Symmetries and Exact Solutions of Einstein-Vacuum Field Equations and Einstein-Maxwell Equations

Thesis

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NISHA GOYAL

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CERTIFICATE

This is to certify that the thesis entitled, **"Symmetries and Exact Solutions of Einstein-Vacuum Field Equations and Einstein-Maxwell Equations"**, submitted by Ms. Nisha Goyal in the fulfillment of the requirements for the award of the degree of Doctor of Philosophy in the School of Mathematics and Computer Applications, Thapar University, Patiala, is a record of candidate's own work carried out by her under my supervision and guidance. The matter presented in this thesis has not been submitted in part or full for the award of any degree in any other University or Institute.

Attestation by supervisor

Dr. Rajesh Kumar Gupta
Assistant Professor
School of Mathematics and Computer Applications
Thapar University
Patiala-147004
INDIA

DECLARATION

It is certified that the thesis is entirely my own and that the ideas and references cited herein have been duly acknowledged.

Nisha Goyal

Attestation by supervisor

Dr. Rajesh Kumar Gupta
Assistant Professor
School of Mathematics and Computer Applications
Thapar University
Patiala-147004
INDIA

DEDICATED

TO

MY PARENTS

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Abstract

The thesis entitled *SYMMETRIES AND EXACT SOLUTIONS OF EINSTEIN-VACUUM FIELD EQUATIONS AND EINSTEIN-MAXWELL EQUATIONS* comprises eight chapters. This thesis is a condensed review of the exact solutions of Einstein field equations and ensuing phenomena. General relativity is a theory of Einstein field equations. Einstein field equations are systems of highly nonlinear partial differential equations. To study general relativity, It is crucial that we analyze partial differential equations in a systematic manner. The solutions to these equations may be found using either numerical or analytical methods.

One of the most important methods of finding analytical solutions of nonlinear problems is through symmetry analysis. **Chapter 1**, presents primarily the review of the related works and methodologies utilized in the thesis. The investigations carried out are confined to the applications of the group-theoretic methods to nonlinear systems governed by partial differential equations. Chapters 2, 3, 5, 6, 7, are based on the applications of Lie classical method while in chapter 4, Symmetry Approach is utilized. In chapter 8, we used extended $\left(\frac{G'}{G}\right)$-expansion method to derive exact solutions.

In **Chapter 2**, the invariance under continuous groups of transformations of a

system of nonlinear partial differential equations derived from nondiagonal Einstein-rosen metric,

$$2(x+y)uu_{xy} - 2(x+y)(u_x u_y - v_x v_y) + u(u_x + u_y) = 0,$$

$$2(x+y)uv_{xy} - 2(x+y)(v_x u_y + u_x v_y) + u(v_x + v_y) = 0,$$

$$\partial_x(\log f) = \frac{\partial_x^2(\log\alpha)}{\partial_x(\log\alpha)} + \frac{1}{4\alpha\alpha_x}T_r A^2,$$

$$\partial_y(\log f) = \frac{\partial_y^2(\log\alpha)}{\partial_y(\log\alpha)} + \frac{1}{4\alpha\alpha_y}T_r B^2,$$

where u and v are arbitrary functions of x and y and $\alpha_x = \frac{\partial\alpha}{\partial x}$, $\alpha_y = \frac{\partial\alpha}{\partial y}$ and the matrices A and B are defined by $A = -\alpha\partial_x g_{ik} g^{kj}$, $B = -\alpha\partial_y g_{ik} g^{kj}$, has been studied. To our knowledge, no exact solution of this system has been reported. To check the integrability of the system, *Painlevé property* is utilized. The systematic investigation of group-invariant reductions is based on the computations of an optimal list of inequivalent subalgebras. In this chapter, the optimal list of one dimensional subalgebras is presented and some exact solutions of systems of Einstein vacuum equations are derived.

Chapter 3 is devoted to the study of system of Einstein-Maxwell equations derived from the non-static cylindrically symmetric metric of Einstein and Rosen:

$$u_{xx} + \frac{1}{x}u_x - u_{tt} = -e^{-2u}(v_x^2 - v_t^2 + w_x^2 - w_t^2),$$

$$v_{xx} + \frac{1}{x}v_x - v_{tt} = 2(u_x v_x - u_t v_t),$$

$$w_{xx} + \frac{1}{x}w_x - w_{tt} = 2(u_x w_x - u_t w_t),$$

$$v_x w_t = w_x v_t,$$

$$k_x = x(u_x^2 + u_t^2) + xe^{-2u}(v_x^2 + v_t^2 + w_x^2 + w_t^2),$$

$$k_t = 2xu_xu_t + 2xe^{-2u}(v_xv_t + w_xw_t)$$

. Group-invariant reductions are carried out using inequivalent one-dimensional subalgebras of system of Einstein-Maxwell equations. This enables reduction of system of PDEs to ODEs. Different exact solutions are constructed.

In **Chapter 4**, a system of Einstein Maxwell equations for the pure magnetostatic fields

$$\phi_{rr} + \frac{1}{r}\phi_r + \phi_{zz} - \frac{e^{2\phi}}{2r^2}(A_r^2 + A_z^2) = 0,$$

$$A_{rr} - \frac{1}{r} + A_{zz} + 2(A_r\phi_r + A_z\phi_z) = 0,$$

$$\Lambda_r = r(\phi_r^2 - \phi_z^2) + \frac{e^{2\phi}}{2r}(A_r^2 - A_z^2),$$

$$\Lambda_z = 2r\phi_r\phi_z + \frac{1}{r}e^{2\phi}A_rA_z$$

, where A is magnetostatic potential and function of r and z only, has been investigated for symmetry reductions and exact solutions. The infinitesimals which leaves system invariant are obtained by using symmetry method based on Fréchet derivatives of differential operators. An optimal system of conjugacy inequivalent subalgebras is then identified with the adjoint action of symmetry group. For each basic vector field in optimal system, the above system os reduced to system of ODEs which is further solved with the aim of deriving certain exact solutions.

In **Chapter 5**, we have investigated the symmetries from the classical point of view of pure magnetic fields:

$$\mu_{\rho\rho} - \mu_{tt} + \frac{\mu_\rho}{\rho} - \frac{e^{\mu}}{\rho^2}(\phi_\rho^2 - \phi_t^2) = 0,$$

$$\phi_{\rho\rho} - \phi_{tt} - \frac{\phi_\rho}{\rho} + \mu_\rho\phi_\rho - \mu_t\phi_t = 0,$$

$$\lambda_\rho = \frac{\rho}{2}(\mu_\rho^2 + \mu_t^2) + \frac{e^{\mu}}{\rho^2}(\phi_\rho^2 + \phi_t^2),$$

$$\lambda_t = \rho\mu_\rho\mu_t + 2\frac{e^\mu}{\rho}e^\mu\phi_\rho\phi_t,$$

and pure electric fields:

$$\mu_{\rho\rho} - \mu_{tt} + \frac{\mu_\rho}{\rho} + e^{-\mu}(\psi_\rho^2 - \psi_t^2) = 0,$$

$$\psi_{\rho\rho} - \psi_{tt} + \frac{\psi_\rho}{\rho} - \mu_\rho\psi_\rho + \mu_t\phi_t = 0,$$

$$\lambda_\rho = \frac{\rho}{2}(\mu_\rho^2 + \mu_t^2) + \rho e^{-\mu}(\psi_\rho^2 + \psi_t^2),$$

$$\lambda_t = \rho\mu_\rho\mu_t + 2\rho e^{-\mu}\psi_\rho\psi_t.$$

Corresponding to each basic vector field, the reductions of the above nonlinear systems to ODEs are obtained. These reduced systems of ODEs are further studied for exact solutions.

Chapter 6 is concerned with the study of Einstein vacuum equations for axially symmetric gravitational fields

$$\frac{\partial^2\psi}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial\psi}{\partial\rho} + \frac{\partial^2\psi}{\partial z^2} = 0,$$

$$\gamma(\rho, z) = \int \rho\left(\left(\frac{\partial\psi}{\partial\rho}\right)^2 + \left(\frac{\partial\psi}{\partial z}\right)^2\right)d\rho + \int 2\rho\frac{\partial\psi}{\partial\rho}\frac{\partial\psi}{\partial z}dz$$

. This system is examined for Lie symmetry group. An optimal system of inequivalent one-dimensional subalgebras of the above system having four basic vector fields is determined. Using the non-equivalent Lie ansätz for each essential vector field, the nonlinear ODEs and exact solution are constructed.

Chapter 7, is based on the exact solutions by classical Lie method and extended $\left(\frac{G'}{G}\right)$-expansion method to the variable coefficients fifth order KdV equation

$$u_t + \alpha(t)u^2u_x + \beta(t)u_xu_{xx} + \gamma(t)uu_{xxx} + \delta(t)u_{xxxxx} = 0,$$

where u is function of x and t, in Chapter 7. The variable coefficient fifth order KdV equation describes, physically important nonlinear equations (Kaup-Kupershmidt (KK), Lax, Sawada-Kotera (SK), CaudreyDoddGibbon (CDG) and Ito equations), so the similarity solutions of these equations can be obtained by substituting particular values of $\alpha(t), \beta(t), \gamma(t)$ and $\delta(t)$. The obtained solutions are also represented graphically in this chapter.

In **Chapter 8**, we have studied the two nonlinear partial differential equations of nano-ionic currents along Microtubules (MTs)

$$\frac{l^3}{3}u_{xxx} + \frac{Z^{\frac{3}{2}}}{l}\left(\Psi G_0 - 2\lambda C_0\right) u u_t + 2u_x + \frac{ZC_0}{l}u_t + \frac{1}{l}\left(RZ^{-1} - G_0 Z\right) u = 0,$$

and

$$R_2 C_0 l^2 u_{xxt} + l^2 u_{xx} + 2R_1 C_0 \lambda u u_t - R_1 C_0 u_t = 0.$$

Exact solutions of these equations are derived by using the $\left(\frac{G'}{G}\right)$-expansion method.

List of Research Papers

1. Nisha Goyal and R. K. Gupta, Symmetries and exact solutions of the nondiagonal Einstein-Rosen metrics, Physica Scripta 85 (2012) 015004 (6pp). **(Imapct Factor 1.204) (SCI)**

2. Nisha Goyal and R. K. Gupta, A class of exact solutions of Einstein field equations, Physica Scripta 85 (2012) 055011 (6pp). **(Imapct Factor 1.204) (SCI)**

3. Nisha Goyal and R. K. Gupta, A Class of new exact solutions of Einstein-Maxwell equations for magnetostatic fields, Chinese Physics B 21 (2012) 090401 (6pp). **(Imapct Factor 1.376) (SCI)**

4. Nisha Goyal and R. K. Gupta, Similarity analysis and new exact solutions of the Einstein-Maxwell Equations for non-static Einstein and Rosen metrics, Proceedings of International Conference on Mathematics and Statistics 2012 (Accepted).

5. Nisha Goyal and R. K. Gupta, On Symmetries and exact solutions of Einstein vacuum equations for axially symmetric gravitational fields, International Journal of Mathematical and Computational Sciences 6 (2012) 104-107.

6. R. K. Gupta and Nisha Goyal, Symmetries and exact solutions of generalized fifth order variable coefficients KdV equation, Journal of Differential Equations (Communicated).

7. R. K. Gupta and Nisha Goyal, Exact solutions for new equations of nano-ionic currents along MTs in biosciences, PRAMANA-A Journal of Physics (Communicated).

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Chapter 1

Introduction

1.1 Introduction

Albert Einstein proposed the theory of relativity. The theory begins with two fundamental postulates. The first states that for all observers moving uniformly relative to each other, the law of physics have the same form. The second is the invariance of the speed of light. General relativity is the theory of gravity proposed by Einstein in 1915. What distinguishes general relativity from other physical theories is the idea that spacetime is no longer a passive stage upon which nature performs. Indeed the curvature of spacetime is coupled to its matter and energy content via the Einstein field equations in a highly nonlinear manner. The Einstein field equations, which play a central role in the Einstein's theory of general relativity, have symmetry consideration as one of the most important mathematical properties apart from their applications and implications for astrophysics and cosmology. This is why these equations have been a subject of extensive and intensive study both by mathematicians and physicists. The behavior of the gravitational field is at present best described by the theory of general relativity. General relativity in the framework of differential manifold is utilized to study various aspects of gravitational field. Increasingly, general relativistic models are being utilized for the analysis of strong gravitational fields; the conventional Newtonian models are not appropriate in the scenario eg. neutron star models.

In this thesis, the Einstein vacuum field equations and Einstein-Maxwell equations in general relativity are studied. To analyze the behavior of the gravitational field, we need to solve the Einstein field equations which describe how curvature and matter are coupled. This system of highly nonlinear partial differential equations is not easily tractable in general and solutions are normally sought via simplifying assumptions on the nature of the matter content or the form of the gravitational potentials. Solutions to the Einstein field equations are listed by Stephani et al. [143] and Krasinski [71]. A number of exact solutions have been discovered with applications in relativistic astrophysics:

(a) The Schwarzschild exterior solution. This describes the gravitational field outside a static spherically symmetric body. Historically, this is the first exact solution of the field equations [131].

(b) The Schwarzschild interior solution. This describes the gravitational field inside a static spherically symmetric body. This solution [132] matches smoothly to the exterior Schwarzschild line element, and is a good model of small stars, where the pressures are not too large.

(c) The Reissner-Nordström [126] solution. This model describes the spacetime outside a charged, static spherically symmetric body.

It is important to note that other exterior solutions, describing important astrophysical phenomena, exist. The most important of these are the Vaidya solution [152], which describes radiating bodies, and the Kerr solution [66] which describes rotating bodies.

The study of exact solutions forms an important area of research in general relativity. They are important because many qualitative features of the gravitational field are obtained by analyzing individual systems. Without exact solutions of the Einstein field equations, it is not possible to consider many of the physical implications of the general field equations because of their nonlinearity. Main focus of the thesis is to derive new exact solutions of the Einstein vacuum and Einstein-Maxwell field equations in this thesis. An extensive literature of exact solutions to the Einstein field equations has been generated over the years; recent reviews are given by Delgaty and Lake [30], Stephani et al. [143] and Krasinski [71]. Most of these solutions, however, do not stand the tests of physical reality. Some of the exact solutions, that qualify all the physical requirements include those of Durgapal and Bannerji [33], Durgapal and Fuloria [34], Finch and Skea [37], Tikekar [147], Maharaj and Leach [92] and Lake [73]. It is interesting to observe that a set of these isotropic solutions may be used as seed solutions to produce anisotropic solutions to the Einstein field equations with the help of a new algorithm proposed by Maharaj and Chaisi [93].

Interior regular charged perfect fluid solutions have been studied by different authors. The original Schwarzschild idea of constant density has also been tested in the charged case for a perfect fluid (Wilson [165], Mehra and Bohra [98], de Felice et al. [29]). Exact solutions of the Einstein-Maxwell field equations are important in the description of relativistic astrophysical processes. These solutions may be utilized to model a charged relativistic star as they are matchable to the Reissner-Nordström exterior at the boundary [133]. It is for this reason that many investigators use a variety of techniques to attain exact solutions. A comprehensive list of Einstein-Maxwell solutions, satisfying a variety of criteria for physical admissability, is provided by Ivanov [61]. It is interesting to observe that, in the presence of charge, the gravitational collapse of a spherically symmetric distribution of matter to a point singularity may be avoided [71]. In this situation the gravitational attraction is counterbalanced by the repulsive Colombian force, in addition to the pressure gradient. Einstein-Maxwell solutions are also important in studies involving the cosmic censorship hypothesis and the formation of naked singularities [64]. The presence of charge affects the values for redshifts, luminosities and maximum mass for stars. Consequently, the Einstein-Maxwell system, for a charged star, has attracted considerable attention in various physical investigations including those of Patel et al. [117] and Sharma et al. [136].

In an attempt to generate exact solutions in some cases, Vaidya and Tikekar [152] proposed that the geometry of the spacelike hypersurfaces generated by t=constant is that of the 3-spheroid. This spheroidal condition provides a clear geometrical interpretation which is not the case in many other exact solutions. Knutsen [68] was the first to consider the pressure gradients of stars with spheroidal geometry and showed that they are negative. Also note that spheroidal geometries exhibit the important physical feature of being stable with respect to radial pulsations (Knutsen [69]). Tikekar [148] comprehensively studied a particular spheroidal geometry and showed that it could be applied to superdense neutron stars with densities in the range of 1014 g cm3. Maharaj and Leach [92] found all spheroidal solutions, for uncharged stars, that could be expressed in terms of elementary functions. Mukherjee et al. [106] showed that it was possible to express the general solution in terms of Gegenbauer functions; an alternate form of the general solution was found by

Gupta and Jasim [46]. These uncharged solutions can be extended to models in the presence of electromagnetic field. Spheroidal models in the presence of an electric field have been extensively studied by Sharma et at. [136], Patel and Koppar [116], Patel et al. [117], and Gupta and Kumar [47]. These investigations have been motivated on the grounds that restricting the geometry of the hypersurfaces $t = constant$ to be spheroidal produces neutral and charged stars which are consistent with observations for dense astronomical objects. Models with spheroidal geometry can be directly related to particular physical intuitions: the maximum mass is in agreement with values for cold compact objects (Sharma et al. [136]); values for densities are consistent with strange matter (Tikekar and Jotania [150]); the equation of state is consistent with a compact X-ray binary pulsar Her X-1 (Sharma and Mukherjee [134]); relevance to equation of state for stars compared of quark-diquark mixtures in equilibrium (Sharma and Mukherjee [135]); and uniform charged dust in equilibrium (Tikekar [147]). Spheroidal geometries are relevant in core-envelope stellar models, core consisting of isotropic fluid and an envelope of anisotropic fluid, as shown by Thomas et al. [146], and Tikekar and Thomas [149]. These references provide a sample as to why the Einstein-Maxwell system, describing the interior of a charged star, has attracted the attention of many researchers. These references indicate that Einstein field equations arise in a variety of applications and deserve clear scrutiny.

1.1.1 The Einstein Field Equations

The theory of general relativity can be interpreted in terms of the union of space and time into a four dimensional spacetime. This spacetime spans space from corner to corner, and time from start to end, and may be visualized as a static four-dimensional chart of universe. A point in spacetime is known as an event and particles exist as world lines in spacetime. A four dimensional Lorentzian manifold **M** with signature $(+,- - -)$ and the metric tensor g_{ab}, which is a function of the position given in coordinates by $x^a, (a, b, ... = 0, 1, 2, 3)$ is considered. If R_{ab} is the Ricci tensor and R is the Ricci scalar, the Einstein field equations (without cosmological constant) are:

$$R_{ab} - \frac{1}{2} g_{ab} R = k T_{ab}, \tag{1.1.1}$$

where T_{ab} is the energy-momentum tensor of the matter and k is the Einstein gravitational constant. These are the basic equations of the general theory of relativity set up by Einstein in 1915.

Since R_{ab} is a nonlinear function of g_{ab} and its derivative, the Einstein field equations are a system of 10 coupled highly nonlinear second order partial differential equations for the 10 independent functions g_{ab} of four spacetime coordinates x^a.

Next, certain basic aspects of differential geometry are considered, which are necessary for later work. The aspects of differential geometry relevant to general relativity are briefly discussed in next section (1.1.2). In (1.1.2), the non vanishing components of the connection coefficients, the Ricci tensor, the Ricci scalar and the Einstein tensor are explicitly calculated for Einstein field equations. The coupling of the Einstein tensor and energy-momentum tensor is used to generate the Einstein field equations.

1.1.2 Differential Geometry

In this section, the basic elements of differential geometry required to obtain Einstein field equations are introduced. The applications of differential geometry to general relativity is dealt with in greater detail by de Felice and Clark [28], Hawking and Eills [52] and Misner et al. [99]. A 4-dimensional differentiable manifold **M** with signature $(-+++)$ as a sapcetime is considered . On the manifold, differentiable structures are defined, which can then be used to model the physics of the spacetime. The manifold is labeled by local coordinates $(x^a) = (x^0, x^1, x^2, x^3)$ where x^0 is the timelike and x^1, x^2, x^3 are spacelike. A manifold is a topological space which locally has the structure of Euclidean space in that it may be converted with coordinates patches. The global structure of **M** may be very different from the Euclidean space.

