^{w_j} - \prod_{j=1}^{n}(a_{qj1})^{w_j} - 1 - \prod_{j=1}^{n}(1-a_{qj4})^{w_j}.$$

Case (i): If $T(\tilde{P}_p) > T(\tilde{P}_q)$ then the ranking of the alternatives is $A_p < A_q$.

Case (ii): If $T(\tilde{P}_p) < T(\tilde{P}_q)$ then the ranking of the alternatives is $A_p > A_q$.

Case (iii): If $T(\tilde{P}_p) = T(\tilde{P}_q)$ then go to Step 9.

Step 9: Check that $G(\tilde{P}_p) > G(\tilde{P}_q)$ or $G(\tilde{P}_p) < G(\tilde{P}_q)$ or $G(\tilde{P}_p) = G(\tilde{P}_q)$, where $G(\tilde{P}_p) = \prod_{j=1}^{n}(a_{pj2})^{w_j} + 1 - \prod_{j=1}^{n}(1 - a_{pj4})^{w_j} - \prod_{j=1}^{n}(a_{pj1})^{w_j} - 1 - \prod_{j=1}^{n}(1 - a_{pj3})^{w_j}$ and $G(\tilde{P}_q) = \prod_{j=1}^{n}(a_{qj2})^{w_j} + 1 - \prod_{j=1}^{n}(1 - a_{qj4})^{w_j} - \prod_{j=1}^{n}(a_{qj1})^{w_j} - 1 - \prod_{j=1}^{n}(1 - a_{qj3})^{w_j}$.

Case (i): If $G(\tilde{P}_p) > G(\tilde{P}_q)$ then the ranking of the alternatives is $A_p < A_q$.

Case (ii): If $G(\tilde{P}_p) < G(\tilde{P}_q)$ then the ranking of the alternatives is $A_p > A_q$.

Case (iii): If $G(\tilde{P}_p) = G(\tilde{P}_q)$ then the ranking of the alternatives is $A_p = A_q$.

6.12 Inappropriateness of Chen and Han's method

In can be easily concluded from Section 3.1 of Chapter 3, that

(i) If $\sum_{j=1}^{n} w_j^l > 1$ then no feasible solution of the IVIFMADMPr will exist. However, if one is interested to find the solution then there is need to increase the total available amount i.e., there is need to replace the weight w_j of the j^{th} attribute with $w_j + \varepsilon_j$ where $\varepsilon_j > 0$.

(ii) If $\sum_{j=1}^{n} w_j^u < 1$ then no feasible solution of the IVIFMADMPr will exist. However, if one is interested to find the solution then to find the feasible solution of IVIFMADM there is need to replace the constraint $\sum_{j=1}^{n} w_j = 1$ with the constraint $\sum_{j=1}^{n} w_j \leq 1$.

(iii) If neither the condition $\sum_{j=1}^{n} w_j^l > 1$ nor the condition $\sum_{j=1}^{n} w_j^u < 1$ will be satisfied. Then there is neither any need to replace w_j of the j^{th} attribute with $w_j + \varepsilon_j$ nor any need to replace the constraint $\sum_{j=1}^{n} w_j = 1$ with the constraint $\sum_{j=1}^{n} w_j \leq 1$.

However, it is obvious from the NLPP (P6.5) that Chen and Han [43] have not considered these conditions to propose the NLPP (P6.5). Furthermore, to find the feasible solution of the IVIFMADMPr, Chen and Han [43] have reduced the lower bound w_j^l and/or increased the upper bound w_j^u. According to Section 3.1 of Chapter 3, this modification physically indicates that the restrictions have been relaxed instead of increasing the amount. Due to all these reasons, it is inappropriate to use the NLPP (P6.5) to find the crisp attribute weight w_j, corresponding to the IVIF weights $\widetilde{w}_j = ([w_{j1}, w_{j2}], [w_{j3}, w_{j4}])$. Hence, it is inappropriate to use Chen and Han's method [43].

6.13 Proposed Jujhar method for solving IVIFMADMPrs having IVIF weights

In all the existing methods [42, 43, 47, 48, 211], firstly the IVIFMADMPrs having IVIF weights have been transformed into a special type of IVIFMADMPrs with partially known attribute weights by replacing the known IVIF weight of the j^{th} attribute $\widetilde{w}_j = ([w_{j1}, w_{j2}], [w_{j3}, w_{j4}])$ with $w_{j1} \leq w_j \leq 1 - w_{j3}$. Then, the transformed IVIFMADMPr with partially known attribute weights has been solved to obtain the ranking of the alternatives. Hence, the Jujhar method, proposed in Chapter 4 with the following modification can be used to solve IVIFMADMPrs having IVIF weights.

"If $\widetilde{w}_j = ([w_{j1}, w_{j2}], [w_{j3}, w_{j4}])$ represents the IVIF weights of the j^{th} attribute. Then, solve the CLPP (P6.11) instead of the CLPP (P2.12) of Chapter 2, used in Case (iii) of Step 4 of the proposed Jujhar method in Chapter 4 to evaluate the optimal attribute weight.

$$max\left[\sum_{j=1}^{n} c_j\, w_j\right]$$

Subject to

$$\begin{cases} w_{j1} \leq w_j \leq 1 - w_{j3}, j = 1,2,\ldots,n \\ w_j \geq 0, j = 1,2,..,n, \\ \sum_{j=1}^{n} w_j = 1, \\ w_p = w_q \;\forall\; c_p = c_q\;. \end{cases} \tag{P6.11}$$

6.14 Exact results of the considered IVIFMADMPrs having IVIF weights

In Section 6.4, Section 6.6 and Section 6.8, some IVIFMADMPrs having IVIF weights were solved by the existing methods [42, 43, 47, 48, 211] and shown that the obtained results are inappropriate. In this section, the exact results of all these IVIFMADMPrs having IVIF weights are obtained by the Jujhar method, proposed in Chapter 4, with modified CLPP (P6.11).

6.14.1 Exact results of first IVIFMADMPr having IVIF weights

Using the proposed Jujhar method, the exact result of the IVIFMADMPr having IVIF attribute weights, considered in the first point of Section 6.4, can be obtained as follows:

Step 1: Since, all the attributes are benefit type. So, according to Step 1 of the Jujhar method, proposed in Section 4.13 of Chapter 4, there is no need to apply Step 1.

Step 2: According to Step 2 of the Jujhar method, proposed in Section 4.13, there is need to calculate $d_{ijk}(\tilde{\alpha}_{ij}, \tilde{\alpha}_{kj})\ \forall\ i = 1,2; j = 1,2$. These values are shown in Table 6.9.

Table 6.9: Values of $\boldsymbol{d_{ijk}(\tilde{\alpha}_{ij}, \tilde{\alpha}_{kj})}$

$d_{111}\left(\begin{pmatrix}[0.40,0.50],\\ [0.30,0.40]\end{pmatrix},\begin{pmatrix}[0.40,0.50],\\ [0.30,0.40]\end{pmatrix}\right) =$ $\frac{1}{4}(0+0+0+0) = 0$	$d_{211}\left(\begin{pmatrix}[0.40,0.60],\\ [0.20,0.40]\end{pmatrix},\begin{pmatrix}[0.40,0.60],\\ [0.20,0.40]\end{pmatrix}\right) =$ $\frac{1}{4}(0+0+0+0) = 0$
$d_{112}\left(\begin{pmatrix}[0.40,0.50],\\ [0.30,0.40]\end{pmatrix},\begin{pmatrix}[0.40,0.60],\\ [0.20,0.40]\end{pmatrix}\right) =$ $\frac{1}{4}(0+0.1+0.1+0) = 0.05$	$d_{212}\left(\begin{pmatrix}[0.40,0.60],\\ [0.20,0.40]\end{pmatrix},\begin{pmatrix}[0.40,0.50],\\ [0.30,0.40]\end{pmatrix}\right) =$ $\frac{1}{4}(0+0.1+0.1+0) = 0.05$
$d_{121}\left(\begin{pmatrix}[0.30,0.40],\\ [0.30,0.40]\end{pmatrix},\begin{pmatrix}[0.40,0.50],\\ [0.30,0.40]\end{pmatrix}\right) =$ $\frac{1}{4}(0+0.1+0.1+0) = 0.05$	$d_{221}\left(\begin{pmatrix}[0.40,0.50],\\ [0.30,0.40]\end{pmatrix},\begin{pmatrix}[0.40,0.60],\\ [0.20,0.40]\end{pmatrix}\right) =$ $\frac{1}{4}(0+0.1+0.1+0) = 0.05$
$d_{122}\left(\begin{pmatrix}[0.30,0.40],\\ [0.30,0.40]\end{pmatrix},\begin{pmatrix}[0.40,0.60],\\ [0.20,0.40]\end{pmatrix}\right) =$ $\frac{1}{4}(0+0+0+0) = 0$	$d_{222}\left(\begin{pmatrix}[0.40,0.50],\\ [0.30,0.40]\end{pmatrix},\begin{pmatrix}[0.40,0.50],\\ [0.30,0.40]\end{pmatrix}\right) =$ $\frac{1}{4}(0+0+0+0) = 0$

Step 3: Using Step 3 of the Jujhar method, proposed in Section 4.13,

$c_1 = \sum_{i=1}^{2}\sum_{k=1}^{2} d_{i1k}(\tilde{\alpha}_{i1}, \tilde{\alpha}_{k1}) = 0 + 0.05 + 0.05 + 0 = 0.10,$

$c_2 = \sum_{i=1}^{2}\sum_{k=1}^{2} d_{i1k}(\tilde{\alpha}_{i1}, \tilde{\alpha}_{k1}) = 0 + 0.05 + 0.05 + 0 = 0.10.$

Step 4: Using Step 4 of the Jujhar method, proposed in Section 4.13, there is need to solve the CLPP (P6.12).

$max(0.10w_1 + 0.10w_2)$

Subject to

$$\begin{cases} 0.0 \leq w_1 \leq 0.8, \\ 0.1 \leq w_2 \leq 0.9, \\ w_1 + w_2 = 1, \\ w_1 \geq 0, w_2 \geq 0. \end{cases} \qquad \text{(P6.12)}$$

It can be easily verified that on solving the CLPP (P6.12) the obtained OAWV is $(w_1, w_2) = (0.50, 0.50)$.

Step 5: On considering the OAWV, $(w_1, w_2) = (0.50, 0.50)$ and using Step 5 of the Jujhar method, proposed in Section 4.13,

$\tilde{P}_1 = ([(0.50)(0.40) + (0.50)(0.40), (0.50)(0.50) + (0.50)(0.60)], [(0.50)(0.30) + (0.50)(0.20), (0.50)(0.40) + (0.50)(0.40)]) = ([0.40, 0.55], [0.25, 0.40])$,

$\tilde{P}_2 = ([(0.50)(0.40) + (0.50)(0.40), (0.50)(0.60) + (0.50)(0.50)], [(0.50)(0.20) + (0.50)(0.30), (0.50)(0.40) + (0.50)(0.40)]) = ([0.40, 0.55], [0.25, 0.40])$.

Step 6: Using Step 6 of the Jujhar method, proposed in Section 4.13, $S(\tilde{P}_1) = 0.15$ and $S(\tilde{P}_2) = 0.15$.

Since, $S(\tilde{P}_1) = S(\tilde{P}_2)$.So, there is need to apply Step 7 of the proposed Jujhar method.

Step 7: Using Step 7 of the Jujhar method, proposed in Section 4.13, $H(\tilde{P}_1) = 0.80$ and $H(\tilde{P}_2) = 0.80$.

Since, $H(\tilde{P}_1) = H(\tilde{P}_2)$.So, there is need to apply Step 8 of the proposed Jujhar method.

Step 8: Using Step 8 of the Jujhar method, proposed in Section 4.13, $T(\tilde{P}_1) = 0$ and $T(\tilde{P}_2) = 0$. Since, $T(\tilde{P}_1) = T(\tilde{P}_2)$.So, there is need to apply Step 9 of the proposed Jujhar method.

Step 9: Using Step 9 of the Jujhar method, proposed in Section 4.13, $G(\tilde{P}_1) = 0.30$ and $G(\tilde{P}_2) = 0.30$.

Since, $G(\tilde{P}_1) = G(\tilde{P}_2)$. Therefore, according to case (iii) of Step 9 of the proposed Jujhar method, the ranking of the alternatives is $A_1 = A_2$.

6.14.2 Exact results of second IVIFMADMPr

Using the proposed Jujhar method, the exact result of the IVIFMADMPr having IVIF attribute weights, considered in the second point of Section 6.4, can be obtained as follows:

Since, Step 1 to Step 3 of the proposed Jujhar method are same as shown in Section 6.14.1. So, to avoid any repetition, the calculations have been started from Step 4.

Step 4: As here $\sum_{j=1}^{n} w_j^u < 1$. Therefore, using Case (i) of Step 4 of the Jujhar method, proposed in Section 4.13, there is need to solve the CLPP (P6.13).

$max(0.20w_1)$

Subject to

$$\begin{cases} 0.10 \le w_1 \le 0.20, \\ 0.10 \le w_2 \le 0.50, \\ w_1 + w_2 \le 1, \\ w_1 = w_2, \\ w_1 \ge 0, w_2 \ge 0. \end{cases} \tag{P6.13}$$

It can be easily verified that on solving the CLPP (P6.13) the obtained non-normalized OAWV is $(w_1, w_2) = (0.20,0.20)$. So, the normalized OAWV is $(w_1, w_2) = \left(\frac{0.20}{0.20+0.20}, \frac{0.20}{0.20+0.20}\right) = (0.50,0.50)$.

Step 5: On considering the OAWV, $(w_1, w_2) = (0.50,0.50)$ and using Step 5 of the Jujhar method, proposed in Section 4.13,

$\tilde{P}_1 = ([(0.50)(0.40) + (0.50)(0.40), (0.50)(0.50) + (0.50)(0.60)], [(0.50)(0.30) + (0.50)(0.20), (0.50)(0.40) + (0.50)(0.40)]) = ([0.40,0.55], [0.25, 0.40])$,

$\tilde{P}_2 = ([(0.50)(0.40) + (0.50)(0.40), (0.50)(0.60) + (0.50)(0.50)], [(0.50)(0.20) + (0.50)(0.30), (0.50)(0.40) + (0.50)(0.40)]) = ([0.40,0.55], [0.25,0.40])$.

Step 6: Using Step 6 of the Jujhar method, proposed in Section 4.13, $S(\tilde{P}_1) = 0.15$ and

$S(\tilde{P}_2) = 0.15$.

Since, $S(\tilde{P}_1) = S(\tilde{P}_2)$.So, there is need to apply Step 7 of the proposed Jujhar method.

Step 7: Using Step 7 of the Jujhar method, proposed in Section 4.13, $H(\tilde{P}_1) = 0.80$ and $H(\tilde{P}_2) = 0.80$.

Since, $H(\tilde{P}_1) = H(\tilde{P}_2)$.So, there is need to apply Step 8 of the proposed Jujhar method.

Step 8: Using Step 8 of the Jujhar method, proposed in Section 4.13, $T(\tilde{P}_1) = 0$ and $T(\tilde{P}_2) = 0$.

Since, $T(\tilde{P}_1) = T(\tilde{P}_2)$.So, there is need to apply Step 9 of the proposed Jujhar method.

Step 9: Using Step 9 of the Jujhar method, proposed in Section 4.13, $G(\tilde{P}_1) = 0.30$ and $G(\tilde{P}_2) = 0.30$.

Since, $G(\tilde{P}_1) = G(\tilde{P}_2)$. Therefore, according to case (iii) of Step 9 of the proposed Jujhar method, the ranking of the alternatives is $A_1 = A_2$.

6.14.3 Exact results of the third IVIFMADMPr

Using the proposed Jujhar method, the exact result of the IVIFMADMPr having IVIF attribute weights, considered in the second point of Section 6.4, can be obtained as follows:

Since, Step 1 to Step 3 of the proposed Jujhar method are same as shown in Section 6.14.1. So, to avoid any repetition, the calculations have been started from Step 4.

Step 4: Since, $\sum_{j=1}^{n} w_j^l > 1$. Therefore, using Case (ii) of Step 4 of the Jujhar method, proposed in Section 4.13, there is need to solve the CLPP (P6.14).

$max(0.20w_1 - \varepsilon_1 - \varepsilon_2)$

Subject to

$$\begin{cases} \frac{60}{120} \le w_1 \le \frac{80}{120}, \\ \frac{60}{120} \le w_2 \le \frac{80}{120}, \\ w_1 + w_2 = 1, \\ w_1 = w_2, \\ 120 = 100 + \varepsilon_1 + \varepsilon_2 \\ 0 \le \varepsilon_1 \le 1, \\ 0 \le \varepsilon_2 \le 1, \\ w_1 \ge 0, w_2 \ge 0, \varepsilon_1 \ge 0, \varepsilon_2 \ge 0. \end{cases} \tag{P6.14}$$

It can be easily verified that on solving the CLPP (P6.14) the obtained OAWV is $(w_1, w_2) = (0.5,0.5)$.

Step 5: On considering the OAWV, $(w_1, w_2) = (0.50,0.50)$ and using Step 5 of the Jujhar method, proposed in Section 4.13,

$\tilde{P}_1 = ([(0.50)(0.40) + (0.50)(0.40), (0.50)(0.50) + (0.50)(0.60)], [(0.50)(0.30) + (0.50)(0.20), (0.50)(0.40) + (0.50)(0.40)]) = ([0.40,0.55], [0.25, 0.40])$,

$\tilde{P}_2 = ([(0.50)(0.40) + (0.50)(0.40), (0.50)(0.60) + (0.50)(0.50)], [(0.50)(0.20) + (0.50)(0.30), (0.50)(0.40) + (0.50)(0.40)]) = ([0.40,0.55], [0.25,0.40])$.

Step 6: Using Step 6 of the Jujhar method, proposed in Section 4.13, $S(\tilde{P}_1) = 0.15$ and $S(\tilde{P}_2) = 0.15$.

Since, $S(\tilde{P}_1) = S(\tilde{P}_2)$. So, there is need to apply Step 7 of the proposed Jujhar method.

Step 7: Using Step 7 of the Jujhar method, proposed in Section 4.13, $H(\tilde{P}_1) = 0.80$ and $H(\tilde{P}_2) = 0.80$.

Since, $H(\tilde{P}_1) = H(\tilde{P}_2)$. So, there is need to apply Step 8 of the proposed Jujhar method.

Step 8: Using Step 8 of the Jujhar method, proposed in Section 4.13, $T(\tilde{P}_1) = 0$ and $T(\tilde{P}_2) = 0$.

Since, $T(\tilde{P}_1) = T(\tilde{P}_2)$. So, there is need to apply Step 9 of the proposed Jujhar method.

Step 9: Using Step 9 of the Jujhar method, proposed in Section 4.13, $G(\tilde{P}_1) = 0.30$ and $G(\tilde{P}_2) = 0.30$.

Since, $G(\tilde{P}_1) = G(\tilde{P}_2)$. Therefore, according to case (iii) of Step 9 of the proposed Jujhar method, the ranking of the alternatives is $A_1 = A_2$.

6.14.4 Exact results of fourth IVIFMADMPr having IVIF weights

Using the proposed Jujhar method, the exact result of the IVIFMADMPr having IVIF attribute weights, considered in the first point of Section 6.6, can be obtained as follows:

Step 1: Since, all the attributes are benefit type. So, according to Step 1 of the Jujhar method, proposed in Section 4.13 of Chapter 4, there is no need to apply Step 1.

Step 2: According to Step 2 of the Jujhar method, proposed in Section 4.13, there is need to calculate $d_{ijk}(\tilde{\alpha}_{ij}, \tilde{\alpha}_{kj})\ \forall\ i = 1,2; j = 1,2$. These values are shown in Table 6.10.

Table 6.10: Values of $d_{ijk}(\tilde{\alpha}_{ij}, \tilde{\alpha}_{kj})$

$d_{111}\left(\begin{pmatrix}[0.10,0.40],\\ {[0.20,0.30]}\end{pmatrix},\begin{pmatrix}[0.10,0.40],\\ {[0.20,0.30]}\end{pmatrix}\right) =$ $\frac{1}{4}(0+0+0+0) = 0$	$d_{211}\left(\begin{pmatrix}[0.01,0.04],\\ {[0.02,0.03]}\end{pmatrix},\begin{pmatrix}[0.01,0.04],\\ {[0.02,0.03]}\end{pmatrix}\right) =$ $\frac{1}{4}(0+0+0+0) = 0$
$d_{112}\left(\begin{pmatrix}[0.10,0.40],\\ {[0.20,0.30]}\end{pmatrix},\begin{pmatrix}[0.02,0.04],\\ {[0.01,0.05]}\end{pmatrix}\right) =$ $\frac{1}{4}(0.08+0.36+0.19+0.25) = 0.22$	$d_{212}\left(\begin{pmatrix}[0.01,0.04],\\ {[0.02,0.03]}\end{pmatrix},\begin{pmatrix}[0.20,0.50],\\ {[0.30,0.40]}\end{pmatrix}\right) =$ $\frac{1}{4}(0.19+0.46+0.28+0.37) = 0.325$
$d_{121}\left(\begin{pmatrix}[0.02,0.04],\\ {[0.01,0.05]}\end{pmatrix},\begin{pmatrix}[0.10,0.40],\\ {[0.20,0.30]}\end{pmatrix}\right) =$ $\frac{1}{4}(0.08+0.36+0.19+0.25) = 0.22$	$d_{221}\left(\begin{pmatrix}[0.20,0.50],\\ {[0.30,0.40]}\end{pmatrix},\begin{pmatrix}[0.01,0.04],\\ {[0.02,0.03]}\end{pmatrix}\right) =$ $\frac{1}{4}(0.19+0.46+0.28+0.37) = 0.325$
$d_{122}\left(\begin{pmatrix}[0.02,0.04],\\ {[0.01,0.05]}\end{pmatrix},\begin{pmatrix}[0.02,0.04],\\ {[0.01,0.05]}\end{pmatrix}\right) =$ $\frac{1}{4}(0+0+0+0) = 0$	$d_{222}\left(\begin{pmatrix}[0.20,0.50],\\ {[0.30,0.40]}\end{pmatrix},\begin{pmatrix}[0.20,0.50],\\ {[0.30,0.40]}\end{pmatrix}\right) =$ $\frac{1}{4}(0+0+0+0) = 0$

Step 3: Using Step 3 of the Jujhar method, proposed in Section 4.13,

$c_1 = \sum_{i=1}^{2}\sum_{k=1}^{2} d_{i1k}(\tilde{\alpha}_{i1}, \tilde{\alpha}_{k1}) = 0 + 0.22 + 0.22 + 0 = 0.44,$

$c_2 = \sum_{i=1}^{2}\sum_{k=1}^{2} d_{i1k}(\tilde{\alpha}_{i1}, \tilde{\alpha}_{k1}) = 0 + 0.325 + 0.325 + 0 = 0.65.$

Step 4: Using Step 4 of the Jujhar method, proposed in Section 4.13, there is need to solve the CLPP (P6.15).

$max(0.44w_1 + 0.65w_2)$

Subject to

$$\begin{cases} 0.0 \leq w_1 \leq 0.8, \\ 0.1 \leq w_2 \leq 0.9, \\ w_1 + w_2 = 1, \\ w_1 \geq 0, w_2 \geq 0. \end{cases} \tag{P6.15}$$

It can be easily verified that on solving the CLPP (P6.15) the obtained OAWV is $(w_1, w_2) = (0.10, 0.90)$.

Step 5: On considering the OAWV, $(w_1, w_2) = (0.10, 0.90)$ and using Step 5 of the Jujhar method, proposed in Section 4.13,

$\tilde{P}_1 = ([(0.10)(0.10) + (0.90)(0.01), (0.10)(0.40) + (0.90)(0.04)], [(0.10)(0.20) + (0.90)(0.02), (0.10)(0.30) + (0.90)(0.03)]) = ([0.019, 0.076], [0.038, 0.057]),$

$\tilde{P}_2 = ([(0.10)(0.02) + (0.90)(0.20), (0.10)(0.04) + (0.90)(0.50)], [(0.10)(0.01) + (0.90)(0.30), (0.10)(0.05) + (0.90)(0.40)]) = ([0.182, 0.454], [0.271, 0.365]).$

Step 6: Using Step 6 of the Jujhar method, proposed in Section 4.13, $S(\tilde{P}_1) = 0$ and $S(\tilde{P}_2) = 0$.

Since, $S(\tilde{P}_1) = S(\tilde{P}_2)$. So, there is need to apply Step 7 of the proposed Jujhar method.

Step 7: Using Step 7 of the Jujhar method, proposed in Section 4.13, $H(\tilde{P}_1) = 0.095$ and $H(\tilde{P}_2) = 0.636$.

Since, $H(\tilde{P}_1) < H(\tilde{P}_2)$. Therefore, according to Step 7 of the proposed Jujhar method, the ranking of the alternatives is $A_1 < A_2$.

6.14.5 Exact results of fifth IVIFMADMPr having IVIF weights

Using the proposed Jujhar method, the exact result of the IVIFMADMPr having IVIF weights, considered in the first point of Section 6.8, can be obtained as follows:

Step 1: Since, all the attributes are benefit type. So, according to Step 1 of the Jujhar method, proposed in Section 4.13 of Chapter 4, there is no need to apply Step 1.

Step 2: According to Step 2 of the Jujhar method, proposed in Section 4.13, there is need to calculate $d_{ijk}\left(\tilde{\alpha}_{ij}, \tilde{\alpha}_{kj}\right) \forall\ i = 1,2; j = 1,2$. These values are shown in Table 6.11.

Table 6.11: Values of $\boldsymbol{d_{ijk}\left(\tilde{\alpha}_{ij}, \tilde{\alpha}_{kj}\right)}$

$d_{111}\left(\begin{pmatrix}[0.10,0.20],\\ {[0.30,0.40]}\end{pmatrix}, \begin{pmatrix}[0.10,0.20],\\ {[0.30,0.40]}\end{pmatrix}\right) =$ $\frac{1}{4}(0+0+0+0) = 0$	$d_{211}\left(\begin{pmatrix}[0.30,0.40],\\ {[0.40,0.50]}\end{pmatrix}, \begin{pmatrix}[0.30,0.40],\\ {[0.40,0.50]}\end{pmatrix}\right) =$ $\frac{1}{4}(0+0+0+0) = 0$
$d_{112}\left(\begin{pmatrix}[0.10,0.20],\\ {[0.30,0.40]}\end{pmatrix}, \begin{pmatrix}[0.30,0.40],\\ {[0.40,0.50]}\end{pmatrix}\right) =$ $\frac{1}{4}(0.2+0.2+0.1+0.1) = 0.15$	$d_{212}\left(\begin{pmatrix}[0.30,0.40],\\ {[0.40,0.50]}\end{pmatrix}, \begin{pmatrix}[0.10,0.20],\\ {[0.30,0.40]}\end{pmatrix}\right) =$ $\frac{1}{4}(0.2+0.2+0.1+0.1) = 0.15$
$d_{121}\left(\begin{pmatrix}[0.30,0.40],\\ {[0.40,0.50]}\end{pmatrix}, \begin{pmatrix}[0.10,0.20],\\ {[0.30,0.40]}\end{pmatrix}\right) =$ $\frac{1}{4}(0.2+0.2+0.1+0.1) = 0.15$	$d_{221}\left(\begin{pmatrix}[0.10,0.20],\\ {[0.30,0.40]}\end{pmatrix}, \begin{pmatrix}[0.30,0.40],\\ {[0.40,0.50]}\end{pmatrix}\right) =$ $\frac{1}{4}(0.2+0.2+0.1+0.1) = 0.15$
$d_{122}\left(\begin{pmatrix}[0.30,0.40],\\ {[0.40,0.50]}\end{pmatrix}, \begin{pmatrix}[0.30,0.40],\\ {[0.40,0.50]}\end{pmatrix}\right) =$ $\frac{1}{4}(0+0+0+0) = 0$	$d_{222}\left(\begin{pmatrix}[0.10,0.20],\\ {[0.30,0.40]}\end{pmatrix}, \begin{pmatrix}[0.10,0.20],\\ {[0.30,0.40]}\end{pmatrix}\right) =$ $\frac{1}{4}(0+0+0+0) = 0$

Step 3: Using Step 3 of the Jujhar method, proposed in Section 4.13,

$c_1 = \sum_{i=1}^{2}\sum_{k=1}^{2} d_{i1k}(\tilde{\alpha}_{i1}, \tilde{\alpha}_{k1}) = 0 + 0.15 + 0.15 + 0 = 0.30,$

$c_2 = \sum_{i=1}^{2}\sum_{k=1}^{2} d_{i1k}(\tilde{\alpha}_{i1}, \tilde{\alpha}_{k1}) = 0 + 0.15 + 0.15 + 0 = 0.30.$

Step 4: Using Step 4 of the Jujhar method, proposed in Section 4.13, there is need to solve the CLPP (P6.16).

$max(0.30w_1 + 0.30w_2)$

Subject to

$$\begin{cases} 0.0 \leq w_1 \leq 0.8, \\ 0.1 \leq w_2 \leq 0.9, \\ w_1 + w_2 = 1, \\ w_1 = w_2 \\ w_1 \geq 0, w_2 \geq 0. \end{cases} \tag{P6.16}$$

It can be easily verified that on solving the CLPP (P6.16) the obtained OAWV is $(w_1, w_2) = (0.50, 0.50)$.

Step 5: On considering the OAWV, $(w_1, w_2) = (0.50, 0.50)$ and using Step 5 of the Jujhar method, proposed in Section 4.13,

$\tilde{P}_1 = ([(0.50)(0.10) + (0.50)(0.30), (0.50)(0.20) + (0.50)(0.40)], [(0.50)(0.30) + (0.50)(0.40), (0.50)(0.40) + (0.50)(0.50)]) = ([0.20, 0.30], [0.35, 0.45])$,

$\tilde{P}_2 = ([(0.50)(0.30) + (0.50)(0.10), (0.50)(0.40) + (0.50)(0.20)], [(0.50)(0.40) + (0.50)(0.30), (0.50)(0.50) + (0.50)(0.40)]) = ([0.20, 0.30], [0.35, 0.45])$.

Step 6: Using Step 6 of the Jujhar method, proposed in Section 4.13, $S(\tilde{P}_1) = -0.15$ and $S(\tilde{P}_2) = -0.15$.

Since, $S(\tilde{P}_1) = S(\tilde{P}_2)$. So, there is need to apply Step 7 of the proposed Jujhar method.

Step 7: Using Step 7 of the Jujhar method, proposed in Section 4.13, $H(\tilde{P}_1) = 0.65$ and $H(\tilde{P}_2) = 0.65$.

Since, $H(\tilde{P}_1) = H(\tilde{P}_2)$. So, there is need to apply Step 8 of the proposed Jujhar method.

Step 8: Using Step 8 of the Jujhar method, proposed in Section 4.13, $T(\tilde{P}_1) = 0$ and $T(\tilde{P}_2) = 0$.

Since, $T(\tilde{P}_1) = T(\tilde{P}_2)$. So, there is need to apply Step 9 of the proposed Jujhar method.

Step 9: Using Step 9 of the Jujhar method, proposed in Section 4.13, $G(\tilde{P}_1) = 0.20$ and $G(\tilde{P}_2) = 0.20$.

Since, $G(\tilde{P}_1) = G(\tilde{P}_2)$. Therefore, according to case (iii) of Step 9 of the proposed

Jujhar method, the ranking of the alternatives is $A_1 = A_2$.

6.14.6 Exact results of sixth IVIFMADMPr having IVIF weights

Using the proposed Jujhar method, the exact result of the IVIFMADMPr having IVIF attribute weights, considered in the second point of Section 6.8, can be obtained as follows:

Since, Step 1 to Step 3 of the proposed Jujhar method are same as shown in Section 6.14.5. So, to avoid any repetition, the calculations have been started from Step 4.

Step 4: Since, $\sum_{j=1}^{n} w_j^u < 1$. Therefore, using Case (i) of Step 4 of the Jujhar method, proposed in Section 4.13, there is need to solve the CLPP (P6.17).

$max(0.30w_1 + 0.30w_2)$

Subject to

$$\left\{\begin{array}{c} 0.0 \leq w_1 \leq 0.20, \\ 0.10 \leq w_1 \leq 0.50, \\ w_1 + w_2 \leq 1, \\ w_1 = w_2 \\ w_1 \geq 0, w_2 \geq 0. \end{array}\right. \qquad \text{(P6.17)}$$

It can be easily verified that on solving the CLPP (P6.17) the obtained non-normalized OAWV is $(w_1, w_2) = (0.20,0.20)$. So, the normalized OAWV is $(w_1, w_2) = \left(\frac{0.20}{0.20+0.20}, \frac{0.20}{0.20+0.20}\right) = (0.50,0.50)$.

Step 5: On considering the OAWV, $(w_1, w_2) = (0.50,0.50)$ and using Step 5 of the Jujhar method, proposed in Section 4.13,

$\tilde{P}_1 = ([(0.50)(0.10) + (0.50)(0.30), (0.50)(0.20) + (0.50)(0.40)], [(0.50)(0.30) + (0.50)(0.40), (0.50)(0.40) + (0.50)(0.50)]) = ([0.20,0.30], [0.35, 0.45])$,

$\tilde{P}_2 = ([(0.50)(0.30) + (0.50)(0.10), (0.50)(0.40) + (0.50)(0.20)], [(0.50)(0.40) + (0.50)(0.30), (0.50)(0.50) + (0.50)(0.40)]) = ([0.20,0.30], [0.35,0.45])$.

Step 6: Using Step 6 of the Jujhar method, proposed in Section 4.13, $S(\tilde{P}_1) = -0.15$ and $S(\tilde{P}_2) = -0.15$.

