THE GEOMETRY OF SLANT LIGHTLIKE SUBMANIFOLDS

A
THESIS

SUBMITTED IN FULFILLMENT OF THE REQUIREMENT FOR THE AWARD OF THE DEGREE OF

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IN

MATHEMATICS

By

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CERTIFICATE

This is to certify that the thesis entitled **"The Geometry of Slant Lightlike Submanifolds"** submitted by Ms. Rashmi, in fulfillment of the requirements for the award of the degree of *Doctor of Philosophy* in the School of Mathematics, Thapar University, Patiala, is a record of the candidate's own research work carried out by her under our supervision and guidance. The matter embodied in this thesis has not been submitted in part or full to any other University or Institute for the award of any degree.

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CANDIDATE'S DECLARATION

I hereby declare that this thesis entitled "**The Geometry of Slant Lightlike Submanifolds**" is an original work done by me for the award of the degree of *Doctor of Philosophy* in Mathematics. I also declare that this thesis or any part of it has not been submitted by me for the award of any degree, diploma, title or recognition and that the references cited herein have been duly acknowledged.

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Dedicated to my Parents

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Rashmi

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ABSTRACT

The present thesis entitled **"The Geometry of Slant Lightlike Submanifolds"** comprises certain investigations carried out by me at the School of Mathematics (SOM), Thapar University, Patiala, under the supervision of Dr. S. S. Bhatia, Professor, School of Mathematics, Thapar University, Patiala and Dr. Rakesh Kumar, Assistant Professor, Department of Basic and Applied Sciences, Punjabi University, Patiala.

The core of differential geometry and modern geometrical dynamics represents the concept of a manifold. Manifolds are the higher-dimensional analogues of surfaces. The local and global properties of smooth manifolds equipped with a metric tensor encodes its geometry. To study the geometric aspects of a manifold, it is more convenient to first embed it into a known manifold and then study the geometry which is induced on it. This approach gives impetus to the study of submanifolds which later developed into a fascinating study of the theory of submanifolds.

The geometry of submanifolds of an almost Hermitian manifold depends upon the behaviour of the tangent bundle of the submanifold with respect to the almost complex structure $\bar{J}$ of the manifold. The action of the almost complex structure give rise to the two well known classes of submanifolds namely, the holomorphic submanifolds and the totally real submanifolds.

Let $\bar{M}$ be an almost Hermitian manifold with almost complex structure $\bar{J}$. Consider M as a submanifold of $\bar{M}$ and let the tangent space and normal space of M at $p \in M$ be denoted by T_pM and $T_pM^{\perp}$, respectively. If T_pM is invariant under the action of $\bar{J}$ for each $p \in M$, that is, if $\bar{J}T_pM = T_pM$, for each $p \in M$, then M is called an invariant (or holomorphic) submanifold of $\bar{M}$. On the other hand, if $\bar{J}T_pM$ is contained in the normal space $T_pM^{\perp}$, for each $p \in M$,

that is, if $\bar{J}T_pM \subset T_pM^{\perp}$, for each $p \in M$, then M is called an anti-invariant (or totally real) submanifold of $\bar{M}$.

In other words, a submanifold of an almost Hermitian manifold is a holomorphic or a totally real submanifold, if and only if, the angle between $\bar{J}X$ and the tangent space T_pM is 0 or $\pi/2$ respectively, for every non zero vector X tangent to any point $p \in M$.

This paved the way for generalization of holomorphic and totally real submanifold to a new class of submanifold, known as slant submanifold, introduced by Chen [23] in 1990. Slant submanifolds are those for which the angle between $\bar{J}X$ and the tangent space T_pM is constant. Thus the theory of slant-submanifolds is an umbrella over the theory of holomorphic submanifolds and totally real submanifolds.

Initially, the slant submanifolds were studied with positive definite metric. This geometry of slant submanifolds could not be used extensively in mathematical physics, since the metric may not always be definite. The developments in theoretical physics require a deeper understanding of the geometric structure of higher dimensional manifolds with indefinite metric. Thus the geometry of manifolds with indefinite metric becomes the topic of interest due to the fact that the signature of the indefinite metric creates an interesting changes in the geometry of manifolds.

From the mid of the 20th century, the Riemannian and semi-Riemannian geometries have been active and fruitful areas of research in differential geometry due to their applications to a variety of subjects in mathematics and physics. In the process of generalization of submanifolds theory from Riemannian manifolds to semi-Riemannian manifolds, lightlike submanifolds arise naturally in the semi-Riemannian category. The theory of submanifolds of Riemannian or semi-

Riemannian manifolds is well known but its counter part lightlike submanifolds is relatively new and is in its developing stage. In the theory of submanifolds of semi-Riemannian manifolds, it is interesting to study the geometry of lightlike submanifolds due to the fact that the intersection of normal vector bundle and the tangent bundle is non-trivial. Thus, the study becomes more interesting and remarkably different from the study of non-degenerate submanifolds. Moreover, the growing importance of lightlike geometry in mathematical physics, in particular, their extensive uses in theory of relativity motivated the geometers to do research on this subject matter.

Sahin [74] clubbed the theory of slant submanifolds with lightlike geometry and introduced slant lightlike submanifolds of an almost Hermitian manifold. The odd dimensional version of almost Hermitian manifolds are also of equal importance in differential geometry.

A very important mechanism for the progress in the field of science and technology is to generalize the existing ideas. The present thesis has been written on the same basis. In present thesis, the geometric aspects of slant lightlike submanifolds of indefinite almost complex manifolds (K$\breve{a}$hler manifolds) and indefinite almost contact manifolds (Sasakian manifolds, Cosymplectic manifolds and Kenmotsu manifolds) have been explored and thereby many existing results have been extended and generalized.

The thesis embodies six chapters. The sequence of chapters is arranged so that the understanding of a chapter stimulates interest in reading the next chapters.

Chapter 1 is introductory and contains most of the pre-requisites of the subsequent chapters of the thesis. In this chapter, apart from setting up the notations and terminologies to be used in the subsequent chapters, some known results interrelated with the work done in the present thesis have also been presented.

Chapter 2 is the core of this thesis and deals with the geometry of totally umbilical slant lightlike submanifolds of indefinite contact manifolds. Primarily, the study emphasizes that every totally contact umbilical slant lightlike submanifold of an indefinite Sasakian manifold is totally contact geodesic slant lightlike submanifold. In case of indefinite Sasakian space form, it has been proved that there does not exist any totally contact umbilical proper slant lightlike submanifold. Further in this chapter, the minimal slant lightlike submanifolds of an indefinite Sasakian manifold have also been characterized.

Since there are significant uses of contact geometry in the various fields of mathematics and physics (for details see Arnold [1], Maclane [57], Nazaikinskii [60]), therefore the study of totally contact umbilical slant lightlike submanifolds have also been generalized for indefinite Cosymplectic manifolds and indefinite Kenmotsu manifolds towards the end of this chapter.

Contents of this chapter have been published in *Bull. Iranian Math. Soc.*, 40(5) (2014), 1135-1151. [67] ***(SCI-indexed), Impact factor - 0.270*** **and *ISRN Geometry*, 2013 (2013), 1-8,** [66] **and in *Tamkang J. Math.*, 46(2) (2015), 179-191.** [70]

Chapter 3 comprises the study of hemi-slant lightlike submanifolds of indefinite Kähler manifolds. The notions of axioms of indefinite hemi-slant 3-planes and indefinite hemi-slant 3-spheres with lightlike submanifolds have been introduced. In this chapter, it has been proved that if an indefinite Kähler manifold satisfies the axioms of indefinite hemi-slant 3-planes and 3-spheres for some slant angle $\theta \in (0, \pi/2)$, then it is an indefinite complex space form.

It has further been proved in this chapter that there do not exist totally umbilical hemi-slant lightlike submanifolds in indefinite Kähler manifolds other than the totally geodesic hemi-slant lightlike submanifolds. Consequently, it has been shown that the induced connection on a totally umbilical hemi-slant lightlike submanifold is a metric connection. A characterization theorem on the non-existence of totally umbilical hemi-slant lightlike submanifold of an indefinite complex space form and some characterization theorems on minimal hemi-slant lightlike submanifolds have also been presented towards the end of this chapter.

Some results of this chapter have been published in *New York J. Math.*, 21 (2015), 191-203. [69] ***(SCI-indexed), Impact factor - 0.418*** **and rest of the results are accepted for publication in *Lobachevskii J. Math.*** [71]

Bishop and O'Neill [11] in 1969 introduced warped product manifolds as a generalization of Riemannian product manifolds. The easiest example of warped product manifolds is surface of revolution. Formally, a warped product manifold $B \times_f F$ of two Riemannian manifolds (B, g_B) and (F, g_F), where g_B and g_F are Riemannian metrics on B and F respectively, is the product manifold $B \times F$, equipped with Riemannian metric $g = g_B + f^2 g_F$, where f is a positive differentiable function on B. Bishop and O'Neill studied these manifolds to study the manifolds of negative curvature. From geometric point of view, this study got momentum, when the study of warped product of CR-submanifolds of Kähler manifolds was introduced by Chen [25, 26]. Warped product manifolds are known to have applications in physics as they provide an excellent setting to model space time.

Chapter 4 is devoted to obtain the necessary and sufficient conditions for a semi-invariant submanifold to be a locally warped product submanifold of invariant and anti-invariant submanifolds of a Cosymplectic manifold in terms of canonical structures. The inequality and equality cases have also been discussed for the squared norm of the second fundamental form in terms of the warping function.

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In **Chapter 5**, results for the non-existence of warped product slant lightlike submanifolds of indefinite Sasakian manifolds have been carried out.

Results of this chapter have been published in *Balkan J. Geom. Appl.*, 20(1) (2015), 98-108. [68] ***(SCI-indexed), Impact factor - 0.806***

O'Neill [61] introduced semi-Riemannian submersions. It is known that in Riemannian manifolds the fibers are always Riemannian manifolds. However, when the manifolds are semi-Riemannian manifolds then the fibers may not be semi-Riemannian manifolds. Sahin [78] in 2011 introduced slant submersions from almost Hermitian manifolds onto Riemannian manifolds. Semi-Riemannian submersions are of interest in physics, owing to their applications in the Yang-Mills theory, Kaluza-Klein theory, supergravity and superstring theories [15, 16, 32, 88]. All these features motivated us to club the theory of lightlike submersions with slant submersions in **Chapter 6**.

In this chapter, slant lightlike submersions from an indefinite almost Hermitian manifold onto a lightlike manifold have been introduced. Further, the geometry of foliations of slant lightlike submersions have been investigated.

Contents of this chapter are accepted for publication in *Ukrainian Math. J.* [72] ***(SCI-indexed), Impact factor - 0.230***

Towards the end, the future directions in this research are outlined. The thesis concludes by listing the Bibliography of various publications cited in this work.

RESEARCH PUBLICATIONS

The following publications are the outcome of the present research work:

1. **Rashmi Sachdeva**, Rakesh Kumar and Satvinder Singh Bhatia, *Slant lightlike submersions from an indefinite almost Hermitian manifold onto a lightlike manifold*, **Ukrainian Math. J. (SCI-indexed), Impact factor - 0.230** (Accepted for publication)

2. **Rashmi Sachdeva**, Rakesh Kumar and Satvinder Singh Bhatia, *Warped product slant lightlike submanifolds of indefinite Sasakian manifolds*, **Balkan J. Geom. Appl.**, **20**(1) (2015), 98-108. **(SCI-indexed), Impact factor - 0.806**

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4. **Rashmi Sachdeva**, Rakesh Kumar and Satvinder Singh Bhatia, *Non existence of totally contact umbilical slant lightlike submanifolds of indefinite Sasakian manifolds*, **Bull. Iranian Math. Soc.**, **40**(5) (2014), 1135-1151. **(SCI-indexed), Impact factor - 0.270**

5. Meraj Ali Khan, Siraj Uddin and **Rashmi Sachdeva**, *Semi-invariant warped product submanifolds of Cosymplectic manifolds*, **J. Inequal. Appl.**, **19** (2012), 1-12. **(SCI-indexed), Impact factor - 0.77**

6. **Rashmi Sachdeva**, Rakesh Kumar and Satvinder Singh Bhatia, *Totally contact umbilical slant lightlike submanifolds of indefinite Kenmotsu manifolds*, **Tamkang J. Math.**, **46**(2) (2015), 179-191. **(SCOPUS-indexed)**

7. **Rashmi Sachdeva**, Rakesh Kumar and Satvinder Singh Bhatia, *The axioms of indefinite hemi-slant planes and spheres with lightlike submanifolds,* **Lobachevskii J. Math.(SCOPUS-indexed)**(Accepted for publication)

8. **Rashmi Sachdeva**, Rakesh Kumar and Satvinder Singh Bhatia, *Nonexistence of totally contact umbilical slant lightlike submanifolds of indefinite Cosymplectic manifolds*, **ISRN Geometry**, **2013** (2013), 1-8.

COMMUNICATED RESEARCH PAPERS

1. **Rashmi Sachdeva**, Rakesh Kumar and Satvinder Singh Bhatia, *Characterization of hemi-slant lightlike submanifolds of indefinite Sasakian manifolds*, **J. Ramanujan Math. Soc. (SCI-indexed)**

2. **Rashmi Sachdeva**, Rakesh Kumar and Satvinder Singh Bhatia, *Totally umbilical slant lightlike submanifolds of indefinite Kähler manifolds,* **Kyungpook Math. J. (SCOPUS-indexed)**

WORKSHOPS / CONFERENCES ATTENDED

1. Attended **ATM School on Geometry and Dynamics**, IISER Mohali during December 17-22, 2014.
2. Presented a paper entitled "Totally umbilical slant lightlike submanifolds" in **International Conference on Algebra, Geometry, Analysis and their Applications** held at Jamia Milia Islamia, Delhi from November 27-29, 2014.
3. Attended **Advanced School and Discussion Meeting on Knot Theory and its applications** at IISER Mohali during December 10-20, 2013.
4. Attended **National Workshop on Differential Geometry** held at Banasthali University, Rajasthan from October 23-27, 2013.
5. Attended **National Workshop on Latex** held at Chitkara University from July 8-10, 2013.
6. Presented a paper entitled "Characterization of slant lightlike submanifolds of an indefinite Sasakian manifold" in **International Conference on Differential Geometry and Relativity** held at Aligarh Muslim University during November 20-22, 2012.
7. Presented a paper entitled "Slant lightlike submanifolds of indefinite almost contact manifolds" in **International Conference on Differential Geometry, Functional Analysis and Applications** held at Jamia Milia Islamia, Delhi during September 8-10, 2012.
8. Attended **Research Promotion Workshop on Introduction to Graph and Geometric Algorithms** held at Thapar University, Patiala from October 28-30, 2010.

LIST OF SYMBOLS

$\mathbb{R}$	Real Numbers
$\mathbb{R}^n$	n-tuples of real numbers
$\bar{M}$	Manifold
M	Submanifold
T_pM	Tangent space at p
TM	Tangent bundle
$RadTM$	Radical distribution
$S(TM)$	Screen distribution
$ltr(TM)$	Lightlike transversal bundle of M
$\perp$	Orthogonal direct sum
h^l	Lightlike second fundamental form of a lightlike submanifold
h^s	Screen transversal second fundamental form of a lightlike submanifold
A_v	Shape operator of M
h^*	Screen second fundamental form
A^*	Screen shape operator of $S(TM)$
D	Distribution
grad f	Gradient of function f
f_*	derivative map of function f
$K(P)$	Sectional curvature of plane P
$\bar{J}$	Almost complex structure on a vector space

Contents

Certificate **i**

Acknowledgements **iv**

Abstract **vi**

Research Publications **xii**

List of Symbols **xvi**

1 INTRODUCTION 20

1.1 Manifolds 21

1.1.1 Semi-Riemannian Manifold 27

1.2 Structures on Manifolds 29

1.3 Submanifolds Theory 35

1.3.1 CR-Submanifolds 37

1.3.2 Slant Submanifolds 39

1.3.3 Lightlike Submanifolds 40

1.3.4 The Induced Geometric Objects on Lightlike Submanifolds 46

1.4 Slant Lightlike Submanifolds 50

1.5 Objectives of the Study 52

1.6 Thesis Organization 52

2 TOTALLY CONTACT UMBILICAL SLANT LIGHTLIKE SUBMANIFOLDS OF INDEFINITE CONTACT MANIFOLDS 56

2.1 Introduction 56

2.2 Slant Lightlike Submanifolds of Indefinite Contact Manifolds . . . 58

2.3 Totally Contact Umbilical Slant Lightlike Submanifolds of Indefinite Sasakian Manifolds 63

2.4 Minimal Slant Lightlike Submanifolds 73

2.5 Totally Contact Umbilical Slant Lightlike Submanifolds of Indefinite Cosymplectic and Kenmotsu Manifolds 75

3 THE AXIOMS OF INDEFINITE SURFACES WITH LIGHTLIKE SUBMANIFOLDS AND TOTALLY UMBILICAL HEMI-SLANT LIGHTLIKE SUBMANIFOLDS 77

3.1 Introduction 77

3.2 The Axioms of Indefinite Hemi-Slant Planes and Spheres with Lightlike Submanifolds 84

3.3 Totally Umbilical Hemi-Slant Lightlike Submanifolds 88

4 SEMI INVARIANT WARPED PRODUCT SUBMANIFOLDS OF COSYMPLECTIC MANIFOLDS 99

4.1 Introduction 99

4.2 Preliminaries 101

4.3 Semi-Invariant Submanifolds of a Cosymplectic Manifold 105

4.4 Semi-Invariant Warped Product Submanifold with Canonical Structures 108

5 WARPED PRODUCT SLANT LIGHTLIKE SUBMANIFOLDS OF INDEFINITE SASAKIAN MANIFOLDS 118

5.1 Introduction 118

5.2 Non-Existence of Warped Product Slant Lightlike Submanifolds . 119

6 SLANT LIGHTLIKE SUBMERSIONS FROM AN INDEFINITE ALMOST HERMITIAN MANIFOLD ONTO A LIGHTLIKE MANIFOLD 126

6.1 Introduction 126

6.2 Lightlike Submersions 129

6.3 Slant Lightlike Submersion 133

Scope for Future Work **142**

Bibliography **143**

Chapter 1

INTRODUCTION

Differential geometry consists of study of curves and surfaces embedded in three dimensional Euclidean space. Manifolds are the generalizations of surfaces to arbitrarily higher dimensional spaces and provide mathematical context for understanding space in all of its manifestations. Manifolds inherit number of local properties of Euclidean space. Today, the tools of manifold theory are indispensible in most major subfields of pure mathematics. Actually, a manifold is an abstract mathematical space, which locally resembles with the Euclidean space, but globally it may have a more complicated structure. For example: the Earth is a manifold: locally it seems to be flat, but when viewed as a whole from the outer space (globally), it is actually spherical. The local and global properties of smooth manifolds equipped with a metric tensor encodes its geometry. Manifolds play an important role in mathematics and physics as they express more complicated structures in terms of well understood properties of simpler Euclidean spaces.

The present chapter is elementary in nature. In this chapter, some basic concepts along with the relevant results which will be frequently used in our subsequent chapters have been presented. For a complete account, one can refer to Bejancu [8], Blair [12, 13], Bognar [14], Chen [21, 24], Duggal and Bejancu [27], Majumdar and Bhattacharyya [58], Matsushima [59], O'Neill [62], Yano and Kon [90] etc.

Before proceeding further, some basic definitions and concepts pertaining to structures on manifolds have been recalled in the following sections:

1.1 Manifolds

Generally speaking, a manifold is a topological space that resembles locally with Euclidean space. Manifolds are the higher-dimensional analogues of surfaces and can be defined mathematically as:

A manifold [90] $\bar{M}$ *of* n*-dimension is a Hausdorff topological space such that for every* $p \in \bar{M}$*, there exists an open neighborhood* U *containing* p *of* $\bar{M}$ *and a homeomorphism* ϕ *of* U *onto an open set in* $\mathbb{R}^n$*.*

For example: The circle S^1 defined by the point (x, y) satisfying $x^2 + y^2 = 1$ in $\mathbb{R}^2$ is a differentiable manifold of dimension 1 and 2-Sphere S^2 defined by the point (x, y, z) satisfying $x^2 + y^2 + z^2 = 1$ is a differentiable manifold of dimension 2. (for detail see [59])

For a manifold $\bar{M}$, a *chart* on $\bar{M}$ is pair (U, ϕ) consisting of subset U of $\bar{M}$ and a 1-1 map ϕ of U onto an open set in $\mathbb{R}^n$. A chart is also called a coordinate pair, where U is called a coordinate neighborhood and ϕ is called a coordinate map. The coordinates on a chart allow one to carry out computations as in Euclidean space, so that many concepts from $\mathbb{R}^n$, such as differentiability, tangent spaces and differential forms carry over to a manifold.

A differentiable (smooth(C^∞)) manifold is a topological manifold with globally defined differentiable (smooth(C^∞)) structure. The formal definition of a differentiable manifold is as below:

Definition 1.1.1. [90] *Let $\bar{M}$ be a Hausdroff space. A differentiable structure on $\bar{M}$ of dimension n is a collection of open chart $\{(U_i, \phi_i)\}_{i\epsilon\Lambda}$ on M where $\phi_i(U_i)_{i\epsilon\Lambda}$ is an open subset of $\mathbb{R}^n$ such that the following conditions are satisfied:*

- *(a)* $\bar{M} = \bigcup_{i\epsilon\Lambda} U_i$*;*
- *(b) For each pair $i, j\epsilon\Lambda$, the mapping ϕ_j o ϕ_i^{-1} is differentiable mapping of $\phi_i(U_i \cap U_j)$ onto $\phi_j(U_j \cap U_i)$;*
- *(c) The collection $\{(U_i, \phi_i)\}_{i\epsilon\Lambda}$ is a maximal family of open chart for which* (*a*) *and* (*b*) *hold, which is known as an atlas* [59].

Let $\bar{M}$ be a smooth manifold of dimension n. A function $f : \bar{M} \to \mathbb{R}$ is said to be *smooth(C^∞)* at a point p in $\bar{M}$, if there is a chart (U, ϕ) about p in $\bar{M}$ such that a function f o ϕ^{-1}, defined on the open subset $\phi(U)$ of $\mathbb{R}^n$, is smooth at $\phi(p)$. The function f is said to be smooth on $\bar{M}$ if it is smooth on every point of $\bar{M}$.

Let p be a point in $\bar{M}$ and $\Im(p)$ denotes the set of all real valued smooth functions, each defined in some neighborhood of p. If $f, g \in \Im(p)$, then $f + g$ and $f.g$ are defined on the intersection of neighborhoods of f and g. λf is defined on the neighborhood of f. If for each $f \in \Im(p)$, there corresponds a real number $v(f)$ satisfying

$$v(\lambda f + \mu g) = \lambda v(f) + \mu(g), \qquad \text{(Linearity)}$$

$$v(fg) = v(f)g(p) + f(p)v(g), \qquad \text{(Leibniz rule)}$$

where λ, $\mu \in \mathbb{R}$ and $f, g \in \Im(p)$, then a linear map $v : \Im(p) \to \mathbb{R}$ satisfying leibniz rule is called a *tangent vector* [59] of $\bar{M}$ at p.

The value $v(f)$, for $f \in \Im(p)$, depends only on the local behaviour of f and if f and g coincide on some neighborhood of p, then $v(f) = v(g)$. For tangent vectors

v, v' of $\bar{M}$ at p and for $\lambda \in \mathbb{R}$, the sum $v + v'$ and the scalar multiple λv are defined as

$$(v + v')(f) = v(f) + v'(f), \quad and \quad (\lambda v)(f) = \lambda(v(f)), \text{ for } f \in \Im(p).$$

The vectors $v + v'$ and λv are also tangent vectors of $\bar{M}$ at p. Hence, by defining the sum and scalar multiple of tangent vectors at p in this manner, the set of all tangent vectors at p becomes a vector space over $\mathbb{R}$, called as a *tangent space* [59] and denoted by $T_p(\bar{M})$. The set of all tangent spaces of a manifold $\bar{M}$ is called a *tangent bundle* [59] of $\bar{M}$ and is denoted by $T\bar{M}$.

Let $(x^1, ..., x^n)$ be the local coordinate system on a subset U of $\bar{M}$ and at point p of U, let

$$\Big(\frac{\partial}{\partial x^i}\Big)_p f = \Big(\frac{\partial f}{\partial x^i}\Big)_p, \quad where \quad i = 1, ..., n.$$

Then $\Big(\frac{\partial}{\partial x^i}\Big)_p$ is essentially a tangent vector at p because the partial derivatives $\frac{\partial}{\partial x^i}|_p$ satisfy the linearity property and Leibniz rule.

If $\bar{M}$ is manifold of dimension n, tangent space $T_p\bar{M}$ is also of dimension n [59] and $\{(\partial/\partial x^1)_p, ..., (\partial/\partial x^n)_p\}$ is a basis of $T_p\bar{M}$.

Let X_p be a tangent vector at a point p of a manifold $\bar{M}$. Then an assignment $X : p \to X_p$ is called a *vector field* [59] on $\bar{M}$. If f is a differentiable function on $\bar{M}$, then Xf is a function on $\bar{M}$ defined by $(Xf)(p) = X_p f$. A vector field X is said to be differentiable, if Xf is differentiable for every differentiable function f. In terms of local coordinates, $(x^1, ..., x^n)$, a vector field X can be expressed as $X = \sum \xi^i(\partial/\partial x^i)$, where ξ^i are functions defined in the coordinate neighborhoods called as components of X, with respect to local coordinates. A vector field X is differentiable, if and only if, its components ξ^i are differentiable. The set of all differentiable vector fields on $\bar{M}$ is denoted by $\chi(\bar{M})$.

Let $T_p^*(\bar{M})$ be the dual space of the tangent space $T_p(\bar{M})$ of $\bar{M}$ at p.

Then each element of $T_p^*(\bar{M})$ is called a *covector* [90] at p. An assignment of a covector at each point p is called a 1-*form* [90]. Let $(U; x^1, ..., x^n)$ be a local coordinate system at p. Then the total differentials $(dx^1)_p, (dx^2)_p, ..., (dx^n)_p$ form a basis of the dual space $T_p^*(\bar{M})$. Thus, in the neighborhood of p, every 1-form ω can uniquely be written as

$$\omega = \Sigma_i f_i dx^i,$$

where f_i are functions in U and are called as components of ω with respect to $x^1, ..., x^n$. The 1-form ω is differentiable, if all f_i are differentiable.

A Riemannian structure on the tangent bundle of a manifold reveals its geometry as it is required to measure the distance and angles on it. Mathematically, a *Riemannian manifold* [59] $(\bar{M}, \bar{g})$ is a real differentiable manifold $\bar{M}$ in which each tangent space is equipped with a positive definite inner product which generates a metric, called the *Riemannian metric* [59] $\bar{g}$. Infact, by a Riemannian metric $\bar{g}$ on a manifold $\bar{M}$, we mean a map $p \to \bar{g}_p$, where $\bar{g}_p$ is a positive definite inner product on $T_p\bar{M}$. This map is required to be smooth (differentiable) in the sense that the function

$$p \to \bar{g}_{ij}(p) = \bar{g}_p(\frac{\partial}{\partial x_i}|_p, \frac{\partial}{\partial x_j}|_p)$$

is smooth for all i, j and for any chart $\bar{M}$. This smoothness condition is same as that for the map $p \to \bar{g}_p(X_p, Y_p)$, for all vector fields X, Y on $\bar{M}$.

An *affine connection* [90] on a manifold $\bar{M}$ is a rule which assigns to each $X \in C^\infty(\bar{M})$, a linear mapping $\bar{\nabla}_X$ of the vector space $C^\infty(\bar{M})$ into itself satisfying the following two conditions:

1. $\bar{\nabla}_{f_1X+f_2Y} = f_1\bar{\nabla}_X + f_2\bar{\nabla}_Y$.
2. $\bar{\nabla}_X(f_1Y) = f_1(\bar{\nabla}_X)Y + (Xf_1)Y$.

Here, the operator $\bar{\nabla}_X$ is called the *covariant differential* [59] along X direction.

A *torsion tensor* [90] of an affine connection is a mapping $T : T\bar{M} \times T\bar{M} \to T\bar{M}$ defined as

$$T(X,Y) = \bar{\nabla}_X Y - \bar{\nabla}_Y X - [X,Y].$$

An affine connection is said to be symmetric or torsion free if its torsion tensor $T = 0$, that is, if $[X,Y] = \bar{\nabla}_X Y - \bar{\nabla}_Y X$.

The following theorem on affine connection due to Yano and Kon [90] is well known.

Theorem 1.1.2. [90] *On a Riemannian manifold $\bar{M}$, there exists one and only one affine connection satisfying:*
(a) The torsion tensor T vanishes, that is, $T(X,Y) = \bar{\nabla}_X Y - \bar{\nabla}_Y X - [X,Y] = 0$.
(b) $\bar{g}$ is parallel, that is, $\bar{\nabla}_X \bar{g} = 0$.

The existence of affine connection in the above theorem can be obtained by using Koszul formula

$$\begin{aligned} 2\bar{g}(\bar{\nabla}_X Y, Z) &= X\bar{g}(Y,Z) + Y\bar{g}(X,Z) - Z\bar{g}(X,Y) \\ &\quad + \bar{g}([X,Y],Z) + \bar{g}([Z,X],Y) + \bar{g}(X,[Z,Y]), \end{aligned} \tag{1.1.1}$$

for any vector field Z on $\bar{M}$. Then the mapping $(X,Y) \to \bar{\nabla}_X Y$ defines an affine connection on $\bar{M}$. The connection $\bar{\nabla}_X Y$ given by (1.1.1) is called the *Levi-Civita connection* [90].

A linear connection $\bar{\nabla}$ on a Riemannian manifold is said to be a Riemannian connection (metric connection), if the Riemannian metric $\bar{g}$ is parallel with respect to $\bar{\nabla}$, that is, if $(\bar{\nabla}_X \bar{g})(Y,Z) = X(\bar{g}(Y,Z)) - \bar{g}(\bar{\nabla}_X Y, Z) - \bar{g}(Y, \bar{\nabla}_X Z) = 0$, for all $X, Y, Z \in \Gamma(T\bar{M})$.

The following result on the existence and uniqueness of metric connections is well known.

Theorem 1.1.3. [90] *On a Riemannian manifold there exists a unique torsion free metric connection.*

Let $(\bar{M}, \bar{g})$ be a Riemannian manifold with Riemannian connection $\bar{\nabla}$. Then the *Riemannian curvature tensor field* [90] $\bar{R}$ in terms of $\bar{\nabla}$ is defined by

$$\bar{R}(X,Y)Z = \bar{\nabla}_X \bar{\nabla}_Y Z - \bar{\nabla}_Y \bar{\nabla}_X Z - \bar{\nabla}_{[X,Y]} Z,$$

for any X, Y and $Z \in T\bar{M}$. The Riemannian curvature tensor field satisfies the following properties.

(i) $\bar{R}(X,Y) = -\bar{R}(Y,X)$, (anti-symmetry)

(ii) $\bar{R}(X,Y)Z + \bar{R}(Y,Z)X + \bar{R}(Z,X)Y = 0$, (*Bianchi's first identity*)

(iii)$\bar{\nabla}_X \bar{R}(Y,Z) + \bar{\nabla}_Y \bar{R}(Z,X) + \bar{\nabla}_Z \bar{R}(X,Y) = 0$. (*Bianchi's second identity*)

Similarly, the *Riemannian curvature tensor field* of covariant degree 4 of $\bar{M}$ can be defined as

$$\bar{R}(X,Y,Z,W) = \bar{g}(\bar{R}(Z,W)Y,X), \quad \forall \quad X,Y,Z,W \in T_p(\bar{M}).$$

Here, $\bar{R}$ is a quadilinear mapping $T_p(\bar{M}) \times T_p(\bar{M}) \times T_p(\bar{M}) \times T_p(\bar{M}) \to \mathbb{R}$ at each point $p \in \bar{M}$ satisfying these three properties:

$$\bar{R}(X,Y,Z,W) = -\bar{R}(Y,X,Z,W).$$

$$\bar{R}(X,Y,Z,W) = -\bar{R}(X,Y,W,Z).$$

$$\bar{R}(X,Y,Z,W) + \bar{R}(X,Z,W,Y) + \bar{R}(X,W,Y,Z) = 0.$$

For each plane $P = span\{X,Y\}$ in the tangent space $T_p\bar{M}$, the *sectional curvature* $K(P)$ is defined by Yano and Kon [90] as

$$K(P) = \bar{R}(X,Y,X,Y) = \bar{g}(\bar{R}(X,Y)Y,X),$$

where $\{X, Y\}$ is an orthonormal basis for P. The sectional curvature $K(P)$ is independent of choice of an orthonormal basis $\{X, Y\}$. If sectional curvature $K(P)$ is constant for all plane P in $T_p\bar{M}$ and for all points p of $\bar{M}$, then $\bar{M}$ is called a *space of constant curvature*. A Riemannian manifold of constant curvature is called a *space form* [90]. A space form is complete simply connected Riemannian manifold of constant curvature.