To study the physics in the manifold **M**, it is necessary to measure the invariant separation of neighboring points. This is done by introducing a systematic, nondegenerate metric tensor field g onto the manifold. The fundamental line element, defining the invariant infinitesimal separation between neighboring points on **M**, is given by

$$ds^2 = g_{ab}dx^a dx^b. \tag{1.1.2}$$

To characterize the curvature of the manifold, it is necessary to introduce additional structure on the manifold. The metric connection Γ, also known as the Christofell symbol of the second kind, is defined in terms of the metric tensor g in (1.1.2), and its derivative by

$$\Gamma^a_{bc} = \frac{1}{2} g^{ad}(g_{cd,b} + g_{db,c} - g_{bc,d}), \tag{1.1.3}$$

where the comma denotes a partial derivative. The metric connection Γ is used to generalize the partial derivative in Minkowski spacetime to the covariant derivative in curved spacetime. For example, the covariant derivative of a (1,0) vector field X is

$$X^a;b = X^a_{,b} + \Gamma^a_{bc}X^c. \tag{1.1.4}$$

Similarly, the covariant derivative of a (2,0) tensor field T is defined by

$$T^{ab}_{;c} = T^{ab}_{,c} + \Gamma^{a}_{cd}T^{db} + \Gamma^{b}_{cd}T^{ad}. \tag{1.1.5}$$

Clearly, the covariant derivative reduces to the partial derivatives in Minkowski spacetime.

The Riemann tensor is a (1,3) tensor field which characterizes the curvature of spacetime. Also known as the curvature tensor vanishes in flat spacetime; there are always nonvanishing components in curved spacetime. It is defined by the non-commutativity of the second covariant derivatives of a vector field X given by

$$X^{a}_{;bc} - X^{a}_{;cb} = R^{a}_{bcd}X^{d}, \tag{1.1.6}$$

which is sometimes called the Ricci identity. On using the definition of the covariant derivative of the vector X in (1.1.6), the Riemann tensor is

$$R^{a}_{bcd} = \Gamma^{a}_{bd,c} - \Gamma^{a}_{bc,d} + \Gamma^{a}_{ec}\Gamma^{e}_{bd} - \Gamma^{a}_{ed}\Gamma^{e}_{bc}, \tag{1.1.7}$$

in terms of the metric connection (1.1.3). The components R^{a}_{bcd} satisfy the following identities

$$\begin{aligned} &R_{abcd} = -R_{bacd} = -R_{abdc} = R_{cdab}, \\ &R_{abcd} + R_{acdb} + R_{adbc} = 0, \\ &R_{abcd;e} + R_{abde;c} + R_{abec;d} = 0. \end{aligned} \tag{1.1.8}$$

On contracting the Riemann tensor (1.1.7), the Ricci tensor is

$$R_{ab} = R^{d}_{adb} = \Gamma^{d}_{ab,d} - \Gamma^{d}_{ad,b} + \Gamma^{e}_{ab}\Gamma^{d}_{ed} - \Gamma^{e}_{ad}\Gamma^{d}_{eb}, \tag{1.1.9}$$

which is a symmetric (0,2) tensor. The Ricci scalar or the curvature scalar is obtained by taking the trace of the Ricci tensor and is given by

$$R = g_{ab}R^{ab} = R^{a}_{a}. \tag{1.1.10}$$

The Einstein tensor

$$G_{ab} = R_{ab} - \frac{1}{2}Rg_{ab}, \tag{1.1.11}$$

is defined in terms of Ricci tensor and Ricci scalar. It is constructed such that it has zero divergence

$$G^{ab}_{;b} = 0, \tag{1.1.12}$$

a result which follows from the equation (1.1.11) and the contracted Bianchi identities. The importance of G in gravity was first recognized by Einstein when developing the field equations for the theory of general relativity.

1.2 Methodology

The main concern of the thesis is to derive exact solutions to the Einstein field equations. The difficulty in obtaining exact solutions lies primarily in the complicated system of nonlinear partial differential equations that arise. It is therefore convenient when the solution of these systems of partial differential equations (PDEs) reduce to the systems of ordinary differential equations (ODEs).

Exact solutions for nonlinear equations are rare, and the methods, which can generate families of them, are not only increasingly popular, but increasingly sought. So far, a number of methods have been proposed to construct the exact solutions; the most effective methods include the classical Lie approach [7,10,13,15,19,21,25,43,47] and [55,58,83,85,94,109,114,137,155], the nonclassical approach [26,38,40], Steinberg's symmetry reduction method [11,142], non local symmetry method [5,41,50], the truncated Painlevé approach [2,24,27,58,122,137,141], the transformation method [20,53,62,79] and [120,165,166,176], the extended tanh method [160,172], the hyperbolic functions expansion methods [59,61,104,154,177], expansion function method

[108], sine-cosine method [156,167], Jacobi-elliptic function method [78], sub ODE method [77], the Isovector method [6,8], the hyperbolic function method [9], the homogenous balance method [154], the $\left(\frac{G'}{G}\right)$-expansion method [74,122,171,175] and F-expansion method [21] and some other methods explained in [13,14,15,57,70,71]. The mathematical techniques which generate a wide range of solutions and applicable to all type of nonlinear differential equations are few. The study of the mathematical properties of the Einstein field equations and the techniques used to solve these equations are important in the context of general relativity. There are different techniques to solve these equations. The approach, which this thesis adopts is the Lie analysis of differential equations, that was first formulated by Lie [83] in his attempt to present a systematic and geometric approach to solve differential equations. From the late 1950s Lie group analysis, also known as Lie group theory or Lie symmetry analysis of differential equations (DEs) and its extensions, has proven to be a powerful technique in finding the closed-form solutions of DEs. The Lie group theory combines analysis and algebra, and was initiated by the Norwegian mathematician Sophus Lie [83]. The theory of Lie groups is a local theory. In simple terms it means that the group of transformations is only defined in the neighborhood of the identity transformation. Lie discovered that the better way to understand Lie groups was to investigate the corresponding vector fields (also known as infinitesimal generators or symmetries) and he was able to relate the theory to its infinitesimal group, i.e. its Lie algebra as it is called today in his honor. Emmy Noether [109] used some of these ideas in the early 1900s and succeeded in proving the correspondence between symmetries of a variational problem and the conservation laws of the Euler Lagrange equations. Chevalley [23] and those that followed

continued to work on Lie groups around the middle of the 20th century. From the late 1950s to the 1960s, Morozov [101] and Mubarakzyanov [102-105] worked on the classification of Lie algebras and others that have followed since such as Patera and Winternitz [118], Turkowski [151] and Mahomed [95] just a few to mention.

Despite efforts to popularize Lie's work through publications, the application of Lie group theory to DEs remained dormant until Ovsiannikov [113] revived it in the late 1950s. Thereafter, the Lie group theory has been applied in many problems (described by linear or nonlinear equations) modeling some physical or abstract phenomena. The basic theory and developments in the Lie group analysis of differential equations can be obtained from [14,15,54,55,111,113] and the literature that followed, some of which are lecture notes and theses [32,83,94,112,153]. The mathematical formulation of symmetries was already present in the theory of algebraic equations as developed by Galois. In fact he introduced and established the concept of group (see Stubhaug [144], p.114). Lie discovered that the concept of symmetry could be used to obtain solutions of DEs. He considered only the local symmetries (point and contact symmetries). Point symmetries depend only upon independent and dependent variables, whereas contact symmetries can also depend upon the first-order derivatives of the dependent variables. From around 1970s until today there has been an interest in exploring nonlocal symmetries. These are the symmetries that depend on integrals of the dependent variables.

Ever since the revival of the application of Lie group theory to DEs, Lie's work has been developed with many topics having being researched and continue to be researched. These include the following: Generation of new solutions from known ones, linearization of ordinary differential equations (ODEs) and PDEs, construction of equivalence group, solving group classification problems, reductions of PDEs (by invariant or similarity solutions), construction of generalized local symmetries and nonlocal symmetries, solving initial and boundary value problems, approximate symmetries, symmetries of stochastic DEs, symmetries of integro differential equations, symmetries of difference equations, symmetries of functional DEs, symmetries of geodesic equations, construction of conservation laws, construction of invariants of algebraic equations and DEs etc.

Roughly, a symmetry is a change or transformation that leaves an object apparently unchanged. For instance any rotation of a circle about its center is a symmetry. Different objects can have different degrees of symmetries: intuitively a circle has more symmetries than a rectangle. Therefore symmetry can be employed as a classification criterion. In the case of DEs a symmetry is an invertible transformation of the dependent and independent variables that does not affect the form of the underlying equation or system of equations. Looking at a DE one can deduce symmetries like translations, scalings and rotations. Certain discrete transformations can also be deduced by inspection. In general finding all the symmetries is a difficult task and requires an algorithmic approach. If we consider symmetries that depend continuously upon a one-parameter and that constitute a group, we can use Lie's algorithm to compute them. In this algorithm the symmetry conditions (also known as the determining equations) upon expansion yield an over-determined system of linear homogeneous PDEs which are solved for symmetries.

Most problems even of no interest generating the determining equations turns out to be tedious. Fortunately, Lie's method for calculating symmetries is algorithmic and can be implemented using packages for symbolic computation (Mathematica, Maple, Reduce, MuPad). In this work the program, Maple for the generation and manipulation of the determining equations for symmetries is used.

The determining equations and the classifying relations for the models are investigated in this work, that may be complicated and difficult to analyze. Therefore we use both the classical Lie approach and the approach based upon the structure of Lie algebras for group classification.

It is well-known in most applications that from a computational standpoint ODEs are easier to solve compared to PDEs. Thus in many applications it is desirable to be able to reduce PDEs to ODEs or at least reduce the number of independent variables. The reduced system is easier to analyze both numerically and analytically than the original system. Therefore, whenever the need arises, the symmetries are used to construct the invariant solutions; the invariant solutions in turn are used to reduce the underlying system of PDEs to the systems of ODEs which could be solved or approximated numerically.

The method employs the use of symmetry transformations of a differential equation to reduce the order of the differential equation. This allows us systematically to study the Einstein field equations which, in different situations, appear as the principal equations in general relativity. Our intention in this thesis is to study some nonlinear system of Einstein field equations and some other physically important nonlinear systems of PDEs.

As mentioned earlier the work in this thesis is based primarily on the applications of following techniques:

(a) Classical Lie method.

(b) Symmetry reduction method.

(c) Painlevé property.

(d) $\left(\frac{G'}{G}\right)$-expansion method.

In the following sections, the relevant concepts of the Lie group of transformations are introduced and then provide the algorithmic descriptions of the techniques which are applied in the later chapters to derive the symmetry group of the systems under investigation. For more details on Lie groups and various theorems, their proofs and other concepts, we refer our reader to Bluman and Cole [14], Bluman and Anco [16] and Olver [112].

1.2.1 Definitions

Definition 1 A transformation of a space $x = (x^1, x^2, ..., x^n)$ is a smooth (C^∞) mapping $x' = \tau(x)$ such that $\tau = (\tau^1, \tau^2, ..., \tau^n)$ is one-one and on-to.

Definition 2 Let r real parameters $\epsilon = (\epsilon^1, \epsilon^2, ..., \epsilon^r)$ lie in a space P. The space P is an r-parameter *Lie group* if there is defined a binary operation $*$ on P such that

1. There is a unique identity element $e \in P$ such that $\epsilon * e = e * \epsilon = \epsilon$ for all $\epsilon \in P$.

2. The operation $*$ is associative: $\epsilon * (\delta * \gamma) = (\epsilon * \delta) * \gamma$ for all $\epsilon, \delta, \gamma \in P$.

3. For every $\epsilon \in P$, there exists an $\epsilon^{-1} \in P$ such that $\epsilon * \epsilon^{-1} = \epsilon^{-1} * \epsilon = e$.

4. Both the binary operation $*$ and the map $\epsilon \longrightarrow \epsilon$ are analytic.

Lie transformation group

The group P of parameters remains in the background; our interest is in transformation groups, i.e., collections of transformations labeled by the parameters ϵ of P.

Definition 3 A Lie transformation group on a space $x = (x^1, x^2, ..., x^n)$ is a collection Ω of smooth transformations τ of x obtained as the homomorphic image of a Lie group of parameters. There is a map $\tau : P \longrightarrow \Omega$ such that

1. $\tau(e)$ is the identity map of x: $\tau(e)(x) = x$ for all x.

2. $\tau(\epsilon) o \tau(\delta) = \tau(\epsilon o \delta)$ for all $\epsilon, \delta \in P$.

3. $\tau(\epsilon^{-1}) = \tau(\epsilon)^{-1}$.

4. The map $x' = F(x; \epsilon) = \tau(\epsilon)(x)$ is a smooth C^∞ in x and ϵ.

One-parameter transformation group

As a special case of transformation group, let the single parameter ϵ be additive.

Definition 4 A one-parameter (ϵ) group acting on a space x is a transformation group on x with the following properties:

1. $\tau(0)$ is the identity transformation on x.

2. $\tau(\epsilon) o \tau(\delta) = \tau(\epsilon + \delta)$.

Infinitesimal operators

The key to practical construction of Lie transformation group is an infinitesimal formulation to the problem, which replaces nonlinear conditions for a group with linear conditions.

Infinitesimal transformations

Consider a one-parameter (ϵ) group of transformations acting on a space $x = (x^1, x^2, ..., x^n)$. In the neighborhood of the identity $\epsilon = 0$, the transformation

$$x' = F(x; \epsilon), \tag{1.2.1}$$

can be expanded as

$$x' = x + \epsilon \xi(x) + O(\epsilon^2), \tag{1.2.2}$$

where $\xi = (\xi^1, \xi^2, ..., \xi^n)$ is given by

$$\xi^i(x) = \frac{\partial F^i}{\partial \epsilon}(x;\epsilon)\big|_{\epsilon=0}. \tag{1.2.3}$$

The quantities ξ^i's are called *infinitesimals* of the one-parameter group: expansion x' represents an *infinitesimal transformation* from the group.

Group operator

The operator X of a one-parameter group with infinitesimal $\xi = (\xi^1, \xi^2, ..., \xi^n)$ is the first order differential operator

$$X = \xi^i(x)\frac{\partial}{\partial x^i}. \tag{1.2.4}$$

Lie algebra of operators

An r-parameter Lie group of transformations has associated r group operators $X_1, X_2, ..., X_r$ which are linearly independent and form an r-dimensional vector space over $\mathbb{R}$. This vector space has the additional structure of being closed under commutation.

Definition 5 Let $X = \xi^i \frac{\partial}{\partial x^i}$ and $Y = \gamma^i \frac{\partial}{\partial x^i}$ be two group operators. The commutator $[X, Y]$ is the first order operator

$$XY - YX = \left(\xi^j \frac{\partial \gamma^i}{\partial x^j} - \gamma^j \frac{\partial \xi^i}{\partial x^j}\right)\frac{\partial}{\partial x^i}. \tag{1.2.5}$$

The commutator $[,]$ bracket has the following properties:

1. Bilinerity: $[X, aY + bZ] = a[X, Y] + b[Y, Z]$, where a, b are real constants.

2. Anticommutativity: $[X, Y] = -[Y, X]$.

3. Jacobi identity: $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$.

Any vector space satisfying these three properties is called a *Lie algebra*, but our Lie algebras are Lie algebras of operators with commutator bracket defined above.

Definition6 The adjoint representation of a Lie group on it's Lie algebra is given as the Lie series

$$Y(\epsilon) = Ad(e^{\epsilon x})Y_0 \tag{1.2.6}$$

which is

$$\sum_{n=0}^{\infty} \frac{\epsilon^n}{n!}\{X^n, Y_0\}, \tag{1.2.7}$$

with .,. being defined recursively:

$$\{X^0, Y_0\} = Y_0, \{X^n, Y_0\} = (-1)^n\{X, \{X^{n-1}, Y_0\}\}. \tag{1.2.8}$$

Alternatively, the adjoint representation can also be calculated by integrating the initial value problem

$$Ad(exp(\epsilon X))Y_0 = Y_0 - \epsilon[X, Y_0] + \frac{\epsilon^2}{2}[X, [X, Y_0]] - ... \tag{1.2.9}$$

Classification of Subgroup and Sub algebras

Let G be a Lie group. An optimal system of r-parameter subgroups is a list of conjugacy inequivalent r-parameter subgroups with the property that any other subgroup is conjugate to precisely one subgroup in the list. Similarly, a list of r-parameter sub algebras forms an optimal system if every r-parameter sub algebra of L is equivalent to a unique member of the list under some element of the adjoint representation.

The problem of finding an optimal system of subgroups is equivalent to that of finding an optimal system of sub algebras. For one dimensional sub algebras, this classification problem is essentially the same as the problem of classifying the orbits of the adjoint representation, since each one-dimensional subalgebra is determined by a nonzero vector in L. This problem is attacked by the naïve approach of taking a general element in the Lie algebra L and subjecting it to its various adjoint transformations so as to "simplify" it as much as possible (Refer to [120]).

1.2.2 Lie Classical Method

A common way of finding exact solutions for a system of nonlinear PDEs is the symmetry reduction by Sophus Lie [83]. This method is entirely algorithmic and allows one to calculate the symmetry group represented by infinitesimal transformations under which solutions of the system are invariant. This method has been applied to hundreds of PDEs and systems of PDEs in order to obtain exact similarity solutions refer to [7,12,14,16,23,42,46,54,76,87,90,100,109,112,136,138,168]. An extensive number of other equations treated by the same method are compiled in the books of Roger and Ames [127] as well as in Ibragimovs survey [55].

The Lie method is also a useful tool for finding exact solutions, constructing new solutions from old ones and characterizing the symmetry properties of PDEs. In particular, we are interested in discovering solutions for well known systems of PDEs.

Everyone who has used Lie's method when studying solutions of PDES knows that this procedure is very time consuming and tedious. Especially, the large number of determining equations resulting from this procedure is hard to handle by a pencil calculation. This was one of the reasons for developing computer-based methods to do the calculations. Various symbolic manipulating programs have been developed in recent years. Today there exists a program in nearly every algebraic language such as MATHEMATICA, REDUCE, MAPLE, MACSYMA and AXIOM. Some of these programs are now capable of deriving completely automatically the symmetries and their algebraic properties of a great number of equations.

To understand how Lie's method works and what information we can gain from it, let us discuss its general procedure. The general case of a nonlinear system of PDEs is describe by

$$\Delta_\nu \left(x, u^{(k)}\right) = 0, \nu = 1, 2, ..., m, \tag{1.2.10}$$

where Δ_ν represents one of the m coupled equations in n independent variables $x = (x^1, x^2, ..., x^n)$ and m dependent variables $u = (u^1, u^2, ..., u^m)$, $u^{(k)}$ denotes all derivatives of the dependent variables u with respect to the independent variables x up to order k. We assume that the Δ_ν are smooth functions in their arguments. The central question of a symmetry analysis of (1.2.10) is under which transformation such a system is invariant. The invariance can be considered following Lie by applying the one-parameter Lie group of point-transformations to (1.2.10):

$$\begin{aligned} (x^i)^* &= x^i + \epsilon \xi^i(x, u) + O(\epsilon^2), \\ (u^\alpha)^* &= u^\alpha + \epsilon \phi^\alpha(x, u) + O(\epsilon^2), \end{aligned} \tag{1.2.11}$$

This means Lies method requires the invariance of (1.2.10) under the transformation (1.2.11). Claiming the invariance of (1.2.10) yields an over determined linear system of PDEs for the infinitesimals ξ^i and ϕ^α. The transformations of the independent and dependent variables are characterized by vector fields or infinitesimal generators given by

$$X = \sum_{i=1}^{n} \xi^i(x, u)\partial_{x^i} + \sum_{\alpha=1}^{m} \phi^\alpha(x, u)\partial_{u^\alpha}. \tag{1.2.12}$$

The mathematical formulation of the invariance criterion for (1.2.10) is

$$X^{(k)} \Delta_\nu \left(x, u^{(k)}\right) |_{\Delta=0} = 0, \tag{1.2.13}$$

where $X^{(k)}$ denotes the kth prolongation of the infinitesimal generator X,

$$X^{(k)} = X + \sum_{\alpha=0}^{m} \sum_{J} \phi_J^\alpha \partial_{u_J^\alpha}, \tag{1.2.14}$$

with the multi-indices $J = (j_1, j_2, ..., j_p)$ and

$$u_J^\alpha = \frac{\partial^p u^\alpha}{\partial x^{j_1} ... \partial x^{j_p}}. \tag{1.2.15}$$

The second summation is extended over all derivatives of u^α up to order k. The higher prolongation elements are determined by the infinitesimals ξ^i and ϕ^α through

$$\phi_J^\alpha = D_J \left(\phi^\alpha(x,u) - \sum_{i=1}^{n} \xi^i(x,u) u_i^\alpha \right) + \sum_{i=1}^{n} \xi^i(x,u) u_{J,i}^\alpha, \tag{1.2.16}$$

$D_i F(x,u)$ is the ith total derivative of a function $F(x,u)$ which is defined by

$$D_i F(x,u) = \partial_{x^i} F(x,u) + \sum \alpha = 1^m u_i^\alpha \partial_{u^\alpha} F(x,u) \tag{1.2.17}$$

The ϕ_J^α can be calculated recursively by a formula known

$$\phi_{J,i}^\alpha = D_i \phi_J^\alpha - \sum_{i=1}^{n} u_{J,i}^\alpha D_i \xi^i. \tag{1.2.18}$$

Once we know the infinitesimals ξ^i and ϕ^α we also know the infinitesimal generator X. With the infinitesimal generator at hand, the explicit transformation acting on the space of independent and dependent variables can be calculated by the so-called Lie series

$$(x^*, u^*) = exp[eX](x,u). \tag{1.2.19}$$

Knowing the explicit transformations in (1.2.19), new solutions can be obtained from old ones. This behavior was already known by Lie in the study of the diffusion equation.

