COMMON FIXED POINT THEOREMS FOR MAPPINGS SATISFYING E.A PROPERTY

A

DISSERTATION SUBMITTED IN FULFILLMENT OF THE REQUIREMENTS FOR THE AWARD OF THE DEGREE OF

MASTER OF SCIENCE

IN

(MATHEMATICS AND COMPUTING)

BY

MANPREET KAUR

ROLL NO - 301103010

Under the supervision of

Dr. S.S. Bhatia

SMCA,

Thapar University, Patiala

SCHOOL OF MATHEMATICS AND COMPUTER APPLICATIONS
THAPAR UNIVERSITY
PATIALA-147001 (PUNJAB)

JULY, 2013.

CERTIFICATE

I hereby certify that the dissertation entitled **"COMMON FIXED POINT THEOREMS FOR MAPPINGS SATISFYING E.A PROPERTY"**, which is being submitted by **Ms. Manpreet Kaur** (Roll no. 301103010), in the fulfillment of the requirements for the award of degree of **MASTER OF SCIENCE** in "Mathematics and Computing", to the School of Mathematics and Computer Applications (SMCA), Thapar University, Patiala, comprises of candidate's own research work carried out under the supervision of **Dr. S.S. Bhatia,** Professor, SMCA, Thapar University, Patiala, during the period from January 2013 to June 2013.

The part of the work presented in this dissertation has not been submitted either in part or in full to this or any other university for the award of any degree.

Manpreet Kaur
(Manpreet Kaur)

This is to certify that the above statement made by the candidate is correct and true to the best of our knowledge.

Dr. S.S. Bhatia
Professor,
SMCA, Thapar University, Patiala
(Supervisor)

Countersigned by:

23/07/2013

Dr. Rajesh Kumar
Associate Professor and Head,
SMCA, Thapar University, Patiala

Dr. S.K. Mohapatra
Dean of Academic Affairs,
Thapar University, Patiala

ACKNOWLEDGEMENT

I wish to express my deep sense of gratitude to my supervisor, Dr. S.S Bhatia, Professor, SMCA, Thapar University, Patiala, for his valuable guidance during the course of this dissertation. I am also grateful to Dr. Rajesh Kumar, Associate Professor and Head, School of Mathematics and Computer Applications, Thapar University, Patiala, for providing all necessary facilities in the department.

I am deeply thankful to Dr. P.K Bajpai, Dean, Research and Sponsored Projects and Dr. S.K. Mohapatra, Dean, Academic Affairs, Thapar University, Patiala, for the support and needful help during the various stages of this dissertation.

I am also very thankful to Ms. Priya Shahi, research scholar in SMCA, who has helped me a lot in understanding the concepts related to this topic. Without her help, this work would indeed have been very difficult for me to complete this dissertation. This work would have incomplete without thanking my friends, who were always there in the hour of need.

Special thanks are gratefully given to my parents for their encouragement and blessings to me at all stages during my studies.

Above all, I pay my reverence to the almighty of GOD.

Manpreet Kaur

Manpreet Kaur

(301103010)

ABSTRACT

The present dissertation entitled, "**COMMON FIXED POINT THEOREMS FOR MAPPINGS SATISFYING E.A PROPERTY**", embodies a brief account of investigations carried out by various authors on existence of fixed points of self-mappings in metric spaces under the supervision of **Dr. S.S. Bhatia,** Professor, School of Mathematics and computer Applications, Thapar University, Patiala.

The aim of this work is to study some results on existence and uniqueness of common fixed points using E.A property. Fixed point theory is a major branch of non-linear functional analysis because of its wide applicability. Various problems in physics, chemistry, biology, economics etc. can be solved by making use of fixed point theorems.

The work presented in this dissertation has been divided into four chapters. The first chapter is introductory. In this chapter, we present a brief account of basic definitions and results which will be used in the later chapters. In second chapter, we present the Banach fixed point theorem and some other fixed point theorems. Also, we present one of its application to solve a differential equation. Towards the end of this chapter, we have studied the Caristi's fixed point theorem [3].

The purpose of the third chapter is to study an interesting property called E.A property introduced by Aamri and Moutawakil [1]. Also, we have studied some common fixed point theorems under strict contractive conditions for mappings satisfying E.A property. In the fourth chapter, we have studied the results given by Imdad, M. and Ali, J. in [7]. The purpose of this chapter is to study how the E.A property replaces the containment condition of ranges of one mapping into the range of other in common fixed point considerations up to a pair of mappings.

References of different publications cited in the present dissertation have been given in the end.

LIST OF SYMBOLS AND NOTATIONS

- $\Re$ Set of real numbers.
- N Set of natural numbers.
- Q Set of rational numbers.
- Z Set of integers.
- C Set of complex numbers.
- $\Re^+$ $[0,\infty]$.
- $\in$ Belongs to.
- $\notin$ Does not belong to.
- $\subset$ Subset.
- $\cup$ Union.
- $\forall$ For all.
- $\Rightarrow$ Implies.
- $\exists$ There exists.
- $[a,b]$ Closed interval.
- (a,b) Open interval.
- A' Set of all limit points.
- $d(A,B)$ Distance between the set A and B.
- X^2 $X \times X$.
- Id_X Identity map.

CONTENTS

CHAPTER-I

INTRODUCTION

1.1 A fixed point of a mapping $f : X \to X$ of a non-empty set X into itself is an $x \in X$ which is mapped into itself (is "kept fixed" by f), that is $f(x) = x$, we say that c is a fixed point of the function $f(x)$ if and only if $f(c) = c$. Not all functions have fixed points. For example, a translation has no fixed point. Fixed point theory implicates the study of conditions on X and or f which guarantees that f always has at least one fixed point as well as the study of methods of approximating fixed points when they do exist.

Due to the wide applicability of fixed point theory, it is considered as a major branch of non-linear functional analysis. Fixed point theory is an interesting area, with a tremendous number of applications in various fields of Mathematics such as the theory of differential and integral equations, solution of system of linear equations and boundary value problems etc. Various problems in physics, chemistry, biology and economics etc. can be solved by making use of fixed point theorems.

Fixed point theorems give the conditions under which maps (single or multivalued) have solutions. The theory itself is a beautiful mixture of analysis (pure and applied), topology and geometry. Over the last 50 years, the theory of fixed points has been revealed as a very powerful and important tool in the study of non-linear phenomena. Many problems in pure and applied mathematics have as their solutions the fixed point of some mapping f and so a number of the procedures in numerical analysis and approximation theory aim to the successive approximations to the fixed point of an appropriate mapping. For example, Newton's method for finding the zero's of a function may be interpreted with the fixed point theorems.

1.2 Definitions and Notations

We now present a brief account of basic definitions, which will be used in the subsequent chapters. These definitions are taken from Jain P. K. and Ahmad, K. [9], Rudin, W. [20].

1.2.1 Metric space

Let X be a non-empty set. A function

$$d : X \times X \to \Re$$

is said to be a metric on X if it satisfies the following conditions:

(i) $d(x, y) \geq 0$, $\quad \forall \; x, y \in X$

(ii) $d(x, y) = 0 \Leftrightarrow x = y$, $\quad \forall \; x, y \in X$

(iii) $d(x, y) = d(y, x)$, $\quad \forall \; x, y \in X$ (Symmetry)

(iv) $d(x, y) \leq d(x, z) + d(z, x)$, $\quad \forall \; x, y, z \in X$ (Triangle inequality)

The ordered pair (X, d) is called a metric space.

Examples:

(i) Let X be the set of all real numbers. For $x, y \in X$, define

$$d(x, y) = |x - y|$$

Then, (X, d) is a metric space. This is called the metric space $\Re$ with the usual metric and we denote it by $\Re_u$.

(ii) Let X be an arbitrary non- empty set. For $x, y \in X$, define d by

$$d(x, y) = \begin{cases} 0, & x = y \\ 1, & x \neq y \end{cases}$$

Then, (X, d) is a metric space. The metric d is called the discrete metric and the space (X, d) is called discrete metric space and is denoted by X_d.

1.2.2 Limit point

Let (X, d) be a metric space and $A \subset X$. A point $x \in X$ is called a limit point (accumulation point or cluster point) of A if each open neighborhood centered on x contains at least one point of A other than x; in other words, if

$$(S_r(x) - \{x\}) \cap A \neq \Phi$$

The set of all limit points of A, denoted by A', is called the derived set of A.

Examples:

Let $\Re_u$ be the usual metric space and $A \subset \Re$.

(i) If $A = [a,b], [a,b), (a,b]$ or (a,b), then $A' = [a,b]$.

(ii) If $A = \left\{1, \frac{1}{2}, \frac{1}{3}, ...\right\}$, then $A' = \{0\}$.

(iii) if $A = Z$, then $A' = \phi$.

1.2.3 Closure of a Set

Let (X,d) be a metric space and $A \subset X$. The set consisting of the points of A and the accumulation points of A is called the closure of A and is denoted by $\overline{A}$.

1.2.4 Open Set

Let (X,d) be a metric space. A set $G \subset X$ is said to be an open set if it is neighborhood of each of its points.

(Equivalently, a set $G \subset X$ is said to be an open set if for each $x \in G$, $\exists$ an $r > 0$ such that $S_r(x) \subset G$.)

Examples:

In the usual metric space $\Re_u$

(i) $\Re$ is an open set.

(ii) (0, 1) is an open set.

(iii) The sets N, Z and Q are not open sets.

1.2.5 Closed Set

Let (X,d) be a metric space and $A \subset X$. The set A is said to be closed if its complement $X - A$ is open.

Examples:

In the usual metric space $\Re_u$, the set

(i) $A = [1,2]$ is closed since $\Re - A = (-\infty,1) \cup (2,\infty)$ is open.

(ii) $A = Q$ is not closed since $\Re - Q$, the set of all irrational numbers, is not open.

(iii) $A = \Re$ is closed since $\Re - A = \Phi$ is open.

1.2.6 Compact Set

A subset K of a metric space X is said to be compact if every open cover of K contains a finite subcover.

Examples:

(i) The discrete metric space X_d, where X is an infinite set, is not compact.

(ii) Consider the set

$$A = \{-1,1\} \cup \left\{-1 - \frac{1}{n}; n \in \mathrm{N}\right\} \cup \left\{1 + \frac{1}{n}; n \in \mathrm{N}\right\}$$

Since $A \subset [-2,2]$, A is bounded. The only limit points of A are -1 and 1, both of which belongs to A. Since A contains all of its limit points, A is closed. A is bounded and closed so A is compact.

1.2.7 Sequence

A function $f : \mathrm{N} \to X$, where X is any set, is called a sequence in X. Because a sequence is uniquely and completely determined by the values $f(n) = x_n$ for $n \in \mathrm{N}$, a sequence is usually denoted by $\{x_n\}$ without explicit reference to f. The value of x_n is called nth value (or term) of the sequence $\{x_n\}$.

Examples:

(i) $\{x_n\} = \left\{\frac{1}{n+1}\right\}$, $n \in \mathrm{N}$ is a sequence. Here $x_1 = \frac{1}{1+1}, x_2 = \frac{1}{2+1},...$ so on.

(ii) $\{x_n\} = (-1)^{n+1}$, $n \in \mathrm{N}$ is a sequence. Here $x_1 = 1, x_2 = -1, x_3 = 1,...$ so on.

1.2.8 Convergent Sequence

Let (X,d) be a metric space. A sequence $\{x_n\}$ in X is said to be convergent if there is a point $x \in X$ such that for each $\in > 0$, $\exists$ a positive integer N such that

$$d(x_n, x) < \in, \quad \forall n \geq N$$

We also say that $\{x_n\}$ converges to x, or that x is the limit of $\{x_n\}$, and we write $x_n \to x$, or

$$\lim_{n\to\infty} x_n = x .$$

If $\{x_n\}$ does not converge, it is said to diverge.

1.2.9 Cauchy Sequence

A sequence $\{x_n\}$ in a metric space (X, d) is said to be Cauchy sequence if for each $\in > 0$, $\exists$ a positive integer N such that

$$d(x_m, x_n) < \in, \quad \forall\ m,n \geq N .$$

Theorem 1.2.10 [14] Every convergent sequence in a metric space is a Cauchy sequence.

Proof. If $x_n \to x$, then for every $\in > 0$, there exists an N such that

$$d(x_n, x) < \frac{\in}{2}, \quad \forall\ n > N .$$

Using triangle inequality for all $m,n > N$, we have

$$d(x_m, x_n) \leq d(x_m, x) + d(x, x_n) < \frac{\in}{2} + \frac{\in}{2} = \in.$$

This proves that $\{x_n\}$ is a Cauchy sequence.

1.2.11 Remark: Every convergent sequence is a Cauchy sequence but the converse is not true as shown by the following example:

Example:

Consider the sequence $\{x_n\}$ in the usual metric space Q_u, where

$$\begin{aligned} x_1 &= .1 \\ x_2 &= .101 \\ x_3 &= .101001 \\ x_4 &= .1010010001 \\ &\cdots \quad \cdots \quad \cdots \\ &\cdots \quad \cdots \quad \cdots \end{aligned}$$

Clearly, $\{x_n\}$ is a Cauchy sequence which does not converge in Q_u.

Theorem 1.2.12 [14] A convergent sequence in X is bounded and its limit is unique.