The following theorem due to Schur [81] is well known.

Theorem 1.1.4. [81] *Let $\bar{M}$ be a connected Riemannian manifold of dimension > 2. If the sectional curvature $K(P)$ depends only on the point p, then $\bar{M}$ is a space of constant curvature.*

1.1.1 Semi-Riemannian Manifold

A semi-Riemannian manifold is a variant of Riemannian manifolds where the metric tensor is allowed to have an indefinite signature. Since the geometry of space-time in general relativity is taken to be endowed with an indefinite metric, so it becomes necessary to study manifolds with structures endowed with indefinite metric. After the work of Bognar [14] on indefinite inner product spaces, the geometry of manifolds with indefinite metric became the topic of interest due to the fact that the signature of an indefinite metric creates interesting changes in the geometry of manifolds and moreover there are wide range of applications of geometry of manifolds with indefinite metric in general theory of relativity.

Let V be an m-dimensional real vector space with a symmetric bilinear mapping $\bar{g} : V \times V \longrightarrow \mathbb{R}$. We say that $\bar{g}$ is *degenerate* [62] on V, if there exists a vector $\xi \neq 0$, of V, such that $\bar{g}(\xi, v) = 0$, for any $v \in V$, otherwise $\bar{g}$ is called *non-degenerate* [62]. Clearly, $\bar{g}$ is non-degenerate if and only if $\bar{g}(u, v) = 0$, for any $v \in V$, implies that $u = 0$. It is important to note that a non-degenerate symmetric bilinear form on V may induce either a non-degenerate or a degenerate symmetric bilinear form on a subspace of V.

Consider a subspace W of V. Then the restriction of $\bar{g}$ on $W \times W$ is also a symmetric bilinear form on W, denoted by $\bar{g}_{|w}$. The dimension q of the largest subspace $W \subset V$ and on which $\bar{g}_{|w}$ is negative definite is called the index of $\bar{g}$ on V and denoted by *ind* V$= q$.

The non-degenerate symmetric bilinear form $\bar{g}$ on V is said to be scalar product (semi-Euclidean metric) and then V is said to be semi-Euclidean space.

The following two are the special cases of metric are important for both the mathematical study and the applications to physics.

(i) If $\bar{g}$ is positive definite metric, then $\bar{g}$ is an inner product (Euclidean metric) and V is an Euclidean space.

(ii) If the index q of $\bar{g}$ is 1, then $\bar{g}$ is Lorentz (Minkowski) metric and V is a Lorentz (Minkowski) space.

In case, if there exists a degenerate $\bar{g}$ on semi-Euclidean space V, then we say that V is a lightlike (degenerate) vector space with respect to $\bar{g}$.

Let $\bar{M}$ *be a smooth manifold and* $p \in \bar{M}$. *A scalar product* $\bar{g}_p$ *(non-degenerate, symmetric and bilinear form) of index* $q \neq 0$ *on tangent space* $T_p(\bar{M})$ *such that* $\bar{g}_p : T_p(\bar{M}) \times T_p(\bar{M}) \to \mathbb{R}$ *is a tensor of type (0, 2). Then, the corresponding tensor field* $\bar{g}$ *is called the indefinite metric tensor and a Riemannian manifold* $\bar{M}$ *endowed with this indefinite metric is called an indefinite Riemannian manifold (or a pseudo-Riemannian manifold or semi-Riemannian manifold)* [62] *and denoted by* $(\bar{M}, \bar{g})$.

Definition 1.1.5. [62] *Let* X *be a tangent vector on* $\bar{M}$ *and* $\bar{g}$ *be an indefinite metric. Then* *(i) X is space-like if $\bar{g}(X,X) > 0$.*

(ii) X is time-like if $\bar{g}(X,X) < 0$.

(iii) X is light-like (degenerate or null) if $\bar{g}(X,X) = 0, X \neq 0$.

The set of all null vectors in $T_p(\bar{M})$ is called null cone at $p \in \bar{M}$.

1.2 Structures on Manifolds

Manifolds with structures are recognized as a powerful technique towards their geometrization. It is well known that a manifold is a topological space, which is locally Euclidean. Therefore $\mathbb{R}^1$ (real line), $\mathbb{R}^2$ (plane), $\mathbb{R}^3$ (space), $\mathbb{R}^n$ (Euclidean Space) are obvious examples of real manifold. Let $\mathbb{C}^n$ be the complex vector space of all n-tuples of complex numbers $(z_1, ..., z_n)$. If we set $z_k = x_k + iy_k,\ x_k,\ y_k \in \mathbb{R}, k = 1, ..., n$, then $\mathbb{C}^n$ can be identified with the real vector space $\mathbb{R}^{2n}$. Since the real and imaginary parts of a holomorphic function are analytic, therefore a holomorphic map between open sets of $\mathbb{C}^n$ or open sets of $\mathbb{R}^{2n}$ is analytic. Hence complex manifold $\mathbb{C}^n$ is equivalent to $\mathbb{R}^{2n}$ real analytic manifold.

Semi-Riemannian manifolds can be classified as (a) almost complex manifolds (even dimensional), (b) almost contact manifolds (odd dimensional). Schouten and Dontzing in 1930 introduced the concept of complex structure and a Hermitian metric on a differentiable manifold and called it a complex manifold. Ehresmann (1950) defined an almost complex structure on an even dimensional differentiable manifold. Calabi and Spencer (1951), Goldberg (1960), Hodge (1951), R.S. Mishra (1969), Tachibana and Yano (1965) etc. studied the different properties of complex and almost complex manifolds.

Let $\bar{M}$ be a C^∞-manifold of even dimension. A tensor field $\bar{J}$ on $\bar{M}$ is said to be an almost complex structure [90] *on $\bar{M}$ if at every point p of $\bar{M}$, $\bar{J}$ is* *an endomorphism of the tangent space* $T_p(\bar{M})$*, such that* $\bar{J}^2X = -X$*, for each vector field* X *on* $\bar{M}$*. A* C^∞*-manifold* $\bar{M}$ *with a fixed almost complex structure* $\bar{J}$ *is called an almost complex manifold* [90] *and denoted by a pair* $(\bar{M}, \bar{J})$*.*

Every complex manifold $\bar{M}$ carries a natural almost complex structure $\bar{J}$. Let $(z_1, ..., z_n)$ be a complex local coordinate system on a neighborhood U of a point p of $\bar{M}$, such that $z_j = x_j + iy_j,\ j = 1, ..., n$. Define an endomorphism $\bar{J}$ on $T_p(\bar{M})$ by $\bar{J}(\partial/\partial x_j) = \partial/\partial y_j$, $\bar{J}(\partial/\partial y_j) = -(\partial/\partial x_j)$. Let $T_p^c(\bar{M})$ be the complexification of $T_p(\bar{M})$. Then we can extend $\bar{J}$ to $T_p^c(\bar{M})$ by $\bar{J}(\partial/\partial z_j) = i(\partial/\partial z_j)$, $\bar{J}(\partial/\partial \bar{z}_j) = -i(\partial/\partial \bar{z}_j)$, where $\partial/\partial z_j = 1/2\{(\partial/\partial x_j) - i(\partial/\partial y_j)\}$ and $\partial/\partial \bar{z}_j = 1/2\{(\partial/\partial x_j) + i(\partial/\partial y_j)\}$. If an element z of $T_p^c(\bar{M})$ is a linear combination of $(\partial/\partial z_j)$ only, then $\bar{J}z = iz$ and if z is a linear combination of $(\partial/\partial \bar{z}_j)$ only, then $\bar{J}z = -iz$. Clearly, this gives $\bar{J}^2 = -I$.

In 1951, Nijenhuis introduced an important tensor, named as Nijenhuis tensor and defined by $N(X, Y) = [X, Y] + \bar{J}[\bar{J}X, Y] + \bar{J}[X, \bar{J}Y] - [\bar{J}X, \bar{J}Y]$ for every $X, Y \in T(M)$. The almost complex structure $\bar{J}$ is said to be integrable, if and only if, the Nijenhuis tensor N vanishes identically.

Since, every complex manifold $\bar{M}$ carries a natural almost complex structure $\bar{J}$, then every complex manifold is almost complex manifold. But the converse is not true in general. In fact, for the converse to be hold it is necessary that the almost complex structure $\bar{J}$ should be integrable. The conditions that an almost complex manifold $\bar{M}$ to be a complex manifold are given by the following theorem due to Yano and Kon [90].

Theorem 1.2.1. [90] *Let* $\bar{M}$ *be an almost complex manifold with almost complex structure* $\bar{J}$*. Then* $\bar{M}$ *is a complex manifold, if and only if,* $\bar{M}$ *admits a linear connection* $\bar{\nabla}$ *such that* $\bar{\nabla}\bar{J} = 0$ *and* $T = 0$*, where* T *denote the torsion of* $\bar{\nabla}$*.*

Let $\bar{M}$ *be an almost complex manifold with almost complex structure* $\bar{J}$*.*

Then, an indefinite Hermitian metric on $\bar{M}$ is an indefinite Riemannian metric $\bar{g}$ such that

$$\bar{g}(\bar{J}X, \bar{J}Y) = \bar{g}(X, Y), \tag{1.2.1}$$

for any vector fields X and Y on $\bar{M}$.

Remark: *Clearly, from (1.2.1), it follows that the index q of $\bar{g}$ is an even number, that is, $q = 2u$. Then, the signature of $\bar{g}$ be $(2u, 2v)$, where $u + v = n$.*

An almost complex manifold with an indefinite Hermitian metric is said to be an indefinite almost Hermitian manifold and a complex manifold with an indefinite Hermitian metric is called an indefinite Hermitian manifold [4].

In 1980, Gray and Hervella [36], presented sixteen classes of an almost Hermitian manifold.

Given an indefinite almost Hermitian manifold with indefinite metric $\bar{g}$, the second fundamental form Φ is defined by $\Phi(X, Y) = \bar{g}(X, \bar{J}Y)$, for all vector fields X and Y on $\bar{M}$. It is obvious that Φ is $\bar{J}$ invariant and skew-symmetric. Using this second fundamental form Φ, we have the following important theorem.

Theorem 1.2.2. [90] *Let $\bar{M}$ be an almost complex manifold with almost complex structure $\bar{J}$ and a Hermitian metric $\bar{g}$. Let $\bar{\nabla}$ be the covariant differentiation of the Riemannian connection defined by $\bar{g}$. Then the following conditions are equivalent:*
(i) $\bar{\nabla}\bar{J} = 0$.
(ii) $\bar{\nabla}\Phi = 0$.
(iii) The almost complex structure has no torsion and the fundamental 2-form Φ is closed, that is, $N = 0$ and $d\Phi = 0$.

In 1933, Kähler presented the idea of a Kählerian structure on a complex manifold. Barros and Romero [4] studied indefinite Kähler manifolds.

Definition 1.2.3. [4] *An indefinite Hermitian metric on an almost complex manifold is said to be an indefinite Kähler metric if the fundamental* 2*-form* Φ *is closed, that is,* $d\Phi = 0$*. A complex manifold equipped with the indefinite Kähler metric is called an indefinite Kähler manifold. In other words, using the Levi-Civita connection* $\bar{\nabla}$ *indefinite almost complex manifold becomes an indefinite Kähler manifold, if* $(\bar{\nabla}_X \bar{J})Y = 0$*, for any* X, Y *in* $T\bar{M}$*.*

Definition 1.2.4. [4] *If the almost complex structure* $\bar{J}$ *of an almost Hermitian manifold* $\bar{M}$ *satisfies* $(\bar{\nabla}_X \bar{J})Y + (\bar{\nabla}_Y \bar{J})X = 0$*, for all* $X, Y \in \Gamma(T\bar{M})$*, then* $\bar{M}$ *is called a nearly Kähler manifold.*

It is well known that every Kähler manifold is nearly Kähler but the converse is not true in general. Like Kähler manifolds, nearly Kähler manifolds also have rich geometrical as well as topological properties. In fact, for the converse to hold it is necessary that the almost complex structure $\bar{J}$ should be integrable. A nearly Kähler structure on a manifold therefore provides an interesting study with different geometric point of view.

If an indefinite Kähler manifold has constant holomorphic sectional curvature c, then it is called an indefinite complex space form and denoted by $\bar{M}(c)$. For an indefinite complex space form, the Riemannian curvature tensor can be expressed as (see [4])

$$\begin{aligned} \bar{R}(X,Y)Z &= \frac{c}{4}\{\bar{g}(Y,Z)X - \bar{g}(X,Z)Y + \bar{g}(\bar{J}Y,Z)\bar{J}X - \bar{g}(\bar{J}X,Z)\bar{J}Y \\ &\quad +2\bar{g}(X,\bar{J}Y)\bar{J}Z\}, \qquad (1.2.2) \end{aligned}$$

for any $X, Y, Z \in \Gamma(T\bar{M})$.

Analogue to almost complex manifolds (even dimensional), the study of almost contact manifolds (odd-dimensional manifold) is of equal importance. The study of an odd-dimensional manifold was initiated by Boothby and Wang in

1958. Gray in 1959 studied odd dimensional manifold from the topological point of view and introduced a structure called contact structure. Sasaki in 1960 and Hsu in 1962 defined and studied almost contact structure and its integrability conditiions with the help of tensor analysis.

Blair [12] in 1976 defined almost contact manifold as:

Definition 1.2.5. [12] *Let $\bar{M}$ be a real $(2n+1)$-dimensional differentiable manifold endowed with an almost contact structure (ϕ, η, V), where ϕ is a tensor field of type $(1,1)$, η is a 1-form and V is a vector field on $\bar{M}$, called a characterstic vector field, satisfying*

$$\phi^2 X = -X + \eta(X)V, \quad \eta \circ \phi = 0, \quad \phi V = 0, \quad \eta(V) = 1. \tag{1.2.3}$$

Then $\bar{M}$ is called an almost contact manifold.

If there exists a semi-Riemannian metric $\bar{g}$ satisfying

$$\bar{g}(\phi X, \phi Y) = \bar{g}(X, Y) - \eta(X)\eta(Y), \quad \bar{g}(X, V) = \eta(X), \tag{1.2.4}$$

for $X, Y \in \Gamma(T\bar{M})$, then $(\phi, \eta, V, \bar{g})$ is called an indefinite almost contact metric structure and $\bar{M}$ is known as an indefinite almost contact manifold.

In the present thesis, the geometrical aspects of the following three classes of almost contact manifolds have been studied:

(a) Sasakian manifold

(b) Cosymplectic manifold

(c) Kenmotsu manifold

An almost contact manifold having contact metric structure $(\phi, \eta, V, \bar{g})$ is called *Sasakian manifold* [12], if and only if

$$(\bar{\nabla}_X \phi)Y = -\bar{g}(X, Y)V + \eta(Y)X \quad \textit{and} \quad \bar{\nabla}_X V = \phi X, \tag{1.2.5}$$

for any $X, Y \in \Gamma(T\bar{M})$, where $\bar{\nabla}$ is the Levi-Civita connection with respect to $\bar{g}$.

If Sasakian manifold has constant ϕ holomorphic sectional curvature c, then it is called an contact space form and denoted by $\bar{M}(c)$. For a contact space form, the expression of Riemannian curvature tensor is given by (see [12])

$$\begin{aligned}\bar{R}(X,Y)Z &= \frac{c+3}{4}\{\bar{g}(Y,Z)X - \bar{g}(X,Z)Y\} + \frac{c-1}{4}\{\eta(X)\eta(Z)Y \\ &-\eta(Y)\eta(Z)X + \bar{g}(X,Z)\eta(Y)V - \bar{g}(Y,Z)\eta(X)V + \bar{g}(\phi Y, Z)\phi X \\ &-\bar{g}(\phi X, Z)\phi Y + 2\bar{g}(\phi X, Y)\phi Z\}, \qquad (1.2.6)\end{aligned}$$

for $X, Y, Z \in \Gamma(T\bar{M})$.

An odd-dimensional counterpart of a Kähler manifold is given by Cosymplectic manifold. Trivial example of a Cosymplectic manifold is given by the product of $2n-$dimensional Kähler manifold with $1-$dimensional manifold.

An almost contact manifold $\bar{M}$ is said to be *Cosymplectic manifold* [13] if

$$(\bar{\nabla}_X \phi)Y = 0 \quad and \quad \bar{\nabla}_X V = 0,$$

for any $X, Y \in \Gamma(T\bar{M})$.

An almost contact metric manifold $\bar{M}$ is called an *Kenmotsu manifold* [13] if

$$(\bar{\nabla}_X \phi)Y = -g(\phi X, Y)V + \eta(Y)\phi X, \quad and \quad \bar{\nabla}_X V = -X + \eta(X)V,$$

for any $X, Y \in \Gamma(T\bar{M})$.

It should be noted that in case, if the index of the metric satisfying (1.2.4), is $q = 0$, then the manifold is the usual almost contact metric manifold and if the index $q \in \{1, 2, ...2n\}$, then manifold $\bar{M}$ is an indefinite almost contact metric manifold. (for detail see [9])

Afterwards, many authors studied different geometric aspects of complex and contact manifolds for definite as well as for indefinite metric. (see [3, 4, 9, 14, 20, 33, 44, 49, 63, 64])

1.3 Submanifolds Theory

To study the geometry of submanifolds, first embed it conveniently into a manifold whose geometry is known and then study the geometry which is induced on it. This approach gave impetus to the study of submanifolds which later developed into a fascinating study of the theory of submanifolds. Further, the theory of submanifolds as a field of differential geometry is as old as differential geometry itself. Hence these facts strengthens the view that the study of submanifolds is a basic branch of geometry.

Chen [21], Ludden [55] and many others have studied the geometry of submanifolds.

Before discussing the theory of submanifolds, we recall the definitions of immersion and embedding.

Definition 1.3.1. [21] *Let f be a differentiable map from manifold M into manifold $\bar{M}$ and let the dimension of M and $\bar{M}$ be n and m respectively $(n < m)$. If at each point p of M, $(f_*)_p$ (the tangential map of f) is one-one, then f is called an immersion of M into $\bar{M}$. If f is an immersion and moreover if f is one-one map from M into $\bar{M}$, then f is called an embedding of M into $\bar{M}$.*

Definition 1.3.2. [21] *If $f : M^n \to \bar{M}^m$ is an immersion, then M is a submanifold of manifold $\bar{M}$.*

If $\bar{M}$ is a Riemannian manifold, then Riemannian metric $\bar{g}$ of $\bar{M}$ induces a Riemannian metric $\bar{g}$ on M and is given by $\bar{g}(f_*X, f_*Y) = g(X, Y)$, where f_* is the Jacobian map of f. This indicates that the geometry of M depends on immersions, also. For every point $p \in M$ the tangent space $T_{f(p)}(\bar{M})$ of $\bar{M}$ admits the following decomposition

$$T_{f(p)}(\bar{M}) = T_pM \oplus T_pM^{\perp},$$

where $T_p(M)$ is tangent space of M at p and $T_pM^{\perp}$ is orthogonal complement of $T_p(M)$ in $T_{f(p)}(\bar{M})$.

The Riemannian connection $\bar{\nabla}$ of $\bar{M}$ induces canonically the connection ∇ on M and $\nabla^{\perp}$ on the normal bundle, governed by the Gauss and Wiengarten formulas as

$$\bar{\nabla}_X Y = \nabla_X Y + h(X,Y) \quad \textit{and} \quad \bar{\nabla}_X U = -A_U X + \nabla^{\perp}_X U,$$

where X and Y are vector fields on M and $U \in TM^{\perp}$, h and A_N are the second fundamental form and the shape operator of M, respectively and are related as

$$\bar{g}(A_U X, Y) = \bar{g}(h(X,Y), U).$$

The geometry of the submanifolds heavily depends on the second fundamental form. Based on this, submanifolds have been divided into following classes:

A submanifold is called a totally umbilical [90] *if its second fundamental form h satisfies*

$$h(X,Y) = \bar{g}(X,Y)\alpha,$$

for all X and $Y \in TM$ and $\alpha = \frac{1}{n}\sum_{i=1}^{n} h(e_i, e_i)$ is the mean curvature vector, where e_i is a local orthonormal basis in $T\bar{M}$.

A submanifold for which the second fundamental form h is identically zero is called a totally geodesic submanifold [90] *and if the mean curvature vector vanishes identically, that is, $\alpha = 0$, then a submanifold is called a minimal submanifold* [90].

Moreover, any submanifold M, which is minimal and totally umbilical is totally geodesic.

Let $(\bar{M}, \bar{g}, \bar{J})$ be an almost Hermitian manifold. Then for any $X \in TM$, we have $g(\bar{J}X, X) = 0$, which implies that $\bar{J}X \perp X$ for each vector field X on

$\bar{M}$. Hence, for a submanifold M of $\bar{M}$ if $X \in T_pM$, then $\bar{J}X$ may or may not belong to T_pM. The submanifolds of almost complex manifolds have additional advantage because of the peculiar behavior of the almost complex structure. The action of the almost complex structure $\bar{J}$ on the tangent vectors of the submanifold of the almost Hermitian manifold gives rise to its classification into invariant and anti-invariant submanifolds. These submanifolds are defined as follows:

Definition 1.3.3. [21] *A submanifold M of an almost Hermitian manifold is said to be an invariant (or holomorphic), if $\bar{J}(T_pM) = T_pM$, for all $p \in M$.*

Definition 1.3.4. [21] *A submanifold M of an almost Hermitian manifold is said to be an anti-invariant (or totally real) if $\bar{J}(T_pM) \subset T_pM^{\perp}$, for all $p \in M$.*

An invariant (or holomorphic) submanifold inherits almost all properties of the ambient manifold and so the study of invariant submanifolds is not so interesting from the point of view of the geometry of the submanifolds. On the other hand the theory of totally real (or anti-invariant) submanifolds has been proved to be a very interesting topic in modern differential geometry. The study of totally real (or anti-invariant) submanifolds was initiated in the early 1970's. Totally real (or anti-invariant) submanifolds have been studied by several geometers like Bagewadi [2], Ludden et al. [56].

In 1978, Bejancu [7] considered a new class of submanifolds of an almost Hermitian manifold known as CR-submanifolds as an umbrella of invariant and anti-invariant submanifolds, that is, the CR-submanifold contains invariant and anti-invariant submanifolds as sub cases.

1.3.1 CR-Submanifolds

Cauchy Riemann (CR)-submanifold provides a single setting to study the invariant and anti-invariant submanifolds of an almost Hermitian manifold. A

CR-submanifold is endowed with two orthogonal complementary distributions such that one is holomorphic and the other is totally real. A distribution D of dimension r on a manifold $\bar{M}$ is an assignment to each point $p \in \bar{M}$, an r-dimensional linear subspace D_p of $T_p\bar{M}$. Because of its importance (as it generalizes both invariant and anti-invariant submanifolds) it attracted the attention of many mathematicians and have become the subject of extensive research.

Let $(\bar{M}, \bar{g})$ be an almost Hermitian manifold having an almost complex structure $\bar{J}$ and M be a Riemannian submanifold immersed in $\bar{M}$. At each point $p \in M$, let D_p be the maximal holomorphic subspace of the tangent space T_pM, that is, $D_p = T_pM \cap \bar{J}T_pM$. If the dimension of D_p is same for all $p \in M$, we get a holomorphic distribution D on M.

Definition 1.3.5. [7] *A submanifold is said to be a CR-submanifold of an almost Hermitian manifold, if there exists a C^∞-holomorphic distribution D on M such that its orthogonal complementary distribution $D^\perp$ is totally real, that is, $\bar{J}D_p^\perp \subseteq T_pM^\perp$, for all $p \ \in \ M$.*

A CR-submanifold M is called proper if neither D nor $D^\perp = \{0\}$. Obviously if $D = \{0\}$, then M is a totally real submanifold and if $D^\perp = \{0\}$, then M is an holomorphic submanifold.

Significant contributions have been made by Bejan [5], Bejancu [7, 8], Chen [22, 25, 26], Shahid [82] and many more, by generalizing some classical results on CR-submanifolds. Since then this subject has attracted number of differential geometers towards this area. These generalization of CR submanifolds give rise to the various notions namely, slant and semi-slant and semi-invariant submanifolds in K*ä*hler as well as in contact setting (cf. [10, 23, 24, 65]).

1.3.2 Slant Submanifolds

A submanifold of an almost Hermitian manifold is an holomorphic or a totally real submanifold, if and only if, the angle between $\bar{J}X$ and tangent space T_pM is 0 or $\pi/2$, respectively, for every non zero vector X tangent to any point $p \in M$. This paved the way for a generalization of holomorphic and totally real submanifold to a new class of submanifold, known as slant submanifold, introduced by Chen [23, 24] in 1990. Slant submanifolds are those for which the angle between $\bar{J}X$ and the tangent space T_pM is constant. Thus, the theory of slant-submanifolds is an umbrella over the theory of holomorphic submanifolds and totally real submanifolds.

Definition 1.3.6. [23] *Let M be a submanifold of an almost Hermitian manifold $\bar{M}$. If for every $p \in M$ and $X \in T_pM$, the angle between the tangent space T_pM and $\bar{J}X_p$ (denoted by $\theta(X_p)$ and called as Wirtinger angle) is constant, then M is called a slant submanifold of $\bar{M}$.*

The following example demonstrates the preceeding definition:

Example [24]: *For any $\alpha > 0$, consider $f : \mathbb{R}^2 \to \mathbb{R}^4$ defined by*

$$f(u, v) = (u\ cos\alpha, u\ sin\alpha, v, 0),$$

then at any point p of $\mathbb{R}^2$, we have

$$df_p = \begin{bmatrix} cos\alpha & 0 \\ sin\alpha & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Let $\{e_1, e_2\}$ be a local orthonormal frame on $\mathbb{R}^2$. Then we can choose

$$e_1 = \frac{df_p(\frac{\partial}{\partial u})}{\|df_p(\frac{\partial}{\partial u})\|} = (cos\alpha \quad sin\alpha \quad 0 \quad 0),$$

and

$$e_2 = \frac{df_p(\frac{\partial}{\partial v})}{\|df_p(\frac{\partial}{\partial v})\|} = (0 \quad 0 \quad 1 \quad 0),$$

where $\|df_p(\frac{\partial}{\partial u})\| = \|df_p(\frac{\partial}{\partial v})\| = 1$. *Let* $\bar{J}$ *be the natural almost complex structure of* $\mathbb{R}^4$. *Then we have*

$$\bar{J}e_1 = (0 \quad 0 \quad cos\alpha \quad sin\alpha)$$

$$\bar{J}e_2 = (-1 \quad 0 \quad 0 \quad 0)$$

Thus, for $i \neq j$, $\langle \bar{J}e_1, e_2 \rangle = cos\alpha, \langle \bar{J}e_2, e_1 \rangle = -cos\alpha, |\langle \bar{J}e_i, e_j \rangle| = cos\alpha > 0$, *which is constant. This implies* $\mathbb{R}^2$ *is slant submanifold of* $\mathbb{R}^4$, *so* f *defines a slant plane with slant angle* α *in* $\mathbb{R}^4$.

Slant submanifolds of a K*ä*hler manifold have been investigated by Chen [23]. Lotta [53] introduced the notion of slant immersion of a Riemannian manifold into an almost contact metric manifold. Some properties about the intrinsic geometry of 3-dimensional non-anti-invariant slant submanifolds of K-contact manifolds have also been studied by Lotta [54]. Later, Cabrerizo et al. [17, 18] studied semi-slant and slant submanifolds of Sasakian manifolds. They studied the special class of three-dimensional slant submanifolds and characterize slant submanifolds of K-contact and Sasakian manifolds. In 2004, Gupta et al. [37] studied slant submanifolds of Kenmotsu manifolds. In 2009, Sahin [76] proved an important result that every totally umbilical proper slant submanifolds of a K*ä*hler manifold is totally geodesic. Later, several geometers studied slant submanifolds. (see [47, 77, 78])

1.3.3 Lightlike Submanifolds

Duggal and Bejancu [27] in 1996 initiated the thought of lightlike geometry. Since then, considerable work has been done in this direction. For a general study of extrinsic geometry of lightlike submanifolds of a semi-Riemannian manifold, refer to two books [27, 31] published in 1996, 2010, respectively.

Duggal and Jin [28] studied a class of totally umbilical lightlike submanifolds in semi-Riemannian manifolds. Further, Jin [43] studied the geometry of lightlike hypersurfaces of an indefinite Sasakian manifold.

The primary difference between the theory of lightlike submanifolds and classical theory of Riemannian or semi-Riemannian submanifolds arises due to the fact that in the first case, the normal vector bundle $TM^{\perp}$ intersects the tangent bundle TM of the submanifold M of $\bar{M}$, whereas in second case $TM \cap TM^{\perp} = \{0\}$. In other words, a vector of a tangent space $T_p\bar{M}$ cannot be decomposed uniquely into a component tangent to T_pM and a component of normal space $T_pM^{\perp}$. Thus, in the usual way, one fails to use the theory of non-degenerate submanifolds to define the induced geometric objects (such as linear connection, second fundamental form, Gauss and Weingarten equations) on the lightlike submanifold.

The used notations and fundamental equations for lightlike submanifolds are referred from the book of Duggal-Bejancu [27].

Let $(\bar{M}, \bar{g})$ be a real $(m+n)$-dimensional semi-Riemannian manifold, where $m > 1$, $n > 1$ and $\bar{g}$ is a semi-Riemannian metric on $\bar{M}$ of constant index $q \in \{1, ..., m+n-1\}$. Hence $\bar{M}$ is never a Riemannian manifold.

Suppose M is a submanifold of $\bar{M}$ of codimension n. Denote g the induced tensor field on M of $\bar{g}$, that is, for any $p \in M$, we have

$$g_p(X_p, Y_p) = \bar{g}_p(X_p, Y_p), \quad \forall\ X_p, Y_p \in T_pM.$$

Consider

$$T_pM^{\perp} = \{V_p \in T_p\bar{M} : \bar{g}_p(V_p, W_p) = 0, \quad \forall\ W_p \in T_pM\}.$$

In case $\bar{g}_p$ is non-degenerate on T_pM then both T_pM and $T_pM^\perp$ are non-degenerate and T_pM and $T_pM^\perp$ are complementary orthogonal vector subspaces of $T_p\bar{M}$. Otherwise, both T_pM and $T_pM^\perp$ are degenerate orthogonal subspaces but no longer complementary subspaces. In this case, there exists a subspace $Rad\ T_pM = T_pM \cap T_pM^\perp$ which is known as radical (null) subspace.

If the mapping

$$Rad\ TM : p \in M \longrightarrow Rad\ T_pM,$$

defines a smooth distribution on M of rank $r > 0$, then the submanifold M of $\bar{M}$ is called an *r-lightlike (r-degenerate, r-null) submanifold* and $Rad\ TM$ is called as *radical distribution* on M.

For the construction of transversal vector bundle of a lightlike submanifold M of $\bar{M}$, we examine the following possible four cases with respect to the dimension and codimension of M and rank of $Rad\ TM$.

1. M is an r-lightlike submanifold if $0 < r < min\{m, n\}$.
2. M is a Coisotropic submanifold if $1 < r = n < m$.
3. M is a Isotropic submanifold if $1 < r = m < n$.
4. M is a Totally lightlike submanifold if $1 < r = n = m$.