Another method to discover new solutions of (1.2.10) is to construct a class of functions which solve (1.2.10) and are invariant under a subgroup of the full symmetry group of (1.2.10). The group invariants can be calculated by solving the characteristic equations

$$\frac{dx^1}{\xi^1(x,u)} = \frac{dx^2}{\xi^2(x,u)} = ... = \frac{dx^n}{\xi^n(x,u)} = \frac{du^1}{\phi^1(x,u)} = \frac{du^2}{\phi^2(x,u)} = ... = \frac{du^m}{\phi^m(x,u)}, \tag{1.2.20}$$

or the invariant surface condition

$$\sum_{i=1}^{n} \xi^i(x,u)u^{\alpha}_{x^i} - \phi^{\alpha}(x,u) = 0 \tag{1.2.21}$$

Then the original system (1.2.10) can be rewritten in terms of group invariants and thus the number of independent variables is reduced.

1.2.3 Symmetry Approach

A technique that has been found an importance in literature on group theoretic methods for the determination of the solutions of a single or system of nonlinear partial differential equations is due to Steinberg [142] and is termed as Symmetry Approach. Though the technique relies heavily on the theory of nonlinear operators yet it has been cast in a form that is easy to utilize by specialists and non-specialists alike. The algorithmic representation of the method makes the concepts clear and straight forward. Further, it bears a close relationship to the method of separation of variables in the case of linear equations. The analytical execution of the technique can be thought of as following of three steps:

(i) Find the symmetries of the differential equations.

(ii) Determine the canonical coordinates for a symmetry or assume a separable form for the differential equation.

(iii) Find the reduced problem in terms of canonical coordinates.

For determining the symmetry operator of a differential equation, we need to proceed as follows:

Suppose that a differential operator L can be written in the form

$$L(u) = \frac{\partial^p u}{\partial t^p} - H(u), \tag{1.2.22}$$

where $u = u(t, x)$ and H may depend on t, x, u and any of u as long as the derivatives of u do not contain more than $(p-1)$ t-derivatives and $(t, x) = (t, x_1, x_2, ..., x_n)$ are $(n+1)$ independent variables.

Consider the symmetry operator, which is quasilinear partial differential equation of first order,

$$S(u) = A(t, x, u)\frac{\partial u}{\partial t} + \sum_{i=1}^{n} B_i(t, x, u)\frac{\partial u}{\partial x_i} + C(t, x, u). \tag{1.2.23}$$

The symmetry operator $S(u)$ is considered only in the sense of infinitesimal symmetries.

Further, define the Fréchet derivative of $L(u)$ by

$$F(L, u, v) = \frac{d}{d\epsilon}L(u + \epsilon v)\Big|_{\epsilon} = 0. \tag{1.2.24}$$

With these definitions in mind, we have to follow the following steps to compute the symmetry operator of differential equations

(i) Compute $F(L, u, v)$.

(ii) Compute $F(L, u, S(u))$.

(iii) Substitute $H(u)$ for $\frac{\partial^p u}{\partial t^p}$ in $F(L, u, S(u))$.

(iv) Set this expression to zero and perform a polynomial expansion, and

(v) Solve the resulting partial differential equations.

Once this resulting set of partial differential equations is solved for the coefficients of $S(u)$, the symmetry operator can be used to find another operator $T(f)$ associated with $S(u)$ and given by

$$T(f) = \sum_{i=1}^{n} B_i(x, u)\frac{\partial f}{\partial x_i} - C(x, u)\frac{\partial f}{\partial u}, f = f(x, u). \tag{1.2.25}$$

1.2.4 The Extended $\left(\frac{G'}{G}\right)$-Expansion Method

In this section, basic idea of the $\left(\frac{G'}{G}\right)$-expansion method [40,89,171,174] for finding traveling wave solutions of nonlinear partial differential equations is described. Suppose that a nonlinear equation, say, in three independent variables x, t and y, is given by

$$P(u, u_x, u_y, u_t, u_{xx}, u_{yy}, u_{tt}, u_{xt}, ...) = 0, \tag{1.2.26}$$

where $u = u(x, t, y)$ is an unknown function, P is a polynomial in $u = u(x, t, y)$ and its various partial derivatives, in which the highest order derivatives and nonlinear terms are involved. In the following steps, the main steps of the modified $\left(\frac{G'}{G}\right)$-expansion method are given:

Step 1. The traveling wave variable

$$u(x, t, y) = u(\xi), \xi = x + y - kt, \tag{1.2.27}$$

permits us to reduce Equation (1.2.26) to an ODE for $u(x, t, y) = u(\xi)$ in the form

$$P(u, -ku', u', k^2u'', ...) = 0, \tag{1.2.28}$$

where prime denotes derivative about ξ.

Step 2. Suppose that the solution of ODE (1.2.28) can be expressed by a polynomial in $\left(\frac{G'}{G}\right)$ as follows:

$$u(\xi) = \sum_{i=0}^{m} a_i \left(\frac{G'(\xi)}{G(\xi)}\right), \tag{1.2.29}$$

where $G = G(\xi)$ satisfies the following second order linear ODE

$$G'' + \lambda G' + \mu G = 0, \tag{1.2.30}$$

where $a_i, (i = 0, 1, 2, ..., m)$, k, λ and μ are constants to be determined later, $a_m \neq 0$. The degree of the polynomial can be determined by balancing the highest order derivative with nonlinear terms.

Step 3. Substituting (1.2.29) into (1.2.30) and using the second order linear ODE (1.2.28) and then equating each coefficient of the resulted polynomial to zero, yields a set of algebraic equations with respect to $a_i, (i = 0, 1, 2, ..., m)$, k λ and μ. Solving the algebraic system, the values of unknowns can be found.

Step 4. Substituting $a_i, (i = 0, 1, 2, ..., m)$, k λ and μ found in Step 3 and the general solutions of Equation (1.2.30) into (1.2.29) we can obtain more traveling wave solutions of the nonlinear PDE (1.2.26). Solutions to Equation (1.2.30) depending on whether $\lambda^2 - 4\mu > 0, \lambda^2 - 4\mu < 0, \lambda^2 - 4\mu = 0,$

$$\frac{G'}{G}=\begin{cases} \frac{\sqrt{\lambda^2-4\mu}}{2}\frac{C_1\cosh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\right)\xi+C_2\sinh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\right)\xi}{C_1\sinh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\right)\xi+C_2\cosh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\right)\xi}-\frac{\lambda}{2}, & \lambda^2-4\mu>0 \\ \frac{\sqrt{\lambda^2-4\mu}}{2}\frac{C_1\cos\left(\frac{4\mu-\sqrt{\lambda^2}}{2}\right)\xi-C_2\sin\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\right)\xi}{C_1\sin\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\right)\xi+C_2\cos\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\right)\xi}-\frac{\lambda}{2}, & \lambda^2-4\mu<0 \\ \frac{C_2}{C_1+C_2\xi}-\frac{\lambda}{2}, & \lambda^2-4\mu=0 \end{cases}$$

1.2.5 Painlevé Analysis

Broadly speaking, Painlevé analysis is the study of the singularity structure of differential equations. Specifically, how the singularities of the solutions depend on the initial conditions of the differential equation?

Definition 2.1. A differential equation has the Painlevé property if all the movable singularities of all its solutions are poles. In the following sections, we discuss the WTC-method for testing PDEs for the Painlevé property. For a more thorough discussion of the Painlevé property, see [2, 24, 27, 139].

Consider a system of M polynomial differential equations,

$$F(u(z), u'(z), ..., u^m(z)) = 0, \tag{1.2.31}$$

where F has components $F_1, F_2, ..., F_M$, the dependent variable $u(z)$ has components $u_1(z), u_2(z), ..., u_M(z)$, the independent variable z has components $z_1, z_2, ..., z_N$ and $u^{(m_i)}(z)$ denotes the collection of mixed derivative terms of order m_i so that the order of the system is $m = \sum_{i=1}^{M} m_i$. If there are any arbitrary coefficients (constants or analytic functions of z) parameterizing the system, we assume they are nonzero.

The algorithm for the Painlevé test is composed of the following three steps:

Step 1 ***(Determine the dominant behavior)***: It is sufficient to substitute

$$u_i(z) = \Phi_i g^{\alpha_i}(z), i = 1, 2, ..., M \tag{1.2.32}$$

where Φ_i is a constant, into (1.2.31) to determine the leading exponents $\alpha_1 \in \{Z, +\}$ which must be a negative integer. In the resulting polynomial system, equating every two possible lowest exponents of $g(z)$ in each equation gives a linear system to determine α_i The linear system is then solved for α_i.

If one or more exponents α_i remain undetermined, we assign integer values to the free α_i so that every equation in (1.2.31) has at least two different terms with equal lowest exponents.

Once α_i is known, we substitute

$$u_i(z) = u_{i,0}(z) g^{\alpha_i}(z), i = 1, 2, .., M \tag{1.2.33}$$

into (1.2.31). We then solve the (typically) nonlinear equation for $u_{i,0}(z)$, which is found by requiring that the leading terms balance. By leading terms, we mean those terms with the lowest exponent of $g(z)$.

If any of the α_i are non-integer, all the α_i are positive, or any of the $u_{i,0}(z) \equiv 0$ then the algorithm terminates.

Step 2 ***(Determine the resonances)***: For each α_i and $u_{i,0}(z)$, we calculate the integers $r_1, r_2, ..., r_m$ for which $u_{i,r_j}(z)$ is an arbitrary function in (1.2.33). To do this, we substitute

$$u_i(z) = u_{i,0}(z)g^{\alpha_i}(z) + u_{i,r}(z)g^{\alpha_i+r}(z) \qquad (1.2.34)$$

into (1.2.31). Then, keeping only the terms with the lowest exponents of g(z); we require that the coefficients of $u_{i,r}(z)$ equate to zero. This is done by computing the roots for r of $det(Q_r) = 0$, where the $M \times M$ matrix Q_r satisfies

$$Q_r u_r = 0, u_r = (u_{1,r}, u_{2,r}, ..., u_{M,r})^T. \qquad (1.2.35)$$

If any of the resonances are non-integer, then the solutions of (1.2.31) have a movable algebraic branch point and the algorithm terminates. If r_m is not in $\{Z, +\}$ then the algorithm terminates; if $r_{m-s+1} = ... = r_m = 0$ and s of the $u_{i,0}(z)$ found in Step 1 are arbitrary, then (1.2.31) has the Painlevé property. If (1.2.31) is parameterized, the values for $r_1, r_2, ..., r_m$ may depend on the parameters, and hence restrict the allowable values for the coefficients. There is always a resonance at -1 which corresponds to the arbitrariness of $g(z)$ and is often called the universal resonance. When there are negative resonances other than -1 then the series solution is not the general solution and further analysis is needed to determine if (1.2.31) passes the Painlevé test.

Step 3 (*Find the constants of integration and check compatibility conditions*): For the system to possess the Painlevé property, the arbitrariness of $u_{i,r}$ must be veried up to the highest resonance level. This is done by substituting

$$u_i(z) = g^{\alpha_i}(z) \sum_{k=0}^{r_m} u_{i,k}(z)g^k(z), i = 1, 2, ..., M \qquad (1.2.36)$$

into (1.2.31), where $u_{i,k}(z)$ are analytic functions of z with $u_{i,0}(z)$ is not equivalent to 0 in a neighborhood of the manifold and α_i is an integer. By equating the coefficients of like powers of $g(z)$ determines the possible values of α_i and defines a recursion relation for $G_{i,k}(z)$. The recursion relation is of the form

$$Q_k u_k = G_k(u_0, u_1, ..., u_{k-1}, g, z), \tag{1.2.37}$$

where Q_k is a $M \times M$ matrix and $u_k = (u_{1,k}, u_{2,k}, ..., u_{M,k})$. For the equation (1.2.31) to have the Painlevé property, the $(M+1) \times M$ augmented matrix $(Q_k|G_k)$ must have rank M when $k \neq r$ and rank $M - s$ when $k = r$, where s is the algebraic multiplicity of r in $det(Q_r) = 0$, $1 \leq k \leq r_m$ and Q_k and G_k are as defined in (1.2.37). If the augmented matrix $(Q_k|G_k)$ is the correct rank, solve the linear system (1.2.37) for $u_{1,k}(z), u_{2,k}(z), ..., u_{M,k}(z)$ and use the results in the linear system at level $k+1$.

If the linear system (1.2.37) does not have a solution, then the solution of (1.2.32) has a movable logarithmic branch point and the algorithm terminates. Often, when (1.2.31) is parameterized, carefully choosing the parameters will resolve the difference in the ranks of Q_k and $(Q_k|Gk)$. If the algorithm does not terminate, then the solutions of (1.2.31) are free of movable algebraic or logarithmic branch points and (1.2.31) has the Painlevé property.

Chapter 2

The Nondiagonal Einstein- Rosen Metrics[1]

2.1 Introduction

The Einstein field equations for the space-times admitting a two-dimensional abelian group of isometries acting orthogonally and transitively on non-null orbits are non linear partial differential equations in two variables. For timelike orbits the equations are elliptic, where as for spacelike orbits the equations are hyperbolic [150]. The space-times admitting two commuting killing vectors can represent interesting physical situations with stationary axial symmetry, planar symmetry or cylindrical symmetry [160]. Since the work by Geroch [44], it has been known that the field equations in the stationary axisymmetric case admit an infinite dimensional group of symmetry transformations.

The Einstein equations in vacuum are:

$$R_{\mu\nu} = 0, \qquad (2.1.1)$$

where $R_{\mu\nu}$ is the Ricci tensor. The particular case when the metric tensor $g_{\mu\nu}$ depends on two variables only is examined, which corresponds to the space-times that admit two commuting killing vector fields, i.e. an abelian two parameter group

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of isometries. In any space-time using the coordinate transformation freedom, $x^\mu = x^\mu(x'^\nu)$, the following constraints on the metric tensor can be found

$$g_{00} = -g_{33}, g_{03} = 0, g_{0a} = 0. \tag{2.1.2}$$

Here and in the following, the Latin indices $a, b, c, ...$ take the values $1, 2, 3, ...$ In these coordinates the space-time becomes:

$$ds^2 = f(dx^2 - dy^2) + g_{ab}dz^a dz^b + 2g_{a3}dz^a dx, \tag{2.1.3}$$

where $f = -g_{00} = g_{33}$. If we now restrict ourselves to the case in which all metric components in (2.1.3) depend on x and y only, the Einstein equations for such metric are still complicated. The situation is different in the particular case in which $g_{a3} = 0$. Since it is not possible to eliminate the metric coefficients g_{a3} by any further coordinate transformation such a simplification should be considered as a real physical constraint. This corresponds to assuming the existence of 2-surface orthogonal to the group of isometries, which is a restriction on two commuting killing vectors. Therefore, from now on, the simplified block diagonal form of the metric (2.1.3) is dealt:

$$ds^2 = f(x, y)(dx^2 - dy^2) + g_{ab}(x, y)dz^1 dz^2. \tag{2.1.4}$$

The stationary axisymmetric gravitational fields corresponds to the analogue of this metric when the independent variables are both spacelike. The metric (2.1.4) was first considered in 1937 by Einstein and Rosen [38] for a diagonal matrix g_{ab}, when the Einstein equations (2.1.1) actually reduces to the linear equation in cylindrical coordinates. The inclusion of the off-diagonal component g_{12} changes the situation drastically and converts the Einstein equations into an essentially nonlinear problem. In the language of weak gravitational waves this corresponds to the appearance of the second polarization state of the wave.

For the stationary analogue of the metric (2.1.4) such a generalization means the rotation has been included. Equations for the metric (2.1.4) were first considered by Kompaneets [77], who noted some of their general properties. In the past, several authors using different simplifying assumptions have obtained a number of exact nontrivial solutions for the metric (2.1.4) or its stationary analogue.

From the physical point of view the metric (2.1.4) and its stationary analogue have many applications in gravitational theory. It suffices to say that to such a class belong the classical solutions of the Robinson-Bondi plane waves [19], the Einstein-Rosen cylindrically wave solutions and their two polarization generalizations [38], the homogenous cosmological models of Binachi types $I-VII$ including the Friedmann-Lemaî̂tre-Robertson-Walker models [135], the Schwarzschild solutions [138], Weyl's axisymmetric solutions [171].

In Einstein equations, the metric tensor in (2.1.4) is denoted by g and the two-dimensional real and symmetric block of the metric tensor (2.1.4) as

$$\begin{bmatrix} g_{11} & g_{12} \\ g_{12} & g_{22} \end{bmatrix}$$

For the determinant of this matrix it is convenient to introduce the notion $\det g = \alpha^2$ and α as nonnegative is considered, i.e., $\alpha \geq 0$. The system of Einstein vacuum equations (2.1.1) for the metric (2.1.4) decomposes into two sets. The first set follows from equations $R_{ab} = 0$, can be written in the form:

$$\begin{aligned} &\tfrac{\partial}{\partial x} u^i_j + \tfrac{\partial}{\partial y} v^i_j = 0, \\ &u^i_j = \alpha(\tfrac{\partial}{\partial x} g_{ik}) g^{kj}, \\ &v^i_j = \alpha(\tfrac{\partial}{\partial y} g_{ik}) g^{kj}, \end{aligned} \tag{2.1.5}$$

and the second set follows from the equations $R_{00} + R_{33} = 0$ and $R_{03} = 0$, which gives the metric coefficient $f(x,y)$ in terms of the metric g, via the relations:

$$\begin{aligned} \partial_x(\log f) &= \tfrac{\partial_x^2(\log\alpha)}{\partial_x(\log\alpha)} + \tfrac{1}{4\alpha\alpha_x}T_rA^2, \\ \partial_y(\log f) &= \tfrac{\partial_y^2(\log\alpha)}{\partial_y(\log\alpha)} + \tfrac{1}{4\alpha\alpha_y}T_rB^2, \end{aligned} \tag{2.1.6}$$

where $\alpha_x = \frac{\partial\alpha}{\partial x}$, $\alpha_y = \frac{\partial\alpha}{\partial y}$ and the matrices A and B are defined by $A = -\alpha\partial_x g_{ik}g^{kj}$, $B = -\alpha\partial_y g_{ik}g^{kj}$.