Proof. Let $x_n \to x$. Then, taking $\in = 1$, we can find an N such that $d(x_n, x) < 1, \forall\, n > N$. Therefore, by the triangle inequality, for all n, we have $d(x_n, x) < 1 + a$, where

$$a = \max\{d(x_1, x), d(x_2, x), ..., d(x_N, x)\}.$$

Which proves that $\{x_n\}$ is bounded. Suppose that $x_n \to x$ and $x_n \to z$, from triangle inequality, we obtain

$$0 \le d(x, z) \le d(x, x_n) + d(x_m, z) \to 0 + 0.$$

Hence, $x = z$, this proves the uniqueness of the limit.

Theorem 1.2.13 [14] Let M be a non-empty subset of a metric space (X, d) and $\overline{M}$ its closure then $x \in \overline{M}$ if and only if there is a sequence (x_n) in M such that $x_n \to x$.

Proof. Suppose $x \in \overline{M}$. If $x \in M$, a sequence of that type is $(x, x, x, ...)$. If $x \notin M$, it is point of accumulation of M. Hence, for each $n = 1, 2, 3, ...$ the ball $B\left(x; \frac{1}{n}\right)$ contains an $x_n \in M$, and $x_n \to x$ as $n \to \infty$.

Conversely, if (x_n) is in M and $x_n \to x$, then $x \in M$ or every neighborhood of x contains points $x_n \neq x$, thus x is a accumulation point of M. From the definition of closure, $x \in M$.

1.2.14 Complete Metric Space

A metric space (X, d) is said to be complete if every Cauchy sequence in X is convergent.

Examples:

(i) The Euclidean space $\Re^n$ is a complete metric space.

(ii) Let $X = \Re$ and $'d'$ be the usual metric on X. Then, (X, d) is a complete metric space.

(iii) The function space $C[a, b]$ is complete; here $[a, b]$ is any given closed interval on $\Re$.

Proof. Suppose (x_m) be any Cauchy sequence in $C[a,b]$. Then, given any $\in > 0$, there is an N such that for all $m,n > N$, we obtain

$$d(x_m, x_n) = \max_{t \in J} |x_m(t) - x_n(t)| < \in, \qquad (1.2.1)$$

where $J = [a,b]$. Hence, for any fixed point $t = t_0 \in J$,

$$|x_m(t_0) - x_n(t_0)| < \in, \; m,n > N .$$

Which proves that $(x_1(t_0), x_2(t_0), x_3(t_0), ...)$ is a Cauchy sequence of real numbers. Since $\Re$ is complete, the sequence converges, say $x_m(t_0) \to x(t_0)$ as $m \to \infty$. In this way, we can associate with each $t \in J$ a unique real number $x(t)$. This defines a function x on J, and we prove that $x \in C[a,b]$ and $x_m \to x$.

Utilizing (1.2.1) with $n \to \infty$, we obtain

$$\max_{t \in J} |x_m(t) - x(t)| \le \in, \qquad (m > N).$$

Therefore, for all $t \in J$, we have

$$|x_m(t) - x(t)| \le \in, \qquad (m > N).$$

This proves that $x_m(t)$ converges to $x(t)$ uniformly on J. Due to the fact that x_m's are continuous on J and the convergence is uniform, the limit function x is continuous on J. Therefore, $x \in C[a,b]$. Also, $x_m \to x$. This proves completeness of $C[a,b]$.

Theorem 1.2.15 [14] A subspace M of a complete metric space X is itself complete if and only if the set M is closed in X.

Proof. Suppose M be complete. From Theorem 1.2.13, for all $x \in \overline{M}$, there is a sequence (x_n) in M which converges to x. As (x_n) is a Cauchy sequence from Theorem 1.2.10 and M is complete, (x_n) converges in M, the limit being unique from Theorem 1.2.12. Therefore, $x \in M$. This proves that M is closed since $x \in \overline{M}$ was arbitrary.

Conversely, suppose M be closed and (x_n) Cauchy in M. Then $x_n \to x \in X$, thits implies $x \in M$, from Theorem 1.2.13, and $x \in M$, since M is closed, that is $M = \overline{M}$. Therefore, the arbitrary Cauchy sequence (x_n) converges in M. Hence, M is complete.

1.2.16 Continuous function

Suppose X and Y are metric spaces, $E \subset X, p \in E$, and f maps E into Y. Then, f is said to be continuous at p if for every $\in > 0$ there exists a $\delta > 0$ such that

$$d_Y(f(x), f(p)) < \in,$$

for all points $x \in E$ for which $d_X(x, p) < \delta$. If f is continuous at every point of E, then f is said to be continuous on E.

Examples:

(i) Let (X,d) be a metric space and $f : X \to X$ be a constant function. Then, f is continuous function.

(ii) Let (X,d) be a discrete metric space then every function $f : X \to Y$ is continuous on X.

1.2.17 Lower Semi Continuous Function

Let (X,d) is a metric space. A function $\Psi : X \to \Re$ is called lower semi continuous at $x_0 \in X$ if

$$\lim_{x \to x_0} \inf \Psi(x) \geq \Psi(x_0).$$

Example:

Let $\Psi : \Re \to \Re$ be a function defined by

$$\Psi(x) = \begin{cases} -1 & if\ x \leq 0 \\ 1 & if\ x > 0 \end{cases}$$

Then, Ψ is lower semi continuous at $x_0 = 0$.

1.2.18 Upper Semi Continuous Function:-

Let (X,d) is a metric space. A function $\Psi : X \to \Re$ is called upper semi continuous at $x_0 \in X$ if

$$\lim_{x \to x_0} \sup \Psi(x) \le \Psi(x_0).$$

Example:

Let $\Psi : \Re \to \Re$ be a function defined by

$$\Psi(x) = \begin{cases} -1 & if \ x < 0 \\ 1 & if \ x \ge 0 \end{cases}$$

Then, Ψ is upper semi continuous function at $x_0 = 0$.

1.2.19 Fixed point [2]

Let (X, d) be a metric space and $T : X \to X$ be a mapping. The point $x \in X$ is called a fixed point of T if x is mapped into itself; i.e.

$$T(x) = x.$$

Examples:

(i) A translation has no fixed point. Let X be a non empty set. We can define a translation $T : X \to X$ as $T(x) = x + c$ where $x \in X$ and 'c' is any constant. Clearly, it has no fixed point.

(ii) The mapping $T : \Re \to \Re$ defined as $T(x) = x^2$ has only two fixed points. Here, 0 and 1 are fixed points.

(iii) The mapping $T : \Re^2 \to \Re^2$ defined as $T(x, y) = (x, 0)$ has infinitely many fixed points.

(iv) The mapping $T : \Re \to \Re$ defined as $T(x) = \dfrac{x}{4}$ has unique fixed point. Clearly, 0 is the fixed point.

1.2.20 Contraction Mapping [2]

Let (X, d) be a metric space. A mapping $T : X \to X$ is said to be Lipschitzian if there exists a constant $\alpha \ge 0$ with

$$d(T(x), T(y)) \le \alpha \, d(x, y) \qquad \forall x, y \in X .$$

The smallest α for which above holds is said to be Lipschitzian constant for T and is denoted by L. If $L < 1$, we say that T is a contraction mapping.

Example:

Let $X=[0,1]$. The mapping $T:X\to X$ given by $T(x)=\frac{x}{2}$ $\forall x\in X$. Then T is a contraction mapping. We have,

$$d(T(x),T(y))=\left|\frac{x}{2}-\frac{y}{2}\right|=\frac{1}{2}|x-y|$$
$$=\frac{1}{2}d(x,y)$$

Thus, $d(Tx,Ty)\le\alpha\, d(x,y)$ where $\alpha=\frac{1}{2}<1$.

1.2.21 Contractive Mapping [19]

Let (X,d) be a metric space. A mapping $T:X\to X$ is called contractive if

$$d(Tx,Ty)<d(x,y) \qquad \forall x,y\in X \text{ and } x\neq y.$$

In 1976, Jungck [10] introduced the following concept of commuting mappings:

1.2.22 Commuting Mapping [10]

Let (X,d) be a metric space. Two self mappings f and g on (X,d) are said to be commuting if

$$fog(x)=gof(x) \qquad \forall x\in X .$$

Example:

Let $X=[1,\infty)$ and $d(x,y)=|x-y|$. Let $f,g:X\to X$ by $f(x)=x^3+x^2+x+\frac{1}{2}$ and $g(x)=x$ for $x\in X$. Then

$$(fog)x=f(g(x))=f(x)=x^3+x^2+x+\frac{1}{2}$$

and

$$(gof)x=g(f(x))=g(x^3+x^2+x+\frac{1}{2})=x^3+x^2+x+\frac{1}{2}$$

Since $(fog)x=(gof)x$, $\forall x\in X$. Thus, f and g are commuting mappings.

In 1982, Sessa [21] introduced the following notion of weak commutativity:

1.2.23 Weakly Commuting Mappings [20]

Two self mappings f and g on a metric space (X,d) are said to be weakly commuting if

$$d(fg(x), gf(x)) \le d(f(x), g(x)) \quad \forall x \in X .$$

Example:

Let $X = [0,1]$ with the usual metric d . Define f , $g : X \to X$ by $f(x) = \frac{x}{2+x}$ and $g(x) = \frac{x}{2}$ $\forall x \in X$.

Then, $d(fg(x), gf(x)) = \left| \frac{x}{(4+2x)} - \frac{x}{(4+x)} \right| = \left| \frac{x^2}{(4+2x)(4+x)} \right|$

$$\le \left| \frac{x^2}{(4+2x)} \right| = \left| \frac{x}{(2+x)} - \frac{x}{2} \right|$$

$$= d(f(x), g(x))$$

Thus, f and g are weakly commuting mappings.

In 1986, Jungck [11] introduced the following concept of compatible maps:

1.2.24 Compatible Mapping [11]

Two self mappings f and g on a metric space (X,d) are said to be compatible if

$$\lim_{n\to\infty} d(fg(x_n), gf(x_n)) = 0 ,$$

whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} g(x_n) = t \text{ , for some } t \in X .$$

Example:

Let $X = [1,\infty]$ and d is the usual metric on X . Define $f, g : X \to X$ by $f(x) = 2x - 1$ *and* $g(x) = x^2$ for all $x \in X$. Consider, the sequence in X as $\{x_n\} = 1 + \frac{1}{n}$.

Then, $f(x_n) = 1 + \frac{2}{n} \to 1$; $g(x_n) = \left(1 + \frac{1}{n}\right)^2 \to 1$; $fg(x_n) \to 1$; $gf(x_n) \to 1$.

Thus, $d(fg(x_n), gf(x_n)) \to 0$ as $n \to \infty$. Hence, f and g are compatible.

1.2.25 Coincidence Point

Let two mappings $f : X \to Y$ and $g : X \to Y$, where X and Y are any two non-empty sets. We say that a point $x \in X$ is a coincidence point of f and g if

$$f(x) = g(x).$$

Example:

Let $X = \Re$ with the usual metric $d(x, y) = |x - y|$ and define $f, g : \Re \to \Re$ by

$$f(x) = \frac{x}{3}, \quad x \in \Re \qquad \text{and} \qquad g(x) = x^2, \quad x \in \Re.$$

Here, 0 and 1/3 are two coincidence points for the maps f and g.

Jungck and Rhoades [13] in 1998, introduced the following concept of weakly compatible maps as follows:

1.2.26 Weakly Compatible Mapping [13]

Two self mappings f and g on a metric space (X, d) are said to be weakly compatible if they commute at their coincidence points; i.e., if $f(x) = g(x)$, for some $x \in X$, then $fg(x) = gf(x)$.

Example:

Let $X = [0,1]$. Define $f, g : X \to X$ by

$$f(x) = \begin{cases} 1 - x, & \text{if } x \in \left[0, \frac{1}{2}\right] \\ 0, & \text{if } x \in \left(\frac{1}{2}, 1\right] \end{cases} \qquad \text{and} \qquad g(x) = \begin{cases} \frac{1}{2}, & \text{if } x \in \left[0, \frac{1}{2}\right] \\ \frac{3}{4}, & \text{if } x \in \left(\frac{1}{2}, 1\right] \end{cases}$$

Since $x = \frac{1}{2}$ is a unique coincidence point of f and g and $fg\left(\frac{1}{2}\right) = f\left(\frac{1}{2}\right) = g\left(\frac{1}{2}\right) = gf\left(\frac{1}{2}\right)$.

Hence, f and g are weakly compatible.

1.2.27 E. A Property [1]

Let f and g be two self mappings of a metric space (X,d). We say that f and g satisfy the property E.A if there exists a sequence $\{x_n\}$ such that

$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} g(x_n) = t \quad \text{for some } t \in X .$$

Example:

Let $X = [0,\infty)$. Define $f, g : X \to X$ by

$$f(x) = \frac{7x}{2} \qquad \text{and} \qquad g(x) = \frac{2x}{3}, \qquad \forall x \in X .$$

Consider, the sequence $\{x_n\} = \frac{1}{n}$. Clearly, $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} g(x_n) = 0$.

Then, f and g satisfy E.A property.

1.2.28 Implicit Relations [18]

Following Popa [18], let Ψ be the family of real lower semi-continuous functions $F(t_1,t_2,...,t_6) : R_+^{\;6} \to R$ satisfying the following conditions:

$F_1 : F$ is non-increasing in the variables t_5 and t_6,

F_2: there exists $h \in (0,1)$ such that for every $u,v \geq 0$ with

$F_{2a} : F(u,v,v,u,u+v,0) \leq 0$ or

$F_{2b} : F(u,v,u,v,0,u+v) \leq 0$

then, we have $u \leq hv$, and

$F_3 : F(u,u,0,0,u,u) > 0, \qquad \forall\, u > 0$.