Case I r-lightlike submanifold ($0 < r < min\{m, n\}$): Consider a complementary distribution $S(TM)$ of $Rad\ TM$ in TM. As M is supposed to be paracompact, such a distribution always exists on M. Clearly, $S(TM)$ is orthogonal to $Rad\ TM$ and non-degenerate with respect to $\bar{g}$. Consider the orthogonal direct sum

$$TM = Rad\ TM \perp S(TM). \tag{1.3.1}$$

Now, consider a vector bundle

$$TM^{\perp} = \cup_{p\in M} T_pM^{\perp}.$$

For a lightlike submanifold M, $TM^{\perp}$ is not complementary to TM in $T\bar{M}|_M$ since $Rad\ TM = TM \cap TM^{\perp}$ is now a distribution on M of rank $r > 0$. To overcome this difficulty, consider a complementary vector bundle $S(TM^{\perp})$ of $Rad\ TM$ in $TM^{\perp}$. It follows that $S(TM^{\perp})$ is also non-degenerate with respect to $\bar{g}$ and $TM^{\perp}$ has the following decomposition

$$TM^{\perp} = Rad\ TM \perp S(TM^{\perp}).$$

$S(TM)$ and $S(TM^{\perp})$ are called as the *screen distribution* and a *screen transversal vector bundle* of M respectively. As $S(TM)$ is a non-degenerate vector subbundle of $T\bar{M}|_M$, therefore

$$T\bar{M}|_M = S(TM) \perp S(TM)^{\perp},$$

where $S(TM)^{\perp}$ is the complementary orthogonal vector bundle of $S(TM)$ in $T\bar{M}|_M$. Clearly, $S(TM^{\perp})$ is a vector subbundle of $S(TM)^{\perp}$ and since both are non-degenerate, therefore we have

$$S(TM)^{\perp} = S(TM^{\perp}) \perp S(TM^{\perp})^{\perp}.$$

For quasi-orthonormal fields of frames on an r-lightlike submanifold, Duggal and Bejancu [27] presented the following results:

Theorem 1.3.7. [27] *Let $(M, g, S(TM), S(TM^{\perp}))$ be an r-lightlike submanifold of $(\bar{M}, \bar{g})$ with $r > 1$. Suppose $\mathcal{U}$ is a coordinate neighborhood of M and $\{\xi_i\}$, $i \in \{1, 2, ..., r\}$ is a basis of $\Gamma(RadTM|_{\mathcal{U}})$. Then there exist smooth sections $\{N_i\}$ of $S(TM^{\perp})^{\perp}|_{\mathcal{U}}$ such that*

$$\bar{g}(N_i, \xi_j) = \delta_{ij}, \quad \bar{g}(N_i, N_j) = 0, \tag{1.3.2}$$

for any $i, j \in \{1, 2, ..., r\}$.

Theorem 1.3.8. [27] *Let $(M, g, S(TM), S(TM^{\perp}))$ be an r-lightlike submanifold of $(\bar{M}, \bar{g})$ with $r > 1$. Then there exists a complementary vector bundle $ltr(TM)$ of $RadTM$ in $S(TM^{\perp})^{\perp}$ such that $\{N_i\}$, $i \in \{1, 2, ..., r\}$ from above theorem is a basis of $\Gamma(ltr(TM)|_{\mathcal{U}})$*

As $ltr(TM)$ is a lightlike vector bundle that has null intersection with TM, we call it a *lightlike transversal vector bundle* of M with respect to the pair $(S(TM), S(TM^{\perp}))$. Consider the vector bundle

$$tr(TM) = ltr(TM) \perp S(TM^{\perp}), \tag{1.3.3}$$

Clearly $tr(TM)$ is of rank n and has null intersection with TM. Thus $tr(TM)$ is a complementary (but never orthogonal) vector bundle to TM in $T\bar{M}|_M$ and $tr(TM)$ is called as *transversal vector bundle* of M. Thus,

$$T\bar{M}|_M = TM \oplus tr(TM) = S(TM) \perp S(TM^{\perp}) \perp (RadTM \oplus ltr(TM)). \tag{1.3.4}$$

Now, we present the example for r-lightlike submanifold.

Example [27]: *Consider a surface M in $\mathbb{R}^4_2$ given by the equations*

$$x^3 = \frac{1}{\sqrt{2}}(x^1 + x^2); \quad x^4 = \frac{1}{2} log(1 + (x^1 - x^2)^2).$$

Then $TM = span\{Z_1, Z_2\}$ and $TM^{\perp} = span\{Z_3, Z_4\}$, where

$$Z_1 = \sqrt{2}(1 + (x^1 - x^2)^2)\frac{\partial}{\partial x^1} + (1 + (x^1 - x^2)^2)\frac{\partial}{\partial x^3} + \sqrt{2}(x^1 - x^2)\frac{\partial}{\partial x^4}$$

$$Z_2 = \sqrt{2}(1 + (x^1 - x^2)^2)\frac{\partial}{\partial x^2} + (1 + (x^1 - x^2)^2)\frac{\partial}{\partial x^3} - \sqrt{2}(x^1 - x^2)\frac{\partial}{\partial x^4}$$

and

$$Z_3 = \frac{\partial}{\partial x^1} + \frac{\partial}{\partial x^2} + \sqrt{2}\frac{\partial}{\partial x^3}$$

$$Z_4 = 2(x^2 - x^1)\frac{\partial}{\partial x^2} + \sqrt{2}(x^2 - x^1)\frac{\partial}{\partial x^3} + (1 + (x^1 - x^2)^2)\frac{\partial}{\partial x^4}.$$

Clearly $RadTM$ is a distribution on M of rank 1 spanned by $\xi = Z_3$. Hence M is a 1-lightlike submanifold of $\mathbb{R}^4_2$. Choose $S(TM)$ and $S(TM^{\perp})$ spanned by Z_2 and Z_4 which are timelike and spacelike respectively. Finally, the lightlike transversal vector bundle is given by

$$ltr(TM) = span\{N = -\frac{1}{2}\frac{\partial}{\partial x^1} + \frac{1}{2}\frac{\partial}{\partial x^2} + \frac{1}{\sqrt{2}}\frac{\partial}{\partial x^3}\},$$

and the transversal bundle is given by $tr(TM) = span\{N, Z_4\}$.

Case II Coisotropic submanifold ($1 < r = n < m$): In this case $RadTM = TM^{\perp}$, that is, $S(TM^{\perp}) = \{0\}$ and thus the former normal bundle from the classical theory of Riemannian submanifolds becomes a distribution on the lightlike submanifold. We call M a *coisotropic submanifold* of $\bar{M}$ and we have

$$TM = S(TM) \perp TM^{\perp}.$$

Thus we obtain

$$T\bar{M}|_M = TM \oplus tr(TM) = S(TM) \perp (TM^{\perp} \oplus ltr(TM)).$$

Case III Isotropic submanifold ($1 < r = m < n$): In this case $RadTM = TM$, that is, $S(TM) = \{0\}$ and we call M a *isotropic submanifold* of $\bar{M}$ and we have

$$TM^{\perp} = TM \perp S(TM^{\perp}).$$

Thus we obtain

$$T\bar{M}|_M = TM \oplus tr(TM) = S(TM^{\perp}) \perp (TM \oplus ltr(TM)).$$

Case IV Totally lightlike submanifold ($1 < r = m = n$): In this case $RadTM = TM = TM^{\perp}$, that is, $S(TM) = \{0\} = S(TM^{\perp})$ and we call M a *totally lightlike submanifold* of $\bar{M}$ and we have

$$T\bar{M}|_M = TM \oplus ltr(TM).$$

1.3.4 The Induced Geometric Objects on Lightlike Submanifolds

Let $(M, g, S(TM), S(TM^{\perp}))$ be m-dimensional lightlike submanifold of $(m+n)$-dimensional semi-Riemannian manifold $(\bar{M}, \bar{g})$. Let $\bar{\nabla}$ and ∇ be the Levi-Civita connection on $\bar{M}$ and induced connection on M, then Gauss and Weingarten formulas are

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y), \quad \bar{\nabla}_X U = -A_U X + \nabla^t_X U, \tag{1.3.5}$$

for all X and $Y \in \Gamma(TM)$ and $U \in \Gamma(tr(TM))$, here $\nabla_X Y$ and $A_U X$ lies in $\Gamma(TM)$ and $\{h(X, Y), \nabla^{\perp}_X U\}$ belongs to $\Gamma(tr(TM))$, respectively. Here the connection ∇ is torsion free on M, the second fundamental form h is symmetric bilinear form on TM and shape operator A_U with respect to U of M is $\Gamma(TM)$-valued bilinear form defined on $\Gamma(tr(TM)) \times \Gamma(TM)$.

According to (1.3.3), let the projection morphisms L and S of $tr(TM)$ on $ltr(TM)$ and $S(TM^{\perp})$, respectively. Then (1.3.5) becomes

$$\bar{\nabla}_X Y = \nabla_X Y + h^l(X, Y) + h^s(X, Y), \tag{1.3.6}$$

$$\bar{\nabla}_X U = -A_U X + D^l_X U + D^s_X U, \tag{1.3.7}$$

where, we put $h^l(X, Y) = L(h(X, Y))$, $h^s(X, Y) = S(h(X, Y))$, $D^l_X U = L(\nabla^{\perp}_X U)$ and $D^s_X U = S(\nabla^{\perp}_X U)$, here h^l is $\Gamma(ltr(TM))$-valued, called the lightlike second fundamental form and h^s is $\Gamma(S(TM^{\perp}))$-valued, called the screen second fundamental form on M. Also note that D^l and D^s does not define linear connections on $tr(TM)$. In particular, we have

$$\bar{\nabla}_X N = -A_N X + \nabla^l_X N + D^s(X, N), \tag{1.3.8}$$

$$\bar{\nabla}_X W = -A_W X + \nabla^s_X W + D^l(X, W), \tag{1.3.9}$$

for any $X \in \Gamma(TM)$, $W \in \Gamma(S(TM^{\perp}))$ and $N \in \Gamma(ltr(TM))$. Using (1.3.3)-(1.3.4) and (1.3.6)-(1.3.9), we obtain

$$\bar{g}(h^s(X,Y),W) + \bar{g}(Y, D^l(X,W)) = g(A_W X, Y), \tag{1.3.10}$$

$$\bar{g}(h^l(X,Y),\xi) + \bar{g}(Y, h^l(X,\xi)) + g(Y, \nabla_X \xi) = 0,$$

$$\bar{g}(N, A_W X) = \bar{g}(D^s(X,N), W),$$

$$\bar{g}(A_N X, N') + \bar{g}(N, A_{N'} X) = 0,$$

$$\bar{g}(A_N X, \bar{P}Y) = \bar{g}(N, \bar{\nabla}_X \bar{P}Y),$$

for any $X, Y \in \Gamma(TM)$, $\xi \in \Gamma(RadTM)$, $N, N' \in \Gamma(ltr(TM))$ and $W \in \Gamma(S(TM^{\perp}))$ where $\bar{P}$ is a projection of TM on $S(TM)$.

In general, the induced linear connection ∇ on M and the transversal linear connection ∇^t on $tr(TM)$ are not metric connections. Using (1.3.6), (1.3.8) and (1.3.9) and considering into mind that $\bar{\nabla}$ is a metric connection, we obtain

$$(\nabla_X g)(Y,Z) = \bar{g}(h^l(X,Z),Y) + \bar{g}(h^l(X,Y),Z)$$

and

$$(\nabla^t_X \bar{g})(V,V') = -\{\bar{g}(A_V X, V') + \bar{g}(A_{V'} X, V)\},$$

for any $X, Y, Z \in \Gamma(TM)$ and $V, V^{'} \in \Gamma(tr(TM))$.

Duggal and Bejancu [27] established the following theorems:

Theorem 1.3.9. [27] *Let M be an r-lightlike submanifold with $r < min\{m,n\}$ or a Coisotropic submanifold of $(\bar{M},\bar{g})$. Then ∇ is a metric connection, if and only if, h^l vanishes identically on M.*

Theorem 1.3.10. [27] *Let M be an r-lightlike submanifold with $r < min\{m, n\}$ or an Isotropic submanifold of $(\bar{M}, \bar{g})$. Then the following assertions are equivalent:*

(i) ∇^t *is a metric linear connection on* $tr(TM)$.

(ii) D^s *is a metric Otsuki connection on* $tr(TM)$.

(iii) A_W *are* $\Gamma(S(TM))$*-valued linear operator.*

(iv) $D^s(X, LV) = 0$*, for any* $X \in \Gamma(TM)$*,* $V \in \Gamma(tr(TM))$.

(v) $ltr(TM)$ *is parallel with respect to* ∇^t.

Next, let $\bar{P}$ be the projection morphism of TM on $S(TM)$. Then by using (1.3.1), we have

$$\nabla_X \bar{P}Y = \nabla^*_X \bar{P}Y + h^*(X, \bar{P}Y), \tag{1.3.11}$$

$$\nabla_X \xi = -A^*_\xi X + \nabla^{*t}_X \xi, \tag{1.3.12}$$

for any $X, Y \in \Gamma(TM)$ and $\xi \in \Gamma(RadTM)$, where $\{\nabla^*_X \bar{P}Y, A^*_\xi X\}$ and $\{h^*(X, \bar{P}Y),\ \nabla^{*t}_X \xi\}$ belong to $\Gamma(S(TM))$ and $\Gamma(RadTM)$, respectively. Here, it is important to note that both ∇^* and ∇^{*t} are metric linear connections on complementary distributions $S(TM)$ and $RadTM$, respectively. On the other hand, h^* and A^* are $\Gamma(RadTM)$-valued and $\Gamma(S(TM))$-valued bilinear forms on $\Gamma(TM) \times \Gamma(S(TM))$ and $\Gamma(RadTM) \times \Gamma(TM)$, respectively and called as the second fundamental forms of complementary distributions $S(TM)$ and $RadTM$ respectively. For any $\xi \in \Gamma(RadTM)$, consider a linear operator as

$$A^*_\xi : \Gamma(TM) \to \Gamma(S(TM)); \quad A^*_\xi X = A^*(\xi, X), \quad \forall \quad X \in \Gamma(TM),$$

called the shape operator of $S(TM)$ with respect to ξ. The second fundamental form and the shape operator of a non-degenerate submanifold of a semi-Riemannian are related by means of the metric tensor field. Contrary to this situation, in case of lightlike submanifolds there are interrelations between geometric objects induced by $tr(TM)$ on one side and geometric objects induced by $S(TM)$ on the other side. More precisely, by using (1.3.6), (1.3.7), (1.3.11) and (1.3.12), we obtain

$$g(A^*_\xi X, \bar{P}Y) = \bar{g}(h^l(X, \bar{P}Y), \xi) \quad and \quad \bar{g}(A_N X, \bar{P}Y) = \bar{g}(h^*(X, \bar{P}Y), N), \tag{1.3.13}$$

for all X and $Y \in \Gamma(TM), \xi \in \Gamma(RadTM)$ and $N \in \Gamma(ltr(TM))$.

As h^l is symmetric therefore from (1.3.13), it follows that the shape operator of $S(TM)$ is a self-adjoint operator on $S(TM)$, that is, we have $g(A^*_\xi \bar{P}X, \bar{P}Y) = g(\bar{P}X, A^*_\xi \bar{P}Y)$.

If $\bar{R}$ and R stands for the curvature tensors of $\bar{\nabla}$ and ∇, respectively then by direct calculations, we have (for details see [27])

$$\begin{aligned} \bar{R}(X,Y)Z \;&= R(X,Y)Z + A_{h^l(X,Z)}Y - A_{h^l(Y,Z)}X + A_{h^s(X,Z)}Y \\ &\quad - A_{h^s(Y,Z)}X + (\nabla_X h^l)(Y,Z) - (\nabla_Y h^l)(X,Z) \\ &\quad + D^l(X, h^s(Y,Z)) - D^l(Y, h^s(X,Z)) + (\nabla_X h^s)(Y,Z) \\ &\quad - (\nabla_Y h^s)(X,Z) + D^s(X, h^l(Y,Z)) \\ &\quad - D^s(Y, h^l(X,Z)), \end{aligned} \tag{1.3.14}$$

$$\begin{aligned} \bar{R}(X,Y)N \;&= R^l(X,Y)N + h^l(Y, A_N X) - h^l(X, A_N Y) + D^l(X, D^s(Y,N)) \\ &\quad - D^l(Y, D^s(X,N)) + (\nabla_Y A)(N,X) - (\nabla_X A)(N,Y) \\ &\quad + A_{D^s(X,N)}Y - A_{D^s(Y,N)}X + (\nabla_X D^s)(Y,N) \\ &\quad - (\nabla_Y D^s)(X,N) + h^s(Y, A_N X) - h^s(X, A_N Y), \end{aligned}$$

$$\begin{aligned}\bar{R}(X,Y)W \;&= R^s(X,Y)W + h^s(Y,A_WX) - h^s(X,A_WY) + D^s(X,D^l(Y,W))\\ &-D^s(Y,D^l(X,W)) + (\nabla_YA)(W,X) - (\nabla_XA)(W,Y)\\ &+A_{D^l(X,W)}Y - A_{D^l(Y,W)}X + (\nabla_XD^l)(Y,W)\\ &-(\nabla_YD^l)(X,W) + h^l(Y,A_WX) - h^l(X,A_WY),\end{aligned}$$

where

$$(\nabla_Xh^s)(Y,Z) = \nabla^s_Xh^s(Y,Z) - h^s(\nabla_XY,Z) - h^s(Y,\nabla_XZ),$$

$$(\nabla_Xh^l)(Y,Z) = \nabla^l_Xh^l(Y,Z) - h^l(\nabla_XY,Z) - h^l(Y,\nabla_XZ), \tag{1.3.15}$$

for any $X, Y, Z \in \Gamma(TM),\ N \in \Gamma(ltr(TM))$ *and* $W \in \Gamma(S(TM^\perp))$.

The Codazzi equation is given by

$$\begin{aligned}(\bar{R}(X,Y)Z)^{tr} \;&= (\nabla_Xh^l)(Y,Z) - (\nabla_Yh^l)(X,Z) + D^l(X,h^s(Y,Z))\\ &-D^l(Y,h^s(X,Z)) + (\nabla_Xh^s)(Y,Z) - (\nabla_Yh^s)(X,Z)\\ &+D^s(X,h^l(Y,Z)) - D^s(Y,h^l(X,Z)).\end{aligned} \tag{1.3.16}$$

1.4 Slant Lightlike Submanifolds

Sahin clubbed the concept of slant submanifolds with lightlike geometry and presented a lightlike notion of slant submanifolds known as slant lightlike submanifolds of an almost Hermitian manifold in [74]. To introduce the notion of slant lightlike submanifolds, a Riemannian distribution is required. For such distribution, Sahin [74] proved the following lemma:

Lemma 1.4.1. [74] *Let M be an r-lightlike submanifold of an indefinite Hermitian manifold $\bar{M}$ of index $2r$. Suppose that $\bar{J}RadTM$ is a distribution on M such that $RadTM \cap \bar{J}RadTM = \{0\}$. Then any complementary distribution to $\bar{J}RadTM \oplus \bar{J}ltr(TM)$ in $S(TM)$ is Riemannian.*

In the light of above lemma, Sahin [74] defined slant lightlike submanifolds as

Definition 1.4.2. [74] *Let M be an r-lightlike submanifold of an indefinite Hermitian manifold $\bar{M}$ of index $2r$. Then M is a slant lightlike submanifold of $\bar{M}$ if the following conditions are satisfied*

- *(A) $Rad(TM)$ is a distribution on M such that $\bar{J}RadTM \cap RadTM = \{0\}$.*
- *(B) For each non-zero vector field tangent to D at $p \in U \subset M$, the angle $\theta(X)$ between $\bar{J}X$ and the vector space D_p is constant, that is, it is independent of the choice of $p \in U \subset M$ and $X \in D_p$, where D is complementary distribution to $\bar{J}RadTM \oplus \bar{J}ltr(TM)$ in the screen distribution $S(TM)$.*

This constant angle $\theta(X)$ is called slant angle of the distribution D. A slant lightlike submanifold is said to be proper if $D \neq \{0\}$ and $\theta \neq 0, \frac{\pi}{2}$.

Using the definition of slant lightlike submanifold, the tangent bundle TM of M is decomposed as

$$TM = RadTM \perp S(TM) = RadTM \perp (\bar{J}RadTM \oplus \bar{J}ltr(TM)) \perp D.$$

Following this field many researchers started working along this line and presented numerous interesting results on slant lightlike submanifolds taking ambient spaces as indefinite Kähler manifolds [4, 85], indefinite Sasakian manifolds [30, 43, 80, 84], indefinite Kenmotsu manifolds [38] and indefinite S-manifolds [51].

In present thesis, the geometric aspects of slant lightlike submanifolds of indefinite almost complex manifolds (Kähler manifolds) and indefinite almost contact manifolds (Sasakian manifolds, Cosymplectic manifolds and Kenmotsu manifolds) have been explored.

1.5 Objectives of the Study

In specific terms, the objectives of this study are as follows :

1. To explore the geometric aspects of slant lightlike submanifolds of indefinite almost complex manifolds.
2. To investigate the geometric aspects of slant lightlike submanifolds of indefinite Sasakian manifolds.
3. To explore the geometric aspects of slant lightlike submanifolds of indefinite almost contact manifolds like indefinite Cosymplectic manifolds and indefinite Kenmotsu manifolds.

1.6 Thesis Organization

The present thesis consists of six chapters and each chapter is divided into various sections. The mathematical relations obtained in the text have been labeled with double decimal numbering. The first figure denotes the chapter number, second represents the section and the third point out the number of the Definition (or Lemma/Proposition/Theorem/Corollary/Remark) as the case may be. For example, Theorem 1.2.3 refers to the third theorem of second section in the first chapter. The work carried out in this Thesis is described as follows: **Chapter 1** is introductory and contains most of the pre-requisites of the subsequent chapters of the thesis. In this chapter, apart from setting up the notations and terminologies to be used in the subsequent chapters, some known results interrelated with the work done in the present thesis have also been presented.

Chapter 2 is the core of this thesis and deals with the geometry of totally umbilical slant lightlike submanifolds of indefinite contact manifolds. Primarily, the study emphasizes that every totally contact umbilical slant lightlike submanifold of an indefinite Sasakian manifold is totally contact geodesic slant lightlike submanifold. In case of indefinite Sasakian space form, it has been proved that there does not exist any totally contact umbilical proper slant lightlike submanifold. Further in this chapter, the minimal slant lightlike submanifolds of an indefinite Sasakian manifold have also been characterized.

Since there are significant uses of contact geometry in the various fields of mathematics and physics (for details see Arnold [1], Maclane [57], Nazaikinskii [60]), therefore the study of totally contact umbilical slant lightlike submanifolds have also been generalized for indefinite Cosymplectic manifolds and indefinite Kenmotsu manifolds towards the end of this chapter.

Contents of this chapter have been published in *Bull. Iranian Math. Soc.*, 40(5) (2014), 1135-1151. [67] ***(SCI-indexed), Impact factor - 0.270* and *ISRN Geometry*, 2013 (2013), 1-8,** [66] **and in *Tamkang J. Math.*, 46(2) (2015), 179-191.** [70]

Chapter 3 comprises the study of hemi-slant lightlike submanifolds of indefinite K*ä*hler manifolds. The notions of axioms of indefinite hemi-slant 3-planes and indefinite hemi-slant 3-spheres with lightlike submanifolds have been introduced. In this chapter, it has been proved that if an indefinite K*ä*hler manifold satisfies the axioms of indefinite hemi-slant 3-planes and 3-spheres for some slant angle $\theta \in (0, \pi/2)$, then it is an indefinite complex space form.

It has further been proved in this chapter that there do not exist totally umbilical hemi-slant lightlike submanifolds in indefinite K*ä*hler manifolds other than the totally geodesic hemi-slant lightlike submanifolds. Consequently, it has been shown that the induced connection on a totally umbilical hemi-slant lightlike submanifold is a metric connection. A characterization theorem on the non-existence of totally umbilical hemi-slant lightlike submanifold of an indefinite complex space form and some characterization theorems on minimal hemi-slant lightlike submanifolds have also been presented towards the end of this chapter.

Some results of this chapter have been published in *New York J. Math.*, 21 (2015), 191-203. [69] ***(SCI-indexed), Impact factor - 0.418*** **and rest of the results are accepted for publication in *Lobachevskii J. Math.*** [71]

Bishop and O'Neill [11] in 1969 introduced warped product manifolds as a generalization of Riemannian product manifolds. The easiest example of warped product manifolds is surface of revolution. Formally, a warped product manifold $B \times_f F$ of two Riemannian manifolds (B, g_B) and (F, g_F), where g_B and g_F are Riemannian metrics on B and F respectively, is the product manifold $B \times F$, equipped with Riemannian metric $g = g_B + f^2 g_F$, where f is a positive differentiable function on B. Bishop and O'Neill studied these manifolds to study the manifolds of negative curvature. From geometric point of view, this study got momentum, when the study of warped product of CR-submanifolds of Kähler manifolds was introduced by Chen [25, 26]. Warped product manifolds are known to have applications in physics. They provide an excellent setting to model space time near a black hole or a massive star.

Chapter 4 is devoted to obtain the necessary and sufficient conditions for a semi-invariant submanifold to be a locally warped product submanifold of invariant and anti-invariant submanifolds of a Cosymplectic manifold in terms of canonical structures. The inequality and equality cases have also been discussed for the squared norm of the second fundamental form in terms of the warping function.

Contents of this chapter have been published in *J. Inequal. Appl.*, 19 (2012), 1-12. [48] ***(SCI-indexed), Impact factor - 0.77***

In **Chapter 5**, results for the non-existence of warped product slant lightlike submanifolds of indefinite Sasakian manifolds have been carried out.

Results of this chapter have been published in *Balkan J. Geom. Appl.*, 20(1) (2015), 98-108. [68] ***(SCI-indexed), Impact factor - 0.806***

O'Neill [61] introduced semi-Riemannian submersions. It is known that in Riemannian manifolds the fibers are always Riemannian manifolds. However, when the manifolds are semi-Riemannian manifolds then the fibers may not be semi-Riemannian manifolds. Sahin [78] in 2011 introduced slant submersions from almost Hermitian manifolds onto Riemannian manifolds. Semi-Riemannian submersions are of interest in physics, owing to their applications in the Yang-Mills theory, Kaluza-Klein theory, supergravity and superstring theories [15, 16, 32, 88]. All these features motivated us to club the theory of lightlike submersions with slant submersions in **Chapter 6**.

In this chapter, slant lightlike submersions from an indefinite almost Hermitian manifold onto a lightlike manifold have been introduced. Further, the geometry of foliations of slant lightlike submersions have been investigated.

Contents of this chapter are accepted for publication in *Ukrainian Math. J.* [72] ***(SCI-indexed), Impact factor - 0.230***

Towards the end, the future directions in this research are outlined. The thesis concludes by listing the Bibliography of various publications cited in this work.

Chapter 2

TOTALLY CONTACT UMBILICAL SLANT LIGHTLIKE SUBMANIFOLDS OF INDEFINITE CONTACT MANIFOLDS

2.1 Introduction

A contact manifold is an analogue of an almost Hermitian manifold, in odd dimensions. Every almost contact manifold admits a Riemannian metric tensor field, which plays an analogue role to an almost Hermitian metric tensor field. Since, contact geometry has vital role in different fields of mathematics like: the theory of differential equations, mechanics, phase spaces of dynamical systems and many more (see Arnold [1], Maclane [57], Nazaikinskii [60]), therefore the odd dimensional version of almost Hermitian manifolds are of equal importance in differential geometry.

The slant submanifolds of odd dimension, that is, contact manifolds have

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Contents of this chapter have been published in ***Bull. Iranian Math. Soc.***, 40(5) (2014), 1135-1151 ***(SCI-indexed), Impact factor - 0.270*** and ***ISRN Geometry***, 2013 (2013), 1-8 and in ***Tamkang J. Math.***, 46(2) (2015), 179-191.

been discussed by various geometers like Cabrerizo et al. [18], Gupta et al. [37] and Lotta [53]. The notion of slant lightlike submanifolds of indefinite contact manifolds, in particular, indefinite Sasakian manifolds was introduced by Sahin and Yildirim [80] in which necessary and sufficient condition for their existence has been obtained. Gupta et al. [38, 39] played a crucial role in the development of the theory of slant lightlike submanifolds of indefinite Kenmotsu manifolds and indefinite Cosymplectic manifolds. Jain et al. [42] in 2013 proved that the totally contact umbilical GCR-lightlike submanifolds of indefinite Cosymplectic manifolds do not exist.

The motive of this chapter is to study the totally contact umbilical slant lightlike submanifolds of indefinite contact manifolds and their space forms in different settings by taking ambient space as indefinite Sasakian manifold, indefinite Cosymplectic manifold and indefinite Kenmotsu manifold etc. Minimal slant lightlike submanifolds for these manifolds have also been discussed in this chapter.

The contents of this chapter have been divided into four sections. In section 2.2, slant lightlike submanifolds of almost contact manifolds with indefinite metric are focussed. In section 2.3, the non-existence of totally contact umbilical slant lightlike submanifolds of an indefinite Sasakian manifold and that of an indefinite Sasakian space form have been studied. (see Theorem 2.3.6 and 2.3.7). Further, the minimal slant lightlike submanifolds of an indefinite Sasakian manifold have been characterized in section 2.4.

Towards the end of this chapter, the results pertaining to the non-existence of totally contact umbilical proper slant lightlike submanifolds of indefinite Cosymplectic and indefinite Kenmotsu manifolds and their space forms have been presented.

2.2 Slant Lightlike Submanifolds of Indefinite Contact Manifolds

To define the notion of slant submanifolds, one needs to consider the angle between two vector fields. A lightlike submanifold has two distributions namely the radical distribution and the screen distribution. The radical distribution is totally lightlike and therefore it is not possible to define angle between two vector fields of the radical distribution. On the other hand, Sahin and Yildirim [80] used the two vector fields of screen distribution to introduce slant lightlike submanifolds of an indefinite Sasakian manifold due to the fact that the screen distribution is non-degenerate. Thus to define slant notion, one will have to choose a Riemannian screen distribution on lightlike submanifold. Towards this direction, Sahin and Yildirim [80] gave the following lemmas:

Lemma 2.2.1. [80] *Let M be an r-lightlike submanifold of an indefinite Sasakian manifold of constant index $2q$ and $\phi RadTM$ be a distribution on M such that $RadTM \cap \phi RadTM = \{0\}$. Then $\phi ltr(TM)$ is a subbundle of the screen distribution $S(TM)$ and $\phi ltr(TM) \cap \phi RadTM = \{0\}$.*

Lemma 2.2.2. [80] *Let M be an r-lightlike submanifold of an indefinite Sasakian manifold of constant index $2r$ and $\phi RadTM$ be a distribution on M such that $RadTM \cap \phi RadTM = \{0\}$. Then a distribution complementary to $\phi ltr(TM) \oplus \phi(RadTM)$ in screen distribution $S(TM)$ is Riemannian.*

Using Lemmas 2.2.1 and 2.2.2, Sahin and Yildirim [80] defined slant lightlike submanifolds of indefinite Sasakian manifolds as below:

Definition 2.2.3. [80] *An r-lightlike submanifold M of an indefinite Sasakian manifold $\bar{M}$ of constant index $2r$ is said to be a slant lightlike submanifold if it satisfies the following conditions:*

- *(A) $RadTM$ is a distribution on M such that $\phi RadTM \cap RadTM = \{0\}$.*
- *(B) For each non zero vector field X tangent to $\bar{D} = D \perp \{V\}$ at $p \in U \subset M$, if X and V are linearly independent, then the angle $\theta(X)$ between ϕX and the vector space $\bar{D}_p$ is constant, that is, it is independent of the choice of $X \in \bar{D}_p$ and $p \in U$, where $\bar{D}$ is complementary distribution to $\phi ltr(TM) \oplus \phi RadTM$ in screen distribution $S(TM)$.*

The constant angle $\theta(X)$ is said to be the slant angle of the distribution $\bar{D}$. The slant lightlike submanifold is called proper if $\bar{D} \neq \{0\}$ and $\theta \neq 0, \pi/2$.

It is well known that a submanifold M is invariant or anti-invariant if $\phi T_pM \subset T_pM$ or $\phi T_pM \subset T_pM^{\perp}$, respectively for any $p \in M$. Thus, from the definition of slant submanifolds, M is invariant or anti-invariant, accordingly, if $\theta(X) = 0$ or $\theta(X) = \frac{\pi}{2}$, respectively.

For a slant lightlike submanifold, tangent bundle TM is decomposed as

$$\begin{aligned} TM &= RadTM \perp S(TM) \\ &= RadTM \perp (\phi RadTM \oplus \phi ltr(TM)) \perp \bar{D}, \end{aligned} \tag{2.2.1}$$

where $\bar{D} = D \perp \{V\}$. Therefore, for any $X \in \Gamma(TM)$, ϕX can1 be expressed as

$$\phi X = TX + FX, \tag{2.2.2}$$

where TX and FX are the tangential and transversal components of ϕX respectively. Similarly, for any $U \in \Gamma(tr(TM))$, ϕU can be expressed as

$$\phi U = BU + CU, \tag{2.2.3}$$

where BU and CU are the tangential and transversal components of ϕU respectively.

