MULTIPLE ITEM INVENTORY MODELS WITH VARIOUS DEMAND FUNCTIONS AND CONSTRAINTS

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CERTIFICATE

I hereby certify that the work which is being presented in the thesis entitled **"Multi-item Inventory Models with various Demand Functions and Constraints"** in partial fulfillment of the requirements for the award of degree of Master of Science, School of Mathematics, Thapar University, Patiala is an authentic record of my own work carried out under the supervision of Dr. Mahesh Kumar Sharma.
The matter presented in this thesis has not been submitted for the award of any other degree of this or any other university.

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ABSTRACT

In real situations often inventory manager have to hold thousand of items in an inventory. In these situations single item inventory models do not help them to manage the inventory properly. So multi item inventory models are often necessary for the inventory holders. In today's market demand of customers is not constant, however it depends on the selling price and also inventory holders have to face different restrictions like limited space in inventory, limited investment and restriction on average inventory level. In present work different models are studied with different demand functions depending on the selling price of items and models with different constraints.

The whole work is divided into four chapters.

Chapter 1 is introductory in which some classic inventory models have been discussed and literature related to the work have also been discussed.

In Chapter 2, a review of research paper entitled "Multi-item EOQ model while demand is sales price and price break sensitive" studied by Pal *et al.* (2012) has been reviewed and some observations have been made.

Based on the observations given in Chapter 2, an attempt has been made in Chapter 3 to find the solution of such type problems using KT conditions, however in place of the models considered by Pal *et al.* (2012), the simple models have been considered.

In Chapter 4, three different multi item inventory models have been solved in which the objective function is same as in model I of chapter 3 subject to limited warehouse storage, limited investment on inventory and restriction on average inventory level constraints separately.

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Chapter 1

Introduction

The word *inventory* refers to any kind of resource having economic value and is maintained to fulfil the present and future needs of an organization. Fred Hansman, defined inventory as: *An idle resource of any kind provided such resource has economic value.* Each organization has some type of inventory planning and control system. State and federal governments, schools, and every manufacturing and production organization are concerned with inventory planning and control. For example a bank has methods to control its inventory of cash, a hospital has methods to control blood supplies and medicines etc. Studying how organizations control their inventory is equivalent to studying how they achieve their objectives by supplying goods and services to their customers. Inventory control has two major objectives. The first objective is to maximize the level of customer service by avoiding shortage in stocks. Shortage in stock causes missed deliveries, backlogged orders, lost sales and unhappy customers. The second objective of inventory control is to promote efficiency in production or purchasing by minimizing the purchasing cost along with providing good level of service to the customers.

Many of the classical inventory models concern with single-item model. The multi-item inventory models are more realistic than the single item model. It is difficult to study an inventory with multiple items by using classical single item model. Further, in the multi-item-case main objective is the cost savings, which result from a collective order of several items. The single-item-models when used for multiple item inventory one item waits until a certain cost-saving order quantity is reached for other item, and there is a fixed order time for all items. Most models and software developed or published concentrate on single-item inventory control. However, retailers are responsible for the management of thousands of items in an inventory and single-item models do not help them to manage large number of items.

In this chapter classical inventory models have been discussed.

1.1 Costs Involved In Inventory Problems

The costs play an important role in making a decision to maintain the inventory in the organization. These costs are as follow :

1. **Purchasing Cost:** Purchasing cost is price of single unit of inventory item. When the item is offered at a discount if the order size exceeds a certain amount, which is a factor in deciding how much to order.

2. **Holding Cost (Carrying cost):** It is the price incurred for carrying (or holding) inventory items in the stock. This includes the storage cost for providing space to store the items in inventory, inventory handling cost for payment of salaries to employees and insurance cost against possible loss from fire or other type of damage.

3. **Setup cost (Ordering Cost):** Setup cost includes all costs that do not vary with size of the order but incurred each time an order is placed.

4. **Shortage cost:** It is the penalty cost when we run out of stock. It includes potential loss of income and the more subjective cost of loss in customer's goodwill.

5. **Total inventory cost:** If price discounts are available, then we should formulate total inventory cost by taking sum of purchasing cost, Inventory Holding Cost, Shortage Cost and Setup cost. Thus, the total inventory cost is given by

   Total inventory cost=Purchasing Cost + Holding Cost + Setup cost + Shortage cost
   When the price discounts are not offered and shortages are not allowed than Total inventory cost is given by
   Total inventory cost = Holding Cost + Setup cost

Which is given by figure 1.1

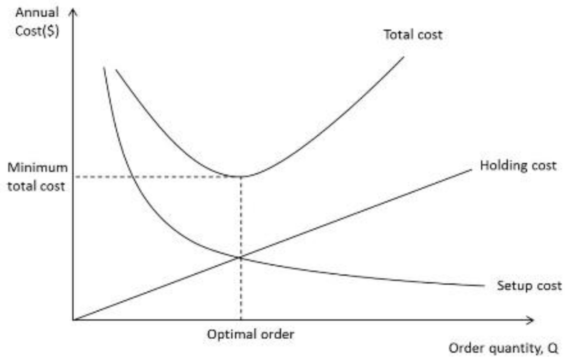

Figure 1.1: The total inventory cost.

Apart from costs, the other variables which play an important role in decision making are as follows:

1. **Demand :** The size of the demand is the number of units required in each period. It is not necessarily equal to the amount sold because demand may remain unfulfilled due to shortage or delays. The demand of the items may be either deterministic or probabilistic. In the deterministic case, the demand over a period is known. This known demand may be fixed or variable with time. Such demand is known as static and dynamic respectively. When the demand over a period is uncertain but can be predicted by a probability distribution, we say it is a case of the probabilistic demand. A probabilistic demand may be stationary or non-stationary over time.

2. **Lead Time :** It is the time lag between the placement and receipt of an order. It can be deterministic or probabilistic. If both demand and leadtime are deterministic, one needs to order in advance by a time equal to lead-time. However, if lead-time is probabilistic, it is very difficult to answer - when to order?

3. **Cycle Time :** The cycle time is time between placements of two orders. It is denoted by T. It can be determined in one of the two ways.

(a) **Continuous Review :** In this case, an order of fixed size is placed every time the inventory level reaches at a pre-specified level, called reorder level.

(b) **Periodic Review :** Here the orders are placed at equal interval of time. This is also called the *fixed order interval system*.

1.2 Basic Economic Order Quantity (EOQ) Model

EOQ model is one of the oldest and most commonly known techniques. This model was first developed by Ford Harris and R.Wilson independently in 1915. The objective is to determine economic order quantity, y which minimizes the total cost of an inventory system when demand occurs at a constant rate. The model is developed under following assumptions:

1. This model deals with single item.
2. The demand rate is known and constant.
3. Quantity discounts are not available.
4. The ordering cost is constant.
5. Shortages are not allowed and lead time is known and is constant.
6. The inventory holding cost per inventory unit per time unit is known and constant during the period under review.

Notations:

$h =$ Holding cost (Cost per inventory unit per unit time)

$y =$ Order quantity (number of units)

$D =$ Demand rate (units per unit time)

$t_0 =$ Ordering cycle length (time units)

$K =$ Setup cost associated with placement of an order (Cost per order)

Where total cost per unit time (TCU) is computed as

$$\begin{aligned} TCU(y) &= \text{Setup cost per unit time} + \text{Holding cost per unit time} \\ &= \frac{KD}{y} + \frac{hy}{2} \end{aligned}$$

The optimum solution that is order quantity y is determined by minimizing $TCU(y)$ with respect to y. Assuming y is continuous, a necessary condition for finding the optimal value of y is

$$\frac{dTCU(y)}{dy} = -\frac{KD}{y^2} + \frac{h}{2} = 0$$

The condition is also sufficient because $TCU(y)$ is a convex function. The solution of the equation yields the EOQ ($i.e$ economic ordered quantity) y^* as

$$y^* = \sqrt{\frac{2KD}{h}} \tag{1.1}$$

Thus, the optimum inventory policy for the model is

$$\text{Order } y^* = \sqrt{\tfrac{2KD}{h}} \text{ units after every } t_0^* = \tfrac{y^*}{D} \text{ time units.}$$

In real situation it is not possible that new orders are received instantly. But, a positive lead time, L, may occur between the placement and the receipt of an order. In this case, the reorder point occurs when the inventory level drops to LD units. Assume that the lead time L is less than the cycle length t_0^*, which may not be the case in general. For this situation we define the effective lead time as

$$L_e = L - nt_0^*$$

where n is the largest integer not exceeding $\frac{L}{t_0^*}$. Thus, the reorder point occurs at $L_e D$ units, and the inventory policy can be restated as

Order the quantity y^* whenever the inventory level drops to LD time units.

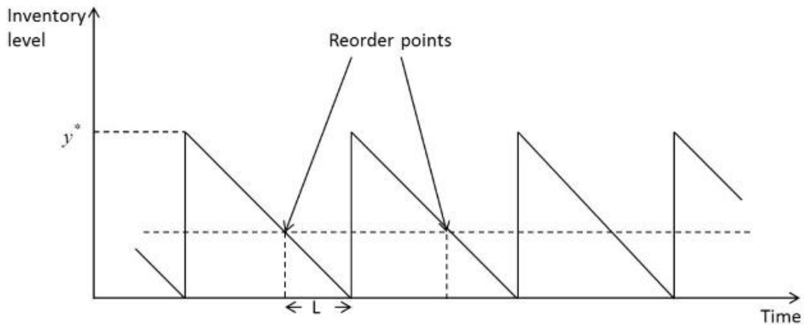

Figure 1.2: Reorder points

1.3 EOQ Model With Price Breaks

In this section, we will study inventory management when the unit purchasing cost decreases with the order quantity y. In other words, a discount is given by the seller if the buyer purchases a large number of units. The unit purchasing price, c, is given as

$$c = \begin{cases} c_1, & \text{if } 0 < y_1 < x_1 \\ c_2, & \text{if } x_1 \le y_2 < x_2 \\ c_3, & \text{if } x_2 \le y_3 \end{cases}$$

s.t $c_3 < c_2 < c_1$ Where x_1 and x_2 are limits (Price break points as shown in figure (1.2)) Therefore, Total cost per unit time is given by

$$TCU(y) = \begin{cases} TCU(y_1) = Dc_1 + \dfrac{KD}{y_1} + \dfrac{h_1 y_1}{2}, & 0 < y_1 < x_1 \\ TCU(y_2) = Dc_2 + \dfrac{KD}{y_2} + \dfrac{h_2 y_2}{2}, & x_1 \le y_2 < x_2 \\ TCU(y_3) = Dc_3 + \dfrac{KD}{y_3} + \dfrac{h_3 y_3}{2}, & x_2 \le y_3 \end{cases}$$

Here Holding cost $h_i = c_i.r$ for $i = 1, 2, 3$. Where r=per cent of rupee value of inventory (per cent per unit time) The optimal solution can be determined by algorithm given below:

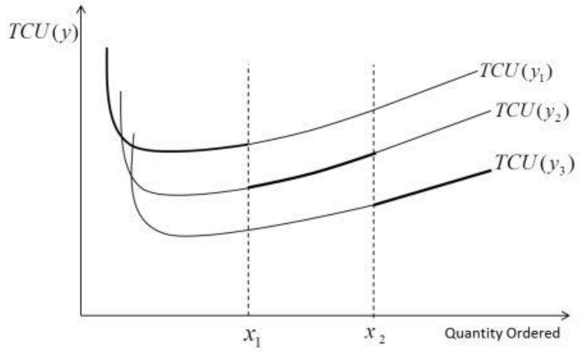

Figure 1.3: Price Breaks

Algorithm

1. Consider the lowest price ($i.e$ c_3) and find y_3^* by using basic EOQ formula given in equation (1.1).

   (a) If $y_3 \geq x_2$, then y_3^* is optimal ordered quantity.