Example:

Define $F(t_1,t_2,...,t_6) : R_+^{\;6} \to R$ as

$$F(t_1,t_2,...,t_6) = t_1 - k \max\left\{t_2, t_3, t_4, \frac{1}{2}(t_5 + t_6)\right\}, \text{ where } k \in (0,1).$$

Now, we show that the given function is implicit function.

F_1: Here, we need to show that F is non-increasing in the variables t_5 and t_6.

For this, suppose $t_5' > t_5$ and $t_6' > t_6$,

This implies that $\frac{t_5' + t_6'}{2} > \frac{t_5 + t_6}{2}$,

Now, consider $F(t_1, t_2, ..., t_5', t_6') = t_1 - k \max\left\{t_2, t_3, t_4, \frac{1}{2}\left(t_5' + t_6'\right)\right\}$

$$< F(t_1, t_2, ..., t_5, t_6)$$

Hence, F is non-increasing in the variable t_5 and t_6.

F_2: Let $u, v \geq 0$. Assume that $F_{2a} : F(u, v, v, u, u+v, 0) \leq 0$

$$\Rightarrow u - k \max\left\{v, v, u, \frac{1}{2}(u+v)\right\} \leq 0$$

$$\Rightarrow u - k \max\left\{v, u, \frac{1}{2}(u+v)\right\} \leq 0$$

$$\Rightarrow u \leq k \max\left\{v, u, \frac{1}{2}(u+v)\right\}$$

If $u \geq v$, then $u \leq ku < u$, a contradiction. Thus $u < v$

$$\Rightarrow u \leq kv, \qquad \text{where } k \in (0,1)$$

F_3 : We need to prove that $F(u, u, 0, 0, u, u) > 0, \qquad \forall\, u > 0$

Consider $F(u, u, 0, 0, u, u) = u - k \max\left\{u, 0, 0, \frac{1}{2}(u+u)\right\}$

$$= u - k \max\{u, u\}$$

$$= u - ku$$

$$= (1-k)u > 0 \qquad \forall u > 0 \qquad (\because k \in (0,1))$$

Hence, all the conditions of implicit functions are satisfied thereby implying that the given function is implicit function.

Chapter second is devoted to the Banach contraction principle and some of the related basic fixed point theorems. Further in this chapter, we present one of its application to solve a differential equation. Towards the end, we have studied Caristi's fixed point theorem [3].

The aim of the chapter three is to study E.A property given by Aamri and Moutawakil [1]. Also, we have studied some new common fixed point theorems under strict contractive conditions for mappings satisfying E.A property given in [1].

Whereas, in the fourth chapter, we have studied how the E.A property replaces the containment condition of range of one mapping into the range of other in common fixed point considerations up to a pair of mappings. The idea of implicit functions due to Popa [18] has been utilized due to their unifying power.

CHAPTER-II

BANACH FIXED POINT THEOREM AND ITS APPLICATION TO DIFFERENTIAL EQUATIONS

2.1 Introduction

The Banach fixed point theorem is an important tool in the theory of metric spaces. In 1922, Banach [4] used the Lipschitzian concept to prove a fixed point theorem, which ensured the existence and uniqueness of fixed point of certain self-maps of metric spaces. This result of Banach [4] is known as Banach fixed point theorem or contraction mapping principle. The theorem also gives an iterative process by which we can obtain approximations to the fixed point. Banach fixed point theorem plays an important role in several branches of Mathematics. For instance, it has been used to show the existence of solutions of linear algebraic equations, ordinary differential equations, intergral equations and many other fields of Mathematics. Due to its importance for mathematical theory, Banach fixed point theorem has been extended in many directions.

In this chapter, we present Banach fixed point theorem and some other related fixed point theorems. Also, we present one of its application to solve a differential equation. Towards the end, we have studied the Caristi's fixed point theorem [3].

2.1.1 Fixed Point [2]

Let (X,d) be a metric space and $T:X\to X$ be a mapping. The point $x\in X$ is called a fixed point of T if x is mapped into itself; i.e.

$$T(x)=x.$$

Examples:

(i) A translation has no fixed point. Let X be a non empty set. We can define a translation $T:X\to X$ as $T(x)=x+c$ where $x\in X$ and 'c' is any constant. Clearly, it has no fixed point.

(ii) The mapping $T:\Re \to \Re$ defined as $T(x)=x^2$ has two fixed points. Here, 0 and 1 are fixed points.

(iii) The mapping $T:\Re^2 \to \Re^2$ defined as $T(x,y)=(x,0)$ has infinitely many fixed points.

(iv) The mapping $T:\Re \to \Re$ defined as $T(x)=\frac{x}{4}$ has unique fixed point. Clearly, 0 is the fixed point.

2.1.2 Contraction Mapping [2]

Let (X,d) be a metric space. A mapping $T:X \to X$ is said to be Lipschitzian if there exists a constant $\alpha \geq 0$ with

$$d(T(x),T(y)) \leq \alpha\, d(x,y) \qquad \forall x,y \in X .$$

The smallest α for which above holds is said to be Lipschitzian constant for T and is denoted by L. If $L<1$, we say that T is a contraction mapping.

Example:

Let $X=[0,1]$. The mapping $T:X \to X$ given by $T(x)=\frac{x}{2}$ $\forall x \in X$. Then T is a contraction mapping.

We have,

$$\begin{aligned} d(T(x),T(y)) &= \left|\frac{x}{2}-\frac{y}{2}\right| = \frac{1}{2}|x-y| \\ &= \frac{1}{2}d(x,y) \end{aligned}$$

Thus, $d(Tx,Ty) \leq \alpha\, d(x,y)$, where $\alpha = \frac{1}{2} < 1$.

2.1.3 Contractive Mapping [19]

Let (X,d) be a metric space. A mapping $T:X \to X$ is called contractive if

$$d(Tx,Ty) < d(x,y) \qquad \forall x,y \in X \text{ and } x \neq y .$$

2.1.4 Complete Metric Space [20]

A metric space (X,d) is said to be complete if every Cauchy sequence in X is convergent.

Examples:

(i) Let $X=\Re$ and $'d'$ be the usual metric on X. Then, (X,d) is a complete metric space.

(ii) The Euclidean space $\Re^n$ is a complete metric space.

2.2 Banach Fixed Point Theorem or Contraction Mapping Principle [9]

Theorem 2.2.1 Let (X,d) be a complete metric space and let $T:X\to X$ be a contraction on X. Then, T has a unique fixed point in X.

Proof: Let $\alpha\in[0,1)$ be the Lipschitz constant such that

$$d(Tx,Ty)\le \alpha d(x,y),\ \forall\, x,y\in X$$

We prove the theorem in various steps.

Step (i). We define a sequence $\{x_n\}\subset X$ in this way:

Consider any point $x_0\in X$ and inductively define the sequence $\{x_n\}$ of points in X as:

$$\begin{aligned}
x_1 &= Tx_0\\
x_2 &= Tx_1 = T^2x_0\\
x_3 &= Tx_2 = T^3x_0\\
&\dots\ \dots\ \dots\ \dots\ \dots\ \dots\\
&\dots\ \dots\ \dots\ \dots\ \dots\ \dots\\
x_n &= Tx_{n-1} = T^nx_0
\end{aligned}$$

Clearly, $\{x_n\}$ is the sequence of images of x_0 under repeated application of T.

Step (ii). $\{x_n\}$ is Cauchy sequence in X. Suppose $m<n$. Then,

$$\begin{aligned}
d(x_m,x_n) &= d(T^mx_0,T^nx_0)\\
&\le \alpha\, d(T^{m-1}x_0,T^{n-1}x_0)\\
&\dots\ \dots\ \dots\ \dots\ \dots\ \dots\\
&\dots\ \dots\ \dots\ \dots\ \dots\ \dots
\end{aligned}$$

$$\leq \alpha^m d(x_0, T^{n-m}x_0)$$
$$\leq \alpha^m [d(x_0, Tx_0) + d(Tx_0, T^2x_0) + ... + d(T^{n-m-1}x_0, T^{n-m}x_0)]$$

(by Triangle inequality)

$$\leq \alpha^m [d(x_0, Tx_0) + \alpha d(x_0, Tx_0) + ... + \alpha^{n-m-1} d(x_0, Tx_0)]$$
$$= \alpha^m [1 + \alpha + \alpha^2 + ... + \alpha^{n-m-1}] d(x_0, Tx_0)$$
$$\leq \frac{\alpha^m}{1-\alpha} d(x_0, Tx_0), \quad (0 \leq \alpha < 1)$$
$$\to 0$$

as m (and hence n) $\to \infty$. Hence $\{x_n\}$ is a Cauchy sequence.

Step (iii). Since X is complete and $\{x_n\}$ is a Cauchy sequence in X, $\exists\, x \in X$ such that $x_n \to x$.

Step (iv). Now, we need to show that x is a fixed point of T. We have

$$d(x, Tx) \leq d(x, x_n) + d(x_n, Tx) \qquad \text{(by Triangle inequality)}$$

$$\leq d(x, x_n) + \alpha d(x_{n-1}, x) \qquad (\because x_n = Tx_{n-1})$$

$$\to 0, \; as \; n \to \infty$$

$$\Rightarrow \quad d(x, Tx) = 0$$

$$\Rightarrow \quad x = Tx$$

Hence x is a fixed point of T. Thus, we obtain the existence of fixed point.

Step (v). In this step, we verify the uniqueness of fixed point**.** Let if possible, x and y be two fixed points of T in X. Then $Tx = x$ and $Ty = y$. Now, we have

$$d(x, y) = d(Tx, Ty) \leq \alpha d(x, y)$$

$$\Rightarrow \quad d(x, y) = 0$$

$$\Rightarrow \quad x = y$$

This completes the proof of the theorem.

2.2.2 Remark: The condition of completeness in Theorem 2.2.1 cannot be relaxed as shown by the following example:

Example:

Take $X = \left(0, \frac{1}{2}\right]$ equipped with the metric of absolute value. Clearly, this is an incomplete metric space. Note that the mapping $T : X \to X$ given by $T(x) = x^2$ is a contraction but T has no fixed point. In fact $T(0) = 0 \notin X$ *and* $T(1) = 1 \notin X$.

2.2.3 Remark: The condition of T being a contraction cannot be replaced by weaker one; namely, contractive as shown by the following example:

Example:

Consider the complete metric space $X = [0, \infty)$ equipped with the metric of absolute value and consider the mapping $T : X \to X$ given by $T(x) = \frac{1}{1+x^2}$

Consider, $d(T(x), T(y)) = \left|\frac{1}{1+x^2} - \frac{1}{1+y^2}\right| = \left|\frac{y^2 - x^2}{(1+x^2)(1+y^2)}\right|$

$$< |x - y| = d(x, y)$$

Hence, the mapping T satisfies $d(T(x), T(y)) < d(x, y)$ and hence T is contractive map, while T is not a contraction and T has no fixed point.

Now, we prove two more fixed point theorems under different hypotheses.

Theorem 2.2.2 [9] Let (X, d) be a compact metric space and $T : X \to X$ a contractive map (not necessarily contraction). Then, T has a unique fixed point in X .

Proof: First of all we show the existence of fixed point. For this define $f : X \to \Re$ by

$$f(x) = d(x, Tx)$$

We first establish that f is continuous. Suppose $\in > 0$ be given. Then

$$|f(x) - f(y)| = |d(x, Tx) - d(y, Ty)|$$

$$\leq |d(x,Tx) - d(Tx,y)| + |d(Tx,y) - d(y,Ty)|$$

(by Triangle inequality)

$$\leq d(x,y) + d(Tx,Ty)$$

$$\leq 2d(x,y) \qquad (\because T \text{ is contractive})$$

$$< \in, \text{ provided } d(x,y) < \delta = \frac{\in}{2}.$$

Since f is continuous and X is compact, $\exists x \in X$ such that $f(x) \leq f(y), \forall y \in X$; i.e., f attains its minimum at x. Now, we prove that x is a fixed point of T.

Let, if possible, x be not a fixed point of T. Then $Tx \neq x$. Since $f(x) \leq f(y), \forall y \in X$, taking $y = Tx$, we have

$$f(x) \leq f(Tx)$$

$$\Rightarrow \quad d(x,Tx) \leq d(Tx,T(Tx))$$

$$< d(x,Tx) \qquad (\because T \text{ is contractive and } Tx \neq x)$$

which is a contradiction . Hence, x is a fixed point of T.

Now, we proceed to show the uniqueness of fixed point of T. Let if possible, x and y be two fixed points of T in X. Then $Tx = x$ and $Ty = y$. Now, we have

$$d(x,y) = d(Tx,Ty) < d(x,y)$$

This is impossible. Thus, $d(x,y) = 0$, thereby it implying that $x = y$.

Theorem 2.2.3 [9] Let (X,d) be a compact metric space, T a contractive map of X into itself. Then, for any $x_0 \in X$, the successive iterates $Tx_0, T^2x_0, \ldots, T^nx_0, \ldots$ converges to the unique fixed point of T.