Considering (2.2.1), let P_1, P_2, Q_1, Q_2 and $\bar{Q}_2$ denote the projection morphisms on $RadTM$, $\phi RadTM$, $\phi ltr(TM)$, D and $\bar{D} = D \perp V$, respectively. Then any $X \in \Gamma(TM)$ can be expressed as:

$$X = P_1X + P_2X + Q_1X + \bar{Q}_2X, \tag{2.2.4}$$

where $\bar{Q}_2X = Q_2X + \eta(X)V$. Apply ϕ to (2.2.4) both sides and this gives

$$\phi X = \phi P_1X + \phi P_2X + FQ_1X + TQ_2X + FQ_2X. \tag{2.2.5}$$

Then, using (2.2.2) and (2.2.3), it yields

$$\phi P_1X = TP_1X \in \Gamma(\phi RadTM), \quad \phi P_2X = TP_2X \in \Gamma(RadTM),$$

$$FP_1X = FP_2X = 0, \quad TQ_2X \in \Gamma(D), \quad FQ_1X \in \Gamma(ltr(TM)).$$

We now prove some lemmas which will be used for the proof of our main results:

Lemma 2.2.4. *Let M be a slant lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$. Then $FQ_2X \in \Gamma(S(TM^\perp))$, for any $X \in \Gamma(TM)$.*

Proof. From (1.3.2) and (1.3.3), it is clear that $FQ_2X \in \Gamma(S(TM^\perp))$ if $g(FQ_2X, \xi) = 0$, for any $\xi \in \Gamma(Rad(TM))$. Thus using

$$g(FQ_2X, \xi) = g(\phi Q_2X - TQ_2X, \xi) = g(\phi Q_2X, \xi) = -g(Q_2X, \phi\xi) = 0,$$

the assertion follows. □

Lemma 2.2.4 implies that $F(D_p)$ is a subspace of $S(TM^\perp)$. Thus, there exists an *invariant* subspace μ_p of $T_p\bar{M}$ such that

$$S(T_pM^\perp) = F(D_p) \perp \mu_p, \tag{2.2.6}$$

therefore,

$$T_p\bar{M} = S(T_pM)\perp\{Rad(T_pM) \oplus ltr(T_pM)\}\perp\{F(D_p)\perp\mu_p\}. \tag{2.2.7}$$

On differentiating (2.2.5) and using (1.3.6)-(1.3.9), (2.2.2) and (2.2.3), we have

$$(\nabla_X T)Y = A_{FQ_1Y}X + A_{FQ_2Y}X + Bh(X,Y) - g(X,Y)V + \eta(Y)X, \tag{2.2.8}$$

and

$$\begin{aligned} D^s(X,FQ_1Y) + D^l(X,FQ_2Y) \; &= F\nabla_X Y - h(X,TY) + Ch^s(X,Y) \\ &\quad -\nabla^s_X FQ_2Y - \nabla^l_X FQ_1Y. \end{aligned} \tag{2.2.9}$$

Further, using the Sasakian property of $\bar{\nabla}$ with (1.3.5), the following lemmas hold.

Lemma 2.2.5. *For a slant lightlike submanifold M of an indefinite Sasakian manifold $\bar{M}$, the following equations hold:*

$$(\nabla_X T)Y = A_{FY}X + Bh(X,Y) - g(X,Y)V + \eta(Y)X, \tag{2.2.10}$$

$$(\nabla^t_X F)Y = Ch(X,Y) - h(X,TY), \tag{2.2.11}$$

where $X, Y \in \Gamma(TM)$ and

$$(\nabla_X T)Y = \nabla_X TY - T\nabla_X Y, \quad (\nabla^t_X F)Y = \nabla^t_X FY - F\nabla_X Y. \tag{2.2.12}$$

Lemma 2.2.6. *Let M be a slant lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$. Then*

$$(\nabla_X B)U = A_{CU}X - TA_UX - g(X,U)V, \quad (\nabla^t_X C)U = -FA_UX - h(X,BU), \tag{2.2.13}$$

where $U \in \Gamma(tr(TM))$, $X \in \Gamma(TM)$ and

$$(\nabla_X B)U = \nabla_X BU - B\nabla^t_X U, \quad (\nabla^t_X C)U = \nabla^t_X CU - C\nabla^t_X U. \tag{2.2.14}$$

In the next theorem, we have obtained the conditions for integrability of various distributions of slant lightlike submanifold of an indefinite Sasakian manifold.

Theorem 2.2.7. *Let M be a slant lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$. Then*

(i) The distribution $\bar{D}$ is integrable, if and only if, $h(X,TY) = h(Y,TX)$, $D^l(X,FQ_2Y) = D^l(Y,FQ_2X)$ and $\nabla^s_X FQ_2Y = \nabla^s_Y FQ_2X$, for any $X,Y \in \Gamma(\bar{D})$.

(ii) The distribution $\phi(ltr(TM))$ is integrable, if and only if, $A_{FQ_1Y}X = A_{FQ_1X}Y$, for any $X,Y \in \Gamma(\phi(ltr(TM)))$.

Proof. Using (2.2.9), we have

$$F\nabla_X Y = D^l(X,FQ_2Y) + h(X,TY) + \nabla^s_X FQ_2Y - Ch^s(X,Y),$$

for any $X,Y \in \Gamma(\bar{D})$. Interchange the role of X and Y then subtract the resulting equation from this, gives (i).
Let $X,Y \in \Gamma(\phi(ltr(TM)))$. Then use of (2.2.10) and (2.2.12) implies

$$-T\nabla_X Y = A_{FQ_1Y}X + Bh(X,Y),$$

This further implies

$$T[X,Y] = A_{FQ_1X}Y - A_{FQ_1Y}X,$$

this completes the proof of (ii). □

The following theorem due to Sahin and Yildirim [80] is important for our subsequent study.

Theorem 2.2.8. [80] *Let M be a lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$. Then M is a slant lightlike submanifold if and only if* *(i)* $\phi(RadTM)$ *is a distribution on* M *such that* $\phi RadTM \cap Rad(TM) = \{0\}$.

(ii) $\bar{D}$ *is a distribution such that*

$$\bar{D} = \{X \in \Gamma(\bar{D}) : T^2X = -\lambda(X - \eta(X)V\}, \tag{2.2.15}$$

is complementary to $\phi ltr(TM) \oplus \phi RadTM$, *where* $\lambda = -cos^2\theta$.

On the parallel lines, the following result holds true for the slant lightlike submanifold of an indefinite Sasakian manifold.

Theorem 2.2.9. *Let* M *be a lightlike submanifold of an indefinite Sasakian manifold* $\bar{M}$. *Then* M *is a slant lightlike submanifold if and only if*

(i) $\phi(RadTM)$ *is a distribution on* M *such that* $\phi RadTM \cap Rad(TM) = \{0\}$.

(ii) for any non-zero tangent vector field $X \in \Gamma(\bar{D})$, *there exists a constant* $\mu \in [-1, 0]$ *such that*

$$BFX = \mu(X - \eta(X)V), \tag{2.2.16}$$

where $\bar{D}$ *is a distribution complementary to* $\phi ltr(TM) \oplus \phi RadTM$ *in* TM *and* $\mu = -sin^2\theta$.

2.3 Totally Contact Umbilical Slant Lightlike Submanifolds of Indefinite Sasakian Manifolds

In this section, the main results of this chapter has been presented followed by some definitions which will be used for proving them.

Definition 2.3.1. [28]*: A lightlike submanifold* M *of a semi-Riemannian manifold* $\bar{M}$ *is said to be totally umbilical in* $\bar{M}$ *if there exists a smooth transversal* *vector field $\alpha \in \Gamma(tr(TM))$ on M, known as transversal curvature vector field of M, such that*

$$h(X,Y) = \alpha g(X,Y), \tag{2.3.1}$$

for any $X, Y \in \Gamma(TM)$. Clearly, M is totally umbilical if and only if there exists smooth vector fields $\alpha^l \in \Gamma(ltr(TM))$ and $\alpha^s \in \Gamma(S(TM^\perp))$, on each coordinate neighborhood, such that

$$h^l(X,Y) = \alpha^l g(X,Y), \quad D^l(X,W) = 0, \quad h^s(X,Y) = \alpha^s g(X,Y). \tag{2.3.2}$$

A lightlike submanifold is called totally geodesic if $h(X,Y) = 0$, for all $X, Y \in \Gamma(TM)$, that is, a lightlike submanifold is totally geodesic if $\alpha^l = 0$ and $\alpha^s = 0$.

Definition 2.3.2. [90] *Let M be a submanifold of a Sasakian manifold $\bar{M}$ such that the characteristic vector field V is tangent to M. If the second fundamental form h of M is of the form*

$$h(X,Y) = [g(X,Y) - \eta(X)\eta(Y)]\alpha + \eta(X)h(Y,V) + \eta(Y)h(X,V), \tag{2.3.3}$$

for $X, Y \in \Gamma(TM)$, where α is a vector field transversal to M, then M is said to be a totally contact umbilical submanifold of $\bar{M}$. If moreover $\alpha = 0$, then it is said to be totally contact geodesic.

Similarly, a lightlike submanifold M of an indefinite Sasakian manifold $\bar{M}$ is said to be totally contact umbilical lightlike submanifold if

$$h^l(X,Y) = [g(X,Y) - \eta(X)\eta(Y)]\alpha_L + \eta(X)h^l(Y,V) + \eta(Y)h^l(X,V), \tag{2.3.4}$$

$$h^s(X,Y) = [g(X,Y) - \eta(X)\eta(Y)]\alpha_S + \eta(X)h^s(Y,V) + \eta(Y)h^s(X,V), \tag{2.3.5}$$

where $\alpha_L \in \Gamma(ltr(TM))$ and $\alpha_S \in \Gamma(S(TM^\perp))$.

The following lemmas have also been used to claim our main results.

Lemma 2.3.3. *Let M be a slant lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$. Then*

$$g(T\bar{Q}_2X, T\bar{Q}_2Y) = cos^2\theta\{g(\bar{Q}_2X, \bar{Q}_2Y) - \eta(\bar{Q}_2X)\eta(\bar{Q}_2Y)\} \tag{2.3.6}$$

and

$$g(F\bar{Q}_2X, F\bar{Q}_2Y) = sin^2\theta\{g(\bar{Q}_2X, \bar{Q}_2Y) - \eta(\bar{Q}_2X)\eta(\bar{Q}_2Y)\} \tag{2.3.7}$$

for any $X, Y \in \Gamma(TM)$.

Proof. From (1.2.4) and (2.2.2), we obtain

$$g(T\bar{Q}_2X, T\bar{Q}_2Y) = -g(\bar{Q}_2X, T^2\bar{Q}_2Y).$$

for any $X, Y \in \Gamma(TM)$. Then using the Theorem 2.2.8, the assertion follows. □

Lemma 2.3.4. *Let M be a totally contact umbilical slant lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$. Then $g(\nabla_X X, \phi\xi) = 0$, for any $X \in \Gamma(D)$ and $\xi \in \Gamma(Rad(TM))$.*

Proof. Let $X \in \Gamma(D)$, this implies $X = Q_2X$. Then using (1.2.5), (1.3.6), (1.3.9) and taking into consideration the property that M is a totally contact umbilical slant lightlike submanifold, it yields

$$\begin{aligned} g(\nabla_X X, \phi\xi) &= \bar{g}(\bar{\nabla}_X X, \phi\xi) \\ &= -\bar{g}(\bar{\nabla}_X TQ_2X, \xi) - \bar{g}(\bar{\nabla}_X FQ_2X, \xi) \\ &= -g(h^l(X, TQ_2X), \xi) - \bar{g}(D^l(X, FQ_2X), \xi) \\ &= -\bar{g}(D^l(X, FQ_2X), \xi), \end{aligned} \tag{2.3.8}$$

Use of (1.2.4), (2.2.2), (2.3.4) and $X \in \Gamma(D)$ gives $h^l(X, TQ_2X) = \{g(X, TQ_2X)\}\alpha_L = 0$. Since $\eta(\xi) = 0$ and $\eta(Q_2X) = 0$, hence by replacing

W by FQ_2X and Y by ξ in (1.3.10) and then taking into consideration that M is a totally contact umbilical slant lightlike submanifold of an indefinite Sasakian manifold, we infer that

$$\begin{aligned}\bar{g}(D^l(X, FQ_2X), \xi) &= -\bar{g}(h^s(X, \xi), FQ_2X) \\ &= -g(X, \xi)g(\alpha_S, FQ_2X) = 0. \qquad (2.3.9)\end{aligned}$$

Therefore from (2.3.8) and (2.3.9), the result follows. □

Now, we present the main results of this section.

Theorem 2.3.5. *Let M be a totally contact umbilical slant lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$. Then at least one of the following statements is true*

(a) M is an anti-invariant submanifold.

(b) $D = \{0\}$.

(c) If M is a proper slant submanifold, then $\alpha_S \in \Gamma(\mu)$.

Proof. Let $X = Q_2X \in \Gamma(D)$. Then use of (2.3.3) and totally contact umbilicity of M, implies

$$h(TQ_2X, TQ_2X) = g(TQ_2X, TQ_2X)\alpha,$$

therefore by taking into account (1.3.5), (2.3.6) and the above equation, it infers that

$$\bar{\nabla}_{TQ_2X}TQ_2X - \nabla_{TQ_2X}TQ_2X = cos^2\theta[g(Q_2X, Q_2X)]\alpha.$$

Using (2.2.2) and taking into account that $\bar{M}$ is a Sasakian manifold, it follows that

$$\begin{aligned}&\phi\bar{\nabla}_{TQ_2X}Q_2X - g(TQ_2X, TQ_2X)V - \bar{\nabla}_{TQ_2X}FQ_2X - \nabla_{TQ_2X}TQ_2X \\ &= cos^2\theta[g(Q_2X, Q_2X)]\alpha.\end{aligned}$$

Then the use of (1.3.6)-(1.3.9) and (2.3.6), implies

$$\phi\nabla_{TQ_2X}Q_2X + \phi h^l(TQ_2X, X) + \phi h^s(TQ_2X, X) + A_{FQ_2X}TQ_2X - \nabla^s_{TQ_2X}$$
$$FQ_2X - D^l(TQ_2X, FQ_2X) - \nabla_{TQ_2X}TQ_2X = cos^2\theta[g(Q_2X, Q_2X)](\alpha + V).$$

Further, the use of (2.2.2), (2.2.3), (2.3.4) and (2.3.5) yields

$$T\nabla_{TQ_2X}Q_2X + F\nabla_{TQ_2X}Q_2X + g(TQ_2X, X)\phi\alpha^l + g(TQ_2X, X)B\alpha^s +$$
$$g(TQ_2X), X)C\alpha^s + A_{FQ_2X}TQ_2X - \nabla^s_{TQ_2X}FQ_2X - D^l(TQ_2X, FQ_2X)$$
$$-\nabla_{TQ_2X}TQ_2X = cos^2\theta[g(Q_2X, Q_2X)](\alpha + V).$$

Equating the transversal components both sides, we get

$$F\nabla_{TQ_2X}Q_2X + g(TQ_2X, X)C\alpha^s - \nabla^s_{TQ_2X}FQ_2X - D^l(TQ_2X, FQ_2X)$$
$$= cos^2\theta[g(Q_2X, Q_2X)]\alpha. \tag{2.3.10}$$

On the other hand, by taking the covariant derivative with respect to TQ_2X on both sides of (2.3.7), for $X = Y \in \Gamma(D)$, implies that

$$g(\nabla^s_{TQ_2X}FQ_2X, FQ_2X) = sin^2\theta g(\nabla_{TQ_2X}Q_2X, Q_2X). \tag{2.3.11}$$

Now, on taking the inner product with respect to FQ_2X on both sides of (2.3.10), it follows that

$$\begin{aligned} g(F\nabla_{TQ_2X}Q_2X, FQ_2X) \quad & -g(\nabla^s_{TQ_2X}FQ_2X, FQ_2X) \\ & = cos^2\theta[g(Q_2X, Q_2X)]g(\alpha_S, FQ_2X). \end{aligned}$$

The following result is immediate from (2.3.7) and (2.3.11)

$$cos^2\theta[g(Q_2X, Q_2X)]g(\alpha_S, FQ_2X) = 0. \tag{2.3.12}$$

Thus, from (2.3.12), it follows that either $\theta = \pi/2$ or $Q_2X = 0$ or $\alpha_S \in \Gamma(\mu)$. Hence the proof is complete. □

Theorem 2.3.6. *Every totally contact umbilical proper slant lightlike submanifold of an indefinite Sasakian manifold is totally contact geodesic.*

Proof. Let M be a totally contact umbilical slant lightlike submanifold and $X{=}Q_2X \in \Gamma(D)$. Then using (2.3.3), we obtain

$$h(TQ_2X, TQ_2X) = g(TQ_2X, TQ_2X)\alpha,$$

further, using (2.3.6), we get

$$\begin{aligned} h(TQ_2X, TQ_2X) &= cos^2\theta[g(Q_2X, Q_2X) - \eta(Q_2X)\eta(Q_2X)]\alpha \\ &= cos^2\theta[g(Q_2X, Q_2X)]\alpha. \qquad (2.3.13) \end{aligned}$$

The use of (1.2.3) and (2.2.9), for any $X \in \Gamma(D)$, gives

$$\begin{aligned} h(TQ_2X, TQ_2X) &= F\nabla_{TQ_2X}X + Ch(TQ_2X, X) - \nabla^s_{TQ_2X}FQ_2X \\ &\quad -D^l(TQ_2X, FQ_2X). \qquad (2.3.14) \end{aligned}$$

Since, M is a totally contact umbilical slant lightlike submanifold, therefore, $Ch(TQ_2X, X) = g(TQ_2X, X)C\alpha = 0$, further from (2.3.13) and (2.3.14), it follows that

$$cos^2\theta[g(Q_2X, Q_2X)]\alpha = F\nabla_{TQ_2X}X - \nabla^s_{TQ_2X}FQ_2X - D^l(TQ_2X, FQ_2X). \qquad (2.3.15)$$

Now, taking the scalar product with respect to FQ_2X on both sides of (2.3.15), this equation takes the form

$$cos^2\theta[g(Q_2X, Q_2X)]\bar{g}(\alpha_S, FQ_2X) = \bar{g}(F\nabla_{TQ_2X}X, FQ_2X) - \bar{g}(\nabla^s_{TQ_2X}FQ_2X, FQ_2X),$$

Further the equation (2.3.7) implies that

$$\begin{aligned} cos^2\theta[g(Q_2X, Q_2X)]\bar{g}(\alpha_S, FQ_2X) &= sin^2\theta[g(\nabla_{TQ_2X}X, Q_2X)] \\ &\quad -\bar{g}(\nabla^s_{TQ_2X}FQ_2X, FQ_2X). \qquad (2.3.16) \end{aligned}$$

Now, for any $X = Q_2X \in \Gamma(D)$, (2.3.7) implies

$$g(FQ_2X, FQ_2X) = sin^2\theta[g(Q_2X, Q_2X)].$$

On taking the covariant derivative with respect to $\bar{\nabla}_{TQ_2X}$ of this equation, we get

$$\bar{g}(\nabla^s_{TQ_2X}FQ_2X, FQ_2X) = sin^2\theta[g(\nabla_{TQ_2X}Q_2X, Q_2X)]. \tag{2.3.17}$$

Further, using (2.3.17) in (2.3.16), implies

$$cos^2\theta[g(Q_2X, Q_2X)]\bar{g}(\alpha_S, FQ_2X) = 0. \tag{2.3.18}$$

Since g is a Riemannian metric on D and M is a proper slant lightlike submanifold, therefore $\bar{g}(\alpha_s, FQ_2X) = 0$. Thus using the Lemma 2.2.4 and the equation (2.2.6), result in

$$\alpha_S \in \Gamma(\mu). \tag{2.3.19}$$

By the use of Sasakian character of $\bar{M}$, that is, $\bar{\nabla}_X\phi Y = \phi\bar{\nabla}_XY - g(X, Y)V$, for any $X, Y \in \Gamma(D)$ and (2.3.3), implies

$$\begin{aligned}&\nabla_XTQ_2Y + g(X, TQ_2Y)\alpha - A_{FQ_2Y}X + \nabla^s_XFQ_2Y + D^l(X, FQ_2Y)\\&= T\nabla_XY + F\nabla_XY + g(X, Y)\phi\alpha - g(X, Y)V.\end{aligned} \tag{2.3.20}$$

On taking the scalar product with respect to $\phi\alpha_S$ on both sides of (2.3.20) and then by using the fact that μ is an invariant subbundle of $T\bar{M}$, (2.3.20) reduces to

$$\bar{g}(\nabla^s_XFQ_2X, \phi\alpha_S) = g(Q_2X, Q_2Y)g(\alpha_S, \alpha_S). \tag{2.3.21}$$

Again, using the Sasakian character of $\bar{M}$, that is, $\bar{\nabla}_X\phi\alpha_S = \phi\bar{\nabla}_X\alpha_S$, it follows that

$$\begin{aligned}-A_{\phi\alpha_S}X + \nabla^s_X\phi\alpha_S + D^l(X, \phi\alpha_S) &= -TA_{\alpha_S}X - FA_{\alpha_S}X + B\nabla^s_X\alpha_S\\&\quad +C\nabla^s_X\alpha_S + \phi D^l(X, \alpha_S).\end{aligned} \tag{2.3.22}$$

On taking the scalar product with respect to FQ_2Y on both sides of (2.3.22) and then by using invariant character of μ, that is, $C\nabla^s_X\alpha_S \in \Gamma(\mu)$ with (1.2.3) and (2.3.7), we get

$$\bar{g}(\nabla^s_X\phi\alpha_S, FQ_2Y) = -g(FA_{\alpha_S}X, FQ_2Y) = -sin^2\theta[g(A_{\alpha_S}X, Q_2Y)]. \quad (2.3.23)$$

Since $\bar{\nabla}$ is a metric connection, therefore

$$(\bar{\nabla}_Xg)(FQ_2Y, \phi\alpha_S) = 0,$$

this further implies that

$$\bar{g}(\nabla^s_XFQ_2Y, \phi\alpha_S) = \bar{g}(\nabla^s_X\phi\alpha_S, FQ_2Y).$$

Therefore, use of (2.3.23) gives

$$\bar{g}(\nabla^s_XFQ_2Y, \phi\alpha_S) = -sin^2\theta[g(A_{\alpha_S}X, Q_2Y)]. \quad (2.3.24)$$

From (2.3.21) and (2.3.24), we have

$$g(Q_2X, Q_2Y)g(\alpha_S, \alpha_S) = -sin^2\theta g[(A_{\alpha_S}X, Q_2Y)], \quad (2.3.25)$$

On using (1.3.10), equation (2.3.25) reduces to

$$\begin{aligned} g(Q_2X, Q_2Y)g(\alpha_S, \alpha_S) &= -sin^2\theta[\bar{g}(h^s(Q_2X, Q_2Y), \alpha_S)] \\ &= -sin^2\theta[g(Q_2X, Q_2Y)]g(\alpha_S, \alpha_S), \quad (2.3.26) \end{aligned}$$

This implies that

$$(1 + sin^2\theta)[g(Q_2X, Q_2Y)]g(\alpha_S, \alpha_S) = 0.$$

Since g is a Riemannian metric on D and M is proper, therefore

$$\alpha_S = 0. \quad (2.3.27)$$

Now, using the Sasakian character of $\bar{M}$, that is, $\bar{\nabla}_X \phi X = \phi \bar{\nabla}_X X$, for any $X \in \Gamma(D)$, implies that

$$\nabla_X TQ_2X + h(X, TQ_2X) - A_{FQ_2X}X + \nabla^s_X FQ_2X + D^l(X, FQ_2X) = T\nabla_X X$$
$$+F\nabla_X X + Bh(X, X) + Ch(X, X).$$

Further, totally contact umbilicity of M gives $h(X, TQ_2X) = 0$ and on comparing the tangential components gives

$$\nabla_X TQ_2X - A_{FQ_2X}X = T\nabla_X X + Bh(X, X). \tag{2.3.28}$$

On taking the scalar product with respect to $\phi\xi \in \Gamma(\phi Rad(TM))$ on both sides of (2.3.28) and considering the Lemma 2.3.4, it follows that

$$g(A_{FQ_2X}X, \phi\xi) + \bar{g}(h^l(Q_2X, Q_2X), \xi) = 0. \tag{2.3.29}$$

Now (1.3.8) yields

$$\bar{g}(h^s(X, \phi\xi), FQ_2X) + \bar{g}(\phi\xi, D^l(X, FQ_2X)) = g(A_{FQ_2X}X, \phi\xi).$$

Taking into account that M is a totally contact umbilical slant lightlike submanifold and using (2.3.5), (2.3.27) in this equation, then we obtain

$$g(A_{FQ_2X}X, \phi\xi) = 0. \tag{2.3.30}$$

Using (2.3.30) in (2.3.29), we obtain $\bar{g}(h^l(Q_2X, Q_2X), \xi) = 0$, then use of (2.3.4), implies

$$g(Q_2X, Q_2X)\bar{g}(\alpha_L, \xi) = 0.$$

Since D is Riemannian, therefore $\bar{g}(\alpha_L, \xi) = 0$. Further, the use of (1.3.2) implies

$$\alpha_L = 0. \tag{2.3.31}$$

Thus from (2.3.27) and (2.3.31), the proof is complete. □

Next, we have investigated the totally contact umbilicity of proper slant lightlike submanifolds of an indefinite Sasakian space form.

If Sasakian manifold has constant ϕ holomorphic sectional curvature c, then it is called an contact space form and denoted by $\bar{M}(c)$. For a contact space form, the expression of Riemannian curvature tensor is (see [49])

$$\begin{aligned}\bar{R}(X,Y)Z \ &= \frac{c+3\epsilon}{4}\{\bar{g}(Y,Z)X - \bar{g}(X,Z)Y\} + \frac{c-\epsilon}{4}\{\eta(X)\eta(Z)Y \\ &-\eta(Y)\eta(Z)X + \bar{g}(X,Z)\eta(Y)V - \bar{g}(Y,Z)\eta(X)V + \bar{g}(\phi Y,Z)\phi X \\ &-\bar{g}(\phi X,Z)\phi Y - 2\bar{g}(\phi X,Y)\phi Z\}. \qquad (2.3.32)\end{aligned}$$

for $X, Y, Z \ \in \ \Gamma(T\bar{M})$.

Theorem 2.3.7. *There does not exist totally contact umbilical proper slant lightlike submanifold of an indefinite Sasakian space form $\bar{M}(c)$ such that $c \neq \epsilon$.*

Proof. Let M be a totally contact umbilical proper lightlike submanifold of $\bar{M}(c)$ such that $c \neq \epsilon$. Then use of (2.3.32) gives

$$\bar{g}(\bar{R}(X,\phi X)Z,\xi) = -\frac{c-\epsilon}{2}g(\phi X,\phi X)g(\phi Z,\xi),$$

for any $X \in \Gamma(D)$, $Z \in \Gamma(\phi ltr(TM))$ and $\xi \in \Gamma(RadTM)$, respectively. Further by using (1.2.4), implies

$$\bar{g}(\bar{R}(X,\phi X)Z,\xi) = -\frac{c-\epsilon}{2}g(Q_2X,Q_2X)g(\phi Z,\xi). \qquad (2.3.33)$$

Also, the use of (2.3.3) and (1.3.14), gives

$$\bar{g}(\bar{R}(X,\phi X)Z,\xi) = \bar{g}((\nabla_X h^l)(\phi X,Z),\xi) - \bar{g}((\nabla_{\phi X} h^l)(X,Z),\xi). \qquad (2.3.34)$$

Making use of (2.3.4) and (1.3.15), the above equation becomes

$$(\nabla_X h^l)(\phi X,Z) = -g(\nabla_X \phi X,Z)\alpha_L - g(TQ_2X,\nabla_X Z)\alpha_L. \qquad (2.3.35)$$

Similarly

$$(\nabla_{\phi X}h^l)(X,Z) = -g(\nabla_{\phi X}X,Z)\alpha_L - g(X,\nabla_{\phi X}Z)\alpha_L. \tag{2.3.36}$$

Putting (2.3.35) and (2.3.36) in (2.3.34), we get

$$\bar{g}(\bar{R}(X,\phi X)Z,\xi) = -g(\nabla_X\phi X,Z)\bar{g}(\alpha_L,\xi) - g(\phi X,\nabla_X Z)\bar{g}(\alpha_L,\xi)$$
$$+g(\nabla_{\phi X}X,Z)\bar{g}(\alpha_L,\xi) + g(X,\nabla_{\phi X}Z)\bar{g}(\alpha_L,\xi). \tag{2.3.37}$$

Now, from (1.2.5), we have

$$g(\phi X,\nabla_X Z) = -\bar{g}(\bar{\nabla}_X\phi X,Z) = -g(\nabla_X\phi X,Z), \tag{2.3.38}$$

and

$$g(X,\nabla_{\phi X}Z) = -\bar{g}(\bar{\nabla}_{\phi X}X,Z) = -g(\nabla_{\phi X}X,Z). \tag{2.3.39}$$

Thus using (2.3.38) and (2.3.39) in (2.3.37), we get

$$\bar{g}(\bar{R}(X,\phi X)Z,\xi) = 0. \tag{2.3.40}$$

Hence, by making use of (2.3.40) in (2.3.33), it follows that

$$\frac{c-\epsilon}{2}g(Q_2X,Q_2X)g(\phi Z,\xi) = 0. \tag{2.3.41}$$

Since D is a Riemannian and $g(\phi Z,\xi) \neq 0$, therefore (2.3.41) implies that $c = \epsilon$. This contradiction completes the proof. □

2.4 Minimal Slant Lightlike Submanifolds

In this section, we establish a result for a proper slant lightlike submanifold of an indefinite Sasakian manifold to be minimal.

A general notion of minimal lightlike submanifold of a semi-Riemannian manifold $\bar{M}$ was introduced by Bejan and Duggal [6] as:

Definition 2.4.1. [6] *A lightlike submanifold* $(M, g, S(TM))$ *isometrically immersed in a semi-Riemannian manifold* $(\bar{M}, \bar{g})$ *is minimal if*

(i) $h^s = 0$ *on* $Rad(TM)$ *and*

(ii) trace $h = 0$, *where trace is written with respect to* g *restricted to* $S(TM)$.

Let $\{\xi_1, ..., \xi_r, \phi\xi_1, ..., \phi\xi_r, V, e_1, ..., e_q, \phi N_1, ..., \phi N_r\}$, be a quasi orthonormal basis of M such that $\{\xi_1, ..., \xi_r\}$, $\{\phi\xi_1, ..., \phi\xi_r\}$, $\{e_1, ..., e_q\}$ and $\{\phi N_1, ..., \phi N_r\}$ form a basis of $Rad(TM)$, $\phi(Rad(TM))$, D and $\phi(ltr(TM))$ respectively.

The following definition of irrotational lightlike submanifold due to Duggal and Sahin [29] is required to prove our next result.

Definition 2.4.2. [29] *A lightlike submanifold* M *of a semi-Riemannian manifold* $\bar{M}$ *is said to be irrotational if and only if* $\bar{\nabla}_X \xi \in \Gamma(TM)$ *for any* $X \in \Gamma(TM)$ *and* $\xi \in \Gamma(Rad(TM))$.

In the next theorem, we have obtained the necessary and sufficient condition for an irrotational slant lightlike submanifold of an indefinite Sasakian manifold to be minimal.

Theorem 2.4.3. *Let* M *be an irrotational slant lightlike submanifold of an indefinite Sasakian manifold* $\bar{M}$. *Then it is minimal if and only if*

$$trace\ A_{W_k}|_{S(TM)} = 0 \text{ and } trace\ A^*_{\xi_i}|_{S(TM)} = 0,$$

where $\{W_k\}_{k=1}^{l}$ *and* $\{\xi_i\}_{i=1}^{r}$ *is a basis* $S(TM^\perp)$ *and* $Rad(TM)$, *respectively.*

Proof. For a characteristic vector field V, we have $\bar{\nabla}_V V = 0$, then using this fact with (1.3.6), we get $h^l(V, V) = 0 = h^s(V, V) = 0$. Since M is an irrotational lightlike submanifold, therefore using definition we have $h^s(X, \xi) = 0$. Thus h^s

vanishes on $Rad(TM)$. Hence M is minimal if and only if trace $h = 0$ on $S(TM)$, that is, M is minimal if and only if

$$\sum_{i=1}^{r} h(\phi\xi_i, \phi\xi_i) + \sum_{i=1}^{r} h(\phi N_i, \phi N_i) + \sum_{j=1}^{q} h(e_j, e_j) = 0.$$

By making use of (1.3.10) and (1.3.13), implies

$$\begin{aligned} \textstyle\sum_{i=1}^{r} h(\phi\xi_i, \phi\xi_i) = & \sum_{i=1}^{r}\{\frac{1}{r}\sum_{a=1}^{r} g(A^*_{\xi_a}\phi\xi_i, \phi\xi_i)N_a \\ & +\frac{1}{l}\sum_{k=1}^{l} g(A_{W_k}\phi\xi_i, \phi\xi_i)W_k\}. \end{aligned} \tag{2.4.1}$$

Similarly

$$\begin{aligned} \textstyle\sum_{i=1}^{r} h(\phi N_i, \phi N_i) & = \sum_{i=1}^{r}\{\frac{1}{r}\sum_{a=1}^{r} g(A^*_{\xi_a}\phi N_i, \phi N_i)N_a \\ & +\frac{1}{l}\sum_{k=1}^{l} g(A_{W_k}e_j, e_j)W_k\}, \end{aligned} \tag{2.4.2}$$

and

$$\sum_{j=1}^{q} h(e_j, e_j) = \sum_{j=1}^{q}\{\frac{1}{r}\sum_{i=1}^{r} g(A^*_{\xi_i}e_j, e_j)N_i + \frac{1}{l}\sum_{k=1}^{l} g(A_{W_k}e_j, e_j)W_k\}. \tag{2.4.3}$$

Thus our assertion follows from (2.4.1)-(2.4.3). □

2.5 Totally Contact Umbilical Slant Lightlike Submanifolds of Indefinite Cosymplectic and Kenmotsu Manifolds

In this section, the non-existence of totally contact umbilical slant lightlike submanifolds of indefinite Cosymplectic and indefinite Kenmotsu manifolds have been explored. All the results of this section are similar to that of section 2.3 for indefinite Sasakian manifolds. For the sake of completeness, the statements of all main results have been presented as under. For details see ([66, 70]).