Since, $S(\tilde{P}_1) = S(\tilde{P}_2)$. So, there is need to apply Step 7 of the proposed Jujhar method.

Step 7: Using Step 7 of the Jujhar method, proposed in Section 4.13, $H(\tilde{P}_1) = 0.65$ and $H(\tilde{P}_2) = 0.65$.

Since, $H(\tilde{P}_1) = H(\tilde{P}_2)$. So, there is need to apply Step 8 of the proposed Jujhar method.

Step 8: Using Step 8 of the Jujhar method, proposed in Section 4.13, $T(\tilde{P}_1) = 0$ and $T(\tilde{P}_2) = 0$.

Since, $T(\tilde{P}_1) = T(\tilde{P}_2)$. So, there is need to apply Step 9 of the proposed Jujhar method.

Step 9: Using Step 9 of the Jujhar method, proposed in Section 4.13, $G(\tilde{P}_1) = 0.20$ and $G(\tilde{P}_2) = 0.20$.

Since, $G(\tilde{P}_1) = G(\tilde{P}_2)$. Therefore, according to case (iii) of Step 9 of the proposed Jujhar method, the ranking of the alternatives is $A_1 = A_2$.

6.14.7 Exact results of the seventh IVIFMADMPr having IVIF weights

Using the proposed Jujhar method, the exact result of the IVIFMADMPr having IVIF attribute weights, considered in the third point of Section 6.8, can be obtained as follows:

Since, Step 1 to Step 3 of the proposed Jujhar method are same as shown in Section 6.14.5. So, to avoid any repetition, the calculations have been started from Step 4.

Step 4: Since, $\sum_{j=1}^{n} w_j^l > 1$. Therefore, using Case (ii) of Step 4 of the Jujhar method, proposed in Section 4.13, there is need to solve the CLPP (P6.18).

$max(0.30w_1 - 0.30w_2 - \varepsilon_1 - \varepsilon_2)$

Subject to

$$\begin{cases} \frac{60}{120} \le w_1 \le \frac{80}{120}, \\ \frac{60}{120} \le w_2 \le \frac{80}{120}, \\ w_1 + w_2 = 1, \\ w_1 = w_2, \\ 120 = 100 + \varepsilon_1 + \varepsilon_2 \\ 0 \le \varepsilon_1 \le 1, \\ 0 \le \varepsilon_2 \le 1, \\ w_1 \ge 0, w_2 \ge 0, \varepsilon_1 \ge 0, \varepsilon_2 \ge 0. \end{cases} \qquad \text{(P6.18)}$$

It can be easily verified that on solving the CLPP (P6.18) the obtained OAWV is $(w_1, w_2) = (0.50, 0.50)$.

Step 5: On considering the OAWV, $(w_1, w_2) = (0.50, 0.50)$ and using Step 5 of the Jujhar method, proposed in Section 4.13,

$\tilde{P}_1 = ([(0.50)(0.10) + (0.50)(0.30), (0.50)(0.20) + (0.50)(0.40)], [(0.50)(0.30) + (0.50)(0.40), (0.50)(0.40) + (0.50)(0.50)]) = ([0.20, 0.30], [0.35, 0.45])$,

$\tilde{P}_2 = ([(0.50)(0.30) + (0.50)(0.10), (0.50)(0.40) + (0.50)(0.20)], [(0.50)(0.40) + (0.50)(0.30), (0.50)(0.50) + (0.50)(0.40)]) = ([0.20, 0.30], [0.35, 0.45])$.

Step 6: Using Step 6 of the Jujhar method, proposed in Section 4.13, $S(\tilde{P}_1) = -0.15$ and $S(\tilde{P}_2) = -0.15$.

Since, $S(\tilde{P}_1) = S(\tilde{P}_2)$. So, there is need to apply Step 7 of the proposed Jujhar method.

Step 7: Using Step 7 of the Jujhar method, proposed in Section 4.13, $H(\tilde{P}_1) = 0.65$ and $H(\tilde{P}_2) = 0.65$.

Since, $H(\tilde{P}_1) = H(\tilde{P}_2)$. So, there is need to apply Step 8 of the proposed Jujhar method.

Step 8: Using Step 8 of the Jujhar method, proposed in Section 4.13, $T(\tilde{P}_1) = 0$ and $T(\tilde{P}_2) = 0$.

Since, $T(\tilde{P}_1) = T(\tilde{P}_2)$. So, there is need to apply Step 9 of the proposed Jujhar method.

Step 9: Using Step 9 of the Jujhar method, proposed in Section 4.13, $G(\tilde{P}_1) = 0.20$ and $G(\tilde{P}_2) = 0.20$.

Since, $G(\tilde{P}_1) = G(\tilde{P}_2)$. Therefore, according to case (iii) of Step 9 of the proposed Jujhar method, the ranking of the alternatives is $A_1 = A_2$.

6.14.8 Exact results of eighth IVIFMADMPr having IVIF weights

Using the proposed Jujhar method, the exact result of the IVIFMADMPr having IVIF attribute weights, considered in the fourth point of Section 6.8, can be obtained as follows:

Step 1: Since, all the attributes are benefit type. So, according to Step 1 of the Jujhar method, proposed in Section 4.13 of Chapter 4, there is no need to apply Step 1.

Step 2: According to Step 2 of the Jujhar method, proposed in Section 4.13, there is need to calculate $d_{ijk}(\tilde{\alpha}_{ij},\tilde{\alpha}_{kj})\ \forall\ i = 1,2; j = 1,2$. These values are shown in Table 6.12.

Table 6.12: Values of $d_{ijk}(\tilde{\alpha}_{ij},\tilde{\alpha}_{kj})$

$d_{111}\left(\begin{pmatrix}[0.10,0.30],\\ [0.20,0.40]\end{pmatrix},\begin{pmatrix}[0.10,0.30],\\ [0.20,0.40]\end{pmatrix}\right)=$ $\frac{1}{4}(0+0+0+0)=0$	$d_{211}\left(\begin{pmatrix}[0.10,0.50],\\ [0.20,0.40]\end{pmatrix},\begin{pmatrix}[0.10,0.50],\\ [0.20,0.40]\end{pmatrix}\right)=$ $\frac{1}{4}(0+0+0+0)=0$
$d_{112}\left(\begin{pmatrix}[0.10,0.30],\\ [0.20,0.40]\end{pmatrix},\begin{pmatrix}[0.10,0.30],\\ [0.10,0.50]\end{pmatrix}\right)=$ $\frac{1}{4}(0+0+0.1+0.1)=0.05$	$d_{212}\left(\begin{pmatrix}[0.10,0.50],\\ [0.20,0.40]\end{pmatrix},\begin{pmatrix}[0.10,0.50],\\ [0.20,0.40]\end{pmatrix}\right)=$ $\frac{1}{4}(0+0+0+0)=0$
$d_{121}\left(\begin{pmatrix}[0.02,0.04],\\ [0.01,0.05]\end{pmatrix},\begin{pmatrix}[0.10,0.30],\\ [0.20,0.40]\end{pmatrix}\right)=$ $\frac{1}{4}(0+0+0.1+0.1)=0.05$	$d_{221}\left(\begin{pmatrix}[0.10,0.50],\\ [0.20,0.40]\end{pmatrix},\begin{pmatrix}[0.10,0.50],\\ [0.20,0.40]\end{pmatrix}\right)=$ $\frac{1}{4}(0+0+0+0)=0$
$d_{122}\left(\begin{pmatrix}[0.02,0.04],\\ [0.01,0.05]\end{pmatrix},\begin{pmatrix}[0.10,0.30],\\ [0.10,0.50]\end{pmatrix}\right)=$ $\frac{1}{4}(0+0+0+0)=0$	$d_{222}\left(\begin{pmatrix}[0.10,0.50],\\ [0.20,0.40]\end{pmatrix},\begin{pmatrix}[0.10,0.50],\\ [0.20,0.40]\end{pmatrix}\right)=$ $\frac{1}{4}(0+0+0+0)=0$

Step 3: Using Step 3 of the Jujhar method, proposed in Section 4.13,

$c_1 = \sum_{i=1}^{2}\sum_{k=1}^{2} d_{i1k}(\tilde{\alpha}_{i1},\tilde{\alpha}_{k1}) = 0 + 0.05 + 0.05 + 0 = 0.10,$

$c_2 = \sum_{i=1}^{2}\sum_{k=1}^{2} d_{i1k}(\tilde{\alpha}_{i1},\tilde{\alpha}_{k1}) = 0+0+0+0 = 0.$

Step 4: Using Step 4 of the Jujhar method, proposed in Section 4.13, there is need to solve the CLPP (P6.19).

$max(0.10w_1)$

Subject to

$$\begin{cases} 0 \le w_1 \le 1, \\ 0 \le w_2 \le 1, \\ w_1 + w_2 = 1, \\ w_1 \ge 0, w_2 \ge 0. \end{cases} \tag{P6.19}$$

It can be easily verified that on solving the CLPP (P6.19) the obtained OAWV is $(w_1,w_2) = (1,0)$.

Step 5: On considering the OAWV, $(w_1,w_2) = (1,0)$ and using Step 5 of the Jujhar method, proposed in Section 4.13,

$\tilde{P}_1 = ([(1)(0.10)+(0)(0.10),(1)(0.30)+(0)(0.50)],[(1)(0.20)+(0)(0.20),(1)(0.40)$
$+(0)(0.40)]) = ([0.10,0.30],[0.20,0.40]),$

$\tilde{P}_2 = ([(1)(0.10)+(0)(0.10),(1)(0.30)+(0)(0.50)],[(1)(0.10)+(0)(0.20),(1)(0.50)$
$+(0)(0.40)]) = ([0.10,0.30],[0.10,0.50]).$

Step 6: Using Step 6 of the Jujhar method, proposed in Section 4.13, $S(\tilde{P}_1) = -0.10$ and $S(\tilde{P}_2) = -0.10$.

Since, $S(\tilde{P}_1) = S(\tilde{P}_2)$. So, there is need to apply Step 7 of the proposed Jujhar method.

Step 7: Using Step 7 of the Jujhar method, proposed in Section 4.13, $H(\tilde{P}_1) = 0.50$ and $H(\tilde{P}_2) = 0.50$.

Since, $H(\tilde{P}_1) = H(\tilde{P}_2)$. So, there is need to apply Step 8 of the proposed Jujhar method.

Step 8: Using Step 8 of the Jujhar method, proposed in Section 4.13, $T(\tilde{P}_1) = 0$ and $T(\tilde{P}_2) = -0.20$.

Since, $T(\tilde{P}_1) > T(\tilde{P}_2)$. Therefore, according to Step 8 of the proposed Jujhar method, the ranking of the alternatives is $A_1 > A_2$.

6.15 Conclusions

The inappropriateness of the existing methods [42, 43, 47, 48, 211] are pointed out. Also, the proposed Jujhar method is used to solve the IVIFMADMPrs having completely known IVIF attribute weights with some modifications. Furthermore, the proposed Jujhar method has been illustrated with the help of some IVIFMADMPrs having IVIF attribute weights.

Chapter 7

Discussion on "Evolving a linear programming technique for MAGDM problems with IVIF information"[6]

Hajiagha et al. [98] proposed a linear programming technique for solving IVIF MAGDM problems. In this technique, firstly an IVIFDM $\widetilde{D} = \left(\left[a_{ij1}, a_{ij2}\right], \left[a_{ij3}, a_{ij4}\right]\right)_{m\times n}$ is obtained by aggregating the IVIFDMs $\widetilde{D}^k = \left(\left[a_{ij1}^k, a_{ij2}^k\right], \left[a_{ij3}^k, a_{ij4}^k\right]\right)_{m\times n}, k = 1,2,\dots,r,$ where, $\left(\left[a_{ij1}^k, a_{ij2}^k\right], \left[a_{ij3}^k, a_{ij4}^k\right]\right)_{m\times n}$ represents the rating value of the i^{th}- alternative over the j^{th}-attribute provided by the k^{th}-decision-maker. Then, the IVIF LPPs $(Ps); s = 1,2,\dots,m$ are constructed.

$$Max\left(\sum_{j=1}^{n}\left(\left[a_{sj1}, a_{sj2}\right], \left[a_{sj3}, a_{sj4}\right]\right) \times w_j\right)$$

Subject to

$$\begin{aligned} &\sum_{j=1}^{n}\left(\left[a_{ij1}, a_{ij2}\right], \left[a_{ij3}, a_{ij4}\right]\right) \times w_j \le \tilde{1}; i = 1,2,\dots,m,\\ &\sum_{j=1}^{n} w_j = 1,\\ &w_j \ge 0; j = 1,2,\dots,n. \end{aligned} \qquad (Ps)$$

where, $\tilde{1} = ([0.90,0.95],[0.01,0.05])$ and w_j represents the normalized weight of the j^{th} attribute.

After that each IVIF LPP $(Ps), s = 1,2,\dots,m$ is transformed into its equivalent CLPP $(P's), s = 1,2,\dots,m.$

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$$Min\left(\sum_{j=1}^{n} y_j\, Ln\left((1-a_{sj1})(1-a_{sj2})a_{sj3}a_{sj4}\right)\right)$$

Subject to

$$\begin{aligned}
&\sum_{j=1}^{n} y_j\, Ln(1-a_{ij1}) \geq t\, Ln(1-0.90),\\
&\sum_{j=1}^{n} y_j\, Ln(1-a_{ij2}) \geq t\, Ln(1-0.95),\\
&\quad\sum_{j=1}^{n} y_j\, Ln(a_{ij3}) \geq t\, Ln(0.01),\\
&\quad\sum_{j=1}^{n} y_j\, Ln(a_{ij4}) \geq t\, Ln(0.05),\\
&\qquad\sum_{j=1}^{n} y_j = 1,\\
&\qquad t, y_j \geq 0; j = 1,2,\dots,n.
\end{aligned} \qquad (P's)$$

Finally, the optimal value $\sum_{j=1}^{n} y_j\, Ln\left((1-a_{sj1})(1-a_{sj2})a_{sj3}a_{sj4}\right); s = 1,2,\dots,m$ of each CLPP $(P's)$, $s = 1,2,\dots,m$ is used to rank the alternatives.

The aim of this chapter is

(i) To point out the inappropriateness of the existing method [98] for solving IVIFMAGDM problems having completely unknown weights.

(ii) To make the researchers aware that it is inappropriate to propose a simplex method for solving the IVIF LPP (Ps) without transforming it into the CLPP $(P's)$, inspired by Khan et al.'s simplex method [122], for resolving the inappropriateness of the existing method [98] due to the following facts:

"*Khan et al.'s claim [123] "No mathematical incorrect assumptions have been considered in the existing simplex method [122] for solving a fuzzy LPP without transforming it into a CLPP" against Bhardwaj and Kumar's claim [16] is not valid. In fact, several mathematical incorrect assumptions, considered by Khan et al. [122], are not pointed out by Bhardwaj and Kumar [16] e.g., Bhardwaj and Kumar have not pointed out that some of the elements of the existing optimal simplex table [122] are not TFNs.*"

Furthermore, to make the researchers aware that it is not possible to resolve the inappropriateness of the existing methods [98]."

7.1 Inappropriateness of Hajiagha et al.'s method

The following example clearly indicates that it is inappropriate to find the solution of IVIF LPPs (Ps) with the help of the transformed CLPPs $(P's)$.

If it is assumed that the $(i,j)^{th}$ element of Table 7.1, represented by an IVIFS $\tilde{\alpha}_{ij} = \left([a_{ij1}, a_{ij2}], [a_{ij3}, a_{ij4}]\right)$, represents the rating value of the i^{th}-alternative over the j^{th}-benefit attribute.

Table 7.1: Rating Values

Attributes→ ↓ Alternatives	G_1	G_2
A_1	$\left(\left[\frac{1}{10}, \frac{2}{10}\right], \left[\frac{3}{10}, \frac{4}{10}\right]\right)$	$\left(\left[\frac{1}{5}, \frac{1}{3}\right], \left[\frac{1}{5}, \frac{1}{3}\right]\right)$
A_2	$\left(\left[\frac{3}{20}, \frac{13}{85}\right], \left[\frac{1}{5}, \frac{3}{5}\right]\right)$	$\left(\left[\frac{1}{5}, \frac{1}{3}\right], \left[\frac{1}{5}, \frac{1}{3}\right]\right)$

It is obvious that the rating value of both the alternatives A_1 and A_2 over the attribute G_2 is same i.e., $\left(\left[\frac{1}{5}, \frac{1}{3}\right], \left[\frac{1}{5}, \frac{1}{3}\right]\right)$. Therefore, the ranking of the alternatives A_1 and A_2 will depend only upon the rating values of A_1 and A_2 over the attribute G_1. i.e., if the rating value of the alternative A_1 over G_1 i.e., $\left(\left[\frac{1}{10}, \frac{2}{10}\right], \left[\frac{3}{10}, \frac{4}{10}\right]\right)$ will be greater than the rating value of the alternative A_2 over the same attribute G_1 i.e., $\left(\left[\frac{3}{20}, \frac{13}{85}\right], \left[\frac{1}{5}, \frac{3}{5}\right]\right)$. Then the relation will be $A_1 > A_2$, otherwise, $A_1 < A_2$.

It is pertinent to mention that as $\left(\left[\frac{1}{10}, \frac{2}{10}\right], \left[\frac{3}{10}, \frac{4}{10}\right]\right) \neq \left(\left[\frac{3}{20}, \frac{13}{85}\right], \left[\frac{1}{5}, \frac{3}{5}\right]\right)$. Therefore, the relation $A_1 = A_2$ is not possible.

In this section, this problem is solved by the existing linear programming technique [98] and shown that the obtained relation is $A_1 = A_2$, which is incorrect.

Using the linear programming technique, proposed by Hajiagha et al. [98], the ranking of the alternatives A_1 and A_2 can be obtained as follows:

Since, there are two alternatives and each alternative has two attributes i.e., $m = 2$ and $n = 2$. Therefore, corresponding to the CLPPs (P_s'), the CLPPs (P_1') and (P_2') will be obtained.

$$Min\left(\sum_{j=1}^{2} y_j\, Ln\left((1-a_{1j})(1-b_{1j})c_{1j}d_{1j}\right)\right)$$

Subject to

$$\begin{aligned}
&\textstyle\sum_{j=1}^{2} y_j\, Ln(1-a_{1j}) \geq t\, Ln(1-b_1),\\
&\textstyle\sum_{j=1}^{2} y_j\, Ln(1-a_{2j}) \geq t\, Ln(1-b_1),\\
&\textstyle\sum_{j=1}^{2} y_j\, Ln(1-b_{1j}) \geq t\, Ln(1-b_2),\\
&\textstyle\sum_{j=1}^{2} y_j\, Ln(1-b_{2j}) \geq t\, Ln(1-b_2),\\
&\qquad\textstyle\sum_{j=1}^{2} y_j\, Ln(c_{1j}) \geq t\, Ln(b_3)\,,\\
&\qquad\textstyle\sum_{j=1}^{2} y_j\, Ln(c_{2j}) \geq t\, Ln(b_3),\\
&\qquad\textstyle\sum_{j=1}^{2} y_j\, Ln(d_{1j}) \geq t\, Ln(b_4)\,,\\
&\qquad\textstyle\sum_{j=1}^{2} y_j\, Ln(d_{2j}) \geq t\, Ln(b_4),\\
&\qquad\qquad\textstyle\sum_{j=1}^{2} y_j = 1,\\
&\qquad\quad t, y_j \geq 0; j = 1,2.
\end{aligned} \qquad (P_1')$$

and

$$Min\left(\sum_{j=1}^{2} y_j\, Ln\left((1-a_{2j})(1-b_{2j})c_{2j}d_{2j}\right)\right)$$

Subject to

Constraints of the CLPP (P_1'). (P_2')

Furthermore, as in the considered IVIFDM, $a_{11} = \frac{1}{10}$, $b_{11} = \frac{2}{10}$, $c_{11} = \frac{3}{10}$, $d_{11} = \frac{4}{10}$, $a_{12} = \frac{1}{5}$, $b_{12} = \frac{1}{3}$, $c_{12} = \frac{1}{5}$, $d_{12} = \frac{1}{3}$, $a_{21} = \frac{3}{20}$, $b_{21} = \frac{13}{85}$, $c_{21} = \frac{1}{5}$, $d_{21} = \frac{3}{5}$, $a_{22} = \frac{1}{5}$, $b_{22} = \frac{1}{3}$, $c_{22} = \frac{1}{5}$, $d_{22} = \frac{1}{3}$. Therefore, on putting these values in (P_1') and (P_2'), these problems will be transformed into the CLPPs (P7.1) and (P7.2) respectively.

$$Min(-2.4487y_1 - 3.3382y_2)$$

Subject to

$$\begin{aligned}
&-0.1053y_1 - 0.2231y_2 \geq t\,Ln(1-b_1),\\
&-0.1625y_1 - 0.2231y_2 \geq t\,Ln(1-b_1),\\
&-0.2231y_1 - 0.4054y_2 \geq t\,Ln(1-b_2),\\
&-0.1660y_1 - 0.4054y_2 \geq t\,Ln(1-b_2),\\
&\quad -1.2039y_1 - 1.6094y_2 \geq t\,Ln(b_3),\\
&\quad -1.6094y_1 - 1.6094y_2 \geq t\,Ln(b_3),\\
&\quad -0.9162y_1 - 1.0986y_2 \geq t\,Ln(b_4)\,,\\
&\quad -0.5108y_1 - 1.0986y_2 \geq t\,Ln(b_4),\\
&\qquad\qquad \textstyle\sum_{j=1}^{2} y_j = 1,\\
&\qquad\qquad t, y_j \geq 0; j = 1,2.
\end{aligned} \tag{P7.1}$$

and

$Min(-2.4487y_1 - 3.3382y_2)$

Subject to

Constraints of the CLPP (P7.1) (P7.2)

It is obvious that both the problems (P7.1) and (P7.2) are same. Therefore, the optimal values of both these CLPPs will also be same and hence, according to the existing linear programming technique [98], $A_1 = A_2$, which is mathematically incorrect.

7.2 Challenges for resolving the inappropriateness of Hajiagha et al.'s method

Since, most of the researchers are well aware about the concept of simplex method. So, to resolve the inappropriateness of Hajiagha et al. method [98], inspired by the existing simplex method [122] for solving a FLPP without transforming into a crisp LPP, one may think to develop a simplex method for solving IVIF LPP (P_s) without transforming it into the CLPP (P'_s). However, after a deep study the following facts have been observed.

(i) Khan et al.'s claim [123] "No mathematical incorrect assumptions have been considered in the existing simplex method [122] for solving a fuzzy LPP without transforming it into a CLPP" against Bhardwaj and Kumar's claim [16] is not valid. In fact, several mathematical incorrect assumptions, considered by Khan et al. [122], are not pointed out by Bhardwaj and Kumar [16] e.g., Bhardwaj and Kumar have not pointed out that some of the elements of the existing optimal simplex table [122] are not TFNs.

(ii) It is not possible to propose a simplex method for solving a FLPP without transforming it into a CLPP.

(iii) It is not possible to propose a simplex method for solving an IVIF LPP (P_s) without transforming it into a CLPP.

The aim of this section is to show that these observations are valid.

7.2.1 Validity of the first observation

In this section, the inappropriateness of Khan et al.'s method [122] have been explained in detailed manner.

7.2.1.1 A brief review about Khan et al.'s work [122]

Khan et al. [122] proposed a method for solving FFLPPs. All the steps of the method, proposed by Khan et al. [122], are same as well-known simplex method for solving a CLPP. The only difference between the well-known simplex method for solving a CLPP and the method, proposed by Khan et al. [122], are as follows.

(i) Khan et al. [122] have represented the elements and variables in the simplex table as TFNs instead of real numbers.

(ii) To check the optimality as well as to find the entering variable, there is need to check that a TFN is a non-negative TFN or a non-positive TFN. In the literature, different methods have been proposed to check that a TFN is a non-negative TFN or a non-positive TFN.

Khan et al. [122] have adopted the following method to check the nature of a TFN, $\tilde{T} = (u, v, w)$.

Find $(\tilde{T}) = u + w - \sigma$, where, $\sigma = \frac{w-u}{6}$.

Case (i): If $R(\tilde{T}) \geq 0$ then $\tilde{T}$ is a non-negative TFN.

Case (ii): If $R(\tilde{T}) \leq 0$ then $\tilde{T}$ is a non-positive TFN.

Furthermore, in the literature, different methods have been proposed to find the minimum of TFNs. Khan et al. [122] have adopted the following method to find the minimum of two TFNs, $\tilde{T}_1 = (u_1, v_1, w_1)$ and $\tilde{T}_2 = (u_2, v_2, w_2)$.

Check that $R(\tilde{T}_1) > R(\tilde{T}_2)$ or $R(\tilde{T}_1) < R(\tilde{T}_2)$ or $R(\tilde{T}_1) = R(\tilde{T}_2)$ where, $R(\tilde{T}_i) = u_i + w_i - \sigma_i$, $\sigma_i = \frac{w_i - u_i}{6}, i = 1,2$.

Case (i): If $R(\tilde{T}_1) > R(\tilde{T}_2)$ then $\tilde{T}_1 > \tilde{T}_2$ i.e., $minimum\ \{\tilde{T}_1, \tilde{T}_2\} = \tilde{T}_2$.

Case (ii): If $R(\tilde{T}_1) < R(\tilde{T}_2)$ then $\tilde{T}_1 < \tilde{T}_2$ i.e., $minimum\ \{\tilde{T}_1, \tilde{T}_2\} = \tilde{T}_1$.

Case (iii): If $R(\tilde{T}_1) = R(\tilde{T}_2)$ then $\tilde{T}_1 = \tilde{T}_2$ i.e., $minimum\ \{\tilde{T}_1, \tilde{T}_2\} = \tilde{T}_1, \tilde{T}_2$.

Bhardwaj and Kumar [16] claimed that Khan et al. [122] have used the relations

(i) $R\left(\frac{\tilde{T}_1}{\tilde{T}_2}\right) = \frac{R(\tilde{T}_1)}{R(\tilde{T}_2)}$.

(ii) $\tilde{T}_1 - \tilde{T}_1 = (0, 0, 0)$.

(iii) $\frac{\tilde{T}_1}{\tilde{T}_1} = (1, 1, 1)$.

in their proposed method. Whereas, in actual case,

(i) $R\left(\frac{\tilde{T}_1}{\tilde{T}_2}\right) \neq \frac{R(\tilde{T}_1)}{R(\tilde{T}_2)}$.

(ii) $\tilde{T}_1 - \tilde{T}_1 \neq (0, 0, 0)$.

(iii) $\frac{\tilde{T}_1}{\tilde{T}_1} \neq (1, 1, 1)$.

Therefore, the method, proposed by Khan et al. [122], is not valid.

Khan et al. [123] replied that they have not used any of these relations in their proposed method [ ]. Hence, Khan et al. [122] method is valid.

7.2.1.2 Validity of the claim

The following clearly indicates that Khan et al. [122] have considered mathematical incorrect assumptions in their proposed method.

(i) Khan et al. [122] have defined as well as used the following multiplication operation of two TFNs.

If $\tilde{T}_1 = (u_1, v_1, w_1)$ and $\tilde{T}_2 = (u_2, v_2, w_2)$ are two TFNs. Then, $\tilde{T}_1 . \tilde{T}_2 =$

$(u_1, v_1, w_1).(u_2, v_2, w_2) = (u_1u_2, v_1v_2, w_1w_2)$.

Khan et al. [122] have assumed that this multiplication operation is valid for all TFNs. However, in actual case, this multiplication is valid only if $u_1 \geq 0$ and $u_2 \geq 0$.

For example, if $\tilde{T}_1 = (u_1, v_1, w_1) = (-2, -1, 3)$ and $\tilde{T}_2 = (u_2, v_2, w_2) = (1, 4, 5)$ i.e., $u_1 \ngeq 0$ and $u_2 \geq 0$ then according to this multiplication

$$\tilde{T}_1.\tilde{T}_2 = (u_1u_2, v_1v_2, w_1w_2) = (-2, -4, 15)$$

It is pertinent to mention that the obtained number $(-2, -4, 15)$ is not a TFN as for a TFN (u, v, w), the condition $u \leq v \leq w$ should always be satisfied.

(ii) Khan et al. [122] have defined as well as used the following division operation of two TFNs. If $\tilde{T}_1 = (u_1, v_1, w_1)$ and $\tilde{T}_2 = (u_2, v_2, w_2)$ are two TFNs. Then, $\frac{\tilde{T}_1}{\tilde{T}_2} = \frac{(u_1, v_1, w_1)}{(u_2, v_2, w_2)} = \left(\frac{u_1}{w_2}, \frac{v_1}{v_2}, \frac{w_1}{u_2}\right)$.

Khan et al. [122] have assumed that this division operation is valid for all TFNs. However, in actual case, this division is valid only if $u_1 \geq 0$ and $u_2 > 0$.

For example, if $\tilde{T}_1 = (u_1, v_1, w_1) = (-3, -2, -1)$ and $\tilde{T}_2 = (u_2, v_2, w_2) = (1, 4, 5)$ i.e., $u_1 \ngeq 0$ and $u_2 > 0$ then according to this division, $\frac{\tilde{T}_1}{\tilde{T}_2} = \left(\frac{u_1}{w_2}, \frac{v_1}{v_2}, \frac{w_1}{u_2}\right) = \left(-\frac{3}{5}, -\frac{1}{2}, -1\right)$.

It is pertinent to mention that the obtained number $\left(-\frac{3}{5}, -\frac{1}{2}, -1\right)$ is not a TFN as for a TFN (u, v, w), the condition $u \leq v \leq w$ should always be satisfied.

(iii)The following clearly indicates that Khan et al. [122] have used the mathematical incorrect assumptions (i) and (ii) in their proposed method.

Khan et al. [122] solved the FLPP (P7.3) by their proposed method and claimed that Table 7.2 represents the optimal table of the FLPP (P7.3).

$Max\ \tilde{z} = (2, 5, 8)\tilde{x}_1 + \left(3, \frac{37}{6}, 10\right)\tilde{x}_2 + \left(5, \frac{34}{3}, 15\right)\tilde{x}_3$

Subject to

$$\begin{cases} (2,5,8)\tilde{x}_1 + \left(3,\frac{41}{6},10\right)\tilde{x}_2 + \left(5,\frac{31}{3},18\right)\tilde{x}_3 \leq \left(6,\frac{50}{3},30\right) \\ \left(4,\frac{32}{3},12\right)\tilde{x}_1 + \left(5,\frac{73}{6},20\right)\tilde{x}_2 + \left(7,\frac{105}{6},30\right)\tilde{x}_3 \leq (10,30,50) \\ (3,5,7)\tilde{x}_1 + (5,15,20)\tilde{x}_2 + (5,10,15)\tilde{x}_3 \leq \left(2,\frac{145}{6},30\right) \\ \tilde{x}_1,\tilde{x}_2,\tilde{x}_3 \geq 0 \end{cases} \qquad \text{(P7.3)}$$

Table 7.2: Optimal table of FLPP (P7.3)

	$\tilde{x}_1$	$\tilde{x}_2$	$\tilde{x}_3$	$\tilde{x}_4$	$\tilde{x}_5$	$\tilde{x}_6$	$\tilde{b}$
$\tilde{x}_4$	$\left(-\frac{8}{5},-\frac{1}{6},1\right)$	$\boldsymbol{\left(-3,-\frac{281}{36},-10\right)}$	$(0,0,0)$	$(1,1,1)$	$(0,0,0)$	$\left(-\frac{6}{5},-\frac{31}{30},-1\right)$	$\left(-12,-\frac{299}{36},0\right)$
$\tilde{x}_5$	$\left(-2,\frac{23}{12},\frac{11}{5}\right)$	$\boldsymbol{\left(-5,\frac{101}{8},-8\right)}$	$(0,0,0)$	$(0,0,0)$	$(1,1,1)$	$\left(-2,-\frac{105}{60},-\frac{7}{5}\right)$	$\left(-20,-\frac{885}{72},8\right)$
$\tilde{x}_3$	$\left(\frac{1}{5},\frac{1}{2},\frac{7}{5}\right)$	$\left(\frac{1}{3},\frac{17}{12},4\right)$	$(1,1,1)$	$(0,0,0)$	$(0,0,0)$	$\left(\frac{1}{15},\frac{1}{10},\frac{1}{5}\right)$	$\left(1,\frac{29}{12},6\right)$
$\tilde{z}$	$\left(-7,\frac{2}{3},19\right)$	$\left(-\frac{25}{3},\frac{4}{9},57\right)$	$(0,0,0)$	$(0,0,0)$	$(0,0,0)$	$\left(\frac{1}{3},\frac{34}{30},3\right)$	$\left(5,\frac{986}{36},90\right)$

It is well known fact that for a TFN, $\tilde{T} = (u,v,w)$, the condition $u \leq v \leq w$ should always be satisfied. While, it is obvious that

(i) For the second element of first row i.e., $\left(-3,-\frac{281}{36},-10\right)$, this condition is not satisfying i.e., $\left(-3,-\frac{281}{36},-10\right)$ is not a TFN.

(ii) For the second element of second row i.e., $\left(-5,\frac{101}{8},-8\right)$, this condition is not satisfying i.e., $\left(-5,\frac{101}{8},-8\right)$ is not a TFN.

7.2.1.3 Exact multiplication and division operations of TFNs

In this section, the exact multiplication and division operations of TFNs are presented [115].

(i) If $\tilde{T}_1 = (u_1,v_1,w_1)$ and $\tilde{T}_2 = (u_2,v_2,w_2)$ are two TFNs such that u_1 and u_2 are any real numbers. Then, $\tilde{T}_1.\tilde{T}_2 = \begin{pmatrix} minimum\{u_1u_2,u_1w_2,w_1u_2,w_1w_2\},v_1v_2, \\ maximium\{u_1u_2,u_1w_2,w_1u_2,w_1w_2\} \end{pmatrix}$.