Proof: From Theorem 2.2.2, we obtain that x is the unique fixed point of T. Then,

$$d(T^{n+1}x_0, x) = d(T(T^nx_0), Tx)$$

$$< d(T^nx_0, x) \qquad (\because T \text{ is contractive })$$

This is true for all $n \in \mathrm{N}$. Thus, $\{d(T^{n+1}x_0, x)\}$ is a decreasing sequence of non-negative real numbers and so converges to its infimum. Suppose $\lim_{n\to\infty} d(T^n x_0, x) = \lambda$.

Now, $\{T^n x_0\}$ being a sequence of points of a compact metric space X, $\exists$ subsequence $\{T^{n_k} x_0\}_{k=1}^{\infty}$ which converges to some point $y \in X$. Therefore

$$d(y, x) = \lim_{k\to\infty} d(T^{n_k} x_0, x) = \lambda$$

If $\lambda \neq 0$, then $y \neq x$ and so

$$\lambda = d(y, x) > d(Ty, Tx) \qquad (\because T \text{ is contractive })$$

$$= d(Ty, x) \qquad (\because Tx = x)$$

$$= \lim_{k\to\infty} d(T(T^{n_k} x_0), x)$$

$$= \lim_{k\to\infty} d(T^{n_k+1} x_0, x)$$

$$= \lambda$$

which is impossible. Thus, $\lambda = 0$. This verifies that

$$\lim_{n\to\infty} T^n x_0 = x.$$

Theorem 2.2.4 [14] Let (X, d) be a complete metric space and $T : X \to X$ be a mapping. If T^m is contraction on X for some positive integer m, then T has a unique fixed point.

Proof: Let us assume that $B = T^m$. It is given that B is a contraction. By Theorem 2.2.1, the mapping B has a unique fixed point, say $\hat{x}$, that is, $B\hat{x} = \hat{x}$. Hence $B^n \hat{x} = \hat{x}$. Again, from Theorem 2.2.1, we have

$$\lim_{n\to\infty} B^n x = \hat{x}, \text{ for all } x \in X$$

In particular, taking $x = T\hat{x}$, since $B^n = T^{nm}$, we obtain

$$\hat{x} = \lim_{n\to\infty} B^n T\,\hat{x} = \lim_{n\to\infty} TB^n\,\hat{x}$$

$$= \lim_{n\to\infty} T\hat{x}$$

$$= T\hat{x}$$

This shows that $\hat{x}$ is a fixed point of T. Also, since every fixed point of T is a fixed point of B. Hence T cannot have more than one fixed point. This completes the proof.

2.3 Application to Differential Equations

The most interesting applications of fixed point theorems arise when the underlying metric space is a function space. Now, we discuss the existence and uniqueness of differential equations. We shall use Banach fixed point theorem to prove the famous Picard's theorem, which play an important role in the theory of differential equations.

2.3.1 Picard's Existence and Uniqueness Theorem (Ordinary differential equations) ([14], p. 314-317)

Let f be continuous on a rectangle $R = \{(t,x) : |t - t_0| \le a, |x - x_0| \le b\}$ and thus bounded on R, say

$$|f(t,x)| \le c\,,\ \forall\ (t,x) \in R \tag{2.3.1}$$

Suppose that f satisfy a Lipschitz condition on R with respect to its second argument, that is, let there is a constant k called as Lipschitz constant such that for all $(t,x), (t,v) \in R$

$$|f(t,x) - f(t,v)| \le k|x - v|. \tag{2.3.2}$$

Then, the initial value problem

$$\frac{dx}{dt} = f(t,x), \qquad x(t_0) = x_0 \tag{2.3.3}$$

has a unique solution. This solution exists on an interval $[t_0 - \beta, t_0 + \beta]$, where

$$\beta < \min\left\{a, \frac{b}{c}, \frac{1}{k}\right\}. \tag{2.3.4}$$

Proof. Suppose $C(J)$ be the metric space of all real-valued continuous functions on an interval $J = [t_0 - \beta, t_0 + \beta]$ with metric d defined by

$$d(x, y) = \max_{t \in J} |x(t) - y(t)|.$$

$C(J)$ is a complete metric space from (1.2.14 (iii)). Let $\tilde{C}$ be the subspace of $C(J)$ consisting of all those functions $x \in C(J)$ that satisfy

$$|x(t) - x_0| \le c\beta. \tag{2.3.5}$$

It is easy to see that $\tilde{C}$ is closed in $C(J)$ from Theorem 1.2.15. Due to the fact that $\tilde{C}$ is closed in $C(J)$, we obtain that $\tilde{C}$ is complete.

Using integration, equation (2.3.3) can be written as $x = Tx$, where $T : \tilde{C} \to \tilde{C}$ is defined by

$$Tx(t) = x_0 + \int_{t_0}^{t} f(\tau, x(\tau))d\tau. \tag{2.3.6}$$

Now, in view of (2.3.4), we have $c\beta < b$. So, if $x \in \tilde{C}$, then $\tau \in J$. Therefore, T is defined for all $x \in \tilde{C}$ and $(\tau, x(\tau)) \in R$. Due to the continuity of f on R, the integral in (2.3.6) exists. As T maps $\tilde{C}$ into itself, utilizing (2.3.6) and (2.3.1), we get

$$|Tx(t) - x_0| = \left| \int_{t_0}^{t} f(\tau, x(\tau))d\tau \right| \le c|t - t_0| \le c\beta.$$

Now, we proceed to show that T is a contraction on $\tilde{C}$. Using the Lipschitz condition (2.3.2), we obtain

$$|Tx(t) - Tv(t)| = \left| \int_{t_0}^{t} [f(\tau, x(\tau)) - f(\tau, v(\tau))]d\tau \right|$$

$$\le |t - t_0| \max_{\tau \in J} k|x(\tau) - v(\tau)|$$

$$\le k\beta\, d(x, v).$$

As the above inequality does not depend on t, we can take the maximum on the left and we get

$$d(Tx,Tv) \le \alpha\, d(x,v) \text{, where } \alpha = k\beta .$$

Using (2.3.4) we obtain that $\alpha = k\beta < 1$. Thus, T is a contraction on $\tilde{C}$. Now, from Theorem 2.2.1, it follows that T has a unique fixed point $x \in \tilde{C}$. Therefore, x is a continuous function on J such that $x = Tx$. So, from equation (2.3.6), we infer that

$$x(t) = x_0 + \int_{t_0}^{t} f(\tau, x(\tau))d\tau . \qquad (2.3.7)$$

Since $(\tau, x(\tau)) \in R$, where f is continuous, (2.3.7) may be differentiated. Hence x is even differentiated and satisfies (2.3.3). Conversely, every solution (2.3.3) must satisfy (2.3.7). This completes the proof of the theorem.

2.4 Caristi's fixed point theorem [3]

To prove Caristi's fixed point theorem, we need the following important results.

Proposition 2.4.1 [3] Let X be a complete metric space and $\phi : X \to (-\infty, \infty]$ a bounded below lower semi continuous function. Suppose that $\{x_n\}$ is a sequence in X such that

$$d(x_n, x_{n+1}) \le \phi(x_n) - \phi(x_{n+1}), \quad \forall n \in \mathrm{N}_0 = \mathrm{N} \cup \{0\}$$

Then $\{x_n\}$ converges to a point $v \in X$ and $d(x_n, v) \le \phi(x_n) - \phi(v)$ for all $n \in \mathrm{N}_0$.

Proof. By the given hypothesis, we have

$$d(x_n, x_{n+1}) \le \phi(x_n) - \phi(x_{n+1}), \quad n \in \mathrm{N}_0 ,$$

It implies that $\{\phi(x_n)\}$ is a decreasing sequence. Moreover, for $m \in \mathrm{N}_0$

$$\begin{aligned}\sum_{n=0}^{m} d(x_n, x_{n+1}) &\le d(x_0, x_1) + d(x_1, x_2) + \ldots + d(x_m, x_{m+1}) \\ &\le \phi(x_0) - \phi(x_{m+1}) \\ &\le \phi(x_0) - \inf_{n \in \mathrm{N}_0} \phi(x_n)\end{aligned}$$

Letting $n \to \infty$, we obtain

$$\sum_{n=0}^{\infty} d(x_n, x_{n+1}) < \infty$$

This implies that $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, there exists $v \in X$ such that $\lim_{n\to\infty} x_n = v$. Let $m, n \in \mathrm{N}_0$ with $m > n$. Then

$$d(x_n, x_m) \le \sum_{i=n}^{m-1} d(x_i, x_{i+1})$$

$$\le \phi(x_n) - \phi(x_m)$$

Letting $m \to \infty$, we have

$$d(x_n, v) \le \phi(x_n) - \lim_{m\to\infty} \phi(x_m)$$

$$\le \phi(x_n) - \phi(v), \quad \forall\ n \in \mathrm{N}_0.$$

Theorem 2.4.2 [3] Let X be a complete metric space and $\phi : X \to (-\infty, \infty]$ a proper, bounded below and lower semi continuous function. Suppose that, for each $u \in X$ with $\inf_{x\in X} \phi(x) < \phi(u)$ there exists a $v \in X$ such that

$$u \ne v \quad \text{and} \quad d(u,v) \le \phi(u) - \phi(v).$$

Then there exists an $x_0 \in X$ such that $\phi(x_0) = \inf_{x\in X} \phi(x)$.

Proof. Let $\inf_{x\in X} \phi(x) < \phi(y)$, $\forall\, y \in X$. Suppose $u_0 \in X$ with $\phi(u_0) < \infty$. If $\inf_{x\in X} \phi(x) = \phi(u_0)$, then nothing to prove. So, assume $\inf_{x\in X} \phi(x) < \phi(u_0)$, and there exists a $u_1 \in X$ such that $u_0 \ne u_1$ and $d(u_0, u_1) \le \phi(u_0) - \phi(u_1)$.

Let us define a sequence $\{u_n\}$ in X, starting with u_0. Suppose $u_{n-1} \in X$. Then choose $u_n \in S_n$, where

$$S_n = \{w \in X : d(u_{n-1}, w) \le \phi(u_{n-1}) - \phi(w)\}$$

such that

$$\phi(u_n) \le \inf_{w\in S_n} \phi(w) + \frac{1}{2}\left\{\phi(u_{n-1}) - \inf_{w\in S_n} \phi(w)\right\} \tag{2.4.1}$$

Since $u_n \in S_n$, we obtain

$$d(u_{n-1}, u_n) \le \phi(u_{n-1}) - \phi(u_n), \qquad n \in \mathrm{N}.$$

In view of Proposition 2.4.1, it implies that $u_n \to v \in X$ and $d(u_{n-1}, v) \le \phi(u_{n-1}) - \phi(v)$. By hypothesis there exists a $z \in X$ such that $z \ne v$ and $d(v, z) \le \phi(v) - \phi(z)$.

Consider,

$$\begin{aligned}\phi(z) &\le \phi(v) - d(v,z) \\ &\le \phi(v) - d(v,z) + \phi(u_{n-1}) - \phi(v) - d(u_{n-1},v) \\ &= \phi(u_{n-1}) - \left[d(v,z) + d(u_{n-1},v)\right] \\ &\le \phi(u_{n-1}) - d(u_{n-1},z). \qquad \text{(using Triangle inequality)}\end{aligned}$$

It implies that $z \in S_n$. It follows from (2.4.1) that

$$2\phi(u_n) - \phi(u_{n-1}) \le \inf_{w \in S_n} \phi(w) \le \phi(z).$$

Thus,

$$\phi(z) \le \phi(v) \le \lim_{n\to\infty} \phi(u_n) \le \phi(z),$$

a contradiction. So, there exists a point $x_0 \in X$ such that $\phi(x_0) = \inf_{x\in X} \phi(x)$.

Theorem 2.4.3 (Caristi's fixed point theorem) [3]

Let X be a complete metric space and $\phi : X \to (-\infty,\infty]$ a proper, bounded below and lower semi continuous function. Let $T : X \to X$ be a mapping such that

$$d(x,Tx) \le \phi(x) - \phi(Tx) \qquad \text{for all } x \in X \tag{2.4.2}$$

Then there exists a point $v \in X$ such that $v = Tv$ and $\phi(v) < \infty$.

Proof. Since ϕ is a proper, then there exists $u \in X$ such that $\phi(u) < \infty$. Let

$$C = \{x \in X : d(u,x) \le \phi(u) - \phi(x)\}$$

Then, C is a non-empty closed subset of X. Now, we proceed to show that C is invariant under T. For each $x \in C$, we have

$$d(u,x) \le \phi(u) - \phi(x)$$

and hence, in view of (2.4.2), we have

$$\begin{aligned}\phi(Tx) &\le \phi(x) - d(x,Tx) \\ &\le \phi(x) - d(x,Tx) + \phi(u) - \phi(x) - d(u,x) \\ &= \phi(u) - \left[d(x,Tx) + d(u,x)\right] \\ &\le \phi(u) - d(u,Tx),\end{aligned}$$

and it implies that $Tx \in C$

Suppose, for contradiction, $x \ne Tx$ for all $x \in C$. Then, for each $x \in C$, there exists $w \in C$ such that

$$x \ne w \text{ and } d(x,w) \le \phi(x) - \phi(w)$$

By Theorem 2.4.2 there exists an $x_0 \in C$ with $\phi(x_0) = \inf_{x \in C} \phi(x)$. Hence, for such an $x_0 \in C$, we obtain

$$
\begin{aligned}
0 < d(x_0, Tx_0) &\le \phi(x_0) - \phi(Tx_0) \qquad\qquad (\inf_{x \in C} \phi(x) = \phi(x_0) \le \phi(Tx_0)) \\
&\le \phi(Tx_0) - \phi(Tx_0) \\
&= 0,
\end{aligned}
$$

a contradiction. This completes the proof of the theorem.