The dynamics of the system is thus essentially determined by equations (2.1.5). The equations (2.1.5) are a set of nonlinear partial differential equations relating to the induced metric g_{ij}. In equations (2.1.6), derivatives are operated on only the scalar function $\log f$. In order to obtain solutions, first the nonlinear system (2.1.5) is solved and then system of differential equations given by (2.1.6). Although, the integration of ODEs of system (2.1.6) is not difficult, yet it is hard to solve the system (2.1.5) because of its nonlinearity. Here, in this chapter by using the standard normalization where $\alpha = \det g = (x+y)^2$, compatible with equation (2.1.4) and parameterizing g as follows:

$$\begin{aligned} g_{11} &= \tfrac{(x+y)}{u}, \\ g_{12} &= \tfrac{(x+y)v}{u}, \\ g_{22} &= \tfrac{(x+y)(u^2+v^2)}{u}, \end{aligned} \tag{2.1.7}$$

where u and v are functions of x and y only, the system (2.1.5) for the Einstein vacuum equations become

$$\begin{aligned} 2(x+y)uu_{xy} - 2(x+y)(u_xu_y - v_xv_y) + u(u_x+u_y) = 0, \\ 2(x+y)uv_{xy} - 2(x+y)(v_xu_y + u_xv_y) + u(v_x+v_y) = 0, \end{aligned} \tag{2.1.8}$$

where the lower suffix denotes the partial differentiation w.r.t to the corresponding variable and the metric function f can be obtained by using system (2.1.6)

$$\begin{aligned} \partial_x(\log f) &= \tfrac{\partial_x^2(\log\alpha)}{\partial_x(\log\alpha)} + \tfrac{1}{4\alpha\alpha_x}T_rA^2, \\ \partial_y(\log f) &= \tfrac{\partial_y^2(\log\alpha)}{\partial_y(\log\alpha)} + \tfrac{1}{4\alpha\alpha_y}T_rB^2, \end{aligned} \tag{2.1.9}$$

where $\alpha_x = \frac{\partial\alpha}{\partial x}$, $\alpha_y = \frac{\partial\alpha}{\partial y}$ and the matrices A and B are defined by $A = -\alpha\partial_x g_{ik}g^{kj}$, $B = -\alpha\partial_y g_{ik}g^{kj}$.

The Lie classical method is utilized to the further investigation of nonlinear system (2.1.8), here in this chapter. The method has yielded quite an exhaustive study and enabled us to recover some important results too. Firstly, Painlevé Property is applied to the nonlinear system (2.1.8) to check the integrability of this system in Section (2.2). In Section (2.3), the derivation of the set of equations for the determination of the infinitesimals of the invariance group of system (2.1.8) are presented. Section (2.4) has been devoted to the study of new exact solutions of the Einstein vacuum field equations for nondiagonal Einstein-Rosen metric.

2.2 Painlevé Analysis

The Painlevé property (PP) of a PDE in N independent variables, is the absence of movable critical singularities near any non characteristic manifold and it is integrable if at least one of the following properties holds.

1. Its general solution can be obtained, and it is an explicit closed form expression, possibly presenting movable critical singularities.

2. It is linearizable.

3. For $N > 1$, it possesses an auto-B$\ddot{a}$cklund transformation (BT) which, if $N = 2$, depends on an arbitrary complex constant, the B$\ddot{a}$cklund parameter.

4. It possesses a BT to another integrable PDE.

The "Painlevé conjecture", as originally formulated by Ablowitz [1], states that when all the ordinary differential equations (ODEs) obtained by exact similarity transform from a given partial differential equation have the Painlevé property, then the partial differential equation is "integrable". The above definition of "Painlevé property" allows this conjecture to be stated directly for a partial differential equation. According to the Ablowitz, Ramani and Segur (ARS) [1] conjecture, Painlevé analysis is an explicit test of whether or not a given PDE may be of Inverse Scattering Transform (IST) class; namely, reduce it to an ODE, and determine whether the ODE is of P-type. The ARS algorithm proceeds in three steps, dealing with the dominant behaviors, the resonances and check compatibility conditions respectively. The behavior of solutions of system (2.1.8) at a movable singular manifold

$$\phi(x,y) = 0, \tag{2.2.1}$$

is determined by a leading-order ($k = 0$) analysis in the Laurent series solution

$$\begin{aligned} u(x,y) &= (\phi(x,y))^{\alpha_1} \sum_{k=0}^{\infty} u_k(x,y)(\phi(x,y))^k, \\ v(x,y) &= (\phi(x,y))^{\alpha_2} \sum_{k=0}^{\infty} v_k(x,y)(\phi(x,y))^k, \end{aligned} \tag{2.2.2}$$

whereby one makes the ansätz

$$\begin{aligned} u(x,y) &= u_0(x,y)[\phi(x,y)]^{\alpha_1}, \\ v(x,y) &= v_0(x,y)[\phi(x,y)]^{\alpha_2}, \end{aligned} \tag{2.2.3}$$

and balances the dominant terms. This gives

$$\alpha_1 = -1, \alpha_2 = -1 \tag{2.2.4}$$

and

$$u_0(x,y) = \pm\iota v_0(x,y), \tag{2.2.5}$$

where $v_0(x,y)$ is an arbitrary function. Since $v_0(x,y)$ is arbitrary. For the branch $\alpha_1 = -1, \alpha_2 = -1$, on substituting

$$\begin{aligned} u(x,y) &= u_0(x,y)g^{-1}(x,y) + u_r(x,y)g^{r-1}(x,y), \\ v(x,y) &= \pm\iota(u_0(x,y)g^{-1}(x,y) + u_r(x,y)g^{r-1}(x,y)), \end{aligned} \tag{2.2.6}$$

into system (2.1.8), the resonances for this branch are $r_1 = -1$ and $r_2 = 0$ and this branch has the Painlevé property by default. Hence, Painlevé property shows that system (2.1.8) is integrable.

In next section, symmetries of this nonlinear system are found by using Lie classical method. These symmetries are further used to derive exact solutions of the system (2.1.8) – (2.1.9).

2.3 Determination of Symmetries

In this section, Lie group method is performed to the system (2.1.8). Firstly, a one parameter Lie group of infinitesimal transformations is considered

$$\begin{aligned} x &\to x + \epsilon\xi_1(x, y, u, v) + O(\epsilon^2), \\ y &\to y + \epsilon\xi_2(x, y, u, v) + O(\epsilon^2), \\ u &\to u + \epsilon\phi_1(x, y, u, v) + O(\epsilon^2), \\ v &\to v + \epsilon\phi_2(x, y, u, v) + O(\epsilon^2), \end{aligned} \tag{2.3.1}$$

with small parameter $\epsilon \leq 1$. The vector field associated with the above group of transformations can be written as

$$V = \xi_1(x, y, u, v)\partial_x + \xi_2(x, y, u, v)\partial_y + \phi_1(x, y, u, v)\partial_u + \phi_2(x, y, u, v)\partial_v. \tag{2.3.2}$$

The symmetry group of the system (2.1.8) is generated by the vector field of the form (2.3.2). Applying the second prolongation $Pr^{(2)}V$ of V to system (2.1.8), the coefficient functions ξ_1, ξ_2, ϕ_1 and ϕ_2 can be found that must satisfy the symmetry conditions

$$\begin{aligned} &2xu\phi_1^{xy} + 2y\phi_1 u_{xy} + 2\xi_1 uu_{xy} + 2\xi_2 uu_{xy} - 2xu_x\phi_1^y - 2xu_y\phi_1^x + 2xv_x\phi_2^y + 2xv_y\phi_2^x \\ &-2yu_x\phi_1^y - 2yu_y\phi_1^x + 2yv_x\phi_2^y + 2yv_y\phi_2^x - 2\xi_1 u_x u_y + 2\xi_1 v_x v_y - 2\xi_2 u_x u_y + 2\xi_2 v_x v_y \\ &+u\phi_1^x + u\phi_1^y + \phi_1 u_x + \phi_1 u_y = 0, \end{aligned} \tag{2.3.3}$$

and

$$\begin{aligned}&2xu\phi_2^{xy} + 2x\phi_1 v_{xy} + 2yu\phi_2^{xy} + 2y\phi_1 v_{xy} + 2\xi_1 uv_{xy} + 2\xi_2 uv_{xy} - 2xv_x\phi_1^y - 2xu_y\phi_2^x\\ &-2xu_x\phi_2^y - 2xv_y\phi_1^x - 2yv_x\phi_1^y - 2yv_x\phi_2^x - 2yu_x\phi_2^y - 2yv_x\phi_1^x - 2\xi_1 v_x u_y - 2\xi_1 u_x v_y\\ &-2\xi_2 v_x u_y - 2\xi_2 u_x v_y + u\phi_2^x + u\phi_2^y + \phi_1 v_x + \phi_1 v_y = 0\end{aligned} \tag{2.3.4}$$

where $\phi_1^x, \phi_1^y, \phi_2^x, \phi_2^y, \phi_1^{xy}$ and ϕ_2^{xy} and are coefficients of $Pr^{(2)}V$

$$Pr^{(2)}V = Pr^1 V + \phi_1^{xy} + \phi_2^{xy}, \tag{2.3.5}$$

and, furthermore,

$$\begin{aligned}&\phi_1^x = D_x\phi_1 - u_x D_x\xi_1 - u_y D_x\xi_2,\\ &\phi_1^{xy} = D_y D_x\phi_1 - u_x D_y D_x\xi_1 - u_{xy} D_x\xi_1 - u_{xx} D_y\xi_1 - u_y D_y D_x\xi_2 - u_{yy} d_x\xi_2 - u_{xy} D_y\xi_2,\end{aligned} \tag{2.3.6}$$

where D_x and D_y are the total derivatives with respect to x and y respectively.

Substituting u_{xy} and v_{xy} into (2.3.3)-(2.3.4) respectively and equating the coefficients of the various monomials in first, second and other partial derivatives with respect to x and y and various powers of u, the following determining equations are derived

$$\begin{aligned}&\xi_{1u} = 0, \xi_{1v} = 0, \xi_{1y} = 0, \xi_{1xx} = 0,\\ &\phi_{1u} = \tfrac{\phi_1}{u}, \phi_{1v} = 0, \phi_{1x} = 0, \phi_{1y} = 0,\\ &\phi_{2u} = 0, \phi_{2v} = \tfrac{\phi_1}{u}, \phi_{2x} = 0, \phi_{2y} = 0,\\ &\xi_2 = (x+y)\xi_{1x} - \xi_1, \xi_{2u} = 0, \xi_{2v} = 0, \xi_{2x} = 0,\\ &\phi_{1v} = \tfrac{u}{2}\phi_{2vv}, \phi_{1x} = \tfrac{u((x+y)\xi_{1x} - \xi_2 - \xi_1)}{2(x+y)^2}, \phi_{1y} = \tfrac{u((x+y)\xi_{2y} - \xi_2 - \xi_1)}{2(x+y)^2}.\end{aligned} \tag{2.3.7}$$

Solution of this system gives the following forms for the infinitesimal elements ξ_1, ξ_2, ϕ_1 and ϕ_2:

$$\begin{aligned}&\xi_1 = ax + b,\\ &\xi_2 = ay - b,\\ &\phi_1 = lu,\\ &\phi_2 = lv + m.\end{aligned} \tag{2.3.8}$$

where a, b, l and m are arbitrary constants.

2.4 Optimal System of Generators

As is well known, the Lie group theoretic method plays an important role in finding exact solutions and performing symmetry reductions of differential equations. Since any linear combination of infinitesimal generators is also an infinitesimal generator, there are always infinitely many different symmetry subgroups for the differential equation. So, a mean of determining which subgroups would give essentially different types of solutions is necessary and significant for a complete understanding of the invariant solutions. As any transformation in the full symmetry group maps a solution to another solution, it is sufficient to find invariant solutions which are not related by transformations in the full symmetry group, this has led to the concept of an optimal system [111-113]. The problem of finding an optimal system of subgroups is equivalent to that of finding an optimal system of subalgebras. For one-dimensional subalgebras, this classification problem is essentially the same as the problem of classifying the orbits of the adjoint representation. This problem is attacked by the naive approach of taking a general element in the Lie algebra and subjecting it to various adjoint transformations so as to simplify it as much as possible. The idea of using the adjoint representation to classify group-invariant solutions was due to Ovsiannikov [112].

The Lie algebra of infinitesimal symmetries of the system (2.3.8) is spanned by the following four vector fields:

$$\{V_1 = \frac{\partial}{\partial x} - \frac{\partial}{\partial y}, \quad V_2 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} \quad V_3 = \frac{\partial}{\partial v}, \quad V_4 = u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v}\}.$$

The commutation relations of this Lie algebra are

$$[V_1, V_2] = -[V_2, V_1] = V_1, \qquad [V_3, V_4] = -[V_4, V_3] = V_3$$

and $[V_i, V_j] = 0, \quad \forall i, j.$

The adjoint action is given by the Lie series

$$Ad(exp(\epsilon V_i)V_j) = V_j - \epsilon[V_i, V_j] + \frac{\epsilon^2}{2}[V_i, [V_i, V_j]] - ..., \tag{2.4.1}$$

where $[V_i, V_j]$ is the commutator for the Lie algebra and ϵ is a parameter. The adjoint table is listed in table (2.1), where the $(i, j)th$ entry gives $Ad(exp(\epsilon V_i)V_j)$.

Adjoint Table (2.1)

Ad	V_1	V_2	V_3	V_4
V_1	V_1	$V_2 - \epsilon V_1$	V_3	V_4
V_2	$V_1 e^\epsilon$	V_2	V_3	V_4
V_3	V_1	V_2	V_3	$V_4 - \epsilon V_3$
V_4	V_1	V_2	$V_3 e^\epsilon$	V_4

To obtain the one-dimensional optimal system of vector fields, we follow the procedure of Olver, Ovsiannikov [111, 112]. A general element, i.e., linear combination of vector fields $V = a_1V_1 + a_2V_2 + a_3V_3 + a_4V_4$ is taken, and try to simplify as many of the coefficients a_i as possible through judicious applications of adjoint map on this element.

First suppose that $a_4 \neq 0$. Scaling V if necessary, it is assumed that $a_4 = 1$ and acted on V by $Ad(exp(a_3V_3))$ to make the coefficient of V_3 vanish:

$V' = Ad(exp(a_3V_3))V = V_4 + a_2'V_2 + a_1'V_1,$

for certain scalars a_2', a_1' depending on a_2, a_1. Similarly, with appropriate adjoint action on V', the coefficient of V_1 can also be vanish. Finally, simplified form $V_4 + a_2V_2$ is obtained for some a_2. The reader can check that no further simplification of the coefficients is possible. Therefore, V is equivalent to $V_4 + aV_2$ under the adjoint representation. In other words, every one-dimensional subalgebra generated with $a_4 \neq 0$ is equivalent to the subalgebra spanned by $V_4 + aV_2$.

The remaining one-dimensional subalgebras are spanned by vectors of the above form with $a_4 = 0$ and if $a_3 \neq 0$, scale to make $a_3 = 1$, and then V_3 is acted upon by $Ad(exp(a_1 V_1))$, so that V is equivalent to $V' = V_3 + a_2' V_2$ for some a_2'.

Similarly, when $a_4 = a_3 = 0, a_2 \neq 0$, V is equivalent to V_2. For $a_4 = a_3 = a_2 = 0, a_1 \neq 0$, V is equivalent to V_1. Recapitulating, an optimal system of one-dimensional subalgebras is deduced spanned by following basic fields.

(i) V_1

(ii) V_2

(iii) $V_3 + \alpha V_2$

(iv) $V_4 + \beta V_2$, where α and β are arbitrary constants.

It seems reasonable now to construct Lie ansätze and to seek exact solutions of the nonlinear system (2.1.8). With this in mind, consider its Lie symmetry generated by the basic operators in the optimal system.

According to the general procedure it is necessary to solve the Lagrange system

$$\frac{dx}{a} = \frac{dy}{b} = \frac{du}{-l} = \frac{dv}{-m}. \tag{2.4.2}$$

On solving the system (2.4.2) for the various operators in the optimal system, a set of non-equivalent Lie ansätze for the functions is obtained. In Table-2.2, the Lie ansätze for all the essential fields comprising the optimal system are presented.

Table 2.2: Similarity Reductions of System (2.1.8) to ODEs

Essential fields	*Similarity variable*(ζ)	*Similarity solution*(u,v)	*ReducedODEs*
V_1	$x+y$	$u=F(\zeta), v=G(\zeta)$	$2\zeta FF''-2\zeta(F'^2-G'^2)+2FF'=0,$ $2\zeta FG''-4\zeta(F'G')+2FG'=0$
V_2	$\frac{x}{y}$	$u=F(\zeta), v=G(\zeta)$	$2(\zeta+1)F(-\zeta F''-F')$ $-2(\zeta+1)(-\zeta F'^2+\zeta G'^2)$ $+F(F'-\zeta F')=0,$ $2(\zeta+1)F(-\zeta G''-G')$ $+4(\zeta+1)\zeta F'G'+F(G'-\zeta G')=0$
$V_3+\alpha V_2$	$\frac{x}{y}$	$u=F(\zeta), v=\log(y^{\frac{1}{\alpha}}G(\zeta))$	$2(\zeta+1)F(-\zeta F''-F')$ $-2(\zeta+1)(-\zeta F'^2-\frac{G'}{\alpha G}+\zeta G'^2)$ $+F(F'-\zeta F')=0,$ $2(\zeta+1)F(\frac{\zeta G'^2}{G}-\zeta G''-G')$ $-2(\zeta+1)(-2\zeta F'G'-\frac{F'G}{\alpha})$ $+F(G'+\frac{G}{\alpha}-\zeta G')=0$
$V_4+\beta V_2$	$\frac{x}{y}$	$u=y^{\frac{1}{\beta}}F(\zeta), v=y^{\frac{1}{\beta}}G(\zeta)$	$2(\zeta+1)F(-\zeta F''-F'+\frac{F'}{\beta})$ $-2(\zeta+1)(-\zeta F'^2-\frac{GG'}{\beta}+\zeta G'^2-\zeta F'^2)$ $+F(F'-\zeta F'+\frac{F}{\beta})=0,$ $2(\zeta+1)F(\frac{G'}{\beta}-\zeta G''-G')$ $-2(\zeta+1)(-2\zeta F'G'+\frac{F'G}{\beta}+\frac{FG'}{\beta})$ $+F(G'+\frac{G}{\beta}-\zeta G')=0$

2.5 Solutions of the Einstein Vacuum Equations

In this section, exact solutions of the reduced ODEs are obtained by using similarity variables and similarity solutions obtained in Table 1. The general solution of these equations involves three variables; one becomes the new independent variable ζ and the other two are F and G, as dependent variables. These solutions are obtained by using characteristic equations (2.4.2).

Vector field V_1

From this vector field, solution of system (2.1.8) are:

$$u(x,y) = F(\zeta) = \frac{4e^{\frac{c_2}{c_1}}}{(x+y)^{\frac{1}{c_1}}\left(4c_1^2 + \frac{c_1^2(e^{\frac{c_2}{c_1}})^2}{((x+y)^{\frac{1}{c_1}})^2}\right)}, \tag{2.5.1}$$

$$u(x,y) = F(\zeta) = \frac{4(x+y)^{\frac{1}{c_5}}}{e^{\frac{c_6}{c_5}}\left(4c_5^2 + \frac{c_5^2((x+y)^{\frac{1}{c_5}})^2}{(e^{\frac{c_6}{c_5}})^2}\right)}, \tag{2.5.2}$$

and

$$v(x,y) = G(\zeta) = \frac{c_4}{\left(c_1^2((x+y)^{\frac{1}{c_1}})^2 + 4c_1^2(e^{\frac{c_2}{c_1}})^2\right)} + c_3, \tag{2.5.3}$$

$$v(x,y) = G(\zeta) = \frac{c_8}{\left(4c_5^2((x+y)^{\frac{1}{c_5}})^2 + c_1^2(e^{\frac{c_8}{c_5}})^2\right)} + c_7, \tag{2.5.4}$$

where $c_1, c_2, c_3, c_4, c_5, c_6, c_7$ and c_8 are arbitrary constants, which is the general solution of the system (2.1.8).