Theorem 2.5.1. *Let M be a totally contact umbilical slant lightlike submanifold of an indefinite Cosymplectic manifold (or indefinite Kenmotsu manifold) $\bar{M}$. Then at least one of the following statements is true*

(a) M is an anti-invariant submanifold.

(b) $D = \{0\}$.

(c) if M is a proper slant submanifold, then $\alpha_S \in \Gamma(\mu)$.

Theorem 2.5.2. *Every totally contact umbilical proper slant lightlike submanifold M of an indefinite Cosymplectic manifold (or indefinite Kenmotsu manifold) $\bar{M}$ is totally contact geodesic.*

Theorem 2.5.3. *There do not exist totally contact umbilical proper slant lightlike submanifolds of an indefinite Cosymplectic space form $\bar{M}(c)$ such that $c \neq 0$ (or indefinite Kenmotsu space form $\bar{M}(c)$ such that $c \neq -1$.)*

Theorem 2.5.4. *A totally contact umbilical proper slant lightlike submanifold M of an indefinite Kenmotsu manifold $\bar{M}$ is minimal if and only if trace $A_{W_k} = 0$ and trace $A^*_{\xi_i} = 0$ on D, where $\{W_k\}_{k=1}^{l}$ and $\{\xi_i\}_{i=1}^{r}$ is a basis of $S(TM^{\perp})$ and $Rad(TM)$, respectively.*

Theorem 2.5.5. *Let M be an irrotational slant lightlike submanifold of an indefinite Cosymplectic manifold (or indefinite Kenmotsu manifold) $\bar{M}$. Then M is minimal if and only if trace $A_{W_k}|_{S(TM)} = 0$ and trace $A^*_{\xi_i}|_{S(TM)} = 0$.*

Chapter 3

THE AXIOMS OF INDEFINITE SURFACES WITH LIGHTLIKE SUBMANIFOLDS AND TOTALLY UMBILICAL HEMI-SLANT LIGHTLIKE SUBMANIFOLDS

3.1 Introduction

Umbilical submanifolds are the simplest submanifolds after the totally geodesic ones and their knowledge sheds light on the geometry of the ambient space. For a tangent plane at a given point, if there exists a totally geodesic submanifold (or totally umbilical submanifold with parallel mean curvature vector field, respectively) such that point lies in the submanifold and if the tangent space to the submanifold at that point is the chosen tangent plane, then the manifold is

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said to satisfy the axiom of planes (or spheres, respectively). Different axioms of planes can be defined by choosing different types of tangent planes.

The axiom of planes for Riemannian manifolds were introduced by Cartan [19] as:

A Riemannian manifold $\bar{M}$ *of dimension* $m \geq 3$ *satisfies the axiom of* k*-planes if for every* k*-dimensional linear subspace* T *of* $T_p(\bar{M})$ *and for every point* p *in* $\bar{M}$*, there exists a* k*-dimensional totally geodesic submanifold* M *of* $\bar{M}$ *containing* p *such that* $T_pM = T$*.*

Cartan [19] proved the following theorems concerning the axiom of k-planes.

Theorem 3.1.1. [19]*: A Riemannian manifold of dimension* $m \geq 3$ *satisfies the axiom of* k*-planes for some* k*,* $2 \leq k < m$ *if and only if it is a real space form.*

Theorem 3.1.2. [19]*: A Riemannian manifold* $\bar{M}$ *has constant sectional curvature if and only if it satisfies the axiom of* k*-planes.*

In the process of generalization, Leung and Nomizu [52] introduced the axiom of k-spheres as below:

A Riemannian manifold $\bar{M}$ *of dimension* $m \geq 3$ *satisfies the axiom of* k*-spheres if for each point* p *in* $\bar{M}$ *and for every linear subspace* T *of* $T_p\bar{M}$*, there exists a* k*-dimensional totally umbilical submanifold* M *of* $\bar{M}$ *with parallel mean curvature vector field such that* $T_pM = T$ *and* $p \in M$*.*

Using the axiom of k-spheres, Leung and Nomizu [52] gave the following characterization theorem.

Theorem 3.1.3. [52] *If a Riemannian manifold* $\bar{M}$ *of dimension* $m \geq 3$ *satisfies the axiom of* k*-spheres for some* $k, 2 \leq k < m$*, then* $\bar{M}$ *has constant sectional curvature.*

Graves and Nomizu [34] generalized the above notions of axioms of planes and spheres from Riemannian manifolds to indefinite Riemannian manifolds.

Later, Kumar et al. [50] defined the axioms of planes and spheres for semi-Riemannian manifolds with lightlike submanifolds and proved the following result:

Definition 3.1.4. [50] *A semi-Riemannian manifold $\bar{M}$ of dimension $m+n \geq 3$ satisfies the axiom of k-planes (k-spheres) if for each point p in $\bar{M}$ and for every k-dimensional linear subspace T of $T_p\bar{M}$, there exists a k-dimensional totally geodesic lightlike submanifold (totally umbilical lightlike submanifold with parallel mean curvature vector field) M of $\bar{M}$ such that $T_pM = T$, $2 \leq k < m+n$ and $p \in M$.*

Theorem 3.1.5. [50] *Let $(\bar{M}, \bar{g})$ be a semi-Riemannian manifold of dimension $m+n \geq 3$ and $(M, g, S(TM), S(TM)^{\perp})$ be a lightlike submanifold of $\bar{M}$. If $\bar{M}$ satisfies the axiom of k-planes (k-spheres) for some k, $2 \leq k < m+n$, then $\bar{M}$ is a real space form.*

In 2006, Sahin [73] gave the following lemma and using this lemma Haider et al. [40] introduced hemi-slant lightlike submanifold of an Kähler manifold.

Lemma 3.1.6. [73] *Let M be a $2q$-lightlike submanifold of an indefinite Kähler manifold $\bar{M}$ with constant index $2q$ such that $2q < dim(\bar{M})$. Then the screen distribution $S(TM)$ of lightlike submanifold M is Riemannian.*

Definition 3.1.7. ***(Hemi-slant lightlike Submanifolds)***[40]*: Let M be a lightlike submanifold of an indefinite Kähler manifold $\bar{M}$ of contant index $2q$. Then the submanifold is said to be hemi-slant lightlike submanifold of $\bar{M}$ if it satisfies the following conditions:*

(i) $RadTM$ is a distribution on M such that $\bar{J}(RadTM) = ltr(TM)$.

(ii) For every non-zero vector field X tangent to $S(TM)$ and for any $p \in U \subset M$, the angle $\theta(X)$, called slant angle, between $\bar{J}X$ and the vector space $S(TM)$ is constant.

Hemi-slant lightlike submanifold is called proper if screen distribution is non-zero and $\theta \neq 0, \frac{\pi}{2}$.

Hence, using the definition of a hemi-slant lightlike submanifold M, any $X \in \Gamma(TM)$ written as

$$\bar{J}X = TX + FX, \tag{3.1.1}$$

where TX is the tangential component of $\bar{J}X$ and FX is the screen transversal component of $\bar{J}X$.

Let $(M, g, S(TM), S(TM^{\perp}))$ be an hemi-slant lightlike submanifold of an indefinite Kähler manifold. Using the decomposition (1.3.1), we denote the projection morphisms on the distributions $S(TM)$ and $RadTM$ as P and Q, respectively. Then for any $X \in \Gamma(TM)$, we have

$$X = PX + QX. \tag{3.1.2}$$

Applying $\bar{J}$ to (3.1.2), we get

$$\bar{J}X = \bar{J}PX + \bar{J}QX$$

or equivalently

$$\bar{J}X = TPX + FPX + FQX, \tag{3.1.3}$$

where

$$TPX \in \Gamma(S(TM)), \quad FPX \in \Gamma(S(TM^{\perp})), \quad FQX \in \Gamma(ltr(TM)). \tag{3.1.4}$$

Similarly, for any $U \in \Gamma(tr(TM))$, we have

$$\bar{J}U = BU + CU, \tag{3.1.5}$$

where BU and CU are the tangential and the transversal component of $\bar{J}U$, respectively. Differentiating (3.1.3) and using (1.3.6)-(1.3.9) and (3.1.5), we have

$$(\nabla_X T)PY = A_{FPY}X + A_{FQY}X + Bh^s(X,Y) + \bar{J}h^l(X,Y) \tag{3.1.6}$$

$$(\nabla_X F)PY = Ch^s(X,Y) - h^s(X,TPY) - D^s(X,FQY) \tag{3.1.7}$$

$$(\nabla_X F)QY = -h^l(X,TPY) - D^l(X,FPY) \tag{3.1.8}$$

where

$$(\nabla_X T)PY = \nabla_X TPY - TP\nabla_X Y$$

$$(\nabla_X F)PY = \nabla^s_X FPY - FP\nabla_X Y$$

$$(\nabla_X F)QY = \nabla^l_X FQY - FQ\nabla_X Y,$$

for all $X, Y \in \Gamma(TM)$.

Further, Haider et al. [40] proved the following characterization theorems for hemi-slant lightlike submanifolds similar to the characterization theorems for slant lightlike submanifolds of indefinite Hermitian manifolds given by Sahin [74].

Theorem 3.1.8. [40] *The necessary and sufficient condition for a $2q$-lightlike submanifold M of an indefinite Kähler manifold $\bar{M}$ of index $2q$ to be hemi-slant lightlike submanifold is that*

(A) $\bar{J}(ltr(TM))$ *is a distribution on M,*

(B) for any vector field X tangent to M, there exists a constant $\lambda \in [-1, 0]$ such that

$$(PT)^2X = \lambda X, \tag{3.1.9}$$

$X \in \Gamma(TM)$ *and* $\lambda = -\cos^2\theta$.

Theorem 3.1.9. [40] *The necessary and sufficient condition for a $2q$-lightlike submanifold M of an indefinite Kähler manifold $\bar{M}$ of index $2q$ to be hemi-slant lightlike submanifold is that*

(C) $\bar{J}(ltr(TM))$ *is a distribution on* M*,*

(D) for any vector field X *tangent to* M*, there exists a constant* $\mu \in [-1, 0]$ *such that*

$$BFPX = \mu PX, \tag{3.1.10}$$

for every $X \in \Gamma(TM)$*, where* $\mu = -\sin^2\theta$*,* θ *is the slant angle of* M*.*

Corollary 3.1.10. [40] *Let* M *be a hemi-slant lightlike submanifold of an indefinite Kähler manifold of constant index* $2q$*. Then for every* $X, Y \in \Gamma(TM)$*, we have*

$$g(TPX, TPX) = \cos^2\theta g(PX, PX), \tag{3.1.11}$$

and

$$\bar{g}(FPX, FPX) = \sin^2\theta g(PX, PX), \tag{3.1.12}$$

for any $X, Y \in \Gamma(TM)$*.*

We have generated the following example which demonstrates the concept of hemi-slant lightlike submanifold.

Example: *Let* $\bar{M} = (\Re^8_2, \bar{g})$ *be a semi-Riemannian manifold, where* $\Re^8_2$ *is a semi-Euclidean space having signature* $(-, -, +, +, +, +, +, +)$ *with respect to the canonical basis* $\{\partial x_1, \partial x_2, \partial x_3, \partial x_4, \partial x_5, \partial x_6, \partial x_7, \partial x_8\}$*. Let* M *be a submanifold of* $\Re^8_2$ *given as*

$$X(s, t, u, v) = (s, t, \sin u, \cos u, -u \sin v, -u \cos v, t, s),$$

then the tangent bundle is spanned by $Z_1 = \partial x_1 + \partial x_8, \quad Z_2 = \partial x_2 + \partial x_7,$

$Z_3 = \cos u \partial x_3 - \sin u \partial x_4 - \sin v \partial x_5 - \cos v \partial x_6, \quad Z_4 = -u \cos v \partial x_5 + u \sin v \partial x_6.$

Clearly, M *is a 2-lightlike submanifold with* $RadTM = span\{Z_1, Z_2\}$ *and* $S(TM) = span\{Z_3, Z_4\}$ *and* M *is Riemannian. It is easy to prove that* $S(TM)$

is a slant distribution having slant angle $\theta = \pi/4$.

Moreover, the screen transversal bundle $S(TM^{\perp})$ *is spanned by*

$$W_1 = \sin u\partial x_3 + \cos u\partial x_4, \quad W_2 = \sec u\partial x_3 + \sin v\partial x_5 + \cos v\partial x_6.$$

The transversal lightlike bundle $ltr(TM)$ *is spanned by*

$$N_1 = \frac{1}{2}(-\partial x_1 + \partial x_8), \quad N_2 = \frac{1}{2}(-\partial x_2 + \partial x_7).$$

Hence M *is a hemi-slant lightlike submanifold of* $\bar{M}$.

In this chapter, we have extended the theory of hemi-slant lightlike submanifolds of indefinite K$\ddot{a}$hler manifolds and we have clubbed this theory with notion of axioms of planes and spheres.

It should be noted that several axioms of submanifolds have been studied by many authors and are used to characterize the real and complex space forms. The aim of this chapter is to introduce axioms of indefinite hemi-slant 3-planes and 3-spheres. It has been proved that if an indefinite K$\ddot{a}$hler manifold satisfies the axioms of indefinite hemi-slant 3-planes and 3-spheres for some slant angle $\theta \in (0, \pi/2)$, then it is an indefinite complex space form. Further, in this chapter it has been established that there do not exist totally umbilical hemi-slant lightlike submanifolds in indefinite K$\ddot{a}$hler manifolds other than the totally geodesic hemi-slant lightlike submanifolds. Consequently, it has also been shown that the induced connection of a totally umbilical hemi-slant lightlike submanifold is a metric connection. A characterization theorem on the non-existence of totally umbilical hemi-slant lightlike submanifold of an indefinite complex space form and some characterization theorems on minimal hemi-slant lightlike submanifolds have also been presented towards the end of this chapter.

The contents of this chapter have been divided into two sections. In section 3.2, axioms of indefinite hemi-slant 3-planes and 3-spheres have been introduced and it has been proved that if an indefinite Kähler manifold satisfies these axioms for some slant angle $\theta \in (0, \pi/2)$, then it is an indefinite complex space form. Some results related to totally umbilical and minimal hemi-slant lightlike submanifolds have also been given in section 3.3.

3.2 The Axioms of Indefinite Hemi-Slant Planes and Spheres with Lightlike Submanifolds

In this section, we present axioms of indefinite hemi-slant 3-planes and 3-spheres with lightlike submanifolds and prove some important results related to it.

Before defining axioms of indefinite hemi-slant 3-planes and hemi-slant 3-spheres with lightlike submanifolds the definitions of slant plane and hemi-slant 3 plane due to Tastan [86] are given:

Definition 3.2.1. [86] *Let T be a 2-dimensional plane. The angle $\theta \in [0, \pi/2]$ between T and $\bar{J}T$ is $\cos\theta = |\bar{g}(X, \bar{J}Y)|$, where $\{X, Y\}$ is an orthonormal basis of T. If θ is constant, then T is called a slant-plane and θ is called a slant angle of T.*

Definition 3.2.2. [86] *A 3-plane T in $T_p\bar{M}$ is said to be hemi-slant if it contains a nonzero vector $Z \in T_p\bar{M}$ and a slant 2-plane with slant angle $\theta \in (0, \pi/2)$ such that $\bar{J}Z$ is perpendicular to T with $\bar{J}Z \perp T$, in which case $T = D^\theta \oplus \{Z\}$, where D^θ is the corresponding slant 2-plane.*

Tastan [86] proposed the axiom of hemi-slant 3-spheres defined as below:

Definition 3.2.3. [86] ***(Axiom of hemi-slant** 3**-spheres)**: An almost Hermitian manifold $\bar{M}$ is said to satisfy the axiom of hemi-slant 3-spheres if for each point $p \in \bar{M}$ and each hemi-slant 3-plane T in $T_p\bar{M}$, there exists a 3-dimensional totally umbilical submanifold M such that $p \in M$ and $T_pM = T$.*

Before presenting the axioms of indefinite hemi-slant 3-planes and indefinite hemi-slant 3-spheres with lightlike submanifolds, let us note the following.

Remark: *Let $\bar{M}$ be a $2q$-lightlike submanifold of an indefinite Kähler manifold $\bar{M}$ with constant index $2q$. Then the screen distribution $S(TM)$ of lightlike submanifold is Riemannian. Let $\{X_1, X_2, ..., X_\alpha, \bar{J}X_1, \bar{J}X_2, ..., \bar{J}X_\alpha\}$ be an orthonormal $\bar{J}$-basis of $S(TM)$. Then, there exists a hemi-slant 3-plane with slant angle θ.*

For example: $T = D^\theta \oplus \{X_3\}$ is a hemi-slant 3-plane having slant angle θ, where $D^\theta = span\{X_1, \cos\theta \bar{J}X_1 + \sin\theta X_2\}$.

Now, we will define the axioms of indefinite hemi-slant 3-planes and indefinite hemi-slant 3-spheres with lightlike submanifolds as follows:

Definition 3.2.4. ***(Axioms of indefinite hemi-slant* 3*-planes (or indefinite hemi-slant* 3*-spheres) with lightlike submanifolds)****: Let $\bar{M}$ be an indefinite almost Hermitian manifold with constant index $2q$. Then $\bar{M}$ is said to satisfy the axiom of indefinite hemi-slant 3-planes or indefinite hemi-slant 3-spheres with lightlike submanifolds, if for each $p \in \bar{M}$ and each hemi-slant 3-plane T in $T_p\bar{M}$, there exists a 3-dimensional totally geodesic lightlike submanifold M or totally umbilical lightlike submanifold M with parallel transversal curvature vector field and an induced metric connection, such that $p \in M$ and $T_pM = T$.*

Definition 3.2.5. *The transversal curvature vector field α of M is said to be parallel in the transversal vector bundle $tr(TM)$ if $\nabla^{\perp}_X \alpha = 0$ for all $X \in \Gamma(TM)$.*

Using (1.3.8), (1.3.9) and (2.3.2), it is clear that for a totally umbilical submanifold, the transversal curvature vector field α is parallel in $tr(TM)$, if and only if,

$$\nabla^l_X \alpha^l = 0, \quad \text{and} \quad \nabla^s_X \alpha^s + D^s(X, \alpha^l) = 0,$$

for any $X \in TM$.

Let M be a totally umbilical lightlike submanifold. Then by using (1.3.14), we have

$$\begin{aligned}\bar{R}(X,Y)Z &= R(X,Y)Z - g(Y,Z)A_\alpha X + g(X,Z)A_\alpha Y \\ &+\{(\nabla_X g)(Y,Z) - (\nabla_Y g)(X,Z)\}\alpha \\ &+g(Y,Z)\{\nabla^l_X \alpha^l + D^s(X,\alpha^l) + \nabla^s_X \alpha^s\} \\ &-g(X,Z)\{\nabla^l_Y \alpha^l + D^s(Y,\alpha^l) + \nabla^s_Y \alpha^s\}, \qquad (3.2.1)\end{aligned}$$

for any $X, Y, Z \in \Gamma(TM)$. This leads to the following lemma:

Lemma 3.2.6. *Let M be a totally umbilical lightlike submanifold of a semi-Riemannian manifold $\bar{M}$ with parallel transversal curvature vector field α. Then*

$$\begin{aligned}\bar{R}(X,Y)Z &= R(X,Y)Z - g(Y,Z)A_\alpha X + g(X,Z)A_\alpha Y + \{(\nabla_X g)(Y,Z) \\ &-(\nabla_Y g)(X,Z)\}\alpha,\end{aligned}$$

for any X, Y and $Z \in \Gamma(TM)$.

The following theorem due to Barros and Romero [4] has been used in the proof of our main result.

Theorem 3.2.7. [4]*: Let $\bar{M}$ be an indefinite Kähler manifold with complex dimension ≥ 2. Then $\bar{M}$ is an indefinite complex space form if and only if $\bar{g}(\bar{R}(X,Y)\bar{J}X, X) = 0$, for every orthonormal vectors X, Y and $\bar{J}X \in T\bar{M}$.*

The main results of this section are as follows:

Theorem 3.2.8. *Let $(\bar{M}, \bar{g})$ be an indefinite Kähler manifold with index $2q$. If $\bar{M}$ satisfies the axiom of indefinite hemi-slant 3-spheres with $\theta \in (0, \pi/2)$, then $\bar{M}$ is an indefinite complex space form.*

Proof. Let X, Y, Z be orthonormal vector fields at arbitrary point $m \in \bar{M}$ such that $\bar{g}(X, \bar{J}Y) = \bar{g}(X, \bar{J}Z) = \bar{g}(Y, \bar{J}Z) = 0$. Let $T = D^{\theta} \oplus \{Y\}$ be an hemi-slant 3-plane with $\theta \in (0, \pi/2)$, where $D^{\theta} = span\{X, \cos\theta \bar{J}X + \sin\theta Z\}$. Let $\bar{M}$ satisfies the axiom of indefinite hemi-slant 3-spheres. Therefore there exists a 3-dimensional totally umbilical lightlike submanifold M with parallel transversal curvature vector field α and an induced metric connection ∇ such that

$$T_m(M) = T, \text{ for any point } m \in \bar{M}.$$

Hence by using the Lemma 3.2.6, for a hemi-slant plane $T = D^{\theta} \oplus \{Y\}$, where $D^{\theta} = \{X, X' = \cos\theta \bar{J}X + \sin\theta Z\}$, the transversal form of $\bar{R}(X, X')X$ is given by

$$(\bar{R}(X, X')X)^{tr} = \{(\nabla_X g)(X', X) - (\nabla_{X'} g)(X, X)\}\alpha.$$

As the induced connection ∇ is metric, therefore we have

$$(\bar{R}(X, (\cos\theta \bar{J}X + \sin\theta X))X)^{tr} = 0,$$

hence we obtain

$$\bar{R}(X, \cos\theta \bar{J}X + \sin\theta X, X, X') = 0. \tag{3.2.2}$$

Similarly for a hemi-slant 3-plane $T' = D'^{\theta} \oplus \{Y\}$ having slant angle $\theta \in (0, \frac{\pi}{2})$, where $D'^{\theta} = span\{X, \cos\theta \bar{J}X - \sin\theta Z\}$, we have

$$\bar{R}(X, \cos\theta \bar{J}X - \sin\theta Z, X, Y) = 0. \tag{3.2.3}$$

Thus by using (3.2.2) and (3.2.3), we obtain $\bar{R}(X, \bar{J}X, X, Y) = 0$. Hence, the result follows from the Theorem 3.2.7. □

Theorem 3.2.9. *Let* $(\bar{M}, \bar{g})$ *be an indefinite Kähler manifold with constant index* $2q$*. If* $\bar{M}$ *satisfies the axiom of indefinite hemi-slant* 3*-planes with* $\theta \in (0, \pi/2)$*, then* $\bar{M}$ *is an indefinite complex space form.*

Proof. Let X, Y, Z be orthonormal vector fields at arbitrary point $m \in \bar{M}$ such that $\bar{g}(X, \bar{J}Y) = \bar{g}(X, \bar{J}Z) = \bar{g}(Y, \bar{J}Z) = 0$. Let $T = D^{\theta} \oplus \{Y\}$ be an hemi-slant 3-plane with $\theta \in (0, \pi/2)$, where $D^{\theta} = span\{X, \cos\theta\bar{J}X + \sin\theta Z\}$. Let $\bar{M}$ satisfies the axiom of indefinite hemi-slant 3-planes. Therefore, using the Definition 3.2.4 for any point $p \in \bar{M}$ and each hemi-slant 3-plane, there exists a 3-dimensional totally geodesic lightlike submanifold M such that

$$T_pM = T.$$

Using (1.3.16), for a hemi-slant plane $T = D^{\theta} \oplus \{Y\}$, where $D^{\theta} = \{X, X' = \cos\theta\bar{J}X + \sin\theta Z\}$, we have

$$(\bar{R}(X, X')X)^{tr} = (\bar{R}(X, \cos\theta\bar{J}X + \sin\theta Z)X)^{tr} = 0.$$

Now by following the same steps as in Theorem 3.2.8, the result follows. □

3.3 Totally Umbilical Hemi-Slant Lightlike Submanifolds

In this section, results related to totally umbilical hemi-slant lightlike submanifolds and minimal hemi-slant lightlike submanifolds of indefinite Kähler manifold have been presented.

The following definition of semi-transversal lightlike submanifold due to Sahin [73] have been used in proving subsequent results.

Definition 3.3.1. [73] *Let M be a lightlike submanifold of an indefinite Kähler manifold $\bar{M}$. Then M is called a semi-transversal lightlike submanifold of $\bar{M}$ if*

(i) $RadTM$ is a distribution on M such that $\bar{J}(RadTM) = ltr(TM)$.

(ii) There exists a real non-null distribution $D \subset S(TM)$ such that

$$S(TM) = D \oplus D^{\perp}, \quad \bar{J}D = D, \quad \bar{J}(D^{\perp}) \subset S(TM^{\perp}),$$

where $D^{\perp}$ is orthogonal complementary to D in $S(TM)$.

The main results of this section are as follows:

Theorem 3.3.2. *Let M be a $2q$-dimensional lightlike submanifold of an indefinite Kähler manifold with index $2q$. Then any Coisotropic semi-transversal lightlike submanifold is a hemi-slant lightlike submanifold with $\theta = 0$. Moreover, for any semi-transversal lightlike submanifold of $\bar{M}$ with $D = \{0\}$ is a hemi-slant submanifold with $\theta = \frac{\pi}{2}$.*

Proof. Let M be a $2q$-dimensional semi-transversal lightlike submanifold of an indefinite Kähler manifold. Then by definition of semi-transversal lightlike submanifold, $RadTM$ is a distribution on M such that $\bar{J}(RadTM) = ltr(TM)$. Moreover, if M is Coisotropic semi-transversal lightlike submanifold, then $S(TM^{\perp}) = \{0\}$, thus $D^{\perp} = \{0\}$. This implies $S(TM) = D$. Since D is invariant with respect to $\bar{J}$, therefore $\theta = 0$. Hence any Coisotropic semi-transversal lightlike submanifold is a hemi-slant lightlike submanifold with $\theta = 0$.

Now, let M be a semi-transversal lightlike submanifold of $\bar{M}$ with $D = \{0\}$. Hence by using the definition of semi-transversal lightlike submanifold, we have $S(TM) = D^{\perp}$, such that $JD^{\perp} \subset S(TM)^{\perp}$. Therefore any semi-transversal lightlike submanifold of $\bar{M}$ with $D = \{0\}$ is a hemi-slant submanifold with $\theta = \frac{\pi}{2}$. □

Theorem 3.3.3. *There does not exist a proper hemi-slant totally lightlike or isotropic submanifold in indefinite Kähler manifolds.*

Proof. Let M be a totally lightlike submanifold of an indefinite Kähler manifold. Then $TM = Rad(TM)$ and hence $S(TM) = \{0\}$. The other assertion follows similarly. □

The following lemmas have been used for the proof of Theorem 3.3.6:

Lemma 3.3.4. *Let M be a hemi-slant lightlike submanifold of an indefinite Kähler manifold. Then $FPX \in \Gamma(S(TM^{\perp}))$, for every $X \in \Gamma(TM)$.*

Proof. From (1.3.2) and (1.3.3), it is clear that $FPX \in \Gamma(S(TM^{\perp}))$, if and only if, $\bar{g}(FPX, \xi) = 0$, for any $\xi \in \Gamma(RadTM)$. Thus, $\bar{g}(FPX, \xi) = \bar{g}(\bar{J}PX - TPX, \xi) = \bar{g}(\bar{J}PX, \xi) = -\bar{g}(PX, \bar{J}\xi) = 0$, which proves the result. □

Moreover, from the (3.1.4), it follows that $F(S(TM))$ is a subspace of $S(TM^{\perp})$. Thus, there exists an invariant subspace μ_p of $T_p\bar{M}$ such that

$$S(T_pM^{\perp}) = F(S(T_pM))\perp\mu_p, \tag{3.3.1}$$

therefore, we have

$$T_p\bar{M} = S(T_pM)\perp\{Rad(T_pM) \oplus ltr(T_pM)\}\perp\{F(S(T_pM))\perp\mu_p\}.$$

Lemma 3.3.5. *Let M be a totally umbilical proper hemi-slant lightlike submanifold of an indefinite Kähler manifold. Then $\nabla_X X \in \Gamma(S(TM))$, for any $X \in \Gamma(S(TM))$.*

Proof. Let $X \in \Gamma(S(TM))$ and $N \in \Gamma(ltr(TM))$ such that $\bar{J}N = \xi$. Now, using (1.3.6), (1.3.9), (3.1.3) and (2.3.2), we obtain

$$\begin{aligned} \bar{g}(\nabla_X X, N) &= \bar{g}(\nabla_X \bar{J}X, \bar{J}N) = \bar{g}(\bar{\nabla}_X TX, \xi) + \bar{g}(\bar{\nabla}_X FX, \xi) \\ &= \bar{g}(h^l(X, TX), \xi) + \bar{g}(D^l(X, FX), \xi) \\ &= g(X, TX)g(\alpha^l, \xi). \end{aligned}$$

Since, $g(X, TX) = g(X, \bar{J}X) = 0$, therefore $\bar{g}(\nabla_X X, N) = 0$. Hence, by making use of (1.3.1) and (1.3.2), the assertion follows. □

Theorem 3.3.6. *Every totally umbilical proper hemi-slant lightlike submanifold of an indefinite Kähler manifold is totally geodesic.*

Proof. Let $X, Y \in \Gamma(S(TM))$. Adding (3.1.7) and (3.1.8), we get

$$\begin{aligned}\nabla^s_X FPY - FP\nabla_X Y - FQ\nabla_X Y \ &= Ch^s(X,Y) - h^s(X,TPY) - h^l(X,TPY) \\ &\quad -D^l(X,FPY).\end{aligned}$$

On replacing X by TX and Y by X, we have

$$\begin{aligned}FP\nabla_{TX}X + FQ\nabla_{TX}X \ &= \nabla^s_{TX}FPX - Ch^s(TX,X) + h^s(TX,TPX) \\ &\quad +h^l(TX,TPX) + D^l(TX,FPX).\end{aligned}$$

Since, M is a totally umbilical hemi-slant lightlike submanifold, therefore using (3.1.11), (2.3.2) and the fact that $h^s(TX,X) = g(TX,X)\alpha^s = \bar{g}(\bar{J}X,X)\alpha^s = 0$, gives

$$cos^2\theta g(X,X)\alpha = FP\nabla_{TX}X + FQ\nabla_{TX}X - \nabla^s_{TX}FPX.$$

Now, taking the scalar product both sides with respect to $FPX \in (S(TM^{\perp}))$, we obtain

$$cos^2\theta g(X,X)\bar{g}(\alpha^s, FPX) = \bar{g}(FP\nabla_{TX}X, FPX) - \bar{g}(\nabla^s_{TX}FPX, FPX),$$

further by using (3.1.12), we get

$$cos^2\theta g(X,X)\bar{g}(\alpha^s, FPX) = sin^2\theta\bar{g}(P\nabla_{TX}X, PX) - \bar{g}(\nabla^s_{TX}FPX, FPX). \tag{3.3.2}$$