CHAPTER-III

COMMON FIXED POINT THEOREMS FOR MAPPINGS SATISFYING E.A PROPERTY

3.1 Introduction

The strict contractive conditions in metric spaces do not ensure the existence of common fixed point unless the space is assumed compact or if the strict contractive conditions are replaced by some stronger conditions as given by Jachymski, J. [8], Jungck, G, Moon, K.B., Park, S., and Rhoades, B.E. [12] and Pant, R.P. [15]. In 1986, Jungck [11] presented the concept of compatible mappings. This notion of compatibility was frequently utilized by various researchers to prove existence theorems in common fixed point theory. Later, Pant [16, 17] initiated the study of common fixed point of non-compatible mappings. Recently, in 2002, Aamri and Moutawakil [1] defined a new property which extended the notion of non-compatible mappings and established common fixed point theorems under strict contractive conditions on a metric space.

The aim of this chapter is to study the E.A property given by Aamri and Moutawakil [1]. Also, we study some common fixed point theorems under strict contractive conditions for mapping satisfying E.A property given in [1].

In 1976, Jungck [10] introduced the following concept of commuting mappings:

3.1.1 Commuting Mapping [10]

Let (X,d) be a metric space. Two self mappings f and g on (X,d) are said to be commuting if

$$fog(x) = gof(x) \qquad \forall x \in X .$$

Example:

Let $X = [1,\infty)$ and $d(x, y) = |x - y|$. Let $f, g : X \to X$ by $f(x) = x^3 + x^2 + x + \frac{1}{2}$ and $g(x) = x$ for $x \in X$. Then

$$(fog)x = f(g(x)) = f(x) = x^3 + x^2 + x + \frac{1}{2}$$

and

$$(gof)x = g(f(x)) = g(x^3 + x^2 + x + \frac{1}{2}) = x^3 + x^2 + x + \frac{1}{2}$$

Since $(fog)x = (gof)x$, $\forall x \in X$. Thus, f and g are commuting mappings.

In 1982, Sessa [21] introduced and extended the notion of commuting mappings to weakly commuting mappings.

3.1.2 Weakly Commuting Mapping [21]

Two self mapping S and T of a metric space (X,d). These self mappings are said to be weakly commuting if

$$d(ST(x),TS(x)) \le d(S(x),T(x)), \forall x \in X$$

Example:

Let $X = [0,1]$ with the usual metric d. Define T, $S: X \to X$ by $S(x) = \frac{x}{2+x}$ and $T(x) = \frac{x}{2}$

Then $d(ST(x),TS(x)) = \left| \frac{x}{(4+2x)} - \frac{x}{(4+x)} \right| = \left| \frac{x^2}{(4+2x)(4+x)} \right|$

$$\le \left| \frac{x^2}{(4+2x)} \right| = \left| \frac{x}{(2+x)} - \frac{x}{2} \right|$$

$$= d(S(x),T(x))$$

Thus, S and T are weakly commuting mappings.

3.1.3 Remark: Two commuting mappings are weakly commuting, but the converse is not true as is shown in the following example:

Example:

Let $X = [0,1]$, with the usual metric d. Define S, $T: X \to X$ by $S(x) = \frac{x}{2+x}$ and $T(x) = \frac{x}{2}, \forall x \in X$.

Then $d(ST(x),TS(x)) = \left| \frac{x}{(4+2x)} - \frac{x}{(4+x)} \right| = \left| \frac{x^2}{(4+2x)(4+x)} \right|$

$$\leq \left|\frac{x^2}{(4+2x)}\right| = \left|\frac{x}{(2+x)} - \frac{x}{2}\right|$$

$$= d(S(x), T(x))$$

Thus, S and T are weakly commuting mappings.

But $(SoT)x = S(T(x)) = S\left(\frac{\frac{x}{2}}{2+\frac{x}{2}}\right) = \left(\frac{\frac{x}{2}}{\frac{4+x}{2}}\right) = \frac{x}{4+x}$

And $(ToS)x = T(S(x)) = T\left(\frac{\frac{x}{2+x}}{2}\right) = \frac{x}{4+2x}$

$\because (SoT)x \neq (ToS)x$

Thus, S and T are not commuting mappings.

Jungck [11] in 1986, introduced the following concept of compatible maps:

3.1.4 Compatible Mapping [11]

Two self mappings S and T of a metric space are said to be compatible if

$$\lim_{n\to\infty} d\,(S(Tx_n), T(Sx_n) = 0$$

whenever (x_n) is a sequence in X such that

$$\lim_{n\to\infty} S(x_n) = \lim_{n\to\infty} T(x_n) = t\,,$$

for some $t \in X$.

Example:

Let $X = [1, \infty]$ and d is the usual metric on X. Define $S, T : X \to X$ by $S(x) = 2x - 1$ *and* $T(x) = x^2$ for all $x \in X$. Consider, the sequence in X as $\{x_n\} = 1 + \frac{1}{n}$.

Then, $S(x_n) = 1 + \frac{2}{n} \to 1$; $T(x_n) = \left(1 + \frac{1}{n}\right)^2 \to 1$; $ST(x_n) \to 1$; $TS(x_n) \to 1$.

Thus, $d(ST(x_n),TS(x_n)) \to 0$ as $n \to \infty$. Hence, S and T are compatible.

3.1.5 Remark: Two weakly commuting mappings are compatible, but the converse is not true as is shown in the following examples.

Examples:

(i) Let $X = \Re$ and d be the metric on X. Let $S,T:(X,d) \to (X,d)$ are self maps defined as $S(x) = x^3$ and $T(x) = 2x^3$.

Here,

$$S(T(x)) = S(2x^3) = 8x^9 \text{ and } T(S(x)) = T(x^3) = 2x^9$$

Therefore,

$$d(ST(x),TS(x)) = \left|8x^9 - 2x^9\right| = 6\left|x^9\right| \text{ and } d(S(x),T(x)) = \left|x^3\right|$$

Clearly, S and T are not weakly commuting as $6\left|x^9\right| > \left|x^3\right|$. On the other hand, the functions S and T are compatible since $d(S(x),T(x)) = \left|x^3\right| \to 0$ iff $d(ST(x),TS(x)) = 6\left|x^9\right| \to 0$. Thus, S and T are compatible but not weakly commuting mappings.

(ii) Let $X = [1,\infty]$ and d is the usual metric on X. Define $S,T: X \to X$ by $S(x) = 2x - 1$ *and* $T(x) = x^2$, for all $x \in X$. Consider, the sequence in X as $\{x_n\} = 1 + \frac{1}{n}$.

Then $S(x_n) = 1 + \frac{2}{n} \to 1$; $\quad T(x_n) = \left(1 + \frac{1}{n}\right)^2 \to 1$; $\quad ST(x_n) \to 1$; $\quad TS(x_n) \to 1$.

Thus, $d(ST(x_n), TS(x_n)) \to 0$ as $n \to \infty$. Hence, S and T are compatible. But $d(ST(x),TS(x)) = 2(x-1)^2 \ \forall\ x \neq 1 \in X$ and $d(S(x),T(x)) = (x-1)^2 \ \forall\, x \neq 1 \in X$; S and T are not weakly commuting.

3.1.6 Coincidence Point

Let $f: X \to Y$ and $g: X \to Y$ be two mappings, where X and Y are any two non-empty sets. We say that a point $x \in X$ is a coincidence point of f and g if

$$f(x) = g(x).$$

Example:

Let $X = \Re$ with the usual metric $d(x,y)=|x-y|$ and define $f,g:\Re\to\Re$ by

$$f(x)=\frac{x}{3},\quad x\in\Re \quad \text{and} \quad g(x)=x^2,\quad x\in\Re.$$

Here, 0 and 1/3 are two coincidence points for the maps f and g.

Jungck and Rhoades [13] in 1998, introduced the following concept of weakly compatible maps:

3.1.7 Weakly Compatible Mapping [13]

Two self mappings T and S of a metric space X are said to be weakly compatible if they commute at their coincidence points; i.e. if $T(x)=S(x)$, for some $x\in X$ then $TS(x)=ST(x)$.

Examples:

(i) Let $X=[0,3]$ with the usual metric $d(x,y)=|x-y|$ and define $T,S:[0,3]\to[0,3]$ as

$$T(x)=\begin{cases} x, & if\ x\in[0,1) \\ 3, & if\ x\in[1,3] \end{cases} \quad \text{and} \quad S(x)=\begin{cases} 3-x, & if\ x\in[0,1) \\ 3, & if\ x\in[1,3] \end{cases}$$

Then, for any $x\in[1,3]$, x is a coincidence point and $ST(x)=TS(x)$ showing that T and S are weakly compatible mappings on [0,3].

(ii) Let $X=\Re$ and define $T,S:\Re\to\Re$ by $T(x)=\frac{x}{3}$, $x\in\Re$ and $S(x)=x^2$, $x\in\Re$. Here, 0 and $\frac{1}{3}$ are two coincidence points for the maps T and S. Note that T and S commute at 0, i.e. $TS(0)=ST(0)=0$, but $TS\left(\frac{1}{3}\right)=T\left(\frac{1}{9}\right)=\frac{1}{27}$ and $ST\left(\frac{1}{3}\right)=S\left(\frac{1}{9}\right)=\frac{1}{81}$ and so, T and S are not weakly compatible maps on $\Re$.

Remark 3.1.8 Two compatible mappings are weakly compatible but the converse is not true as shown by the following example:

Example:

Let $X=[2,20]$ and d be the usual metric on X. Define $T,S:X\to X$ as

$$T(x)=\begin{cases} x, & if\ \ x=2\ or\ x>5 \\ 6, & if\ \ 2<x\le 5 \end{cases} \quad \text{and} \quad S(x)=\begin{cases} x, & if\ \ x=2 \\ 12, & if\ \ 2<x\le 5 \\ x-3, & if\ \ x>5 \end{cases}$$

The mappings T and S are non-compatible since sequence $\{x_n\}$ defined by $\{x_n\}=5+\frac{1}{n},\ n\ge 1$. Then, $T(x_n)=2,\ S(x_n)\to 2, TS(x_n)=6, ST(x_n)=2$. Hence, T and S are not compatible mappings. But they are weakly compatible mappings since they commute at coincidence point at $x=2$ as $TS(2)=ST(2)=2$.

3.1.9 E.A Property [1]

Let T and S be two self mappings of a metric space (X,d). T and S satisfy the E.A property if there exists a sequence (x_n) such that

$$\lim_{n\to\infty} T(x_n)=\lim_{n\to\infty} S(x_n)=t\,,$$

for some $t\in X$.

Examples:

(i) Let $X=[0,1]$. Define $T,S:X\to X$ by

$$T(x)=\begin{cases} 1-x, & if\ \ x\in\left[0,\frac{1}{2}\right] \\ 0, & if\ \ x\in\left(\frac{1}{2},1\right] \end{cases} \quad \text{and} \quad S(x)=\begin{cases} \frac{1}{2}, & if\ \ x\in\left[0,\frac{1}{2}\right] \\ \frac{3}{4}, & if\ \ x\in\left(0,\frac{1}{2}\right] \end{cases}$$

Consider, the sequence $\{x_n\}=\left\{\frac{1}{2}-\frac{1}{n}\right\},\ n\ge 2$, we have $\lim_{n\to\infty} f\left(\frac{1}{2}-\frac{1}{n}\right)=\frac{1}{2}=\lim_{n\to\infty} g\left(\frac{1}{2}-\frac{1}{n}\right)$.

Thus, the pair (T,S) satisfies the E.A property.

(ii) Let $X=[0,+\infty]$. Define $T,S:X\to X$ by

$$T(x)=\frac{x}{2} \quad \text{and} \quad S(x)=\frac{7x}{4},\ \forall\, x\in X$$

Let the sequence $\{x_n\}=\frac{1}{n}$. Clearly, $\lim_{n\to\infty} T(x_n)=\lim_{n\to\infty} S(x_n)=0$.

Thus, T and S satisfy the E.A property.

(iii) Let $X = [2, +\infty)$. Define $T, S : X \to X$ by

$$T(x) = x + 1 \text{ and } S(x) = 2x + 1, \ \forall\, x \in X .$$

Suppose that E.A property holds; then there exists in X a sequence (x_n) satisfying

$$\lim_{n\to\infty} T(x_n) = \lim_{n\to\infty} S(x_n) = t \text{, for some } t \in X .$$

Therefore, $\lim_{n\to\infty}(x_n) = t - 1$ and $\lim_{n\to\infty}(x_n) = \frac{t-1}{2}$.

Then $t = 1$, which is a contradiction since $1 \notin X$. Hence T and S do not satisfy the E.A property.

3.1.10 Remark: Two self mappings S and T of a metric space (X, d) are said to be non-compatible if there exists at least one sequence (x_n) in X such that

$$\lim_{n\to\infty} T(x_n) = \lim_{n\to\infty} S(x_n) = t \text{, for some } t \in X ,$$

But $\lim_{n\to\infty} d(TS(x_n), ST(x_n))$ is either non-zero or non-existent. Therefore, two non-compatible self mappings of a metric space (X, d) satisfy the E.A property.