(ii) If $\tilde{T}_1 = (u_1,v_1,w_1)$ and $\tilde{T}_2 = (u_2,v_2,w_2)$ are two TFNs such that u_1 is any real number and $u_2 > 0$ or $w_2 < 0$. Then, $\frac{\tilde{T}_1}{\tilde{T}_2} = \begin{pmatrix} minimum\left\{\frac{u_1}{u_2}, \frac{u_1}{w_2}, \frac{w_1}{u_2}, \frac{w_1}{w_2}\right\}, \frac{v_1}{v_2}, \\ maximium\left\{\frac{u_1}{u_2}, \frac{u_1}{w_2}, \frac{w_1}{u_2}, \frac{w_1}{w_2}\right\} \end{pmatrix}$.

7.2.1.4 Fuzzy optimal solution of a FLPP by Khan et al.'s method

To point out the mathematical incorrect assumptions, considered by Khan et al. [122], there is need to solve the FLPP (P7.3). Therefore, in this section, the FFLPP (P7.3), considered by Khan et al. [122] to illustrate their proposed method, is solved by Khan et al.'s method [122] in a very detailed manner.

Khan et al. [122] have used the following steps to find the optimal table, represented by Table 7.2, of the FLPP (P7.3).

Step 1: Table 7.3 [122] represents the initial simplex table of the FLPP (P7.3).

Table 7.3: Initial simplex table of the FLPP (P7.3) [122]

	$\tilde{x}_1$	$\tilde{x}_2$	$\tilde{x}_3$	$\tilde{x}_4$	$\tilde{x}_5$	$\tilde{x}_6$	$\tilde{b}$
$\tilde{x}_4$	$(2, 5, 8)$	$\left(3, \frac{41}{6}, 10\right)$	$\left(5, \frac{31}{3}, 18\right)$	$(1, 1, 1)$	$(0, 0, 0)$	$(0, 0, 0)$	$\left(6, \frac{50}{3}, 30\right)$
$\tilde{x}_5$	$\left(4, \frac{32}{3}, 12\right)$	$\left(5, \frac{73}{6}, 20\right)$	$\left(7, \frac{105}{6}, 30\right)$	$(0, 0, 0)$	$(1, 1, 1)$	$(0, 0, 0)$	$(10, 30, 50)$
$\tilde{x}_6$	$(3, 5, 7)$	$(5, 15, 20)$	$(5, 10, 15)$	$(0, 0, 0)$	$(0, 0, 0)$	$(1, 1, 1)$	$\left(2, \frac{145}{6}, 30\right)$
$\tilde{z}$	$(-8, -5, -2)$	$\left(-10, -\frac{37}{6}, -3\right)$	$\left(-15, -\frac{34}{3}, -5\right)$	$(0, 0, 0)$	$(0, 0, 0)$	$(0, 0, 0)$	$(0, 0, 0)$

Step 2: Since the considered problem is of maximization, so the initial fuzzy basic feasible solution, presented in Table 7.3, will be a fuzzy optimal solution if all the values of $\tilde{z}$, corresponding to non-basic variables, will be non-negative TFNs i.e., the TFNs $(-8, -5, -2)$, $\left(-10, -\frac{37}{6}, -3\right)$ and $\left(-15, -\frac{34}{3}, -5\right)$, representing the values of $\tilde{z}$ corresponding to non-basic fuzzy variables $\tilde{x}_1, \tilde{x}_2$ and $\tilde{x}_3$ respectively, will be a non-negative TFNs. As discussed in Section 7.2.1.1, Khan et al. [4] have assumed that a TFN $\tilde{T} = (u, v, w)$ will be non-negative TFN if $R(\tilde{T}) \geq 0$, where $R(\tilde{T}) = u + w - \sigma$, $\sigma = \frac{w-u}{6}$.

Since, $R(-8,-5,-2) = -11$, $R\left(-10,-\frac{37}{6},-3\right) = -14.16$ and $R\left(-15,-\frac{34}{3},-5\right) = -21.66$ i.e., $(-8,-5,-2)$, $\left(-10,-\frac{37}{6},-3\right)$ and $\left(-15,-\frac{34}{3},-5\right)$ are not non-negative TFNs and hence, the optimality condition is not satisfying. So, the initial fuzzy basic feasible solution, presented in Table 7.3, is not a fuzzy optimal solution.

Step 3: Since, the initial fuzzy basic feasible solution, presented in Table 7.3, is not a fuzzy optimal solution, so the fuzzy non-basic variable $\tilde{x}_3$, corresponding to which the value of $R(\tilde{z})$ is most negative, will enter into the basis.

Step 4: Since, all the elements of the column corresponding to the entering variable $\tilde{x}_3$ are non-negative TFNs i.e., $R\left(5,\frac{31}{3},18\right) > 0$, $R\left(7,\frac{105}{6},30\right) > 0$, $R(5,10,15) > 0$. So, any of the fuzzy basic variables $\tilde{x}_4, \tilde{x}_5$ and $\tilde{x}_6$ can leave from the basis. However, according to the minimum ratio rule, out of the fuzzy basic variables $\tilde{x}_i(i = 4,5,6)$ that fuzzy basic variable will leave from the basis corresponding to which $minimum\left\{\frac{b_i}{a_{i3}}, i = 1,2,3\right\}$ will exist i.e., corresponding to which $minimum\left\{\frac{\left(6,\frac{50}{3},30\right)}{\left(5,\frac{31}{3},18\right)}, \frac{(10,30,50)}{\left(7,\frac{105}{6},30\right)}, \frac{\left(2,\frac{145}{6},30\right)}{(5,10,15)}\right\}$ will exist.

Khan et al. [122] have used the following steps to find $minimum\left\{\frac{\left(6,\frac{50}{3},30\right)}{\left(5,\frac{31}{3},18\right)}, \frac{(10,30,50)}{\left(7,\frac{105}{6},30\right)}, \frac{\left(2,\frac{145}{6},30\right)}{(5,10,15)}\right\}$.

Step 4(a): $R\left(6,\frac{50}{3},30\right) = 32, R\left(5,\frac{31}{3},18\right) = 20.83$, $R(10,30,50) = 53.33, R\left(7,\frac{105}{6},30\right) = 33.16$, $R\left(2,\frac{145}{6},30\right) = 27.33$ and $R\ (5,10,15) = 18.33$.

$\Rightarrow \frac{R\left(6,\frac{50}{3},30\right)}{R\left(5,\frac{31}{3},18\right)} = \frac{32}{20.83} = 1.5362$, $\frac{R(10,30,50)}{R\left(7,\frac{105}{6},30\right)} = \frac{53.33}{33.16} = 1.6082$ and $\frac{R\left(2,\frac{145}{6},30\right)}{R(5,10,15)} = \frac{27.33}{18.33} = 1.4909$.

Step 4(b): Since, $minimum\{1.5362, 1.6082, 1.4909\} = 1.4909$, which is corresponding to $\frac{R\left(2,\frac{145}{6},30\right)}{R(5,10,15)}$. Therefore, $minimum\left\{\frac{\left(6,\frac{50}{3},30\right)}{\left(5,\frac{31}{3},18\right)}, \frac{(10,30,50)}{\left(7,\frac{105}{6},30\right)}, \frac{\left(2,\frac{145}{6},30\right)}{(5,10,15)}\right\} = \frac{\left(2,\frac{145}{6},30\right)}{(5,10,15)}$. This indicates that the fuzzy basic variable $\tilde{x}_6$ will leave from the basis.

Step 5: Since, the fuzzy non-basic variable $\tilde{x}_3$ is entering variable (third column of Table 7.3) and the fuzzy basic variable $\tilde{x}_6$ is the leaving variable (third row of Table 7.3). So, in the next simplex table the third element of the third column should be $(1,1,1)$. To obtain the same, Khan et al. [122] have applied the arithmetic operation $\frac{R_3}{(5,10,15)}$.

Khan et al. [122] claimed that after applying this operation the simplex Table 7.3 will be transformed into the simplex Table 7.4.

Table 7.4: First simplex table of the FLPP (P7.3)

	$\tilde{x}_1$	$\tilde{x}_2$	$\tilde{x}_3$	$\tilde{x}_4$	$\tilde{x}_5$	$\tilde{x}_6$	$\tilde{b}$
$\tilde{x}_4$	$(2,5,8)$	$\left(3,\frac{41}{6},10\right)$	$\left(5,\frac{31}{3},18\right)$	$(1,1,1)$	$(0,0,0)$	$(0,0,0)$	$\left(6,\frac{50}{3},30\right)$
$\tilde{x}_5$	$\left(4,\frac{32}{3},12\right)$	$\left(5,\frac{73}{6},20\right)$	$\left(7,\frac{105}{6},30\right)$	$(0,0,0)$	$(1,1,1)$	$(0,0,0)$	$(10,30,50)$
$\tilde{x}_6$	$\left(\frac{1}{5},\frac{1}{2},\frac{7}{5}\right)$	$\left(\frac{1}{3},\frac{17}{12},4\right)$	$(1,1,1)$	$(0,0,0)$	$(0,0,0)$	$\left(\frac{1}{15},\frac{1}{10},\frac{1}{5}\right)$	$\left(1,\frac{29}{12},6\right)$
$\tilde{z}$	$(-8,-5,-2)$	$\left(-10,-\frac{37}{6},-3\right)$	$\left(-15,-\frac{34}{3},-5\right)$	$(0,0,0)$	$(0,0,0)$	$(0,0,0)$	$(0,0,0)$

Step 6: Since, the remaining elements of the column corresponding to $\tilde{x}_3$ in the next simplex table should be $(0,0,0)$. So, to obtain the same, Khan et al. [122] have used the arithmetic operations

$$R_1+\left(-18,-\frac{31}{3},-5\right)R'_3,\ R_2+\left(-30,-\frac{105}{6},-7\right)R'_3,\ R_4+\left(5,\frac{34}{3},15\right)R'_3.$$

Khan et al. [122] have claimed that after applying these operations, the simplex Table 7.4 will be transformed into the simplex Table 7.5 (same as the simplex Table 7.2).

Table 7.5: Optimal simplex table of the FLPP (P7.3)

	$\tilde{x}_1$	$\tilde{x}_2$	$\tilde{x}_3$	$\tilde{x}_4$	$\tilde{x}_5$	$\tilde{x}_6$	$\tilde{b}$
$\tilde{x}_4$	$\left(-\frac{8}{5},-\frac{1}{6},1\right)$	$\left(-3,-\frac{281}{36},-10\right)$	$(0,0,0)$	$(1,1,1)$	$(0,0,0)$	$\left(-\frac{6}{5},-\frac{31}{30},-1\right)$	$\left(-12,-\frac{299}{36},0\right)$
$\tilde{x}_5$	$\left(-2,\frac{23}{12},\frac{11}{5}\right)$	$\left(-5,\frac{101}{8},-8\right)$	$(0,0,0)$	$(0,0,0)$	$(1,1,1)$	$\left(-2,-\frac{105}{60},-\frac{7}{5}\right)$	$\left(-20,-\frac{885}{72},8\right)$

$\tilde{x}_3$	$\left(\frac{1}{5},\frac{1}{2},\frac{7}{5}\right)$	$\left(\frac{1}{3},\frac{17}{12},4\right)$	$(1,1,1)$	$(0,0,0)$	$(0,0,0)$	$\left(\frac{1}{15},\frac{1}{10},\frac{1}{5}\right)$	$\left(1,\frac{29}{12},6\right)$
$\tilde{z}$	$\left(-7,\frac{2}{3},19\right)$	$\left(-\frac{25}{3},\frac{4}{9},57\right)$	$(0,0,0)$	$(0,0,0)$	$(0,0,0)$	$\left(\frac{1}{3},\frac{34}{30},3\right)$	$\left(5,\frac{986}{36},90\right)$

Step 7: It can be easily verified that in the simplex Table 7.5, $R(\tilde{z})$ corresponding to each non-basic variable is a non-negative real number. So, the fuzzy basic feasible solution, obtained in Table 7.5, is a fuzzy optimal solution and hence, Table 7.5 is optimal simplex table of the FFLPP (P7.3).

7.2.1.5 Mathematical incorrect assumptions considered by Khan et al.

Bhardwaj and Kumar [16] pointed out that the following mathematical incorrect assumptions have been considered in Khan et al. method [122].

(i) Khan et al. [122] have assumed that if $\tilde{A}$ and $\tilde{B}$ are two TFNs then $R\left(\frac{\tilde{A}}{\tilde{B}}\right)=\frac{R(\tilde{A})}{R(\tilde{B})}$. While, in actual case $R\left(\frac{\tilde{A}}{\tilde{B}}\right)\neq\frac{R(\tilde{A})}{R(\tilde{B})}$.

(ii) Khan et al. [122] have assumed that if $\tilde{A}$ is a TFN then $\frac{\tilde{A}}{\tilde{A}}=(1,1,1)$ and $\tilde{A}-\tilde{A}=(0,0,0)$. While, in actual case neither $\frac{\tilde{A}}{\tilde{A}}\neq(1,1,1)$ nor $\tilde{A}-\tilde{A}\neq(0,0,0)$.

Khan et al. [123] replied that this claim of Bhardwaj and Kumar [16] is irrelevant as no such mathematical incorrect assumptions have been used in the existing method [122].

However, the following clearly indicates that the claim of Bhardwaj and Kumar [16] is relevant i.e., the mathematical incorrect assumptions, pointed out by Bhardwaj and Kumar [16], have been used in Khan et al. method [122].

(i) It is obvious from Step 4 of Section 7.2.1.4 that Khan et al. [122] have assumed that to find $minimum\left\{\frac{\left(6,\frac{50}{3},30\right)}{\left(5,\frac{31}{3},18\right)},\frac{(10,30,50)}{\left(7,\frac{105}{6},30\right)},\frac{\left(2,\frac{145}{6},30\right)}{(5,10,15)}\right\}$ is equivalent to find $minimum\left\{\frac{R\left(6,\frac{50}{3},30\right)}{R\left(5,\frac{31}{3},18\right)},\frac{R(10,30,50)}{R\left(7,\frac{105}{6},30\right)},\frac{R\left(2,\frac{145}{6},30\right)}{R(5,10,15)}\right\}$. While, in actual case, to find

$minimum\left\{\frac{\left(6,\frac{50}{3},30\right)}{\left(5,\frac{31}{3},18\right)},\frac{(10,30,50)}{\left(7,\frac{105}{6},30\right)},\frac{\left(2,\frac{145}{6},30\right)}{(5,10,15)}\right\}$ is equivalent to find $minimum\left\{R\left(\frac{\left(6,\frac{50}{3},30\right)}{\left(5,\frac{31}{3},18\right)}\right),R\left(\frac{(10,30,50)}{\left(7,\frac{105}{6},30\right)}\right),R\left(\frac{\left(2,\frac{145}{6},30\right)}{(5,10,15)}\right)\right\}$.

This clearly indicates that Khan et al. [122] have assumed the property $R\left(\frac{\tilde{A}}{\tilde{B}}\right)=\frac{R(\tilde{A})}{R(\tilde{B})}$. Whereas, Bhardwaj and Kumar [16] have shown that $R\left(\frac{\tilde{A}}{\tilde{B}}\right)\neq\frac{R(\tilde{A})}{R(\tilde{B})}$.

(ii) It is obvious from Step 5 of Section 7.2.1.4 that Khan et al. [122] have assumed that on applying the operation $\frac{R_3}{(5,10,15)}$ on the elements of third row of simplex Table 7.3, the elements of third row of the simplex Table 7.4 will be obtained i.e., Khan et al. [122] have assumed that

(a) $\frac{(3,5,7)}{(5,10,15)}=\left(\frac{1}{5},\frac{1}{2},\frac{7}{5}\right)$.

(b) $\frac{(5,15,20)}{(5,10,15)}=\left(\frac{1}{3},\frac{17}{12},4\right)$.

(c) $\frac{(5,10,15)}{(5,10,15)}=(1,1,1)$.

(d) $\frac{(0,0,0)}{(5,10,15)}=(0,0,0)$.

(e) $\frac{(0,0,0)}{(5,10,15)}=(0,0,0)$.

(f) $\frac{(1,1,1)}{(5,10,15)}=\left(\frac{1}{15},\frac{1}{10},\frac{1}{5}\right)$.

(g) $\frac{\left(2,\frac{145}{6},30\right)}{(5,10,15)}=\left(1,\frac{\frac{145}{6}}{10},\frac{30}{5}\right)$.

It is obvious that to find these values there is need to use the division operation of TFNs. As pointed out in (ii) of Section 7.2.1.2, Khan et al. [122] have used the following division operation to obtain these elements.

If $\tilde{T}_1=(u_1,v_1,w_1)$ and $\tilde{T}_2=(u_2,v_2,w_2)$ are two TFNs. Then, $\frac{\tilde{T}_1}{\tilde{T}_2}=\left(\frac{u_1}{w_2},\frac{v_1}{v_2},\frac{w_1}{u_2}\right)$.

On using this division operation $\frac{(3,5,7)}{(5,10,15)}=\left(\frac{1}{5},\frac{1}{2},\frac{7}{5}\right)$, $\frac{(0,0,0)}{(5,10,15)}=(0,0,0)$, $\frac{(0,0,0)}{(5,10,15)}=$

$(0,0,0)$ and $\frac{(1,1,1)}{(5,10,15)} = \left(\frac{1}{15}, \frac{1}{10}, \frac{1}{5}\right)$. But, $\frac{(5,10,15)}{(5,10,15)} = \left(\frac{5}{15}, \frac{10}{10}, \frac{15}{5}\right) = \left(\frac{1}{3}, 1, 3\right) \neq (1,1,1)$.

Whereas, Khan et al. [122] have assumed that $\frac{(5,10,15)}{(5,10,15)} = (1,1,1)$ i.e., to obtain this element, Khan et al. [122] have used the operation $\frac{(u_1,v_1,w_1)}{(u_2,v_2,w_2)} = \left(\frac{u_1}{u_2}, \frac{v_1}{v_2}, \frac{w_1}{w_2}\right)$. However, the following example clearly indicates that this division operation is not valid in general.

If $\tilde{T}_1 = (2,3,4)$ and $\tilde{T}_2 = (5,6,7)$ are two TFNs then according to this division operation $\frac{\tilde{T}_1}{\tilde{T}_2} = \left(\frac{2}{5}, \frac{3}{6}, \frac{4}{7}\right)$, which is not a TFN as the necessary condition of a TFN, $\frac{2}{5} \leq \frac{3}{6} \leq \frac{4}{7}$ is not satisfying.

Furthermore, it is pertinent to mention that although on applying the operation $\frac{(u_1,v_1,w_1)}{(u_2,v_2,w_2)} = \left(\frac{u_1}{w_2}, \frac{v_1}{v_2}, \frac{w_1}{u_2}\right)$, the values of $\frac{(5,15,20)}{(5,10,15)}, \frac{\left(2,\frac{145}{6},30\right)}{(5,10,15)}$ should be $\left(\frac{1}{3}, \frac{3}{2}, 4\right)$, $\left(\frac{2}{15}, \frac{29}{12}, 6\right)$.

While, inadvertently, Khan et al. [122] have mentioned that the values of $\frac{(5,15,20)}{(5,10,15)}, \frac{\left(2,\frac{145}{6},30\right)}{(5,10,15)}$ are $\left(\frac{1}{3}, \frac{17}{12}, 4\right)$ and $\left(1, \frac{29}{12}, 6\right)$ respectively.

(iii) It is obvious from Step 6 of Section 7.2.1.4 that Khan et al. [122] have assumed that on applying the operation $R_1 + \left(-18, -\frac{31}{3}, -5\right) R_3'$ on the elements of first row of Table 7.4, the elements of first row of Table 7.4 will be obtained i.e., Khan et al. [122] have assumed that

(a) $(2,5,8) + \left(-18, -\frac{31}{3}, -5\right)\left(\frac{1}{5}, \frac{1}{2}, \frac{7}{5}\right) = \left(-\frac{8}{5}, -\frac{1}{6}, 1\right)$.

(b) $\left(3, \frac{41}{6}, 10\right) + \left(-18, -\frac{31}{3}, -5\right)\left(\frac{1}{3}, \frac{17}{12}, 4\right) = \left(-3, -\frac{281}{36}, -10\right)$.

(c) $\left(5, \frac{31}{3}, 18\right) + \left(-18, -\frac{31}{3}, -5\right)(1,1,1) = (0,0,0)$.

(d) $(1,1,1) + \left(-18, -\frac{31}{3}, -5\right)(0,0,0) = (1,1,1)$.

(e) $(0,0,0) + \left(-18, -\frac{31}{3}, -5\right)(0,0,0) = (0,0,0)$.

(f) $(0,0,0) + \left(-18, -\frac{31}{3}, -5\right)\left(\frac{1}{15}, \frac{1}{10}, \frac{1}{5}\right) = \left(-\frac{6}{5}, -\frac{31}{30}, -1\right)$.

(g) $\left(6,\frac{50}{3},30\right)+\left(-18,-\frac{31}{3},-5\right)\left(1,\frac{29}{12},6\right)=\left(-12,-\frac{299}{36},0\right)$.

It is obvious that to find these values there is need to use the multiplication operation as well as the addition operation of TFNs. As pointed out in (i) of Section 7.2.1.2, that Khan et al. [122] have used the following multiplication operation to obtain these elements.

If $\tilde{T}_1=(u_1,v_1,w_1)$ and $\tilde{T}_2=(u_2,v_2,w_2)$ are two TFNs.

Then, $\tilde{T}_1.\tilde{T}_2=(u_1,v_1,w_1).(u_2,v_2,w_2)=(u_1u_2,v_1v_2,w_1w_2)$.

On using this multiplication operation $\left(-18,-\frac{31}{3},-5\right)\left(\frac{1}{5},\frac{1}{2},\frac{7}{5}\right)=\left(-\frac{18}{5},-\frac{31}{6},-7\right)$.

Furthermore, using the addition operation $(u_1,v_1,w_1)+(u_2,v_2,w_2)=(u_1+u_2,v_1+v_2,w_1+w_2)$, $(2,5,8)+\left(-18,-\frac{31}{3},-5\right)\left(\frac{1}{5},\frac{1}{2},\frac{7}{5}\right)=(2,5,8)+\left(-\frac{18}{5},-\frac{31}{6},-7\right)=\left(-\frac{8}{5},-\frac{1}{6},1\right)$.

It is obvious that the element $\left(-\frac{18}{5},-\frac{31}{6},-7\right)$ is not a TFN as for a TFN $\tilde{T}=(u,v,w)$, the condition $u\leq v\leq w$ should always be satisfied. Therefore, the element $\left(-\frac{8}{5},-\frac{1}{6},1\right)$ is not correct.

Similarly, it can be easily verified that the remaining elements, obtained by Khan et al. [122], are not correct.

7.2.2 Validity of the second observation

To resolve the inappropriateness of Khan et al.'s method [122], there is need to define the subtraction and division of two TFNs in such a manner that $\tilde{A}-\tilde{A}=(0,0,0)$ and $\frac{\tilde{A}}{\tilde{A}}=(1,1,1)$, where $\tilde{A}=(u,v,w)$ is a TFN. However, till now no such operations have been defined in the literature. Therefore, it is not possible to resolve this inappropriateness of Khan et al.'s method [122].

7.2.3 Validity of the third observation

To resolve the inappropriateness of Hajiagha et al.'s method [98], there is need to define the subtraction and division of two IVIFs in such a manner that $\tilde{A} - \tilde{A} = \tilde{0}$ and $\frac{\tilde{A}}{\tilde{A}} = ([1, 1], [0, 0])$, where $\tilde{A} = ([a_{i1}, a_{i2}], [a_{i3}, a_{i4}])$ is an IVIFS. However, till now no such operations have been defined in the literature. Therefore, it is not possible to resolve this inappropriateness of Hajiagha et al.'s method [98].

Chapter 8

Appropriate definition of an IVIFS[7]

The concept of an IVIFS, proposed by Atanassov and Gargov [9], has been used by several researchers in their research work. However, after a deep study, it is observed that the existing definition of an IVIFS is not appropriate. The aim of this chapter is to make the researchers aware about the inappropriateness of the existing definition as well as to propose the appropriate definition of an IVIFS.

8.1 Inappropriateness of the existing definition of an IVIFS

The following clearly indicates that the existing definition of an IVIFS [9] is not appropriate.

(i) Let $\tilde{\alpha}_{ij} = \left(\left[a_{ij1}, a_{ij2}\right], \left[a_{ij3}, a_{ij4}\right]\right) = ([0.1,0.5], [0.2,0.4])$ be an IVIFS. Then, according to existing definition [9], the interval of degree of hesitation $\left[a_{ij5}, a_{ij6}\right]$ can be obtained by subtracting the sum of $\left[a_{ij1}, a_{ij2}\right]$ and $\left[a_{ij3}, a_{ij4}\right]$ from [1,1] i.e., $\left[a_{ij5}, a_{ij6}\right] = [1,1] - \left(\left[a_{ij1}, a_{ij2}\right] + \left[a_{ij3}, a_{ij4}\right]\right) = [1,1] - ([0.1,0.5] + [0.2,0.4]) = [1,1] - [0.3,0.9] = [0.1,0.7]$.

Finally, the interval of degree of membership, non-membership and hesitation of the decision-maker with respect to $\tilde{\alpha}_{ij}$ are [0.1,0.5], [0.2,0.4] and [0.1,0.7] respectively. Now, if it is assumed that the interval of degree of membership [0.1,0.5] and interval of degree of hesitation [0.1,0.7] are known. Then, according to the existing definition [9], the interval of degree of non-membership $\left[a_{ij3}, a_{ij4}\right]$ can be obtained by subtracting the sum of these intervals from [1,1] i.e., according to the existing definition [9], $\left[a_{ij3}, a_{ij4}\right] = [1,1] -$

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[7] The contents of this chapter have been communicated in "Neurocomputing" for the possible publication

$([0.1,0.5]+[0.1,0.7])=[1,1]-[0.2,1.2]=[-0.2,0.8]$.

However, as the lower bound a_{ij3} of the obtained interval is a negative real number. Therefore, the obtained interval does not represent the interval of degree of non-membership. Since, it is the well known fact that the lower bound a_{ij3} of the interval of degree of non-membership $[a_{ij3},a_{ij4}]$ should always be a non-negative real number lying between 0 and 1.

(ii) An IFS $\tilde{\alpha}_{ij}=\langle\mu_{ij},\nu_{ij}\rangle$ is constructed by considering the assumption that the sum of degree of membership μ_{ij}, degree of non-membership ν_{ij} and degree of hesitation h_{ij} will be 1. Therefore, if any two are known then by subtracting their sum from 1, third one can be obtained. For example, if $\mu_{ij}=0.7$ and $\nu_{ij}=0.2$ are known then $h_{ij}=1-(0.7+0.2)=0.1$. Now, if it is assumed that $\mu_{ij}=0.7$ and $h_{ij}=0.1$ are known then $\nu_{ij}=1-(0.7+0.1)=0.2$, which is same as the known degree of non-membership. Similarly, if it is assumed that $\nu_{ij}=0.2$ and $h_{ij}=0.1$ are known then $\mu_{ij}=1-(0.2+0.1)=0.7$, which is same as the known degree of membership.

On the same direction, the intervals $[a_{ij1},a_{ij2}]$, $[a_{ij3},a_{ij4}]$ and $[a_{ij5},a_{ij6}]$ will be valid only if on considering any two intervals, third interval can be obtained. For example, if the interval of degree of membership, non-membership and hesitation of the decision-maker with respect to $\tilde{\alpha}_{ij}$ are $[0.1,0.5]$, $[0.2,0.4]$ and $[0.1,0.7]$ respectively. Then, on considering $[0.1,0.5]$, $[0.2,0.4]$, the interval $[0.1,0.7]$ should be obtained and on considering $[0.1,0.5]$, $[0.1,0.7]$ the interval $[0.2,0.4]$ should be obtained. However, the following clearly indicates that this condition is not satisfying.

(i) According to existing definition [9], on subtracting the sum of $[a_{ij1},a_{ij2}]=[0.1,0.5]$ and $[a_{ij5},a_{ij6}]=[0.1,0.7]$ from $[1,1]$, the interval $[a_{ij3},a_{ij4}]=[0.2,0.4]$ should be obtained. While, it is obvious that $[1,1]-([a_{ij1},a_{ij2}]+[a_{ij5},a_{ij6}])=[1,1]-$

$([0.1,0.5]+[0.1,0.7])=[1,1]-[0.2,1.2]=[-0.2,0.8]\neq[0.2,0.4]$. According to existing definition [9], on subtracting the sum of $[a_{ij3},a_{ij4}]=[0.2,0.4]$ and $[a_{ij5},a_{ij6}]=[0.1,0.7]$ from $[1,1]$, the interval $[a_{ij1},a_{ij2}]=[0.1,0.5]$ should be obtained. While, it is obvious that $[1,1]-([a_{ij3},a_{ij4}]+[a_{ij5},a_{ij6}])=[1,1]-([0.2,0.4]+[0.1,0.7])=[1,1]-[0.3,1.1]=[-0.1,0.7]\neq[0.1,0.5]$.

8.2 Proposed definition of an IVIFS

The set $\tilde{\alpha}_{ij}=([a_{ij1},a_{ij2}],[a_{ij3},a_{ij4}])$, where $[a_{ij1},a_{ij2}]$ and $[a_{ij3},a_{ij4}]$ represents the interval of degree of membership and the degree of non-membership of the decision-maker with respect to set $\tilde{\alpha}_{ij}$ respectively, will represent an IVIFS if a_{ij1},a_{ij2},a_{ij3} and a_{ij4} will satisfy the following conditions.

(i) $0\leq a_{ij1}\leq a_{ij2}\leq 1$,

(ii) $0\leq a_{ij3}\leq a_{ij4}\leq 1$,

(iii) $a_{ij1}+a_{ij4}=a_{ij2}+a_{ij3}$,

(iv) $a_{ij1}+a_{ij4}\leq 1$.

Furthermore, the interval $[1-a_{ij1}-a_{ij4},1-a_{ij1}-a_{ij4}]$ will represent the interval of degree of hesitation.

The conditions $a_{ij1}+a_{ij4}=a_{ij2}+a_{ij3}$, $a_{ij1}+a_{ij4}\leq 1$ and the interval of degree of hesitation are obtained as follows:

It is well known fact that for an IFS the following conditions will always be satisfied:

$1-$ degree of membership $\geq$ degree of non-membership (8.1)

$1-$ degree of non-membership $\geq$ degree of membership (8.2)

The inequalities (8.1) and (8.2) can be transformed into equations (8.3) and (8.4) respectively.

$1 -$ degree of membership=degree of non-membership + degree of hesitation (8.3)

$1 -$ degree of non-membership=degree of membership + degree of hesitation (8.4)

Assuming that the intervals $[a_{ij1}, a_{ij2}], [a_{ij3}, a_{ij4}]$ and $[a_{ij5}, a_{ij6}]$ represents the interval of degree of membership, the interval of degree of non-membership and the interval of degree of hesitation, equations (8.3) and (8.4) can be transformed into the equations (8.5) and (8.6) respectively.

$$[1,1] - [a_{ij1}, a_{ij2}] = [a_{ij3}, a_{ij4}] + [a_{ij5}, a_{ij6}] \tag{8.5}$$

$$[1,1] - [a_{ij3}, a_{ij4}] = [a_{ij1}, a_{ij2}] + [a_{ij5}, a_{ij6}] \tag{8.6}$$

Using the relations, $1 - [a, b] = [1 - b, 1 - a]$ and $[a, b] + [c, d] = [a + c, b + d]$, the equations (8.5) and (8.6) can be transformed into the equations (8.7) and (8.8) respectively.

$$[1 - a_{ij2}, 1 - a_{ij1}] = [a_{ij3} + a_{ij5}, a_{ij4} + a_{ij6}] \tag{8.7}$$

$$[1 - a_{ij4}, 1 - a_{ij3}] = [a_{ij1} + a_{ij5}, a_{ij2} + a_{ij6}] \tag{8.8}$$

Using the relation $[a, b] = [c, d] \Rightarrow a = c, b = d$, the equations (8.7) and (8.8) can be transformed into the equations (8.9) to (8.12) respectively.

$$1 - a_{ij2} = a_{ij3} + a_{ij5} \tag{8.9}$$

$$1 - a_{ij1} = a_{ij4} + a_{ij6} \tag{8.10}$$

$$1 - a_{ij4} = a_{ij1} + a_{ij5} \tag{8.11}$$

$$1 - a_{ij3} = a_{ij2} + a_{ij6} \tag{8.12}$$

On simplifying the equations (8.9) to (8.12), the equations (8.13) to (8.16) are obtained.

$$a_{ij5} = 1 - a_{ij2} - a_{ij3} \tag{8.13}$$

$$a_{ij6} = 1 - a_{ij1} - a_{ij4} \tag{8.14}$$

$$a_{ij5} = 1 - a_{ij1} - a_{ij4} \tag{8.15}$$

$$a_{ij6} = 1 - a_{ij2} - a_{ij3} \tag{8.16}$$

Now, $a_{ij5} = a_{ij5}$

$$\Rightarrow 1 - a_{ij2} - a_{ij3} = 1 - a_{ij1} - a_{ij4}$$

$$\Rightarrow a_{ij2} + a_{ij3} = a_{ij1} + a_{ij4}. \tag{8.17}$$

Also, $a_{ij6} = a_{ij6}$

$$\Rightarrow 1 - a_{ij1} - a_{ij4} = 1 - a_{ij2} - a_{ij3}$$

$$\Rightarrow a_{ij1} + a_{ij4} = a_{ij2} + a_{ij3}. \tag{8.18}$$

Using the equations (8.13) to (8.18) $a_{ij5} = a_{ij6} = 1 - a_{ij1} - a_{ij4}$.