   (b) If $y_3^* < x_2$, then go to step 2.

2. Calculate y_2^* based on c_2.

   (a) If $x_1 \leq y_2^* < x_2$, then compare $TCU(y_2^*)$ and $TCU(x_2)$. But if $TCU(y_2^*) \geq TCU(x_2)$, then $EOQ = x_2$. Otherwise $EOQ = y_2^*$.

   (b) If $y_3^* < x_1$ as well as x_2, then go to step 3.

3. Calculate y_1^* based on price c_1 and compare $TCU(x_1)$, $TCU(x_2)$, and $TCU(y_1^*)$ to find EOQ. The quantity with lowest cost will naturally be the required EOQ.

1.4 Multi-item Inventory Models

1.4.1 Introduction

In real life there are thousands of items in most of the inventories. In such situations the single-item models cannot help us to manage the large number of items. Some times a multi-item model is often necessary, especially when the number of items are very large. A lot of time and work (and therefore cost) may be saved by using multi-item model. Although there are, of course, important items which must be studied individually by using single-item model, but large majority of items (or some less important items) may be dealt with in groups using multi-item models to save time, work and money.

In this model production or supply is instantaneous with no lead time. Demand is uniform and deterministic and shortages are not allowed. Define

$n =$ Total number of items in inventory.

$f_i =$ Storage area required by the i^{th} item.

$W =$ Total available storage space to all items in the inventory.

$D_i =$ Rate of demand for i^{th} item.

$Q_i =$ Ordered quantit for i^{th} item.

$C_i =$ Price per unit of item item i.

$F =$ Total investment for all items.

$h_i =$ Holding cost for i^{th} item.

$K_i =$ Setup cost for i^{th} item.

1.4.2 Multi-item EOQ Model With Warehouse Space Constraint

In real life situations, a retailer often have limited space for their inventory. Under this constraint of limited space they have to take decision about the quantity of all items. So, in this model we minimize the total inventory cost under the constraint that available storage area is limited that is if i^{th} item of inventory requires area f_i than total area required by the all inventory items should less than or equal to total available area. Mathematically problem is given below.

$$\text{Minimize} \qquad TCU(Q_1, Q_2, \ldots, Q_n) = \sum_{i=1}^{n} \left[\frac{D_i K_i}{Q_i} + \frac{Q_i h_i}{2} \right]$$

subject to the space constraint $i.e$

$$\sum_{i=1}^{n} f_i Q_i \leq W$$

and $Q_i \geq 0$ for all $i = 1, 2, \ldots, n$.

1.4.3 Multi-item EOQ Model with Investment Constraint

Some times the retailer has limited investment to spend on the inventory. Thus the decision maker places a limit on the amount of inventory to be carried. Therefore, we will have to minimize the objective function so that the total cost of inventory should not exceed the fixed limit. If C_i is the cost of the i^{th} item and F is the total investment available than mathematically problem is given below.

$$\text{Minimize} \qquad TCU(Q_1, Q_2, \ldots, Q_n) = \sum_{i=1}^{n} \left[\frac{D_i K_i}{Q_i} + \frac{Q_i h_i}{2} \right]$$

subject to the investement constraint $i.e$

$$\sum_{i=1}^{n} C_i Q_i \leq F$$

and $Q_i \geq 0$ for all $i = 1, 2, \ldots, n$.

1.4.4 Multi-item EOQ Model with Average Inventory Level Constraint

In this model, there is condition imposed on average inventory level. Since the average number of units in the inventory of an item is $\frac{Q_i}{2}$ for all $i = 1, 2, \ldots, n$. Therefore the total average number of all items held in the inventory should not exceed the fixed inventory level M. Mathematically problem is given below.

$$\text{Minimize} \qquad TCU(Q_1, Q_2, \ldots, Q_n) = \sum_{i=1}^{n} \left[\frac{D_i K_i}{Q_i} + \frac{Q_i h_i}{2} \right]$$

subject to the investement constraint *i.e*

$$\frac{1}{2} \sum_{i=1}^{n} Q_i \leq M$$

and $Q_i \geq 0$ for all i

These type of models have been considered in subsequent chapters and corresponding solution procedure have been developed.

1.5 Present Work

In Chapter 2, a review of research paper entitled "Multi-item EOQ model while demand is sales price and price break sensitive" studied by Pal *et al.* (2012) has been reviewed and some observations have been made.

Based on the observations given in Chapter 2, an attempt has been made in Chapter 3 to find the solution of such type problems using KT conditions. In this chapter three multi-item inventory models has been studied. In Model I demand depends quadratically on the selling price, in Model II demand is negative power function of selling price and in Model III again demand is negative power function of selling price and purchasing cost is negative power function of demand.

In Chapter 4, three different multi item inventory models have been solved in which the objective function is same as in model I of chapter 3 subject to limited warehouse storage, limited investment on inventory and restriction on average inventory level constraints separately.

1.6 Literature Review

Inventory theory deals with the management of stock levels with the aim that demand for these items is met. Most of the models are developed to answer the two questions, first one is when an order should be placed? and the second is what ordered quantity should be? A lot of single items models had been developed till the date based on different assumptions like variable or probabilistic demand, zero-lead time lost sales and back ordering assumptions when in inventory demand exceeds supply etc. Wang [22] presented a model where he defined an appropriate time-dependent partial backlogging rate and introduces the opportunity cost due to lost sales. The effects of changes in the backlogging parameter and unit opportunity cost on the optimal total cost have been studied and optimal number of replenishment was carried out. Later on Wee *et al.* [19] presents a modified method to compute economic order quantities without derivatives by cost-difference comparisons. Extensions to allow the back orders were done for EOQ/EPQ models. Limiting values on a finite planning horizon were used rather than algebraic manipulations for the cost function comparisons. They used the cost-difference comparisons between two consecutive batch numbers for a finite horizon and the variable of batch size to express cost function rather than cycle length. The convergence of optimal batch size is derived rather than optimal cycle length. For the models with back-orders, they used fill rate to express the proportion with positive inventory in a cycle length.

Sana [18] presented an EOQ model over an infinite horizon for perishable items where demand was price dependent and partial backorder permitted. The rate of deterioration was taken to be time proportional and its was assumed that shortage occurs at starting inventory cycle. It is assumed that the demand rate is price dependent and the deterioration rate is taken to be time proportional. The SFI (Shortage Followed by Inventory) of replenishment is followed. An analytical optimal solution of the integrated average profit function was discussed for various partial backlogging issues. Also, new functions of price-dependent demand and deterioration have been introduced and the analysis of optimal solution have been done from general profit function. Finally, the optimal solution of the integrated profit function was discussed with appropriate numerical examples. The author developed the criterion for the optimal solution for the replenishment schedule , and proved that the optimal ordering policy is unique.

Sana [4] developed a finite time-horizon deterministic EOQ (Economic Order Quantity) model where the rate of demand decreases quadratically with selling price. Prices at different periods were considered as decision variables. The objective was to find the optimal ordering quantity and optimal sales prices that maximizes the vendors total profit. The author divides the time horizon into n equal periods with n different prices which were decision variables. A profit function has been formulated which has been maximized analytically.

Inventory control problems in real world usually involve multiple products. Multi item inventory models are often necessary for inventories holding thousand of items as studied by Lenard and Roy [25] presented the difficulties encountered in the practice of inventory control. It led to the conclusion that a large gap exists between theory and practice in inventory management. They presents the multi-item inventory control by defining the concepts of families and aggregate items. In inventory control there may be a large number of objectives, the most important ones generally concerning the overall service level and the average quantity of items held in stock. These objectives are naturally not determined at the item level but at a higher level where a large number of items are concerned. The author studied that single-item models are not appropriate to this type of management. It is only with a multi-item inventory model able to give a overview of the system that objectives may be set and that policies may be determined with respect to the objectives. Moreover, a lot of time and work (and therefore cost) may be saved by coordinating policies for the different items. Although there are, of course, important items which must be studied individually, large majority of items may be dealt with in groups.

Two item inventory model for deteriorating items with a linear stock dependent demand has been studied by Bhattacharya [9]. Classical inventory models generally dealt with a single-item. But in real world situation, a single-item inventory seldom occurs. It is a common experience that the presence of a second item in an inventory favors the demand of the first and vice-versa; the effect may be different in the two cases. This is why; the companies or the retailers deal with several items and stock them in their showrooms/warehouses. This leads to the idea of a multi-item inventory. Further author showed that from linear demand rate, it follows that more is the inventory, more is demand. They also mentioned that under proper restrictions on the model, a steady state optimal solution can always be calculated.

Haksever and Moussourakis [12] presented a mixed-integer programming model to optimize the two fundamental decisions of inventory management that is how much to order and when to order for ordering multiple inventory items subject to multiple resource constraints. It also determines whether a fixed cycle for all products or an independent cycle for each should be used for a lower total cost.

Brandimarte [16] considered a stochastic version of the classical multi-item capacitated lot sizing problem. Demand uncertainty was explicitly modeled through a scenario tree, resulting in a multi-stage mixed integer stochastic programming model with resource. The author proposed a plant-location-based model formulation and a heuristic solution approach based on a fix-and-relax strategy.

A multi-item EOQ model is developed by Sana [11] when the time varying demand is influenced by enterprises initiatives like advertising media and salesmen effort. He developed a model for deteriorating and ameliorating items with capacity constraint for storage facility. The effect of inflation and time value of money in the profit, cost parameter and associated profit was also considered. The associated profit function was maximized by Euler-Lagrange's method and it was illustrated by varying demands like quadratic, linear and exponential demand functions. The model provides the major contribution on the effect of advertising and salesmen's initiatives on demand - to operations in management practice.

Zhang [10] considered the multi-product newsboy problem with both supplier quantity discounts and a budget constraint, while each feature has been addressed separately. Different from most previous nonlinear optimization models on the topic, the problem was formulated as a mixed integer nonlinear programming model due to price discounts. A Lagrangian relaxation approach is presented to solve the problem. The purpose of this study was to investigate the affect of both a budget constraint and supplier quantity discounts on the optimal order quantities in a multi-product newsboy problem.

Kotab and Fergany [14] derived the analytical solution of the EOQ model of multiple items with both demand-dependent unit cost and leading time using geometric programming approach. The varying purchase and leading time crashing costs were considered to be continuous functions of demand rate and leading time, respectively. The aim of this study was to derive the optimal solution policy of EOQ inventory model and minimize the total cost function based on the values of demand rate, order quantity and leading time using geometric programming technique.

Pal *et al.* [7] studied a three-layer multi-item supply chain involving multiple suppliers, manufacturer and multiple retailers where each finished product is produced by the combination of the fixed percentage of various types of raw materials and each raw material supplier can supply only one material. Here, they consider that the manufacturer delivers finished products to the multiple retailers where each of the retailers sells their multiple products according to their demand in the market. Overall, the total integrated profit of the supply chain is evaluated and is optimized with respect to ordering lot sizes of the raw materials.

A multi-item deterministic economic order quantity model is developed by Pal *et al.* [3] for a retailer. They assumed that the demand rate of the items decreases quadratically with increase in selling price and increases exponentially with increase in level of price breaks. They have considered a restriction on level of price breaks. In this paper, according to the level of price break retailer (or inventory manager) gives discount on selling price of items to customer. If the total selling price of retail mall at any time t is more than the level of price breaks, then the retailer gives discount on selling price to customers. The author formulated a maximizing profit model by considering selling prices, ordering quantities of products and level of price breaks as decision variables with respect to the restriction on total selling price at any time t.