For solution of system (2.1.9) and value of metric function f, arbitrary constants $c_1, c_4 = 1$ and $c_2, c_3 = 0$, are chosen in system (2.5.1)-(2.5.2)

$$f(x,y) = \exp\left(\frac{1}{8x^2+16xy+8y^2+2} - 2\log(x+y) + \log(4x^2+8xy+4y^2+1)\right), \tag{2.5.5}$$

and $f(x,y)$ from (2.5.3) and (2.5.4), by assuming $c_5, c_8 = 1$ and $c_6, c_7 = 0$, is given as

$$\begin{aligned} f(x,y) &= -2\log(x+y) + \tfrac{2}{(32x^2+64xy+32y^2+2)} - \tfrac{2}{63}\log(4+x^2+y^2+2xy) \\ &+ \tfrac{8}{(x^2+y^2+2xy+4)} + \tfrac{65}{63}\log(16x^2+16y^2+32xy+1). \end{aligned} \tag{2.5.6}$$

Vector field V_2

On solving the ODEs corresponding to this vector field, the solution of system (2.1.8) is

$$u(x,y) = F(\zeta) = -\frac{2c_2^2 q(\zeta)^2\sqrt{-c_2c_1^2\zeta e^{r(\zeta)}(e^{c_3})^2}}{c_1^2 c_2(\zeta+1)\sqrt{s(\zeta)}}, \tag{2.5.7}$$

where $q(\zeta), r(\zeta)$ and $s(\zeta)$ are as follows:

$$s(\zeta) = \left(e^{\int \frac{p(\zeta)}{(\zeta+1)^2} d\zeta}\right)^4 \left(e^{\int \frac{p(\zeta)}{\zeta(\zeta+1)^2} d\zeta}\right) \left(e^{\int \frac{\zeta p(\zeta)}{(\zeta+1)^2} d\zeta}\right)^3 \left(e^{c_2 \int \frac{p(\zeta)}{(\zeta+1)^2} d\zeta}\right)^4 \left(e^{c_2 \int \frac{p(\zeta)}{\zeta(\zeta+1)^2} d\zeta}\right)$$
$$\left(e^{c_2 \int \frac{p(\zeta)}{(\zeta+1)^2} d\zeta}\right)^3 \left(e^{\iota c_1 c_2 \int \frac{\sqrt{\zeta} p(\zeta)}{(\zeta+1)^2} d\zeta}\right) \left(e^{\iota c_1 c_2 \int \frac{p(\zeta)}{\sqrt{\zeta}(\zeta+1)^2} d\zeta}\right) s(\zeta)^2, \tag{2.5.8}$$

$$r(\zeta) = \left(e^{\iota c_1 \int \frac{\sqrt{\zeta} p(\zeta)}{(\zeta+1)^2} d\zeta}\right) \left(e^{\iota c_1 \int \frac{p(\zeta)}{\sqrt{\zeta}(\zeta+1)^2} d\zeta}\right), \tag{2.5.9}$$

$$p(\zeta) = \frac{l(\zeta)}{q(\zeta)}, \tag{2.5.10}$$

$$l(\zeta) = \left(\frac{\iota + \sqrt{\zeta}}{-\iota + \sqrt{\zeta}}\right)^{\frac{1}{4} c_1}, \tag{2.5.11}$$

$$q(\zeta) = \left(\frac{\iota + \sqrt{\zeta}}{-\iota + \sqrt{\zeta}}\right)^{\frac{-1}{4} c_1} + \left(\frac{\iota + \sqrt{\zeta}}{-\iota + \sqrt{\zeta}}\right)^{\frac{1}{4} c_1} \tag{2.5.12}$$

and

$$v(x,y) = G(\zeta) = \int e^{f(\zeta)} d\zeta + c_4, \tag{2.5.13}$$

where $f(\zeta)$ is given as

$$\begin{aligned} f(\zeta) = &-2\int \frac{p(\zeta)}{(\zeta+1)^2} d\zeta - \frac{1}{2}\int \frac{p(\zeta)}{\zeta(\zeta+1)^2} d\zeta - \frac{3}{2}\int \frac{\zeta p(\zeta)}{(\zeta+1)^2} d\zeta + \frac{1}{2}\iota c_1 \int \frac{\sqrt{\zeta} p(\zeta)}{(\zeta+1)^2} d\zeta + \\ &\frac{1}{2}\iota c_1 \int \frac{p(\zeta)}{\sqrt{\zeta}(\zeta+1)^2} d\zeta - 2c_2 \int \frac{\sqrt{\zeta} p(\zeta)}{(\zeta+1)^2} d\zeta - \frac{1}{2} c_2 \int \frac{p(\zeta)}{\zeta(\zeta+1)^2} d\zeta - \frac{3}{2} c_2 \int \frac{\zeta p(\zeta)}{(\zeta+1)^2} d\zeta \\ &- \frac{1}{2}\iota c_1 c_2 \int \frac{\sqrt{\zeta} p(\zeta)}{(\zeta+1)^2} d\zeta - \frac{1}{2}\iota c_1 c_2 \int \frac{p(\zeta)}{\sqrt{\zeta}(\zeta+1)^2} d\zeta + c_3, \end{aligned} \tag{2.5.14}$$

where $p(\zeta)$ and $\zeta = \frac{x}{y}$ is given as above and c_1, c_2, c_3 and c_4 are arbitrary constants.

In this case it is difficult to find the value of f.

Vector field $V_3 + \alpha V_2$

Solution of ODEs corresponding to this vector field can be obtainted as:

In this case, from the table we can obtain $F(\zeta)$ in the form of $G(\zeta)$:

$$F(\zeta) = \exp\left(\int \frac{2(\zeta+1)\left(\frac{\zeta G'(\zeta)^2}{G(\zeta)} - \zeta G''(\zeta) - G'(\zeta)\right) + \left(G'(\zeta) + \frac{G(\zeta)}{\alpha} - \zeta G'(\zeta)\right)}{2(\zeta+1)\left(-2\zeta G'(\zeta) - \frac{G(\zeta)}{\alpha}\right)} d\zeta\right), \tag{2.5.15}$$

and by substituting the value of $F(\zeta)$ in the following equation

$$\begin{aligned} &2(\zeta+1)F\left(-\zeta F''(\zeta) - F'(\zeta)\right) - 2(\zeta+1)\left(-\zeta F'(\zeta)^2 - \frac{G'(\zeta)}{\alpha G(\zeta)} + \zeta G'(\zeta)^2\right) \\ &+F(\zeta)\left(F'(\zeta) - \zeta F'(\zeta)\right) = 0, \end{aligned} \tag{2.5.16}$$

which is quite difficult to solve.

Vector field $V_4 + \beta V_2$

Corresponding to this vector field, solution of ODE is

$$F(\zeta) = \frac{c_1(\zeta+1)^{\frac{1}{\beta}}}{(\exp^{\arctan(\sqrt{\zeta})})^2}, G(\zeta) = 0, \tag{2.5.17}$$

and solution of system (2.1.8) corresponding to this vector field are:

$$u(x,y) = y^{\frac{1}{\beta}} \left(\frac{c_1(\frac{x}{y}+1)^{\frac{1}{\beta}}}{(\exp^{\arctan(\sqrt{\frac{x}{y}})})^2} \right), v(x,y) = 0, \tag{2.5.18}$$

where c_1 is arbitrary constants.

and solution of system (2.1.9) in this case is:

$$f(x,y) = \exp\left(\frac{a(x,y)}{2\beta^2\sqrt{x}} \right), \tag{2.5.19}$$

where $a(x,y)$ is given as

$$\begin{aligned} &-4\arctan(\sqrt{\tfrac{x}{y}})\beta\sqrt{x} + \log(x)\beta^2\sqrt{x} - 4\log(x+y)\beta^2\sqrt{x} + 2\log(x+y)\sqrt{x} \\ &+4\sqrt{\tfrac{x}{y}}\sqrt{y}\arctan(\sqrt{\tfrac{y}{x}})\beta + \log(y)\beta^2\sqrt{x}. \end{aligned} \tag{2.5.20}$$

In this case also as above, from Table (2.2), $F(\zeta)$ in the form of $G(\zeta)$ is obtained as:

$$F(\zeta) = \exp\left(\int \frac{2(\zeta+1)\left(\frac{G'(\zeta)}{\beta} - \zeta G''(\zeta) - G'(\zeta)\right) + \left(G'(\zeta) + \frac{G(\zeta)}{\beta} - \zeta G'(\zeta)\right)}{2(\zeta+1)\left(-2\zeta G'(\zeta) + \frac{G(\zeta)}{\beta} + \frac{G'(\zeta)}{\beta}\right)} d\zeta \right), \tag{2.5.21}$$

and by substituting the value of $F(\zeta)$ in the following equation

$$\begin{aligned} &2(\zeta+1)F\left(-\zeta F''(\zeta) - F'(\zeta) + \frac{F'(\zeta)}{\beta}\right) - 2(\zeta+1)\left(-2\zeta F'(\zeta)^2 - \frac{G(\zeta)G'(\zeta)}{\beta} + \zeta G'(\zeta)^2\right) \\ &+F(\zeta)\left(F'(\zeta) - \zeta F'(\zeta) + \frac{F(\zeta)}{\beta}\right) = 0, \end{aligned} \tag{2.5.22}$$

which is quite difficult to solve, so we found only one solution (2.5.18) and (2.5.19) of the nonlinear system (2.1.8) – (2.1.9) in this case.

2.6 Concluding Remarks

Exact solutions of Einstein vacuum field equations with two commuting Killing spacelike vectors are obtained. In literature, the reduced system $(2.1.8)-(2.1.9)$ of Einstein vacuum field equations obtained from nondiagonal Einstein-Rosen Metrics, for deriving exact solutions has not considered yet. Exact solutions obtained in this chapter describe the axially symmetric gravitational fields in general relativity. In this chapter, the integrability of nonlinear system of Einstein vacuum equations has been checked by applying the Painlevé Property. The Lie symmetry method is utilized to investigate the symmetries and invariant solutions of Einstein vacuum field equations. The vector fields of the optimal system lead to reductions of the nonlinear system of PDEs to ODEs. The infinitesimal generators in optimal system are used for reductions and exact solutions. By using these vector fields, group-invariant solutions are found. The exact solution corresponding to the vector field V_1 is general solution for the system $(2.1.8)-(2.1.9)$. This is new exact analytic and general solution of nonlinear system of Einstein vacuum field equations. It is worth to mention here that the authenticity of all the solutions has been checked with the aid of software Maple.

Chapter 3

Einstein-Maxwell Equations [1]

3.1 Introduction

In this chapter, the Einstein-Maxwell (EM) equations for non-static cylindrical symmetric metric are studied. A great deal of work has been done on the EM equations. This metric has been studied by various authors to find solutions and physical interpretations. Some of which are Einstein and Rosen [38], Liang [89], Majumdar [103], McVittie [104], Pirani [127], Rao [132], Weber and Wheeler [170] and Weyl [171]. Consequently, in this chapter, Lie classical approach is carried out to determine the invariance group of transformations admitted by the system of Einstein field equations. This eventually reduces the EM equations, which are coupled nonlinear pdes in four unknowns to a system of odes.

The work put in this chapter has been structured as follows. In section (3.2), the EM equations for non-static cylindrical symmetric metric are briefly outlined. The symmetries of system of EM equations are determined in section (3.3). Finally, in section (3.4) the similarity reductions and similarity solutions of EM equations are presented.

[1] The contents of this chapter are published in the *Physica Scripta* 85 (2012) 055011 (6pp).

3.2 Einstein-Maxwell Equations

The non-static cylindrical symmetric Einstein-Rosen [38] metric is considered:

$$ds^2 = e^{-2u}[e^{2k}(dx^2 - dt^2) + W^2 d\phi^2] + e^{2u}dz^2, \tag{3.2.1}$$

where all metric functions depend only on x and t and one can choose $W = x$; they are related by a complex substitution to the static subcase of the stationary axisymmetric solutions. As we know, the Einstein-Rosen metric is cylindrically symmetric, since it is special case of the general cylindrically symmetric metric. The metric depends on the time coordinate t and spatial coordinate x lies in interval $[0, \infty]$. The field equations in the theory of general relativity for regions of space-time containing electromagnetic fields but no matter are given by [37] as follows:

$$R_a^b = -8\pi E_a^b, \tag{3.2.2}$$

$$E_a^b = -F_{a\sigma}F^{b\sigma} + \frac{1}{4}g_\alpha^\beta F_{\sigma\epsilon}F^{\sigma\epsilon}, \tag{3.2.3}$$

$$F_{ab} = A_{a,b} - A_{b,a}, \tag{3.2.4}$$

$$(\sqrt{(-g)}F^{ab})_{,b} = 0, \tag{3.2.5}$$

where R_a^b is the Ricci tensor, E_a^b is the electromagnetic energy tensor, F_{ab} is the electromagnetic field tensor which satisfies Maxwell's equations for empty space time and A_a is the 4−potential. A comma followed by a lower suffix denotes a partial differentiation with respect to corresponding variable. The existence of killing vector $\zeta = \partial_z$ introduce scalar potentials w and v as

$$\sqrt{\frac{k_0}{2}}\zeta^a F_{ab}^* = w_{,b} - \iota v_{,b}. \tag{3.2.6}$$

If w and v are functions of x and t only, then from (3.2.4), (3.2.5) and (3.2.6), the non-zero components of the field tensor F_{mn} with respect to the metric (3.2.1) are as follows:

$$\begin{aligned}\sqrt{\tfrac{k_0}{2}}F_{zx} &= w_{,x},\\ \sqrt{\tfrac{k_0}{2}}F_{zt} &= w_{,t},\\ \sqrt{\tfrac{k_0}{2}}F_{\phi x} &= We^{-2u}v_{,t},\\ \sqrt{\tfrac{k_0}{2}}F_{\phi t} &= We^{-2u}v_{,x}.\end{aligned} \tag{3.2.7}$$

Now, the field equations for stationary axisymmetric Einstein-Maxwell fields are (Stephani et al. [150], 2003; chap. 22, p.354):

$$u_{xx} + \frac{1}{x}u_x - u_{tt} = -e^{-2u}(v_x^2 - v_t^2 + w_x^2 - w_t^2), \tag{3.2.8}$$

$$v_{xx} + \frac{1}{x}v_x - v_{tt} = 2(u_x v_x - u_t v_t), \tag{3.2.9}$$

$$w_{xx} + \frac{1}{x}w_x - w_{tt} = 2(u_x w_x - u_t w_t), \tag{3.2.10}$$

$$v_x w_t = w_x v_t, \tag{3.2.11}$$

$$k_x = x(u_x^2 + u_t^2) + xe^{-2u}(v_x^2 + v_t^2 + w_x^2 + w_t^2), \tag{3.2.12}$$

$$k_t = 2xu_x u_t + 2xe^{-2u}(v_x v_t + w_x w_t), \tag{3.2.13}$$

where the lower suffix denotes the partial differentiation w.r.t to the corresponding variable. The problem of solving equations (3.2.8) – (3.2.13) is that of determining u, v and w from (3.2.8) – (3.2.11) and then k will be given by (3.2.12) and (3.2.13).

The nonlinear system of PDEs (3.2.8) – (3.2.13) represents mathematically and physically important phenomena for electromagnetic fields and gravitational fields in the theory of general relativity. Hence symmetries and exact solutions of this nonlinear system are of great importance. Group-theoretic techniques are applicable to solve nonlinear systems. To study exact solutions and symmetries for this nonlinear system, Lie classical method has been applied.

3.3 Group Infinitesimals

In this section, Lie's method [90] of infinitesimal transformation groups is applied on system of Einstein-Maxwell equations. On considering a one point group transformations of point-like transformations acting on the space of independent variables (x, t) and on the space of dependent variables (u, v, w), the associated infinitesimal generator is given by

$$X = \xi(x,t,u,v,w)\partial_x + \tau(x,t,u,v,w)\partial_t + \eta(x,t,u,v,w)\partial_u + \phi(x,t,u,v,w)\partial_v + \psi(x,t,u,v,w)\partial_w, \tag{3.3.1}$$

and the second extension of operator X has the general form

$$X^{(2)} = X + \eta^x \frac{\partial}{\partial u_x} + \eta^t \frac{\partial}{\partial u_t} + \eta^{xx} \frac{\partial}{\partial u_{xx}} + \eta^{tt} \frac{\partial}{\partial u_{tt}} + \eta^{xt} \frac{\partial}{\partial u_{xt}} + \phi^x \frac{\partial}{\partial v_x} + \phi^t \frac{\partial}{\partial v_t} + \phi^{xx} \frac{\partial}{\partial v_{xx}} + \phi^{tt} \frac{\partial}{\partial v_{tt}} + \phi^{xt} \frac{\partial}{\partial v_{xt}} + \psi^t \frac{\partial}{\partial w_t} + \psi^{xx} \frac{\partial}{\partial w_{xx}} + \psi^{tt} \frac{\partial}{\partial w_{tt}} + \psi^{xt} \frac{\partial}{\partial w_{xt}}.$$

Operator $X^{(2)}$ is admitted by partial differential equations (3.2.8)-(3.2.11) if and only if

$$\begin{aligned}
&X^{(2)}[u_{xx} + \tfrac{1}{x}u_x - u_{tt} = -e^{-2u}(v_x^2 - v_t^2 + w_x^2 - w_t^2)] \mid_{(u_{xx}+\frac{1}{x}u_x-u_{tt}+e^{-2u}(v_x^2-v_t^2+w_x^2-w_t^2)=0)} = 0,\\
&X^{(2)}[v_{xx} + \tfrac{1}{x}v_x - v_{tt} = 2(u_x v_x - u_t v_t)] \mid_{(v_{xx}+\frac{1}{x}v_x-v_{tt}-2(u_x v_x-u_t v_t)=0)} = 0,\\
&X^{(2)}[w_{xx} + \tfrac{1}{x}w_x - w_{tt} = 2(u_x w_x - u_t w_t)] \mid_{(w_{xx}+\frac{1}{x}w_x-w_{tt}-2(u_x w_x-u_t w_t)=0)} = 0.
\end{aligned} \tag{3.3.2}$$

Then equations (3.2.8)-(3.2.11) become the following invariance conditions

$$\eta^{xx} + x\eta^x - \xi u - \eta^{tt} + 2e^{-2u}(v_x\phi^x - v_t\phi^t + w_x\psi^x - w_t\psi^t - \eta({v_x}^2 - {v_t}^2 + {w_x}^2 - {w_t}^2)) = 0, \tag{3.3.3}$$

$$\phi^{xx} + x\phi^x - \xi u - \phi^{tt} - 2(u_x\phi^x + v_x\eta^x - u_t\phi^t - v_t\eta^t) = 0, \tag{3.3.4}$$

$$\psi^{xx} + x\psi^x - \xi u - \psi^{tt} - 2(u_x\psi^x + v_x\eta^x - u_t\psi^t - v_t\eta^t) = 0, \tag{3.3.5}$$

$$v_x\psi^t + w_t\phi^x - v_t\psi^x - w_x\phi^t = 0, \tag{3.3.6}$$

By substituting equations values of prolonged infinitesimals into equations (3.3.3)-(3.3.6) and then causing to vanish the coefficients of the various monomials in partial derivatives of the dependent variables (u, v, w), systems of differential equations for the unknown function is obtained, i.e., $\xi(x,t,u,v,w), \tau(x,t,u,v,w), \eta(x,t,u,v,w)$, $\phi(x,t,u,v,w)$ and $\psi(x,t,u,v,w)$. The solution of these systems of determining equations yield the following symmetries

$$\begin{aligned} \xi &= ax, \\ \tau &= at + b, \\ \eta &= l, \\ \phi &= lv + mw + n, \\ \psi &= lw - mv. \end{aligned} \tag{3.3.7}$$

with arbitrary constants a, b, l, m and n.

The six-dimensional Lie algebra associated with the system of infinitesimals consists of following six vector fields:

$\{V_1 = \frac{\partial}{\partial t}, V_2 = x\frac{\partial}{\partial x} + t\frac{\partial}{\partial t}, V_3 = \frac{\partial}{\partial v}, V_4 = \frac{\partial}{\partial w}, V_5 = w\frac{\partial}{\partial v} - v\frac{\partial}{\partial w}, V_6 = \frac{\partial}{\partial u} + v\frac{\partial}{\partial v} + w\frac{\partial}{\partial w}.\}$

When the dimension of the Lie algebra, associated with a group of invariant transformations of a PDE, is greater than 1, there are often an infinite number of subgroups. To each s-parameter subgroup there corresponds a family of group- invariant solutions. So, in general, it is quite impossible to determine all possible group-invariant solutions of a PDE. In order to minimize this search, it is useful to construct the optimal system of solutions. The construction of the one-dimensional optimal system of subalgebras can be carried out by using the adjoint transformations as suggested by Ovsiannikov [121]. The optimal system consists of the following six basic vector fields:

$$\langle V_1, V_2, V_3 + \alpha V_2, V_4 + \gamma V_3 + \beta V_2, V_5 + \delta V_2, V_6 + \sigma V_5 + \nu V_2 \rangle$$

where α, β, δ, σ and ν are arbitrary constants and $\gamma = 0, +1, -1$.