Therefore, $[a_{ij5}, a_{ij6}] = [1 - a_{ij1} - a_{ij4}, 1 - a_{ij1} - a_{ij4}]$.

8.3 Validity of the proposed definition of an IVIFS

In the proposed definition of an IVIFS, the interval of degree of hesitation is defined as $[a_{ij5}, a_{ij6}] = [1 - a_{ij1} - a_{ij4}, 1 - a_{ij1} - a_{ij4}]$. Since, in the proposed definition, the condition $1 - a_{ij1} - a_{ij4} \geq 0$ has been considered. So, the value of a_{ij5} will always be greater than or equal to zero. Therefore, to prove that the proposed definition is valid, it is sufficient to prove that if any two among the three $[a_{ij1}, a_{ij2}]$, $[a_{ij3}, a_{ij4}]$ and $[a_{ij5}, a_{ij6}] = [1 - a_{ij1} - a_{ij4}, 1 - a_{ij1} - a_{ij4}]$ are known then the third one should be obtained by using the equations (8.5) and (8.6).

(i) Let the interval of degree of membership $[a_{ij1}, a_{ij2}]$ and the interval of degree of hesitation $[a_{ij5}, a_{ij6}] = [1 - a_{ij1} - a_{ij4}, 1 - a_{ij1} - a_{ij4}]$ be known and $[a, b]$ represents the unknown interval of degree of non-membership. Then, using the equation (8.5),

$$[1,1] - [a_{ij1}, a_{ij2}] = [1 - a_{ij1} - a_{ij4}, 1 - a_{ij1} - a_{ij4}] + [a, b]$$

$$\Rightarrow [1 - a_{ij2}, 1 - a_{ij1}] = [1 - a_{ij1} - a_{ij4} + a, 1 - a_{ij1} - a_{ij4} + b]$$

$$\Rightarrow 1 - a_{ij2} = 1 - a_{ij1} - a_{ij4} + a,\ 1 - a_{ij1} = 1 - a_{ij1} - a_{ij4} + b$$

$$\Rightarrow a = -a_{ij2} + a_{ij1} + a_{ij4},$$

$$b = -a_{ij1} + a_{ij1} + a_{ij4} = a_{ij4}$$

Using the condition (iii), $a_{ij1} + a_{ij4} = a_{ij2} + a_{ij3}$ i.e., $a_{ij3} = -a_{ij2} + a_{ij1} + a_{ij4}$,

$a = a_{ij3}$

$\Rightarrow [a,b] = [a_{ij3}, a_{ij4}]$.

Similarly, the same can also be proved by using the equation (8.6).

(ii) Let the interval of degree of non-membership $[a_{ij3}, a_{ij4}]$ and the interval of degree of hesitation $[a_{ij5}, a_{ij6}] = [1 - a_{ij1} - a_{ij4}, 1 - a_{ij1} - a_{ij4}]$ be known and $[a,b]$ represents the unknown interval of degree of membership. Then, using the equation (8.5),

$[1,1] - [a_{ij3}, a_{ij4}] = [1 - a_{ij1} - a_{ij4}, 1 - a_{ij1} - a_{ij4}] + [a,b]$

$\Rightarrow [1 - a_{ij4}, 1 - a_{ij3}] = [1 - a_{ij1} - a_{ij4} + a, 1 - a_{ij1} - a_{ij4} + b]$

$\Rightarrow 1 - a_{ij4} = 1 - a_{ij1} - a_{ij4} + a, 1 - a_{ij3} = 1 - a_{ij1} - a_{ij4} + b$

$\Rightarrow a = -a_{ij4} + a_{ij1} + a_{ij4} = a_{ij1},$

$b = -a_{ij3} + a_{ij1} + a_{ij4}.$

Using the condition (iii), $a_{ij1} + a_{ij4} = a_{ij2} + a_{ij3}$ i.e., $a_{ij2} = -a_{ij3} + a_{ij1} + a_{ij4}$

$b = a_{ij2}$

$\Rightarrow [a,b] = [a_{ij1}, a_{ij2}]$.

Similarly, the same can also be proved by using the equation (8.6).

8.4 Illustrative examples

In this section, the proposed definition is illustrated with the help of some numerical examples.

Example 8.1 If $\tilde{a}_{ij} = ([a_{ij1}, a_{ij2}], [a_{ij3}, a_{ij4}]) = ([0.10,0.50],[0.20,0.40])$. Then, according to proposed definition, it is not an IVIFS as the condition $a_{ij1} + a_{ij4} = a_{ij2} + a_{ij3}$ is not satisfying. While, according to the existing definition [9], it is an IVIFS. Furthermore, according to existing definition [9], the interval of degree of hesitation will be $[1 - 0.50 - 0.40, 1 - 0.10 - 0.20] = [0.1, 0.7]$.

Now, if it is assumed that the interval of degree of membership $[0.10,0.50]$ and interval of degree of hesitation [0.10,0.70] are known. Then, according to the existing definition [9], the interval of degree of non-membership $[a_{ij3}, a_{ij4}]$ can be obtained by subtracting the sum of these intervals from [1,1] i.e., according to the existing definition [9], $[a_{ij3}, a_{ij4}] = [1,1] - ([0.10,0.50] + [0.10,0.70]) = [1,1] - [0.20,1.20] = [-0.20,0.80]$

This clearly indicates the lower bound a_{ij3} of the interval of degree of non-membership $[a_{ij3}, a_{ij4}]$, obtained by considering the existing definition [9] of an IVIFS, is a negative real number, which contradicts the well known fact that a_{ij3} should always be a non-negative real number lying between 0 and 1.

Example 8.2 If $\tilde{\alpha}_{ij} = \left([a_{ij1}, a_{ij2}], [a_{ij3}, a_{ij4}]\right) = ([0.20,0.70], [0.30,0.80])$. Then, according to existing definition [9], it is not an IVIFS as the condition $a_{ij2} + a_{ij4} = 1.5 \leq 1$ is not satisfying. While, according to proposed definition, it is an IVIFS as all the conditions $0 \leq a_{ij1} \leq a_{ij2} \leq 1$, $0 \leq a_{ij3} \leq a_{ij4} \leq 1$, $a_{ij1} + a_{ij4} = a_{ij2} + a_{ij3}$ and $a_{ij1} + a_{ij4} \leq 1$ or $a_{ij3} + a_{ij2} \leq 1$ are satisfying. Furthermore, as $a_{ij1} = 0.20$, $a_{ij2} = 0.70$, $a_{ij3} = 0.30$ and $a_{ij4} = 0.80$ so using the equation (8.13) or the equation (8.15) i.e., $a_{ij5} = 1 - a_{ij2} - a_{ij3}$ or $a_{ij5} = 1 - a_{ij1} - a_{ij4}$, the obtained value of a_{ij5} is 0 and using the equation (8.14) or the equation (8.16) i.e., $a_{ij6} = 1 - a_{ij1} - a_{ij4}$ or $a_{ij6} = 1 - a_{ij2} - a_{ij3}$, the obtained value of a_{ij6} is also 0. Therefore, the interval of degree of hesitation $[a_{ij5}, a_{ij6}]$ is [0,0].

Now, the following clearly indicates that on considering any two among [0.20,0.70], [0.30,0.80] and [0,0], the third one can be obtained.

(1) Let $[a_{ij1}, a_{ij2}] = [0.2,0.7]$ and $[a_{ij5}, a_{ij6}] = [0,0]$ be known. Then,

$$[1,1] - [a_{ij1}, a_{ij2}] = [a_{ij3}, a_{ij4}] + [a_{ij5}, a_{ij6}]$$

$$\Rightarrow 1 - [0.2,0.7] = [a_{ij3}, a_{ij4}] + [0,0]$$

$$\Rightarrow [0.3,0.8] = [0 + a_{ij3}, 0 + a_{ij4}]$$

$$\Rightarrow 0.3 = 0 + a_{ij3} \text{ and } 0.8 = 0 + a_{ij4}$$

$\Rightarrow a_{ij3} = 0.3$ and $a_{ij4} = 0.8$

$\Rightarrow [a_{ij3}, a_{ij4}] = [0.3,0.8]$.

(2) Let $[a_{ij3}, a_{ij4}] = [0.3,0.8]$ and $[a_{ij5}, a_{ij6}] = [0,0]$ be known. Then

$[1,1] - [a_{ij1}, a_{ij2}] = [a_{ij3}, a_{ij4}] + [a_{ij5}, a_{ij6}]$

$\Rightarrow 1 - [a_{ij1}, a_{ij2}] = [0.3,0.8] + [0,0]$

$\Rightarrow [1 - a_{ij2}, 1 - a_{ij1}] = [0.3,0.8]$

$\Rightarrow 1 - a_{ij2} = 0.3$ and $1 - a_{ij1} = 0.8$

$\Rightarrow a_{ij1} = 0.2$ and $a_{ij2} = 0.7$

$\Rightarrow [a_{ij1}, a_{ij2}] = [0.2,0.7]$.

Example 8.3 If $\tilde{\alpha}_{ij} = ([a_{ij1}, a_{ij2}], [a_{ij3}, a_{ij4}]) = ([0.10,0.50], [0.10,0.50])$. Then, it is an IVIFS according to both existing definition [9] as well as proposed definition. However, according to existing definition [9], the interval of degree of hesitation will be $[1 - 0.50 - 0.50, 1 - 0.10 - 0.10] = [0.00,0.80]$.

Now, if it is assumed that the interval of degree of membership $[0.10,0.50]$ and interval of degree of hesitation $[0.00,0.80]$ are known. Then, according to the existing definition [9], the interval of degree of non-membership can be obtained by subtracting the sum of these intervals from $[1,1]$. i.e., according to the existing definition [9],

$[a_{ij3}, a_{ij4}] = [1,1] - ([0.10,0.50] + [0.00,0.80]) = [1,1] - [0.10,1.30] = [-0.30,0.90]$.

However, as the obtained value of a_{ij3}, obtained by considering the existing definition of an IVIFS [9], is a negative real number, which contradicts the well-known fact that the a_{ij3} should always be a non-negative real number lying between 0 and 1.

While, according to proposed definition, it is an IVIFS as all the conditions $0 \leq a_{ij1} \leq a_{ij2} \leq 1$, $0 \leq a_{ij3} \leq a_{ij4} \leq 1$, $a_{ij1} + a_{ij4} = a_{ij2} + a_{ij3}$ and $a_{ij1} + a_{ij4} \leq 1$ or $a_{ij2} + a_{ij3} \leq 1$ are satisfying. Furthermore, as $a_{ij1} = 0.10$, $a_{ij2} = 0.50$, $a_{ij3} = 0.10$ and $a_{ij4} = 0.50$ so using the equation (8.13) or the equation (8.15) i.e., $a_{ij5} = 1 - a_{ij2} - a_{ij3}$ or $a_{ij5} = 1 - a_{ij1} - a_{ij4}$, the obtained value of a_{ij5} is 0.40 and using the equation (8.14) or the equation (8.16) i.e., $a_{ij6} = 1 - a_{ij1} - a_{ij4}$ or $a_{ij6} = 1 - a_{ij2} - a_{ij3}$, the obtained value of a_{ij6} is also 0.40. Therefore, the interval of degree of hesitation $[a_{ij5}, a_{ij6}]$ is [0.40,0.40].

Now, the following clearly indicates that on considering any two among $[0.10,0.50], [0.10,0.50]$ and $[0.40,0.40]$, the third one can be obtained.

1) Let $[a_{ij1}, a_{ij2}] = [0.10,0.50]$ and $[a_{ij5}, a_{ij6}] = [0.40,0.40]$ be known. Then $[1,1] - [a_{ij1}, a_{ij2}] = [a_{ij3}, a_{ij4}] + [a_{ij5}, a_{ij6}]$

$$\Rightarrow 1 - [0.10,0.50] = [a_{ij3}, a_{ij4}] + [0.40,0.40]$$

$$\Rightarrow [0.50,0.90] = [a_{ij3} + 0.40, a_{ij4} + 0.40]$$

$$\Rightarrow 0.50 = a_{ij3} + 0.40 \text{ and } 0.90 = a_{ij4} + 0.40$$

$$\Rightarrow a_{ij3} = 0.10 \text{ and } a_{ij4} = 0.50$$

$$\Rightarrow [a_{ij3}, a_{ij4}] = [0.10,0.50].$$

2) Let $[a_{ij3}, a_{ij4}] = [0.10,0.50]$ and $[a_{ij5}, a_{ij6}] = [0.40,0.40]$ be known. Then $[1,1] - [a_{ij1}, a_{ij2}] = [a_{ij3}, a_{ij4}] + [a_{ij5}, a_{ij6}]$

$$\Rightarrow 1 - [a_{ij1}, a_{ij2}] = [0.10,0.50] + [0.40,0.40]$$

$$\Rightarrow [1 - a_{ij2}, 1 - a_{ij1}] = [0.50,0.90]$$

$$\Rightarrow 1 - a_{ij2} = 0.50 \text{ and } 1 - a_{ij1} = 0.90$$

$$\Rightarrow a_{ij1} = 0.10 \text{ and } a_{ij2} = 0.50$$

$$\Rightarrow [a_{ij1}, a_{ij2}] = [0.10,0.50].$$

8.5 Solution of a real-life IVIFMADMPr

In this section, the existing IVIFMADMPr [258] having completely known attribute weights have been solved with the following modification:

"Rating value of each alternative over each criterion is represented as the proposed IVIFS instead of the existing IVIFS".

There is a panel of four alternatives for the investment company to invest money in best option $A_i, i = 1,2,3,4$ on the basis of the following three benefit attributes.

(i) G_1: Risk analysis

(ii) G_2: Growth

(iii) G_3: Environmental impact

Furthermore,

(i) The $(i,j)^{th}$ element of the Table 8.1, represented by an IVIFS $\tilde{\alpha}_{ij} = ([a_{ij1}, a_{ij2}], [a_{ij3}, a_{ij4}])$, represents the rating value of the i^{th}-company over the j^{th}-attribute.

Table 8.1: Rating Values

Attributes→ ↓ Alternatives	G_1	G_2	G_3
A_1	$\begin{pmatrix}[0.40,0.50],\\ [0.30,0.40]\end{pmatrix}$	$\begin{pmatrix}[0.40,0.60],\\ [0.20,0.40]\end{pmatrix}$	$\begin{pmatrix}[0.20,0.60],\\ [0.40,0.80]\end{pmatrix}$
A_2	$\begin{pmatrix}[0.20,0.70],\\ [0.30,0.80]\end{pmatrix}$	$\begin{pmatrix}[0.20,0.90],\\ [0.10,0.80]\end{pmatrix}$	$\begin{pmatrix}[0.30,0.40],\\ [0.50,0.60]\end{pmatrix}$
A_3	$\begin{pmatrix}[0.30,0.40],\\ [0.50,0.60]\end{pmatrix}$	$\begin{pmatrix}[0.20,0.20],\\ [0.50,0.50]\end{pmatrix}$	$\begin{pmatrix}[0.10,0.50],\\ [0.40,0.80]\end{pmatrix}$
A_4	$\begin{pmatrix}[0.20,0.40],\\ [0.30,0.50]\end{pmatrix}$	$\begin{pmatrix}[0.20,0.60],\\ [0.40,0.80]\end{pmatrix}$	$\begin{pmatrix}[0.40,0.70],\\ [0.20,0.50]\end{pmatrix}$

(ii) Weights w_1, w_2 and w_3 of the attributes G_1, G_2 and G_3 respectively are $w_1 = w_2 = w_3 = \frac{1}{3}$.

Using the Jujhar method, proposed in Section 4.13 of Chapter 4, the ranking of the alternatives for the considered real-life IVIFMADMPr having completely known attribute weights can be obtained as follows.

Since, as all the attribute weights are known so there is no need to apply Step 1 to Step 4 of the proposed Jujhar method.

Step 5: On considering the OAWV, $(w_1, w_2, w_3) = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ and using Step 5 of Jujhar method, proposed in Section 4.13 of Chapter 4,

$\tilde{P}_1 = \left(\left[\left(\frac{1}{3}\right)(0.40) + \left(\frac{1}{3}\right)(0.40) + \left(\frac{1}{3}\right)(0.20), \left(\frac{1}{3}\right)(0.50) + \left(\frac{1}{3}\right)(0.60) + \left(\frac{1}{3}\right)(0.60)\right],\right.$

$\left.\left[\left(\frac{1}{3}\right)(0.30) + \left(\frac{1}{3}\right)(0.20) + \left(\frac{1}{3}\right)(0.40), \left(\frac{1}{3}\right)(0.40) + \left(\frac{1}{3}\right)(0.40) + \left(\frac{1}{3}\right)(0.80)\right]\right)$

$= ([0.3333, 0.5666], [0.3000, 0.5333]),$

$\tilde{P}_2 = \left(\left[\left(\frac{1}{3}\right)(0.20) + \left(\frac{1}{3}\right)(0.20) + \left(\frac{1}{3}\right)(0.30), \left(\frac{1}{3}\right)(0.70) + \left(\frac{1}{3}\right)(0.90) + \left(\frac{1}{3}\right)(0.40)\right],\right.$

$\left.\left[\left(\frac{1}{3}\right)(0.30) + \left(\frac{1}{3}\right)(0.10) + \left(\frac{1}{3}\right)(0.50), \left(\frac{1}{3}\right)(0.80) + \left(\frac{1}{3}\right)(0.80) + \left(\frac{1}{3}\right)(0.60)\right]\right)$

$= ([0.2333, 0.6666], [0.3000, 0.7333]),$

$\tilde{P}_3 = \left(\left[\left(\frac{1}{3}\right)(0.30) + \left(\frac{1}{3}\right)(0.20) + \left(\frac{1}{3}\right)(0.10), \left(\frac{1}{3}\right)(0.40) + \left(\frac{1}{3}\right)(0.20) + \left(\frac{1}{3}\right)(0.50)\right],\right.$

$\left.\left[\left(\frac{1}{3}\right)(0.50) + \left(\frac{1}{3}\right)(0.50) + \left(\frac{1}{3}\right)(0.40), \left(\frac{1}{3}\right)(0.60) + \left(\frac{1}{3}\right)(0.50) + \left(\frac{1}{3}\right)(0.80)\right]\right)$

$= ([0.2000, 0.3666], [0.4666, 0.6333]),$

$\tilde{P}_4 = \left(\left[\left(\frac{1}{3}\right)(0.20) + \left(\frac{1}{3}\right)(0.20) + \left(\frac{1}{3}\right)(0.40), \left(\frac{1}{3}\right)(0.40) + \left(\frac{1}{3}\right)(0.60) + \left(\frac{1}{3}\right)(0.70)\right],\right.$

$\left.\left[\left(\frac{1}{3}\right)(0.30) + \left(\frac{1}{3}\right)(0.40) + \left(\frac{1}{3}\right)(0.20), \left(\frac{1}{3}\right)(0.50) + \left(\frac{1}{3}\right)(0.80) + \left(\frac{1}{3}\right)(0.50)\right]\right)$

$= ([0.2666, 0.5666], [0.3000, 0.6000]),$

Step 6: Using Step 6 of the Jujhar method, proposed in Section 4.13 of Chapter 4,

$S(\tilde{P}_1) = 0.0333, S(\tilde{P}_2) = -0.0667, S(\tilde{P}_3) = -0.2666$ and $S(\tilde{P}_4) = -0.0334.$

Since, $S(\tilde{P}_1) > S(\tilde{P}_4) > S(\tilde{P}_2) > S(\tilde{P}_3)$. So, according to Step 6 of the Jujhar method, proposed in Section 4.13 of Chapter 4, the ranking of the alternatives is $A_1 > A_4 > A_2 > A_3$.

8.6 Conclusions

It is shown that the existing definition of an IVIFS [9] is not appropriate and the appropriate definition of an IVIFS is proposed. Also, the validity of the proposed definition is discussed. Furthermore, the proposed IVIFS is used to represent the rating value of each alternative over each attribute in an existing real-life IVIFMADMPr [258] having completely known attribute weights with modified data. Finally, the modified real-life IVIFMADMPr having completely known attribute weights is solved by the proposed Jujhar method.

Chapter 9

Appropriate definition of an IVPFS[8]

The concept of an IVPFS, proposed by Peng and Yang [164], has been used by several researchers in their work. However, after a deep study, it is observed that the existing definition of an IVPFS is not appropriate. The aim of this chapter is to make the researchers aware about the inappropriateness of the existing definition as well as to propose the appropriate definition of an IVPFS.

9.1 Inappropriateness of the existing definition of an IVPFS

The following clearly indicates that the existing definition of an IVPFS [164] is not appropriate.

(i) Let $\tilde{\alpha}_{ij} = \left(\left[a_{ij1}, a_{ij2}\right], \left[a_{ij3}, a_{ij4}\right]\right) = ([0.80,0.90],[0.10,0.20])$ be an IVPFS. Then, according to existing definition [164], the interval of degree of hesitation $\left[a_{ij5}, a_{ij6}\right]$ can be obtained by using the expression, $\left[a_{ij5}, a_{ij6}\right] = \left[\sqrt{1-\left(a_{ij2}\right)^2-\left(a_{ij4}\right)^2}, \sqrt{1-\left(a_{ij1}\right)^2-\left(a_{ij3}\right)^2}\right]$ i.e.,

$\left[a_{ij5}, a_{ij6}\right] = \left[\sqrt{1-(0.90)^2-(0.20)^2}, \sqrt{1-(0.80)^2-(0.10)^2}\right] =$

$\left[\sqrt{1-0.81-0.04}, \sqrt{1-0.64-0.01}\right] = \left[\sqrt{0.15}, \sqrt{0.35}\right]$

Finally, the interval of degree of membership, non-membership and hesitation of the decision-maker with respect to $\tilde{\alpha}_{ij}$ are $[0.80,0.90]$, $[0.10,0.20]$ and $\left[\sqrt{0.15}, \sqrt{0.35}\right]$ respectively.

Now, if it is assumed that the interval of degree of membership $[0.80,0.90]$ and the

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[8] The contents of this chapter have been communicated in "Applied Soft Computing" for the possible publication

interval of the degree of hesitation $\left[\sqrt{0.15}, \sqrt{0.35}\right]$ are known. Then, according to the existing definition [164], the interval of the degree of non-membership $\left[a_{ij3}, a_{ij4}\right]$ can be obtained by using the expression, $\left[a_{ij3}, a_{ij4}\right] = \left[\sqrt{1-\left(a_{ij2}\right)^2-\left(a_{ij6}\right)^2}, \sqrt{1-\left(a_{ij1}\right)^2-\left(a_{ij5}\right)^2}\right]$ i.e., $\left[a_{ij3}, a_{ij4}\right] = \left[\sqrt{1-(0.90)^2-\left(\sqrt{0.35}\right)^2}, \sqrt{1-(0.80)^2-\left(\sqrt{0.15}\right)^2}\right] = \left[\sqrt{1-0.81-0.35}, \sqrt{1-0.64-0.15}\right] = \left[\sqrt{-0.16}, \sqrt{0.21}\right]$

However, as the lower bound a_{ij3} of the obtained interval is a negative real number. Therefore, the obtained interval does not represent the interval of degree of non-membership. Since, it is well-known fact that the lower bound a_{ij3} of the interval of degree of non-membership $\left[a_{ij3}, a_{ij4}\right]$ should always be a non-negative real number lying between 0 and 1.

(ii) A PFS $\tilde{\alpha}_{ij} = \langle \mu_{ij}, \nu_{ij} \rangle$ is constructed by considering the assumption that the sum of square of degree of membership μ_{ij}, non-membership ν_{ij} and degree of hesitation h_{ij} will be 1. Therefore, if any two are known then by subtracting the sum of their square from 1, the third one can be obtained. For example, if $\mu_{ij} = 0.70$ and $\nu_{ij} = 0.20$ are known then $h_{ij} = \sqrt{1-(0.70)^2-(0.20)^2} = \sqrt{0.47}$. Now, if it is assumed that $\mu_{ij} = 0.70$ and $h_{ij} = \sqrt{0.47}$ are known then $\nu_{ij} = \sqrt{1-(0.70)^2-\left(\sqrt{0.47}\right)^2} = 0.20$, which is same as the known degree of non-membership. Similarly, if it is assumed that $\nu_{ij} = 0.20$ and $h_{ij} = \sqrt{0.47}$ are known then $\mu_{ij} = \sqrt{1-(0.20)^2-\left(\sqrt{0.47}\right)^2} = 0.70$, which is same as the known degree of membership.

On the same direction, the intervals $\left[a_{ij1}, a_{ij2}\right]$, $\left[a_{ij3}, a_{ij4}\right]$ and $\left[a_{ij5}, a_{ij6}\right]$ will be valid only if on considering any two intervals, third interval can be obtained. For example, if the interval of degree of membership, the interval of degree of non-membership and the interval of degree of hesitation of the decision-maker with respect to $\tilde{\alpha}_{ij}$ are $[0.80,0.90]$, $[0.10,0.20]$ and $\left[\sqrt{0.15},\sqrt{0.35}\right]$, respectively. Then, on considering $[0.80,0.90]$, $\left[\sqrt{0.15},\sqrt{0.35}\right]$ the interval $[0.10,0.20]$ should be obtained and on considering $[0.10,0.20]$, $\left[\sqrt{0.15},\sqrt{0.35}\right]$, the interval $[0.80,0.90]$ should be obtained. However, the following clearly indicates that this condition is not satisfying.

(i) According to existing definition [164], on using the expression $\left[a_{ij3},a_{ij4}\right]=\left[\sqrt{1-\left(a_{ij2}\right)^2-\left(a_{ij6}\right)^2},\sqrt{1-\left(a_{ij1}\right)^2-\left(a_{ij5}\right)^2}\right]$, the interval $\left[a_{ij3},a_{ij4}\right]=[0.10,0.20]$ should be obtained. While, it is obvious that $\left[a_{ij3},a_{ij4}\right]=\left[\sqrt{1-\left(a_{ij2}\right)^2-\left(a_{ij6}\right)^2},\sqrt{1-\left(a_{ij1}\right)^2-\left(a_{ij5}\right)^2}\right]=$

$\left[\sqrt{1-(0.90)^2-\left(\sqrt{0.35}\right)^2},\sqrt{1-(0.80)^2-\left(\sqrt{0.15}\right)^2}\right]=$

$\left[\sqrt{1-0.81-0.35},\sqrt{1-0.64-0.15}\right]=\left[\sqrt{-0.16},\sqrt{0.21}\right]\neq[0.10,0.20]$.

(ii) According to existing definition [164], on using the expression $\left[a_{ij1},a_{ij2}\right]=\left[\sqrt{1-\left(a_{ij4}\right)^2-\left(a_{ij6}\right)^2},\sqrt{1-\left(a_{ij3}\right)^2-\left(a_{ij5}\right)^2}\right]$, the interval $\left[a_{ij1},a_{ij2}\right]=[0.80,0.90]$ should be obtained. While, it is obvious that $\left[a_{ij1},a_{ij2}\right]=$

$\left[\sqrt{1-\left(a_{ij4}\right)^2-\left(a_{ij6}\right)^2},\sqrt{1-\left(a_{ij3}\right)^2-\left(a_{ij5}\right)^2}\right]=$

$\left[\sqrt{1-(0.20)^2-\left(\sqrt{0.35}\right)^2},\sqrt{1-(0.10)^2-\left(\sqrt{0.15}\right)^2}\right]$

$\left[\sqrt{1-0.04-0.35},\sqrt{1-0.01-0.15}\right]=\left[\sqrt{0.61},\sqrt{0.84}\right]\neq[0.80,0.90]$.

9.2 Proposed definition of IVPFS

The set $\tilde{\alpha}_{ij} = \left(\left[a_{ij1}, a_{ij2}\right], \left[a_{ij3}, a_{ij4}\right]\right)$, where $\left[a_{ij1}, a_{ij2}\right]$ and $\left[a_{ij3}, a_{ij4}\right]$ represents the interval of degree of membership and the interval of degree of non-membership of decision-maker of set $\tilde{\alpha}_{ij}$ respectively, will represent an IVPFS if $a_{ij1}, a_{ij2}, a_{ij3}$ and a_{ij4} will satisfy the following conditions.

(i) $0 \leq a_{ij1} \leq a_{ij2} \leq 1,$

(ii) $0 \leq a_{ij3} \leq a_{ij4} \leq 1,$

(iii) $\left(a_{ij1}\right)^2 + \left(a_{ij4}\right)^2 = \left(a_{ij2}\right)^2 + \left(a_{ij3}\right)^2,$

(iv) $\left(a_{ij1}\right)^2 + \left(a_{ij4}\right)^2 \leq 1.$

Furthermore, the interval $\left[\sqrt{1 - \left(a_{ij1}\right)^2 - \left(a_{ij4}\right)^2}, \sqrt{1 - \left(a_{ij1}\right)^2 - \left(a_{ij4}\right)^2}\right]$ will represent the interval of degree of hesitation.

The conditions $\left(a_{ij1}\right)^2 + \left(a_{ij4}\right)^2 = \left(a_{ij2}\right)^2 + \left(a_{ij3}\right)^2$, $\left(a_{ij1}\right)^2 + \left(a_{ij4}\right)^2 \leq 1$ and the interval of degree of hesitation are obtained as follows:

It is well-known fact that for a PFS the following conditions will always be satisfied:

$$1 - (\text{the degree of membership})^2 \geq (\text{the degree of nonmembership})^2 \tag{9.1}$$

$$1 - (\text{the degree of non} - \text{membership})^2 \geq (\text{the degree of membership})^2 \tag{9.2}$$

The inequalities (9.1) and (9.2) can be transformed into equations (9.3) and (9.4) respectively.

$$1 - (\text{the degree of membership})^2 = (\text{the degree of non} - \text{membership})^2 + (\text{the degree of hesitation})^2 \tag{9.3}$$

$$1 - (\text{the degree of non} - \text{membership})^2 = (\text{the degree of membership})^2 + (\text{the degree of hesitation})^2 \tag{9.4}$$

Assuming that the intervals $\left[a_{ij1}, a_{ij2}\right], \left[a_{ij3}, a_{ij4}\right]$ and $\left[a_{ij5}, a_{ij6}\right]$ represents the interval of the degree of membership, non-membership and hesitation respectively, equations (9.3) and (9.4) can be transformed into equations (9.5) and (9.6) respectively.

$$[1,1]-\left[\left(a_{ij1}\right)^2,\left(a_{ij2}\right)^2\right]=\left[\left(a_{ij3}\right)^2,\left(a_{ij4}\right)^2\right]+\left[\left(a_{ij5}\right)^2,\left(a_{ij6}\right)^2\right] \tag{9.5}$$

$$[1,1]-\left[\left(a_{ij3}\right)^2,\left(a_{ij4}\right)^2\right]=\left[\left(a_{ij1}\right)^2,\left(a_{ij2}\right)^2\right]+\left[\left(a_{ij5}\right)^2,\left(a_{ij6}\right)^2\right] \tag{9.6}$$

Using the relations, $1-[a,b]=[1-b,1-a]$ and $[a,b]+[c,d]=[a+c,b+d]$, the equations (9.5) and (9.6) can be transformed into the equations (9.7) and (9.8) respectively.

$$\left[1-\left(a_{ij2}\right)^2,1-\left(a_{ij1}\right)^2\right]=\left[\left(a_{ij3}\right)^2+\left(a_{ij5}\right)^2,\left(a_{ij4}\right)^2+\left(a_{ij6}\right)^2\right] \tag{9.7}$$

$$\left[1-\left(a_{ij4}\right)^2,1-\left(a_{ij3}\right)^2\right]=\left[\left(a_{ij1}\right)^2+\left(a_{ij5}\right)^2,\left(a_{ij2}\right)^2+\left(a_{ij6}\right)^2\right] \tag{9.8}$$

Using the relation $[a,b]=[c,d]\Rightarrow a=c,b=d$, the equations (9.7) and (9.8) can be transformed into the equations (9.9) to (9.12) respectively.

$$1-\left(a_{ij2}\right)^2=\left(a_{ij3}\right)^2+\left(a_{ij5}\right)^2 \tag{9.9}$$

$$1-\left(a_{ij1}\right)^2=\left(a_{ij4}\right)^2+\left(a_{ij6}\right)^2 \tag{9.10}$$

$$1-\left(a_{ij4}\right)^2=\left(a_{ij1}\right)^2+\left(a_{ij5}\right)^2 \tag{9.11}$$

$$1-\left(a_{ij3}\right)^2=\left(a_{ij2}\right)^2+\left(a_{ij6}\right)^2 \tag{9.12}$$

On simplifying the equations (9.9) to (9.12), the equations (9.13) to (9.16) are obtained.

$$\left(a_{ij5}\right)^2=1-\left(a_{ij2}\right)^2-\left(a_{ij3}\right)^2 \tag{9.13}$$

$$\left(a_{ij6}\right)^2=1-\left(a_{ij1}\right)^2-\left(a_{ij4}\right)^2 \tag{9.14}$$

$$\left(a_{ij5}\right)^2=1-\left(a_{ij1}\right)^2-\left(a_{ij4}\right)^2 \tag{9.15}$$

$$\left(a_{ij6}\right)^2=1-\left(a_{ij2}\right)^2-\left(a_{ij3}\right)^2 \tag{9.16}$$

Now, $\left(a_{ij5}\right)^2=\left(a_{ij5}\right)^2$

$$\Rightarrow 1-\left(a_{ij2}\right)^2-\left(a_{ij3}\right)^2=1-\left(a_{ij1}\right)^2-\left(a_{ij4}\right)^2$$

$$\Rightarrow \left(a_{ij2}\right)^2+\left(a_{ij3}\right)^2=\left(a_{ij1}\right)^2+\left(a_{ij4}\right)^2. \tag{9.17}$$

Also, $\left(a_{ij6}\right)^2 = \left(a_{ij6}\right)^2$

$$\Rightarrow 1-\left(a_{ij1}\right)^2-\left(a_{ij4}\right)^2 = 1-\left(a_{ij2}\right)^2-\left(a_{ij3}\right)^2$$

$$\Rightarrow \left(a_{ij1}\right)^2+\left(a_{ij4}\right)^2 = \left(a_{ij2}\right)^2+\left(a_{ij3}\right)^2. \tag{9.18}$$

Using the equations (9.13) to (9.18), $\left(a_{ij5}\right)^2 = \left(a_{ij6}\right)^2 = 1-\left(a_{ij1}\right)^2-\left(a_{ij4}\right)^2$.