Next, taking covariant derivative of (3.1.12) with respect to $\bar{\nabla}_{TX}$, we get

$$\bar{g}(\nabla^s_{TX}FPX, FPX) = sin^2\theta g(\nabla_{TX}PX, PX)$$

and then by using this relation in (3.3.2), we obtain

$$cos^2\theta g(X,X)\bar{g}(\alpha^s, FPX) = 0.$$

Since g is a Riemannian metric on $S(TM)$ and M is a proper hemi-slant lightlike submanifold, therefore we have $\bar{g}(\alpha^s, FPX) = 0$. Thus by using (3.1.4) and (3.3.1), we obtain

$$\alpha^s \in \Gamma(\mu). \tag{3.3.3}$$

Now, by using the Kählerian character of $\bar{M}$ for all $X, Y \in \Gamma(S(TM))$ along with equations (1.3.6), (1.3.9) and (2.3.1), we have

$$\begin{aligned} &\nabla_X TPY + g(X, TPY)\alpha - A_{FPY}X + \nabla^s_X FPY + D^l(X, FPY) = T\nabla_X Y \\ &\quad + F\nabla_X Y + g(X, Y)\bar{J}\alpha \end{aligned}$$

Now, take scalar product with respect to $\bar{J}\alpha^s$ on both sides and on using invariant property of μ with (3.3.3), we obtain

$$\bar{g}(\nabla^s_X FPY, \bar{J}\alpha^s) = g(X, Y)\bar{g}(\alpha^s, \alpha^s). \tag{3.3.4}$$

Since μ is an invariant subspace, therefore by using the Kählerian property of $\bar{M}$ for any $\alpha^s \in \Gamma(\mu)$, we get

$$\begin{aligned} -A_{\bar{J}\alpha^s}X + \nabla^s_X \bar{J}\alpha^s + D^l(X, \bar{J}\alpha^s) &= -TA_{\alpha^s}X - FA_{\alpha^s}X + B\nabla^s_X \alpha^s \\ &\quad + C\nabla^s_X \alpha^s + \bar{J}D^l(X, \alpha^s). \end{aligned}$$

On taking the scalar product with respect to FPY on both sides, we obtain

$$\bar{g}(\nabla^s_X \bar{J}\alpha^s, FPY) = -\bar{g}(FA_{\alpha^s}X, FPY) + \bar{g}(C\nabla^s_X \alpha^s, FPY).$$

From (3.1.5), we know that for any $U \in \Gamma(tr(TM))$, BU and CU are tangential and transversal components of $\bar{J}U$, respectively. Therefore if $U \in \Gamma(ltr(TM))$, then $\bar{J}U = BU \in \Gamma(Rad(TM))$ and $CU = 0$. Moreover, since $S(TM^\perp) = F(S(TM))\perp\mu$, therefore for any $U \in \Gamma(S(TM^\perp))$, $BU \in \Gamma(S(TM))$ and $CU \in \Gamma(\mu)$. Since $\nabla^s_X \alpha_S \in \Gamma(S(TM^\perp))$, therefore $C\nabla^s_X \alpha_S \in \Gamma(\mu)$. Hence, we have

$$\bar{g}(\nabla^s_X \bar{J}\alpha^s, FPY) = -\bar{g}(FA_{\alpha^s}X, FPY) = -sin^2\theta g(A_{\alpha^s}X, PY). \tag{3.3.5}$$

Since $\bar{\nabla}$ is a metric connection, therefore $(\bar{\nabla}_X g)(FPY, \bar{J}\alpha^s) = 0$. This further implies that

$$\bar{g}(\nabla^s_X FPY, \bar{J}\alpha^s) = \bar{g}(\nabla^s_X \bar{J}\alpha^s, FPY),$$

therefore by using (3.3.5), we obtain

$$\bar{g}(\nabla^s_X FPY, \bar{J}\alpha^s) = -sin^2\theta g(A_{\alpha^s}X, PY). \tag{3.3.6}$$

From (3.3.4) and (3.3.6), we have

$$g(X, Y)g(\alpha^s, \alpha^s) = -sin^2\theta g(A_{\alpha^s}X, PY)$$

and using (1.3.10) here, we obtain

$$g(X, Y)g(\alpha^s, \alpha^s) = -sin^2\theta\bar{g}(h^s(X, PY), \alpha^s) = -sin^2\theta g(X, Y)g(\alpha^s, \alpha^s).$$

This implies that

$$(1 + sin)^2\theta g(X, Y)g(\alpha^s, \alpha^s) = 0.$$

Since g is a Riemannian metric on $S(TM)$ and M is a proper hemi slant lightlike submanifold, therefore we get

$$\alpha^s = 0. \tag{3.3.7}$$

Further, using the Kähler character of $\bar{M}$ for any $X \in \Gamma(S(TM))$, we have

$$\nabla_X TX + h(X, TX) - A_{FX}X + \nabla^s_X FX + D^l(X, FX) = T\nabla_X X + F\nabla_X X$$
$$+Bh(X, X) + Ch^s(X, X).$$

Since M is a totally umbilical hemi-slant lightlike submanifold, therefore using $h(X, TX) = 0$ and then by comparing the tangential components, we obtain

$$\nabla_X TX - A_{FX}X = T\nabla_X X + Bh(X, X).$$

Taking the scalar product both sides with respect to $N \in \Gamma(ltr(TM))$ such that $\bar{J}N = \xi \in \Gamma(Rad(TM))$ and using the Lemma 3.3.5, we get

$$\bar{g}(A_{FX}X, N) = \bar{g}(\bar{J}h^l(X, X), N). \tag{3.3.8}$$

Now, using (1.3.6), (1.3.9), (2.3.2) and the Lemma 3.3.5, we have

$$\begin{aligned}\bar{g}(A_{FX}X, N) &= \bar{g}(\bar{J}X, \bar{\nabla}_X N) - \bar{g}(TX, \bar{\nabla}_X N) = \bar{g}(\bar{\nabla}_X X, \xi) + \bar{g}(\bar{\nabla}_X TX, N)\\ &= \bar{g}(h^l(X, X), \xi).\end{aligned}$$

Hence by using this in (3.3.8), we have

$$2\bar{g}(h^l(X, X), \xi) = 0.$$

Since M is a totally umbilical proper hemi-slant lightlike submanifold, therefore we have

$$g(X, X)\bar{g}(\alpha^l, \xi) = 0.$$

Moreover, as g is a Riemannian metric on $S(TM)$ therefore $\bar{g}(\alpha^l, \xi) = 0$ and have by making use of (1.3.2), we obtain that

$$\alpha^l = 0. \tag{3.3.9}$$

Thus, from (3.3.7) and (3.3.9), the proof of the theorem follows. □

Theorem 3.3.7. *Let M be a totally umbilical proper hemi-slant lightlike submanifold of an indefinite Kähler manifold $\bar{M}$. Then the induced connection ∇ is a metric connection on M.*

Proof. Using (2.3.2) and (3.3.9), we have $h^l = 0$ then using the Theorem 2.2 in [27] at page 159, the induced connection ∇ behaves as a metric connection on M. □

Theorem 3.3.8. *There does not exist a totally umbilical proper hemi-slant lightlike submanifold of an indefinite complex space form $\bar{M}(c)$ such that $c \neq 0$.*

Proof. Suppose M be a totally umbilical proper hemi-slant lightlike submanifold of $\bar{M}(c)$ such that $c \neq 0$. Then by using (1.2.2), for any $X \in \Gamma(S(TM))$ and

$\xi, \xi' \in \Gamma(Rad(TM))$, we obtain

$$\bar{g}(\bar{R}(X, \bar{J}X)\xi', \xi) = -\frac{c}{2}g(X, X)g(\bar{J}\xi', \xi). \tag{3.3.10}$$

On the other hand, using (2.3.2) and (1.3.14), we get

$$\bar{g}(\bar{R}(X, \bar{J}X)\xi', \xi) = \bar{g}((\nabla_X h^l)(\bar{J}X, \xi'), \xi) - \bar{g}((\nabla_{\bar{J}X} h^l)(X, \xi'), \xi). \tag{3.3.11}$$

Now, using (2.3.2) and (1.3.15), we have

$$\begin{aligned}(\nabla_X h^l)(\bar{J}X, \xi') &= -g(\nabla_X \bar{J}X, \xi')\alpha^l - g(\bar{J}X, \nabla_X \xi')\alpha^l = \bar{g}(\bar{\nabla}_X TX, \xi') \\ &= \bar{g}(h^l(X, TX), \xi') = g(X, \bar{J}X)\bar{g}(\alpha^l, \xi') = 0. \end{aligned} \tag{3.3.12}$$

Similarly

$$\begin{aligned}(\nabla_{\bar{J}X} h^l)(X, \xi') &= -g(\nabla_{\bar{J}X} X, \xi')\alpha^l - g(X, \nabla_{\bar{J}X} \xi')\alpha^l = \bar{g}(\bar{\nabla}_{\bar{J}X} X, \xi') \\ &= \bar{g}(h^l(\bar{J}X, X), \xi') = g(\bar{J}X, X)\bar{g}(\alpha^l, \xi') = 0. \end{aligned} \tag{3.3.13}$$

Thus, from (3.3.10) to (3.3.13), we obtain

$$\frac{c}{2}g(X, X)g(\bar{J}\xi', \xi) = 0.$$

As $S(TM)$ is Riemannian and using (1.3.2), we have $g(\bar{J}\xi', \xi) \neq 0$, therefore $c = 0$. Hence, the proof follows from the contradiction. □

In the following theorems, we have established some results for a proper hemi-slant lightlike submanifold of an indefinite Kähler manifold to be minimal.

General notions of minimal lightlike submanifold and of an irrorational lightlike submanifold of a semi-Riemannian manifold have already been mentioned in Chapter 2.

Theorem 3.3.9. *Let M be a totally umbilical proper hemi-slant lightlike submanifold of an indefinite Kähler manifold $\bar{M}$. Then M is minimal.*

Proof. The assertion follows directly by using the Theorem 3.3.6. □

The following lemma is required for the proof of our next theorems.

Lemma 3.3.10. *Let M be a proper hemi-slant lightlike submanifold of an indefinite Kähler manifold $\bar{M}$ such that $dim(S(TM)) = dim(S(TM^{\perp}))$. If $\{e_1, ., e_k\}$ is a local orthonormal basis of $S(TM)$, then $\{csc\theta Fe_1, ..., csc\theta Fe_k\}$ is a orthonormal basis of $S(TM^{\perp})$.*

Proof. Since $\{e_1, ..., e_k\}$ is a local orthonormal basis of $S(TM)$ and $S(TM)$ is Riemannian therefore by using (3.1.12), we get

$$\bar{g}(csc\theta Fe_i, csc\theta Fe_j) = csc^2\theta sin^2\theta g(e_i, e_j) = \delta_{ij}.$$

This proves the assertion. □

Theorem 3.3.11. *Let M be an irrotational hemi-slant lightlike submanifold of an indefinite Kähler manifold $\bar{M}$. Then M is minimal if and only if*

$$trace\ A_{W_q}|_{S(TM)} = 0, \quad trace\ A^*_{\xi_j}|_{S(TM)} = 0,$$

where $\{W_q\}_{q=1}^{l}$ and $\{\xi_j\}_{j=1}^{r}$ are the basis of $S(TM^{\perp})$ and $Rad(TM)$, respectively.

Proof. Since M is an irrotational, therefore $h^s(X, \xi) = 0$ for any $X \in \Gamma(TM)$ and $\xi \in \Gamma(Rad(TM))$. Thus h^s vanishes on $Rad(TM)$. Hence M is minimal, if and only if, trace $h = 0$ on $S(TM)$, that is, M is minimal, if and only if, $\sum_{i=1}^{k} h(e_i, e_i) = 0$. Now, using (1.3.10) and (1.3.13), we obtain

$$\sum_{i=1}^{k} h(e_i, e_i) = \sum_{i=1}^{k}\{\frac{1}{r}\sum_{j=1}^{r} g(A^*_{\xi_j}e_i, e_i)N_j + \frac{1}{l}\sum_{q=1}^{l} g(A_{W_q}e_i, e_i)W_q\}. \quad (3.3.14)$$

Thus the assertion follows from (3.3.14). □

Theorem 3.3.12. *Let M be a proper hemi-slant lightlike submanifold of an indefinite Kähler manifold $\bar{M}$. Then M is minimal if and only if*

$$trace\ A_{W_q}|_{S(TM)} = 0, \quad trace\ A^*_{\xi_j}|_{S(TM)} = 0 \quad and \quad \bar{g}(D^l(X,W),Y) = 0,$$

for any $X, Y \in \Gamma(Rad(TM))$, where $\{W_q\}_{q=1}^{l}$ and $\{\xi_j\}_{j=1}^{r}$ are the basis of $S(TM^\perp)$ and $Rad(TM)$, respectively.

Proof. Let $X, Y \in \Gamma(Rad(TM))$. Then by using (1.3.10), it is clear that $h^s = 0$ on $Rad(TM)$, if and only if, $\bar{g}(D^l(X,W),Y) = 0$. Moreover, using the Proposition 3.1 of [6], $h^l = 0$ on $Rad(TM)$. Hence M is minimal, if and only if, $\sum_{i=1}^{k} h(e_i,e_i) = 0$. Using (1.3.10) and (1.3.13), we obtain

$$\sum_{i=1}^{k} h(e_i,e_i) = \sum_{i=1}^{k}\{\frac{1}{r}\sum_{j=1}^{r} g(A^*_{\xi_j}e_i,e_i)N_j + \frac{1}{l}\sum_{q=1}^{l} g(A_{W_q}e_i,e_i)W_q\}.$$

Thus the assertion follows directly. □

Theorem 3.3.13. *Let M be a proper hemi-slant lightlike submanifold of an indefinite Kähler manifold $\bar{M}$ such that $dim(S(TM)) = dim(S(TM^\perp))$. Then M is minimal if and only if*

$$trace\ A_{csc\theta Fe_i}|_{S(TM)} = 0, \quad trace\ A^*_{\xi_j}|_{S(TM)} = 0, \quad and \quad \bar{g}(D^l(X,Fe_i),Y) = 0,$$

for all X and $Y \in \Gamma(Rad(TM))$, where $\{e_i\}_{i=1}^{k}$ is a basis of $S(TM)$.

Proof. Let $\{e_i\}_{i=1}^{k}$ is a basis of $S(TM)$. Then using the Lemma 3.3.10, $\{csc\theta Fe_i\}_{i=1}^{k}$ is a basis of $S(TM^\perp)$. Therefore, we have

$$h^s(X,X) = \sum_{i=1}^{k} \lambda_i csc\theta Fe_i, \tag{3.3.15}$$

for any $X \in \Gamma(TM)$ and for some functions $\lambda_i, i \in \{1,...,k\}$. Now, by using (1.3.10) we have

$$\bar{g}(h^s(X,X),W) = \bar{g}(A_W X,X)$$

for any $X \in \Gamma(S(TM))$. Then using (3.1.12) and (3.3.15), we obtain $\lambda_i = \bar{g}(A_{csc\theta Fe_i}X, X)$ and hence we get

$$h^s(X, X) = \sum_{i=1}^{k} csc\theta Fe_i \bar{g}(A_{csc\theta Fe_i}X, X),$$

for any $X \in \Gamma(S(TM))$. Hence the assertion comes from the Theorem 3.3.13. $\square$

Chapter 4

SEMI INVARIANT WARPED PRODUCT SUBMANIFOLDS OF COSYMPLECTIC MANIFOLDS

4.1 Introduction

Bishop and O'Neill [11] introduced the concept of warped product manifolds by homothetically warping the product metric of a product manifold $B \times F$ onto the fibers of $m \times F$ for each $m \in B$. Such metrics are not only seen in differential geometric studies but are also used to model space-time near black holes or massive stars. For instance, the best relativistic model of the Schwarzschild space-time describing the neighborhood of bodies with large gravitational fields is a warped product manifold. The study of warped product manifolds assumed significance for these kind of applications. The geometrical aspects of these manifolds have been studied by various researchers like Chen [25, 26], Khan et al. [45, 46]. The study of geometry of warped product manifolds got impetus with Chen's work

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Results of this chapter have been published in ***J. Inequal. Appl.***, 19 (2012), 1-12. ***(SCI-indexed), Impact factor-0.77*** on warped product CR-submanifolds of a Kähler manifold [25, 26]. Chen [22] studied warped product CR-submanifolds of the form $M = M_{\perp} \times_f M_T$, where $M_{\perp}$ is an anti-holomorphic submanifold and M_T is a holomorphic submanifold of a Kähler manifold $\bar{M}$ and proved that warped product CR-submanifolds are simply CR-product submanifolds. In this chapter, we consider the semi-invariant warped product submanifolds of the type $M = M_T \times_f M_{\perp}$.

The main objective of this chapter is to obtain the necessary and sufficient conditions for a semi-invariant submanifold to be a locally warped product submanifold of invariant and anti-invariant submanifolds of a Cosymplectic manifold in terms of the canonical structures. The inequality and equality cases are also discussed for the squared norm of the second fundamental form in terms of the warping function.

The contents of this chapter have been divided into three sections. Section 4.2 deals with the preliminaries related to this chapter. In section 4.3, the integrability conditions of the involved distributions of semi-invariant submanifolds of Cosymplectic manifolds have been derived. Further, in this section a necessary and sufficient condition for a semi-invariant submanifold of a Cosymplectic manifold to be a locally Riemannian product of the involved distribution has been established. Since the semi-invariant warped product submanifolds are generalization of locally Riemannian product submanifolds, therefore it is worthwhile to study warped product submanifolds in terms of canonical structures. For this, some characterization results on semi-invariant warped product submanifolds in terms of canonical structures have been obtained in section 4.4. The inequality and equality cases have also been discussed for the squared norm of the second fundamental form in terms of the warping function in this section.

4.2 Preliminaries

Before giving the main results of this chapter, we present some relevant concepts and results which will be required in proving our main results.

An odd dimensional $C^\infty-$manifold $\bar{M}$ is said to have an *almost contact structure* if there exist on $\bar{M}$ a tensor field ϕ of type $(1,1)$, a vector field V and 1-form η satisfying,

$$\phi^2 X = -X + \eta \otimes V, \;\; \phi(V) = 0, \;\; \eta \circ \phi = 0, \;\; \eta(V) = 1.$$

There always exists a Riemannian metric g on an almost contact manifold $\bar{M}$ satisfying following conditions

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad \eta(X) = g(X, V), \tag{4.2.1}$$

where X, Y are vector fields on $\bar{M}$. (for details see [12])

An almost contact structure (ϕ, V, η) is said to be *normal* if the almost complex structure $\bar{J}$ on the product manifold $\bar{M} \times \mathbb{R}$ given by

$$\bar{J}(X, f\frac{d}{dt}) = (\phi X - fV, \eta(X)\frac{d}{dt}),$$

where f is the $C^\infty-$function on $\bar{M} \times \mathbb{R}$ and has no torsion, that is, $\bar{J}$ is integrable. The condition for normality in terms of ϕ, V and η is

$$[\phi, \phi] + 2d\eta \otimes V = 0,$$

on $\bar{M}$, where $[\phi, \phi]$ is the Nijenhuis tensor of ϕ. The fundamental 2-form Φ is defined by $\Phi(X, Y) = g(X, \phi Y)$.

An almost contact metric structure (ϕ, V, η, g) is said to be Cosymplectic, if it is normal and both Φ and η are closed (for details see [12]) and structure equations of a Cosymplectic manifold are given by

$$(\bar{\nabla}_X \phi)Y = 0 \quad and \quad \bar{\nabla}_X V = 0, \tag{4.2.2}$$

for any $X, Y \in T\bar{M}$.

Let M be a submanifold of an almost contact metric manifold $\bar{M}$ with induced metric g and if ∇ and $\nabla^{\perp}$ are the induced connection on the tangent bundle TM and the normal bundle $TM^{\perp}$ of M, respectively, then Gauss and Weingarten formulae are given by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{4.2.3}$$

$$\bar{\nabla}_X U = -A_U X + \nabla^{\perp}_X U, \tag{4.2.4}$$

for each $X, Y \in \Gamma(TM)$ and $U \in \Gamma(TM^{\perp})$, where h and A_U are the second fundamental form and the shape operator, respectively for the immersion of M into $\bar{M}$ and are related as $g(h(X, Y), U) = g(A_U X, Y)$, where g denotes the Riemannian metric on $\bar{M}$ as well as on M. The mean curvature vector α on M is given by $\alpha = \frac{1}{n}\sum_{i=1}^{n} h(e_i, e_i)$, where n is the dimension of M and $\{e_1, e_2, ...e_n\}$ be a local orthonormal frame of vector fields on M. The squared norm of the second fundamental form is defined as

$$\|h\|^2 = \sum_{i,j=1}^{n} g(h(e_i, e_j), h(e_i, e_j)).$$

For any $X \in \Gamma(TM)$, ϕX can be expressed as

$$\phi X = TX + FX, \tag{4.2.5}$$

where TX and FX are the tangential and normal components of ϕX, respectively. Similarly, for any $U \in \Gamma(TM)^{\perp}$, ϕU can be expressed as

$$\phi U = tU + fU, \tag{4.2.6}$$

where tU and fU are the tangential and normal components of ϕU, respectively. The covariant derivatives of the tensor fields T and F are defined as

$$(\nabla_X T)Y = \nabla_X TY - T\nabla_X Y, \qquad (\nabla_X F)Y = \nabla^{\perp}_X FY - F\nabla_X Y, \tag{4.2.7}$$

for any $X, Y \in \Gamma(TM)$.

Let M be a Riemannian manifold isometrically immersed in an almost contact metric manifold $\bar{M}$. Then, there exists a maximal invariant subspace denoted by D_p of the tangent space T_pM of M for every $p \in M$. If the dimension of D_p is same for all values of $p \in M$, then D_p gives an invariant distribution D on M.

A submanifold M of an almost contact metric manifold $\bar{M}$ is called *semi-invariant submanifold* [12] if there exists a differentiable distribution D on M whose orthogonal complementary distribution $D^{\perp}$ is anti-invariant, i.e.,

(i) $TM = D \oplus D^{\perp} \oplus \langle V \rangle$.

(ii) D is an invariant distribution.

(iii) $D^{\perp}$ is an anti-invariant distribution, i.e., $\phi D^{\perp} \subseteq TM^{\perp}$.

A semi-invariant submanifold is called anti-invariant if $D_p = \{0\}$ and invariant submanifold if $D_p^{\perp} = \{0\}$. It is called a proper semi-invariant submanifold if neither $D_p = \{0\}$ nor $D_p^{\perp} = \{0\}$, for each $p \in M$.

Let M be a semi-invariant submanifold of an almost contact metric manifold $\bar{M}$. Then, FT_pM is a subspace of $T_pM^{\perp}$ such that $T_pM^{\perp} = FT_pM \oplus \mu$, where μ is the invariant subspace of $TM^{\perp}$ under ϕ.

The orthogonal projection on TM of a semi-invariant submanifold M of a Cosymplectic manifold are denoted by P_1 and P_2, that is, for any $X \in TM$, we have

$$X = P_1X + P_2X + \eta(X)V, \tag{4.2.8}$$

where $P_1X \in \Gamma(D)$, $P_2X \in \Gamma(D^{\perp})$ and $\eta(X)V \in \langle V \rangle$. It follows that

$$(i)\ TP_2 = 0 \qquad (ii)\ FP_1 = 0 \qquad (iii)\ t(TM^{\perp}) = D^{\perp} \qquad (iv)\ fTM^{\perp} \subseteq \mu. \tag{4.2.9}$$

From (4.2.2) - (4.2.6), we have

$$(\nabla_X T)Y = A_{FY}X + th(X,Y), \tag{4.2.10}$$

$$(\nabla_X F)Y = fh(X,Y) - h(X,TY), \tag{4.2.11}$$

for any $X, Y \in \Gamma(TM)$.

Definition 4.2.1. [12] *A semi-invariant submanifold M is said to be a locally semi-invariant product submanifold if M is locally a Riemannian product of the leaves of distributions D, $D^\perp$ and $\langle V \rangle$.*

Bishop and O'Neill [11] in 1969 introduced the concept of warped product manifolds and proved the following results for warped product manifolds.

Definition 4.2.2. [11] *Let B and F be two Riemannian manifolds with Riemannian metrics g_B and g_F and $f > 0$ a differentiable function on B. Let $\pi : B \times F \to B$ and $\eta : B \times F \to F$ be two projection morphism on the product manifold $B \times F$. Then the warped product manifold $M = B \times_f F$ is the product manifold equipped with the Riemannian metric g, where*

$$g = g_B + f^2 g_F.$$

If X is tangent to $M = B \times_f F$, then we have

$$\|X\|^2 = \|\pi_* X\|^2 + f^2(\pi(X))\|\eta_* X\|^2,$$

where f is known as the warping function of the warped product.

Theorem 4.2.3. [11] *Let $M = B \times_f F$ be a warped product manifold. If $X, Y \in \Gamma(TB)$ and $U, Z \in \Gamma(TF)$ then*

$$\nabla_X Y \in \Gamma(TB).$$

$$\nabla_X U = \nabla_U X = \frac{Xf}{f}U = X(ln\ f)U. \tag{4.2.12}$$

$$\nabla_U Z = -\frac{g(U,Z)}{f}\nabla f,$$

where ∇f is the gradient of f and is defined as

$$g(\nabla f, U) = Uf, \tag{4.2.13}$$

for all $U \in \Gamma(TM)$.

Corollary 4.2.4. [11] *On a warped product manifold $M = B \times_f F$, we have*

(i) B is totally geodesic in M.

(ii) F is totally umbilical in M.

4.3 Semi-Invariant Submanifolds of a Cosymplectic Manifold

In this section, we obtain the integrability conditions for involved distributions in the definition of a semi-invariant submanifold and have investigated the geometric properties of their leaves.

Towards this direction, Shoeb et al. [83], have proved the integrability of distribution $D^{\perp}$ as under:

Theorem 4.3.1. [83] *Let M be a semi-invariant submanifold of a Cosymplectic manifold. Then the anti-invariant distribution $D^{\perp}$ is integrable.*

On the other hand, we have obtained necessary and sufficient conditions for the integrability of distribution D and for the leaves of distribution to be totally geodesic in submanifold M of $\bar{M}$ as under:

Theorem 4.3.2. *Let M be a semi-invariant submanifold of a Cosymplectic manifold $\bar{M}$. Then its invariant distribution D is integrable if and only if $g(h(X, \phi Y), \phi Z) = g(h(\phi X, Y), \phi Z)$, for each $X, Y \in \Gamma(D)$ and $Z \in \Gamma(D^{\perp})$.*

Proof. Theorem follows directly by using (4.2.1), (4.2.2) and (4.2.3). □

Theorem 4.3.3. *If the invariant distribution D of a semi-invariant submanifold M of a Cosymplectic manifold $\bar{M}$ is integrable, then its leaves are totally geodesic in M if and only if $h(X,Y) \in \Gamma(\mu)$, for each $X \in \Gamma(TM)$ and $Y \in \Gamma(D)$.*

Proof. From (4.2.11), we have

$$F\nabla_X Y = fh(X,Y) - h(X,TY),$$

for any $X \in \Gamma(TM)$ and $Y \in \Gamma(D)$. On taking inner product both sides with ϕZ, for any $Z \in \Gamma(D^{\perp})$, we get

$$g(F\nabla_X Y, \phi Z) = -g(h(X,TY), \phi Z).$$

Hence the result follows. □

Using (4.2.10) and the Theorem 4.3.3, the following corollary follows immediately:

Corollary 4.3.4. *The invariant distribution D of a semi-invariant submanifold M of a Cosymplectic manifold $\bar{M}$ is integrable and its leaves are totally geodesic in M if and only if $(\nabla_X T)Y = 0$, for any $X, Y \in \Gamma(D)$.*

The following lemma is required for the proof of our next theorem.

Lemma 4.3.5. *Let M be a semi-invariant submanifold of a Cosymplectic manifold $\bar{M}$. Then the leaf $M_{\perp}$ of $D^{\perp}$ is totally geodesic in M, if and only if, $g(h(X,Z), \phi W) = 0$, for any $X \in \Gamma(D)$ and $Z, W \in \Gamma(D^{\perp})$.*

Proof. Using (4.2.1)-(4.2.4), we obtain

$$g(\nabla_Z W, \phi X) = g(h(X,Z), \phi W),$$

and hence the result follows from above equation. □

In the following theorem, the condition for a semi-invariant submanifold M of a Cosymplectic manifold $\bar{M}$ to be a locally semi-invariant product has been obtained.

Theorem 4.3.6. *A semi-invariant submanifold M of a Cosymplectic manifold $\bar{M}$ is locally a semi-invariant product, if and only if,* $(\nabla_{W_1}T)W_2 = 0$, *for any* $W_1, W_2 \in \Gamma(TM)$.

Proof. If T is parallel, then using (4.2.10), we have

$$A_{FW_2}W_1 = -th(W_1, W_2), \tag{4.3.1}$$

for any W_1, W_2 tangent to M. In particular, if $X \in \Gamma(D)$, then (4.3.1) gives $th(W_1, X) = 0$, that is

$$A_{FZ}X = 0, \tag{4.3.2}$$

for any $Z \in \Gamma(D^{\perp})$. Thus, by the Theorem 4.3.2 and the Lemma 4.3.5, D is integrable and the leaf $M_{\perp}$ of $D^{\perp}$ is totally geodesic in M.

Let M_T be a leaf of D. Now, for any $X, Y \in \Gamma(D)$ and $Z \in \Gamma(D^{\perp})$, using (4.3.2), we have $g(A_{\phi Z}X, Y) = 0$ and using (4.2.1)-(4.2.4), we get

$$g(\nabla_X \phi Y, Z) = 0,$$

which shows that leaf of D is totally geodesic in M and distribution $\langle V \rangle$ is already totally geodesic in M and hence M is locally semi-invariant product.

Conversely, if M is a locally semi-invariant product, then for any $X \in \Gamma(D)$ and $W_1 \in \Gamma(TM)$, we get $\nabla_{W_1}X \in \Gamma(D)$, thus using (4.2.10) and the Theorem 4.3.3, we get

$$(\bar{\nabla}_{W_1}T)Y = 0.$$

Similarly, for any $Z \in \Gamma(D^{\perp})$ and $W_1 \in \Gamma(TM)$, we get $\nabla_{W_1}Z \in \Gamma(D^{\perp})$ then using (4.2.7), we have $(\bar{\nabla}_{W_1}T)Z = 0$ and it is easy to see that $(\bar{\nabla}_{W_1}T)V = 0$. This implies $(\bar{\nabla}_{W_1}T)W_2 = 0$ for $W_1, W_2 \in \Gamma(TM)$ and hence the proof is complete. □

4.4 Semi-Invariant Warped Product Submanifold with Canonical Structures

Throughout this section, M_T and $M_\perp$ denote the invariant and anti-invariant submanifolds, respectively of a Cosymplectic manifold $\bar{M}$. Semi-invariant warped product submanifolds of a Cosymplectic manifold $\bar{M}$ are represented by $M_\perp \times_f M_T$ and $M_T \times_f M_\perp$ and we take M_T tangential to $\langle V \rangle$ throughout this section. The first type of warped product of a Cosymplectic manifold does not exist in the sense of [87]. We in this section, discuss the second type of warped product and obtain some interesting results.

The following lemma have an important role in the proof of our next theorem.

Lemma 4.4.1. *Let $M = M_T \times_f M_\perp$ be a semi-invariant warped product submanifold of a Cosymplectic manifold $\bar{M}$. Then*

$$(\nabla_Z T)X = TX(lnf)Z,$$

$$(\nabla_Y T)Z = g(P_2 Y, Z)\, T(\nabla\ lnf),$$

for X, Y and Z are tangent to M_T, M and $M_\perp$, respectively.

Proof. Let $M = M_T \times_f M_\perp$ be any warped product submanifold. Then by using the Theorem 4.2.3 for any $X \in \Gamma(TM_T)$ and $Z \in \Gamma(TM_\perp)$, we have

$$\nabla_X Z = \nabla_Z X = X(lnf)Z. \tag{4.4.1}$$

Now, using (4.2.7) and (4.4.1), we obtain $(\nabla_Z T)X = TX(lnf)Z$, which proves the first part of the Lemma.

Now, let $Y \in \Gamma(TM)$ then $TY \in \Gamma(TM_T)$, therefore $(\nabla_Y T)Z \in \Gamma(TM_T)$ for each $Z \in \Gamma(TM)$.