3.4 Reduced Systems and Exact Solutions

In this section, a classification of symmetry reductions of system of PDEs (3.2.8)-(3.2.11) by one-dimensional subalgebras (up to conjugacy) of symmetry algebra is given. Some exact invariant solutions are also obtained. The reduction procedure explained below is followed.

For each representative V in optimal system of one-dimensional subalgebras:

(i) Reduce the system of PDEs (3.2.8)-(3.2.11) to system of ODEs.

(ii) Determine normalizer of V and conjugacy classes of one-dimensional subalgebras in the normalizer.

The symmetries determined by elements in (ii) are inherited by the reduced ODE obtained in (i). All possible reductions and solutions (where possible) by one-dimensional subalgebras (up to conjugacy).

(i) *Subalgebra* $\mathfrak{L}_1 = \langle V_1 \rangle$

The differential invariants (and hence the similarity variables) for

$V = V_1 = \frac{\partial}{\partial t}$

are determined by solving the characteristic system

$\frac{dx}{0} = \frac{dt}{1} = \frac{du}{0} = \frac{dv}{0} = \frac{dw}{0}$

which give the similarity variables as

$\zeta = x \; u(x,t) = F(\zeta),\, v(x,t) = G(\zeta),\, w(x,t) = H(\zeta)$

Equations (3.2.8)-(3.2.11) under this transformation reduce to the following system of ODEs:

$$F'' + \frac{F'}{\zeta} + e^{-2F}(G'^2 + H'^2) = 0, \tag{3.4.1}$$

$$G'' + \frac{G'}{\zeta} - 2F'G' = 0, \tag{3.4.2}$$

and

$$H'' + \frac{H'}{\zeta} - 2F'H' = 0, \tag{3.4.3}$$

where $'$ denotes the differentiation w.r.t ζ. Solution of system (3.4.1)-(3.4.3) can be expressed in the form

$$F(\zeta) = \frac{1}{2}\log\left(\frac{-1}{c_1^2\log(\zeta)^2 - 2c_1^2c_3\log(\zeta) + c_1^2c_3^2 + c_2^2\log(\zeta)^2 - 2c_2^2c_3\log(\zeta) + c_2^2c_3^2}\right), \tag{3.4.4}$$

$$G(\zeta) = \frac{c_1}{(c_1^2 + c_2^2)(\log(x) - c_3)}, \tag{3.4.5}$$

$$H(\zeta) = \frac{c_2}{(c_1^2 + c_2^2)(\log(x) - c_3)}, \tag{3.4.6}$$

where c_1, c_2 and c_3 are constants of integration.

Now, the solution of the system (3.2.8)-(3.2.13) is

$$u(x,t) = F(\zeta) = \tfrac{1}{2}\log\left(\tfrac{-1}{c_1^2\log(x)^2 - 2c_1^2c_3\log(x) + c_1^2c_3^2 + c_2^2\log(x)^2 - 2c_2^2c_3\log(x) + c_2^2c_3^2}\right), \tag{3.4.7}$$

$$v(x,t) = G(\zeta) = \frac{c_1}{(c_1^2 + c_2^2)(\log(x) - c_3)}, \tag{3.4.8}$$

and

$$w(x,t) = H(\zeta) = \frac{c_2}{(c_1^2 + c_2^2)(\log(x) - c_3)}, \tag{3.4.9}$$

and

$$k(x,t) = \frac{1}{2\log(x) - 2c_3}. \tag{3.4.10}$$

(ii) *Subalgebra* $\mathfrak{L}_2 = \langle V_2 \rangle$

The invariants associated with the infinitesimal generator V_2 are obtained by integrating the characteristic equation

$\frac{dx}{x} = \frac{dt}{t} = \frac{du}{0} = \frac{dv}{0} = \frac{dw}{0}$

and have the similarity forms

$\zeta = \frac{x}{t}$, $u(x,t) = F(\frac{x}{t})$, $v(x,t) = G(\frac{x}{t})$, $w(x,t) = H(\frac{x}{t})$.

Substitution of these similarity variable in system (3.2.8)-(3.2.11) implies the following system of ODES:

$$F''(\zeta^2 - 1) + F'\left(2\zeta - \frac{1}{\zeta}\right) = e^{-2F}(G'^2(1-\zeta^2) + H'^2(1-\zeta^2)), \tag{3.4.11}$$

$$G''(\zeta^2 - 1) + G'\left(2\zeta - \frac{1}{\zeta}\right) = 2F'G'(\zeta^2 - 1), \tag{3.4.12}$$

$$H''(\zeta^2 - 1) + H'\left(2\zeta - \frac{1}{\zeta}\right) = 2F'H'(\zeta^2 - 1), \tag{3.4.13}$$

where $'$ is differentiation w.r.t ζ. Solution of system of ODEs (3.4.12) and (3.4.13) gives,

$$G''(\zeta)H'(\zeta) - G'(\zeta)H''(\zeta) = 0, \tag{3.4.14}$$

for solving the above system of ODEs, $H(\zeta) = a + bG(\zeta)$ is assumed, where a and b are arbitrary constants. Thus, ODEs (3.4.11)-(3.4.13) are reduced as:

$$F''(\zeta^2 - 1) + F'\left(2\zeta - \frac{1}{\zeta}\right) = e^{-2F}(1 + b^2)G'^2(1-\zeta^2), \tag{3.4.15}$$

and

$$G''(\zeta^2 - 1) + G'\left(2\zeta - \frac{1}{\zeta}\right) = 2F'G'(\zeta^2 - 1). \tag{3.4.16}$$

On solving above ODEs, following solutions are obtained

$$e^{-F(\zeta)} = c_1\sqrt{1+b^2}\operatorname{sech}^{-1}(\zeta) + c_2, \tag{3.4.17}$$

and

$$G(\zeta) = c_1 \int_{c_3}^{\zeta} \frac{e^{2u(\phi)}}{\phi\sqrt{\phi^2 - 1}} \mathrm{d}\phi, \tag{3.4.18}$$

where c_1, c_2 and c_3 are constants of integration. Hence, solution of system of (3.2.8)-(3.2.11) is given as follows:

$$u(x,t) = F(\zeta) = -\log\left(c_1\sqrt{1+b^2}\operatorname{sech}^{-1}(\zeta) + c_2\right), \tag{3.4.19}$$

$$v(x,t) = G(\zeta) = c_1 \int_{c_3}^{\zeta} \frac{e^{2u(\phi)}}{\phi\sqrt{\phi^2-1}} \mathrm{d}\phi, \tag{3.4.20}$$

$$w(x,t) = H(\zeta) = a + b\left(c_1 \int_{c_3}^{\zeta} \frac{e^{2u(\phi)}}{\phi\sqrt{\phi^2-1}} \mathrm{d}\phi\right), \tag{3.4.21}$$

and from (3.2.12) and (3.2.13), $k(x,t)$ can be evaluated which is quite difficult in this case.

Now by setting $t = \iota z$, the nonlinear system (3.2.8)-(3.2.13) reduces to Ernst equations given by system

$$u_{xx} + \frac{1}{x}u_x + u_{zz} = -e^{-2u}(v_x^2 + v_z^2 + w_x^2 + w_z^2), \tag{3.4.22}$$

$$v_{xx} + \frac{1}{x}v_x + v_{zz} = 2(u_x v_x + u_z v_z), \tag{3.4.23}$$

$$w_{xx} + \frac{1}{x}w_x + w_{zz} = 2(u_x w_x + u_z w_z), \tag{3.4.24}$$

$$v_x w_z = w_x v_z, \tag{3.4.25}$$

$$k_x = x(u_x^2 - u_z^2) + xe^{-2u}(v_x^2 - v_z^2 + w_x^2 - w_z^2), \tag{3.4.26}$$

$$k_z = 2xu_x u_z + 2xe^{-2u}(v_x v_z + w_x w_z). \tag{3.4.27}$$

For the vector field V_2, the similarity variable and similarity solutions are

$\zeta = \frac{x}{z}$ and $u(x,z) = F(\zeta)$, $v(x,z) = G(\zeta)$ and $w(x,z) = H(\zeta)$.

for these similarity variables and similarity solutions, the nonlinear system (3.4.22)-(3.4.25) reduces to the following ODEs:

$$F''(\zeta^2+1) + F'\left(2\zeta + \frac{1}{\zeta}\right) + e^{-2F}(G'^2(1+\zeta^2) + H'^2(1+\zeta^2)) = 0, \tag{3.4.28}$$

$$G''(\zeta^2+1) + G'\left(2\zeta + \frac{1}{\zeta}\right) - 2F'G'(\zeta^2+1) = 0, \tag{3.4.29}$$

$$H''(\zeta^2+1) + H'\left(2\zeta + \frac{1}{\zeta}\right) - 2F'H'(\zeta^2+1) = 0, \tag{3.4.30}$$

where $'$ is differentiation w.r.t ζ. Solution of the system of ODEs (3.4.28)-(3.4.30) gives,

$$G''(\zeta)H'(\zeta) - G'(\zeta)H''(\zeta) = 0. \tag{3.4.31}$$

For solving the above system of ODEs, $H(\zeta) = r + sG(\zeta)$ is assumed, where r and s are arbitrary constants. Thus, new ODEs become:

$$F''(\zeta^2 + 1) + F'\left(2\zeta + \frac{1}{\zeta}\right) + e^{-2F}(1 + s^2)G'^2(1 + \zeta^2) = 0, \tag{3.4.32}$$

and

$$G''(\zeta^2 + 1) + G'\left(2\zeta + \frac{1}{\zeta}\right) - 2F'G'(\zeta^2 + 1) = 0. \tag{3.4.33}$$

Solution of these ODEs yields

$$F(\zeta) = -\frac{1}{2}\log\left(-c_4^2\left(\tanh^{-1}\left(\frac{1}{\sqrt{1+\zeta^2}}\right) + c_5\right)^2 (1 + s^2)\right), \tag{3.4.34}$$

and

$$G(\zeta) = -\frac{1}{c_4\,(1 + s^2)\left(\tanh^{-1}\left(\frac{1}{\sqrt{1+\zeta^2}}\right) + c_5\right)} + c_6, \tag{3.4.35}$$

where c_4, c_5 and c_6 are constants of integration. Hence, solution of system of (3.4.22)-(3.4.27) is given as follows:

$$u(x, z) = -\frac{1}{2}\log\left(-c_4^2\left(\tanh^{-1}\left(\frac{1}{\sqrt{1+\left(\frac{x}{z}\right)^2}}\right) + c_5\right)^2 (1 + s^2)\right), \tag{3.4.36}$$

$$v(x, z) = -\frac{1}{c_4\,(1 + s^2)\left(\tanh^{-1}\left(\frac{1}{\sqrt{1+\left(\frac{x}{z}\right)^2}}\right) + c_5\right)} + c_6, \tag{3.4.37}$$

$$w(x, z) = r + s\left(-\frac{1}{c_4\,(1 + s^2)\left(\tanh^{-1}\left(\frac{1}{\sqrt{1+\left(\frac{x}{z}\right)^2}}\right) + c_5\right)} + c_6\right), \tag{3.4.38}$$

and

$$k(x, z) = c_7, \tag{3.4.39}$$

where c_4, c_5, c_6, c_7, r and s are arbitrary constants, which is general exact analytic solution of nonlinear system of Einstein-Maxwell equations. On choosing constants as $c_4 = c_5 = 1, c_6 = -1, c_7 = 0, r = 0$ and $s^2 = -2$, we have $\exp(2u(x,z)) \to 1$ and $\exp(2k(x,z)) \to 1$ as $\zeta \to \infty$ and $v(x,z) \to 0$ and $w(x,z) \to 0$ as $\zeta \to \infty$. This solution represents the external gravitational fields of rotating body. This is asymptotically flat solution.

(iii) *Subalgebra* $\mathfrak{L}_{2,3} = \langle V_3 + \alpha V_2 \rangle$

The associated Langrange's system

$\frac{dx}{\alpha x} = \frac{dt}{\alpha t} = \frac{du}{0} = \frac{dv}{1} = \frac{dw}{0}$

gives the following functional form

$\zeta = \frac{x}{t}$, $u(x,t) = F(\zeta)$, $v(x,t) = \log\left(t^{\frac{1}{\alpha}} G(\zeta)\right)$,$w(x,t) = H(\zeta)$,

From (3.2.11), as above we have $H'(\zeta) = 0$, so system (3.2.8)-(3.2.10) reduces to the following ODEs

$$(1-\zeta^2)F'' + (1-2\zeta^2)\frac{F'}{\zeta} + \exp -2F\left((1-\zeta^2)\frac{G'^2}{G^2} - \frac{1}{\alpha^2} + \frac{2\zeta G'}{\alpha G}\right) = 0, \quad (3.4.40)$$

and

$$(1-\zeta^2)G'' - (1-\zeta^2)\frac{G'^2}{G} + (1-2\zeta^2)\frac{G'}{\zeta} + \frac{G}{\alpha} - \frac{2\zeta F' G}{\alpha} - 2(1-\zeta^2)F'G' = 0, \quad (3.4.41)$$

which is quite difficult to solve.

Following a procedure similar to the above the highly nonlinear system (3.2.8)-(3.2.11) can be reduced to system of ODEs corresponding to basic vector fields (v).

(iv) *Subalgebra* $\mathfrak{L}_{2,3,4} = \langle V_4 + V_3 + \beta V_2 \rangle$

The characteristic equation associated with $V_4 + V_3 + \beta V_2$ is

$\frac{dx}{\beta x} = \frac{dt}{\beta t} = \frac{du}{0} = \frac{dv}{1} = \frac{dw}{1}$

which generates the invariants

$\zeta = \frac{x}{t}$, $u(x,t) = F(\zeta)$, $v(x,t) = \log\left(t^{\frac{1}{\beta}}G(\zeta)\right)$,$w(x,t) = \log\left(t^{\frac{1}{\beta}}H(\zeta)\right)$,

From (3.2.11), as above $H(\zeta) = l + mG(\zeta)$,

so system (3.2.8)-(3.2.10) reduces to the following system of ODEs

$$\begin{aligned}&\beta^2\left(1-\zeta^2\right)F''G^2 + \tfrac{\beta^2 G^2 F'}{\zeta}\left(1-2\zeta^2\right) - \exp^{-2F}G^2\left(1+m^2\right) + 2\beta\zeta\exp^{-2F}GG'\\&\left(1+m^2\right) - \beta^2\exp^{-2F}G'^2\left(1+m^2\right) + \exp^{-2F}\beta^2\zeta^2G'^2\left(1+m^2\right) = 0,\end{aligned} \tag{3.4.42}$$

and

$$\begin{aligned}&\beta GG''\left(-1+\zeta^2\right) + \beta G'^2\left(1-\zeta^2\right) - \tfrac{\beta GG'}{\zeta}(1-2\zeta^2) + 2\beta F'GG'\left(1-2\zeta^2\right)\\&-G^2 + 2\zeta F'G^2 = 0.\end{aligned} \tag{3.4.43}$$

Now equation (4.64) can be solved if $m = \pm\iota$ and so ODE (3.4.42) reduces to

$$F''\left(1-\zeta^2\right) + \frac{F'}{\zeta}\left(1-2\zeta^2\right), \tag{3.4.44}$$

After solving this equation we have

$$u(x,t) = F(\zeta) = c_1 + \arctan\left(\frac{1}{\sqrt{-1+\left(\frac{x}{t}\right)^2}}\right)c_2, \tag{3.4.45}$$

where c_1 and c_2 are arbitrary constants and by substitution of this value in ODE (3.4.43), $v(x,t)$ and $w(x,t)$, can be obtained which are in the form of integration.

(v) *Subalgebra* $\mathfrak{L}_{2,3,4} = \langle V_6 + \sigma V_5 + \nu V_2\rangle$

Here, reduction corresponding to this subalgebra is not obtained because of some difficulties.

3.5 Results and Discussion

Exact solutions to Einstein-Maxwell equations with the technique of the Lie groups are generated. This technique has proved a powerful one in the last decades in different equations and this new application comes to confirm this states. Infinitesimal generators of Lie groups associated with Einstein-Maxwell equations are determined.

Closed form solutions are found that might prove to be interesting physically. Some of the following important remarks are also obtained:

***Remark* 1**: By applying Lie classical approach, it become possible to find vector fields which are used to derive exact solution of nonlinear system (3.2.8) – (3.2.13). Corresponding to the vector field V_2, interesting solution representing exterior gravitations fields of rotating bodies such as galaxies and stars etc. is obtained.

***Remark* 2**: If we consider, $w = constant$ then the nonlinear system reduces to

$$u_{xx} + \frac{1}{x}u_x - u_{tt} = -e^{-2u}(v_x^2 - v_t^2), \tag{3.5.1}$$

$$v_{xx} + \frac{1}{x}v_x - v_{tt} = 2(u_x v_x - u_t v_t), \tag{3.5.2}$$

$$k_x = x(u_x^2 + u_t^2) + xe^{-2u}(v_x^2 + v_t^2), \tag{3.5.3}$$

$$k_t = 2xu_x u_t + 2xe^{-2u}(v_x v_t), \tag{3.5.4}$$

which are Einstein-Maxwell equations in purely electric field and can be solved as above by taking $w = constant$ and by choosing appropriate constants in vector field V_2 and by letting $t = \iota z$, exact solutions of Ernst equations for the above system of Einstein-Maxwell equations, we can obtain new exact analytic solutions. In the same way, when $v = constant$, two more solutions can be obtained from vector field V_2.

***Remark* 3**:The surfaces $v = constant$ and $w = constant$ are null surfaces and solution in this case represent null electromagnetic field in vacuum. The metric (3.2.1) for this solution is plane symmetric in the form of Taub [152] in (1951), represents plane electromagnetic waves.

Chapter 4

Einstein- Maxwell Equations For Magnetostatic Fields [1]

4.1 Introduction

Magnetic fields play an important role in the study of astrophysical objects such as neutron stars, white dwarfs, pulsars, black holes and galaxy formation. In fact, several observations show that there are various scenarios where the magnetic fields and general relativity can not be neglected. One of them is the presence of strong magnetic fields in active galactic nuclei [4, 97]. These nuclei are known to produce more radiation than the rest of the entire galaxy and directly affect its structure and evolution. Another scenario is the production of relativistic collimated jets in the inner regions of accretion discs, which can be explained considering magneto-centrifugal mechanisms [95, 161]. Also, magnetic fields are important in understanding the interplay between magnetic and thermal processes for strongly magnetic neutron stars [3, 56]. At least 10 percent of all neutron stars are born as magnetars, with magnetic fields above $1014G$ [62, 126]. Analytical models that describe these astrophysical objects are often associated with solutions of Einstein's equations [118, 133]. In the search for more realistic models for compact stellar systems,

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the energy-momentum tensor, the source of Einstein's equations, is modified by introducing more complex terms that take into account additional physical properties as, for example, electromagnetic fields.

These fields for deriving solutions were first considered by Bonnor [20], for obtaining solutions corresponding to radial and longitudinal fields. Misra and RadhaKrishna [107], have obtained some solutions in which metric function depend upon one independent variable. Garćia et al. [42], have presented a detailed study of the Counter-Rotating model for generic electrostatic (magnetostatic) axially symmetric thin disks without radial pressure.

In this chapter, new exact analytic solutions of Einstein-Maxwell equations for purely magnetostatic fields in general relativity are derived.

The simplest metric to describe a static axially symmetric spacetime is the Weyl's line element [171]

$$ds^2 = e^{-2\phi}[r^2 d\phi^2 + e^{2\Lambda}(dr^2 + dz^2)] - e^{2\phi} dt^2, \tag{4.1.1}$$

where ϕ and Λ are functions of r and z only. The Einstein-Maxwell field equations for the geometrized units $8\pi G = c = \mu_0 = \epsilon_0 = 0$, are given by

$$R_{ab} = T_{ab}, \tag{4.1.2}$$

$$T_{ab} = F_{ac}F_b^c - \frac{1}{4} g_{ab} F_{cd} F^{cd}, \tag{4.1.3}$$

$$F^{ab}_{;b} = 0, \tag{4.1.4}$$

$$F_{ab} = A_{a,b} - A_{b,a}, \tag{4.1.5}$$

where R_{ab} is the Ricci tensor, T_{ab} is the electromagnetic energy tensor, F_{ab} is the electromagnetic field tensor which satisfies Maxwell's equations for empty space time and $A_a = (\phi, A, 0, 0)$ is the 4−potential. A comma followed by a lower suffix denotes a partial differentiation with respect to corresponding variable.