Therefore, $\left[\left(a_{ij5}\right)^2,\left(a_{ij6}\right)^2\right] = \left[1-\left(a_{ij1}\right)^2-\left(a_{ij4}\right)^2, 1-\left(a_{ij1}\right)^2-\left(a_{ij4}\right)^2\right]$

i.e., $\left[a_{ij5}, a_{ij6}\right] = \left[\sqrt{1-\left(a_{ij1}\right)^2-\left(a_{ij4}\right)^2}, \sqrt{1-\left(a_{ij1}\right)^2-\left(a_{ij4}\right)^2}\right]$.

9.3 Validity of the proposed definition of an IVPFS

In the proposed definition of an IVPFS, the interval of degree of hesitation is defined as $\left[a_{ij5}, a_{ij6}\right] = \left[\sqrt{1-\left(a_{ij1}\right)^2-\left(a_{ij4}\right)^2}, \sqrt{1-\left(a_{ij1}\right)^2-\left(a_{ij4}\right)^2}\right]$. Since, in the proposed definition, the condition $1-\left(a_{ij1}\right)^2-\left(a_{ij4}\right)^2 \geq 0$ has been considered. So, the value of a_{ij5} will always be greater than or equal to zero. Therefore, to prove that the proposed definition is valid, it is sufficient to prove that if any two among the three $\left[a_{ij1}, a_{ij2}\right], \left[a_{ij3}, a_{ij4}\right]$ and $\left[a_{ij5}, a_{ij6}\right] = \left[\sqrt{1-\left(a_{ij1}\right)^2-\left(a_{ij4}\right)^2}, \sqrt{1-\left(a_{ij1}\right)^2-\left(a_{ij4}\right)^2}\right]$ are known then the third one should be obtained by using the equations (9.5) and (9.6).

(i) Let the interval of degree of membership $\left[a_{ij1}, a_{ij2}\right]$ and the interval of degree of hesitation $\left[a_{ij5}, a_{ij6}\right] = \left[\sqrt{1-\left(a_{ij1}\right)^2-\left(a_{ij4}\right)^2}, \sqrt{1-\left(a_{ij1}\right)^2-\left(a_{ij4}\right)^2}\right]$ be known and $[a, b]$ represents the unknown interval of the degree of non-membership. Then, using the equation (9.5),

$$[1,1]-\left[\left(a_{ij1}\right)^2,\left(a_{ij2}\right)^2\right] = \left[1-\left(a_{ij1}\right)^2-\left(a_{ij4}\right)^2, 1-\left(a_{ij1}\right)^2-\left(a_{ij4}\right)^2\right]+\left[a^2, b^2\right]$$

$$\Rightarrow \left[1-\left(a_{ij2}\right)^2, 1-\left(a_{ij1}\right)^2\right] = \left[1-\left(a_{ij1}\right)^2-\left(a_{ij4}\right)^2+a^2, 1-\left(a_{ij1}\right)^2-\left(a_{ij4}\right)^2+b^2\right]$$

$$\Rightarrow 1-(a_{ij2})^2 = 1-(a_{ij1})^2-(a_{ij4})^2+a^2,$$

$$1-(a_{ij1})^2 = 1-(a_{ij1})^2-(a_{ij4})^2+b^2$$

$$\Rightarrow a^2 = -(a_{ij2})^2+(a_{ij1})^2+(a_{ij4})^2,$$

$$b^2 = -(a_{ij1})^2+(a_{ij1})^2+(a_{ij4})^2 = (a_{ij4})^2 \Rightarrow b = a_{ij4}.$$

Using the condition (iii), $(a_{ij1})^2+(a_{ij4})^2 = (a_{ij2})^2+(a_{ij3})^2$ i.e., $(a_{ij3})^2 = -(a_{ij2})^2 + (a_{ij1})^2+(a_{ij4})^2$,

$$a^2 = (a_{ij3})^2 \Rightarrow a = a_{ij3}.$$

$$\text{i.e., } [a,b] = [a_{ij3}, a_{ij4}]$$

Similarly, the same can also be proved by using the equation (9.6).

(ii) Let the interval of degree of non-membership $[a_{ij3}, a_{ij4}]$ and the interval of degree of hesitation, $[a_{ij5}, a_{ij6}] = \left[\sqrt{1-(a_{ij1})^2-(a_{ij4})^2}, \sqrt{1-(a_{ij1})^2-(a_{ij4})^2}\right]$ be known and $[a,b]$ represents the unknown interval of degree of membership. Then, using the equation (9.5),

$$[1,1]-\left[(a_{ij3})^2,(a_{ij4})^2\right] = \left[1-(a_{ij1})^2-(a_{ij4})^2, 1-(a_{ij1})^2-(a_{ij4})^2\right]+[a^2,b^2]$$

$$\Rightarrow \left[1-(a_{ij4})^2, 1-(a_{ij3})^2\right] = \left[1-(a_{ij1})^2-(a_{ij4})^2+a^2, 1-(a_{ij1})^2-(a_{ij4})^2+b^2\right]$$

$$\Rightarrow 1-(a_{ij4})^2 = 1-(a_{ij1})^2-(a_{ij4})^2+a^2, 1-(a_{ij3})^2 = 1-(a_{ij1})^2-(a_{ij4})^2+b^2$$

$$\Rightarrow a^2 = -(a_{ij4})^2+(a_{ij1})^2+(a_{ij4})^2 = (a_{ij1})^2 \Rightarrow a = a_{ij1},$$

$$b^2 = -(a_{ij3})^2+(a_{ij1})^2+(a_{ij4})^2.$$

Using the condition (iii), $(a_{ij1})^2+(a_{ij4})^2 = (a_{ij2})^2+(a_{ij3})^2$ i.e., $(a_{ij2})^2 = -(a_{ij3})^2+(a_{ij1})^2+(a_{ij4})^2$

$b^2 = (a_{ij2})^2 \Rightarrow b = a_{ij2}$.

i.e., $[a, b] = [a_{ij1}, a_{ij2}]$.

Similarly, the same can also be proved by using the equation (9.6).

9.4 Illustrative examples

In this section, the proposed definition is illustrated with the help of some numerical examples.

Example 9.1 If $\tilde{\alpha}_{ij} = ([a_{ij1}, a_{ij2}], [a_{ij3}, a_{ij4}]) = ([0.80, 0.90], [0.10, 0.20])$. Then, according to proposed definition, it is not an IVPFS as the condition $(a_{ij1})^2 + (a_{ij4})^2 = (a_{ij2})^2 + (a_{ij3})^2$ is not satisfying. While, according to the existing definition [164], it is an IVPFS. Furthermore, according to existing definition [164], the interval of degree of hesitation will be

$$\left[\sqrt{1-(0.90)^2-(0.20)^2}, \sqrt{1-(0.80)^2-(0.10)^2}\right]$$

$$= \left[\sqrt{1-0.81-0.04}, \sqrt{1-0.64-0.01}\right] = \left[\sqrt{0.15}, \sqrt{0.35}\right].$$

Now, if it is assumed that the interval of degree of membership $[0.80, 0.90]$ and interval of degree of hesitation $[\sqrt{0.15}, \sqrt{0.35}]$ are known. Then, according to the existing definition [164], the interval of degree of non-membership $([a_{ij3}, a_{ij4}])$ can be obtained by subtracting the sum of square of these intervals from $[1,1]$ i.e., according to the existing definition, $[a_{ij3}, a_{ij4}] = \left[\sqrt{1-(0.90)^2-(\sqrt{0.35})^2}, \sqrt{1-(0.80)^2-(\sqrt{0.15})^2}\right] =$

$$\left[\sqrt{1-0.81-0.35}, \sqrt{1-0.64-0.15}\right] = \left[\sqrt{-0.16}, \sqrt{0.21}\right] \neq [0.10, 0.20].$$

This clearly indicates that according to existing definition [164], the lower bound a_{ij3} of the interval of degree of non-membership $[a_{ij3}, a_{ij4}]$ is a negative real number, which contradicts the well known fact that a_{ij3} should always be a non-negative real number lying between 0 and 1.

Example 9.2 If $\tilde{\alpha}_{ij} = ([a_{ij1}, a_{ij2}], [a_{ij3}, a_{ij4}]) = ([0.40, 0.60], [0.70, \sqrt{0.69}])$ then according to existing definition [164], it is not an IVPFS as the condition $\left(a_{ij2}\right)^2 + \left(a_{ij4}\right)^2 = 1.05 \leq 1$ is not satisfying. However, according to proposed definition, it is an IVPFS as all the conditions $0 \leq a_{ij1} \leq a_{ij2} \leq 1$, $0 \leq a_{ij3} \leq a_{ij4} \leq 1$, $\left(a_{ij1}\right)^2 + \left(a_{ij4}\right)^2 = \left(a_{ij2}\right)^2 + \left(a_{ij3}\right)^2$ and $\left(a_{ij1}\right)^2 + \left(a_{ij4}\right)^2 \leq 1$ or $\left(a_{ij2}\right)^2 + \left(a_{ij3}\right)^2 \leq 1$ are satisfying. Furthermore, as $a_{ij1} = 0.40$, $a_{ij2} = 0.60$, $a_{ij3} = 0.70$ and $a_{ij4} = \sqrt{0.69}$ so using equation (9.13) or equation (9.15) i.e., $\left(a_{ij5}\right)^2 = 1 - \left(a_{ij2}\right)^2 - \left(a_{ij3}\right)^2$ or $\left(a_{ij5}\right)^2 = 1 - \left(a_{ij1}\right)^2 - \left(a_{ij4}\right)^2$, the obtained value of a_{ij5} is 0.15 and using the equation (9.14) or the equation (9.16) i.e., $\left(a_{ij6}\right)^2 = 1 - \left(a_{ij1}\right)^2 - \left(a_{ij4}\right)^2$ or $\left(a_{ij6}\right)^2 = 1 - \left(a_{ij2}\right)^2 - \left(a_{ij3}\right)^2$, the obtained value of a_{ij6} is also 0.15. Therefore, the interval of degree of hesitation $\left[a_{ij5}, a_{ij6}\right]$ is $\left[\sqrt{0.15}, \sqrt{0.15}\right]$.

Now, the following clearly indicates that on considering any two among $[0.40, 0.60]$, $\left[0.70, \sqrt{0.69}\right]$ and $\left[\sqrt{0.15}, \sqrt{0.15}\right]$, the third one can be obtained.

(i) Let $\left[a_{ij1}, a_{ij2}\right] = [0.40, 0.60]$ and $\left[a_{ij5}, a_{ij6}\right] = \left[\sqrt{0.15}, \sqrt{0.15}\right]$ be known. Then

$$[1,1] - \left[\left(a_{ij1}\right)^2, \left(a_{ij2}\right)^2\right] = \left[\left(a_{ij3}\right)^2, \left(a_{ij4}\right)^2\right] + \left[\left(a_{ij5}\right)^2, \left(a_{ij6}\right)^2\right]$$

$$\Rightarrow 1 - [(0.40)^2, (0.60)^2] = \left[\left(a_{ij3}\right)^2, \left(a_{ij4}\right)^2\right] + \left[\left(\sqrt{0.15}\right)^2, \left(\sqrt{0.15}\right)^2\right]$$

$$\Rightarrow [0.64, 0.84] = \left[\left(a_{ij3}\right)^2 + \left(\sqrt{0.15}\right)^2, \left(a_{ij4}\right)^2 + \left(\sqrt{0.15}\right)^2\right]$$

$$\Rightarrow 0.64 = 0.15 + \left(a_{ij3}\right)^2 \text{ and } 0.84 = 0.15 + \left(a_{ij4}\right)^2$$

$$\Rightarrow \left(a_{ij3}\right)^2 = 0.49 \text{ and } \left(a_{ij4}\right)^2 = 0.69$$

$$\Rightarrow \left[a_{ij3}, a_{ij4}\right] = \left[0.7, \sqrt{0.69}\right].$$

(ii) Let $\left[a_{ij3}, a_{ij4}\right] = \left[0.7, \sqrt{0.69}\right]$ and $\left[a_{ij5}, a_{ij6}\right] = \left[\sqrt{0.15}, \sqrt{0.15}\right]$ be known. Then

$$[1,1] - \left[\left(a_{ij3}\right)^2, \left(a_{ij4}\right)^2\right] = \left[\left(a_{ij1}\right)^2, \left(a_{ij2}\right)^2\right] + \left[\left(a_{ij5}\right)^2, \left(a_{ij6}\right)^2\right]$$

$$\Rightarrow 1 - \left[(0.7)^2, \left(\sqrt{0.69}\right)^2\right] = \left[\left(a_{ij1}\right)^2, \left(a_{ij2}\right)^2\right] + \left[\left(\sqrt{0.15}\right)^2, \left(\sqrt{0.15}\right)^2\right]$$

$$\Rightarrow [0.31,0.51] = \left[0.15 + \left(a_{ij1}\right)^2, 0.15 + \left(a_{ij2}\right)^2\right]$$

$$\Rightarrow 0.31 = 0.15 + \left(a_{ij1}\right)^2 \text{ and } 0.51 = 0.15 + \left(a_{ij2}\right)^2$$

$$\Rightarrow \left(a_{ij1}\right)^2 = 0.16 \text{ and } \left(a_{ij2}\right)^2 = 0.36$$

$$\Rightarrow \left[\left(a_{ij1}\right)^2, \left(a_{ij2}\right)^2\right] = [0.16,0.36] \text{ i.e., } \left[a_{ij1}, a_{ij2}\right] = [0.40, 0.60].$$

Example 9.3 If $\tilde{\alpha}_{ij} = \left(\left[a_{ij1}, a_{ij2}\right], \left[a_{ij3}, a_{ij4}\right]\right) = ([0.10,0.50], [0.10,0.50])$. Then, it is an IVPFS according to both existing as well as proposed definition. However, according to existing definition [164], the interval of degree of hesitation will be

$$\left[\sqrt{1-(0.50)^2-(0.50)^2}, \sqrt{1-(0.10)^2-(0.10)^2}\right]$$

$$= \left[\sqrt{1-0.25-0.25}, \sqrt{1-0.01-0.01}\right] = \left[\sqrt{0.50}, \sqrt{0.98}\right].$$

Now, if it is assumed that the interval of degree of membership $[0.10,0.50]$ and interval of the degree of hesitation $\left[\sqrt{0.50}, \sqrt{0.98}\right]$ are known. Then, according to the existing definition [164], the interval of the degree of non-membership can be obtained by subtracting the sum of the square of these intervals from [1,1]. i.e., according to the existing definition [164],

$$\left[a_{ij3}, a_{ij4}\right] = \left[\sqrt{1-(0.50)^2-\left(\sqrt{0.98}\right)^2}, \sqrt{1-(0.10)^2-\left(\sqrt{0.98}\right)^2}\right] =$$

$$\left[\sqrt{1-0.25-0.98}, \sqrt{1-0.01-0.50}\right] = \left[\sqrt{1-1.23}, \sqrt{1-0.51}\right] = \left[\sqrt{-0.23}, \sqrt{0.49}\right] \neq$$

$[0.10,0.50]$.

This clearly indicates that according to existing definition [164], the obtained value of a_{ij3} is a negative real number, which contradicts the well known fact that the a_{ij3} should always be a non-negative real number lying between 0 and 1.

However, according to proposed definition, it is an IVPFS as all the conditions $0 \leq a_{ij1} \leq a_{ij2} \leq 1$, $0 \leq a_{ij3} \leq a_{ij4} \leq 1$, $\left(a_{ij1}\right)^2 + \left(a_{ij4}\right)^2 = \left(a_{ij2}\right)^2 + \left(a_{ij3}\right)^2$ and $\left(a_{ij1}\right)^2 + \left(a_{ij4}\right)^2 \leq 1$ or $\left(a_{ij2}\right)^2 + \left(a_{ij3}\right)^2 \leq 1$ are satisfying. Furthermore, as $a_{ij1} = 0.10, a_{ij2} = 0.50$,

$a_{ij3} = 0.10$ and $a_{ij4} = 0.50$ so using the equation (9.13) or the equation (9.15) i.e., $\left(a_{ij5}\right)^2 = 1 - \left(a_{ij2}\right)^2 - \left(a_{ij3}\right)^2$ or $\left(a_{ij5}\right)^2 = 1 - \left(a_{ij1}\right)^2 - \left(a_{ij4}\right)^2$, the obtained value of a_{ij5} is $\sqrt{0.74}$ and using the equation (9.14) or the equation (9.16) i.e., $\left(a_{ij6}\right)^2 = 1 - \left(a_{ij1}\right)^2 - \left(a_{ij4}\right)^2$ or $\left(a_{ij6}\right)^2 = 1 - \left(a_{ij2}\right)^2 - \left(a_{ij3}\right)^2$, the obtained value of a_{ij6} is also $\sqrt{0.74}$. Therefore, the interval of degree of hesitation $\left[a_{ij5}, a_{ij6}\right]$ is $\left[\sqrt{0.74}, \sqrt{0.74}\right]$.

Now, the following clearly indicates that on considering any two among $[0.10,0.50], [0.10,0.50]$ and $\left[\sqrt{0.74}, \sqrt{0.74}\right]$, the third one can be obtained.

(1) Let $\left[a_{ij1}, a_{ij2}\right] = [0.10,0.50]$ and $\left[a_{ij5}, a_{ij6}\right] = \left[\sqrt{0.74}, \sqrt{0.74}\right]$ be known. Then,

$$[1,1] - \left[\left(a_{ij1}\right)^2, \left(a_{ij2}\right)^2\right] = \left[\left(a_{ij3}\right)^2, \left(a_{ij4}\right)^2\right] + \left[\left(a_{ij5}\right)^2, \left(a_{ij6}\right)^2\right]$$

$$\Rightarrow 1 - [0.01,0.25] = \left[\left(a_{ij3}\right)^2, \left(a_{ij4}\right)^2\right] + [0.74,0.74]$$

$$\Rightarrow [0.75,0.99] = \left[\left(a_{ij3}\right)^2 + 0.74, \left(a_{ij4}\right)^2 + 0.74\right]$$

$$\Rightarrow 0.75 = \left(a_{ij3}\right)^2 + 0.74 \text{ and } 0.99 = \left(a_{ij4}\right)^2 + 0.74$$

$$\Rightarrow a_{ij3} = 0.10 \text{ and } a_{ij4} = 0.50$$

$$\Rightarrow \left[a_{ij3}, a_{ij4}\right] = [0.10,0.50].$$

(2) Let $\left[a_{ij3}, a_{ij4}\right] = [0.10,0.50]$ and $\left[a_{ij5}, a_{ij6}\right] = \left[\sqrt{0.74}, \sqrt{0.74}\right]$ be known. Then,

$$[1,1] - \left[\left(a_{ij1}\right)^2, \left(a_{ij2}\right)^2\right] = [0.01,0.25] + \left[\left(a_{ij5}\right)^2, \left(a_{ij6}\right)^2\right]$$

$$\Rightarrow 1 - \left[\left(a_{ij1}\right)^2, \left(a_{ij2}\right)^2\right] = [0.01,0.25] + [0.74,0.74]$$

$$\Rightarrow \left[1 - \left(a_{ij2}\right)^2, 1 - \left(a_{ij1}\right)^2\right] = [0.75,0.99]$$

$$\Rightarrow 1 - \left(a_{ij2}\right)^2 = 0.75 \text{ and } 1 - \left(a_{ij1}\right)^2 = 0.99$$

$$\Rightarrow a_{ij1} = 0.10 \text{ and } a_{ij2} = 0.50$$

$$\Rightarrow \left[a_{ij1}, a_{ij2}\right] = [0.10,0.50].$$

9.5 Solution of a real-life IVPFMADMPr

In this section, the existing IVPFMADMPr [258] having completely known attribute weights have been solved with the following modification:

“Rating value of each alternative over each criterion is represented as the proposed IVPFS instead of the existing IVPFS”.

There is a panel of four alternatives for the investment company to invest money in best option $A_i, i = 1,2,3,4$ on the basis of the following three benefit attributes.

(i) G_1: Risk analysis

(ii) G_2: Growth

(iii) G_3: Environmental impact

Furthermore,

(i) The $(i,j)^{th}$ element of the Table 9.1, represented by an IVPFS $\tilde{\alpha}_{ij} = \left(\left[a_{ij1}, a_{ij2}\right], \left[a_{ij3}, a_{ij4}\right]\right)$, represents the rating value of the i^{th}-company over the j^{th}-attribute.

Table 9.1: Rating Values

Attributes→ ↓ Alternatives	G_1	G_2	G_3
A_1	$\begin{pmatrix}[0.40,0.60],\\ [0.70,\sqrt{0.69}]\end{pmatrix}$	$\begin{pmatrix}[0.10,0.50],\\ [0.10,0.50]\end{pmatrix}$	$\begin{pmatrix}[0.10,0.40],\\ [0.70,0.80]\end{pmatrix}$
A_2	$\begin{pmatrix}[0.20,0.50],\\ [0.20,0.50]\end{pmatrix}$	$\begin{pmatrix}[\sqrt{0.18},0.50],\\ [0.30,0.40]\end{pmatrix}$	$\begin{pmatrix}[0.20,0.70],\\ [0.60,0.90]\end{pmatrix}$
A_3	$\begin{pmatrix}[0.20,0.70],\\ [0.60,0.90]\end{pmatrix}$	$\begin{pmatrix}[0.10,0.40],\\ [0.70,0.80]\end{pmatrix}$	$\begin{pmatrix}[0.20,0.50],\\ [0.20,0.50]\end{pmatrix}$

A_4	$\begin{pmatrix}[0.10,0.40],\\ [0.70,0.80]\end{pmatrix}$	$\begin{pmatrix}[0.40,0.60],\\ [0.70,\sqrt{0.69}]\end{pmatrix}$	$\begin{pmatrix}[0.30,0.40],\\ [0.30,0.40]\end{pmatrix}$

(ii) Weights w_1, w_2 and w_3 of the attributes G_1, G_2 and G_3 respectively are $w_1 = w_2 = w_3 = \frac{1}{3}$.

Using the Jorawar method, proposed in Section 5.4 of Chapter 5, the ranking of the alternatives for the considered real-life IVPFMADMPr having completely known attribute weights can be obtained as follows.

Since, all the attribute weights are known so there is no need to apply Step 1 to Step 4 of the proposed Jorawar method.

Step 5: On considering the OAWV, $(w_1, w_2, w_3) = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ and using Step 5 of the Jujhar method, proposed in Section 5.4 of Chapter 5,

$$\tilde{P}_1 = \left(\left[\left(\frac{1}{3}\right)(0.40) + \left(\frac{1}{3}\right)(0.10) + \left(\frac{1}{3}\right)(0.10), \left(\frac{1}{3}\right)(0.60) + \left(\frac{1}{3}\right)(0.50) + \left(\frac{1}{3}\right)(0.40)\right],\right.$$

$$\left.\left[\left(\frac{1}{3}\right)(0.70) + \left(\frac{1}{3}\right)(0.10) + \left(\frac{1}{3}\right)(0.70), \left(\frac{1}{3}\right)\left(\sqrt{0.69}\right) + \left(\frac{1}{3}\right)(0.50) + \left(\frac{1}{3}\right)(0.80)\right]\right)$$

$$= ([0.1999, 0.4998], [0.4999, 0.7100]),$$

$$\tilde{P}_2 = \left(\left[\left(\frac{1}{3}\right)(0.20) + \left(\frac{1}{3}\right)\left(\sqrt{0.18}\right) + \left(\frac{1}{3}\right)(0.20), \left(\frac{1}{3}\right)(0.50) + \left(\frac{1}{3}\right)(0.50) + \left(\frac{1}{3}\right)(0.70)\right],\right.$$

$$\left.\left[\left(\frac{1}{3}\right)(0.20) + \left(\frac{1}{3}\right)(0.30) + \left(\frac{1}{3}\right)(0.60), \left(\frac{1}{3}\right)(0.50) + \left(\frac{1}{3}\right)(0.40) + \left(\frac{1}{3}\right)(0.90)\right]\right)$$

$$= ([0.2745, 0.5665], [0.3664, 0.5998]),$$

$$\tilde{P}_3 = \left(\left[\left(\frac{1}{3}\right)(0.20) + \left(\frac{1}{3}\right)(0.10) + \left(\frac{1}{3}\right)(0.20), \left(\frac{1}{3}\right)(0.70) + \left(\frac{1}{3}\right)(0.40) + \left(\frac{1}{3}\right)(0.50)\right],\right.$$

$$\left.\left[\left(\frac{1}{3}\right)(0.60) + \left(\frac{1}{3}\right)(0.70) + \left(\frac{1}{3}\right)(0.20), \left(\frac{1}{3}\right)(0.90) + \left(\frac{1}{3}\right)(0.80) + \left(\frac{1}{3}\right)(0.50)\right]\right)$$

$$= ([0.1665, 0.5332], [0.4998, 0.7331]),$$

$$\tilde{P}_4 = \left(\left[\left(\frac{1}{3}\right)(0.10) + \left(\frac{1}{3}\right)(0.40) + \left(\frac{1}{3}\right)(0.30), \left(\frac{1}{3}\right)(0.40) + \left(\frac{1}{3}\right)(0.60) + \left(\frac{1}{3}\right)(0.40)\right],\right.$$

$$\left.\left[\left(\frac{1}{3}\right)(0.70) + \left(\frac{1}{3}\right)(0.70) + \left(\frac{1}{3}\right)(0.30), \left(\frac{1}{3}\right)(0.80) + \left(\frac{1}{3}\right)\left(\sqrt{0.69}\right) + \left(\frac{1}{3}\right)(0.40)\right]\right)$$

$= ([0.2665, 0.4665], [0.5665, 0.6767])$,

Step 6: Using Step 6 of the Jorawar method, proposed in Section 5.4 of Chapter 5,

$S(\tilde{P}_1) = -0.2321, S(\tilde{P}_2) = -0.0488, S(\tilde{P}_3) = -0.2376$ and $S(\tilde{P}_4) = -0.2451$.

Since, $S(\tilde{P}_2) > S(\tilde{P}_1) > S(\tilde{P}_3) > S(\tilde{P}_4)$. So, according to Step 6 of the Jorawar method, proposed in Section 5.4 of Chapter 5, the ranking of the alternatives is $A_2 > A_1 > A_3 > A_4$.

9.6 Conclusions

It is shown that the existing definition of an IVPFS [164] is not appropriate and the appropriate definition of an IVPFS is proposed. Also, the validity of the proposed definition is discussed. Furthermore, the proposed IVPFS is used to represent the rating value of each alternative over each attribute in an existing real-life IVPFMADMPr [164] having completely known attribute weights. Finally, the modified real-life IVPFMADMPr having completely known attribute is solved by the proposed Jorawar method.

Chapter 10

Future Scope[9]

The following problems may be considered as challenging open research problems.

(1) Tao et al. [196] proposed the expressions (10.1) and (10.2) to evaluate the sum and the product respectively of two IFSs $\alpha_1 = \langle \mu_{\alpha_1}, \nu_{\alpha_1} \rangle$ and $\alpha_2 = \langle \mu_{\alpha_2}, \nu_{\alpha_2} \rangle$.

(i) $$\alpha_1 \oplus_c \alpha_2 = \langle 1 - \phi^{-1}[\phi(1-\mu_{\alpha_1}) + \phi(1-\mu_{\alpha_2})], \phi^{-1}[\varphi(\nu_{\alpha_1}) + \phi(\nu_{\alpha_2})] \rangle \tag{10.1}$$

(ii) $$\alpha_1 \otimes_c \alpha_2 = \langle \phi^{-1}[\phi(\mu_{\alpha_1}) + \phi(\mu_{\alpha_2})], 1 - \phi^{-1}[\phi(1-\nu_{\alpha_1}) + \phi(1-\nu_{\alpha_2})] \rangle \tag{10.2}$$

Also, using the operational law (10.1), Tao et al. [196], proposed the IFCAAO (10.3)

$$\oplus_{c\,i=1}^{\;n} w_i \alpha_i = \langle 1 - \phi^{-1}\left(\sum_{i=1}^{n} w_i \phi\left(1-\mu_{\alpha_i}\right)\right), \phi^{-1}\left(\sum_{i=1}^{n} w_i \phi\left(\nu_{\alpha_i}\right)\right) \rangle \tag{10.3}$$

where, ϕ is a strictly decreasing function such that $\phi(1) = 0, \phi(0) = \infty, \phi^{-1}(0) = 1$ and $\phi^{-1}(\infty) = 0$.

If $\alpha_1, \alpha_2, \ldots, \alpha_n$ are n IFSs then the expressions (10.1) and (10.2) will be transformed into the expressions (10.4) and (10.5) respectively.

(i) $$\oplus_{c\,i=1}^{\;n} \alpha_i = \langle 1 - \phi^{-1}\left(\phi(1-\mu_{\alpha_1}) + \phi(1-\mu_{\alpha_2}) + \cdots + \phi(1-\mu_{\alpha_n})\right), \phi^{-1}\left(\phi(\nu_{\alpha_1}) + \phi(\nu_{\alpha_2}) + \cdots + \phi(\nu_{\alpha_n})\right) \rangle \tag{10.4}$$

(ii) $$\otimes_{c\,i=1}^{\;n} \alpha_i = \langle \phi^{-1}\left(\phi(\mu_{\alpha_1}) + \phi(\mu_{\alpha_2}) + \cdots + \phi(\mu_{\alpha_n})\right), 1 - \phi^{-1}\left(\phi(1-\nu_{\alpha_1}) + \phi(1-\nu_{\alpha_2}) + \cdots + \phi(1-\nu_{\alpha_n})\right) \rangle \tag{10.5}$$

However, the following examples clearly indicates that

(i) The expression (10.4), to evaluate the sum of IFSs, can be used only if $\alpha_i \neq \langle 1,0 \rangle$ for any i.

(ii) The expression (10.5), to evaluate the product of IFSs, can be used only if $\alpha_i \neq \langle 0,1 \rangle$

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[9] Some contents of this chapter have been published in "Computers and Industrial Engineering 135 (2019) 314-316 and Cognitive Computation (2020) doi.org/10.1007/s12559-020-09746-0"

for any i.

The following example clearly indicates that if $\alpha_i = \langle 1,0 \rangle$ then $\oplus c_{i=1}^{n} \alpha_i = \langle 1,0 \rangle$ i.e, $\oplus c_{i=1}^{n} \alpha_i$ is independent from the remaining IFSs α_i. Hence, $\oplus c_{i=1}^{n} \alpha_i$ can be used only if $\alpha_i \neq \langle 1,0 \rangle$ for any i.

Example 10.1 Let $\alpha_1 = \langle 1,0 \rangle, \alpha_2 = \langle 0.4,0.3 \rangle, \alpha_3 = \langle 0.5,0.2 \rangle$. Then,

$\alpha_1 \oplus \alpha_2 \oplus \alpha_3 = \langle 1 - \phi^{-1}[\phi(1-\mu_{\alpha_1}) + \phi(1-\mu_{\alpha_2}) + \phi(1-\mu_{\alpha_3})], \phi^{-1}[\phi(\nu_{\alpha_1}) + \phi(\nu_{\alpha_2}) + \phi(\nu_{\alpha_3})] \rangle$

$= \langle 1 - \phi^{-1}[\phi(1-1) + \phi(1-0.4) + \phi(1-0.5)], \phi^{-1}[\phi(0) + \phi(0.3) + \phi(0.2)] \rangle$

$= \langle 1 - \phi^{-1}[\phi(0) + \phi(0.6) + \phi(0.5)], \phi^{-1}[\phi(0) + \phi(0.3) + \phi(0.2)] \rangle$

$= \langle 1 - \phi^{-1}[\infty + \phi(0.6) + \phi(0.5)], \phi^{-1}[\infty + \phi(0.3) + \phi(0.2)] \rangle$

$= \langle 1 - \phi^{-1}[\infty], \phi^{-1}[\infty] \rangle$

$= \langle 1 - 0,0 \rangle = \langle 1,0 \rangle$.

On the basis of the above discussion, it can be easily concluded that the expression (10.4) can be used only if $\alpha_i \neq \langle 1,0 \rangle$ for any i. Therefore, the IFCAAO (10.3) can be used only if $\alpha_i \neq \langle 1,0 \rangle$ for any i.

To overcome this limitation of IFCAAO can be considered as challenging open research problem.

(2) Wei [226] proposed the expression (10.6) to evaluate the product of two PIFSs $\alpha_1 = (\mu_{\alpha_1}, \eta_{\alpha_1}, \nu_{\alpha_1})$ and $\alpha_2 = (\mu_{\alpha_2}, \eta_{\alpha_2}, \nu_{\alpha_2})$.

$$\alpha_1 \otimes \alpha_2 = \left(\mu_{\alpha_1}\mu_{\alpha_2}, 1 - (1 - \eta_{\alpha_1})(1 - \eta_{\alpha_2}), 1 - (1 - \nu_{\alpha_1})(1 - \nu_{\alpha_2})\right) \tag{10.6}$$

In future, the other researchers may use the expression (10.6) to evaluate the product of two PIFSs in their research work. However, the following example clearly indicates that expression (10.6) is not valid.