Further, for any $X \in \Gamma(TM_T)$, we have

$$g((\nabla_Y T)Z, X) = -g(Z, \nabla_Y TX).$$

Using (4.2.8), the above equation reduces to

$$g((\nabla_Y T)Z, X) = -g(Z, \nabla_{P_1Y+P_2Y+\eta(Y)V} TX) = -g(Z, \nabla_{P_2Y} TX) - \eta(Y)g(Z, \nabla_V TX).$$

Using (4.4.1), the second term of right hand side is identically zero, then the above equation takes the form

$$g((\nabla_Y T)Z, X) = -g(Z, \nabla_{P_2Y} TX) = -g(Z, TX(lnf)P_2Y) = -TX(lnf)g(Z, P_2Y),$$

further using (4.2.13), we get $g((\nabla_Y T)Z, X) = g(T\nabla lnf, X)g(Z, P_2Y)$, i.e.,

$$(\nabla_Y T)Z = T(\nabla lnf)g(Z, P_2Y).$$

This completes the proof of the lemma. □

We, now prove the following characterization theorem in terms of canonical structure T.

Theorem 4.4.2. *A proper semi-invariant submanifold of a Cosymplectic manifold $\bar{M}$ is locally a semi-invariant warped product submanifold if and only if*

$$(\nabla_{W_1} T)W_2 = (TW_2\nu)P_2W_1 + g(P_2W_1, P_2W_2)\phi\nabla\nu, \tag{4.4.2}$$

for each $W_1, W_2 \in \Gamma(TM)$ and ν is a C^∞-function on M satisfying $W\nu = 0$, for any $W \in \Gamma(D^\perp)$.

Proof. Let $M = M_T \times_f M_\perp$ be a semi-invariant warped product submanifold of a Cosymplectic manifold. Then, from (4.2.7) and (4.2.8), we have

$$(\nabla_{W_1} T)W_2 = (\nabla_{W_1} T)P_1W_2 + (\nabla_{W_1} T)P_2W_2 + \eta(W_1)(\nabla_{W_1} T)V.$$

Again, by using (4.2.7) and (4.2.8), this equation becomes

$$(\nabla_{W_1}T)W_2 = (\nabla_{P_1W_1}T)P_1W_2 + (\nabla_{P_2W_1}T)P_1W_2 + (\nabla_{W_1}T)P_2W_2. \tag{4.4.3}$$

Now, from the Lemma 4.4.1, we have

$$(\nabla_{P_2W_1}T)P_1W_2 = TW_2(lnf)P_2W_1$$

and

$$(\nabla_{W_1}T)P_2W_2 = g(P_2W_1, P_2W_2)T(\nabla lnf).$$

On substituting these relations in (4.4.3), we get

$$(\nabla_{W_1}T)W_2 = (TW_2\nu)P_2W_1 + g(P_2W_1, P_2W_2)\phi(\nabla\nu),$$

this proves the necessary part.

Conversely, suppose that M be a semi-invariant submanifold of a Cosymplectic manifold $\bar{M}$ and satisfying (4.4.2) then $(\nabla_X T)Y = 0$, for each $X, Y \in \Gamma(D)$, then by using the Corollary 4.3.4, D is integrable and each leave M_T of D is totally geodesic in M. Moreover, from (4.4.2), we get

$$g((\nabla_Z T)X, W) = TX(\nu)g(Z, W),$$

for $X \in \Gamma(D)$ and $Z, W \in \Gamma(D^\perp)$. Now from (4.2.2), (4.2.5) and (4.2.7) this equation takes the form

$$g(\bar{\nabla}_Z X, \phi W) = -TX(\nu)g(Z, W).$$

Using the Cosymplectic character of $\bar{M}$ and the equation (4.2.3), we infer

$$g(\nabla_Z X, \phi W) = -TX(\nu)g(Z, W).$$

By the use of (4.2.13), we get

$$g(\nabla_Z W, \phi X) = g(T\nabla\nu, X)g(Z, W). \tag{4.4.4}$$

Now, let $M_\perp$ be a leaf of $D^\perp$ and h' be the second fundamental form of the immersion of $M_\perp$ into M. Then $g(h'(Z,W),X) = g(\nabla_Z W, X)$.

Using (4.4.4), we get

$$g(h'(Z,W),\phi X) = -g(\nabla\nu, \phi X)g(Z,W),$$

or

$$h'(Z,W) = -g(Z,W)\nabla\nu.$$

This shows that $M_\perp$ is a totally umbilical submanifold in M with non-vanishing mean curvature $\nabla\mu$. Also, as $W\nu = 0$, for all $W \in \Gamma(D^\perp)$, i.e., then mean curvature vector of $M_\perp$ is parallel and the leaves of $D^\perp$ are extrinsic spheres in M. Hence, from a result of Hiepko [41], the submanifold M is locally a semi-invariant warped product submanifold M_T and $M_\perp$, with warping function $f = e^\nu$. $\square$

Remark: *Theorem 4.4.2 is the generalization of the Theorem 4.3.6, and shows the effect of ∇T, when the submanifold is a semi-invariant warped product submanifold.*

We, now prove the following theorem in terms of canonical structure F:

Theorem 4.4.3. *Let M be a semi-invariant submanifold of a Cosymplectic manifold $\bar{M}$. Then it is locally semi-invariant warped product submanifold, if and only if*

$$g((\nabla_{W_1}F)W_2, \phi W) = -P_1W_2(\nu)g(W_1, W), \tag{4.4.5}$$

for $W_1, W_2 \in \Gamma(TM)$ and $W \in \Gamma(D^\perp)$, where ν is a $C^\infty-$function on M such that $Z\nu = 0$, for all $Z \in \Gamma(D^\perp)$.

Proof. Let $M = M_T \times_f M_\perp$ be a semi-invariant warped product submanifold of Cosymplectic manifold $\bar{M}$. Then M_T and $M_\perp$ are totally geodesic and totally umbilical submanifold in M, respectively. Moreover, for $X \in \Gamma(D)$ and $Z \in \Gamma(D^{\perp})$, we have

$$\nabla_X Z = \nabla_Z X = (Xlnf)Z.$$

From the (4.2.8), we have

$$\begin{aligned}(\nabla_{W_1}F)W_2 &= (\nabla_{P_1W_1+P_2W_1+\eta(W_1)V}F)W_2\\ &= (\nabla_{P_1W_1}F)W_2 + (\nabla_{P_2W_1}F)W_2 + \eta(W_1)(\nabla_V F)W_2.\end{aligned}$$

Again by using (4.2.8), we infer

$$\begin{aligned}(\nabla_{W_1}F)W_2 = (\nabla_{P_1W_1}F)P_1W_2 &+ (\nabla_{P_1W_1}F)P_2W_2 + \eta(W_2)(\nabla_{P_1W_1}F)V + (\nabla_{P_2W_1}F)P_1W_2\\ &+ (\nabla_{P_2W_1}F)P_2W_2 + \eta(W_2)(\nabla_{P_2W_1}F)V + \eta(W_1)(\nabla_V F)P_1W_2\\ &+ \eta(W_1)(\nabla_V F)P_2W_2 + \eta(W_1)\eta(W_2)(\nabla_V F)V.\end{aligned}$$

In the view of (4.2.2), (4.2.3) and (4.2.11), this equation reduces to

$$(\nabla_{W_1}F)W_2 = (\nabla_{P_1W_1}F)P_1W_2 + (\nabla_{P_1W_1}F)P_2W_2 + (\nabla_{P_2W_1}F)P_1W_2 + (\nabla_{P_2W_1}F)P_2W_2.$$

On taking inner product with ϕW on both sides, we get

$$\begin{aligned}g((\nabla_{W_1}F)W_2, \phi W) &= g((\nabla_{P_1W_1}F)P_1W_2 + (\nabla_{P_1W_1}F)P_2W_2 + (\nabla_{P_2W_1}F)P_1W_2\\ &\quad + (\nabla_{P_2W_1}F)P_2W_2, \phi W),\end{aligned}$$

for any $W \in \Gamma(D^{\perp})$.

Using (4.2.9), (4.2.11) and the fact that $P_1W_1 \in \Gamma(D)$, $P_1W_1 \in \Gamma(D^{\perp})$, for any $W_1 \in \Gamma(TM)$, the above equation reduces to

$$\begin{aligned}g((\nabla_{W_1}F)W_2, \phi W) &= g(fh(P_1W_1, P_1W_2), \phi W) - g(h(P_1W_1, TP_1W_2), \phi W)\\ &\quad + g(fh(P_1W_1, P_2W_2), \phi W) + g(fh(P_2W_1, P_1W_2), \phi W)\\ &\quad + g(fh(P_2W_1, P_2W_2), \phi W) - g(h(P_2W_1, TP_1W_2), \phi W).\end{aligned}$$

From (4.2.1), we get

$$g(\nabla_{W_1}F)W_2, \phi W) = -g(h(P_1W_1, TP_1W_2) + h(P_2W_1, TP_1W_2, \phi W),$$

using (4.2.3), we obtain

$$g(\nabla_{W_1}F)W_2, \phi W) = -g(\bar{\nabla}_{P_1W_1}\phi P_1W_2, \phi W) - g(\bar{\nabla}_{P_2W_1}\phi P_1W_2, \phi W).$$

Using the covariant differentiation property of ϕ and the fact that $P_1W_2 \in \Gamma(D)$ and $P_2W_2 \in \Gamma(D^{\perp})$, for any $W_2 \in \Gamma(TM)$, then from (4.2.1), we obtain

$$g(\nabla_{W_1}F)W_2, \phi W) = g(P_1W_2, \bar{\nabla}_{P_1W_1}W) - g(\bar{\nabla}_{P_2W_1}P_1W_2, W).$$

Again using (4.2.3), we get

$$g((\nabla_{W_1}F)W_2, \phi W) = g(P_1W_2, \nabla_{P_1W_1}W) - g(\nabla_{P_2W_1}P_1W_2, W)$$

The first term of right-hand side of this equation is zero due to (4.4.1) and then using the fact that $P_1W_2 \in \Gamma(D)$ and $W \in \Gamma(D^{\perp})$, we obtain

$$\begin{aligned} g((\nabla_{W_1}F)W_2, \phi W) &= -g(P_1W_2(lnf)P_2W_1, W) \\ &= -P_1W_2(lnf)g(P_2W_1, W) \\ &= -P_1W_2(lnf)g(W_1, W) \\ &= -P_1W_2(\nu)g(W_1, W), \end{aligned}$$

which proves the necessary part.

Conversely, suppose that M be a semi-invariant submanifold of a Cosymplectic manifold satisfying (4.4.5). Then, it is easy to see that

$$g((\nabla_X F)Y, \phi W) = 0,$$

for each X, $Y \in \Gamma(D)$ and $W \in \Gamma(D^{\perp})$. By using (4.2.11), we have

$$g(h(X, \phi Y), \phi W) = 0.$$

Therefore by the Theorem 4.3.2 and Theorem 4.3.3, distribution D is integrable and its leaves are totally geodesic in M. Let $Z \in \Gamma(D^{\perp})$, by using (4.4.5), we obtain

$$g((\nabla_Z F)X, \phi W) = -X(\nu)g(Z, W)$$

Using (4.2.11), we have

$$g(h(\phi X, Z), \phi W) = X(\nu)g(Z, W). \tag{4.4.6}$$

Let $M_{\perp}$ be a leaf of $D^{\perp}$ and h' be the second fundamental form of the immersion of $M_{\perp}$ into M and ∇' is the induced connection on $M_{\perp}$. Then by the Gauss formula, we have

$$\nabla_Z W = \nabla'_Z W + h'(Z, W). \tag{4.4.7}$$

Now using (4.2.2) and (4.2.3), for any $Z, W \in \Gamma(D^{\perp})$ and $X \in \Gamma(D)$, we obtain

$$g(h(Z, X), \phi W) = g(\phi X, \nabla_Z W).$$

Using (4.4.7), we have

$$g(h(Z, X), \phi W) = g(h'(Z, W), \phi X). \tag{4.4.8}$$

By (4.4.6) and (4.4.8), we have

$$g(h'(Z, W), X) = -X(\nu)g(Z, W),$$

or

$$h'(Z, W) = -g(Z, W)\nabla\nu,$$

hence $M_{\perp}$ is totally umbilical in M with mean curvature vector $\nabla\nu$.

Moreover, the mean curvature is parallel on $M^{\perp}$, as $Z\nu = 0$ for all $Z \in \Gamma(D^{\perp})$.

This proves that $M_{\perp}$ is an extrinsic sphere. Hence, from a result of Hiepko [41], M is locally a warped product submanifold of $\bar{M}$. $\square$

We, now discuss squared norm of second fundamental form in terms of warping function. The main results are as under:

Theorem 4.4.4. *Let* $M = M_T \times_f M_\perp$ *be a semi-invariant warped product submanifold of a Cosymplectic manifold of* $\bar{M}$*. Then*

(a) $h_{\phi D^\perp}(\phi X, Z) = (Xlnf)\phi Z$.

(b) $g(h(\phi X, Z), \phi h(X, Z)) = \|h_\mu(X, Z)\|^2$, *for any* $X \in \Gamma(D)$ *and* $Z \in \Gamma(D^\perp)$.

Proof. Using Gauss formula, we have

$$h(\phi X, Z) = \phi\nabla_Z X + \phi h(X, Z) - \nabla_Z \phi X,$$

for any $X \in \Gamma(D)$ and $Z \in \Gamma(D^\perp)$. Further, by using (4.4.1), we get

$$h(\phi X, Z) = (Xlnf)\phi Z + \phi h(X, Z) - (\phi Xlnf)Z. \tag{4.4.9}$$

On equating the tangential components of above equation, we obtain

$$(\phi Xlnf)Z = \phi h(X, Z),$$

On taking inner product both sides with respect to $W \in \Gamma(D^\perp)$, we get

$$g(h(X, Z), \phi W) = (-\phi Xlnf)g(Z, W),$$

or equivalently

$$h_{\phi D^\perp}(X, Z) = (-\phi Xlnf)\phi Z.$$

Replacing X by ϕX, we obtain

$$h_{\phi D^\perp}(\phi X, Z) = (Xlnf)\phi Z,$$

which proves the part (a) of the theorem.

Now, on comparing the normal components of (4.4.9), we obtain

$$h(\phi X, Z) = (Xlnf)\phi Z + \phi h_\mu(X, Z),$$

or

$$h(\phi X, Z) - \phi h_\mu(X, Z) = (Xlnf)\phi Z.$$

On taking inner product on both sides with respect to $\phi h(X, Z)$ of above equation, we get

$$g(h(\phi X, Z), \phi h(X, Z)) = \|h_\mu(X, Z)\|^2,$$

which proves the part (b) of the Theorem. □

Theorem 4.4.5. *Let $M = M_T \times_f M_\perp$ be a semi-invariant warped product submanifold of a Cosymplectic manifold $\bar{M}$. Then*

(i) The squared norm of the seccond fundamental form satisfies

$$\|h\|^2 \geq 2q\|\nabla lnf\|^2,$$

where ∇lnf is the gradient of the function lnf and q is the dimension of anti-invariant distribution $M_\perp$.

(ii) If the equality holds identically, then M_T is totally geodesic submanifold of $\bar{M}$, $M_\perp$ is a totally umbilical submanifold of $\bar{M}$ and M is minimal.

Proof. Consider a local orthonormal frame of vector fields $\{X_1, X_2, ...X_p, X_{p+1} = \phi X_1, ..., X_{2p} = \phi X_p, X_{2p+1} = V\}$ on M_T and $\{Z_1, Z_2, ...Z_q\}$ on $M_\perp$. Then by using the definition of squared norm of mean curvature vector, we have

$$\begin{aligned} \|h\|^2 &= \sum_{i,j=1}^{2p+1} g(h(X_i, X_j), h(X_i, X_j)) + \sum_{i=1}^{2p+1}\sum_{r=1}^{q} g(h(X_i, Z_r), h(X_i, Z_r)) \\ &+ \sum_{r,s=1}^{q} g(h(Z_r, Z_s), h(Z_r, Z_s)) \qquad (4.4.10) \end{aligned}$$

or,

$$\|h\|^2 \geq \sum_{i=1}^{2p}\sum_{r=1}^{q} g(h(X_i, Z_r), h(X_i, Z_r)).$$

Taking into consideration part (a) of the Theorem 4.4.4, we get

$$\|h\|^2 \geq \sum_{i=1}^{2p}\sum_{r=1}^{q} (\phi X_i lnf)^2 g(Z_r, Z_r),$$

or

$$\|h\|^2 \geq 2q\|\nabla lnf\|^2.$$

which proves the part (i).

If the equality sign holds, then obviously from (4.4.10) and part (a) of the Theorem 4.4.4, we have

$$h(D, D) = 0, \quad h(D^{\perp}, D^{\perp}) = 0 \quad and \quad h(D, D^{\perp}) \in \Gamma(\phi D^{\perp}). \tag{4.4.11}$$

As M_T is a totally geodesic submanifold of M, the first condition of equation (4.4.11) implies that M_T is a totally geodesic in $\bar{M}$. Moreover, $M_{\perp}$ is a totally umbilical in M, the second condition of equation (4.4.11) implies that $M_{\perp}$ is a totally umbilical in $\bar{M}$, and also it clear from (4.4.11) that M is a minimal in $\bar{M}$. □

Chapter 5

Warped product slant lightlike submanifolds of indefinite Sasakian manifolds

5.1 Introduction

In the previous chapter, we have studied semi-invariant warped product submanifolds of a Cosymplectic manifold. In the present chapter, we study another important class of warped product submanifolds namely warped product slant lightlike submanifolds of indefinite Sasakian manifolds. Bishop and O'Neill [11] introduced the notion of warped product manifolds in order to construct a large variety of manifolds of negative curvature. From geometric point of view, this study got momentum, when the study of warped product of CR-submanifolds of K*ä*hler manifolds was introduced by Chen [25, 26]. Although, there are significant applications of warped product submanifolds in general theory of relativity, but a very limited specific information is available on its lightlike case. This motivated

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Contents of this chapter have been published in ***Balkan J. Geom. Appl.*** 20(1) (2015), 98-108. ***(SCI-indexed), Impact factor - 0.806*** the geometers to carry out work on the geometry of the warped product slant lightlike submanifolds.

The motive of this chapter is to explore warped product slant lightlike submanifolds of indefinite Sasakian manifolds.

In this chapter, results for the non-existence of warped product slant lightlike submanifolds of indefinite Sasakian manifolds have been carried out in section 5.2.

The basic definitions and results concerning to slant lightlike submanifolds of an indefinite Sasakian manifold have already been presented in Chapter 2.

5.2 Non-Existence of Warped Product Slant Lightlike Submanifolds

In this section, the warped product slant lightlike submanifolds of indefinite Sasakian manifolds have been explored. Concerning this, some of the basic concepts and results of warped product submanifolds have already been presented in Chapter 4.

Before presenting main results of this section, we recall some definitions required for the proofs of our main theorems.

Definition 5.2.1. [30] *Let* $(M, g, S(TM), S(TM^{\perp}))$ *be a lightlike submanifold of an indefinite Sasakian manifold* $\bar{M}$*, where characteristic vector field* V *is tangent to* M*. Then* M *is called a contact Screen Cauchy Riemann lightlike submanifold if:*

(i) There exist real non-null distributions $D \subset S(TM)$ *and* $D^{\perp}$ *such that*

$$S(TM) = D \oplus D^{\perp} \perp \{V\}, \quad \phi D^{\perp} \subset (S(TM^{\perp})), \quad D \cap D^{\perp} = \{0\}.$$

where $D^{\perp}$ is orthogonal complementary to $D \perp \{V\}$ in $S(TM)$.

(ii) The distributions D and $Rad(TM)$ are invariant with respect to ϕ.

A contact Screen Cauchy Riemann lightlike submanifold of an indefinite Sasakian manifold is Screen real or holomorphic lightlike submanifold if and only if $D = \{0\}$ or $D^{\perp} = \{0\}$, respectively.

Definition 5.2.2. [73] *A lightlike submanifold of an indefinite Kähler manifold $\bar{M}$ is called a transversal lightlike submanifold of $\bar{M}$ if $\bar{J}(Rad(TM)) = ltr(TM)$ and $\bar{J}(S(TM)) \subseteq S(TM^{\perp})$.*

Now onwards, let M_{θ} denote a proper slant lightlike submanifold, M_T as a holomorphic Screen Cauchy-Riemann lightlike submanifold and $M_{\perp}$ as a transversal lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$.

The main results of this section are the following theorems:

Theorem 5.2.3. *Let $\bar{M}$ be an indefinite Sasakian manifold. Then there does not exist warped product submanifold $M = M_{\theta} \times_f M_T$ of $\bar{M}$ such that M_{θ} is a proper slant lightlike submanifold of $\bar{M}$ and M_T is a holomorphic Screen Cauchy-Riemann (SCR) lightlike submanifold of $\bar{M}$.*

Proof. Let X, linearly independent of V, be tangent to $D \subset S(TM)$ of a holomorphic SCR-lightlike submanifold M_T and $Z \in \Gamma(D^{\theta})$ of a slant lightlike submanifold M_{θ}. Then by using (4.2.12), we have

$$g(\nabla_{\phi X} Z, X) = Z(ln\ f) g(\phi X, X) = 0.$$

Therefore using (1.3.6), (1.2.3) - (1.2.5) and (2.2.2), we get

$$\begin{aligned} 0 &= \bar{g}(\bar{\nabla}_{\phi X} Z, X) = -\bar{g}(\phi Z, \bar{\nabla}_{\phi X} \phi X) \\ &= \bar{g}(\bar{\nabla}_{\phi X} TZ, \phi X) - \bar{g}(FZ, \bar{\nabla}_{\phi X} \phi X) \\ &= \bar{g}(\nabla_{\phi X} TZ, \phi X) - \bar{g}(FZ, h^s(\phi X, \phi X)). \end{aligned}$$

Further by virtue of (4.2.12), we obtain

$$TZ(ln\ f)g(X,X) = \bar{g}(h^s(\phi X, \phi X), FZ).$$

Use of polarization identity implies

$$TZ(ln\ f)g(X,Y) = \bar{g}(h^s(\phi X, \phi Y), FZ), \tag{5.2.1}$$

for any X, Y, linearly independent of V, tangent to $D \subset S(TM)$ of a holomorphic SCR-lightlike submanifold M_T and $Z \in \Gamma(D^\theta)$ of a slant lightlike submanifold M_θ. On the other hand, using (1.3.9) and (4.2.12), we have

$$\begin{aligned} g(A_{FZ}\phi X, \phi Y) &= -g(\bar{\nabla}_{\phi X} FZ, \phi Y) = \bar{g}(Z, \bar{\nabla}_{\phi X} Y) - \bar{g}(TZ, \bar{\nabla}_{\phi X}\phi Y) \\ &= -\bar{g}(\bar{\nabla}_{\phi X} Z, Y) + \bar{g}(\bar{\nabla}_{\phi X} TZ, \phi Y) \\ &= -Z(ln\ f)g(\phi X, Y) + TZ(ln\ f)g(X,Y). \end{aligned}$$

Now, using (1.3.10), we have $\bar{g}(h^s(\phi X, \phi Y), FZ) = g(A_{FZ}\phi X, \phi Y)$, therefore we obtain

$$\bar{g}(h^s(\phi X, \phi Y), FZ) = -Z(ln\ f)g(\phi X, Y) + TZ(ln\ f)g(X,Y). \tag{5.2.2}$$

Thus, (5.2.1) and (5.2.2) imply that $Z(ln\ f)g(\phi X, Y) = 0$, for any X, Y, linearly independent of V, tangent to $D \subset S(TM)$ of a holomorphic SCR-lightlike submanifold M_T and $Z \in \Gamma(D^\theta)$ of a slant lightlike submanifold M_θ. Since $M_T \neq \{0\}$ is a Riemannian and invariant, therefore, we obtain

$$Zln\ f = 0,$$

this shows that, f is constant. Hence, the proof is complete. □

Theorem 5.2.4. *Let $\bar{M}$ be an indefinite Sasakian manifold. Then there does not exist warped product submanifold $M = M_T \times_f M_\theta$ in $\bar{M}$ such that M_T is a holomorphic SCR-lightlike submanifold and M_θ is a proper slant lightlike submanifold of $\bar{M}$.*

Proof. Let X, linearly independent of V, be tangent to $D \subset S(TM)$ of a holomorphic SCR-lightlike submanifold M_T and $Z \in \Gamma(D^\theta)$ of a slant lightlike submanifold M_θ. Then using (4.2.12), we have $g(\nabla_{TZ}X, Z) = X(ln\ f)g(TZ, Z) = 0$. This further using with (1.3.9), (1.3.10) and (2.3.6) implies that

$$\begin{aligned}
0 &= \bar{g}(\bar{\nabla}_{TZ}X, Z) = -\bar{g}(\phi X, \bar{\nabla}_{TZ}TZ) - \bar{g}(\phi X, \bar{\nabla}_{TZ}FZ) \\
&= g(\nabla_{TZ}\phi X, TZ) + g(\phi X, A_{FZ}TZ) \\
&= g(\nabla_{TZ}\phi X, TZ) + g(h^s(\phi X, TZ), FZ) \\
&= \phi X(ln\ f)g(TZ, TZ) + \bar{g}(h^s(\phi X, TZ), FZ) \\
&= \phi X(ln\ f).cos^2\theta g(Z, Z) + \bar{g}(h^s(\phi X, TZ), FZ).
\end{aligned}$$

Replace X by ϕX, we get

$$X(ln\ f)cos^2\theta g(Z, Z) + \bar{g}(h^s(X, TZ), FZ) = 0. \tag{5.2.3}$$

After replacing Z by TZ and then by using (2.2.15) and (2.3.6), we obtain

$$\bar{g}(h^s(X, Z), FTZ) = X(ln\ f)cos^2\theta g(Z, Z).$$

On the other hand, use of (1.3.6), (2.2.2), (2.2.15), (2.3.6) and (4.2.12), for any X, linearly independent of V, tangent to $D \subset S(TM)$ of a holomorphic SCR-lightlike submanifold M_T and $Y, Z \in \Gamma(D^\theta)$ of a slant lightlike submanifold M_θ, implies

$$\begin{aligned}
\bar{g}(h^s(TZ, X), FY) &= -\bar{g}(TZ, \bar{\nabla}_X\phi Y) + \bar{g}(TZ, \bar{\nabla}_X TY) \\
&= \bar{g}(T^2Z, \bar{\nabla}_X Y) + \bar{g}(FTZ, \bar{\nabla}_X Y) + \bar{g}(TZ, \nabla_X TY) \\
&= -cos^2\theta X(ln\ f)g(Z, Y) + \bar{g}(FTZ, h^s(X, Y)) \\
&\quad + X(ln\ f)g(TZ, TY) \\
&= \bar{g}(FTZ, h^s(X, Y)).
\end{aligned}$$

Put $Y = Z$, we get

$$\bar{g}(h^s(TZ, X), FZ) = \bar{g}(FTZ, h^s(X, Z)). \tag{5.2.4}$$

Thus, from (5.2.3) and (5.2.4), we have

$$X(ln\ f)cos^2\theta g(Z,Z)=0.$$

Since D^θ is a proper slant and Z is non-null, therefore $X(ln\ f)=0$. This proves our assertion. □

Theorem 5.2.5. *Let $\bar{M}$ be an indefinite Sasakian manifold. Then there does not exist warped product submanifold $M=M_\perp\times_f M_\theta$ of $\bar{M}$ such that $M_\perp$ is a transversal lightlike submanifold and M_θ is a proper slant lightlike submanifold of $\bar{M}$.*

Proof. Let $Z\in\Gamma(D^\theta)$ of a slant lightlike submanifold M_θ and X independent of V and tangent to $S(TM)$ of a transversal lightlike submanifold $M_\perp$. Then by using (1.3.9), (1.2.3) - (1.2.5), (2.2.2), (2.3.6) and (4.2.12), we have

$$\begin{aligned} g(A_{\phi X}TZ,Z) &= \bar{g}(\bar{\nabla}_{TZ}X,\phi Z)=g(\nabla_{TZ}X,TZ)+\bar{g}(h^s(TZ,X),FZ)\\ &= X(ln\ f)g(TZ,TZ)+\bar{g}(h^s(TZ,X),FZ)\\ &= X(ln\ f)cos^2\theta g(Z,Z)+\bar{g}(h^s(TZ,X),FZ). \end{aligned}$$

Using (1.3.10) in the left hand side of above equation, we obtain

$$\bar{g}(h^s(TZ,Z),\phi X)=X(ln\ f)cos^2\theta g(Z,Z)+\bar{g}(h^s(TZ,X),FZ). \tag{5.2.5}$$

Replace Z by TZ in (5.2.5) and then by using (2.2.15) and (2.3.6), we get

$$\bar{g}(h^s(Z,TZ),\phi X)=-X(ln\ f)cos^2\theta g(Z,Z)+\bar{g}(h^s(Z,X),FTZ).$$

On the other hand, using (1.3.9), (1.2.3)-(1.2.5), (2.2.2), (2.2.15) and (4.2.12), we have

$$\begin{aligned} g(A_{FZ}X, TZ) &= -\bar{g}(\bar{\nabla}_X FZ, TZ) = \bar{g}(\bar{\nabla}_X Z, \phi TZ) + \bar{g}(\bar{\nabla}_X TZ, TZ) \\ &= \bar{g}(\bar{\nabla}_X Z, T^2 Z) + \bar{g}(\bar{\nabla}_X Z, FTZ) + g(\nabla_X TZ, TZ) \\ &= -cos^2\theta g(\nabla_X Z, Z) + \bar{g}(h^s(X, Z), FTZ) + X(ln\ f)g(TZ, TZ) \\ &= -cos^2\theta X(ln\ f)g(Z, Z) + \bar{g}(h^s(X, Z), FTZ) \\ &\quad +X(ln\ f)cos^2\theta g(Z, Z) \\ &= \bar{g}(h^s(X, Z), FTZ). \end{aligned}$$

Hence by making use of (1.3.10), we obtain

$$\bar{g}(h^s(TZ, X), FZ) = \bar{g}(h^s(X, Z), FTZ). \tag{5.2.6}$$

Thus, (5.2.5) and (5.2.6), implies that

$$2X(ln\ f)cos^2\theta g(Z, Z) = 0.$$

Since D^θ is Riemannian and M_θ is a proper slant lightlike submanifold, therefore we have $X(ln\ f) = 0$. Hence, f is constant and the assertion follows. □

Remark: *From the Theorem 5.2.3, Theorem 5.2.4 and Theorem 5.2.5, it is clear that there do not exist warped product lightlike submanifolds of the following forms*
$M = M_\theta \times_f M_T$
$M = M_T \times_f M_\theta$
$M = M_\perp \times_f M_\theta$
Now onwards, we call $M = M_\theta \times_f M_\perp$ *as a warped product slant lightlike submanifold, where* M_θ *is a proper slant lightlike submanifold and* $M_\perp$ *is a transversal lightlike submanifold of an indefinite Sasakian manifold* $\bar{M}$.

Theorem 5.2.6. *Let* $M = M_\theta \times_f M_\perp$ *be a warped product slant lightlike submanifold of an indefinite Sasakian manifold* $\bar{M}$ *such that* $M_\perp$ *is a transversal lightlike* *submanifold and* M_θ *is a proper slant lightlike submanifold of* $\bar{M}$*. Then*

$$g(h^s(X,Y),\bar{J}Z) = -TX(ln\ f)g(Y,Z),$$

for any $X \in \Gamma(D^\theta)$ *of a slant lightlike submanifold* M_θ *and* Y, Z *are independent of* V *and tangent to* $S(TM)$ *of transversal lightlike submanifold* $M_\perp$*.*

Proof. For any $X \in \Gamma(D^\theta)$ of a slant lightlike submanifold M_θ and Y, Z are independent of V, tangent to $S(TM)$ of transversal lightlike submanifold $M_\perp$, using (1.3.9), (1.2.3)-(1.2.5) and (2.2.2), we have

$$g(h^s(TX,Y),\phi Z) = g(\bar{\nabla}_Y TX,\phi Z) = g(\nabla_Y X, Z) + g(\bar{\nabla}_Y \phi FX, Z).$$

Since $F(D^\theta) \subset S(TM^\perp)$, by taken into consideration Lemma 2.2.4 and μ is invariant, therefore using (2.2.3), we have $\phi FX = BFX$ and $CFX = 0$. Hence using (2.2.16) and (4.2.12), we obtain

$$g(h^s(TX,Y),\phi Z) = X(ln\ f)g(Y,Z) - sin^2\theta g(\bar{\nabla}_Y X, Z).$$

Again using (4.2.12), we have

$$g(h^s(TX,Y),\phi Z) = (1 - sin^2\theta)X(ln\ f)g(Y,Z) = cos^2\theta X(ln\ f)g(Y,Z).$$

Replacing X by TX and then using (2.2.15), the assertion follows. □

Chapter 6

Slant lightlike submersions from an indefinite almost Hermitian manifold onto a lightlike manifold

6.1 Introduction

Immersions and submersions, which are the special tools in Differential Geometry, also play a fundamental role in Riemannian Geometry.