The Einstein-Maxwell equations in presence of purely magnetostatic fields are equivalent to the system:

$$\phi_{rr} + \frac{1}{r}\phi_r + \phi_{zz} - \frac{e^{2\phi}}{2r^2}(A_r^2 + A_z^2) = 0, \tag{4.1.6}$$

$$A_{rr} - \frac{1}{r} + A_{zz} + 2(A_r\phi_r + A_z\phi_z) = 0, \tag{4.1.7}$$

$$\Lambda_r = r(\phi_r^2 - \phi_z^2) + \frac{e^{2\phi}}{2r}(A_r^2 - A_z^2), \tag{4.1.8}$$

$$\Lambda_z = 2r\phi_r\phi_z + \frac{1}{r}e^{2\phi}A_rA_z, \tag{4.1.9}$$

where A is magnetostatic potential and function of r and z only.

For solving the non linear system (4.1.6)-(4.1.9) of PDEs, first the system (4.1.6)-(4.1.7) is solved for obtaining values of $\phi(r, z)$ and $A(r, z)$. After that, by substituting these values of ϕ and A into the system (4.1.8)-(4.1.9), the values of $\Lambda(r, z)$ are obtained.

In section (4.2), the symmetries of system (4.1.6)-(4.1.7), in the generalized form are derived, which are then used to obtain associated Lie algebra of vector fields. Section (4.3), deals with the determination of the transformation group for the reduction of system of nonlinear PDEs to system of ODEs. Exact solutions are also obtained in this section. In section (4.4), discussion on derived results is given.

4.2 Symmetry Group

In order to determine the Lie group of transformations of system (4.1.6)-(4.1.7), the symmetry reduction method is exploited in a similar manner as explained in Chapter 1.

Let the system (4.1.6)-(4.1.7) be considered as a manifold $\bar{N} = (N_1, N_2)$

$$N_1(\phi, A) \equiv \phi_{rr} + \frac{1}{r}\phi_r + \phi_{zz} - \frac{e^{2\phi}}{2r^2}(A_r^2 + A_z^2) = 0, \tag{4.2.1}$$

$$N_2(\phi, A) \equiv A_{rr} - \frac{1}{r} + A_{zz} + 2(A_r\phi_r + A_z\phi_z) = 0, \tag{4.2.2}$$

in the space of variables $\bar{X} = (r, z)$,$\bar{\eta} = (\phi, A)$. The one-parameter group of local point transformations that leaves system (4.1.6)-(4.1.7) invariant corresponds to the vector fields of the form

$$V = P(\bar{X}, \bar{\eta})\frac{\partial}{\partial r} + Q(\bar{X}, \bar{\eta})\frac{\partial}{\partial z} - R(\bar{X}, \bar{\eta})\frac{\partial}{\partial \phi} - S(\bar{X}, \bar{\eta})\frac{\partial}{\partial A}. \tag{4.2.3}$$

The group infinitesimals P, Q, R and S are to be found under the following conditions

$$F_i(N_i, \bar{\eta}, \bar{S}) \mid_{\bar{N}=0} = \bar{0}, \tag{4.2.4}$$

for $i = 1, 2$.

In equation (4.2.4), $F_i(N_i, \bar{\eta}, \bar{S})$ denotes the Fréchet derivative of N_i at $\bar{\eta} = (\phi, A)$ in the direction of the quasi-linear symmetry operator $\bar{S} = (S_1, S_2)$ and is defined by

$$\bar{F}[\bar{N}, \bar{\eta}, \bar{S}] = \frac{d}{d\epsilon}\bar{F}[\bar{N}(\bar{\eta} + \epsilon\bar{S})] \mid_{\epsilon=0} . \tag{4.2.5}$$

The symmetry operator $\bar{S} = (S_1, S_2)$ has the following form:

$$S_1(\phi) \equiv P(\bar{X}, \bar{\eta})\phi_r + Q(\bar{X}, \bar{\eta})\phi_z + R(\bar{X}, \bar{\eta}), \tag{4.2.6}$$

$$S_2(A) \equiv P(\bar{X}, \bar{\eta})A_r + Q(\bar{X}, \bar{\eta})A_z + S(\bar{X}, \bar{\eta}), \tag{4.2.7}$$

Equation (4.2.5) is used to find the Fréchet derivative of each of the two nonlinear operators defined through equations (4.2.1) and (4.2.2), and the following expressions are derived:

$$F_1(\bar{N_1}, \bar{\eta}, \bar{S}) = [S_1]_{rr} + \frac{1}{r}[S_1]_r + [S_1]_{zz} - \frac{e^{2[S_1]}}{2r^2}\big(A_r[S_2]_z + A_z[S_2]_z + 2[S_1](A_r^2 + A_z^2)\big), \tag{4.2.8}$$

$$F_2(\bar{N}_2,\bar{\eta},\bar{S}) = [S_1]_{rr} - \frac{1}{r}[S_1]_r + [S_1]_{zz} + 2(A_r[S_1]_r + \phi_r[S_2]_r + A_z[S_1]_z + \phi_z[S_1]_z). \quad (4.2.9)$$

Substituting the values of S_1 and S_2 from (4.2.6) and (4.2.7) in equations (4.2.8) and (4.2.9), when expanded, results in the polynomial expressions in various partial derivatives of ϕ and A with respect to the spatial variable. The calculations to obtain these derivatives are tedious, simplified set of determining equations for the group infinitesimals P, Q, R and S is listed here. From (4.2.8), after equating the coefficients of various derivative terms to zero, the following determining equations are obtained

$$\begin{aligned} &Q_\phi = 0, Q_A = 0, Q_r = 0, Q_{zz} = 0, R_z = 0, R_\phi = 0, R_A = 0, R_r = 0, \\ &S_z = 0, S_\phi = 0, S_A = Q_z - R, S_r = 0, P = rQ_z. \end{aligned} \quad (4.2.10)$$

From (4.2.9) following set of determining equations has been obtained keeping in view the consequences on the infinitesimals effected by the sets of equations (4.2.10):

$$\begin{aligned} &P_z = -Q_r, P_\phi = 0, P_A = 0, P_r = Q_z, Q_{zz} = -Q_{rr}, R_z = -\tfrac{Q_r}{2r}, \\ &R_A = -\tfrac{S_{AA}}{2}, R_r = \tfrac{-P+rQ_z}{2r^2}. \end{aligned} \quad (4.2.11)$$

Now solving these sets of determining equations (4.2.10) and (4.2.11) for the infinitesimals P, Q, R and S, the following form of the generalized symmetries is obtained:

$$\begin{aligned} P &= ar, \\ Q &= az + b, \\ R &= a - l, \\ S &= lA + m. \end{aligned} \quad (4.2.12)$$

where a, b, l and m are arbitrary constants.

The Lie algebra associated with the system of infinitesimals consists of following four vector fields:

$$\{V_1 = \frac{\partial}{\partial z}, \quad V_2 = \frac{\partial}{\partial A}, \quad V_3 = -\frac{\partial}{\partial \phi} + A\frac{\partial}{\partial A}, \quad V_4 = r\frac{\partial}{\partial r} + z\frac{\partial}{\partial z} + \frac{\partial}{\partial \phi}.\}$$

The commutation relations of these vector fields is

$[V_1, V_4] = -[V_4, V_1] = V_1, \quad [V_2, V_3] = -[V_3, V_2] = V_2$

and $[V_i, V_j] = 0$ for all other i and j.

The Adjoint Table for the Lie algebra G, determined by V_1, V_2, V_3, V_4 is shown by Table (3.1).

Adjoint Table (3.1)

Ad	V_1	V_2	V_3	V_4
V_1	V_1	V_2	V_3	$V_4 - \epsilon V_1$
V_2	V_1	V_2	$V_3 - \epsilon V_2$	V_4
V_3	V_1	$V_2 e^{\epsilon}$	V_3	V_4
V_4	$V_1 e^{\epsilon}$	V_2	V_3	V_4

As already mentioned in chapter 2, that one may obtain the reduced system of ODEs from any linear combination of generators. Since there exist infinite possibilities for such combinations, the symmetry algebra of system (4.1.6)-(4.1.7) into conjugacy inequivalent sub algebra under the adjoint action of the symmetry group is classified. This leads to determine an optimal system of non-equivalent (non-conjugate) one dimensional sub algebras of the symmetry algebra of the system (4.1.6)-(4.1.7). To obtain the optimal system, general element $V = a_1V_1 + a_2V_2 + a_3V_3 + a_4V_4$ is considered and subject it to various adjoint transformations to simplify it as much as possible. Following basic fields which form an optimal system are deduced.

$$\{V_1, \quad V_2, \quad V_2 + V_1, \quad V_2 - V_1, \quad V_3, \quad V_3 + V_1, \quad V_3 - V_1, \quad V_4 + \alpha V_3\},$$

where α is arbitrary constant. Because of the discrete symmetry $(r, z, \phi, A) \rightarrow (-r, z, \phi, A)$, will map $V_2 - V_1$ and $V_3 - V_1$ to $V_2 + V_1$ and $V_3 + V_1$ respectively, and therefore, in the optimal system, we confine ourselves to remaining six essential vector fields of the optimal system, while neglecting the other two.

4.3 Exact Solutions of Einstein-Maxwell Equations for Magnetostatic Fields

It seems reasonable now to construct Lie ansätze and to seek exact solutions of the nonlinear system (4.1.6)-(4.1.9). With this in mind, consider its Lie symmetry generated by the basic operators in the optimal system.

Reduction (i) V_1

Solving the characteristic equation $\frac{dr}{0} = \frac{dz}{1} = \frac{d\phi}{0} = \frac{dA}{0}$, the similarity reduction is

$\zeta = r$, $\phi(r, z) = F(\zeta)$, $A(r, z) = G(\zeta)$.

the resulting system of ODEs is as follows:

$$F''(\zeta) + \frac{F'(\zeta)}{r} - \frac{e^{2F(\zeta)}G'^2(\zeta)}{2r^2} = 0, \tag{4.3.1}$$

$$G''(\zeta) - \frac{G'}{\zeta} + 2F'(\zeta)G'(\zeta) = 0. \tag{4.3.2}$$

The solution of the reduced system of ODEs is as follows:

$$F(\zeta) = \frac{1}{2}\log\zeta - \frac{1}{2}\log\left(\frac{e^{-2c_1}}{2c_2^2\cosh\left(\frac{\log\zeta - c_3}{c_2}\right)^2\zeta}\right) + c_1, \tag{4.3.3}$$

and

$$G(\zeta) = \frac{\sinh\left(\frac{\log\zeta - c_3}{2c_2}\right)}{e^{2c_1}c_2\cosh\left(\frac{\log\zeta - c_3}{2c_2}\right)} + c_4. \tag{4.3.4}$$

Hence, the solution of system (4.1.6)-(4.1.9) is given by:

$$\phi(r, z) = \frac{1}{2}\log r - \frac{1}{2}\log\left(\frac{e^{-2c_1}}{2c_2^2\cosh\left(\frac{\log r - c_3}{c_2}\right)^2 r}\right) + c_1 \tag{4.3.5}$$

$$A(r, z) = \frac{\sinh\left(\frac{\log r - c_3}{2c_2}\right)}{e^{2c_1}c_2\cosh\left(\frac{\log r - c_3}{2c_2}\right)} + c_4, \tag{4.3.6}$$

and

$$\Lambda(r, z) = \log r - c_3 + 2\log\left(\cosh\left(\frac{\log r - c_3}{2c_2}\right)\right) + \frac{\log r}{4c_2^2} - \frac{c_3}{4c_2^2}, \tag{4.3.7}$$

where c_1, c_2, c_3 and c_4 are arbitrary constants.

Reduction (ii) V_2

The similarity variables associated with this basic vector field are $r = constant$ and $z = constant$ therefore $A(r, z)$ and $\phi(r, z)$ are also constant functions.

Reduction (iii) $V_2 + V_1$

By considering the linear combination of V_2 and V_1, the vector field $V = V_1 + V_2$ becomes $V = \frac{\partial}{\partial z} + \frac{\partial}{\partial A}$, so similarity variables are $\zeta = r$, $\phi(r, z) = F(\zeta)$, $A(r, z) = z + G(\zeta)$.

The corresponding ODEs for $F(\zeta)$ and $G(\zeta)$ are given by

$$F''(\zeta) + \frac{F'(\zeta)}{\zeta} - \frac{e^{2F(\zeta)}(1 + G'(\zeta)^2)}{2\zeta^2} = 0, \tag{4.3.8}$$

$$G''(\zeta) - \frac{G'(\zeta)}{\zeta} + 2F'(\zeta)G'(\zeta) = 0. \tag{4.3.9}$$

On solving second ODE, $G'(\zeta)$ is

$$G'(\zeta) = c_1 \zeta e^{-2F(\zeta)}. \tag{4.3.10}$$

By substituting this value of $G'(\zeta)$ into ODE (4.3.8), we obtain

$$F''(\zeta) + \frac{F'(\zeta)}{\zeta} - \frac{e^{2F(\zeta)}(1 + c_1^2 \zeta^2 e^{-4F(\zeta)})}{2\zeta^2} = 0, \tag{4.3.11}$$

where c_1 is arbitrary constant. it is quite difficult to solve this ODE.

Reduction (iv) V_3

Corresponding to this vector field, only constant solution is obtained.

Reduction (v) $V_3 + V_1$

For this vector field, the forms of the similarity variable and similarity solution are as follows:

$\zeta = r$, $\phi(r, z) = -z + F(\zeta)$, $A(r, z) = e^z G(\zeta)$.

Using these substitutions, system (4.1.6)-(4.1.7) reduces to

$$2\zeta^2 F''(\zeta) + 2\zeta F'(\zeta) - e^{2F(\zeta)}(G'^2(\zeta) + G^2(\zeta)) = 0, \tag{4.3.12}$$

$$\zeta G''(\zeta) - G'(\zeta) - \zeta G(\zeta) + 2\zeta F'(\zeta)G'(\zeta) = 0. \tag{4.3.13}$$

Now, after the substitution of $G(\zeta) = e^{-F(\zeta)}H(\zeta)$ and $F'(\zeta) = M(\zeta)$, the ODEs reduced to simpler form:

$$2\zeta^2 M'(\zeta)+2\zeta M(\zeta)-M(\zeta)^2 H(\zeta)^2+2M(\zeta)H(\zeta)H'(\zeta)-H'(\zeta)^2-H(\zeta)^2 = 0, \tag{4.3.14}$$

$$\zeta M'(\zeta)H(\zeta) + \zeta M(\zeta)^2 H(\zeta) - \zeta H''(\zeta) - M(\zeta)H(\zeta) + H'(\zeta) + \zeta H(\zeta) = 0. \tag{4.3.15}$$

For the solution of above ODEs, three cases are taken depending upon $H(\zeta)$

case-(i): When $H(\zeta) = 0$

This is not physically interesting case.

case-(ii) and (iii): When $H(\zeta) = \pm\sqrt{2}\iota\zeta$, the above ODEs, reduced into single equation

$$\zeta^2 M'(\zeta) - \zeta M(\zeta) + \zeta^2 M(\zeta)^2 + 1 + \zeta^2 = 0. \tag{4.3.16}$$

Solution of above ODE is:

$$M(\zeta) = \frac{-c_1\zeta Y_1(\zeta) + J_0(\zeta) - \zeta J_1(\zeta) + c_1 Y_0(\zeta)}{\zeta(c_1 Y_0(\zeta) + J_0(\zeta))}. \tag{4.3.17}$$

Now, the solution of system (4.1.6)-(4.1.9) is:

$$\phi(r, z) = -z + \int \frac{-c_1 r Y_1(r) + J_0(r) - rJ_1(r) + c_1 Y_0(r)}{r(c_1 Y_0(r) + J_0(r))} dr, \tag{4.3.18}$$

$$A(r, z) = e^z e^{-\int \frac{-c_1 r Y_1(r) + J_0(r) - rJ_1(r) + c_1 Y_0(r)}{r(c_1 Y_0(r) + J_0(r))} dr} \pm\sqrt{2}\iota r, \tag{4.3.19}$$

and

$$\Lambda(r, z) = \log(r) + 2\log(c_1 Y_0(r) + J_0(r)) - 2z, \tag{4.3.20}$$

where c_1 is arbitrary constant. $J_v(r)$ and $Y_v(r)$ are the modified Bessel functions of the first and second kinds, respectively. They satisfy the modified Bessel equation:

$$r^2Y'' + rY' - (r^2 + v^2)Y = 0. \tag{4.3.21}$$

Reduction (vi) $V_4 + \alpha V_3$

For $\alpha = 1$, the similarity variable $\zeta = \frac{r}{z}$ and new dependent variables are $\phi(r,z) = F(\zeta)$, $A(r,z) = zG(\zeta)$.

Using of these new dependent variables into system (4.1.6)-(4.1.7), leads to the following system of reduced ODEs with independent variable ζ:

$$F''(\zeta)(1+\zeta^2)+\frac{F'(\zeta)}{\zeta}(1+2\zeta^2)-\frac{1}{2}e^{2F(\zeta)}\frac{\left(G'(\zeta)^2(1+\zeta^2)+G(\zeta)^2-2\zeta G(\zeta)G'(\zeta)\right)}{\zeta^2}=0, \tag{4.3.22}$$

$$G''(\zeta)(1+\zeta^2) - \frac{G'(\zeta)}{\zeta} + 2F'(\zeta)G'(\zeta)(1+\zeta^2) - 2\zeta F'(\zeta)G(\zeta) = 0. \tag{4.3.23}$$

Now, after the substitution of $G(\zeta) = e^{-F(\zeta)}H(\zeta)$ and $F'(\zeta) = M(\zeta)$, above ODEs become

$$\begin{aligned}&-2\zeta^2M'(\zeta) - 2\zeta M(\zeta) - 2\zeta^4M'(\zeta) - 4\zeta^3M(\zeta) + M(\zeta)^2H(\zeta)^2(1+\zeta^2) + H'(\zeta)^2\\ &(1+\zeta^2) - 2M(\zeta)H(\zeta)H'(\zeta)(1+\zeta^2) + H(\zeta)^2 + 2\zeta M(\zeta)H(\zeta)^2 - 2\zeta H(\zeta)H'(\zeta) = 0,\end{aligned} \tag{4.3.24}$$

and

$$\begin{aligned}&\zeta M'(\zeta)H(\zeta) + \zeta M(\zeta)^2H(\zeta) - \zeta H''(\zeta) + \zeta^3M'(\zeta)H(\zeta) - \zeta^3H''(\zeta) - M(\zeta)H(\zeta)\\ &+H'(\zeta) + 2\zeta^2M(\zeta)H(\zeta) - \zeta^3M(\zeta)^2H(\zeta) = 0.\end{aligned} \tag{4.3.25}$$

One more solution is also derived by using the following similarity variable and similarity solutions: $\zeta = \frac{r}{z}$, $\phi(r,z) = F(\zeta)$, $A(r,z) = G(\zeta)$.