Example 10.2 Let $\alpha_1 = (\mu_{\alpha_1}, \eta_{\alpha_1}, \nu_{\alpha_1}) = (0.55,0.15,0.15)$ and $\alpha_2 = (\mu_{\alpha_2}, \eta_{\alpha_2}, \nu_{\alpha_2}) =$

(0.15,0.70,0.10) be two PIFSs. Then, $\alpha_1 \otimes \alpha_2 = \left(\mu_{\alpha_1}\mu_{\alpha_2}, 1-\left(1-\ \eta_{\alpha_1}\right)\left(1-\eta_{\alpha_2}\right), 1-\left(1-\nu_{\alpha_1}\right)\left(1-\nu_{\alpha_2}\right)\right) = \left((0.55)(0.15), 1-(1-0.15)(1-0.70), 1-(1-0.15)(1-0.10)\right) = (0.082, 0.745, 0.235)$.

It is obvious that, $0.082 +\ 0.745 + 0.235 = 1.062$ is greater than 1. So, $\alpha_1 \otimes \alpha_2$ is not PIFS as for a PIFS the following necessary conditions should be satisfied.

(i) $0 \leq \mu_{\alpha_1}\mu_{\alpha_2} \leq 1$.

(ii) $0 \leq 1-\left(1-\ \eta_{\alpha_1}\right)\left(1-\eta_{\alpha_2}\right) \leq 1$.

(iii) $0 \leq\ 1-\left(1-\nu_{\alpha_1}\right)\left(1-\nu_{\alpha_2}\right) \leq 1$.

(iv) $\mu_{\alpha_1}\mu_{\alpha_2} + 1-\left(1-\ \eta_{\alpha_1}\right)\left(1-\eta_{\alpha_2}\right) +\ 1-\left(1-\nu_{\alpha_1}\right)\left(1-\nu_{\alpha_2}\right) \leq 1$.

Therefore, the expression (10.6) is not valid.

If in the existing expression (10.6) the term $1-\left(1-\ \eta_{\alpha_1}\right)\left(1-\eta_{\alpha_2}\right)$ is replaced with the term $\eta_{\alpha_1}\eta_{\alpha_2}$. Then, the expression (10.6), proposed by Wei [226] to evaluate the product of two PIFSs $\alpha_1 = \left(\mu_{\alpha_1}, \eta_{\alpha_1}, \nu_{\alpha_1}\right)$ and $\alpha_2 = \left(\mu_{\alpha_2}, \eta_{\alpha_2}, \nu_{\alpha_2}\right)$, will be transformed into the modified expression (10.7).

$$\alpha_1 \otimes \alpha_2 = \left(\mu_{\alpha_1}\mu_{\alpha_2}, \eta_{\alpha_1}\eta_{\alpha_2}, 1-\left(1-\nu_{\alpha_1}\right)\left(1-\nu_{\alpha_2}\right)\right) \tag{10.7}$$

Since,

$$\begin{aligned}
&\mu_{\alpha_1}\mu_{\alpha_2} + \eta_{\alpha_1}\eta_{\alpha_2} + \left(\ 1-\left(1-\nu_{\alpha_1}\right)\left(1-\nu_{\alpha_2}\right)\right) \\
&\leq \left(\mu_{\alpha_1} + \eta_{\alpha_1}\right)\left(\mu_{\alpha_2} + \eta_{\alpha_2}\right) + 1 - \left(1-\nu_{\alpha_1}\right)\left(1-\nu_{\alpha_2}\right) \\
&\leq \left(1-\nu_{\alpha_1}\right)\left(1-\nu_{\alpha_2}\right) + 1 - \left(1-\nu_{\alpha_1}\right)\left(1-\nu_{\alpha_2}\right) \quad \left(\begin{matrix} \because\ \mu_{\alpha_1} + \eta_{\alpha_1} + \nu_{\alpha_1} \leq 1 \\ \mu_{\alpha_2} + \eta_{\alpha_2} + \nu_{\alpha_2} \leq 1 \end{matrix}\right) \\
&\leq 1
\end{aligned}$$

So, one may claim that the modified expression (10.7) is valid. But, if

(i) $\mu_{\alpha_i} = 0$ for any i, then the first term of the expression (10.7) will always be zero.

(ii) $\eta_{\alpha_i} = 0$ for any i, then the second term of the expression (10.7) will always be zero.

(iii) $\nu_{\alpha_i} = 1$ for any i, then the third term of the expression (10.7) will always be 1.

Therefore, it is inappropriate to use the modified expression (10.7) to propose any type of picture fuzzy aggregation operators.

To define an appropriate operation of two PIFSs may be considered as challenging open research problem.

(3) Garg [83] proposed a method to solve MCDM problems under picture fuzzy environment.

To propose the method, firstly, Garg [83] proposed the expressions (10.8) and (10.9) to evaluate the sum and product respectively of two PIFSs $\alpha_1 = \langle \mu_1, \eta_1, \nu_1 \rangle$ and $\alpha_2 = \langle \mu_2, \eta_2, \nu_2 \rangle$.

$$\alpha_1 \oplus \alpha_2 = \langle h^{-1}\big(h(\mu_1) + h(\mu_2)\big), g^{-1}\big(g(\eta_1) + g(\eta_2)\big), g^{-1}\big(g(\nu_1) + g(\nu_2)\big)\rangle \tag{10.8}$$

$$\alpha_1 \otimes \alpha_2 = \langle g^{-1}\big(g(\mu_1) + g(\mu_2)\big), h^{-1}\big(h(\eta_1) + h(\eta_2)\big), h^{-1}\big(h(\nu_1) + h(\nu_2)\big)\rangle \tag{10.9}$$

where, g is a decreasing function generated from t-norm as $T(x, y) = g^{-1}\big(g(x) + g(y)\big)$ such that, $g(1) = 0$ and h generates the t-conorm as $S(x, y) = h^{-1}\big(h(x) + h(y)\big)$, $h(t) = g(1 - t)$.

In future, the other researchers may use the expression (10.9), to propose various types of picture fuzzy weighted geometric aggregation operators and/ or in their research work.

However, the following clearly indicates that the expression (10.9) is not valid

Example 10.3 Let $\alpha_1 = \langle \mu_1, \eta_1, \nu_1 \rangle = \langle 0.55, 0.15, 0.15 \rangle$ and $\alpha_2 = \langle \mu_2, \eta_2, \nu_2 \rangle = \langle 0.15, 0.70, 0.10 \rangle$ be two PIFSs. Then, $\alpha_1 \otimes \alpha_2 = \langle \mu_1\mu_2, 1 - (1 - \eta_1)(1 - \eta_2), 1 - (1 - \nu_1)(1 - \nu_2)\rangle = \langle (0.55)(0.15), 1 - (1 - 0.15)(1 - 0.70), 1 - (1 - 0.15)(1 - 0.10)\rangle = \langle 0.0825, 0.745, 0.235 \rangle$.

It is obvious that, $0.0825 + 0.745 + 0.235 = 1.0625$ is greater than 1. So, $\alpha_1 \otimes \alpha_2$ is not a PFN. Therefore, the expression (10.9) is not valid.

If in the existing expression (10.9) the term $h^{-1}\big(h(\eta_1)+h(\eta_2)\big)$ is replaced with the term $g^{-1}\big(g(\eta_1)+g(\eta_2)\big)$. Then, the expression (10.9), may be transformed into the modified expression (10.10).

$$\alpha_1 \otimes \alpha_2 = \langle g^{-1}\big(g(\mu_1)+g(\mu_2)\big), g^{-1}\big(g(\eta_1)+g(\eta_2)\big), h^{-1}\big(h(\nu_1)+h(\nu_2)\big)\rangle \qquad (10.10)$$

Since, the co-domain of the functions g^{-1} and h^{-1} is the closed interval $[0,1]$. So, the conditions, $0 \le g^{-1}\big(g(\mu_1)+g(\mu_2)\big) \le 1$, $0 \le g^{-1}\big(g(\eta_1)+g(\eta_2)\big) \le 1$ and $0 \le h^{-1}\big(h(\nu_1)+h(\nu_2)\big) \le 1$ will always be satisfied. Also,

$$g^{-1}\big(g(\mu_1)+g(\mu_2)\big)+g^{-1}\big(g(\eta_1)+g(\eta_2)\big)+h^{-1}\big(h(\nu_1)+h(\nu_2)\big)$$
$$\le g^{-1}\big(g(\mu_1+\eta_1)+g(\mu_2+\eta_2)\big)+h^{-1}\big(h(\nu_1)+h(\nu_2)\big)$$
$$= 1-h^{-1}\big(g(\mu_1+\eta_1)+g(\mu_2+\eta_2)\big)+h^{-1}\big(h(\nu_1)+h(\nu_2)\big)$$
$$= 1-h^{-1}\big(g(1-\nu_1)+g(1-\nu_2)\big)+h^{-1}\big(h(\nu_1)+h(\nu_2)\big)$$
$$= 1-h^{-1}\big(h(\nu_1)+h(\nu_2)\big)+h^{-1}\big(h(\nu_1)+h(\nu_2)\big)$$
$$= 1$$

Therefore, one may claim that the modified expression (10.10) is valid. But, if

(i) $\mu_{\alpha_i}=0$ for any i, then the first term of expression (10.10) will always be zero.

(ii) $\eta_{\alpha_i}=0$ for any i, then the second term of expression (10.10) will always be zero.

(iii) $\nu_{\alpha_i}=1$ for any i, then the third term of expression (10.10) will always be 1.

Therefore, it is inappropriate to use the modified expression (10.10) to propose any type of picture fuzzy aggregation operators.

To define an appropriate operation of two PIFSs may be considered as challenging open research problem.

(4) Jana et al. [106] pointed out that till now no one has used the Dombi t-norm and the Dombi t-conorm operations to define the arithmetic aggregation operators of PIFSs. To fill this gap

Jana et al. [106] proposed the Dombi *t*-norm operation (10.11) and the Dombi *t*-conorm operation (10.12) of PIFSs $\tilde{p}_1 = \langle \hat{\mu}_1, \hat{\eta}_1, \hat{\nu}_1 \rangle$ and $\tilde{p}_2 = \langle \hat{\mu}_2, \hat{\eta}_2, \hat{\nu}_2 \rangle$.

$$\tilde{p}_1 \oplus \tilde{p}_2 = \left\langle 1 - \frac{1}{1+\left\{\left(\frac{\hat{\mu}_1}{1-\hat{\mu}_1}\right)^{\mathfrak{R}} + \left(\frac{\hat{\mu}_2}{1-\hat{\mu}_2}\right)^{\mathfrak{R}}\right\}^{\frac{1}{\mathfrak{R}}}}, \frac{1}{1+\left\{\left(\frac{1-\hat{\eta}_1}{\hat{\eta}_1}\right)^{\mathfrak{R}} + \left(\frac{1-\hat{\eta}_2}{\hat{\eta}_2}\right)^{\mathfrak{R}}\right\}^{\frac{1}{\mathfrak{R}}}}, \frac{1}{1+\left\{\left(\frac{1-\hat{\nu}_1}{\hat{\nu}_1}\right)^{\mathfrak{R}} + \left(\frac{1-\hat{\nu}_2}{\hat{\nu}_2}\right)^{\mathfrak{R}}\right\}^{\frac{1}{\mathfrak{R}}}} \right\rangle \quad (10.11)$$

$$\tilde{p}_1 \otimes \tilde{p}_2 = \left\langle \frac{1}{1+\left\{\left(\frac{1-\hat{\mu}_1}{\hat{\mu}_1}\right)^{\mathfrak{R}} + \left(\frac{1-\hat{\mu}_2}{\hat{\mu}_2}\right)^{\mathfrak{R}}\right\}^{\frac{1}{\mathfrak{R}}}}, 1 - \frac{1}{1+\left\{\left(\frac{\hat{\eta}_1}{1-\hat{\eta}_1}\right)^{\mathfrak{R}} + \left(\frac{\hat{\eta}_2}{1-\hat{\eta}_2}\right)^{\mathfrak{R}}\right\}^{\frac{1}{\mathfrak{R}}}}, 1 - \frac{1}{1+\left\{\left(\frac{\hat{\nu}_1}{1-\hat{\nu}_1}\right)^{\mathfrak{R}} + \left(\frac{\hat{\nu}_2}{1-\hat{\nu}_2}\right)^{\mathfrak{R}}\right\}^{\frac{1}{\mathfrak{R}}}} \right\rangle (10.12)$$

where, $\mathfrak{R} \geq 1$ and $\lambda > 0$.

Also, using (10.11) and (10.12), Jana et al. [106] proposed various types of picture fuzzy Dombi weighted averaging operators and picture fuzzy Dombi weighted geometric operators respectively. Furthermore, using the proposed picture fuzzy Dombi weighted averaging/geometric operators, Jana et al. [106] proposed a method for solving multiple attribute decision-making problems under picture fuzzy environment.

In future, the other researchers may use the operation (10.12) and/or the picture fuzzy Dombi weighted geometric operators in their research work and/or to solve real-life MADMPrs under picture fuzzy environment.

However, the following clearly indicates that the expression (10.12) is not valid

Example 10.4 Let $\tilde{p}_1 = \langle \hat{\mu}_1, \hat{\eta}_1, \hat{\nu}_1 \rangle = \langle 0,1,0 \rangle$ and $\tilde{p}_2 = \langle \hat{\mu}_2, \hat{\eta}_2, \hat{\nu}_2 \rangle = \langle 0,0,1 \rangle$ be two PIFSs. Then,

$$\tilde{p}_1 \otimes \tilde{p}_2 = \left\langle \frac{1}{1+\left\{\left(\frac{1-\hat{\mu}_1}{\hat{\mu}_1}\right)^{\mathfrak{R}} + \left(\frac{1-\hat{\mu}_2}{\hat{\mu}_2}\right)^{\mathfrak{R}}\right\}^{\frac{1}{\mathfrak{R}}}}, 1 - \frac{1}{1+\left\{\left(\frac{\hat{\eta}_1}{1-\hat{\eta}_1}\right)^{\mathfrak{R}} + \left(\frac{\hat{\eta}_2}{1-\hat{\eta}_2}\right)^{\mathfrak{R}}\right\}^{\frac{1}{\mathfrak{R}}}}, 1 - \frac{1}{1+\left\{\left(\frac{\hat{\nu}_1}{1-\hat{\nu}_1}\right)^{\mathfrak{R}} + \left(\frac{\hat{\nu}_2}{1-\hat{\nu}_2}\right)^{\mathfrak{R}}\right\}^{\frac{1}{\mathfrak{R}}}} \right\rangle$$

$$= \left\langle \frac{1}{1+\left\{\left(\frac{1-0}{0}\right)^{\mathfrak{R}} + \left(\frac{1-0}{0}\right)^{\mathfrak{R}}\right\}^{\frac{1}{\mathfrak{R}}}}, 1 - \frac{1}{1+\left\{\left(\frac{1}{1-1}\right)^{\mathfrak{R}} + \left(\frac{0}{1-0}\right)^{\mathfrak{R}}\right\}^{\frac{1}{\mathfrak{R}}}}, 1 - \frac{1}{1+\left\{\left(\frac{0}{1-0}\right)^{\mathfrak{R}} + \left(\frac{1}{1-1}\right)^{\mathfrak{R}}\right\}^{\frac{1}{\mathfrak{R}}}} \right\rangle$$

$$= \left\langle \frac{1}{1+\{\infty+\infty\}^{\frac{1}{\Re}}}, 1-\frac{1}{1+\{\infty+0\}^{\frac{1}{\Re}}}, 1-\frac{1}{1+\{0+\infty\}^{\frac{1}{\Re}}} \right\rangle = \left\langle \frac{1}{1+\infty}, 1-\frac{1}{1+\infty}, 1-\frac{1}{1+\infty} \right\rangle$$

$= \langle 0,1,1 \rangle$.

It is obvious that, $0+1+1=2$ is greater than 1. So, $\tilde{p}_1 \otimes \tilde{p}_2$ is not PIFS. Therefore, the operation (10.12) is not valid.

If in the existing operation (10.12) the term $1-\frac{1}{1+\left\{\left(\frac{\hat{\eta}_1}{1-\hat{\eta}_1}\right)^{\Re}+\left(\frac{\hat{\eta}_2}{1-\hat{\eta}_2}\right)^{\Re}\right\}^{\frac{1}{\Re}}}$ is replaced with the term $\frac{1}{1+\left\{\left(\frac{1-\hat{\eta}_1}{\hat{\eta}_1}\right)^{\Re}+\left(\frac{1-\hat{\eta}_2}{\hat{\eta}_2}\right)^{\Re}\right\}^{\frac{1}{\Re}}}$. Then, the Dombi *t*-conorm operation (10.12), will be transformed into the modified Dombi *t*-conorm operation (10.13).

$$\tilde{p}_1 \otimes \tilde{p}_2 = \left\langle \frac{1}{1+\left\{\left(\frac{1-\hat{\mu}_1}{\hat{\mu}_1}\right)^{\Re}+\left(\frac{1-\hat{\mu}_2}{\hat{\mu}_2}\right)^{\Re}\right\}^{\frac{1}{\Re}}}, \frac{1}{1+\left\{\left(\frac{1-\hat{\eta}_1}{\hat{\eta}_1}\right)^{\Re}+\left(\frac{1-\hat{\eta}_2}{\hat{\eta}_2}\right)^{\Re}\right\}^{\frac{1}{\Re}}}, 1-\frac{1}{1+\left\{\left(\frac{\hat{\nu}_1}{1-\hat{\nu}_1}\right)^{\Re}+\left(\frac{\hat{\nu}_2}{1-\hat{\nu}_2}\right)^{\Re}\right\}^{\frac{1}{\Re}}} \right\rangle \qquad (10.13)$$

It can be easily verified that the conditions $0 \leq \frac{1}{1+\left\{\left(\frac{1-\hat{\mu}_1}{\hat{\mu}_1}\right)^{\Re}+\left(\frac{1-\hat{\mu}_2}{\hat{\mu}_2}\right)^{\Re}\right\}^{\frac{1}{\Re}}} \leq 1$, $0 \leq \frac{1}{1+\left\{\left(\frac{1-\hat{\eta}_1}{\hat{\eta}_1}\right)^{\Re}+\left(\frac{1-\hat{\eta}_2}{\hat{\eta}_2}\right)^{\Re}\right\}^{\frac{1}{\Re}}} \leq 1$ and $0 \leq 1-\frac{1}{1+\left\{\left(\frac{\hat{\nu}_1}{1-\hat{\nu}_1}\right)^{\Re}+\left(\frac{\hat{\nu}_2}{1-\hat{\nu}_2}\right)^{\Re}\right\}^{\frac{1}{\Re}}} \leq 1$ will always be satisfied. Also,

$$\frac{1}{1+\left\{\left(\frac{1-\hat{\mu}_1}{\hat{\mu}_1}\right)^{\Re}+\left(\frac{1-\hat{\mu}_2}{\hat{\mu}_2}\right)^{\Re}\right\}^{\frac{1}{\Re}}} + \frac{1}{1+\left\{\left(\frac{1-\hat{\eta}_1}{\hat{\eta}_1}\right)^{\Re}+\left(\frac{1-\hat{\eta}_2}{\hat{\eta}_2}\right)^{\Re}\right\}^{\frac{1}{\Re}}} + 1-\frac{1}{1+\left\{\left(\frac{\hat{\nu}_1}{1-\hat{\nu}_1}\right)^{\Re}+\left(\frac{\hat{\nu}_2}{1-\hat{\nu}_2}\right)^{\Re}\right\}^{\frac{1}{\Re}}}$$

$$\leq \frac{1}{1+\left\{\left(\frac{1-\hat{\mu}_1-\hat{\eta}_1}{\hat{\mu}_1+\hat{\eta}_1}\right)^{\Re}+\left(\frac{1-\hat{\mu}_2-\hat{\eta}_2}{\hat{\mu}_2+\hat{\eta}_2}\right)^{\Re}\right\}^{\frac{1}{\Re}}} + 1-\frac{1}{1+\left\{\left(\frac{\hat{\nu}_1}{1-\hat{\nu}_1}\right)^{\Re}+\left(\frac{\hat{\nu}_2}{1-\hat{\nu}_2}\right)^{\Re}\right\}^{\frac{1}{\Re}}}$$

$$\leq \frac{1}{1+\left\{\left(\frac{\hat{\nu}_1}{1-\hat{\nu}_1}\right)^{\Re}+\left(\frac{\hat{\nu}_2}{1-\hat{\nu}_2}\right)^{\Re}\right\}^{\frac{1}{\Re}}} + 1-\frac{1}{1+\left\{\left(\frac{\hat{\nu}_1}{1-\hat{\nu}_1}\right)^{\Re}+\left(\frac{\hat{\nu}_2}{1-\hat{\nu}_2}\right)^{\Re}\right\}^{\frac{1}{\Re}}}$$

≤ 1.

Therefore, one may conclude that the modified expression (10.13) is valid. But, if

(i) $\mu_{\alpha_i} = 0$ for any i, then the first term of expression (10.13) will always be zero.

(ii) $\eta_{\alpha_i} = 0$ for any i, then the second term of expression (10.13) will always be zero.

(iii) $\nu_{\alpha_i} = 1$ for any i, then the third term of expression (10.13) will always be 1.

Therefore, it is inappropriate to use the modified expression (10.13) to propose any type of picture fuzzy aggregation operators.

To define an appropriate operation of two PFSs may be considered as challenging open research problem.

(5) Garg [82] solved an IVPFMADMPr by the existing TOPSIS method [82] and shown that the obtained preference order of the alternatives is not correct. Garg [82] also proposed the following method for solving same type of problems.

Let there be m-alternatives $A_1, A_2, \ldots, A_m$ and each alternative has n-atttribute $G_1, G_2, \ldots, G_n$. Also, let the evaluation of the decision-maker for the i^{th}-alternative A_i over the j^{th}-attribute G_j be represented by an IVPFS $\tilde{\alpha}_{ij} = \left(\left[a_{ij1}, a_{ij2}\right], \left[a_{ij3}, a_{ij4}\right]\right)$.

Furthermore, let the weight w_j of the j^{th}-attribute satisfies the following restrictions

(i) $w_j \geq 0$

(ii) $\sum_{j=1}^{n} w_j = 1$.

Then, using the Garg's IVPFMADM method [82], the preference order for all the alternatives can be obtained as follows:

Step 1: Check that all the criteria are of same type or not i.e., check that all the criteria are benefit type criteria or cost type criteria.

Case (i): If all the criteria are of same type then go to Step 2.

Case (ii): If some criteria are cost type criteria and the remaining are benefit type criteria then convert the j^{th} cost type criteria into benefit type criteria by replacing all the elements $\tilde{\alpha}_{ij} = \left(\left[a_{ij1}, a_{ij2}\right], \left[a_{ij3}, a_{ij4}\right]\right)$ of the j^{th} column of the IVPFDM $\widetilde{D} = \left(\tilde{\alpha}_{ij}\right)_{m\times n}$ with $\tilde{\alpha}_{ij} = \left(\left[a_{ij3}, a_{ij4}\right], \left[a_{ij1}, a_{ij2}\right]\right)$ and go to Step 2.

Step 2: Using the expression (10.14), transform each IVPF element $\tilde{\alpha}_{ij} = ([a_{ij1}, a_{ij2}], [a_{ij3}, a_{ij4}])$ of the IVPFDM $\widetilde{D} = (\tilde{\alpha}_{ij})_{m\times n} = ([a_{ij1}, a_{ij2}], [a_{ij3}, a_{ij4}])$ into the crisp element d_{ij}

$$d_{ij} = \left(\frac{\left(a_{ij1}^2 - a_{ij3}^2\right)\left(1+\sqrt{1-(a_{ij1})^2-(a_{ij3})^2}\right)+\left(a_{ij2}^2 - a_{ij4}^2\right)\left(1+\sqrt{1-(a_{ij2})^2-(a_{ij4})^2}\right)}{2}\right). \tag{10.14}$$

Step 3: Using the expressions (10.15) and (10.16), find the values of

$$d(A_i, 1) = \sqrt{\sum_{j=1}^{n}\left(w_j(1-d_{ij})^2\right)^2}, i = 1,2,\ldots,m \tag{10.15}$$

$$d(A_i, -1) = \sqrt{\sum_{j=1}^{n}\left(w_j(d_{ij}+1)^2\right)^2}, i = 1,2,\ldots,m \tag{10.16}$$

Step 4: Using the expression (10.17), find the values of $CC(A_i) = \frac{d(A_i,-1)}{d(A_i,1)+d(A_i,-1)}, i = 1,2,\ldots,m.$ (10.17)

Step 5: Check that $CC(A_p) > CC(A_q)$ or $CC(A_p) < CC(A_q)$ or $CC(A_p) = CC(A_q)$.

Case (i): If $CC(A_p) > CC(A_q)$ then $A_p > A_q$.

Case (ii): If $CC(A_p) < CC(A_q)$ then $A_p < A_q$.

Case (iii): If $CC(A_p) = CC(A_q)$ then $A_p = A_q$.

However, the following example clearly indicates that it is inappropriate to use Garg's IVPFMADM method [82] and hence, to resolve the inappropriateness of Garg's method [82] may be considered as challenging open research problem.

Let us consider an IVPFMCDM problem having two alternatives A_1 and A_2 and each alternative has two benefit criteria G_1 and G_2. Also, Table 10.1 represents the IVPFDM of the considered IVPFMCDM problem.

Table 10.1 Rating values

Attributes→ ↓ Alternatives	G_1	G_2
A_1	$([0.30,0.60],[0.30,0.60])$	$([0.20,0.60],[0.20,0.60])$
A_2	$([0.20,0.70],[0.20,0.70])$	$([0.20,0.60],[0.20,0.60])$

It is obvious that the rating value of both the alternatives A_1 and A_2 over the criteria G_2 is same i.e., $([0.20,0.60],[0.20,0.60])$. Therefore, the ranking of the alternatives A_1 and A_2 will depend only upon the rating values of A_1 and A_2 over the criterion G_1 i.e., if the rating value of the alternative A_1 over G_1 i.e., $([0.30,0.60],[0.30,0.60])$ will be greater than the rating value of the alternative A_2 over the same criterion G_1 i.e., $([0.20,0.70],[0.20,0.70])$. Then, the relation will be $A_1 > A_2$ otherwise the relation will be $A_1 < A_2$.

It is pertinent to mention that as $([0.30,0.60],[0.30,0.60]) \neq ([0.20,0.70],[0.20,0.70])$. Therefore, the relation $A_1 = A_2$ is not possible. However, the following clearly indicates that on solving this problem by Garg's IVPFMCDM method [82], the obtained relation is $A_1 = A_2$, which is obviously incorrect.

Using the Garg's IVPFMCDM method [82], the preference order of the alternatives A_1 and A_2 can be obtained as follows:

Step 1: Since both the considered criteria G_1 and G_2 are benefit criteria. So there is no need to apply Step 1 of the Garg's IVPFMCDM method [82].

Step 2: According to Step 2 of Garg's method [82], there is need to calculate $d_{ij}\ \forall\ i = 1,2,\ldots,4; j = 1,2,\ldots,4$. These values are shown in Table 10.2.

Table 10.2: Values of d_{ij}

$d_{11} = 0$	$d_{12} = 0$
$d_{21} = 0$	$d_{22} = 0$

Step 3: Using Step 3 of Garg's IVPFMCDM method [82],

$d(A_1, 1) = \sqrt{(w_1(1 - 0.0)^2)^2 + (w_2(1 - 0.0)^2)^2} = \sqrt{w_1{}^2 + w_2{}^2}$,

$d(A_2, 1) = \sqrt{(w_1(1 - 0.0)^2)^2 + (w_2(1 - 0.0)^2)^2} = \sqrt{w_1{}^2 + w_2{}^2}$,

$d(A_1, -1) = \sqrt{(w_1(0.0 + 1)^2)^2 + (w_2(0.0 + 1)^2)^2} = \sqrt{w_1{}^2 + w_2{}^2}$,

$d(A_2, -1) = \sqrt{(w_1(0.0 + 1)^2)^2 + (w_2(0.0 + 1)^2)^2} = \sqrt{w_1{}^2 + w_2{}^2}$.

Step 4: Using Step 4 of Garg's IVPFMCDM method [82],

(i) $CC(A_1) = \frac{d(A_1,-1)}{d(A_1,1)+d(A_1,-1)} = \frac{\sqrt{w_1{}^2+w_2{}^2}}{\sqrt{w_1{}^2+w_2{}^2}+\sqrt{w_1{}^2+w_2{}^2}} = \frac{1}{2}$

(ii) $CC(A_1) = \frac{d(A_1,-1)}{d(A_1,1)+d(A_1,-1)} = \frac{\sqrt{w_1{}^2+w_2{}^2}}{\sqrt{w_1{}^2+w_2{}^2}+\sqrt{w_1{}^2+w_2{}^2}} = \frac{1}{2}$

Step 5: Since $CC(A_1) = CC(A_2)$. So according to Step 5 of Garg's IVPFMCDM method [82], $A_1 = A_2$.

Therefore, according to Step 5 of the Garg's IVPFMCDM method [82], $A_1 = A_2$, which is mathematically incorrect.

(6) Yue [260] proposed a geometric approach for solving IVIFMAGDM problems. There are seven steps in this approach. In Step 1 to Step 3 of this approach, the IVIFMAGDM problem is transformed into IVIFMADMPr and in Step 4 to Step 7 of this approach the ranking of the alternatives, for the transformed IVIFMADM, is obtained. It is obvious that if one would like to solve an IVIFMADMPr with the help of Yue's geometric approach. Then, there is need to use only Step 4 to Step 7 of this approach. However, the following examples clearly indicates that, it is inappropriate to use Yue's approach [260]. Hence, to

resolve the inappropriateness of Yue's approach [260] may be considered as challenging open research problem.

Let us consider that two students A_1 and A_2 secure marks a_1 and a_2 in the subject S_1. While, the equal marks (say, a) in the subject S_2. Then, to find the ranking of these students is a MADMPr.

Since, the marks of A_1 and A_2 in S_2 are equal. So, the ranking of A_1 and A_2 will depend only upon the marks in the subject S_1 i.e.,

(i) If $a_1 > a_2$ then A_1 is superior to A_2.

(ii) If $a_1 < a_2$ then A_2 is superior to A_1.

(iii) A_1 and A_2 can never be equivalent as $a_1 \neq a_2$.

On the same direction, if Table 10.3 represents the interval-valued intuitionistic fuzzy decision matrix of an IVIFMADMPr having two alternatives A_1, A_2 and two attributes G_1, G_2. Then, the alternative A_1 and A_2 can never be equivalent as $\left(\left[0,\frac{1}{2}\right],\left[0,\frac{1}{2}\right]\right) \neq \left(\left[0,\frac{1}{3}\right],\left[0,\frac{1}{3}\right]\right)$.

Table 10.3: Rating values

Attributes→ ↓ Alternatives	G_1	G_2
A_1	$\left(\left[0,\frac{1}{2}\right],\left[0,\frac{1}{2}\right]\right)$	$\left(\left[0,\frac{1}{2}\right],\left[0,\frac{1}{2}\right]\right)$
A_2	$\left(\left[0,\frac{1}{3}\right],\left[0,\frac{1}{3}\right]\right)$	$\left(\left[0,\frac{1}{2}\right],\left[0,\frac{1}{2}\right]\right)$

While, following clearly indicates that on solving the considered IVIFMADMPr by Yue's approach [260], the relation A_1 equivalent to A_2 is obtained i.e., the ranking of the alternatives A_1 and A_2, obtained by the Yue's approach [260], is not valid.

Using Step 4 to Step 7 of the geometric approach [260], the ranking of the alternatives

A_1 and A_2 can be obtained as follows:

According to Step 4 to Step 7 of the geometric approach [260], to find the ranking of the alternatives A_1 and A_2, there is need to compare the relative closeness, $RC_1 = r_{11} + r_{12} = \frac{P(\tilde{y}_{11} \geq \tilde{y}_1^-)}{P(\tilde{y}_1^+ \geq \tilde{y}_{11}) + P(\tilde{y}_{11} \geq \tilde{y}_1^-)} + \frac{P(\tilde{y}_{12} \geq \tilde{y}_2^-)}{P(\tilde{y}_2^+ \geq \tilde{y}_{12}) + P(\tilde{y}_{12} \geq \tilde{y}_2^-)}$ and $RC_2 = r_{21} + r_{22} = \frac{P(\tilde{y}_{21} \geq \tilde{y}_1^-)}{P(\tilde{y}_1^+ \geq \tilde{y}_{21}) + P(\tilde{y}_{21} \geq \tilde{y}_1^-)} + \frac{P(\tilde{y}_{22} \geq \tilde{y}_2^-)}{P(\tilde{y}_2^+ \geq \tilde{y}_{22}) + P(\tilde{y}_{22} \geq \tilde{y}_2^-)}$, where $P(\tilde{\alpha} \geq \tilde{\beta}) = P\left(([\mu_{\tilde{\alpha}}^l, \mu_{\tilde{\alpha}}^u], [\nu_{\tilde{\alpha}}^l, \nu_{\tilde{\alpha}}^u]) \geq \left([\mu_{\tilde{\beta}}^l, \mu_{\tilde{\beta}}^u], [\nu_{\tilde{\beta}}^l, \nu_{\tilde{\beta}}^u]\right)\right) = \frac{1}{2}\left(P(\mu_{\tilde{\alpha}} \geq \mu_{\tilde{\beta}}) + P(\nu_{\tilde{\beta}} \geq \nu_{\tilde{\alpha}})\right)$.