Barron and Sana [6] proposed an economic order quantity inventory model of multi-items in a two-layer supply chain where demand is sensitive to promotional effort. In this inventory model, the supplier offerd a delay period to the retailer for paying the outstanding amount of the purchasing cost for the finished products. The profit functions of the supplier and the retailer were formulated by considering the setup cost, holding cost, selling price, and promotional costs. They also compared collaborative and non-collaborative systems in terms of their average profits.

Chapter 2

Multi-item EOQ Model while Demand is Sales Price and Price Break Sensitive (Review of Research Paper)

2.1 Introduction

Pal *et al.* (2012) developed a multi-item deterministic economic order quantity model for a retailer. It is assumed that the demand rate of the items decreases quadratically with increase in selling price and increases exponentially with increase in level of price breaks with a restriction on level of price breaks. In this model, according to the level of price break retailer (or inventory manager) gives discount on selling price of items to customer. If the total selling price of retailer is more than the level of price breaks, then the retailer gives discount on selling price to customers. The author formulated a maximizing profit model by considering selling prices, ordering quantities of products and level of price breaks as decision variables with respect to the restriction on total selling price. The model is solved by using the method of Lagrangian multiplier. In this chapter above mentioned paper has been reviewed and some observations have been made and an attempt has been made to rewrite some of the expressions.

2.2 Fundamental assumptions and notation

2.2.1 Assumptions

The following assumptions were considered by B.Pal *et al.* (2012).

1. The model is multiple item model.
2. Ordered quantity and selling price for a product at each period is a decision variable.
3. Demand rate for a product at each period is dependent on selling price and level of price breaks.
4. If total selling price of the retailer at any time is more than the level of price breaks, then retailer offers a percentage of discount on price to the customers.
5. Holding cost and ordering cost for the products are different.

2.2.2 Notations

C_i = Unit purchasing cost for i^{th} item.
p_i = Unit selling price for i^{th} item.
B = Price breaks level.
$R(.)$ = Demand rate.
h_i = Holding Cost per unit item per unit time
C_{s_i} = Setup cost for i^{th} item.
Q_i = Quantity ordered for i^{th} item

2.3 Mathematical formulation and analysis of the model

B. Pal *et al.* (2012) developed the model by considering the demand dependent on selling price and level of price breaks.
Mathematically, demand function is given by

$$R(p_i, B) = a_i - b_i p_i - c_i p_i^2 + \alpha e^{-\beta B}$$

where $a_i, b_i, c_i, \alpha, \beta$ are all suitable positive constants and $a_i >> b_i >> c_i$ for $i = 1, 2, \ldots, n$. The total profit has been maximized with respect to discount criteria. Retailer offered discount to the customers on the basis that wether selling price crosses the price break level or not. According to this, the model was divided in the following two cases.

2.4 Case I

In this case the total selling price did not crosse the level of price breaks at any time t.

Here, the criteria of discount was not satisfied by the the total selling price of retail mall at any time t. So, retailer had not offered any discount to the customers. Total profit of retail mall per unit time was

$$E\pi(Q_i, p_i, B) = \sum_{i=1}^{n} \left[p_i R(p_i, B) - \frac{h_i Q_i}{2} - \frac{R(p_i, B) C_{s_i}}{Q_i} - C_i R(p_i, B) \right] \tag{2.1}$$

subject to total selling price at time t of vendor less than level of price breaks

$$\sum_{i=1}^{n} p_i R(p_i, B) < B \tag{2.2}$$

Solution Methodology :

To obtain the optimal solution of the problem given in (2.1) and (2.2) the method of Lagrangian multiplier has been used. The Lagrangian function L is given by

$$L(Q_i, p_i, B, \lambda) = \sum_{i=1}^{n} \left[p_i R(p_i, B) - \frac{h_i Q_i}{2} - \frac{R(p_i, B) C_{s_i}}{Q_i} - C_i R(p_i, B) \right] + \lambda \left[B - \sum_{i=1}^{n} p_i R(p_i, B) \right] \tag{2.3}$$

Here, L is the function of $2n + 2$ variables $p_1, p_2, \ldots, p_n, Q_1, Q_2, \ldots, Q_n, B$ and λ.

Necessary Conditions:

The necessary conditions for the maximum of $E\pi(Q_i, p_i, B)$ (where $i = 1, 2, \ldots, n$) give

$$\frac{\partial L}{\partial p_i} = (1 - \lambda)\big[R(p_i, B) - p_i(b_i + 2c_i p_i)\big] + \left(\frac{C_{s_i}}{Q_i} + C_i \right)(b_i + 2c_i p_i) = 0 \tag{2.4}$$

$$\frac{\partial L}{\partial Q_i} = -\frac{h_i}{2} + \frac{R(p_i, B) C_{s_i}}{Q_i^2} = 0 \tag{2.5}$$

$$\frac{\partial L}{\partial B} = \alpha\beta e^{-\beta B} \sum_{i=1}^{n} \left[\frac{C_{s_i}}{Q_i} - (1-\lambda)p_i + C_i \right] = 0 \tag{2.6}$$

$$\frac{\partial L}{\partial \lambda} = B - \sum_{i=1}^{n} p_i R(p_i, B) < 0 \tag{2.7}$$

Equations (2.5) and (2.6) give respectively

$$R(p_i, B) = \frac{h_i Q_i^2}{2C_{s_i}} \tag{2.8}$$

and

$$Q_i = \frac{C_{s_i}}{(\lambda - 1)p_i - C_i} \tag{2.9}$$

where $i = 1, 2, \ldots, n.$

Using values from equations (2.8) and (2.9) in equation (2.4), they have

$$(1-\lambda)\left[\frac{h_i C_{s_i}}{2((\lambda-1)p_i - C_i)^2} - p_i(b_i + 2c_i p_i) \right] + (\lambda - 1)p_i(b_i + 2c_i p_i) = 0$$

$$\Rightarrow 4((\lambda-1)p_i - C_i)^2 p_i(b_i + 2c_i p_i) = h_i C_{s_i} \tag{2.10}$$

where $i = 1, 2, \ldots, n$

They obtained the values of prices (p_i) by assuming the value of Lagrangian multiplier (λ) from Eq. (2.10). Using the values of p_i and λ in equation (2.8) and (2.9) the values of ordering lot sizes (Q_i) and level of price breaks (B) are obtained.

Sufficient condition:

A sufficient condition for $E\pi(Q_i, p_i, B)$ to have a relative maximum at $\{Q_i^*, p_i^*, B^*\}$ is that each root of the polynomial z_k defined by the following determinant equation be negative.

$$\Delta = \begin{vmatrix} L_{p_1p_1} - z & L_{p_1p_2} & \dots & L_{p_1p_n} & L_{p_1Q_1} & L_{p_1Q_2} & \dots & L_{p_1Q_n} & L_{p_1B} & g_{p_1} \\ L_{p_2p_1} & L_{p_2p_2} - z & \dots & L_{p_2p_n} & L_{p_2Q_1} & L_{p_2Q_2} & \dots & L_{p_2Q_n} & L_{p_2B} & g_{p_2} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ L_{p_np_1} & L_{p_np_2} & \dots & L_{p_np_n} - z & L_{p_nQ_1} & L_{p_nQ_2} & \dots & L_{p_nQ_n} & L_{p_nB} & g_{p_n} \\ L_{Q_1p_1} & L_{Q_1p_2} & \dots & L_{Q_1p_n} & L_{Q_1Q_1} - z & L_{Q_1Q_2} & \dots & L_{Q_1Q_n} & L_{Q_1B} & g_{Q_1} \\ L_{Q_2p_1} & L_{Q_2p_2} & \dots & L_{Q_2p_n} & L_{Q_2Q_1} & L_{Q_2Q_2} - z & \dots & L_{Q_2Q_n} & L_{Q_2B} & g_{Q_2} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ L_{Q_np_1} & L_{Q_np_2} & \dots & L_{Q_np_n} & L_{Q_nQ_1} & L_{Q_nQ_2} & \dots & L_{Q_nQ_n} - z & L_{Q_nB} & g_{Q_n} \\ L_{Bp_1} & L_{Bp_2} & \dots & L_{Bp_n} & L_{BQ_1} & L_{BQ_2} & \dots & L_{BQ_n} & L_{BB} - z & g_B \\ g_{p_1} & g_{p_2} & \dots & g_{p_n} & g_{Q_1} & g_{Q_2} & \dots & g_{Q_n} & g_B & 0 \end{vmatrix} \tag{2.11}$$

Where $X = \{p_1, p_2, \dots, p_n, Q_1, Q_2, \dots, Q_n, B\}$

$$\begin{aligned} L_{p_ip_j} &= \frac{\partial^2 L}{\partial p_i \partial p_j}(X^*, \lambda^*), \;\; L_{Q_iQ_j} = \frac{\partial^2 L}{\partial Q_i \partial Q_j}(X^*, \lambda^*), \\ L_{p_iQ_j} &= \frac{\partial^2 L}{\partial p_i \partial Q_j}(X^*, \lambda^*), \;\; L_{p_iB} = \frac{\partial^2 L}{\partial p_i \partial B}(X^*, \lambda^*), \\ L_{BB} &= \frac{\partial^2 L}{\partial B^2}(X^*, \lambda^*), \;\; g_{p_i} = \frac{\partial g}{\partial p_i}(X^*), \\ g_{Q_i} &= \frac{\partial g}{\partial Q_i}(X^*), \text{ and } \; g_B = \frac{\partial g}{\partial B}(X^*) \end{aligned}$$

Also, we have $L_{p_ip_j} = L_{Q_iQ_j} = L_{p_iQ_j} = 0$ for $i \neq j$ and $i, j = 1, 2, \dots, n$.

It has been mentioned that it is not possible to prove analytically that all the roots of the equation (2.11) for $\Delta = 0$ were negative; hence authors showed it numerically.

2.5 Case II

In this case total selling price of retail mall at any time t crosses the level of price breaks. Here, the criteria of discount was satisfied by total selling price of retail at a time t. So, the retailer provided a percentage of discount on selling price to the customers.

Total profit of retail mall per unit time is

$$E\pi(Q_i,p_i,B)_1=\sum_{i=1}^{n}\left[(1-\mu_i)p_iR((1-\mu_i)p_i,B)-\frac{h_iQ_i}{2}-\frac{R((1-\mu_i)p_i,B)C_{s_i}}{Q_i}-C_iR((1-\mu_i)p_i,B)\right] \tag{2.12}$$

subject to total selling price at time t of vendor more than level of price breaks.

$$\sum_{i=1}^{n}p_iR(p_i,B)\geq B \tag{2.13}$$

where μ_i is percentage of discount on price.