Example:

Let $X = [0,1]$ with the usual metric space d i.e., $d(x, y) = |x - y|$. Define $T, S : X \to X$ by

$$T(x) = \begin{cases} 1 - x, & if \ x \in \left[0, \frac{1}{2}\right] \\ 0, & if \ x \in \left(\frac{1}{2}, 1\right] \end{cases} \quad \text{and} \quad S(x) = \begin{cases} \frac{1}{2}, & if \ x \in \left[0, \frac{1}{2}\right] \\ \frac{3}{4}, & if \ x \in \left(0, \frac{1}{2}\right] \end{cases}$$

Consider, the sequence $\{x_n\} = \left\{\frac{1}{2} - \frac{1}{n}\right\}$, $n \geq 2$, we have $\lim_{n\to\infty} T\left(\frac{1}{2} - \frac{1}{n}\right) = \frac{1}{2} = \lim_{n\to\infty} S\left(\frac{1}{2} - \frac{1}{n}\right)$.

Thus, T and S satisfies property E.A. But $\lim_{n\to\infty} d\left(TS\left(\frac{1}{2} - \frac{1}{n}\right), ST\left(\frac{1}{2} - \frac{1}{n}\right)\right) \neq 0$. Hence, T and S are non-compatible mappings.

3.2 Main results

Now, we study the main results of Aamri and Moutawakil [1].

Theorem 3.2.1 Suppose S and T be two weakly compatible self mappings of a metric space (X, d) such that

(i) T and S satisfy the E.A property,

(ii) $d(Tx, Ty) < \max\{d(Sx, Sy), [d(Tx, Sx) + d(Ty, Sy)]/2,$

$$[d(Ty, Sx) + d(Tx, Sy)]/2\}, \forall x \neq y \in X$$

(iii) $TX \subset SX$.

If SX or TX is a complete subspace of X, then T and S have a unique common fixed point.

Proof. Given that T and S satisfy the E.A property, then there exists in X a sequence (x_n) satisfying

$$\lim_{n\to\infty} T(x_n) = \lim_{n\to\infty} S(x_n) = t, \text{ for some } t \in X.$$

Suppose that SX is complete. Then, $\lim_{n\to\infty} S(x_n) = Sa$, for some $a \in X$. Also, $\lim_{n\to\infty} T(x_n) = Sa$. Now, we prove that $Ta = Sa$. Let $Ta \neq Sa$. Condition (ii) implies

$$d(Tx_n, Ta) < \max\{d(Sx_n, Sa), [d(Tx_n, Sx_n) + d(Ta, Sa)]/2$$

$$[d(Ta, Sx_n) + d(Tx_n, Sa)]/2\}.$$

Taking limit $n \to \infty$, we get

$$d(Sa, Ta) \leq \max\{d(Sa, Sa), [d(Ta, Sa) + d(Sa, Sa)]/2,$$

$$[d(Ta, Sa) + d(Sa, Sa)]/2\}$$

$$\leq d(Ta, Sa)/2;$$

This is a contradiction. Hence, $Ta = Sa$

Since T and S are weakly compatible, $STa = TSa$ and therefore, $TTa = TSa = STa = SSa$

Now, we will show that Ta is a common fixed point of T and S. Suppose that $Ta \neq TTa$. Then

$$d(Ta, TTa) \leq \max\{d(Sa, STa), [d(Ta, Sa) + d(TTa, STa)]/2$$

$$[d(TTa, Sa) + d(Ta, STa)]/2\}$$

$$\leq \max\{d(Ta, TTa), d(TTa, Ta)\} = d(Ta, TTa)$$

This is a contradiction. Hence, $TTa = Ta$ and $STa = TTa = Ta$. Since $TX \subset SX$, we can similarly prove the theorem when TX is complete subspace of X. Uniqueness of the common fixed point follows easily.

Now, we give an example to support the above theorem.

Example:

Let $X = [1, +\infty)$ with the usual metric space $d(x, y) = |x - y|$. Define $T, S : X \to X$ by

$$T(x) = 2x - 1 \text{ and } S(x) = x^2, \ \forall x \in X. \text{ Then}$$

(i) T and S satisfy the E.A property for the sequence $\{x_n\} = 1 + \frac{1}{n}, \ n = 1, 2, \ldots,$

(ii) T and S are weakly compatible,

(iii) T and S satisfy for all $x \neq y$

$$d(Tx, Ty) < \max\{d(Sx, Sy), [d(Tx, Sx) + d(Ty, Sy)]/2,$$

$$[d(Ty, Sx) + d(Tx, Sy)]/2\},$$

(iv) $T1 = S1 = 1$.

Corollary 3.2.2 Let S and T be two non-compatible weakly compatible self mappings of a metric space (X, d) such that

(i) $d(Tx, Ty) < \max\{d(Sx, Sy), [d(Tx, Sx) + d(Ty, Sy)]/2,$

$$[d(Ty, Sx) + d(Tx, Sy)]/2, \forall x \neq y \in X,$$

(ii) $TX \subset SX$.

If SX or TX is a complete subspace of X, then T and S have a unique common fixed point.

Proof. Since S and T are two non-compatible self mappings of a metric space (X,d). Then by Remark 3.1.11, self mappings S and T satisfy the E.A property and hence proof follows from Theorem 3.2.1.

Corollary 3.2.3 Suppose S and T be two weakly compatible self mappings of a metric space (X,d). Suppose that there exists a mapping $\Phi : X \to \Re^{+}$ such that

(i) $d(Sx,Tx) < \Phi(Sx) - \Phi(Tx), \ \forall x \in X$,

(ii) $d(Tx,Ty) < \max\{d(Sx,Sy),[d(Ty,Sx)+d(Tx,Sy)]/2\}, \forall x \neq y \in X$

(iii) $TX \subset SX$.

If SX or TX is a complete subspace of X, then T and S have a unique common fixed point.

Proof: Let $x_0 \in X$. Choose $x_1 \in X$ such that $Tx_0 = Sx_1$. Choose $x_2 \in X$ such that $Tx_1 = Sx_2$. In general, choose $x_n \in X$ such that $Tx_{n-1} = Sx_n$. Then

$$d(Sx_n, Sx_{n+1}) = d(Sx_n, Tx_n) \le \Phi(Sx_n) - \Phi(Tx_n) = \Phi(Sx_n) - \Phi(Sx_{n+1})$$

Let the nonnegative real sequence (a_n) defined by $a_n = \Phi(Sx_n), \ n = 1,2,...$. It is easy to see that the sequence (a_n) is non-increasing and bounded below. Therefore, (a_n) is an convergent sequence. On the other hand, we have

$$d(Sx_n, Sx_{n+m}) \le a_n - a_{n+m}$$

which implies that the sequence (Sx_n) is a Cauchy sequence in SX . Suppose that SX is a complete subspace of X. Then, there exists $t \in SX$ such that $\lim_{n\to\infty} S(x_n) = t$. Also, we have $\lim_{n\to\infty} T(x_n) = t$.

Subsequently, T and S satisfy the E.A property. From (ii), it implies that

$$d(Tx,Ty) < \max\{d(Sx,Sy),[d(Tx,Sx)+d(Ty,Sy)]/2,$$

$$[d(Ty,Sx)+d(Tx,Sy)]/2\}, \forall x \neq y \in X$$

Hence, all the conditions of Theorem 3.2.3 are satisfied. Thus, T and S have a unique common fixed point.

In 1976, Caristi [6] proved that a self mapping T of a complete metric space (X,d) has a fixed point if there exists a lower semi-continuous function $\Phi : X \to \Re^+$ satisfying

$$d(x,Tx) \leq \Phi(x) - \Phi(Tx).$$

However, it may be observed that T will have a fixed point if it satisfies the above inequality for arbitrary Φ and its graph is closed.

Setting $S = Id_X$ in Corollary 3.2.3, we get the following results:

Corollary 3.2.4 Suppose T be self mapping of a complete metric space (X,d).

Suppose that there exists a mapping $\Phi : X \to \Re^+$ such that

(i) $d(x,Tx) \leq \Phi(x) - \Phi(Tx), \forall x \in X$,

(ii) $d(Tx,Ty) < \max\{d(x,y),[d(x,Ty)+d(y,Tx)]/2\}, \forall x \neq y \in X$.

Then T has a unique fixed point.

If we take $T = Id_X$ in Corollary 3.2.3, we have the following result:

Corollary 3.2.5 Let S be surjective self mapping of a complete metric space (X,d). Suppose that there exists a mapping $\Phi : X \to \Re^+$ such that

(i) $d(x,Sx) \leq \Phi(Sx) - \Phi(x), \forall x \in X$,

(ii) $d(x,y) < \max\{d(Sx,Sy),[d(y,Sx)+d(x,Sy)]/2\}, \forall x \in X$.

Then S has a unique fixed point.

The next theorem involves a function F, where $F : \Re^+ \to \Re^+$ satisfies the following conditions:

(F_1) F is nondecreasing on $\Re^+$,

(F_2) $0 < F(t) < t$, for each $t \in (0,+\infty)$.

Theorem 3.2.6 Let A, B, T and S be self mapping of a metric space (X, d) such that

(i) $d(Ax, By) \le F(\max\{d(Sx, Ty), d(Sx, By), d(Ty, By)\}), \forall (x, y) \in X^2$,

(ii) (A, S) and (B, T) are weakly compatibles,

(iii) (A, S) or (B, T) satisfy the E.A property,

(iv) $AX \subset TX$ and $BX \subset SX$.

If the range of the one of the mapping A, B, T or S is a complete subspace of X, then A, B, T and S have a unique common fixed point.

Proof. Let the pair (B, T) satisfies the E.A property. From the definition of the E.A property there exists a sequence (x_n) in X such that $\lim_{n\to\infty} B(x_n) = \lim_{n\to\infty} T(x_n) = t$, for some $t \in X$. As $BX \subset SX$, there exists in X a sequence (y_n) such that $B(x_n) = S(y_n)$. Therefore, $\lim_{n\to\infty} S(y_n) = t$. Now, we show that $\lim_{n\to\infty} A(y_n) = t$. From (i), we have

$$d(Ay_n, Bx_n) \le F(\max\{d(Sy_n, Tx_n), d(Sy_n, Bx_n), d(Tx_n, Bx_n)\})$$

$$\le F(\max\{d(Bx_n, Tx_n), 0, d(Tx_n, Bx_n)\})$$

$$\le F(d(Tx_n, Bx_n)) \le d(Tx_n, Bx_n).$$

So, $\lim_{n\to\infty} d(Ay_n, Bx_n) = 0$. Since $d(Ay_n, t) \le d(Ay_n, Bx_n) + d(Bx_n, t)$. We obtain that $\lim_{n\to\infty} A(y_n) = t$. Due to the fact that SX is a complete subspace of X, we have $t = Su$, for some $u \in X$. Subsequently, we have $\lim_{n\to\infty} A(y_n) = \lim_{n\to\infty} B(x_n) = \lim_{n\to\infty} T(x_n) = \lim_{n\to\infty} S(y_n) = Su$. In view of (i), we get

$$d(Au, Bx_n) \le F(\max\{d(Su, Tx_n), d(Su, Bx_n), d(Tx_n, Bx_n)\})$$

Taking $n \to \infty$ and using (F_2), it implies that $Au = Su$. The weak compatibility of A and S implies that $ASu = SAu$ and then $AAu = ASu = SAu = SSu$.

On the other hand, since $AX \subset TX$, therefore exists $v \in X$ such that $Au = Tv$. We show that $Tv = Bv$. Utilizing (i), we get

$$d(Au,Bv) \le F(\max\{d(Su,Tv), d(Su,Bv), d(Tv,Bv)\})$$

$$\le F(\max\{d(Au,Bv), d(Au,Bv)\})$$

$$\le F(d(Au,Bv))$$

which is a contradiction. This infers that $Au = Su = Tv = Bv$. The weak compatibility of B and T implies that $BTv = TBv$ and $TTv = TBv = BTv = BBv$.

Now, we show that Au is a common fixed point of A, B, T and S. From (i), it implies

$$d(Au, AAu) = d(AAu, Bv)$$

$$\le F(\max\{d(SAu,Tv), d(SAu,Bv), d(Tv,Bv)\})$$

$$\le F(\max\{d(AAu,Au), d(AAu,Au)\})$$

$$\le F(d(AAu,Au)).$$

So, $Au = AAu = SAu$ and Au is a common fixed point of A and S. Similarly, we show that Bv is a common fixed point of B and T. Since $Au = Bv$, we conclude that Au is a common fixed point of A, B, T and S. Similarly, we can prove when TX is assumed to be a complete subspace of X. The cases in which AX or BX, respectively, is complete since $AX \subset TX$ and $BX \subset SX$. If $Au = Bu = Tu = Su = u$ and $Av = Bv = Tv = Sv = v$, then (i) gives

$$d(u,v) = d(Au,Bv) \le F(\max\{d(Su,Tv), d(Su,Bv), d(Tv,Bv)\})$$

$$\le F(d(u,v)).$$

Therefore $u = v$ and the common fixed point is unique. This completes the proof of the theorem.

Taking $T = S$ in Theorem 3.2.6, we obtain the following result.

Corollary 3.2.7 Let A, B and S be self mappings of a metric space (X,d) such that

(i) $d(Ax,By) < F(\max\{d(Sx,Sy), d(Sx,By), d(Sy,By)\}), \forall (x,y) \in X^2$,

(ii) (A,S) and (B,S) are weakly compatible,

(iii) (A,S) or (B,S) satisfy the E.A property,

(iv) $AX \subset SX$ and $BX \subset SX$.