It is obvious that to find the values of RC_1 and RC_2, there is need to calculate the IVIFPIS $Y_+ = (\tilde{y}_1^+, \tilde{y}_2^+) = \left(([\tau_1^{+l}, \tau_1^{+u}], [\upsilon_1^{+l}, \upsilon_1^{+u}]), ([\tau_2^{+l}, \tau_2^{+u}], [\upsilon_2^{+l}, \upsilon_2^{+u}])\right)$, the IVIFNIS $Y_- = (\tilde{y}_1^-, \tilde{y}_2^-) = \left(([\tau_1^{-l}, \tau_1^{-u}], [\upsilon_1^{-l}, \upsilon_1^{-u}]), ([\tau_2^{-l}, \tau_2^{-u}], [\upsilon_2^{-l}, \upsilon_2^{-u}])\right)$, and the possibility degrees $P(\tilde{y}_{11} \geq \tilde{y}_1^-), P(\tilde{y}_1^+ \geq \tilde{y}_{11}), P(\tilde{y}_{12} \geq \tilde{y}_2^-), P(\tilde{y}_2^+ \geq \tilde{y}_{12}), P(\tilde{y}_{21} \geq \tilde{y}_1^-), P(\tilde{y}_1^+ \geq \tilde{y}_{21})$, $P(\tilde{y}_{22} \geq \tilde{y}_2^-)$ and $P(\tilde{y}_2^+ \geq \tilde{y}_{22})$.

These values can be calculated as follows:

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Since $\tilde{y}_{11} = ([\tau_{11}^l, \tau_{11}^u], [\upsilon_{11}^l, \upsilon_{11}^u]) = \left(\left[0, \frac{1}{2}\right], \left[0, \frac{1}{2}\right]\right)$, $\tilde{y}_{12} = ([\tau_{12}^l, \tau_{12}^u], [\upsilon_{12}^l, \upsilon_{12}^u]) = \left(\left[0, \frac{1}{2}\right], \left[0, \frac{1}{2}\right]\right)$, $\tilde{y}_{21} = ([\tau_{21}^l, \tau_{21}^u], [\upsilon_{21}^l, \upsilon_{21}^u]) = \left(\left[0, \frac{1}{3}\right], \left[0, \frac{1}{3}\right]\right)$ and

$\tilde{y}_{22} = ([\tau_{22}^l, \tau_{22}^u], [\upsilon_{22}^l, \upsilon_{22}^u]) = \left(\left[0, \frac{1}{2}\right], \left[0, \frac{1}{2}\right]\right)$. So, using the existing expression [260],

(i) The IVIFNIS, $\tilde{y}_1^- = ([\tau_1^{-l}, \tau_1^{-u}], [\upsilon_1^{-l}, \upsilon_1^{-u}])$

$= \left(\left[\min(\tau_{11}^l, \tau_{21}^l), \min(\tau_{11}^u, \tau_{21}^u)\right], \left[\max(\upsilon_{11}^l, \upsilon_{21}^l), \max(\upsilon_{11}^u, \upsilon_{21}^u)\right]\right) = \left(\left[0, \frac{1}{3}\right], \left[0, \frac{1}{2}\right]\right)$.

(ii) The IVIFNIS, $\tilde{y}_2^- = ([\tau_2^{-l}, \tau_2^{-u}], [\upsilon_2^{-l}, \upsilon_2^{-u}])$

$= \left(\left[\min(\tau_{12}^l, \tau_{22}^l), \min(\tau_{12}^u, \tau_{22}^u)\right], \left[\max(\upsilon_{12}^l, \upsilon_{22}^l), \max(\upsilon_{12}^u, \upsilon_{22}^u)\right]\right) = \left(\left[0, \frac{1}{2}\right], \left[0, \frac{1}{2}\right]\right)$.

(iii) The IVIFPIS, $\tilde{y}_1^+ = ([\tau_1^{+l}, \tau_1^{+u}], [\upsilon_1^{+l}, \upsilon_1^{+u}])$

$$= \left(\left[\max\left(\tau_{11}^{l}, \tau_{21}^{l}\right), \max(\tau_{11}^{u}, \tau_{21}^{u})\right], \left[\min\left(\upsilon_{11}^{l}, \upsilon_{21}^{l}\right), \ \min(\upsilon_{11}^{u}, \upsilon_{21}^{u})\right]\right) = \left(\left[0, \tfrac{1}{2}\right], \left[0, \tfrac{1}{3}\right]\right).$$

(iv) The IVIFPIS, $\tilde{y}_2^{+} = \left([\tau_2^{+l}, \tau_2^{+u}], [\upsilon_2^{+l}, \upsilon_2^{+u}]\right)$

$$= \left(\left[\max\left(\tau_{12}^{l}, \tau_{22}^{l}\right), \max(\tau_{12}^{u}, \tau_{22}^{u})\right], \left[\min\left(\upsilon_{12}^{l}, \upsilon_{22}^{l}\right), \ \min(\upsilon_{12}^{u}, \upsilon_{22}^{u})\right]\right) = \left(\left[0, \tfrac{1}{2}\right], \left[0, \tfrac{1}{2}\right]\right).$$

Values of the possibility degrees

Using the existing expression [260],

$$P([a,b] \geq [c,d]) = \frac{l_{over}^{([c,d],[a,b])}}{l_{overall}^{([c,d],[a,b])}} = \frac{l_{over}^{([c,d],[a,b])}}{2\left((b-a)+(d-c)\right)}$$

where,

$$l_{over}^{([c,d],[a,b])} = \begin{cases} (b-d)+(d-c)+(b-a)+(a-c); & c \leq a < d < b \\ 2\left((b-a)+(d-c)\right) & ; c < d \leq a < b \\ (d-c)+(b-a)+(a-c) & ; c \leq a < b \leq d \\ (b-d)+(d-c)+(d-a) & ; a \leq c < d \leq b \\ 2(b-c) & ; a < c \leq b \leq d \\ 2\left((b-a)+(d-c)\right) & ; a < b \leq c < d \end{cases}$$

(i) $P(\tilde{y}_{11} \geq \tilde{y}_1^{-}) = \frac{1}{2}\left(P\left(\left[0, \frac{1}{2}\right] \geq \left[0, \frac{1}{3}\right]\right) + P\left(\left[0, \frac{1}{2}\right] \geq \left[0, \frac{1}{2}\right]\right)\right) = \frac{1}{2}(0.6 + 0.5) = 0.55.$

(ii) $P(\tilde{y}_1^{+} \geq \tilde{y}_{11}) = \frac{1}{2}\left(P\left(\left[0, \frac{1}{2}\right] \geq \left[0, \frac{1}{2}\right]\right) + P\left(\left[0, \frac{1}{2}\right] \geq \left[0, \frac{1}{3}\right]\right)\right) = \frac{1}{2}(0.5 + 0.6) = 0.55.$

(iii) $P(\tilde{y}_{12} \geq \tilde{y}_2^{-}) = \frac{1}{2}\left(P\left(\left[0, \frac{1}{2}\right] \geq \left[0, \frac{1}{2}\right]\right) + P\left(\left[0, \frac{1}{2}\right] \geq \left[0, \frac{1}{2}\right]\right)\right) = \frac{1}{2}(0.5 + 0.5) = 0.5.$

(iv) $P(\tilde{y}_2^{+} \geq \tilde{y}_{12}) = \frac{1}{2}\left(P\left(\left[0, \frac{1}{2}\right] \geq \left[0, \frac{1}{2}\right]\right) + P\left(\left[0, \frac{1}{2}\right] \geq \left[0, \frac{1}{2}\right]\right)\right) = \frac{1}{2}(0.5 + 0.5) = 0.5.$

(v) $P(\tilde{y}_{21} \geq \tilde{y}_1^{-}) = \frac{1}{2}\left(P\left(\left[0, \frac{1}{3}\right] \geq \left[0, \frac{1}{3}\right]\right) + P\left(\left[0, \frac{1}{2}\right] \geq \left[0, \frac{1}{3}\right]\right)\right) = \frac{1}{2}(0.5 + 0.6) = 0.55.$

(vi) $P(\tilde{y}_1^{+} \geq \tilde{y}_{21}) = \frac{1}{2}\left(P\left(\left[0, \frac{1}{2}\right] \geq \left[0, \frac{1}{3}\right]\right) + P\left(\left[0, \frac{1}{3}\right] \geq \left[0, \frac{1}{3}\right]\right)\right) = \frac{1}{2}(0.6 + 0.5) = 0.55.$

(vii) $P(\tilde{y}_{22} \geq \tilde{y}_2^{-}) = \frac{1}{2}\left(P\left(\left[0, \frac{1}{2}\right] \geq \left[0, \frac{1}{2}\right]\right) + P\left(\left[0, \frac{1}{2}\right] \geq \left[0, \frac{1}{2}\right]\right)\right) = \frac{1}{2}(0.5 + 0.5) = 0.5.$

(viii) $P(\tilde{y}_2^{+} \geq \tilde{y}_{22}) = \frac{1}{2}\left(P\left(\left[0, \frac{1}{2}\right] \geq \left[0, \frac{1}{2}\right]\right) + P\left(\left[0, \frac{1}{2}\right] \geq \left[0, \frac{1}{2}\right]\right)\right) = \frac{1}{2}(0.5 + 0.5) = 0.5.$

Elements of the relative closeness matrix

Using the existing expression [260], $r_{ij} = \frac{P(\tilde{y}_{ij} \geq \tilde{y}_j^-)}{P(\tilde{y}_j^+ \geq \tilde{y}_{ij}) + P(\tilde{y}_{ij} \geq \tilde{y}_j^-)}$,

(i) $r_{11} = \frac{P(\tilde{y}_{11} \geq \tilde{y}_1^-)}{P(\tilde{y}_1^+ \geq \tilde{y}_{11}) + P(\tilde{y}_{11} \geq \tilde{y}_1^-)} = \frac{0.55}{0.55+0.55} = 0.55.$

(ii) $r_{12} = \frac{P(\tilde{y}_{12} \geq \tilde{y}_2^-)}{P(\tilde{y}_2^+ \geq \tilde{y}_{12}) + P(\tilde{y}_{12} \geq \tilde{y}_2^-)} = \frac{0.50}{0.50+0.50} = 0.50.$

(iii) $r_{21} = \frac{P(\tilde{y}_{21} \geq \tilde{y}_1^-)}{P(\tilde{y}_1^+ \geq \tilde{y}_{21}) + P(\tilde{y}_{21} \geq \tilde{y}_1^-)} = \frac{0.55}{0.55+0.55} = 0.55.$

(iv) $r_{22} = \frac{P(\tilde{y}_{22} \geq \tilde{y}_2^-)}{P(\tilde{y}_2^+ \geq \tilde{y}_{22}) + P(\tilde{y}_{22} \geq \tilde{y}_2^-)} = \frac{0.50}{0.50+0.50} = 0.50.$

Therefore, the relative closeness matrix, $R = \begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{pmatrix} = \begin{pmatrix} 0.55 & 0.50 \\ 0.55 & 0.50 \end{pmatrix}$.

Values of the relative closeness

(i) Using the values of r_{11} and r_{12},

$RC_1 = r_{11} + r_{12} = 0.55 + 0.50 = 1.05.$

(ii) Using the values of r_{21} and r_{22},

$RC_2 = r_{21} + r_{22} = 0.55 + 0.50 = 1.05.$

Ranking of the alternatives

It is obvious that $RC_1 = RC_2$. Therefore, according to Step 7 of the existing approach [260], $\tilde{A}_1$ and $\tilde{A}_2$ are equivalent. While, as discussed earlier that for the considered problem $\tilde{A}_1$ and $\tilde{A}_2$ can never be equivalent. This clearly indicates that the existing geometric approach [260] is not valid in its present form.

(7) Chen [32] proposed an inclusion-based TOPSIS method for solving IVIFMAGDM problems. There are nine steps in this method. In Step 1 to Step 6 of this method, the IVIFMAGDM problem is transformed into an IVIFMADMPr and in Step 7 to Step 9 of this method, the obtained IVIFMADMPr is solved to find the ranking of the alternatives. It is obvious that if one want to solve a IVIFMADMPr with the help of Chen's inclusion-based TOPSIS method. Then, there is need to use only Step 7 to Step 9 of this method.

However, the following examples clearly indicates that, it is inappropriate to use Chen's method [32]. Hence to resolve the inappropriateness of Chen's method [32] may be considered as challenging open research problem.

Let us consider an IVIFMADMPr having two alternatives A_1 and A_2 and each alternative has two benefit attribute's G_1 and G_2. Furthermore, let Table 10.4 represents the IVIFDM of the considered problem.

Table 10.4: Rating value

Attributes→ ↓ Alternatives	G_1	G_2
A_1	$\left(\left[0,\frac{1}{2}\right],\left[0,\frac{1}{2}\right]\right)$	$\left(\left[0,\frac{1}{2}\right],\left[0,\frac{1}{2}\right]\right)$
A_2	$\left(\left[0,\frac{1}{3}\right],\left[0,\frac{1}{3}\right]\right)$	$\left(\left[0,\frac{1}{2}\right],\left[0,\frac{1}{2}\right]\right)$

It is obvious that the rating value of both the alternatives A_1 and A_2 over the attribute G_2 is same i.e., $\left(\left[0,\frac{1}{2}\right],\left[0,\frac{1}{2}\right]\right)$. Therefore, the ranking of the alternatives A_1 and A_2 will depend only upon the rating values of A_1 and A_2 over the attribute G_1. i.e., if the rating value of the alternative A_1 over G_1 i.e., $\left(\left[0,\frac{1}{2}\right],\left[0,\frac{1}{2}\right]\right)$ will be greater than the rating value of the alternative A_2 over the same attribute G_1 i.e., $\left(\left[0,\frac{1}{3}\right],\left[0,\frac{1}{3}\right]\right)$. Then the relation will be $A_1 > A_2$. Otherwise, $A_1 < A_2$.

It is pertinent to mention that as $\left(\left[0,\frac{1}{2}\right],\left[0,\frac{1}{2}\right]\right) \neq \left(\left[0,\frac{1}{3}\right],\left[0,\frac{1}{3}\right]\right)$. Therefore, the relation $A_1 = A_2$ is not possible.

This problem is solved by the existing inclusion-based TOPSIS method (Step 7 to Step 9) [32], and shown that the obtained relation is $A_1 = A_2$, which is incorrect.

Using Step 7 to Step 9 of the inclusion-based TOPSIS method [32], the ranking of the alternatives A_1 and A_2 can be obtained as follows:

Step 1: Using Step 7 of the inclusion-based TOPSIS method [32], $\tilde{\alpha}_{*1}^{-} = \left(\left[0, \frac{1}{3}\right], \left[0, \frac{1}{2}\right]\right)$, $\tilde{\alpha}_{*2}^{-} = \left(\left[0, \frac{1}{2}\right], \left[0, \frac{1}{2}\right]\right)$, $\tilde{\alpha}_{*1}^{+} = \left(\left[0, \frac{1}{2}\right], \left[0, \frac{1}{3}\right]\right)$ and $\tilde{\alpha}_{*2}^{+} = \left(\left[0, \frac{1}{2}\right], \left[0, \frac{1}{2}\right]\right)$.

Step 2: Using Step 8 of the inclusion-based TOPSIS method [32],

(i) $P(\tilde{\alpha}_{11} \supseteq \tilde{\alpha}_{*1}^{-}) = \frac{1}{2}\left(P^{-}(\tilde{\alpha}_{11} \supseteq \tilde{\alpha}_{*1}^{-}) + P^{+}(\tilde{\alpha}_{11} \supseteq \tilde{\alpha}_{*1}^{-})\right) = \frac{1}{2}(0.1428 + 1) = 0.5714.$

(ii) $P(\tilde{\alpha}_{*1}^{+} \supseteq \tilde{\alpha}_{11}) = \frac{1}{2}\left(P^{-}(\tilde{\alpha}_{*1}^{+} \supseteq \tilde{\alpha}_{11}) + P^{+}(\tilde{\alpha}_{*1}^{+} \supseteq \tilde{\alpha}_{11})\right) = \frac{1}{2}(0.1428 + 1) = 0.5714.$

(iii) $P(\tilde{\alpha}_{12} \supseteq \tilde{\alpha}_{*2}^{-}) = \frac{1}{2}\left(P^{-}(\tilde{\alpha}_{12} \supseteq \tilde{\alpha}_{*2}^{-}) + P^{+}(\tilde{\alpha}_{12} \supseteq \tilde{\alpha}_{*2}^{-})\right) = \frac{1}{2}(0 + 1) = 0.5.$

(iv) $P(\tilde{\alpha}_{*2}^{+} \supseteq \tilde{\alpha}_{12}) = \frac{1}{2}\left(P^{-}(\tilde{\alpha}_{*2}^{+} \supseteq \tilde{\alpha}_{12}) + P^{+}(\tilde{\alpha}_{*2}^{+} \supseteq \tilde{\alpha}_{12})\right) = \frac{1}{2}(0 + 1) = 0.5.$

(v) $P(\tilde{\alpha}_{21} \supseteq \tilde{\alpha}_{*1}^{-}) = \frac{1}{2}\left(P^{-}(\tilde{\alpha}_{21} \supseteq \tilde{\alpha}_{*1}^{-}) + P^{+}(\tilde{\alpha}_{21} \supseteq \tilde{\alpha}_{*1}^{-})\right) = \frac{1}{2}(0.25 + 0.8571) = 0.5535.$

(vi) $P(\tilde{\alpha}_{*1}^{+} \supseteq \tilde{\alpha}_{21}) = \frac{1}{2}\left(P^{-}(\tilde{\alpha}_{*1}^{+} \supseteq \tilde{\alpha}_{21}) + P^{+}(\tilde{\alpha}_{*1}^{+} \supseteq \tilde{\alpha}_{21})\right) = \frac{1}{2}(0.25 + 0.8571) = 0.5535.$

(vii) $P(\tilde{\alpha}_{22} \supseteq \tilde{\alpha}_{*2}^{-}) = \frac{1}{2}\left(P^{-}(\tilde{\alpha}_{22} \supseteq \tilde{\alpha}_{*2}^{-}) + P^{+}(\tilde{\alpha}_{22} \supseteq \tilde{\alpha}_{*2}^{-})\right) = \frac{1}{2}(0 + 1) = 0.5.$

(viii) $P(\tilde{\alpha}_{*2}^{+} \supseteq \tilde{\alpha}_{22}) = \frac{1}{2}\left(P^{-}(\tilde{\alpha}_{*2}^{+} \supseteq \tilde{\alpha}_{22}) + P^{+}(\tilde{\alpha}_{*2}^{+} \supseteq \tilde{\alpha}_{22})\right) = \frac{1}{2}(0 + 1) = 0.5.$

Therfore, $CC(\tilde{A}_1) = \frac{P(\tilde{\alpha}_{11} \supseteq \tilde{\alpha}_{*1}^{-}) \times w_1 + P(\tilde{\alpha}_{12} \supseteq \tilde{\alpha}_{*2}^{-}) \times w_2}{\left(P(\tilde{\alpha}_{11} \supseteq \tilde{\alpha}_{*1}^{-}) + P(\tilde{\alpha}_{*1}^{+} \supseteq \tilde{\alpha}_{11})\right) \times w_1 + \left(P(\tilde{\alpha}_{12} \supseteq \tilde{\alpha}_{*2}^{-}) + P(\tilde{\alpha}_{*2}^{+} \supseteq \tilde{\alpha}_{12})\right) \times w_2}$

$= \frac{0.5714 \times w_1 + 0.5 \times w_2}{(0.5714 + 0.5714) \times w_1 + (0.5 + 0.5) \times w_2} = 0.5,$

$CC(\tilde{A}_2) = \frac{P(\tilde{\alpha}_{21} \supseteq \tilde{\alpha}_{*1}^{-}) \times w_1 + P(\tilde{\alpha}_{22} \supseteq \tilde{\alpha}_{*2}^{-}) \times w_2}{\left(P(\tilde{\alpha}_{21} \supseteq \tilde{\alpha}_{*1}^{-}) + P(\tilde{\alpha}_{*1}^{+} \supseteq \tilde{\alpha}_{21})\right) \times w_1 + \left(P(\tilde{\alpha}_{22} \supseteq \tilde{\alpha}_{*2}^{-}) + P(\tilde{\alpha}_{*2}^{+} \supseteq \tilde{\alpha}_{22})\right) \times w_2}$

$= \frac{0.5535 \times w_1 + 0.5 \times w_2}{(0.5535 + 0.5535) \times w_1 + (0.5 + 0.5) \times w_2} = 0.5.$

Step 3: Since, $CC(\tilde{A}_1) = CC(\tilde{A}_2)$, therefore, according to Step 9 of the inclusion-based TOPSIS method [32], $A_1 = A_2$.

While, as discussed earlier that for the considered problem $A_1 = A_2$ is not possible. This clearly indicates that the existing inclusion-based TOPSIS method [32] is not valid in its present form.

(8) Chen [26, 31] proposed an inclusion-based LINMAP method for solving IVIFMAGDM problems. In the method, proposed by Chen [26, 31], there are fifteen steps. In Step 1 to Step 7 of this method, an IVIFMAGDM problem is transformed into an IVIFMADMPr and from Step 8 to Step 15, the obtained IVIFMADMPr is solved to obtain the ranking of the alternatives.

However, the following examples clearly indicates that, it is inappropriate to use Chen's method [26, 31]. Hence, to resolve the inappropriateness of Chen's method [26, 31] may be considered as challenging open research problem.

Let us consider an IVIFMADMPr having two alternatives A_1 and A_2 and each alternative has two benefit attributes G_1 and G_2. Furthermore, let Table 10.5 represents the IVIFDM of the considered problem.

Table 10.5: Rating values

Attributes→ ↓ Alternatives	G_1	G_2
A_1	$\left(\left[\frac{1}{3},\frac{1}{2}\right],\left[\frac{1}{3},\frac{1}{2}\right]\right)$	$\left(\left[\frac{1}{5},\frac{1}{3}\right],\left[\frac{1}{5},\frac{1}{3}\right]\right)$
A_2	$\left(\left[\frac{1}{4},\frac{1}{3}\right],\left[\frac{1}{4},\frac{1}{3}\right]\right)$	$\left(\left[\frac{1}{5},\frac{1}{3}\right],\left[\frac{1}{5},\frac{1}{3}\right]\right)$

It is obvious that the rating value of both the alternatives A_1 and A_2 over the attribute G_2 is same i.e., $\left(\left[\frac{1}{5},\frac{1}{3}\right],\left[\frac{1}{5},\frac{1}{3}\right]\right)$. Therefore, the ranking of the alternatives A_1 and A_2 will depend only upon the rating values of A_1 and A_2 over the attribute G_1. i.e., if the rating value of the alternative A_1 over G_1 i.e., $\left(\left[\frac{1}{3},\frac{1}{2}\right],\left[\frac{1}{3},\frac{1}{2}\right]\right)$ will be greater than the rating value of the alternative

A_2 over the same attribute G_1 i.e., $\left(\left[\frac{1}{4},\frac{1}{3}\right],\left[\frac{1}{4},\frac{1}{3}\right]\right)$. Then, the relation will be $A_1 > A_2$. Otherwise, $A_1 < A_2$.

It is pertinent to mention that as $\left(\left[\frac{1}{3},\frac{1}{2}\right],\left[\frac{1}{3},\frac{1}{2}\right]\right) \neq \left(\left[\frac{1}{4},\frac{1}{3}\right],\left[\frac{1}{4},\frac{1}{3}\right]\right)$. Therefore, the relation $A_1 = A_2$ is not possible.

However, the following clearly indicates that on solving the considered IVIFMAGDM problem by the existing inclusion-based LINMAP method [26, 31] the obtained relation is $A_1 = A_2$, which is incorrect.

Using Step 7 to Step 15 of the inclusion-based LINMAP method [26, 31], the ranking of the alternatives A_1 and A_2 can be obtained as follows:

According to Step 7 to Step 15 of the inclusion-based LINMAP method [26, 31], to find the rank of the alternatives A_1 and A_2, there is need to compare the comprehensive inclusion-based index $\overline{CI}(A_1) = I(\tilde{a}_{11}) \times \overline{w}_1 + I(\tilde{a}_{12}) \times \overline{w}_2$ and $\overline{CI}(A_2) = I(\tilde{a}_{21}) \times \overline{w}_1 + I(\tilde{a}_{22}) \times \overline{w}_2$.

Since, the considered attributes are benefit attributes. Therefore, using the existing expression [26, 31], $I(\tilde{a}_{11}) = \frac{P(\tilde{a}_{11} \supseteq \tilde{a}_{*1}^{-})}{P(\tilde{a}_{*1}^{+} \supseteq \tilde{a}_{11}) + P(\tilde{a}_{11} \supseteq \tilde{a}_{*1}^{-})}$, $I(\tilde{a}_{12}) = \frac{P(\tilde{a}_{12} \supseteq \tilde{a}_{*2}^{-})}{P(\tilde{a}_{*2}^{+} \supseteq \tilde{a}_{12}) + P(\tilde{a}_{12} \supseteq \tilde{a}_{*2}^{-})}$, $I(\tilde{a}_{21}) = \frac{P(\tilde{a}_{21} \supseteq \tilde{a}_{*1}^{-})}{P(\tilde{a}_{*1}^{+} \supseteq \tilde{a}_{21}) + P(\tilde{a}_{21} \supseteq \tilde{a}_{*1}^{-})}$ and $I(\tilde{a}_{22}) = \frac{P(\tilde{a}_{22} \supseteq \tilde{a}_{*2}^{-})}{P(\tilde{a}_{*2}^{+} \supseteq \tilde{a}_{22}) + P(\tilde{a}_{22} \supseteq \tilde{a}_{*2}^{-})}$, where $P(\tilde{a}_1 \supseteq \tilde{a}_2) = \frac{1}{2}\left(P^{-}(\tilde{a}_1 \supseteq \tilde{a}_2) + P^{+}(\tilde{a}_1 \supseteq \tilde{a}_2)\right)$.

It is obvious that to find $I(\tilde{a}_{11})$, $I(\tilde{a}_{12})$, $I(\tilde{a}_{21})$ and $I(\tilde{a}_{22})$, there is need to calculate the IVIFPIS, $\tilde{a}_{*j}^{+} = \left(\left[\mu_{*j}^{+l}, \mu_{*j}^{+u}\right], \left[\nu_{*j}^{+l}, \nu_{*j}^{+u}\right]\right)$ $j = 1,2$ IVIFNIS, $\tilde{a}_{*j}^{-} = \left(\left[\mu_{*j}^{-l}, \mu_{*j}^{-u}\right], \left[\nu_{*j}^{-l}, \nu_{*j}^{-u}\right]\right)$, $j = 1,2$ the lower comparison inclusion possibilities, $P^{-}(\tilde{a}_{11} \supseteq \tilde{a}_{*1}^{-})$, $P^{-}(\tilde{a}_{*1}^{+} \supseteq \tilde{a}_{11})$, $P^{-}(\tilde{a}_{12} \supseteq \tilde{a}_{*2}^{-})$, $P^{-}(\tilde{a}_{*2}^{+} \supseteq \tilde{a}_{12})$, $P^{-}(\tilde{a}_{21} \supseteq \tilde{a}_{*1}^{-})$, $P^{-}(\tilde{a}_{*1}^{+} \supseteq \tilde{a}_{21})$, $P^{-}(\tilde{a}_{22} \supseteq \tilde{a}_{*2}^{-})$, $P^{-}(\tilde{a}_{*2}^{+} \supseteq \tilde{a}_{22})$ and the upper comparison inclusion possibilities, $P^{+}(\tilde{a}_{11} \supseteq \tilde{a}_{*1}^{-})$, $P^{+}(\tilde{a}_{*1}^{+} \supseteq \tilde{a}_{11})$, $P^{+}(\tilde{a}_{12} \supseteq \tilde{a}_{*2}^{-})$, $P^{+}(\tilde{a}_{*2}^{+} \supseteq \tilde{a}_{12})$, $P^{+}(\tilde{a}_{21} \supseteq \tilde{a}_{*1}^{-})$, $P^{+}(\tilde{a}_{*1}^{+} \supseteq \tilde{a}_{21})$, $P^{+}(\tilde{a}_{22} \supseteq \tilde{a}_{*2}^{-})$, $P^{+}(\tilde{a}_{*2}^{+} \supseteq$

$\tilde{a}_{22})$.

These values can be calculated as follows:

Values of the interval-valued intuitionistic fuzzy positive and negative-ideal solutions

Since $\tilde{a}_{11} = \left([\mu_{11}^{l}, \mu_{11}^{u}], [\nu_{11}^{l}, \nu_{11}^{u}]\right) = \left(\left[\frac{1}{3}, \frac{1}{2}\right], \left[\frac{1}{3}, \frac{1}{2}\right]\right)$, $\tilde{a}_{12} = \left([\mu_{12}^{l}, \mu_{12}^{u}], [\nu_{12}^{l}, \nu_{12}^{u}]\right) = \left(\left[\frac{1}{5}, \frac{1}{3}\right], \left[\frac{1}{5}, \frac{1}{3}\right]\right)$, $\tilde{a}_{21} = \left([\mu_{21}^{l}, \mu_{21}^{u}], [\nu_{21}^{l}, \nu_{21}^{u}]\right) = \left(\left[\frac{1}{4}, \frac{1}{3}\right], \left[\frac{1}{4}, \frac{1}{3}\right]\right)$ and $\tilde{a}_{22} = \left([\mu_{22}^{l}, \mu_{22}^{u}], [\nu_{22}^{l}, \nu_{22}^{u}]\right) = \left(\left[\frac{1}{5}, \frac{1}{3}\right], \left[\frac{1}{5}, \frac{1}{3}\right]\right)$. Therefore, using the existing expression [26, 31]

(i) The IVIFNIS,

$$\tilde{a}_{*1}^{-} = \left([\mu_{*1}^{-l}, \mu_{*1}^{-u}], [\nu_{*1}^{-l}, \nu_{*1}^{-u}]\right)$$

$$= \left(\left[\min\left(\mu_{11}^{l}, \mu_{21}^{l}\right), \min(\mu_{11}^{u}, \mu_{21}^{u})\right], \left[\max\left(\nu_{11}^{l}, \nu_{21}^{l}\right),\ \max(\nu_{11}^{u}, \nu_{21}^{u})\right]\right) = \left(\left[\frac{1}{4}, \frac{1}{3}\right], \left[\frac{1}{3}, \frac{1}{2}\right]\right)$$

(ii) The IVIFNIS,

$$\tilde{a}_{*2}^{-} = \left([\mu_{*2}^{-l}, \mu_{*2}^{-u}], [\nu_{*2}^{-l}, \nu_{*2}^{-u}]\right)$$

$$= \left(\left[\min\left(\mu_{12}^{l}, \mu_{22}^{l}\right), \min(\mu_{12}^{u}, \mu_{22}^{u})\right], \left[\max\left(\nu_{12}^{l}, \nu_{22}^{l}\right),\ \max(\nu_{12}^{u}, \nu_{22}^{u})\right]\right) = \left(\left[\frac{1}{5}, \frac{1}{3}\right], \left[\frac{1}{5}, \frac{1}{3}\right]\right)$$

(iii) The IVIFPIS,

$$\tilde{a}_{*1}^{+} = \left([\mu_{*1}^{+l}, \mu_{*1}^{+u}], [\nu_{*1}^{+l}, \nu_{*1}^{+u}]\right)$$

$$= \left(\left[\max\left(\mu_{11}^{l}, \mu_{21}^{l}\right), \max(\mu_{11}^{u}, \mu_{21}^{u})\right], \left[\min\left(\nu_{11}^{l}, \nu_{21}^{l}\right),\ \min(\nu_{11}^{u}, \nu_{21}^{u})\right]\right) = \left(\left[\frac{1}{3}, \frac{1}{2}\right], \left[\frac{1}{4}, \frac{1}{3}\right]\right)$$

(iv) The IVIFPIS,

$$\tilde{a}_{*2}^{+} = \left([\mu_{*2}^{+l}, \mu_{*2}^{+u}], [\nu_{*2}^{+l}, \nu_{*2}^{+u}]\right)$$

$$= \left(\left[\max\left(\mu_{12}^{l}, \mu_{22}^{l}\right), \max(\mu_{12}^{u}, \mu_{22}^{u})\right], \left[\min\left(\nu_{12}^{l}, \nu_{22}^{l}\right),\ \min(\nu_{12}^{u}, \nu_{22}^{u})\right]\right) = \left(\left[\frac{1}{5}, \frac{1}{3}\right], \left[\frac{1}{5}, \frac{1}{3}\right]\right).$$

Values of the lower comparison inclusion possibilities

Using the existing expression for the lower inclusion comparison possibility [26, 31],

$P^{-}\left(\left([\mu_{1}^{l}, \mu_{1}^{u}], [\nu_{1}^{l}, \nu_{1}^{u}]\right) \supseteq \left([\mu_{2}^{l}, \mu_{2}^{u}], [\nu_{2}^{l}, \nu_{2}^{u}]\right)\right) = \max\left\{1 - \max\left\{\frac{(1-\nu_{2}^{l})-\mu_{1}^{l}}{(1-\mu_{1}^{l}-\nu_{1}^{u})+(1-\mu_{2}^{u}-\nu_{2}^{l})}, 0\right\}, 0\right\}$, we have,

(i) $P^{-}(\tilde{a}_{11} \supseteq \tilde{a}_{*1}^{-}) = P^{-}\left(\left(\left[\frac{1}{3}, \frac{1}{2}\right], \left[\frac{1}{3}, \frac{1}{2}\right]\right) \supseteq \left(\left[\frac{1}{4}, \frac{1}{3}\right], \left[\frac{1}{3}, \frac{1}{2}\right]\right)\right) = 0.$

(ii) $P^{-}(\tilde{a}_{*1}^{+} \supseteq \tilde{a}_{11}) = P^{-}\left(\left(\left[\frac{1}{3}, \frac{1}{2}\right], \left[\frac{1}{4}, \frac{1}{3}\right]\right) \supseteq \left(\left[\frac{1}{3}, \frac{1}{2}\right], \left[\frac{1}{3}, \frac{1}{2}\right]\right)\right) = 0.$