Solution Methodology:

The author applied the method of Lagrangian multipliers to optimize the profit function of retailer. In this case, the Lagrange function L_1 is defined as

$$L_1(Q_i,p_i,B,\lambda)=\sum_{i=1}^{n}\left[(1-\mu_i)p_iR(p_i,B)-\frac{h_iQ_i}{2}-\frac{R((1-\mu_i)p_i,B)C_{s_i}}{Q_i}-C_iR((1-\mu_i)p_i,B)\right] + \lambda\left[\sum_{i=1}^{n}p_iR(p_i,B)-B\right] \tag{2.14}$$

Necessary Conditions: The necessary conditions for optimum solution are

$$\frac{\partial L_1}{\partial p_i}=(1+\lambda)\left[R((1-\mu_i)p_i,B)-(1-\mu_i)^2p_i(b_i+2c_i(1-\mu_i)p_i)\right] + \left(\frac{C_{s_i}}{Q_i}+C_i\right)(1-\mu_i)(b_i+2c_i(1-\mu_i)p_i)=0 \tag{2.15}$$

$$\frac{\partial L_1}{\partial Q_i}=-\frac{h_i}{2}+\frac{R((1-\mu_i)p_i,B)C_{s_i}}{Q_i^2}=0 \tag{2.16}$$

$$\frac{\partial L_1}{\partial B}=\alpha\beta e^{-\beta B}\sum_{i=1}^{n}\left[\frac{C_{s_i}}{Q_i}-(1+\lambda-\mu_i)p_i+C_i\right]=0 \tag{2.17}$$

$$\frac{\partial L_1}{\partial \lambda}=\sum_{i=1}^{n}p_iR(p_i,B)-B\geq 0 \tag{2.18}$$

From equations (2.16) and (2.17), author concludes that,

$$R((1-\mu_i)p_i,B)=\frac{h_iQ_i^2}{2C_{s_i}} \tag{2.19}$$

$$Q_i = \frac{C_{s_i}}{(1+\lambda)p_i - C_i} \tag{2.20}$$

$$R(p_i, B) = \frac{h_i Q_i^2}{2C_{s_i}} - \mu_i b_i p_i - (2\mu_i - \mu_i^2)c_i p_i^2 \tag{2.21}$$

where $i = 1, 2, \ldots, n$.

From equation (2.15), (2.19) and (2.20), they had

$$(1+\lambda-\mu_i)h_i C_{s_i} + \{\lambda p_i(c_i p_i(4 - 2\mu_i + \mu_i^2) - 2b_i\mu_i)\}((1+\lambda-\mu_i)p_i - C_i^2) = 0 \tag{2.22}$$

where $i = 1, 2, \ldots, n$.

They obtained the values of prices (p_i) by assuming the value of Lagrangian multiplier (λ) from Eq. (2.22). Using the values of p_i and λ in equation (2.20) and (2.21) the values of ordering lot sizes (Q_i) and level of price breaks (B) are obtained.

Sufficient condition:

A sufficient condition for $E\pi(Q_i, p_i, B)$ to have a relative maximum at $\{Q_i^*, p_i^*, B^*\}$ is that each root of the polynomial z_k defined by the determinant equation (2.11) be negative. They proved numerically that all the roots of equation (2.11) for $\Delta = 0$ were negative, because it was not possible to prove it analytically.

2.6 Observations

On reviewing the solution methodology of the problem the following observations have been made.

2.6.1 Case I

1. From equation (2.2), it is clear that constraint is strict inequality type, so it is not possible to apply existing techniques for the solution of this problem.

2. In necessary conditions for the maximum of $E\pi(Q_i, p_i)$ there was no restriction on Lagrange's multiplier, if they consider it inequality constraint and used the extended

Lagrange's multiplier method then there should be restriction on Lagrange's multiplier (λ).

3. The authors used only positive values of Lagrange's multiplier in numerical example then the expression for equation (2.7) should be

$$\sum_{i=1}^{n} p_i R(p_i, B) - B < 0 \tag{2.23}$$

or the values of λ should be less than or equals to zero.

4. The derivative of lagrangian function (L) with respect to price break level B (equation (2.6)) is not correct. The correct expression for equation (2.6) should be as

$$\frac{\partial L}{\partial B} = \alpha\beta e^{-\beta B} \sum_{i=1}^{n} \left[\frac{C_{s_i}}{Q_i} - (1-\lambda)p_i + C_i \right] + \lambda = 0 \tag{2.24}$$

5. Expression for Q_i given in equation (2.6) is also not correct. The expression should be

$$Q_i = \frac{C_{s_i}}{(1-\lambda)p_i - C_i} \tag{2.25}$$

Even the above expression (2.25) is correct only for $\lambda = 0$. If we use the equation (2.24) in place of (2.6) the explicit expression for Q_i is not possible.

Similar observations can be made in the case II as in case I expect that the constraint is inequality constraint in place of strict inequality.

2.7 Future Scope

Based on above observations a different type of solution methodology is required for optimal solution of this inventory model. In chapter 3 an attempt has been made to overcome these observations by using a simple demand function (neglecting exponential term in demand function given by Pal *et al.*(2012).)

Chapter 3

Multi-item Inventory Model with Price Break and Demand Depends on Selling Price

3.1 Introduction

In chapter 2 a multi-item EOQ model "Multi-item EOQ model while demand is sales price and price break sensitive" has been discussed and some observations have been made on solution methodology given by Pal *et al.* (2012). Based on those observations an attempt has been made to find the optimal solution of such type of problems using KT conditions. However in place of the model considered by Pal *et al.* (2012) a simple model has been considered in which exponential term in demand function is neglected. In this chapter we considered the following three models.

In the first model it has been considered that demand of customers depends quadratically on the selling price only and the price break is fixed.

In second model, the demand is dependent as negative power function on the selling price only and the price break is fixed. The demand of customers decreases the negative power of selling price with increase in it.

In third model, again demand is dependent on selling price as negative power function of it and the purchasing cost price is dependent as the negative power function of demand.

In these models the total profit from the selling of products has been maximized and it is assumed that level of price break is fixed such that if total selling price of all items is more than this fixed price break level then retailer will give discount on selling price of each item to the customers.

3.2 Mathematical Formulation

The same assumptions and notations considered by B. Pal *et al.* (2012) as given in section (2.1.1) and (2.1.2) respectively in chapter 2 are used.

3.3 Model I

The model has been divided into two cases on the basis that total selling price is less than equals to or greater than equals to price break level.

3.3.1 Case I

In this case total selling price of the retailer is not more than the fixed level of price break. So here, retailer does not offer any discount on selling price to the customers. Mathematically problem is given below

$$\text{Maximize} \qquad E\pi = \sum_{i=1}^{n}\left[p_i R(p_i) - \frac{h_i Q_i}{2} - \frac{R(p_i) C_{s_i}}{Q_i} - C_i R(p_i)\right] \tag{3.1}$$

subject to the constraint

$$\sum_{i=1}^{n} p_i R(p_i) \leq B \tag{3.2}$$

where the demand depends on the selling price. The demand of customers decreases quadratically with increase in selling price and vice versa. The demand is given by the following equation.

$$R(p_i) = a_i - b_i p_i - c_i p_i^2$$

Where a_i, b_i, c_i are suitable positive constants and $a_i >> b_i >> c_i$ for $i = 1, 2, \ldots, n$.

Solution Methodology

Step 1: To solve the problem, find the unconstrained solution by using following equations.

$$\frac{\partial E\pi}{\partial p_i} = (a_i - b_i p_i - c_i p_i^2) + (b_i + 2c_i p_i)\left[-p_i + \frac{C_{s_i}}{Q_i} + C_i\right] = 0 \tag{3.3}$$

$$\frac{\partial E\pi}{\partial Q_i} = -\frac{h_i}{2} + \frac{(a_i - b_i p_i - c_i p_i^2)\, C_{s_i}}{Q_i^2} = 0 \tag{3.4}$$

where $i = 1, 2, \ldots, n$.

By solving these equations, the ordered quantities Q_i and selling prices p_i for $i = 1, 2, \ldots, n$ are obtained.

Step 2: If this solution satisfy the constraint (3.2), then we are done. Otherwise, the constraint must be activated and go to step 3.

Step 3: Then use the KT conditions to determine the constrained optimal solution.

Necessary Conditions:

The necessary conditions for maximization of problem given in equation (3.1) and (3.2) are

$$\lambda \leq 0 \tag{3.5}$$

$$\nabla E\pi(Q_i, p_i) + \lambda \nabla \left(\sum_{i=1}^{n} p_i R(p_i) - B\right) = 0 \tag{3.6}$$

$$\lambda \left(\sum_{i=1}^{n} p_i R(p_i) - B\right) = 0 \tag{3.7}$$

$$\sum_{i=1}^{n} p_i R(p_i) - B \leq 0 \tag{3.8}$$

Where λ is Lagrangian multiplier.

From equation (3.6) following equations are obtained.

$$(\lambda + 1)\left[\left(a_i - b_i p_i - c_i p_i^2\right) - p_i(b_i + 2c_i p_i)\right] + (b_i + 2c_i p_i)\left[\frac{C_{s_i}}{Q_i} + C_i\right] = 0 \tag{3.9}$$

and

$$-\frac{h_i}{2} + \frac{(a_i - b_i p_i - c_i p_i^2)\, C_{s_i}}{Q_i^2} = 0 \tag{3.10}$$

From equation (3.10), we get

$$Q_i = \left(\frac{2\,(a_i - b_i p_i - c_i p_i^2)\, C_{s_i}}{h_i}\right)^{\frac{1}{2}} \tag{3.11}$$

From equations (3.11) and (3.9), we get

$$(\lambda+1)[(a_i - b_i p_i - c_i p_i^2) - p_i(b_i + 2c_i p_i)] \\ + (b_i + 2c_i p_i)\left[\left(\frac{C_{s_i} h_i}{2\,(a_i - b_i p_i - c_i p_i^2)}\right)^{\frac{1}{2}} + C_i\right] = 0 \tag{3.12}$$

where $i = 1, 2, \ldots, n$.

By solving above equations (3.12) and (3.11) we get the values of p_i, Q_i and λ.

Sufficient Condition:

The KT conditions are also sufficient if the objective function that is $E\pi$ is concave and solution space is a convex function.

3.3.2 Numerical Example

Let us consider a retail mall with three items with the following parameter values where fixed price break Level(B) is \$13,000.

	Item 1	Item 2	Item 3
Parameters of demand function	$a_1 = 170, b_1 = 1, c_1 = 0.005$	$a_2 = 146, b_2 = 1.1, c_2 = 0.006$	$a_3 = 129, b_3 = 0.9, c_3 = 0.004$
h_i	\$0.5	\$0.6	\$0.45
C_i	\$9	\$7	\$8
C_{s_i}	\$150	\$200	\$140

Firstly while computing unconstrained solution from equation (3.3) and (3.4), the values of p_i and Q_i for $i = 1, 2, \ldots, n$ are obtained as given by the following table.

	Item 1	Item 2	Item 3
Selling price per unit item *i.e* p_i	\$63.054	\$51.13	\$56.74
Ordering lot size per order *i.e* Q_i	228.56	222.22	201.19
Total Profit		$E\pi = \$10,807.78$	

Since unconstrained solution satisfied the constraint, so there is no need to activate the constraint.