Let (B, g_B) and (F, g_F) be two C^∞-Riemannian manifolds of dimension m and n, respectively. A surjective C^∞-map $f : (B, g_B) \to (F, g_F)$ is a C^∞-*submersion* [32] if it has maximal rank at any point of B, i.e., each derivative map f_* of f is onto.

Let $x \in F, f^{-1}(x)$ is a k-dimensional submanifold of B called a *fiber*, where $k = m - n$

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Putting $\mathcal{V}_p = ker f_{*p}$, for any $p \in B$, we obtain an integrable distribution $\mathcal{V}$ which corresponds to the foliation of B determined by the fibres of f, since each $\mathcal{V}_p$, coincides with the tangent space of $f^{-1}(x)$ at p, $f(p) = x$. Each $\mathcal{V}_p$ is called the *vertical space* at p, $\mathcal{V}$ is the vertical distribution and the sections of $\mathcal{V}$ are called *vertical vector fields.*

Let $\mathcal{H}$ be the complementary distribution of $\mathcal{V}$ determined by the Riemannian metric g_B. So, at any $p \in B$, one has the orthogonal decomposition

$$T_p(B) = \mathcal{V}_p \oplus \mathcal{H}_p$$

$\mathcal{H}_p$ is called the *horizontal space* at p. The sections of the horizontal distribution $\mathcal{H}$ are the *horizontal vector fields.*

For any $E \in \chi(B)$, vE and hE denote the vertical and the horizontal components of E, respectively.

A C^∞-submersion $f : (B, g_B) \to (F, g_F)$ is called a Riemannian submersion [32] if, at each point p of B, f_{*p} preserves the length of the horizontal vectors.

A vector field X on B is called *basic* if X is horizontal and f-related to a vector field X_* on F, i.e., $f_* X_x = X_* f(x)$ for all $x \in B$.

Thus $X \to X_*$ is a one-to-one correspondence between basic vector fields on B and arbitrary vector fields on B.

The following lemma due to [32] is well known.

Lemma 6.1.1. [32] *Let $f : (B, g_B) \to (F, g_F)$ be a Riemannian submersion, and denote by ∇ and $\bar{\nabla}$ the Levi-Civita connections of B and F, respectively. If X, Y are basic vector fields, f-related to X_*, Y_*, then:*

(i) $g_B(X, Y) = g_F(X_*, Y_*)\ o\ f$,

(ii) $h[X, Y]$ is the basic vector field f-related to $[X_*, Y_*]$

(iii) $h(\nabla_X Y)$ *is the basic vector field* f*-related to* $\bar{\nabla}_{X_*} Y_*$

(iv) for any vertical vector field V*,* $[X, V]$ *is vertical.*

Riemannian submersions between Riemannian manifolds have been studied by O'Neill [61] and Gray [35]. Later Watson [89] defined almost Hermitian submersions between almost Hermitian manifolds. Semi-Riemannian submersions were introduced by O'Neill [62] in 1983. As it is known that, when B and F are Riemannian manifolds then the fibers of the submersion f are always Riemannian manifolds. However, when the manifolds are semi-Riemannian manifolds then the fibers may not be semi-Riemannian manifolds. Therefore, Sahin [75] introduced a screen lightlike submersion from a lightlike manifold onto a semi-Riemannian manifold. Further, Sahin and Gündüzalp [79] introduced a lightlike submersion from a semi-Riemannian manifold onto a lightlike manifold.

Almost Hermitian submersions between almost Hermitian manifolds were defined by Watson [89]. As a generalization of almost Hermitian submersions, Sahin [78] introduced slant submersions from almost Hermitian manifolds onto Riemannian manifolds. It is well-known that semi-Riemannian submersions are of interest in physics, owing to their applications in the Yang-Mills theory, Kaluza-Klein theory, supergravity and superstring theories. (for details see (Bourguignon and Lawson [15, 16], Falcitelli [32], Visinescu [88])).

Motivated from the definition of Slant Submersions and significant applications of Riemannian Submersions in mathematical physics, the theory of lightlike submersions have been clubbed with slant submersions.

The main objective of this chapter is to introduce a slant lightlike submersion from an indefinite Hermitian manifold onto a lightlike manifold and to investigate the various geometric aspects of slant lightlike submersion.

The contents of this chapter are divided into two sections. Section 6.2, deals with fundamental notations and equations of lightlike submersions required in our subsequent discussions. In section 6.3, slant lightlike submersions from an indefinite almost Hermitian manifold onto a lightlike manifold has been introduced. The geometry of foliations of slant lightlike submersions have also been investigated in this section.

6.2 Lightlike Submersions

Let (B, g_B) be a semi-Riemannian manifold and (F, g_F) an r-lightlike manifold. Consider $f : B \to F$ a smooth submersion, then $f^{-1}(q)$ is a submanifold of B having dimension as dim$B$ - dim$F$, for $q \in F$. Then kernel of the derivative map f_* at the point p is given by (for details see [79])

$$Kerf_* = \{X \in T_p(B) : f_*(X) = 0\},$$

and $(Kerf_*)^\perp$ is given by

$$(Kerf_*)^\perp = \{Y \in T_p(B) : g_B(Y, X) = 0, \forall X \in Kerf_*\}.$$

Since $T_p(B)$ is a semi-Riemannian manifold, therefore $(Kerf_*)^\perp$ may not be complementary to $Kerf_*$, so assume $\triangle = Kerf_* \cap (Kerf_*)^\perp \neq \{0\}$. Then the following four cases arise for submersions.

Case 1. When $0 < dim\triangle < min\{dim(Kerf_*)^\perp\ , dim(Kerf_*)\}$:

In this case, $\triangle$ is the null subspace (or called radical subspace) of $T_p(B)$. The non-degenerate complementary subspace of $\triangle$ in $Kerf_*$ is denoted by $S(Kerf_*)$, therefore we have

$$Kerf_* = \triangle \perp S(Kerf_*),$$

and

$$(Kerf_*)^\perp = \triangle \perp S(Kerf_*)^\perp,$$

where $S(Kerf_*)^\perp$ is a complementary subspace of $\triangle$ in $(Kerf_*)^\perp$. The non-degeneracy of $S(Kerf_*)^\perp$ in $T_p(B)$ implies that

$$T_p(B) = S(Kerf_*)\perp(S(Kerf_*))^\perp,$$

where $(S(Kerf_*))^\perp$ is the complementary subspace of $S(Kerf_*)$ in $T_p(B)$. Since $S(Kerf_*)$ and $(S(Kerf_*))^\perp$ are non-degenerate, therefore

$$(S(Kerf_*))^\perp = S(Kerf_*)^\perp\perp(S(Kerf_*)^\perp)^\perp.$$

Then from [27], we can construct a quasi-orthonormal basis of T_PB along $Kerf_*$ such that

$$g(N_i, N_j) = g(\xi_i, \xi_j) = 0, \quad g(\xi_i, N_j) = \delta_{ij},$$

$$g(W_\alpha, N_j) = g(W_\alpha, \xi_j) = 0, \quad g(W_\alpha, W_\beta) = \epsilon_\alpha \delta_{\alpha\beta},$$

where $\{\xi_i\}$, $\{N_i\}$ and $\{W_\alpha\}$ are the basis of $\triangle$, $(S(Kerf_*)^\perp)^\perp$ and $S(Kerf_*)^\perp$ respectively. Denote the set of vector fields $\{N_i\}$ by $ltr(Kerf_*)$ and consider

$$tr(Kerf_*) = ltr(Kerf_*)\perp S(Kerf_*)^\perp.$$

Let $\mathcal{V} = Kerf_*$ denotes the vertical space of $T_p(B)$ and $\mathcal{H} = tr(Kerf_*)$ denotes the horizontal space of $T_p(B)$.

Therefore $T_p(B)$ can be written as

$$T_p(B) = \mathcal{V}_p \oplus \mathcal{H}_p.$$

Considering above details, Sahin [79] introduced lightlike submersion from a semi-Riemannian manifold to a lightlike manifold is defined as below.

Definition 6.2.1. [79] *Let* (B, g_B) *and* (F, g_F) *be a semi-Riemannian manifold and an* r*-lightlike manifold, respectively. Let* $f : B \to F$ *be a submersion such that* *(i)* $dim\triangle = dim\{(Kerf_*) \cap (Kerf_*)^{\perp}\} = r$, *where* $0 < r < min\{dim(Kerf_*^{\perp}),$ $dim(Kerf_*)\}$.

(ii) f_* *preserves the length of horizontal vectors, that is,*

$$g_B(X, Y) = g_F(f_*X, f_*Y), \quad for \quad X, Y \in \Gamma(\mathcal{H})$$

Then f *is called an* r*-lightlike submersion.*

The other cases of Submersions are as follows:

Case 2. When $dim\triangle = dim(Kerf_*) < dim(Kerf_*)^{\perp}$. Then $\mathcal{V} = \triangle$ and $\mathcal{H} = S(Kerf_*)^{\perp} \perp ltr(Kerf_*)$ and f is called an isotropic submersion.

Case 3. When $dim\triangle = dim(Kerf_*)^{\perp} < dim(Kerf_*)$. Then $\mathcal{V} = S(Kerf_*) \perp \triangle$ and $\mathcal{H} = ltr(Kerf_*)$ and f is called a co-isotropic submersion.

Case 4. When $dim\triangle = dim(Kerf_*) = dim(Kerf_*)^{\perp}$. Then $\mathcal{V} = \triangle$ and $\mathcal{H} = ltr(Kerf_*)$ and f is called a totally lightlike submersion.

We, now prove the following theorem which is useful to define slant lightlike Submersion from an indefinite almost Hermitian manifold to a lightlike manifold.

Theorem 6.2.2. *Let* $f : B \to F$ *be an* r*-lightlike submersion from an indefinite almost Hermitian manifold* $(B, g_B, \bar{J}_1)$ *where* g_B *is a semi-Riemannian metric of index* $2r$ *to an* r*-lightlike manifold* (F, g_F)*. Let* $\bar{J}\triangle$ *be a distribution on* M *such that* $\bar{J}\triangle \cap \triangle = 0$*. Then any complementary distribution to* $\bar{J}\triangle \oplus \bar{J}ltr(Kerf_*)$ *in* $S(Kerf_*)$ *is Riemannian.*

Proof. Let $\bar{J}ltr(Kerf_*)$ is invariant with respect to $\bar{J}$, therefore $1 = g(N, \xi) = g(\bar{J}N, \bar{J}\xi) = 0$, for any $\xi \in \Gamma(Rad(Kerf_*))$ and $N \in \Gamma(ltr(Kerf_*))$, which leads to a contradiction as $g(N, \xi) = 1$.

Also $\bar{J}ltr(Kerf_*)$ does not belong to $S(Kerf_*)^{\perp}$, since $S(Kerf_*)^{\perp}$ is orthogonal to $S(Kerf_*)$, therefore $0 = g(\bar{J}\xi, \bar{J}N) = g(\xi, N) = 1$. Thus $\bar{J}ltr(Kerf_*)$ is distribution on M.

Moreover, $\bar{J}ltr(Kerf_*)$ does not belong to $\triangle$, if $\bar{J}N \in \Gamma(\triangle)$, then $\bar{J}^2N = -N \in \Gamma(\bar{J}\triangle)$, which is a contradiction.

Similarly, $\bar{J}ltr(Kerf_*)$ does not belong to $\bar{J}\triangle$. Hence $\bar{J}ltr(Kerf_*) \subset S(Kerf_*)$ such that $\bar{J}\triangle \cap \bar{J}ltr(Kerf_*) = \{0\}$.

Let D be a distribution complementary to $\bar{J}\triangle \oplus \bar{J}ltr(Kerf_*)$ in $S(Kerf_*)$. Then, for local quasi orthonormal frames on B, $\{\xi_1, ..., \xi_r, \bar{J}\xi_1, ..., \bar{J}\xi_r, N_1, ..., N_r, \bar{J}N_1, ..., \bar{J}N_r\}$ form orthonormal basis of $\triangle \oplus \bar{J}\triangle \oplus ltr(Kerf_*) \oplus \bar{J}ltr(kerf_*)$. Now define $\{U_1, ..., U_{2r}, V_1, ..., V_{2r}\}$ as

$$
\begin{array}{ll}
U_1 = \frac{1}{\sqrt{2}}(\xi_1 + N_1) & U_2 = \frac{1}{\sqrt{2}}(\xi_1 - N_1) \\
U_3 = \frac{1}{\sqrt{2}}(\xi_2 + N_2) & U_4 = \frac{1}{\sqrt{2}}(\xi_2 - N_2) \\
... & ... \\
... & ... \\
U_{2r-1} = \frac{1}{\sqrt{2}}(\xi_r + N_r) & U_{2r} = \frac{1}{\sqrt{2}}(\xi_r - N_r)
\end{array}
$$

$$
\begin{array}{ll}
V_1 = \frac{1}{\sqrt{2}}(\bar{J}\xi_1 + \bar{J}N_1) & V_2 = \frac{1}{\sqrt{2}}(\bar{J}\xi_1 - \bar{J}N_1) \\
V_3 = \frac{1}{\sqrt{2}}(\bar{J}\xi_2 + \bar{J}N_2) & V_4 = \frac{1}{\sqrt{2}}(\bar{J}\xi_2 - \bar{J}N_2) \\
... & ... \\
... & ... \\
V_{2r-1} = \frac{1}{\sqrt{2}}(\bar{J}\xi_r + \bar{J}N_r) & V_{2r} = \frac{1}{\sqrt{2}}(\bar{J}\xi_r - \bar{J}N_r).
\end{array}
$$

Hence $Span\{\xi_i, \bar{J}\xi_i, N_i, \bar{J}N_i\}$, i.e, $\bar{J}\triangle \oplus \triangle \oplus \bar{J}ltr(Kerf_*) \oplus ltr(Kerf_*)$ is a non-degenerate space of constant index $2r$ on B.

Now, since

$$ind(TB) = ind(ltr(Kerf_*) \oplus \triangle) + ind(\bar{J}ltr(Kerf_*) \oplus \bar{J}\triangle) + ind(D),$$

therefore, $2r = 2r + index(S(Kerf_*^{\perp}) \perp D)$.

Hence, $S(Kerf_*^{\perp}) \perp D$ or D is Riemannian. □

6.3 Slant Lightlike Submersion

In this section, we introduce slant lightlike submersions from an almost Hermitian manifold onto a lightlike manifold by using the Theorem 6.2.2 and the definition of slant lightlike submanifolds of indefinite Hermitian manifolds (see Definition 1.4.2) of chapter 1 due to Sahin [74].

Definition 6.3.1. *Let $(B^{2m}, g_B, \bar{J})$ be an indefinite almost Hermitian manifold with semi-Riemannian metric g_B of constant index $2r$, $0 < r < m$ and (F, g_F) be an r-lightlike submanifold. If $f : B \to F$ be an r-lightlike submersion, then it is called slant lightlike submersion if:*

(a) $\bar{J}\triangle$ is a distribution in $Kerf_*$ such that $\triangle \cap \bar{J}\triangle = \{0\}$.

(b) The angle $\theta_p(X)$ between D and $\bar{J}X$ is constant, for non zero vector field X tangent to D, where distribution D is complementary to $\bar{J}\triangle \oplus \bar{J}ltr(Kerf_*)$ in $S(Kerf_*)$.

From the Definition 6.3.1, we have the following decomposition.

$$\begin{aligned} T_pB &= \mathcal{V}_p \oplus \mathcal{H}_p \\ &= \{\triangle \perp (\bar{J}\triangle \oplus \bar{J}ltr(kerf_*) \perp D\} \oplus \{f(D) \perp \mu \perp ltr(Kerf_*)\} \end{aligned}$$

where μ is orthogonal complementary subbundle to $f(D)$ in $S(Kerf_*)$. Then any $X \in \mathcal{V}_p$ can be written as

$$\bar{J}X = \phi X + \omega X, \qquad (6.3.1)$$

where ϕX and ωX are the tangential and the transversal components of $\bar{J}X$ respectively. Similarly any $V \in \mathcal{H}_p$, $\bar{J}V$ can be expressed as

$$\bar{J}V = BV + CV, \tag{6.3.2}$$

where BV and CV are the tangential and the transversal components of $\bar{J}V$ respectively.

Let π_1, π_2, π_3 and π_4 the projection morphisms from TB to $\triangle$, $\bar{J}\triangle$, $\bar{J}ltr(Kerf_*)$ and D respectively. Then

$$X = \pi_1 X + \pi_2 X + \pi_3 X + \pi_4 X, \tag{6.3.3}$$

for any $X \in \mathcal{V}_p$.

Applying $\bar{J}$ to (6.3.3), we obtain

$$\bar{J}X = \bar{J}\pi_1 X + \bar{J}\pi_2 X + \phi\pi_4 X + \omega\pi_4 X + \omega\pi_3 X, \tag{6.3.4}$$

for any $X \in \mathcal{V}_p$. Then clearly

$$\bar{J}\pi_1 X = \phi\pi_1 X \in \Gamma(\bar{J}\triangle), \quad \bar{J}\pi_2 X = \phi\pi_2 X \in \Gamma(\triangle), \quad \omega\pi_1 X = 0, \quad \omega\pi_2 X = 0,$$

$$\phi\pi_4 X \in \Gamma(D), \quad \omega\pi_4 X \in \Gamma(f(D)), \quad \phi\pi_3 X = 0, \quad \omega\pi_3 X \in \Gamma(ltr(Kerf_*)).$$

Therefore, we can write

$$\phi X = \phi\pi_1 X + \phi\pi_2 X + \phi\pi_4 X. \tag{6.3.5}$$

Since the geometry of Riemannian submersions is characterized by O'Neill's tensors $\mathcal{T}$ and $\mathcal{A}$, therefore Sahin [79] defined these tensors for lightlike submersions as

$$\mathcal{T}_X Y = h\nabla_{\nu X}\nu Y + \nu\nabla_{\nu X} hY, \tag{6.3.6}$$

$$\mathcal{A}_X Y = \nu\nabla_{hX} hY + h\nabla_{hX}\nu Y, \tag{6.3.7}$$

where ∇ is the Levi-Civita connection of g_B.

Since in lightlike submersions the vertical and the horizontal subspaces are not orthogonal, therefore $\mathcal{T}$ and $\mathcal{A}$ are not skew-symmetric contrary to the Riemannian Submersions case where $\mathcal{T}$ and $\mathcal{A}$ are symmetric. $\mathcal{T}$ and $\mathcal{A}$ both reverse the horizontal and vertical subspaces and moreover $\mathcal{T}$ has symmetry property, that is

$$\mathcal{T}_X Y = \mathcal{T}_Y X. \tag{6.3.8}$$

Using (6.3.6) and (6.3.7), we have the following lemma which will be used in the proof of our main results.

Lemma 6.3.2. *Let $f : B \to F$ be a slant lightlike submersion, where B and F are as in the Definition 6.3.1. Then*

(i) $\nabla_U V = \mathcal{T}_U V + \nu \nabla_U V$,

(ii) $\nabla_V X = h\nabla_V X + \mathcal{T}_V X$,

(iii) $\nabla_X V = \mathcal{A}_X V + \nu \nabla_X V$,

(iv) $\nabla_X Y = h\nabla_X Y + \mathcal{A}_X Y$,

for any $U, V \in \Gamma(Kerf_*)$ and $X, Y \in \Gamma(tr(Kerf_*))$.

Moreover, using the Lemma 6.3.2 along with (6.3.1) and (6.3.2), gives the following lemma:

Lemma 6.3.3. *Let $f : B \to F$ be a slant lightlike submersion. Then*

$$(\nabla_X \omega)Y = C\mathcal{T}_X Y - \mathcal{T}_X \phi Y, \quad (\nabla_X \phi)Y = B\mathcal{T}_X Y - \mathcal{T}_X \omega Y, \tag{6.3.9}$$

where

$$(\nabla_X \omega)Y = h\nabla_X \omega Y - \omega \nabla_X Y, \quad (\nabla_X \phi)Y = \mathcal{V}\nabla_X \phi Y - \phi \mathcal{V}\nabla_X Y,$$

for any $X, Y \in (ker f_*)$.

We now, prove two characterization theorems for the existence of slant lightlike submersions.

Theorem 6.3.4. *Let* f *be a lightlike submersion from an indefinite almost Hermitian manifold* $(B, g_B, \bar{J})$, *where* g_B *is a semi-Riemannian metric of index* $2r$, *onto an* r*-lightlike manifold* (F, g_F). *Then* f *is a proper slant lightlike submersion if and only if*

- *(i)* $\bar{J}(ltr(Ker f_*))$ *is a distribution on* B.
- *(ii) for any* $X \in \Gamma(Ker f_*)$ *there exists a constant* $\lambda \in [-1, 0]$ *such that*

$$\phi^2 \pi_4 X = \lambda \pi_4 X. \tag{6.3.10}$$

Moreover, in this case, $\lambda = -cos^2\theta$.

Proof. Let f be a slant lightlike submersion, then $\bar{J}\triangle$ is a distribution on $S(Ker f_*)$. Thus from Theorem 6.2.2, $\bar{J}(ltr(Ker f_*))$ is a distribution on B.

Now, the slant angle $\theta(\pi_4 X)$ between $\bar{J}\pi_4 X$ and D_p is constant and given by

$$\cos\theta(\pi_4 X) = \frac{g(\bar{J}\pi_4 X, \phi\pi_4 X)}{|\bar{J}\pi_4 X||\phi\pi_4 X|} = -\frac{g(\pi_4 X, \phi^2\pi_4 X)}{|\pi_4 X||\phi\pi_4 X|}. \tag{6.3.11}$$

On the other hand, $\cos\theta(\pi_4 X)$ is also given by

$$\cos\theta(\pi_4 X) = \frac{|\phi\pi_4 X|}{|\bar{J}\pi_4 X|}. \tag{6.3.12}$$

By using (6.3.11) and (6.3.12), we get

$$\cos^2\theta(\pi_4 X) = -\frac{g(\pi_4 X, \phi^2\pi_4 X)}{|\pi_4 X|^2}. \tag{6.3.13}$$

Since, the angle $\theta(\pi_4 X)$ is constant on D, therefore $\phi^2\pi_4 X = \lambda\pi_4 X$, where $\lambda = -\cos^2\theta$. Hence the assertion (ii) follows.

Conversely, (i) implies that $\bar{J}\triangle$ is a distribution on $S(Kerf_*)$. Hence by using Theorem 6.2.2, any complementary distribution to $\bar{J}\triangle \oplus \bar{J}ltr(Kerf_*)$ in $S(Kerf_*)$ is Riemannian. This completes the proof. $\square$

Corollary 6.3.5. *Let $f : B \to F$ be a proper slant lightlike submersion with slant angle θ. Then*

$$g_B(\phi X, \phi Y) = cos^2\theta g_B(X, Y)$$

$$g_B(\omega X, \omega Y) = sin^2\theta g_B(X, Y). \tag{6.3.14}$$

for all $X, Y \in \Gamma(Kerf_*)$

Theorem 6.3.6. *Let $f : B \to F$ be a lightlike submersion. Then it called a proper slant lightlike submersion if and only if*

- *(i) $\bar{J}(ltr(Kerf_*))$ is a distribution on B.*
- *(ii) there exists a constant $\nu \in [-1, 0]$ such that*

$$B\omega\pi_4 X = \nu\pi_4 X, \tag{6.3.15}$$

  for any vector field X tangent to B, where $\nu = -\sin^2\theta$.

Proof. Let f be a slant lightlike submersion. Then $\bar{J}(ltr(kerf_*))$ is a distribution on B.

Applying $\bar{J}$ to (6.3.4) and using (6.3.1)-(6.3.3), we infer

$$-X = -\pi_1 X - \pi_2 X + \phi^2\pi_4 X + \omega\phi\pi_4 X + B\omega\pi_3 X + B\omega\pi_4 X.$$

Comparing the components of the distribution D on both sides of this equation, we get

$$-\pi_4 X = \phi^2 \pi_4 X + B\omega\pi_4 X, \tag{6.3.16}$$

On making use of (6.3.10), we get (6.3.15). Hence the assertion (ii) follows.

Conversely, using (6.3.15) and (6.3.16), we have

$$\phi^2 \pi_4 X = -\cos^2\theta \pi_4 X.$$

Hence proof follows from the Theorem 6.3.4. □

We, now present a characterization theorem for the transversal component ωX of $\bar{J}X$ to be parallel. Further, we establish a necessary and sufficient condition for a vertical distribution to be totally geodesic foliation in B.

Theorem 6.3.7. *Let $f : B \to F$ be a slant lightlike submersion. If ω is parallel with respect to ∇, then we have*

$$\mathcal{T}_{\phi X}\phi X = -cos^2\theta \mathcal{T}_X X, \quad \mathcal{T}_{\phi X}\phi X = -\mathcal{T}_X X, \quad \textit{and} \quad \mathcal{T}_{\phi X}\phi X = 0, \tag{6.3.17}$$

for every $X \in \Gamma(D)$, $X \in \Gamma(\bar{J}\triangle\perp\triangle)$ and $X \in \Gamma(\bar{J}(ltr(Kerf_*))$, respectively.

Proof. Let ω be parallel. Then from (6.3.9), we have

$$C\mathcal{T}_X Y = \mathcal{T}_X \phi Y$$

for $X, Y \in \Gamma(TB)$.

Now, interchanging the role of X and Y, we get

$$C\mathcal{T}_Y X = \mathcal{T}_Y \phi X.$$

On subtracting these two equations, we have

$$C\mathcal{T}_X Y - C\mathcal{T}_Y X = \mathcal{T}_X \phi Y - \mathcal{T}_Y \phi X.$$

Using (6.3.8), we derive

$$\mathcal{T}_X \phi Y = \mathcal{T}_Y \phi X.$$

On substituting Y by ϕX, we get

$$\mathcal{T}_X \phi^2 X = \mathcal{T}_{\phi X} \phi X.$$

Thus using the Theorem 6.3.4 with the fact that $\phi^2 X = -X$, for every $X \in \Gamma(\triangle \perp \bar{J}\triangle)$ and $\phi X = 0$ for any $X \in \Gamma(\bar{J}(ltr(Kerf_*))$, (6.3.17) follows. $\square$

We now investigate the geometry of the leaves of the distribution $\Gamma(\mathcal{V})$.

Theorem 6.3.8. *Let f be a lightlike submersion from an indefinite Kähler manifold (B, g_B), where g_B is a semi-Riemannian metric of index $2r$, onto an r-lightlike manifold (F, g_F). Then the distribution $\mathcal{V}$ defines a totally geodesic foliation on B if and only if*

$$\omega(\nu \nabla_X \phi Y + \mathcal{T}_X \omega Y) + C(\mathcal{T}_X \phi Y + h \nabla_X \omega Y) = 0,$$

for all $X, Y \in \mathcal{V}$.

Proof. Let $X, Y \in \Gamma(\mathcal{V})$. Then using the Lemma (6.3.2) with (6.3.1) and (6.3.2), we obtain

$$\begin{aligned} \nabla_X Y &= -\bar{J}\nabla_X \bar{J}Y = -\bar{J}(\mathcal{T}_X \phi Y + \nu \nabla_X \phi Y + \mathcal{T}_X \omega Y + h \nabla_X \omega Y) \\ &= -(B\mathcal{T}_X \phi Y + C\mathcal{T}_X \phi Y + \phi \nu \nabla_X \phi Y + \omega \nu \nabla_X \phi Y + \phi \mathcal{T}_X \omega Y + \omega \mathcal{T}_X \omega Y \\ &\quad + Bh\nabla_X \omega Y + Ch\nabla_X \omega Y) \qquad (6.3.18) \end{aligned}$$

Hence $\nabla_X Y \in \Gamma(\mathcal{V})$ if and only if $\omega(\nu \nabla_X \phi Y + \mathcal{T}_X \omega Y) + C(\mathcal{T}_X \phi Y + h \nabla_X \omega Y) = 0$. $\square$

Corollary 6.3.9. *Let f be a lightlike submersion from an indefinite Kähler manifold (B, g_B), where g_B is a semi-Riemannian metric of constant index $2r$, onto* *an r-lightlike manifold* (F, g_F)*. Then B is a locally product Riemannian manifold if and only if*

$$\omega(\nu\nabla_X\phi Y + \mathcal{T}_X\omega Y) + C(\mathcal{T}_X\phi Y + h\nabla_X\omega Y) = 0,$$

$$\phi(\nu\nabla_U BV + \mathcal{A}_U CV) + B(\mathcal{A}_U BV + h\nabla_U CV) = 0,$$

for $X, Y \in \Gamma(\mathcal{V})$ *and* $U, V \in \Gamma(\mathcal{H})$*.*

Further, we prove that the orthogonal complementary subbundle μ of $f(D)$ in $S(Kerf_*)$ is holomorphic with respect to $\bar{J}$ and find its dimension.

Theorem 6.3.10. *Let* $f : B \to F$ *be a proper slant lightlike submersion. Then* μ *is holomorphic (invariant) with respect to* $\bar{J}$*.*

Proof. Using (6.3.1), for any $V \in \Gamma(\mu)$ and $\omega X \in \Gamma(f(D))$, we have $g_B(\bar{J}V, \omega X) = -g_B(\bar{J}V, \phi X)$. By virtue of the Theorem 6.3.4, we get

$$g_B(\bar{J}V, \omega X) = -cos^2\theta g_B(V, X) + g_B(V, \omega\phi X) = 0$$

Similarly,

$$g_B(\bar{J}V, Y) = -g_B(V, \bar{J}Y) = 0$$

for any $Y \in \Gamma(Kerf_*)$. Also for any $N \in \Gamma(ltr(Kerf_*))$,

$$g_B(\bar{J}V, N) = -g_B(V, \bar{J}N) = 0.$$

This completes the proof. □

Theorem 6.3.11. *Let f be a proper slant lightlike submersion from an almost Hermitian manifold* $(B^m, g_B, \bar{J})$ *onto an r-lightlike manifold* (F^n, g_F)*, where* g_B *is a semi-Riemannian metric of index* $2r$*. Then* $dim(\mu) = 2n - m + 2r$*. If* $\mu = \{0\}$*, then* $n = \frac{m-2r}{2}$*.*

Proof. Since $dimD = m - n - 3r$ and $dimS(Kerf_*^{\perp}) = n - r$, therefore $dim\mu = 2n - m + 2r$. Moreover B is almost Hermitian manifold so its dimension m, which is even and hence dimension of μ is even. □

Theorem 6.3.12. *Let f be a lightlike submersion from an indefinite almost Hermitian manifold $(B^m, g_B, \bar{J})$, where g_B is a semi-Riemannian metric of index $2r$, onto an r-lightlike manifold (F^n, g_F). Let $\{e_1, ..., e_{m-n-3r}\}$ be a local orthonormal basis of D then $\{csc\theta\omega e_1, ..., csc\theta\omega e_{m-n-3r}\}$ is a local orthonormal basis of D.*

Proof. Since $\{e_1, ..., e_{m-n-3r}\}$ be a local orthonormal basis of $\Gamma(D)$ and D is Riemannian, therefore, using (6.3.14), we have

$$g_B(csc\theta\omega e_i, csc\theta\omega e_j) = csc^2\theta sin^2\theta g_B(e_i, e_j) = \delta_{ij},$$

this proves the theorem. □

Hence, we have the following result similar to the above theorem.

Corollary 6.3.13. *If $\{e_1, ..., e_{\frac{m-n-3r}{2}}\}$ are unit vector fields in D then $\{e_1, \sec\theta\phi e_1, e_2, \sec\theta\phi e_2, ..., e_{\frac{m-n-3r}{2}}, \sec\theta\phi e_{\frac{m-n-3r}{2}}\}$ is a local orthonormal basis of D.*

SCOPE FOR FUTURE WORK

Based on the present research, It is suggested that further research can be stimulated on the geometry of slant lightlike submanifolds of nearly Kähler manifolds, trans-Sasakian manifolds, *S*-manifolds and LP trans-Sasakain manifolds, etc.

In the present thesis, we have introduced the notion of slant lightlike submersion. Similarly, we can investigate hemi-slant lightlike submersions and its geometry.

In semi-Riemannian context, a minimal isometric immersion is a particular harmonic map. So, one can explore the relation between the classical minimality with the minimality introduced in lightlike case. In the light of this, harmonicity of slant lightlike submersions can be investigated.

Further, we have studied the warped product slant lightlike submanifolds of indefinite Sasakian manifolds in the present work. So, doubly warped product slant lightlike submanifolds and their goemetric inequalities can be explored.

Recently Shukla and Yadav [85] introduced the notion of semi-slant lightlike submanifolds of indefinite Kähler manifolds. Various geometric aspects like totally geodesicity, totally umbilicity and minimality can be explored for semi-slant lightlike submanifolds for contact manifolds with indefinite metric.

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