If the range of the one of the mapping A, B or S is a complete subspace of X , then A, B and S have a unique fixed point.

CHAPTER-IV

COMMON FIXED POINT THEOREMS FOR A PAIR OF MAPPINGS USING IMPLICIT FUNCTIONS

4.1 Introduction

In 1976, Jungck, G. [10] gave a generalized version of Banach fixed point theorem as:

Theorem 4.1.1 Let T be a continuous mapping of a complete metric space (X,d) into itself. Then T has a fixed point in X if there exist $\alpha \in (0,1)$ and a mapping $S: X \to X$ which commutes with T and satisfies $S(X) \subset T(X)$ and $d(Sx, Sy) \leq \alpha\, d(Tx, Ty),\ \forall x, y \in X$.

As seen in Theorem 4.1.1, many common fixed point theorems involve conditions on commutativity, continuity, completeness and suitable containment of ranges of the involved maps besides an appropriate contraction condition. The researchers of this field are aimed at weakening one or more of these conditions.

In order to accommodate many contraction conditions, Popa [18] in 1999 gave the concept of implicit functions. The notion of implicit functions is very fruitful due to their unifying power. Later, Imdad and Javid [7] utilized implicit functions to prove a general fixed point theorem for a pair of mappings without requiring the containment of ranges. Also, Imdad and Javid [7] unified several fixed point theorems by using an implicit function.

The purpose of this chapter is to study the results proved by Imdad and Javid [7]. Imdad and Javid [7] proved that how E.A property replaces the containment condition of ranges of one mapping into the range of other in common fixed point theorems. Also, we can use the modification suggested in [7] to all common fixed point theorems for a pair of mappings in the literature.

4.1.2 Compatible Mappings [11]

A pair of self-mappings (T, I) of a metric space (X,d) is said to be compatible if

$$\lim_{n\to\infty} d(TI(x_n), IT(x_n)) = 0,$$

whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} T(x_n) = \lim_{n\to\infty} I(x_n) = t, \text{ for some } t \in X$$

Example:

Let $X = \Re$ and d be the metric on X. Let $T, I : (X,d) \to (X,d)$ are self maps defined as

$$T(x) = x^3 \text{ and } I(x) = 2 - x$$

Here, $T(I(x_n)) = T(2 - x_n) = (2 - x_n)^3$ and $I(T(x_n)) = I(x_n^{\ 3}) = 2 - x_n^{\ 3}$.

Therefore, $d(T(x_n), I(x_n)) = \left|x_n^{\ 3} - 2 + x_n\right| = \left|(x_n - 1)(x_n^{\ 2} + x_n + 2)\right| = \left|x_n - 1\right|\left|x_n^{\ 2} + x_n + 2\right| \to 0$

if and only if $x_n \to 1$

and $d(TI(x_n), IT(x_n)) = \left|(8 - x_n^{\ 3} - 12x_n + 6x_n^{\ 2}) - (2 + x_n^{\ 3})\right| = \left|6x_n^{\ 2} - 12x_n + 6\right| = 6\left|x_n - 1\right|^2 = 0$

iff $x_n \to 1$.

Thus, T and I are compatible mappings.

4.1.3 Non-compatible Mappings [1]

A pair of self-mappings (T, I) of a metric space (X,d) is said to be noncompatible if there exists at least one sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} T(x_n) = \lim_{n\to\infty} I(x_n) = t, \text{ for some } t \in X.$$

But $\lim_{n\to\infty} d(TIx_n, ITx_n)$ is either non-zero or non-existent.

Example:

Let $X = [0,1]$ with the usual metric space d i.e., $d(x,y) = |x - y|$. Define $T, I : X \to X$ by

$$T(x) = \begin{cases} 1 - x, & if \ \ x \in \left[0, \frac{1}{2}\right] \\ 0, & if \ \ x \in \left(\frac{1}{2}, 1\right] \end{cases} \quad \text{and} \quad I(x) = \begin{cases} \frac{1}{2}, & if \ \ x \in \left[0, \frac{1}{2}\right] \\ \frac{3}{4}, & if \ \ x \in \left(0, \frac{1}{2}\right] \end{cases}$$

Consider, the sequence $\{x_n\}=\left\{\frac{1}{2}-\frac{1}{n}\right\}$, $n\geq 2$, we have $\lim_{n\to\infty} T\left(\frac{1}{2}-\frac{1}{n}\right)=\frac{1}{2}=\lim_{n\to\infty} I\left(\frac{1}{2}-\frac{1}{n}\right)$.

Thus, T and I satisfies E.A property. But $\lim_{n\to\infty} d\left(TI\left(\frac{1}{2}-\frac{1}{n}\right), IT\left(\frac{1}{2}-\frac{1}{n}\right)\right)\neq 0$. Hence, T and I are non-compatible mappings.

4.1.4 E.A Property [1]

A pair of self-mappings (T,I) of a metric space (X,d) is said to satisfy the E.A property if there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} T(x_n)=\lim_{n\to\infty} I(x_n)=t \text{, for some } t\in X$$

Clearly, a pair of compatible as well as noncompatible self-mappings of a metric space (X,d) satisfies the E.A property.

4.1.5 Remark: A pair of mappings satisfying E.A property need not follow the pattern of containment of range of one map into the range of other as utilized in common fixed point considerations but still it relaxes such requirements as shown by the following example:

Example:

Consider $X=[-1,1]$ with the usual metric. Two self-mappings T and I on X defined by

$$T(x)=\begin{cases}\frac{1}{2}, & if\ \ x=-1,\\ \frac{x}{4}, & if\ \ -1<x<1,\\ \frac{3}{5}, & if\ \ x=1,\end{cases} \qquad I(x)=\begin{cases}\frac{1}{2}, & if\ \ x=-1,\\ \frac{x}{2}, & if\ \ -1<x<1,\\ \frac{-1}{2}, & if\ \ x=1.\end{cases}$$

Consider, the sequence $\{x_n\}=\frac{1}{n}$. Clearly, $\lim_{n\to\infty} T(x_n)=\lim_{n\to\infty} I(x_n)=0$.

Then, T and I satisfy E.A property.

Also, $T(X)=\left\{\frac{1}{2},\frac{3}{5}\right\}\cup\left(\frac{-1}{4},\frac{1}{4}\right)$ and $I(X)=\left[\frac{-1}{2},\frac{1}{2}\right]$. It is clear that neither $T(X)$ is contained in $I(X)$ nor $I(X)$ is contained in $T(X)$.

4.2 Implicit Relation

Following Popa [18], let Ψ be the family of real lower semi-continuous functions $F(t_1,t_2,...,t_6):R_+^{\ 6}\to R$ satisfying the following conditions:

$F_1:F$ is non-increasing in the variables t_5 and t_6,

F_2: there exists $h\in(0,1)$ such that for every $u,v\geq 0$ with

$F_{2a}:F(u,v,v,u,u+v,0)\leq 0$ or

$F_{2b}:F(u,v,u,v,0,u+v)\leq 0$

then, we have $u\leq hv$, and

$F_3:F(u,u,0,0,u,u)>0,\ \forall\ u>0$.

Example 4.2.1 Define $F(t_1,t_2,...,t_6):R_+^{\ 6}\to R$ as

$$F(t_1,t_2,...,t_6)=t_1-k\ \max\left\{t_2,t_3,t_4,\frac{1}{2}(t_5+t_6)\right\},\text{ where } k\in(0,1).$$

Now, we show that the given function is implicit function.

F_1: Here, we need to show that F is non-increasing in the variables t_5 and t_6.

For this, suppose $t_5' > t_5$ and $t_6' > t_6$,

This implies that $\dfrac{t_5'+t_6'}{2} > \dfrac{t_5+t_6}{2}$,

Now, consider $F(t_1,t_2,...,t_5',t_6')= t_1-k\max\left\{t_2,t_3,t_4,\frac{1}{2}\left(t_5'+t_6'\right)\right\}$

$$<F(t_1,t_2,...,t_5,t_6)$$

Hence, F is non-increasing in the variable t_5 and t_6.

F_2:. Let $u,v\geq 0$. Assume that $F_{2a}:F(u,v,v,u,u+v,0)\leq 0$

$$\Rightarrow u - k \max\left\{v, v, u, \frac{1}{2}(u+v)\right\} \le 0$$

$$\Rightarrow u - k \max\left\{v, u, \frac{1}{2}(u+v)\right\} \le 0$$

$$\Rightarrow u \le k \max\left\{v, u, \frac{1}{2}(u+v)\right\}$$

If $u \ge v$, then $u \le ku < u$, a contradiction. Thus, $u < v$

$$\Rightarrow u \le kv, \qquad \text{where } k \in (0,1)$$

F_3 : We need to prove that $F(u,u,0,0,u,u) > 0, \qquad \forall\, u > 0$

Consider, $F(u,u,0,0,u,u) = u - k \max\left\{u, 0, 0, \frac{1}{2}(u+u)\right\}$

$$= u - k \max\{u, u\}$$

$$= u - ku$$

$$= (1-k)u > 0 \qquad \forall u > 0 \qquad (\because k \in (0,1))$$

Hence, all the conditions of implicit functions are satisfied thereby implying that the given function is implicit function.

Example 4.2.2 Define $F(t_1, t_2, ..., t_6) : R_+^6 \to R$ as

$$F(t_1, t_2, ..., t_6) = t_1 - h \max\left\{t_2, t_3, t_4, \frac{t_5}{2}, \frac{t_6}{2}\right\}, \text{ where } h \in (0,1).$$

Now, we show that the given function is implicit function.

F_1: Here, we need to show that F is non-increasing in the variables t_5 and t_6.

For this, suppose $t_5' > t_5$ and $t_6' > t_6$

This implies that $\frac{t_5'}{2} > \frac{t_5}{2}$ and $\frac{t_6'}{2} > \frac{t_6}{2}$

Now, consider $F(t_1, t_2, ..., t_5', t_6') = t_1 - h\max\left\{t_2, t_3, t_4, \frac{t_5'}{2}, \frac{t_6'}{2}\right\}$

$$< F(t_1, t_2, ..., t_5, t_6)$$

Hence, F is non-increasing in the variable t_5 and t_6.

F_2: Let $u, v \geq 0$. Assume that $F_{2a} : F(u, v, v, u, u+v, 0) \leq 0$

$$\Rightarrow u - h\max\left\{v, v, u, \frac{1}{2}(u+v)\right\} \leq 0$$

if $u \geq v$, then $u \leq hu < u$, a contradiction. Thus, $u < v$

$$\Rightarrow u \leq hv, \qquad \text{where } h \in (0,1)$$

F_3 : We need to prove that $F(u, u, 0, 0, u, u) > 0, \qquad \forall\ u > 0$

Consider, $F(u, u, 0, 0, u, u) = u - h\max\{u, 0, 0, u, u\}$

$$= u - hu$$

$$= u(1-h) > 0, \qquad \forall u > 0, \text{ where } h \in (0,1)$$

Hence, all the conditions of implicit functions are satisfied thereby implying that the given function is implicit function.

Example 4.2.3 Define $F(t_1, t_2, ..., t_6) : R_+^6 \to R$ as

$$F(t_1, t_2, ..., t_6) = t_1 - k\max\left\{t_2, t_3, t_4, t_5, \left(\frac{1}{2}\right)t_6\right\}, \text{ where } k \in (0,1)$$

Now, we show that the given function is implicit function.

F_1: Here, we need to show that F is non-increasing in the variables t_5 and t_6.

For this, suppose $t_5' > t_5$ and $t_6' > t_6$,

This implies that $\left(\frac{1}{2}\right)t_6' > \left(\frac{1}{2}\right)t_6$,

Now, consider $F(t_1, t_2, ..., t_5', t_6') = t_1 - k\max\left\{t_2, t_3, t_4, t_5', \left(\frac{1}{2}\right)t_6'\right\}$

$$< F(t_1, t_2, ..., t_6)$$

Hence, F is non-increasing in the variable t_5 and t_6.

F_2: Let $u, v \geq 0$. Assume that $F_{2a}: F(u, v, v, u, u+v, 0) \leq 0$

$$\Rightarrow u - k\max\left\{v, v, u, \left(\frac{1}{2}\right)(u+v), 0\right\} \leq 0$$

if $u \geq v$, then $u \leq ku < u$, a contradiction. Thus $u < v$

$$\Rightarrow u \leq kv = hv, \qquad \text{where } h = k \in (0,1)$$

F_3 : We need to prove that $F(u, u, 0, 0, u, u) > 0, \qquad \forall\ u > 0$

Consider, $F(u, u, 0, 0, u, u) = u - k\max\left\{u, 0, 0, 0, \left(\frac{1}{2}\right)u\right\}$

$$= u(1-k) > 0, \text{ for all } u > 0$$

Hence, all the conditions of implicit functions are satisfied thereby implying that the given function is implicit function.

Example 4.2.4 Define $F(t_1, t_2, ..., t_6): R_+^6 \rightarrow R$ as

$$F(t_1, t_2, ..., t_6) = t_1^2 - a\left\{t_2^2 + t_6\left(\left(\frac{1}{2}\right)(t_3 + t_4) + t_3 t_5\right)\right\}, \text{ where } a \in (0,1).$$

Now, we show that the given function is implicit function.