Check for optimality:

Since the hessian matrix given below

$$H = \begin{bmatrix} \frac{\partial^2 E\pi}{\partial p_1^2} & \frac{\partial^2 E\pi}{\partial p_1 \partial p_2} & \frac{\partial^2 E\pi}{\partial p_1 \partial p_3} & \frac{\partial^2 E\pi}{\partial p_1 \partial Q_1} & \frac{\partial^2 E\pi}{\partial p_1 \partial Q_2} & \frac{\partial^2 E\pi}{\partial p_1 \partial Q_3} \\ \frac{\partial^2 E\pi}{\partial p_2 \partial p_1} & \frac{\partial^2 E\pi}{\partial p_2^2} & \frac{\partial^2 E\pi}{\partial p_2 \partial p_3} & \frac{\partial^2 E\pi}{\partial p_2 \partial Q_1} & \frac{\partial^2 E\pi}{\partial p_2 \partial Q_2} & \frac{\partial^2 E\pi}{\partial p_2 \partial Q_3} \\ \frac{\partial^2 E\pi}{\partial p_3 \partial p_1} & \frac{\partial^2 E\pi}{\partial p_3 \partial p_2} & \frac{\partial^2 E\pi}{\partial p_3^2} & \frac{\partial^2 E\pi}{\partial p_3 \partial Q_1} & \frac{\partial^2 E\pi}{\partial p_3 \partial Q_2} & \frac{\partial^2 E\pi}{\partial p_3 \partial Q_3} \\ \frac{\partial^2 E\pi}{\partial Q_1 \partial p_1} & \frac{\partial^2 E\pi}{\partial Q_1 \partial p_2} & \frac{\partial^2 E\pi}{\partial Q_1 \partial p_3} & \frac{\partial^2 E\pi}{\partial Q_1^2} & \frac{\partial^2 E\pi}{\partial Q_1 \partial Q_2} & \frac{\partial^2 E\pi}{\partial Q_1 \partial Q_3} \\ \frac{\partial^2 E\pi}{\partial Q_2 \partial p_1} & \frac{\partial^2 E\pi}{\partial Q_2 \partial p_2} & \frac{\partial^2 E\pi}{\partial Q_2 \partial p_3} & \frac{\partial^2 E\pi}{\partial Q_2 \partial Q_1} & \frac{\partial^2 E\pi}{\partial Q_2^2} & \frac{\partial^2 E\pi}{\partial Q_2 \partial Q_3} \\ \frac{\partial^2 E\pi}{\partial Q_3 \partial p_1} & \frac{\partial^2 E\pi}{\partial Q_3 \partial p_2} & \frac{\partial^2 E\pi}{\partial Q_3 \partial p_3} & \frac{\partial^2 E\pi}{\partial Q_3 \partial Q_1} & \frac{\partial^2 E\pi}{\partial Q_3 \partial Q_2} & \frac{\partial^2 E\pi}{\partial Q_3^2} \end{bmatrix} \tag{3.13}$$

is negative semi definite and positive semi definite for objective function and constraint respectively. So the objective function is concave and solution space is convex, hence the solution given in above table is optimal.

3.3.3 Case II

In this case total selling price of retailer is more than the level of price break. The retailer offers the discount on selling price to the customers. Let μ_i is the percentage of discount for the i^{th} item. The objective function in this case is given as follows.

$$E\pi_1(Q_i, p_i) = \sum_{i=1}^{n} \left[(1-\mu_i)p_i R((1-\mu_i)p_i) - \frac{h_i Q_i}{2} - \frac{R((1-\mu_i)p_i)C_{s_i}}{Q_i} - C_i R((1-\mu_i)p_i) \right] \tag{3.14}$$

subject to the constraint that total selling price at any time t is more than the level of price break.

$$\sum_{i=1}^{n} p_i R(p_i) \geq B$$

$$\Rightarrow B \leq \sum_{i=1}^{n} p_i R(p_i) \tag{3.15}$$

Solution Methodology:

Step 1: To solve the problem, find the unconstrained solution first by using following equations.

$$\frac{\partial E\pi_1}{\partial Q_i} = (1-\mu_i)R((1-\mu_i)p_i + (1-\mu_i)(b_i + 2c_i(1-\mu_i)p_i)\left(-(1-\mu_i)p_i + \frac{C_{s_i}}{Q_i} + C_i\right) = 0 \tag{3.16}$$

$$\frac{\partial E\pi_1}{\partial Q_i} = -\frac{h_i}{2} + \frac{R((1-\mu_i)p_i)C_{s_i}}{Q_i^2} = 0 \tag{3.17}$$

where $i = 1, 2, \ldots, n$

By solving these equations, the ordered quantities Q_i and selling prices p_i for $i = 1, 2, \ldots, n$ are obtained.

Step 2: If this solution satisfies the constraint (3.15), then we are done. Otherwise constraint must be activated and go to step 3.

Step 3: Use the KT conditions to determine the constrained optimal solution.

Necessary Conditions:

The necessary conditions for maximization of problem given in equations (3.14) and (3.15) are

$$\lambda \leq 0 \tag{3.18}$$

$$\nabla E\pi_1(Q_i, p_i) + \lambda\nabla\left(B - \sum_{i=1}^{n} p_i R(p_i)\right) = 0 \tag{3.19}$$

$$\lambda\left[B - \sum_{i=1}^{n} p_i R(p_i)\right] = 0 \tag{3.20}$$

$$B - \sum_{i=1}^{n} p_i R(p_i) \leq 0 \tag{3.21}$$

Where λ is Lagrangian multiplier.

From equation (3.19), we have

$$-(1-\mu_i)^2 p_i(b_i + 2c_i(1-\mu_i)p_i) + (1-\mu_i)R((1-\mu_i)p_i) + (1-\mu_i)(b_i + 2c_i(1-\mu_i)p_i)\left(\frac{C_{s_i}}{Q_i} + C_i\right) + \lambda[R(p_i) + p_i(b_i + 2c_i p_i)] = 0 \tag{3.22}$$

and

$$-\frac{h_i}{2} + \frac{R((1-\mu_i)p_i)C_{s_i}}{Q_i^2} = 0 \tag{3.23}$$

From equation (3.23), we have

$$Q_i = \left(\frac{2R((1-\mu_i)p_i)C_{s_i}}{h_i}\right)^{\frac{1}{2}} \tag{3.24}$$

From equations (3.22) and (3.24), we have

$$\begin{aligned} &-(1-\mu_i)^2 p_i(b_i + 2c_i(1-\mu_i)p_i) + (1-\mu_i)R((1-\mu_i)p_i) \\ &+ (1-\mu_i)(b_i + 2c_i(1-\mu_i)p_i)\left(\left(\frac{h_i C_{s_i}}{2R((1-\mu_i)p_i)}\right)^{\frac{1}{2}} + C_i\right) \\ &+ \lambda[R(p_i) + p_i(b_i + 2c_i p_i)] = 0 \end{aligned} \tag{3.25}$$

where $i = 1, 2, \ldots, n$.

By solving above equations (3.25) and (3.24) we get the values of p_i, Q_i and λ.

Sufficient Condition:

It has been proved numerically that the hessian matrix of $E\pi(Q_i, p_i)$ is negative semi definite and the hessian matrix of the constraint $B - \sum_{i=1}^{n} p_i R(p_i) \leq 0$ is positive definite.

3.3.4 Numerical Example

Let us consider a retail mall with three items with the following parameter values where fixed price break level(B) is \$12,000.

	Item 1	Item 2	Item 3
Parameters of demand function	$a_1 = 161, b_1 = 1, c_1 = 0.005$	$a_2 = 166, b_2 = 1.1, c_2 = 0.006$	$a_3 = 135, b_3 = 0.9, c_3 = 0.006$
h_i	\$0.5	\$0.6	\$0.45
C_i	\$9	\$7	\$8
C_{s_i}	\$300	\$350	\$320
μ_i	10%	12%	9%

Firstly while computing unconstrained solution from equation (3.16) and (3.17), the values of p_i and Q_i for $i = 1, 2, \ldots, n$ are obtained as given in following table.

	Item 1	Item 2	Item 3
Selling price per unit item *i.e* p_i	\$67.54	\$63.84	\$59.12
Ordering lot size per order *i.e* Q_i	313.18	315.41	313.75
Total Profit		$E\pi = \$11,109.23$	

Since unconstrained solution satisfied the constraint, so there is no need to activate the constraint.

Check for optimality:

Since the hessian matrix (3.13) is negative semi definite and positive semi definite for objective function and constraint respectively. So the objective function is concave and solution space is convex, hence the solution given in above table is optimal.

3.4 Model II

On the basis that wether total selling price satisfies the criteria of discount or not the model has divided into two cases.

3.4.1 Case I

When total selling price of the retailer does not crosses the fixed level of price break. In this case retailer does not offer any discount to the customers.

The objective function is given below.

$$\text{Maximize} \qquad E\pi(Q_i, p_i) = \sum_{i=1}^{n} \left[p_i R(p_i) - \frac{h_i Q_i}{2} - \frac{R(p_i) C_{s_i}}{Q_i} - R(p_i) C_i \right] \tag{3.26}$$

subject to the constraint

$$\sum_{i=1}^{n} p_i R(p_i) \leq B \tag{3.27}$$

where the demand of customers is dependent as the negative power of selling price. Mathematically, demand of customers depends on the selling price as:

$$R(p_i) = \alpha p_i^{-\epsilon}$$

Where $\alpha > 0$ and $\epsilon > 1$

Solution Methodology:

Step 1: To solve the problem, find the unconstrained solution first by using following equations.

$$\frac{\partial E\pi}{\partial p_i} = \alpha(1-\epsilon)p_i^{-\epsilon} + \alpha\epsilon p_i^{-\epsilon-1}\left[\frac{C_{s_i}}{Q_i} + C_i\right] = 0 \tag{3.28}$$

$$\frac{\partial E\pi}{\partial Q_i} = -\frac{h_i}{2} + \frac{\alpha p_i^{-\epsilon} C_{s_i}}{Q_i^2} = 0 \tag{3.29}$$

where $i = 1, 2, \ldots, n$.

By solving these equations, the ordered quantities Q_i and selling prices p_i for $i = 1, 2, \ldots, n$ are obtained.

Step 2: If this solution satisfy the constraint (given in equation (3.27)), then we are done. Otherwise constraint must be activated and go to step 3.

Step 3: Then use the KT conditions to determine the constrained optimal solution.

Necessary Conditions:

The necessary (KT) conditions for maximization of problem given in equations (3.26) and (3.27) are

$$\lambda \leq 0 \tag{3.30}$$

$$\nabla E\pi(Q_i, p_i) + \lambda\nabla\left(\alpha\sum_{i=1}^{n} p_i^{-\epsilon+1} - B\right) = 0 \tag{3.31}$$

$$\lambda\left(\alpha\sum_{i=1}^{n} p_i^{-\epsilon+1} - B\right) = 0 \tag{3.32}$$

$$\alpha\sum_{i=1}^{n} p_i^{-\epsilon+1} - B \leq 0 \tag{3.33}$$

Where λ is the Lagrangian multiplier.

From equation (3.31), we have

$$\alpha(1+\lambda)(1-\epsilon)p_i^{-\epsilon} + \alpha\epsilon p_i^{-\epsilon-1}\left[\frac{C_{s_i}}{Q_i} + C_i\right] = 0 \tag{3.34}$$

and

$$-\frac{h_i}{2} + \frac{\alpha p_i^{-\epsilon} C_{s_i}}{Q_i^2} = 0 \tag{3.35}$$

From equation (3.35), we have

$$Q_i = \left(\frac{2\alpha p_i^{-\epsilon} C_{s_i}}{h_i}\right)^{\frac{1}{2}} \tag{3.36}$$

From equation equation (3.36) and (3.34), we have

$$\alpha(1+\lambda)(1-\epsilon)p_i^{-\epsilon} + \alpha\epsilon p_i^{-\epsilon-1}\left[\left(\frac{h_i C_{s_i}}{2\alpha p_i^{-\epsilon}}\right)^{\frac{1}{2}} + C_i\right] = 0 \tag{3.37}$$

where $i = 1, 2, \ldots, n$.

By solving above equations (3.37) and (3.36) we get the values of p_i, Q_i and λ.

Sufficient Condition:

The KT conditions are also sufficient if the objective function that is $E\pi(Q_i, p_i)$ is concave and solution space that is constraint is a convex function.

3.4.2 Numerical Example

Let us consider a retail mall with three items with the following parameter values where fixed price break level(B)=\$3500, $\alpha = 55600, \epsilon = -2.5$

	Item 1	Item 2	Item 3
h_i	\$0.5	\$0.6	\$0.45
C_i	\$9	\$7	\$8
C_{s_i}	\$150	\$200	\$140

Firstly while computing unconstrained solution from equation (3.28) and (3.29), the values of p_i and Q_i for $i = 1, 2, \ldots, n$ are obtained as given in following table.