(iii) $P^{-}(\tilde{a}_{12} \supseteq \tilde{a}_{*2}^{-}) = P^{-}\left(\left(\left[\frac{1}{5}, \frac{1}{3}\right], \left[\frac{1}{5}, \frac{1}{3}\right]\right) \supseteq \left(\left[\frac{1}{5}, \frac{1}{3}\right], \left[\frac{1}{5}, \frac{1}{3}\right]\right)\right) = 0.3571.$

(iv) $P^{-}(\tilde{a}_{*2}^{+} \supseteq \tilde{a}_{12}) = P^{-}\left(\left(\left[\frac{1}{5}, \frac{1}{3}\right], \left[\frac{1}{5}, \frac{1}{3}\right]\right) \supseteq \left(\left[\frac{1}{5}, \frac{1}{3}\right], \left[\frac{1}{5}, \frac{1}{3}\right]\right)\right) = 0.3571.$

(v) $P^{-}(\tilde{a}_{21} \supseteq \tilde{a}_{*1}^{-}) = P^{-}\left(\left(\left[\frac{1}{4}, \frac{1}{3}\right], \left[\frac{1}{4}, \frac{1}{3}\right]\right) \supseteq \left(\left[\frac{1}{4}, \frac{1}{3}\right], \left[\frac{1}{3}, \frac{1}{2}\right]\right)\right) = 0.4445.$

(vi) $P^{-}(\tilde{a}_{*1}^{+} \supseteq \tilde{a}_{21}) = P^{-}\left(\left(\left[\frac{1}{3}, \frac{1}{2}\right], \left[\frac{1}{4}, \frac{1}{3}\right]\right) \supseteq \left(\left[\frac{1}{4}, \frac{1}{3}\right], \left[\frac{1}{4}, \frac{1}{3}\right]\right)\right) = 0.4445.$

(vii) $P^{-}(\tilde{a}_{22} \supseteq \tilde{a}_{*2}^{-}) = P^{-}\left(\left(\left[\frac{1}{5}, \frac{1}{3}\right], \left[\frac{1}{5}, \frac{1}{3}\right]\right) \supseteq \left(\left[\frac{1}{5}, \frac{1}{3}\right], \left[\frac{1}{5}, \frac{1}{3}\right]\right)\right) = 0.3571.$

(viii) $P^{-}(\tilde{a}_{*2}^{+} \supseteq \tilde{a}_{22}) = P^{-}\left(\left(\left[\frac{1}{5}, \frac{1}{3}\right], \left[\frac{1}{5}, \frac{1}{3}\right]\right) \supseteq \left(\left[\frac{1}{5}, \frac{1}{3}\right], \left[\frac{1}{5}, \frac{1}{3}\right]\right)\right) = 0.3571.$

Values of the upper comparison inclusion possibilities

Using the existing expression for the upper inclusion comparison possibility [26, 31], $P^{+}\left(\left([\mu_1^l, \mu_1^u], [\nu_1^l, \nu_1^u]\right) \supseteq \left([\mu_2^l, \mu_2^u], [\nu_2^l, \nu_2^u]\right)\right) = \max\left\{1 - \max\left\{\frac{(1-\nu_2^u)-\mu_1^u}{(1-\mu_1^u-\nu_1^l)+(1-\mu_2^l-\nu_2^u)}, 0\right\},\right.$ $\left.0\right\}$, we have,

(i) $P^{+}(\tilde{a}_{11} \supseteq \tilde{a}_{*1}^{-}) = P^{+}\left(\left(\left[\frac{1}{3}, \frac{1}{2}\right], \left[\frac{1}{3}, \frac{1}{2}\right]\right) \supseteq \left(\left[\frac{1}{4}, \frac{1}{3}\right], \left[\frac{1}{3}, \frac{1}{2}\right]\right)\right) = 1.$

(ii) $P^{+}(\tilde{a}_{*1}^{+} \supseteq \tilde{a}_{11}) = P^{+}\left(\left(\left[\frac{1}{3}, \frac{1}{2}\right], \left[\frac{1}{4}, \frac{1}{3}\right]\right) \supseteq \left(\left[\frac{1}{3}, \frac{1}{2}\right], \left[\frac{1}{3}, \frac{1}{2}\right]\right)\right) = 1.$

(iii) $P^{+}(\tilde{a}_{12} \supseteq \tilde{a}_{*2}^{-}) = P^{+}\left(\left(\left[\frac{1}{5}, \frac{1}{3}\right], \left[\frac{1}{5}, \frac{1}{3}\right]\right) \supseteq \left(\left[\frac{1}{5}, \frac{1}{3}\right], \left[\frac{1}{5}, \frac{1}{3}\right]\right)\right) = 0.6428.$

(iv) $P^{+}(\tilde{a}_{*2}^{+} \supseteq \tilde{a}_{12}) = P^{+}\left(\left(\left[\frac{1}{5}, \frac{1}{3}\right], \left[\frac{1}{5}, \frac{1}{3}\right]\right) \supseteq \left(\left[\frac{1}{5}, \frac{1}{3}\right], \left[\frac{1}{5}, \frac{1}{3}\right]\right)\right) = 0.6428.$

(v) $P^{+}(\tilde{a}_{21} \supseteq \tilde{a}_{*1}^{-}) = P^{+}\left(\left(\left[\frac{1}{4}, \frac{1}{3}\right], \left[\frac{1}{4}, \frac{1}{3}\right]\right) \supseteq \left(\left[\frac{1}{4}, \frac{1}{3}\right], \left[\frac{1}{3}, \frac{1}{2}\right]\right)\right) = 0.75.$

(vi) $P^{+}(\tilde{a}_{*1}^{+} \supseteq \tilde{a}_{21}) = P^{+}\left(\left(\left[\frac{1}{3}, \frac{1}{2}\right], \left[\frac{1}{4}, \frac{1}{3}\right]\right) \supseteq \left(\left[\frac{1}{4}, \frac{1}{3}\right], \left[\frac{1}{4}, \frac{1}{3}\right]\right)\right) = 0.75.$

(vii) $P^{+}(\tilde{a}_{22} \supseteq \tilde{a}_{*2}^{-}) = P^{+}\left(\left(\left[\frac{1}{5},\frac{1}{3}\right],\left[\frac{1}{5},\frac{1}{3}\right]\right) \supseteq \left(\left[\frac{1}{5},\frac{1}{3}\right],\left[\frac{1}{5},\frac{1}{3}\right]\right)\right) = 0.6428.$

(viii) $P^{+}(\tilde{a}_{*2}^{+} \supseteq \tilde{a}_{22}) = P^{+}\left(\left(\left[\frac{1}{5},\frac{1}{3}\right],\left[\frac{1}{5},\frac{1}{3}\right]\right) \supseteq \left(\left[\frac{1}{5},\frac{1}{3}\right],\left[\frac{1}{5},\frac{1}{3}\right]\right)\right) = 0.6428.$

Values of the inclusion comparison possibilities

Using the existing expression for the inclusion comparison possibility [26, 31], $P(\tilde{a}_1 \supseteq \tilde{a}_2) = \frac{1}{2}\left(P^{-}(\tilde{a}_1 \supseteq \tilde{a}_2) + P^{+}(\tilde{a}_1 \supseteq \tilde{a}_2)\right)$, we have

(i) $P(\tilde{a}_{11} \supseteq \tilde{a}_{*1}^{-}) = \frac{1}{2}\left(P^{-}(\tilde{a}_{11} \supseteq \tilde{a}_{*1}^{-}) + P^{+}(\tilde{a}_{11} \supseteq \tilde{a}_{*1}^{-})\right) = 0.5.$

(ii) $P(\tilde{a}_{*1}^{+} \supseteq \tilde{a}_{11}) = \frac{1}{2}\left(P^{-}(\tilde{a}_{*1}^{+} \supseteq \tilde{a}_{11}) + P^{+}(\tilde{a}_{*1}^{+} \supseteq \tilde{a}_{11})\right) = 0.5.$

(iii) $P(\tilde{a}_{12} \supseteq \tilde{a}_{*2}^{-}) = \frac{1}{2}\left(P^{-}(\tilde{a}_{12} \supseteq \tilde{a}_{*2}^{-}) + P^{+}(\tilde{a}_{12} \supseteq \tilde{a}_{*2}^{-})\right) = 0.4999.$

(iv) $P(\tilde{a}_{*2}^{+} \supseteq \tilde{a}_{12}) = \frac{1}{2}\left(P^{-}(\tilde{a}_{*2}^{+} \supseteq \tilde{a}_{12}) + P^{+}(\tilde{a}_{*2}^{+} \supseteq \tilde{a}_{12})\right) = 0.4999.$

(v) $P(\tilde{a}_{21} \supseteq \tilde{a}_{*1}^{-}) = \frac{1}{2}\left(P^{-}(\tilde{a}_{21} \supseteq \tilde{a}_{*1}^{-}) + P^{+}(\tilde{a}_{21} \supseteq \tilde{a}_{*1}^{-})\right) = 0.5972.$

(vi) $P(\tilde{a}_{*1}^{+} \supseteq \tilde{a}_{21}) = \frac{1}{2}\left(P^{-}(\tilde{a}_{*1}^{+} \supseteq \tilde{a}_{21}) + P^{+}(\tilde{a}_{*1}^{+} \supseteq \tilde{a}_{21})\right) = 0.5972.$

(vii) $P(\tilde{a}_{22} \supseteq \tilde{a}_{*2}^{-}) = \frac{1}{2}\left(P^{-}(\tilde{a}_{22} \supseteq \tilde{a}_{*2}^{-}) + P^{+}(\tilde{a}_{22} \supseteq \tilde{a}_{*2}^{-})\right) = 0.4999.$

(viii) $P(\tilde{a}_{*2}^{+} \supseteq \tilde{a}_{22}) = \frac{1}{2}\left(P^{-}(\tilde{a}_{*2}^{+} \supseteq \tilde{a}_{22}) + P^{+}(\tilde{a}_{*2}^{+} \supseteq \tilde{a}_{22})\right) = 0.4999.$

Values of the inclusion-based index

(i) Putting the values of $P(\tilde{a}_{11} \supseteq \tilde{a}_{*1}^{-})$ and $P(\tilde{a}_{*1}^{+} \supseteq \tilde{a}_{11})$,

$I(\tilde{a}_{11}) = \frac{P(\tilde{a}_{11} \supseteq \tilde{a}_{*1}^{-})}{P(\tilde{a}_{*1}^{+} \supseteq \tilde{a}_{11}) + P(\tilde{a}_{11} \supseteq \tilde{a}_{*1}^{-})}$, we have, $I(\tilde{a}_{11}) = \frac{0.5}{0.5+0.5} = 0.5.$

(ii) Putting the values of $P(\tilde{a}_{12} \supseteq \tilde{a}_{*2}^{-})$ and $P(\tilde{a}_{*2}^{+} \supseteq \tilde{a}_{12})$,

$I(\tilde{a}_{12}) = \frac{P(\tilde{a}_{12} \supseteq \tilde{a}_{*2}^{-})}{P(\tilde{a}_{*2}^{+} \supseteq \tilde{a}_{12}) + P(\tilde{a}_{12} \supseteq \tilde{a}_{*2}^{-})}$, we have, $I(\tilde{a}_{12}) = \frac{0.4999}{0.4999+0.4999} = 0.5.$

(iii) Putting the values of $P(\tilde{a}_{21} \supseteq \tilde{a}_{*1}^{-})$ and $P(\tilde{a}_{*1}^{+} \supseteq \tilde{a}_{21})$,

$I(\tilde{a}_{21}) = \frac{P(\tilde{a}_{21} \supseteq \tilde{a}_{*1}^{-})}{P(\tilde{a}_{*1}^{+} \supseteq \tilde{a}_{21}) + P(\tilde{a}_{21} \supseteq \tilde{a}_{*1}^{-})}$, we have, $I(\tilde{a}_{21}) = \frac{0.5972}{0.5972+0.5972} = 0.5.$

(iv) Putting the values of $P(\tilde{a}_{22} \supseteq \tilde{a}_{*2}^{-})$ and $P(\tilde{a}_{*2}^{+} \supseteq \tilde{a}_{22})$,

$I(\tilde{a}_{22}) = \frac{P(\tilde{a}_{22} \supseteq \tilde{a}_{*2}^{-})}{P(\tilde{a}_{*2}^{+} \supseteq \tilde{a}_{22}) + P(\tilde{a}_{22} \supseteq \tilde{a}_{*2}^{-})}$, we have, $I(\tilde{a}_{22}) = \frac{0.4999}{0.4999+0.4999} = 0.5$.

Values of the comprehensive inclusion-based index

(i) Putting the values of $I(\tilde{a}_{11})$ and $I(\tilde{a}_{12})$,

$\overline{CI}(\tilde{A}_1) = I(\tilde{a}_{11}) \times \overline{w}_1 + I(\tilde{a}_{12}) \times \overline{w}_2$, we have,

$\overline{CI}(\tilde{A}_1) = 0.5 \times \overline{w}_1 + 0.5 \times \overline{w}_2 = 0.5(\overline{w}_1 + \overline{w}_2) = 0.5$.

(ii) Putting the values of $I(\tilde{a}_{21})$ and $I(\tilde{a}_{22})$,

$\overline{CI}(\tilde{A}_2) = I(\tilde{a}_{21}) \times \overline{w}_1 + I(\tilde{a}_{12}) \times \overline{w}_2$, we have,

$\overline{CI}(\tilde{A}_2) = 0.5 \times \overline{w}_1 + 0.5 \times \overline{w}_2 = 0.5(\overline{w}_1 + \overline{w}_2) = 0.5$.

It is obvious that the values of $\overline{CI}(\tilde{A}_1)$ and $\overline{CI}(\tilde{A}_2)$ are independent from the values of $\overline{w}_1$ and $\overline{w}_2$. Therefore, there is no need to apply Step 13 and Step 14 of the existing inclusion-based LINMAP method [26, 31] to find the optimal values of $\overline{w}_1$ and $\overline{w}_2$ for the considered problem.

Ranking of the alternatives

It is obvious from that $\overline{CI}(\tilde{A}_1) = \overline{CI}(\tilde{A}_2)$. Therefore, according to Step 15 of the existing method [ ], $\tilde{A}_1 = \tilde{A}_2$. While, as discussed earlier that for the considered problem $\tilde{A}_1 = \tilde{A}_2$ is not possible. This clearly indicates that the existing inclusion-based LINMAP method [ ] is not valid in its present form.

(9) Oztaysi et al. [158] proposed a multi-expert IVIFMADM method. To show the applicability of the method, Oztaysi et al. [158], applied this method to select the alternative fuel technology of a utility company in the U.S.A. In future, the other researchers may use the same method to select the alternative fuel technology of other utility companies. However, the following examples clearly indicates that, it is inappropriate to use Oztaysi et al.'s method [158]. Hence to resolve the inappropriateness of Oztaysi et al.'s method [158] may be considered as challenging open research problem.

Let us consider an IVIFMADMPr having two alternatives A_1 and A_2 and each alternative has two attributes G_1 and G_2. Furthermore, let Table 10.6 represents the IVIFDM of the considered problem.

Table 10.6: Rating value

Attributes→ ↓ Alternatives	G_1	G_2
A_1	$\left(\left[0,\frac{1}{2}\right],\left[0,\frac{1}{2}\right]\right)$	$\left(\left[0,\frac{1}{3}\right],\left[0,\frac{1}{3}\right]\right)$
A_2	$\left(\left[0,\frac{1}{2}\right],\left[0,\frac{1}{2}\right]\right)$	$\left(\left[0,\frac{1}{2}\right],\left[0,\frac{1}{2}\right]\right)$

It is obvious that the rating value of both the alternatives A_1 and A_2 over the attribute G_1 is same i.e., $\left(\left[0,\frac{1}{2}\right],\left[0,\frac{1}{2}\right]\right)$. Therefore, the ranking of the alternatives A_1 and A_2 will depend only upon the rating values of A_1 and A_2 over the attribute G_2. i.e., if the rating value of the alternative A_1 and A_2 over G_2 i.e., $\left(\left[0,\frac{1}{3}\right],\left[0,\frac{1}{3}\right]\right)$ will be greater than the rating value of the alternative A_2 over the same attribute G_2 i.e., $\left(\left[0,\frac{1}{2}\right],\left[0,\frac{1}{2}\right]\right)$. Then the relation will be $A_1 > A_2$. Otherwise, $A_1 < A_2$. It is pertinent to mention that as $\left(\left[0,\frac{1}{2}\right],\left[0,\frac{1}{2}\right]\right) \neq \left(\left[0,\frac{1}{3}\right],\left[0,\frac{1}{3}\right]\right)$. Therefore, the relation $A_1 = A_2$ is not possible. In this section, this problem is solved by the existing method [158], and shown that it is not possible to conclude that $A_1 > A_2$ or $A_1 < A_2$ or $A_1 = A_2$.

However, the following clearly indicates that on solving this IVIFMADMPr by the existing method [158], the relation $A_1 = A_2$ is obtained, which is mathematically incorrect. Using the existing method [158], the ranking of the alternatives for the considered IVIFMADMPr can be obtained as follows:

Step 1: Since, it is assumed that there is only one decision-maker in the considered

IVIFMADMPr. Therefore, using Step 1 of the existing method [158], the IVIFPIS $PIS_1 =$

$$\begin{pmatrix} \left(\left[max(0,0), max\left(\frac{1}{2},\frac{1}{2}\right)\right], \left[min(0,0), min\left(\frac{1}{2},\frac{1}{2}\right)\right]\right), \\ \left(\left[max(0,0), max\left(\frac{1}{3},\frac{1}{2}\right)\right], \left[min(0,0), min\left(\frac{1}{3},\frac{1}{2}\right)\right]\right) \end{pmatrix} = \left(\left(\left[0,\frac{1}{2}\right], \left[0,\frac{1}{2}\right]\right), \left(\left[0,\frac{1}{2}\right], \left[0,\frac{1}{3}\right]\right)\right)$$

and IVIFNIS $NIS_1 = \begin{pmatrix} \left(\left[min(0,0), min\left(\frac{1}{2},\frac{1}{2}\right)\right], \left[max(0,0), max\left(\frac{1}{2},\frac{1}{2}\right)\right]\right), \\ \left(\left[min(0,0), min\left(\frac{1}{3},\frac{1}{2}\right)\right], \left[max(0,0), max\left(\frac{1}{3},\frac{1}{2}\right)\right]\right) \end{pmatrix} =$

$\left(\left(\left[0,\frac{1}{2}\right], \left[0,\frac{1}{2}\right]\right), \left(\left[0,\frac{1}{3}\right], \left[0,\frac{1}{2}\right]\right)\right)$.

Step 2: Using Step 10.2 of the existing method [158]:

(i) The separation measure between the first alternative and PIS_1 i.e.,

$$D_1^* = \sqrt{\frac{1}{2}\begin{pmatrix} w_1\left((0-0)^2 + \left(\frac{1}{2}-\frac{1}{2}\right)^2 + (0-0)^2 + \left(\frac{1}{2}-\frac{1}{2}\right)^2 + (0-0)^2 + (1-1)^2\right) + \\ w_2\left((0-0)^2 + \left(\frac{1}{2}-\frac{1}{2}\right)^2 + (0-0)^2 + \left(\frac{1}{2}-\frac{1}{2}\right)^2 + (0-0)^2 + (1-1)^2\right) \end{pmatrix}}$$

$$= \sqrt{\frac{1}{2}\left(w_1(0) + w_2(0)\right)} = 0.$$

(ii) The separation measure between the second alternative and PIS_1 i.e.,

$$D_2^* = \sqrt{\frac{1}{2}\begin{pmatrix} w_1\left((0-0)^2 + \left(\frac{1}{3}-\frac{1}{2}\right)^2 + (0-0)^2 + \left(\frac{1}{3}-\frac{1}{3}\right)^2 + \left(\frac{1}{3}-\frac{1}{6}\right)^2 + (1-1)^2\right) + \\ w_2\left((0-0)^2 + \left(\frac{1}{2}-\frac{1}{2}\right)^2 + (0-0)^2 + \left(\frac{1}{2}-\frac{1}{3}\right)^2 + \left(0-\frac{1}{6}\right)^2 + (1-1)^2\right) \end{pmatrix}}$$

$$= \sqrt{\frac{1}{2}\left(\frac{1}{18}(w_1 + w_2)\right)} = 0.1666.$$

Step 3: Using Step 10.3 of the existing method [158],

(i) The separation measure between the first alternative and NIS_1 i.e.,

$$D_1^- = \sqrt{\frac{1}{2}\left(w_1\left((0-0)^2 + \left(\frac{1}{2}-\frac{1}{2}\right)^2 + (0-0)^2 + \left(\frac{1}{2}-\frac{1}{2}\right)^2 + (0-0)^2 + (1-1)^2\right) + w_2\left((0-0)^2 + \left(\frac{1}{2}-\frac{1}{2}\right)^2 + (0-0)^2 + \left(\frac{1}{2}-\frac{1}{2}\right)^2 + (0-0)^2 + (1-1)^2\right)\right)}$$

$$= \sqrt{\frac{1}{2}\left(w_1(0) + w_2(0)\right)} = 0.$$

(ii) The separation measure between the second alternative and NIS_1 i.e.,

$$D_2^- = \sqrt{\frac{1}{2}\left(w_1\left((0-0)^2 + (0-0)^2 + (0-0)^2 + \left(\frac{1}{3}-\frac{1}{2}\right)^2 + \left(\frac{1}{3}-\frac{1}{6}\right)^2 + (1-1)^2\right) + w_2\left((0-0)^2 + \left(\frac{1}{2}-\frac{1}{3}\right)^2 + (0-0)^2 + \left(\frac{1}{2}-\frac{1}{2}\right)^2 + \left(0-\frac{1}{6}\right)^2 + (1-1)^2\right)\right)}$$

$$= \sqrt{\frac{1}{2}\left(\frac{1}{18}(w_1 + w_2)\right)} = 0.1666.$$

Step 4: Using Step 10.5 of the existing method [158],

(i) $U_1 = \frac{D_1^-}{D_1^- + D_1^*} = \frac{0}{0}$.

(ii) $U_2 = \frac{D_2^-}{D_2^- + D_2^*} = \frac{0.1666}{0.1666+0.1666} = 0.5$.

Step 5: According to Step 10.5 of the existing method [158],

Case (i) $A_1 > A_2$ if $U_1 > U_2$

Case (ii) $A_1 < A_2$ if $U_1 < U_2$

Case (iii) $A_1 = A_2$ if $U_1 = U_2$.

However, it is obvious that the obtained value of U_1 is $\frac{0}{0}$, which is an indeterminate value instead of a real number. Therefore, it is not possible to check that $U_1 > U_2$ or $U_1 < U_2$ or $U_1 = U_2$ and hence, it is not possible to conclude that $A_1 > A_2$ or $A_1 < A_2$ or $A_1 = A_2$ i.e., the existing method [158], fails to rank the alternatives A_1 and A_2.

(10) Kahraman et al. [111] proposed the subtraction and division operations of IVIFSs. Using these operations, Kahraman et al. [111] also proposed an EDAS method for solving IVIFMADMPrs. In future, other researchers may use the operations and the EDAS method in their research work. However, in actual case, the operations, proposed by Kahraman et al. [ ], are not valid and hence, the EDAS method is also not valid.

Kahraman et al. [111] proposed the following subtraction operation $\tilde{A} \ominus \tilde{B}$ and the division operation $\frac{\tilde{A}}{\tilde{B}}$ between two interval valued intuitionistic fuzzy sets $\tilde{A} = \left(\left[\mu^L_{\tilde{A}}, \mu^U_{\tilde{A}}\right], \left[\nu^L_{\tilde{A}}, \nu^U_{\tilde{A}}\right]\right)$ and $\tilde{B} = \left(\left[\mu^L_{\tilde{B}}, \mu^U_{\tilde{B}}\right], \left[\nu^L_{\tilde{B}}, \nu^U_{\tilde{B}}\right]\right)$.

(i) If $\mu^L_{\tilde{A}} \geq \mu^L_{\tilde{B}}, \mu^U_{\tilde{A}} \geq \mu^U_{\tilde{B}}, \nu^L_{\tilde{A}} \leq \nu^L_{\tilde{B}}, \nu^U_{\tilde{A}} \leq \nu^U_{\tilde{B}},\ \nu^L_{\tilde{B}} > 0, \nu^U_{\tilde{B}} > 0, \nu^L_{\tilde{A}}\left(1 - \mu^L_{\tilde{B}}\right) \leq \nu^L_{\tilde{B}}\left(1 - \mu^L_{\tilde{A}}\right), \nu^U_{\tilde{A}}\left(1 - \mu^U_{\tilde{B}}\right) \leq \nu^U_{\tilde{B}}\left(1 - \mu^U_{\tilde{A}}\right)$.

Then, $\tilde{A} \ominus \tilde{B} = \left(\left[\frac{\mu^L_{\tilde{A}} - \mu^L_{\tilde{B}}}{1 - \mu^L_{\tilde{B}}}, \frac{\mu^U_{\tilde{A}} - \mu^U_{\tilde{B}}}{1 - \mu^U_{\tilde{B}}}\right], \left[\frac{\nu^L_{\tilde{A}}}{\nu^L_{\tilde{B}}}, \frac{\nu^U_{\tilde{A}}}{\nu^U_{\tilde{B}}}\right]\right)$.

Otherwise, $\widetilde{A} \ominus \tilde{B} = ([0,0\ ], [1,1])$.

(ii) If $\mu^L_{\tilde{A}} \leq \mu^L_{\tilde{B}}, \mu^U_{\tilde{A}} \leq \mu^U_{\tilde{B}}, \nu^L_{\tilde{A}} \geq \nu^L_{\tilde{B}}, \nu^U_{\tilde{A}} \geq \nu^U_{\tilde{B}},\ \mu^L_{\tilde{B}} > 0, \mu^U_{\tilde{B}} > 0, \mu^L_{\tilde{A}}\left(1 - \nu^L_{\tilde{B}}\right) \leq \mu^L_{\tilde{B}}\left(1 - \nu^L_{\tilde{A}}\right), \mu^U_{\tilde{A}}\left(1 - \nu^U_{\tilde{B}}\right) \leq \mu^U_{\tilde{B}}\left(1 - \nu^U_{\tilde{A}}\right)$.

Then, $\frac{\tilde{A}}{\tilde{B}} = \left(\left[\frac{\mu^L_{\tilde{A}}}{\mu^L_{\tilde{B}}}, \frac{\mu^U_{\tilde{A}}}{\mu^U_{\tilde{B}}}\right], \left[\frac{\nu^L_{\tilde{A}} - \nu^L_{\tilde{B}}}{1 - \nu^L_{\tilde{B}}}, \frac{\nu^U_{\tilde{A}} - \nu^U_{\tilde{B}}}{1 - \nu^U_{\tilde{B}}}\right]\right)$.

Otherwise, $\frac{\tilde{A}}{\tilde{B}} = ([0,0\ ], [1,1])$.

The following clearly indicates that the subtraction and the division operations, proposed by Kahraman et al. [111], are not valid.

1. Let $\tilde{A} = \left[\mu^L_{\tilde{A}}, \mu^U_{\tilde{A}}\right], \left[\nu^L_{\tilde{A}}, \nu^U_{\tilde{A}}\right] = [0.1,0.4], [0.01,0.03]$ and $\tilde{B} = \left[\mu^L_{\tilde{B}}, \mu^U_{\tilde{B}}\right], \left[\nu^L_{\tilde{B}}, \nu^U_{\tilde{B}}\right] = [0.1,0.4], [0.1,0.5]$ be two interval valued intitionistic fuzzy sets.

   Since, for the considered interval valued intuitionistic fuzzy sets all the conditions $\mu^L_{\tilde{A}} \geq \mu^L_{\tilde{B}}, \mu^U_{\tilde{A}} \geq \mu^U_{\tilde{B}}, \nu^L_{\tilde{A}} \leq \nu^L_{\tilde{B}}, \nu^U_{\tilde{A}} \leq \nu^U_{\tilde{B}}\ , \nu^L_{\tilde{B}} > 0, \nu^U_{\tilde{B}} > 0,\ \nu^L_{\tilde{A}}\left(1 - \mu^L_{\tilde{B}}\right) \leq \nu^L_{\tilde{B}}\left(1 - \mu^L_{\tilde{A}}\right)$ and $\nu^U_{\tilde{A}}\left(1 - \mu^U_{\tilde{B}}\right) \leq \nu^U_{\tilde{B}}\left(1 - \mu^U_{\tilde{A}}\right)$ are satisfying. Therefore, using the subtraction operation $\tilde{A} \ominus \tilde{B} = \left(\left[\frac{\mu^L_{\tilde{A}} - \mu^L_{\tilde{B}}}{1 - \mu^L_{\tilde{B}}}, \frac{\mu^U_{\tilde{A}} - \mu^U_{\tilde{B}}}{1 - \mu^U_{\tilde{B}}}\right], \left[\frac{\nu^L_{\tilde{A}}}{\nu^L_{\tilde{B}}}, \frac{\nu^U_{\tilde{A}}}{\nu^U_{\tilde{B}}}\right]\right)$, proposed by Kahraman et al. [111], $\tilde{A} \ominus \tilde{B} = \left(\left[\frac{0.1-0.1}{1-0.1}, \frac{0.4-0.4}{1-0.4}\right], \left[\frac{0.01}{0.1}, \frac{0.03}{0.5}\right]\right) = ([0,0], [0.1,0.06])$.

   It is obvious that $\tilde{A} \ominus \tilde{B} = \left[\mu^L_{\tilde{A}\ominus\tilde{B}}, \mu^U_{\tilde{A}\ominus\tilde{B}}\right], \left[\nu^L_{\tilde{A}\ominus\tilde{B}}, \nu^U_{\tilde{A}\ominus\tilde{B}}\right] = ([0,0], [0.1,0.06])$ is not an IVIFS as for $\nu^L_{\tilde{A}\ominus\tilde{B}} = 0.1$ and $\nu^U_{\tilde{A}\ominus\tilde{B}} = 0.06$, the necessary conditions $\nu^L_{\tilde{A}\ominus\tilde{B}} \leq \nu^U_{\tilde{A}\ominus\tilde{B}}$ is not satisfying.

This clearly indicates that the subtraction operation, proposed by Kahraman et al. [111], is not valid.

2. Let $\tilde{A} = [\mu^L_{\tilde{A}}, \mu^U_{\tilde{A}}], [\nu^L_{\tilde{A}}, \nu^U_{\tilde{A}}] = [0.01, 0.03], [0.1, 0.4]$ and $\tilde{B} = [\mu^L_{\tilde{B}}, \mu^U_{\tilde{B}}], [\nu^L_{\tilde{B}}, \nu^U_{\tilde{B}}] = [0.1, 0.5], [0.1, 0.4]$ be two IVIFSs.

   Since, for the considered IVIFSs all the conditions $\mu^L_{\tilde{A}} \leq \mu^L_{\tilde{B}}, \mu^U_{\tilde{A}} \leq \mu^U_{\tilde{B}}, \nu^L_{\tilde{A}} \geq \nu^L_{\tilde{B}}, \nu^U_{\tilde{A}} \geq \nu^U_{\tilde{B}}$, $\mu^L_{\tilde{B}} > 0, \mu^U_{\tilde{B}} > 0$, $\mu^L_{\tilde{A}}(1 - \nu^L_{\tilde{B}}) \leq \mu^L_{\tilde{B}}(1 - \nu^L_{\tilde{A}})$ and $\mu^U_{\tilde{A}}(1 - \nu^U_{\tilde{B}}) \leq \mu^U_{\tilde{B}}(1 - \nu^U_{\tilde{A}})$ are satisfying.

Therefore, using the division operation $\frac{\tilde{A}}{\tilde{B}} = \left(\left[\frac{\mu^L_{\tilde{A}}}{\mu^L_{\tilde{B}}}, \frac{\mu^U_{\tilde{A}}}{\mu^U_{\tilde{B}}}\right], \left[\frac{\nu^L_{\tilde{A}} - \nu^L_{\tilde{B}}}{1 - \nu^L_{\tilde{B}}}, \frac{\nu^U_{\tilde{A}} - \nu^U_{\tilde{B}}}{1 - \nu^U_{\tilde{B}}}\right]\right)$, proposed by Kahraman et al. [ ], $\frac{\tilde{A}}{\tilde{B}} = \left(\left[\frac{0.01}{0.1}, \frac{0.03}{0.5}\right], \left[\frac{0.1 - .1}{0.1 - .1}, \frac{0.4 - 0.4}{0.4 - 0.4}\right]\right) = ([0.1, 0.06], [0, 0])$.

It is obvious that $\frac{\tilde{A}}{\tilde{B}} = \left(\left[\frac{\mu^L_{\tilde{A}}}{\mu^L_{\tilde{B}}}, \frac{\mu^U_{\tilde{A}}}{\mu^U_{\tilde{B}}}\right], \left[\frac{\nu^L_{\tilde{A}} - \nu^L_{\tilde{B}}}{1 - \nu^L_{\tilde{B}}}, \frac{\nu^U_{\tilde{A}} - \nu^U_{\tilde{B}}}{1 - \nu^U_{\tilde{B}}}\right]\right) = ([0.1, 0.06], [0, 0])$ is not an IVIFS as for $\mu^L_{\frac{\tilde{A}}{\tilde{B}}} = 0.1$ and $\mu^U_{\frac{\tilde{A}}{\tilde{B}}} = 0.06$ the necessary condition $\mu^L_{\frac{\tilde{A}}{\tilde{B}}} \leq \mu^U_{\frac{\tilde{A}}{\tilde{B}}}$ is not satisfying.

This clearly indicates that the division operation, proposed by Kahraman et al. [111], is not valid.

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