F_1: Here, we need to show that F is non-increasing in the variables t_5 and t_6.

For this, suppose $t_5' > t_5$ and $t_6' > t_6$,

This implies that $t_6'\left(\left(\frac{1}{2}\right)(t_3+t_4)+t_3t_5'\right) > t_6\left(\left(\frac{1}{2}\right)(t_3+t_4)+t_3t_5\right)$,

Now, consider $F(t_1,t_2,...,t_5',t_6') = t_1^2 - a\left\{t_2^2 + t_6'\left(\left(\frac{1}{2}\right)(t_3+t_4)+t_3t_5'\right)\right\}$

$$< F(t_1,t_2,...,t_6)$$

Hence, F is non-increasing in the variable t_5 and t_6.

F_2: Let $u,v \geq 0$. Assume that $F_{2a}: F(u,v,v,u,u+v,0) \leq 0$

$$\Rightarrow u^2 - av^2 \leq 0$$

$$\Rightarrow u^2 \leq av^2$$

$$\Rightarrow u \leq a^{\frac{1}{2}}v = hv \text{, where } h = a^{\frac{1}{2}} < 1.$$

F_3 : We need to prove that $F(u,u,0,0,u,u) > 0, \quad \forall\, u > 0$.

Consider, $F(u,u,0,0,u,u) = u^2(1-a) > 0, \quad \forall\, u > 0$.

Hence, all the conditions of implicit functions are satisfied thereby implying that the given function is implicit function.

Example 4.2.5 Define $F(t_1,t_2,...,t_6): R_+^6 \to R$ as

$$F(t_1,t_2,...,t_6) = t_1^3 - at_1^2t_2 - bt_1t_3t_4 - ct_5^2t_6 - dt_5t_6^2 \text{ ,}$$

where $a > 0,\ b,c,d \geq 0,\ a+c+d < 1$ and $a+b < 1$.

Now, we show that the given function is implicit function.

F_1: Here, we need to show that F is non-increasing in the variables t_5 and t_6.

For this, suppose $t_5' > t_5$ and $t_6' > t_6$, Since $t_5' > t_5 \Rightarrow t_5'^2 > t_5^2$ and $t_6' > t_6 \Rightarrow t_6'^2 > t_6^2$

$$\Rightarrow t_5'^2 t_6' > t_5^2 t_6' > t_5^2 t_6 \qquad\qquad \Rightarrow t_5' t_6'^2 > t_5' t_6^2 > t_5 t_6^2$$

$$\Rightarrow t_5'^2 t_6' > t_5^2 t_6 \qquad\qquad \Rightarrow t_5' t_6'^2 > t_5 t_6^2$$

Thus $t_5'^2 t_6' > t_5^2 t_6$ and $t_5' t_6'^2 > t_5 t_6^2$

Now, consider $F(t_1, t_2, ..., t_5', t_6') = t_1^3 - at_1^2 t_2 - bt_1 t_3 t_4 - ct_5'^2 t_6' - dt_5' t_6'^2$

$$< F(t_1, t_2, ..., t_5, t_6)$$

Hence, F is non-increasing in the variable t_5 and t_6.

F_2: Let $u, v \geq 0$. Assume that $F_{2a} : F(u, v, v, u, u+v, 0) \leq 0$

$$\Rightarrow u^3 - au^2 v - bu^2 v \leq 0$$

$$\Rightarrow u^3 - u^2 (a+b) v \leq 0$$

$$\Rightarrow u^2 (u - (a+b) v) \leq 0$$

$$\Rightarrow u - (a+b) v \leq 0 \qquad (\because u > 0)$$

$$\Rightarrow u \leq (a+b) v = hv \qquad (h = a+b < 1)$$

F_3 : We need to prove that $F(u, u, 0, 0, u, u) > 0, \quad \forall\ u > 0$

Consider, $F(u, u, 0, 0, u, u) = u^3 - au^3 - cu^3 - du^3$

$$= u^3 - u^3 (a + c + d)$$

$$= u^3 (1 - (a + c + d)) > 0 \qquad \forall u > 0 \ \ (h = (a + c + d) < 1)$$

Hence, all the conditions of implicit functions are satisfied thereby implying that the given function is implicit function.

4.3 Results

Now, we study the general common fixed point theorem proved by Imdad, M. and Ali, J. in [7] for a pair of mappings without requiring the containments of ranges. In this theorem, Imdad, M. and Ali, J. [7] utilized a suitable implicit function given by Popa [18].

Theorem 4.3.1 Let T and I be self-mappings of a metric space (X,d) such that

(i) T and I satisfy E.A property,

(ii) $\forall x, y \in X \quad and \quad F \in \Psi$,

$$F(d(Tx,Ty), d(Ix,Iy), d(Ix,Tx), d(Iy,Ty), d(Ix,Ty), d(Iy,Tx)) \leq 0, \qquad (4.3.1)$$

(iii) $I(X)$ is a complete subspace of X.

Then ,

(a) the pair (T, I) has a point of coincidence,

(b) the pair (T, I) has a common fixed point provided it is weakly compatible.

Proof: Since T and I satisfy E.A property, then there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} T(x_n) = \lim_{n\to\infty} I(x_n) = t \text{, for some } t \in X .$$

Due to the fact that $I(X)$ is a complete subspace of X, every convergent sequence of points of $I(X)$ has a limit in $I(X)$. Therefore,

$$\lim_{n\to\infty} I(x_n) = t = Ia = \lim_{n\to\infty} T(x_n) \text{, for some } a \in X .$$

thereby implying that $t = Ia \in I(X)$. Now, we proceed to show that $Ia = Ta$. Suppose $Ia \neq Ta$, then $d(Ta, Ia) > 0$. Utilizing (4.3.1)

$$F(d(Ta,Tx_n), d(Ia,Ix_n), d(Ia,Ta), d(Ix_n,Tx_n), d(Ia,Tx_n), d(Ix_n,Ta)) \leq 0$$

Letting $n \to \infty$, we obtain

$$F(d(Ta,t), d(Ia,t), d(Ia,Ta), d(t,t), d(Ia,t), d(t,Ta)) \leq 0$$

Further, using the fact that $Ia = t$ and by the definition of implicit functions (F_{2b}), it implies that $d(Ta, Ia) \leq 0$. Thus, $Ta = Ia$ which proves that a is a coincidence point of T and I.

Due to the fact that T and I are weakly compatible, we have

$$It = ITa = TIa = Tt$$

Now, we prove that $Tt = t$. If $Tt \neq t$, then $d(Tt, t) > 0$. Again utilizing (4.3.1)

$$F(d(Tt,Ta), d(It,Ia), d(It,Tt), d(Ia,Ta), d(It,Ta), d(Ia,Tt)) \leq 0$$

or

$$F(d(Tt,t), d(Tt,t), 0, 0, d(Tt,t), d(Tt,t)) \leq 0,$$

This contradicts F_3. Therefore $Tt = t$ which proves that t is a common fixed point of T and I.

Remark 4.3.2 Theorem 4.3.1 is a generalized and improved form of Theorem 4.1.1 due to Jungck [10] without any continuity requirement besides relaxing the containment of the range of one map into the range of the other (i.e $TX \subset IX$). Also the commutativity requirement is reduced to points of coincidence along with replacement of the completeness of the space with a natural condition.

4.4 Fixed Point Theorems for Families of Mappings

Now, we study a common fixed point theorem for two finite families of mappings as an application of Theorem 4.3.1 proved in [7].

Theorem 4.4.1 Let $\{T_1, T_2, ..., T_m\}$ and $\{I_1, I_2, ..., I_p\}$ be two finite families of self-mappings of a metric space (X, d) with $T = T_1 T_2 ... T_m$ and $I = I_1 I_2 ... I_p$ satisfying E.A property and condition (4.3.1). If $I(X)$ is a complete subspace of X, then

(i) (T, I) has a point of coincidence.

Moreover, if $T_i T_j = T_j T_i$, $I_k I_l = I_l I_k$ and $T_i I_k = I_k T_i$ $\forall i, j \in I_1 = \{1, 2, ..., m\}$ and $k, l \in I_2 = \{1, 2, ..., p\}$, then $(\forall i \in I_1$ and $k \in I_2)$ T_i and I_k have a common fixed point.

Proof: Due to the fact that T and I satisfy all the conditions of Theorem 4.3.1, we obtain conclusion (i). So using the componentwise commutativity of various pairs, we can show that $TI = IT$ and thereby implying that the pair (T, I) is weakly compatible. As all the conditions of Theorem 4.3.1 (for mappings T and I) are satisfied so we obtain the existence of unique common fixed point, say t. Now, we proceed to show that t remains to be fixed point of all the component maps. For proving this, we consider

$$T(T_i t) = ((T_1, T_2, ..., T_m)T_i)t = (T_1, T_2, ..., T_{m-1})((T_m T_i)t) = (T_1 ... T_{m-1})(T_i T_m t)$$

$$= ... = T_1 T_i (T_2 T_3 T_4 ... T_m t) = T_i T_1 (T_2 T_3 ... T_m t) = T_i (Tt) = T_i t \text{ .}$$

Similarly, we can prove that

$$T(I_k t) = I_k (Tt) = I_k t \qquad I(I_k t) = I_k (It) = I_k t$$

and $$I(T_i t) = T_i (It) = T_i t \ ,$$

which proves that $T_i t \ and \ I_k t$ are other fixed points of the pair (T, I) for all $i \ and \ k$. Now, we need to prove the uniqueness of common fixed point of the pair separately. As t is a unique common fixed point for the pair (T, I), thus

$$t = T_i t = I_k t$$

which proves that t is a common fixed point of $T_i \ and \ I_k$ for all $i \ and \ k$.

By setting $T_i = F(\forall i) \ and \ I_k = IX$ (identity map) in Theorem 4.3.1, we get the following fixed point theorem which can be viewed as a variant of Bryant's Theorem [5].

Corollary 4.4.2 Let F be a self-mapping of a metric space (X, d) such that there exists some $m \in N$ satisfying

$$F(d(F^m(x,)F^m(y)), d(x, y), d(x, F^m(x)), d(y, F^m(y)), d(x, F^m(y)), d(y, F^m(x))) \leq 0$$

$\forall x, y \in X \ and \ F \in \Psi$. If $F^m(X)$ is a complete subspace of X, then F has a unique fixed point.

References

[1] Aamri, M., El Moutawakil, D., Some new common fixed point theorems under strict contractive conditions, **J. Math. Anal. Appl**., 270 (2002),181–188.

[2] Agarwal, R.P., Meehan, M. and O' Regan, D., Fixed point theory and applications, **Cambridge University Press,** (2001).

[3] Agarwal, R.P., O' Regan, D. and Sahu, D.R., Fixed Point Theory for Lipschitzian-type Mappings with Applications, **Springer Dordrecht Heidelberg London New York**, 6 (2009).

[4] Banach, S., Sur les operations dans les ensembles abstraits et leur application aux equations, intergrales, **Fund Math**., 3 (1922),160.

[5] Bryant, V.W., A remark on fixed point theorem for iterated mappings, **Amer. Math. Monthly**, 75, (1968) 399–400.

[6] Caristi, J., Fixed point theorems for mapping satisfying inwardness conditions, ***Trans.*** **Amer. Math. Soc.**, 215, (1976) 241–251.

[7] Imdad, M. and Ali, J., Jungck's common fixed point theorems and E.A property, **Acta Mathematica Sinicia, English Series**, 24, (2008) 87-94.

[8] Jachymski, J., Common fixed point theorems for some families of maps, **Indian J. Pure Appl. Math**., 25, (1994) 925–937.

[9] Jain, P.K. and Ahmad, K., Metric spaces, **Narosa Publication**, (2004).

[10] Jungck, G., Commuting mappings and fixed points, **Amer. Math. Monthly**, 83, (1976) 261- 263.

[11] Jungck, G., Compatible mappings and common fixed points, **Internat. J. Math. Math. Sci**., 9(4), (1986) 771–779.

[12] Jungck, G., Moon, K.B and Park, S., Rhoades, B.E., On generalization of the Meir–Keeler type contraction maps: Corrections, **J. Math. Anal. Appl**., 180, (1993) 221–222.

[13] Jungck, G. and Rhoades, B. E., Fixed point for set valued functions without continuity, **Indian J.Pure Appl. Math. 29(3),** (1998), 227–238.

[14] Kreyszig, E., Introductory functional analysis with applications, **John Wiley and Sons, Inc.**, (1989).

[15] Pant, R.P., Common fixed points of contractive maps, **J. Math. Anal. Appl**., 226, (1998) 251–258.

[16] Pant, R. P., A common fixed point theorem under a new condition, **Indian J. Pure Appl. Math.,** 30(2), (1999) 147–152.

[17] Pant, R.P., *R*-weak commutativity and common fixed points, **Soochow J. Math**., 25, (1999) 37–42.

[18] Popa, V., Some fixed point theorems for compatible mappings satisfying an implicit relation, **Demonstratio Math.,** 32(1), (1999) 157–163.

[19] Rhoades, B. E., Contractive definitions revisited, **Contemp. Math**., 21, (1983) 189–205.

[20] Rudin, W., Principle of mathematical analysis, **McGraw – Hill**., (1976).

[21] Sessa, S., On a weak commutativity condition of mappings in fixed point considerations, **Publ. Inst. Math. Beograd.,** 32(46), (1982) 149–153.