	Item 1	Item 2	Item 3
Selling price per unit item $i.e$ p_i	\$16.43	\$13.02	\$14.45
Ordering lot size per order $i.e$ Q_i	174.58	246.15	208.75
Total Profit	$E\pi = \$1047.61$		

Since unconstrained solution satisfied the constraint, so there is no need to activate the constraint.

Check for optimality:

Since the hessian matrix (3.13) is negative semi definite and positive semi definite for objective function and constraint respectively. So the objective function is concave and solution space is convex, hence the solution given in above table is optimal.

3.4.3 Case II

When the total selling price of retailer crosses the level of price break. In this case, total selling price is more than the fixed price break level B and the retailer offers the discount to the customers. Let μ_i is the percentage of discount for the i^{th} item. The objective function in this case is given as follows.

$$E\pi_1(Q_i, p_i) = \sum_{i=1}^{n}\left[\alpha(1-\mu_i)^{-\epsilon+1}p_i^{-\epsilon+1} - \frac{h_iQ_i}{2} - \frac{\alpha(1-\mu_i)^{-\epsilon}p_i^{-\epsilon}C_{s_i}}{Q_i} - \alpha(1-\mu_i)^{-\epsilon}p_i^{-\epsilon}C_i\right] \quad (3.38)$$

subject to the constraint that total selling price at any time t is more than the level of price break.

$$\alpha\sum_{i=1}^{n} p_i^{-\epsilon+1} \geq B$$

$$\Rightarrow B \leq \alpha\sum_{i=1}^{n} p_i^{-\epsilon+1} \quad (3.39)$$

Solution Methodology:

Step 1: To solve the problem, find the unconstrained solution first by using following equations.

$$\frac{\partial E\pi_1}{\partial Q_i} = \alpha(1-\epsilon)(1-\mu_i)^{-\epsilon+1}p_i^{-\epsilon} + \alpha\epsilon(1-\mu_i)^{-\epsilon}p_i^{-\epsilon-1}\left[\frac{C_{s_i}}{Q_i} + C_i\right] \quad (3.40)$$

$$\frac{\partial E\pi_1}{\partial Q_i} = -\frac{h_i}{2} + \frac{\alpha(1-\mu_i)^{-\epsilon}p_i^{-\epsilon}C_{s_i}}{Q_i^2} = 0 \quad (3.41)$$

where $i = 1, 2, \ldots, n$

By solving these equations, the ordered quantities Q_i and selling prices p_i for $i = 1, 2, \ldots, n$ are obtained.

Step 2: If this solution satisfies the constraint (equation (3.39)), then we are done. Otherwise constraint must be activated and go to step 3.

Step 3: Use the KT conditions to determine the constrained optimal solution.

Necessary Conditions:

The necessary conditions for maximization of problem given in equations (3.38) and (3.39) are

$$\lambda \leq 0 \tag{3.42}$$

$$\nabla E\pi_1(Q_i, p_i) + \lambda \nabla \left(B - \alpha \sum_{i=1}^{n} p_i^{-\epsilon+1} \right) = 0 \tag{3.43}$$

$$\lambda \left[B - \alpha \sum_{i=1}^{n} p_i^{-\epsilon+1} \right] = 0 \tag{3.44}$$

$$B - \alpha \sum_{i=1}^{n} p_i^{-\epsilon+1} \leq 0 \tag{3.45}$$

Where λ is the Lagrangian multiplier.

From equation (3.43), we have

$$\begin{aligned} &\alpha(1-\epsilon)(1-\mu_i)^{-\epsilon+1} p_i^{-\epsilon} + \alpha\epsilon(1-\mu_i)^{-\epsilon} p_i^{-\epsilon-1} \left[\frac{C_{s_i}}{Q_i} + C_i \right] \\ &+ \lambda[\alpha p_i^{-\epsilon}(\epsilon - 1)] = 0 \end{aligned} \tag{3.46}$$

and

$$-\frac{h_i}{2} + \frac{\alpha(1-\mu_i)^{-\epsilon} p_i^{-\epsilon} C_{s_i}}{Q_i^2} = 0 \tag{3.47}$$

From equation (3.47), we have

$$Q_i = \left(\frac{2\alpha(1-\mu_i)^{-\epsilon} p_i^{-\epsilon} C_{s_i}}{h_i} \right)^{\frac{1}{2}} \tag{3.48}$$

From equations (3.48) and (3.46), we have

$$\begin{aligned} &\alpha(1-\epsilon)(1-\mu_i)^{-\epsilon+1} p_i^{-\epsilon} + \alpha\epsilon(1-\mu_i)^{-\epsilon} p_i^{-\epsilon-1} \left[\left(\frac{h_i C_{s_i}}{2\alpha(1-\mu_i)^{-\epsilon} p_i^{-\epsilon}} \right)^{\frac{1}{2}} + C_i \right] \\ &+ \lambda[\alpha p_i^{-\epsilon}(\epsilon - 1)] = 0 \end{aligned} \tag{3.49}$$

where $i = 1, 2, \ldots, n$.

By solving above equations (3.49) and (3.48) we get the values of p_i, Q_i and λ.

Sufficient Condition:

It has been proved numerically that the hessian matrix of $E\pi(Q_i, p_i)$ is negative semi definite and the hessian matrix of the constraint $B - \alpha \sum_{i=1}^{n} p_i^{-\epsilon+1} \leq 0$ is positive definite.

3.4.4 Numerical Example

Let us consider a retail mall with three items with the following parameter values where fixed price break level(B)=$5,000, $\alpha = 146000$ and $\epsilon = 2.5$

	Item 1	Item 2	Item 3
h_i	$0.5	$0.6	$0.45
C_i	$9	$7	$8
C_{s_i}	$300	$350	$320
μ_i	10%	12%	9%

Firstly while computing unconstrained solution from equation (3.40) and (3.41), the values of p_i and Q_i for $i = 1, 2, \ldots, n$ are obtained as given in following table.

	Item 1	Item 2	Item 3
Selling price per unit item *i.e* p_i	$18.03	$14.48	$15.79
Ordering lot size per order *i.e* Q_i	63.80	542.09	515.09
Total Profit		$E\pi = \$2,409$	

Since unconstrained solution satisfied the constraint, so there is no need to activate the constraint.

Check for optimality:

Since the hessian matrix (3.13) is negative semi definite and positive semi definite for objective function and constraint respectively. So the objective function is concave and solution space is convex, hence the solution given in above table is optimal.

3.5 Model III

3.5.1 Case I

When total selling price of the retailer does not crosses the fixed level of price break. In this case retailer does not offer any discount to the customers. Therefore the objective function that is profit is given below.

$$\text{Maximize} \qquad E\pi(Q_i, p_i) = \sum_{i=1}^{n}\left[p_iR(p_i) - \frac{h_iQ_i}{2} - \frac{p_iR(p_i)C_{s_i}}{Q_i} - R(p_i)C(R(p_i))\right] \tag{3.50}$$

subject to constraint

$$\sum_{i=1}^{n} p_iR(p_i) \leq B \tag{3.51}$$

Where the demand depends on the selling price and also purchasing cost price depends on the demand. Purchasing cost price also dependent on the negative power of demand rate. The demand and purchasing cost price are given below:

$$R(p_i) = ap_i^{-\alpha}$$

$$C(R_i) = bR_i^{-\beta}$$

Where $a > 0$, $\alpha > 1$, $b > 0$ and $0 < \beta < 1$

Solution Methodology:

Step 1: To solve the problem, find the unconstrained solution first by using following equations.

$$\frac{\partial E\pi}{\partial p_i} = a(-\alpha+1)p_i^{-\alpha} + \frac{a\alpha p_i^{-\alpha-1}C_{s_i}}{Q_i} - b\alpha(\beta-1)a^{-\beta+1}p_i^{\alpha(\beta-1)-1} = 0 \tag{3.52}$$

$$\frac{\partial E\pi}{\partial Q_i} = -\frac{h_i}{2} + \frac{ap_i^{-\alpha}C_{s_i}}{Q_i^2} = 0 \tag{3.53}$$

where $i = 1, 2, \ldots, n$

By solving these equations, the ordered quantities Q_i and selling prices p_i for $i = 1, 2, \ldots, n$ are obtained.

Step 2: If this solution satisfies the constraint (equation (3.51)), then we are done. Otherwise constraint must be activated and go to step 3.

Step 3: Then use the KT conditions to determine the constrained optimal solution.

Necessary Conditions:

The necessary conditions for maximization of problem given in equations (3.50) and (3.51) are

$$\lambda \leq 0 \tag{3.54}$$

$$\nabla E\pi(Q_i, p_i) + \lambda\nabla\left(a\sum_{i=1}^{n} p_i^{-\alpha+1} - B\right) = 0 \tag{3.55}$$

$$\lambda\left[a\sum_{i=1}^{n} p_i^{-\alpha+1} - B\right] = 0 \tag{3.56}$$

$$a\sum_{i=1}^{n} p_i^{-\alpha+1} - B \leq 0 \tag{3.57}$$

Where λ is the Lagrangian multiplier.

From equation (3.55), we have

$$\begin{aligned}(1+\lambda)(a(-\alpha+1)p_i^{-\alpha}) + \frac{a\alpha p_i^{-\alpha-1}C_{s_i}}{Q_i} \\ - b\alpha(\beta-1)a^{-\beta+1}p_i^{\alpha(\beta-1)-1} = 0\end{aligned} \tag{3.58}$$

and

$$-\frac{h_i}{2} + \frac{ap_i^{-\alpha}C_{s_i}}{Q_i^2} = 0 \tag{3.59}$$

From equation (3.59), we have

$$Q_i = \left(\frac{2ap_i^{-\alpha}C_{s_i}}{h_i}\right)^{\frac{1}{2}} \tag{3.60}$$

From equation equation (3.58) and (3.60), we have

$$(1+\lambda)(a(-\alpha+1)p_i^{-\alpha}) + \alpha a^{\frac{1}{2}}p_i^{-\frac{\alpha}{2}-1}\left(\frac{C_{s_i}h_i}{2}\right)^{\frac{1}{2}} - b\alpha(\beta-1)a^{-\beta+1}p_i^{\alpha(\beta-1)-1} = 0 \tag{3.61}$$

where $i = 1, 2, \ldots, n$.

By solving above equations (3.61) and (3.60) we get the values of p_i, Q_i and λ.

Sufficient Condition:

The KT conditions are also sufficient if the objective function that is $E\pi(Q_i, p_i)$ is concave and solution space that is constraint is a convex function.

3.5.2 Numerical Example

Let us consider a retail mall with three items with the following parameter values where fixed price break level(B)=$10,500, $\alpha = 2.5, \epsilon = 0.2, a = 500000$ and $b = 5$

	Item 1	Item 2	Item 3
h_i	\$0.5	\$0.6	\$0.45
C_{s_i}	\$150	\$200	\$140

Firstly while computing unconstrained solution from equation (3.31) and (3.32), the values of p_i and Q_i for $i = 1, 2, \ldots, n$ are obtained but these values does not satisfied the constraint. Therefore by activating constraint and then using KT conditions optimal solution has been determined. By assuming value of $\lambda = -0.872676$ and using in equations (3.50) and (3.49), we have values of p_i and Q_i given in following table.

	Item 1	Item 2	Item 3
Selling price per unit item *i.e* p_i	\$26.01958648	\$33.58926351	\$24.33879064
Ordering lot size per order *i.e* Q_i	294.74	225.78	326.28
Total Profit	$E\pi = \$9,336.34$		

Check for optimality:

Since the hessian matrix (3.13) is negative semi definite and positive semi definite for objective function and constraint respectively. So the objective function is concave and solution space is convex, hence the solution given in above table is optimal.

3.5.3 Case II

When the total selling price of retailer crosses the level of price break. In this case, total selling price is more than the fixed price break level B and the retailer offers the discount to the customers. Let μ_i is the percentage of discount for the i^{th} item. The objective function in this case is given as follows.

$$E\pi_1(Q_i, p_i) = \sum_{i=1}^{n}\left[(1-\mu_i)p_iR((1-\mu_i)p_i) - \frac{h_iQ_i}{2} - \frac{R((1-\mu_i)p_i)C_{s_i}}{Q_i} - C_iR((1-\mu_i)p_i)\right] \quad (3.62)$$

Subject to the constraint that total selling price at any time t is more than the level of price break.

$$\sum_{i=1}^{n} p_iR(p_i) \geq B$$

$$\Rightarrow B \leq \sum_{i=1}^{n} p_i R(p_i) \tag{3.63}$$

Solution Methodology:

Step 1: To solve the problem, find the unconstrained solution first by using following equations.

$$\begin{aligned}\frac{\partial E\pi_1}{\partial p_i} &= a(1-\alpha)(1-\mu_i)^{-\alpha+1}p_i^{-\alpha} + \frac{a\alpha(1-\mu_i)^{-\alpha}p_i^{-\alpha-1}C_{s_i}}{Q_i} \\ &+ b\alpha(1-\beta)a^{-\beta+1}(1-\mu_i)^{\alpha(\beta-1)}p_i^{\alpha(\beta-1)-1} = 0\end{aligned} \tag{3.64}$$

$$\frac{\partial E\pi_1}{\partial p_i} = -\frac{h_i}{2} + \frac{a(1-\mu_i)^{-\alpha}p_i^{-\alpha}C_{s_i}}{Q_i^2} = 0 \tag{3.65}$$

where $i = 1, 2, \ldots, n$

By solving these equations, the ordered quantities Q_i and selling prices p_i for $i = 1, 2, \ldots, n$ are obtained.

Step 2: If this solution satisfies the constraint (equation (3.63)), then we are done. Otherwise constraint must be activated and go to step 3.

Step 3: Use the KT conditions to determine the constrained optimal solution.

Necessary Conditions:

The necessary conditions for maximization of problem given in equations (3.62) and (3.63) are

$$\lambda \leq 0 \tag{3.66}$$

$$\nabla E\pi_1(Q_i, p_i) + \lambda \nabla \left(B - a\sum_{i=1}^{n} p_i^{-\alpha+1} \right) = 0 \tag{3.67}$$

$$\lambda \left[B - \sum_{i=1}^{n} a p_i^{-\alpha+1} \right] = 0 \tag{3.68}$$

$$B - \sum_{i=1}^{n} a p_i^{-\alpha+1} \leq 0 \tag{3.69}$$

Where λ is the Lagrangian multiplier.

From equation (3.67), we have

$$\begin{aligned}&a(1-\alpha)(1-\mu_i)^{-\alpha+1}p_i^{-\alpha} + \frac{a\alpha(1-\mu_i)^{-\alpha}p_i^{-\alpha-1}C_{s_i}}{Q_i} \\ &+ b\alpha(1-\beta)a^{-\beta+1}(1-\mu_i)^{\alpha(\beta-1)}p_i^{\alpha(\beta-1)-1} + \lambda[a p_i^{-\alpha}(\alpha-1)] = 0\end{aligned} \tag{3.70}$$

and

$$-\frac{h_i}{2}+\frac{a(1-\mu_i)^{-\alpha}p_i^{-\alpha}C_{s_i}}{Q_i^2}=0 \tag{3.71}$$

From equation (3.71), we have

$$Q_i=\left(\frac{2a(1-\mu_i)^{-\alpha}p_i^{-\alpha}C_{s_i}}{h_i}\right)^{\frac{1}{2}} \tag{3.72}$$

From equations (3.72) and (3.70), we get

$$\begin{aligned} a(1-\alpha)(1-\mu_i)^{-\alpha+1}p_i^{-\alpha}+\alpha a^{\frac{1}{2}}(1-\mu_i)^{-\frac{1}{2}}p_i^{-\frac{\alpha}{2}-1}\left(\frac{C_{s_i}h_i}{2}\right)^{\frac{1}{2}} \\ +b\alpha(1-\beta)a^{-\beta+1}(1-\mu_i)^{\alpha(\beta-1)}p_i^{\alpha(\beta-1)-1}+\lambda[ap_i^{-\alpha}(\alpha-1)]=0 \end{aligned} \tag{3.73}$$

where $i=1,2,\ldots,n$.

By solving above equations (3.73) and (3.72) we get the values of p_i, Q_i and λ.

Sufficient Condition:

It has been proved numerically that the hessian matrix of $E\pi(Q_i,p_i)$ is negative semi definite and the hessian matrix of the constraint $B-\alpha\sum_{i=1}^{n}p_i^{-\epsilon+1}\leq 0$ is positive definite.

3.5.4 Numerical Example

Let us consider a retail mall with three items with the following parameter values where fixed price break level(B)=$20,000, $a=54,600$, $\alpha=1.5$, $b=30$ and $\beta=0.1$

	Item 1	Item 2	Item 3
h_i	$0.5	$0.6	$0.45
C_i	$9	$7	$8
C_{s_i}	$300	$350	$320
μ_i	10%	12%	9%

Firstly while computing unconstrained solution from equation (3.64) and (3.65), the values of p_i and Q_i for $i=1,2,\ldots,n$ are obtained as given in following table.

	Item 1	Item 2	Item 3
Selling price per unit item *i.e* p_i	\$64.35	\$68.16	\$62.77
Ordering lot size per order *i.e* Q_i	385.55	370.30	424.11
Total Profit		$E\pi = \$14,020.18$	

Since unconstrained solution satisfied the constraint, so there is no need to activate the constraint.

Check for optimality:

Since the hessian matrix (3.13) is negative semi definite and positive semi definite for objective function and constraint respectively. So the objective function is concave and solution space is convex, hence the solution given in above table is optimal.

3.6 Conclusion

In this study the different multi-item inventory models are considered and corresponding solution procedure have been obtained. In Model I demand is dependent as the quadratic function of the selling price. There are many practical situations for this type of demand function. In many practical situations, the inventory manager of a inventory has the opportunities to decrease or increase the price of the items before the end of the season. It is real fact that some items like harvested food, grains etc, are less available in the different seasons, that compel the inventory manager to increase the price at different period of season.

In Model II demand for each item is the negative power function of the selling price of that item. The change of the rate of demand is very high for a negative power function of price. Practically there are rare real life situation where the change of rate of demand is very high.

In Model III unit cost depends on the demand function. This implies that the decision maker (or inventory holder) employs better equipment and more resources for the production of this product, that is the decision maker will focus on production management as the demand increases. This cost function can be typically found in the industries which produce a successful technologically advanced products such as computer industry. Therefore the decision maker will justify the more efficient production processes as the demand increases.

Chapter 4

Multi-item Inventory Model in which Demand Depends on Selling Price under Storage, Investment and Average Inventory Level Constraint.

4.1 Introduction

In this chapter, three multi-item inventory models have been considered. In these models the objective function is the total profit obtained from an inventory has been considered same for all three models and is similar to the objective function considered in chapter 3, the demand is quadratically dependent on selling price for all models. The objective function is maximized under storage, investment and average inventory level constraint separately. In all the three models KT conditions are used to find the optimal solutions.

In first model total profit is subject to the limited storage space for all items in the inventory that is total area required by all the items should be less than the available storage area.

In second model total profit is subject to the limited investment available. In this model inventory holder places a limit on the amount of investment to be carried that is the total purchasing cost should be less than or equals to the available investment.

In third model there is constraint on average inventory level. The average inventory level should be less than the pre-fixed level of inventory.

4.2 Mathematical Formulation

4.2.1 Assumptions

The following assumptions are made to develop the model:

1. Each model is multiple item model.

2. Production or supply is instantaneous with no lead time.

3. Ordered quantity and selling price p_i for a product at each period are decision variables.

4. Demand rate for a product at each period is dependent on selling price quadratically.

5. Shortages are not allowed.

6. Holding cost and ordering cost for the products are different.

4.2.2 Notations

C_i = Unit purchasing cost for i^{th} item.

p_i = Unit selling price for i^{th} item.

$R(.)$ = Demand rate.

h_i = Holding Cost per unit item per unit time

C_{s_i} = Setup cost for i^{th} item.

Q_i = Quantity ordered for i^{th} item

f_i = Floor area or Storage space required per unit of item.

W = Available total space for all inventory items in the inventory.

F = Total Investment limit for all items in the inventory.

M = Fixed Average Inventory Model.

4.2.3 Demand and Objective function

For all three models we have used same demand and objective function but in each case constraint is totally different.
Mathematically, demand function is given as

$$R(p_i) = a_i - b_i p_i - c_i p_i^2$$

Where a_i, b_i, c_i are suitable positive constants and $a_i >> b_i >> c_i$ for $i = 1, 2, \ldots, n$.
Objective function that is total profit is given below

$$E\pi = \sum_{i=1}^{n} \left[p_i R(p_i) - \frac{h_i Q_i}{2} - \frac{R(p_i) C_{s_i}}{Q_i} - C_i R(p_i) \right]$$

4.3 Model I

In this model objective function that is total profit is subject to the constraint limited storage area. Mathematically, problem is given as

$$\text{Maximize} \qquad E\pi = \sum_{i=1}^{n} \left[p_i R(p_i) - \frac{h_i Q_i}{2} - \frac{R(p_i) C_{s_i}}{Q_i} - C_i R(p_i) \right] \tag{4.1}$$

subject to

$$\sum_{i=1}^{n} f_i Q_i \leq W \tag{4.2}$$

Solution Methodology:

Step 1: To solve the problem, find the unconstrained solution first by using following equations.

$$\frac{\partial E\pi}{\partial p_i} = R(p_i) + (b_i + 2c_i p_i) \left[-p_i + \frac{C_{s_i}}{Q_i} + C_i \right] = 0 \tag{4.3}$$

$$\frac{\partial E\pi}{\partial Q_i} = -\frac{h_i}{2} + \frac{R(p_i) C_{s_i}}{Q_i^2} = 0 \tag{4.4}$$

where $i = 1, 2, \ldots, n$.

By solving these equations ordered quantity Q_i and selling prices p_i for $i = 1, 2, \ldots, n$ are obtained.

Step 2: If this solution satisfies the constraint (equation (4.2)), then we are done. Otherwise constraint must be activated and go to step 3.

Step 3: Then use the KT conditions to determine the constrained optimal values of the order quantities.

Necessary Conditions:

The necessary conditions for maximization of problem given in equations (4.1) and (4.2) are

$$\lambda \leq 0 \tag{4.5}$$

$$\nabla E\pi(Q_i, p_i) + \lambda \nabla \left(\sum_{i=1}^{n} f_i Q_i - W \right) \tag{4.6}$$

$$\lambda \left[\sum_{i=1}^{n} f_i Q_i - W \right] = 0 \tag{4.7}$$

$$\sum_{i=1}^{n} f_i Q_i - W \leq 0 \tag{4.8}$$

Where λ is the Lagrangian multiplier.

From equation (4.6), we have

$$R(p_i) + (b_i + 2c_i p_i) \left[-p_i + \frac{C_{s_i}}{Q_i} + C_i \right] = 0 \tag{4.9}$$

$$-\frac{h_i}{2} + \frac{R(p_i) C_{s_i}}{Q_i^2} + \lambda f_i = 0 \tag{4.10}$$

From equation (4.10), we have

$$Q_i = \left(\frac{2R(p_i) C_{s_i}}{h_i - 2\lambda f_i} \right)^{\frac{1}{2}} \tag{4.11}$$

Using equation (4.11) and (4.9), we have

$$R(p_i) + (b_i + 2c_i p_i) \left[-p_i + \left(\frac{(h_i - 2\lambda f_i) C_{s_i}}{2R(p_i)} \right)^{\frac{1}{2}} + C_i \right] = 0 \tag{4.12}$$

where $i = 1, 2, \ldots, n$.

By solving above equations (4.12) and (4.11) we get the values of p_i, Q_i and λ.

Sufficient Condition:

The KT conditions are also sufficient if the objective function that is $E\pi(Q_i, p_i)$ is concave and solution space that is constraint is a convex function.

4.3.1 Numerical Example

Let us consider a retail mall with three items with the following parameter values where total available storage space (W) is $650 sq.ft$

	Item 1	Item 2	Item 3
Parameters of demand function	$a_1 = 170, b_1 = 1, c_1 = 0.005$	$a_2 = 146, b_2 = 1.1, c_2 = 0.006$	$a_3 = 129, b_3 = 0.9, c_3 = 0.004$
h_i	\$0.5	\$0.6	\$0.45
C_i	\$9	\$7	\$8
C_{s_i}	\$150	\$200	\$140
$f_i(sq.ft)$	0.70	0.80	0.40

Firstly while computing unconstrained solution from equation (4.1) and (4.2), the values of p_i and Q_i for $i = 1, 2, \dots, n$ are obtained as given by the following table.

	Item 1	Item 2	Item 3
Selling price per unit item $i.e$ p_i	\$63.054	\$51.13	\$56.74
Ordering lot size per order $i.e$ Q_i	228.56	222.22	201.19
Total Profit		$E\pi = \$10,807.78$	

Since unconstrained solution satisfied the constraint, so there is no need to activate the constraint.

Check for optimality:

Since the hessian matrix (3.13) is negative semi definite and positive semi definite for objective function and constraint respectively. So the objective function is concave and solution space is convex, hence the solution given in above table is optimal.

4.4 Model II

In this model objective function that is total profit is subject to the constraint limited investment available. Mathematically problem is

$$\text{Maximize} \qquad E\pi = \sum_{i=1}^{n} \left[p_i R(p_i) - \frac{h_i Q_i}{2} - \frac{R(p_i) C_{s_i}}{Q_i} - C_i R(p_i) \right] \tag{4.13}$$

subject to

$$\sum_{i=1}^{n} C_i Q_i \leq F \tag{4.14}$$

Solution Methodology:

Step 1: To solve the problem, find the unconstrained solution first by using following equations.

$$\frac{\partial E\pi}{\partial p_i} = R(p_i) + (b_i + 2c_i p_i) \left[-p_i + \frac{C_{s_i}}{Q_i} + C_i \right] = 0 \tag{4.15}$$

$$\frac{\partial E\pi}{\partial Q_i} = -\frac{h_i}{2} + \frac{R(p_i) C_{s_i}}{Q_i^2} = 0 \tag{4.16}$$

where $i = 1, 2, \ldots, n$.

By solving these equations ordered quantity Q_i and selling prices p_i for $i = 1, 2, \ldots, n.$ are obtained.

Step 2: If this solution satisfies the constraint (equation (4.2)), then we are done. Otherwise constraint must be activated and go to step 3.

Step 3: Then use the KT conditions to determine the constrained optimal values of the order quantities.

Necessary Conditions:

The necessary conditions for maximization of problem given in equations (4.13) and (4.14) are

$$\lambda \leq 0 \tag{4.17}$$

$$\nabla E\pi(Q_i, p_i) + \lambda \nabla \left(\sum_{i=1}^{n} Q_i C_i - F \right) = 0 \tag{4.18}$$

$$\lambda \left[\sum_{i=1}^{n} C_i Q_i - F \right] = 0 \tag{4.19}$$

$$\sum_{i=1}^{n} C_i Q_i - F \leq 0 \tag{4.20}$$

Where λ is the Lagrangian multiplier.

From equation (4.18), we have

$$R(p_i) + (b_i + 2c_i p_i)\left[-p_i + \frac{C_{s_i}}{Q_i} + C_i\right] = 0 \tag{4.21}$$

and

$$-\frac{h_i}{2} + \frac{R(p_i)C_{s_i}}{Q_i^2} + \lambda C_i = 0 \tag{4.22}$$

From equation (4.22) we have

$$Q_i = \left(\frac{2R(p_i)C_{s_i}}{h_i - 2\lambda C_i}\right)^{\frac{1}{2}} \tag{4.23}$$

Using equation (4.23) and (4.21), we have

$$R(p_i) + (b_i + 2c_i p_i)\left[-p_i + \left(\frac{(h_i - 2\lambda C_i)C_{s_i}}{2R(p_i)}\right)^{\frac{1}{2}} + C_i\right] = 0 \tag{4.24}$$

where $i = 1, 2, \ldots, n$.

By solving above equations (4.24) and (4.23) we get the values of p_i, Q_i and λ.

Sufficient Condition:

The KT conditions are also sufficient if the objective function that is $E\pi(Q_i, p_i)$ is concave and solution space that is constraint is a convex function.

4.4.1 Numerical Example

Let us consider a retail mall with three items with the following parameter values where total investment limit (F) is $\$8,000$

	Item 1	Item 2	Item 3
Parameters of demand function	$a_1 = 161, b_1 = 1, c_1 = 0.005$	$a_2 = 166, b_2 = 1.1, c_2 = 0.006$	$a_3 = 135, b_3 = 0.9, c_3 = 0.006$
h_i	\$0.5	\$0.6	\$0.45
C_i	\$9	\$7	\$8
C_{s_i}	\$300	\$350	\$320

Firstly while computing unconstrained solution from equation (4.11) and (4.12), the values of p_i and Q_i for $i = 1, 2, \ldots, n$ are obtained as given in following table.

	Item 1	Item 2	Item 3
Selling price per unit item *i.e* p_i	\$60.79	\$56.18	\$53.80
Ordering lot size per order *i.e* Q_i	312.61	315.04	313.19
Total Profit		$E\pi = \$11,109.23$	

Since unconstrained solution satisfied the constraint, so there is no need to activate the constraint.

Check for optimality:

Since the hessian matrix (3.13) is negative semi definite and positive semi definite for objective function and constraint respectively. So the objective function is concave and solution space is convex, hence the solution given in above table is optimal.

4.5 Model III

In this model objective function that is total profit is subject to the constraint that average inventory level should less than the fixed inventory level . Mathematically

$$\text{Maximize} \qquad E\pi = \sum_{i=1}^{n} \left[p_i R(p_i) - \frac{h_i Q_i}{2} - \frac{R(p_i) C_{s_i}}{Q_i} - C_i R(p_i) \right] \tag{4.25}$$

subject to

$$\frac{1}{2} \sum_{i=1}^{n} Q_i \leq M \tag{4.26}$$

Solution Methodology:

Step 1: To solve the problem, find the unconstrained solution first by using following equations.

$$\frac{\partial E\pi}{\partial p_i} = R(p_i) + (b_i + 2c_i p_i) \left[-p_i + \frac{C_{s_i}}{Q_i} + C_i \right] = 0 \tag{4.27}$$

$$\frac{\partial E\pi}{\partial Q_i} = -\frac{h_i}{2} + \frac{R(p_i) C_{s_i}}{Q_i^2} = 0 \tag{4.28}$$

where $i = 1, 2, \ldots, n$.

By solving these equations ordered quantity Q_i and selling prices p_i for $i = 1, 2, \ldots, n$ are obtained.

Step 2: If this solution satisfies the constraint (equation (4.26)), then we are done. Otherwise constraint must be activated and go to step 3.

Step 3: Then use the KT conditions to determine the constrained optimal values of the order quantities.

Necessary Conditions:

The necessary conditions for maximization of problem given in equations (4.25) and (4.26) are

$$\lambda \leq 0 \tag{4.29}$$

$$\nabla E\pi(Q_i, p_i) + \lambda \nabla \left(\frac{1}{2} \sum_{i=1}^{n} Q_i - M \right) \tag{4.30}$$

$$\lambda \left[\frac{1}{2} \sum_{i=1}^{n} Q_i - M \right] = 0 \tag{4.31}$$

$$\frac{1}{2} \sum_{i=1}^{n} Q_i - M \leq 0 \tag{4.32}$$

Where λ is the Lagrangian multiplier.

From (4.30), we have

$$R(p_i) + (b_i + 2c_i p_i) \left[-p_i + \frac{C_{s_i}}{Q_i} + C_i \right] = 0 \tag{4.33}$$

and

$$-\frac{h_i}{2} + \frac{R(p_i) C_{s_i}}{Q_i^2} + \frac{\lambda}{2} = 0 \tag{4.34}$$

From equation (4.34), we have

$$Q_i = \left(\frac{2R(p_i) C_{s_i}}{h_i - \lambda} \right)^{\frac{1}{2}} \tag{4.35}$$

Using equations (4.35) and (4.33), we have

$$R(p_i) + (b_i + 2c_i p_i) \left[-p_i + \left(\frac{(h_i - \lambda) C_{s_i}}{2R(p_i)} \right)^{\frac{1}{2}} + C_i \right] = 0 \tag{4.36}$$

where $i = 1, 2, \ldots, n$.

By solving above equations (4.36) and (4.35) we get the values of p_i, Q_i and λ.

Sufficient Condition:

The KT conditions are also sufficient if the objective function that is $E\pi(Q_i, p_i)$ is concave and solution space that is constraint is a convex function.

4.5.1 Numerical Example

Let us consider a retail mall with three items with the following parameter values where fixed average inventory level (M) is 500

	Item 1	Item 2	Item 3
Parameters of demand function	$a_1 = 180, b_1 = 1, c_1 = 0.005$	$a_2 = 152, b_2 = 1.1, c_2 = 0.006$	$a_3 = 134, b_3 = 0.9, c_3 = 0.004$
h_i	\$0.5	\$0.6	\$0.45
C_i	\$9	\$7	\$8
C_{s_i}	\$300	\$250	\$320

Firstly while computing unconstrained solution from equation (4.21) and (4.22), the values of p_i and Q_i for $i = 1, 2, \ldots, n$ are obtained as given in following table.

	Item 1	Item 2	Item 3
Selling price per unit item *i.e* p_i	\$65.77	\$52.67	\$58.50
Ordering lot size per order *i.e* Q_i	333.36	253.98	310.22
Total Profit		$E\pi = \$11,750.89$	

Since unconstrained solution satisfied the constraint, so there is no need to activate the constraint.

Check for optimality:

Since the hessian matrix (3.13) is negative semi definite and positive semi definite for objective function and constraint respectively. So the objective function is concave and solution space is convex, hence the solution given in above table is optimal.

4.6 Conclusion

In this chapter, three multi-item inventory models have been studied, the objective function has been considered as the total profit obtained from an inventory same for all three models. The demand is quadratically dependent on selling price for all models. The objective function is maximized under storage, investment and average inventory level constraint separately. The solution procedure is developed using KT conditions. In the practical situation, there are various types of competitions among the retailers or inventory holders who are holding same type of inventory items. Also they have to face several problem like limited space in warehouse(or inventory storage), investment limitations and inventory level limitation etc.

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