Using the above variables in system of PDEs (4.1.6)-(4.1.7) which reduces to a system of ODEs given below:

$$\frac{F''(\zeta)}{F(\zeta)}(1+\zeta^2)-\frac{F'^2(\zeta)}{F^2(\zeta)}(1+\zeta^2)+\frac{F'(\zeta)}{\zeta F'(\zeta)}(1+2\zeta^2)-1-\frac{1}{2}F^2\zeta G'^2(\zeta)(1+\zeta^2)=0, \tag{4.3.26}$$

and

$$G''(\zeta)(1+\zeta^2) - \frac{G'(\zeta)}{\zeta} + \frac{2F'(\zeta)G'(\zeta)}{F(\zeta)}(1+\zeta^2). \tag{4.3.27}$$

Solving the above system of ODEs, value of $F(\zeta)$ and $G(\zeta)$is

$$F(\zeta) = \frac{\sqrt{(2)}c_2\zeta\left(2c_1^2 + \left(e^{\frac{\tanh^{-1}\left(\frac{1}{\sqrt{(1+\zeta^2)}}\right)}{\sqrt{2}c_2}}\right)^2\left(e^{\frac{c_3}{\sqrt{2}c_2}}\right)^2\right)}{4\left(e^{\frac{\tanh^{-1}\left(\frac{1}{\sqrt{(1+\zeta^2)}}\right)}{\sqrt{2}c_2}}\right)\left(e^{\frac{c_3}{\sqrt{2}c_2}}\right)} \tag{4.3.28}$$

$$G(\zeta) = \frac{4\sqrt{2}c_1}{c_2\left(2c_1^2 + e^{\frac{\left(\tanh^{-1}\left(\frac{1}{\sqrt{(1+\zeta^2)}}\right)+c_3\right)\sqrt{(2)}}{c_2}}\right)}. \tag{4.3.29}$$

Hence, the solution of nonlinear system (4.1.6)-(4.1.9) is given as:

$$\phi(r,z) = \log\left(z\frac{\sqrt{(2)}c_2\left(\frac{r}{z}\right)\left(2c_1^2 + \left(e^{\frac{\tanh^{-1}\left(\frac{1}{\sqrt{(1+\frac{r}{z}^2)}}\right)}{\sqrt{2}c_2}}\right)^2\left(e^{\frac{c_3}{\sqrt{2}c_2}}\right)^2\right)}{4\left(e^{\frac{\tanh^{-1}\left(\frac{1}{\sqrt{(1+\frac{r}{z}^2)}}\right)}{\sqrt{2}c_2}}\right)\left(e^{\frac{c_3}{\sqrt{2}c_2}}\right)}\right), \tag{4.3.30}$$

$$A(r,z) = \frac{4\sqrt{2}c_1}{c_2\left(2c_1^2 + e^{\frac{\left(\tanh^{-1}\left(\frac{1}{\sqrt{(1+\frac{r}{z}^2)}}\right)+c_3\right)\sqrt{(2)}}{c_2}}\right)}, \tag{4.3.31}$$

and

$$\Lambda(r,z) = -\frac{1}{2c_2^2}\left(\int p(r,z)dr - 2\int q(r,z)dz\right), \tag{4.3.32}$$

where $p(r,z) = \frac{l(r,z)}{m(r,z)}$

$$\begin{aligned} l(r,z) = &\left(-z^2 + 2c_2^2r^2 + 2\sqrt{2}z^2c_2\sqrt{\tfrac{r^2+z^2}{z^2}} + r^2 - 2c_2^2z^2\right)j(r,z) \\ &-4c_2^2c_1^2z^2 - 4c_1^2c_2^2r^2 + 2c_1^2r^2 - 4\sqrt{2}c_1^2c_2z^2\sqrt{\tfrac{r^2+z^2}{z^2}} - 2c_1^2z^2, \end{aligned} \tag{4.3.33}$$

$$m(r,z) = r(r^2+z^2)(2c_1^2 + j(r,z)) \tag{4.3.34}$$

$$q(r,z)=\frac{\left(\left(\sqrt{2}c_2\sqrt{\frac{r^2+z^2}{z^2}}-1\right)j(r,z)-2c_1^2-2\sqrt{2}c_2c_1^2\sqrt{\frac{r^2+z^2}{z^2}}\right)z}{(r^2+z^2)(2c_1^2+j(r,z))}, \tag{4.3.35}$$

where

$$j(r,z)=e^{\frac{\left(\tanh^{-1}\left(\frac{1}{\sqrt{\frac{r^2+z^2}{z^2}}}+c_3\right)\sqrt{2}\right)}{c_2}} \tag{4.3.36}$$

with c_1, c_2 and c_3 arbitrary constants.

4.4 Conclusion and Outlook

New exact solutions of Einstein-Maxwell equations in general relativity corresponding to the magnetostatic fields are derived. The symmetry method based on Fréchet derivatives is used to obtain new exact analytic solutions of nonlinear system of PDEs. The symmetries of Einstein Maxwell equation are exploited to derive some ansätz leading to the reduction of variables, where the analytic solutions are easier to obtain by considering the optimal system of conjugacy inequivalent subgroups. The solutions (4.3.5)-(4.3.7), (4.3.18)-(4.3.20) and (4.3.30)-(4.3.32) are new exact analytic solutions. The obtained solutions depend upon both of independent variables r and z, might to be interesting for further applications.

Chapter 5

The Einstein-Maxwell Equations For Non-Static Einstein And Rosen Metrics[1]

5.1 Introduction

The metric

$$ds^2 = e^{\lambda-\mu}(dt^2 - d\rho^2) - \rho^2 e^{-\mu} d\Phi^2 - e^{\mu} dz^2, \qquad (5.1.1)$$

where λ and μ are functions of ρ and t only. This metric has been investigated by Einstein and Rosen [38], Weber and Wheeler [170], Bonnor [20] and others in connection with gravitational radiance in empty space-time.

Misra and Radhakrishna [107], by considering the field equations for regions containing an electromagnetic field but no matter given as follows:

$$\begin{aligned} R^{\beta}_{\alpha} &= -8\pi E^{\beta}_{\alpha}, \\ (\sqrt{-g}F^{\mu\nu})_{,\nu} &= 0, \end{aligned} \qquad (5.1.2)$$

where

$$E^{\beta}_{\alpha} = -F_{\mu\alpha}F^{\mu\beta} + \frac{1}{4}g^{\beta}_{\alpha}F^{\sigma\epsilon}_{\sigma\epsilon}, \qquad (5.1.3)$$

[1]The contents of this chapter are accepted in *Proceedings of International Conference on Mathematics and Statistics* .

and

$$F_{\mu\nu} = A_{\mu,\nu} - A_{\nu,\mu}. \tag{5.1.4}$$

Here R_α^β is the contracted curvature tensor, E_α^β is the electromagnetic energy tensor, $F_{\alpha\beta}$ is the electromagnetic field tensor and A_α the 4-potential. A comma followed by the lower suffix denotes a partial differentiation with respect to the corresponding variable. The charge-current density has been put equal to zero.

The non-zero components of the Ricci tensor for the metric (5.1.1) are

$$\begin{aligned} R_1^1 &= \tfrac{1}{2}e^{\lambda-\mu}\left(-\lambda_{11} + \lambda_{44} + \mu_{11} - \mu_{44} - \mu_1^2 + \tfrac{\lambda_1+\mu_1}{\rho}\right), \\ R_2^2 &= -R_3^3 = \tfrac{e^{\lambda-\mu}}{2}\left(\mu_{11} - \mu 44 + \tfrac{\mu_1}{\rho}\right), \\ R_4^4 &= \tfrac{e^{\lambda-\mu}}{2}\left(-\lambda_{11} + \lambda_{44} + \mu_{11} - \mu_{44} + \mu_4^2 + \tfrac{\mu_1-\lambda_1}{\rho}\right), \\ R_1^4 &= -R_4^1 = \tfrac{e^{\lambda-\mu}}{2}\left(\mu_1\mu_4 - \tfrac{\lambda_4}{\rho}\right). \end{aligned} \tag{5.1.5}$$

The lower suffixes 1 and 4 after the unknown functions denote a partial differentiation with respect to ρ and t respectively. Also ρ, Φ, z and t correspond respectively to x^1, x^2, x^3 and x^4.

From (5.1.5), $R_2^2 + R_3^3 = 0$,

and consequently from (5.1.2) and (5.1.3), it follows that

$E_2^2 + E_3^3 = g^{11}g^{44}(F_{14})^2 - g^{22}g^{33}(F_{23})^2 = 0.$

Since g^{11}, g^{22} and g^{33} are all negative and g^{44} are positive, it is clear that the above equation will hold only if

$$F_{14} = F_{23} = 0. \tag{5.1.6}$$

The remaining components of the $F_{\alpha\beta}$ can, by virtue of (5.1.4), be derived from any two components of the 4-potential, namely A_2 and A_3, where

$$A_2 = \frac{\phi}{\sqrt{8\pi}}, A_3 = \frac{\psi}{\sqrt{8\pi}}, \tag{5.1.7}$$

so that

$$F_{12} = \frac{-\phi_1}{\sqrt{8\pi}}, F_{13} = \frac{\psi_1}{\sqrt{8\pi}}, F_{24} = \frac{\phi_4}{\sqrt{8\pi}}, F_{34} = \frac{\psi_4}{\sqrt{8\pi}}. \tag{5.1.8}$$

From (5.1.3), (5.1.6) and (5.1.8), the remaining independent components of the electromagnetic energy tensor are found to be

$$\begin{aligned} E_4^4 &= -E_1^1 = \tfrac{1}{16\pi} e^{-\lambda} \big[\tfrac{e^{2\mu}}{\rho^2} (\phi_1^2 + \phi_4^2) + \psi_1^2 + \psi_4^2 \big], \\ E_2^2 &= -E_3^3 = \tfrac{1}{16\pi} e^{-\lambda} \big[- \tfrac{e^{2\mu}}{\rho^2} (\phi_1^2 - \phi_4^2) + \psi_1^2 - \psi_4^2 \big], \\ E_1^4 &= \tfrac{1}{8\pi} e^{-\lambda} \big[\tfrac{e^{2\mu}}{\rho^2} \phi_1 \phi_4 + \psi_1 \psi_4 \big], \\ E_2^3 &= \tfrac{1}{8\pi} e^{-\lambda} \big[- \phi_1 \psi_1 + \phi_4 \psi_4 \big], \end{aligned} \tag{5.1.9}$$

The field equations (5.1.2) with the help of (5.1.5) and (5.1.9) can be written in the form:

$$\lambda_{11} - \lambda_{44} + \frac{1}{2}(\mu_1^2 - \mu_4^2) - \frac{e^{\mu}}{\rho^2}(\phi_1^2 - \phi_4^2) - e^{-\mu}(\psi_1^2 - \psi_4^2) = 0, \tag{5.1.10}$$

$$\mu_{11} + \frac{\mu}{\rho}\mu_1 - \mu_{44} - \frac{e^{\mu}}{\rho^2}(\phi_1^2 - \phi_4^2) - e^{-\mu}(\psi_1^2 - \psi_4^2) = 0, \tag{5.1.11}$$

$$\lambda_1 = \frac{\rho}{2}(\mu_1^2 + \mu_4^2) + \frac{e^{\mu}}{\rho^2}(\phi_1^2 + \phi_4^2) + e^{-\mu}(\psi_1^2 + \psi_4^2), \tag{5.1.12}$$

$$\lambda_4 = \rho\mu_1\mu_4 + 2\frac{e^{\mu}}{\rho}\phi_1\phi_+ 2\rho e^{-\mu}\psi_1\psi_4, \tag{5.1.13}$$

$$\phi_1\psi_1 = \phi_4\psi_4, \tag{5.1.14}$$

and

$$\phi_{11} - \phi_{44} - \frac{\phi_1}{\rho} + \mu_1\phi_1 - \mu_4\phi_4, \tag{5.1.15}$$

$$\psi_{11} - \psi_{44} + \frac{\psi_1}{\rho} - \mu_1\psi_1 + \mu_4\psi_4 \tag{5.1.16}$$

The field equations which determine λ, μ, ϕ and ψ are (5.1.10)-(5.1.16).

It can be easily seen that condition of integrability of (5.1.12) and (5.1.13) is satisfied, i.e.,

$$\lambda_{14} = \lambda_{41}, \tag{5.1.17}$$

Also (5.1.10) is identically satisfied , if (5.1.11), (5.1.12), (5.1.13) and (5.1.14) hold. Hence the problem of solving equations (5.1.10)-(5.1.16) reduces to that of determining μ, ϕ and ψ from (5.1.11), (5.1.14), (5.1.15) and (5.1.16), and then to determine λ from (5.1.12) and (5.1.13).

In view of the nonlinear character of the expression involved, it is extremely difficult to find exact solutions of the field equations. In this chapter, exact solutions of the system (5.1.10)-(5.1.16) are obtained in the following cases:

(i) Pure Magnetic Fields;

(ii) Pure Electric Fields.

To find these exact solutions, Lie classical method is used. In section (5.2), pure magnetic fields are studied. In Section (5.3), Lie Classical Method is utilized to obtain symmetries, which are further used to obtain associated Lie algebra of vector fields. Section (5.4), deals with the reduction of system of nonlinear PDEs to system of ODEs. Exact solutions of pure magnetic fields have also been obtained in this section. Section (5.5) is devoted to pure electric fields. Reductions and new exact solutions of pure electric fields are also derived in this section. In section (5.7), some concluding remarks are noted.

5.2 Pure Magnetic Fields

In this section for pure magnetic fields, i.e., when $\psi = 0$, the above system reduces to the following system of nonlinear PDEs become:

$$\mu_{\rho\rho} - \mu_{tt} + \frac{\mu_\rho}{\rho} - \frac{e^\mu}{\rho^2}(\phi_\rho^2 - \phi_t^2) = 0, \tag{5.2.1}$$

$$\phi_{\rho\rho} - \phi_{tt} - \frac{\phi_\rho}{\rho} + \mu_\rho\phi_\rho - \mu_t\phi_t = 0, \tag{5.2.2}$$

$$\lambda_\rho = \frac{\rho}{2}(\mu_\rho^2 + \mu_t^2) + \frac{e^\mu}{\rho^2}(\phi_\rho^2 + \phi_t^2), \tag{5.2.3}$$

$$\lambda_t = \rho\mu_\rho\mu_t + 2\frac{e^\mu}{\rho}e^\mu\phi_\rho\phi_t, \tag{5.2.4}$$

exact solutions of above system are obtained. Lie classical method is applied to the system (5.2.1)-(5.2.2) and found values of μ and ϕ, and the by substituting these values in (5.2.3) and (5.3.4), value of λ is obtained.

5.3 Similarity Variables and Similarity Solutions

In this section Lie classical method [82] to the system (5.2.1)-(5.2.2) is applied by considering the one-parameter Lie group of infinitesimal transformations in $\rho, t, \mu, \phi, \eta^1$ and η^2 . This transformation leaves invariant the following set

$$S_\Delta \equiv \{\mu,(\rho,t),\phi(\rho,t) : \Delta_1(\mu,\phi) = 0, \Delta_2(\mu,\phi) = 0\} \tag{5.3.1}$$

of solutions of the system (5.2.1)-(5.2.2), where $\Delta_1 = \mu_{\rho\rho} - \mu_{tt} + \frac{\mu_\rho}{\rho} - \frac{e^\mu}{\rho^2}(\phi_\rho^2 - \phi_t^2)$ and $\Delta_2 = \phi_{\rho\rho} - \phi_{tt} - \frac{\phi_\rho}{\rho} + \mu_\rho\phi_\rho - \mu_t\phi_t$. This yields an over determined, linear system of equations for the infinitesimals $\xi_1(\rho,t), \tau_1(\rho,t), \eta^1(\rho,t)$ and $\eta^2(\rho,t)$. The associated Lie algebra of infinitesimal symmetries is the set of vector fields of the form

$$\Gamma = \xi_1\partial_\rho + \tau_1\partial_t + \eta^1\partial_\mu + \eta^2\partial_\phi, \tag{5.3.2}$$

where $\partial_\rho = \frac{\partial}{\partial_\rho}$ etc. The set S_Δ is invariant under the one-parameter transformations provided that $Pr^{(2)}(\Delta) \mid_{\Delta=0}= 0$, where $Pr^{(2)}\Gamma$ is the second prolongation of the vector field Γ, which is explicitly given in terms of $\xi_1, \tau_1, \eta^1, \eta^2$. This procedure yields an over determined system of linear PDEs. Solving this system of PDEs, the

Lie symmetries of the system are

$$\begin{aligned} \xi_1 &= a\rho, \\ \tau_1 &= at + b, \\ \eta^1 &= 2a - 2c, \\ \eta^2 &= c\phi + d, \end{aligned} \tag{5.3.3}$$

where a, b, c and d are arbitrary constants.

Having determined the infinitesimals, the symmetry variables are found by solving the invariant surface conditions

$$\begin{aligned} \Psi_1 &\equiv \xi_1\mu_\rho + \tau_1\mu_t - \eta^1 = 0, \\ \Psi_2 &\equiv \xi_1\phi_\rho + \tau_1\mu_t - \eta^2 = 0. \end{aligned} \tag{5.3.4}$$

The associated vector fields are

$$\Gamma_1 = \frac{\partial}{\partial t}, \quad \Gamma_2 = \frac{\partial}{\partial \phi}, \quad \Gamma_3 = -2\frac{\partial}{\partial \mu} + \phi\frac{\partial}{\partial \phi}, \quad \Gamma_4 = \rho\frac{\partial}{\partial \rho} + t\frac{\partial}{\partial t} + 2\frac{\partial}{\partial \mu}$$

The commutator of these four vector fields is another first order operator given by the following relations

$$[\Gamma_1, \Gamma_4] = -[\Gamma_4, \Gamma_1] = V_1, [\Gamma_2, \Gamma_3] = -[\Gamma_3, \Gamma_2] = \Gamma_2$$

and $[\Gamma_i, \Gamma_j] = 0 \quad \forall i, j$

The adjoint action is given by the Lie series

$Ad(exp(\epsilon\Gamma_1)\Gamma_4) = \Gamma_4 - \epsilon\Gamma_1, \quad Ad(exp(\epsilon\Gamma_2)\Gamma_3) = \Gamma_3 - \epsilon\Gamma_2,$

$Ad(exp(\epsilon\Gamma_3)\Gamma_2) = \Gamma_2 e^\epsilon, \quad Ad(exp(\epsilon\Gamma_4)\Gamma_1) = \Gamma_1 e^\epsilon$

and $Ad(exp(\epsilon\Gamma_i)\Gamma_j) = \Gamma_j \quad \forall i, j$

Firstly, an optimal system is constructed and then embarked upon the various reductions associated with generators in the optimal system. A general element $\Gamma = a_1\Gamma_1 + a + 2\Gamma_2 + a_3\Gamma_3 + a_4\Gamma_4$ of symmetry algebra is considered and then subject it to various adjoint transformations to simplify it as much as possible (refer to [112]). The optimal system described by Ovsiannikov [112] consists of the following four basic vector fields:

$$\{(i)\ \Gamma_1, \quad (ii)\ \Gamma_2 + \alpha\Gamma_1, \quad (iii)\ \Gamma_3 + \beta\Gamma_1, \quad (iv)\ \Gamma_4 + \gamma\Gamma_3\}$$

The similarity variables and forms can be obtained by solving characteristic equations:

$$\frac{d\rho}{\xi} = \frac{dt}{\tau} = \frac{d\mu}{\eta^1} = \frac{d\phi}{\eta^2}. \tag{5.3.5}$$

The general solution of these equations involves three variables; one becomes the new independent variable ζ and the other two, say F and G, play the role of new dependent variable.

5.4 Exact Solutions of Einstein-Maxwell Equations with Pure Magnetic Fields

Generator (i)

Using the generator Γ_1 in the optimal system, the similarity variable and similarity solution are:

$\zeta = \rho$, $\mu(\rho, t) = F(\zeta)$, $\phi(\rho, t) = G(\zeta)$.

Using these substitutions in system (5.2.1) – (5.2.2), the reduced system of ODEs is:

$$F''(\zeta) + \frac{F'(\zeta)}{\zeta} - \frac{e^{F(\zeta)}G'(\zeta)^2}{\zeta^2} = 0, \tag{5.4.1}$$

and

$$G''(\zeta) - \frac{G'(\zeta)}{\zeta} + F'(\zeta)G'(\zeta) = 0. \tag{5.4.2}$$

On solving this above system of ODEs, the solution is given follows:

$$F(\zeta) = \log(\zeta) + \log(2) - \log\left(\frac{e^{c_1}}{c_2^2 \cosh\left(\frac{\log(\zeta)-c_3}{c_2}\right)^2 \zeta}\right) + c_1, \tag{5.4.3}$$

and

$$G(\zeta) = \frac{\sinh\left(\frac{\log(\zeta)-c_3}{2c_2}\right)}{e^{c_1} c_2 \cosh\left(\frac{\log(\zeta)-c_3}{2c_2}\right)} + c_4. \tag{5.4.4}$$

Now, solution of the system (5.2.1)-(5.2.2) is:

$$\mu(\rho,t)=\log(\rho)+\log(2)-\log\left(\frac{\exp c_1}{c_2^2\cosh\left(\frac{\log(\rho)-c_3}{c_2}\right)^2\rho}\right)+c_1, \tag{5.4.5}$$

and

$$\phi(\rho,t)=\frac{\sinh\left(\frac{\log(\rho)-c_3}{2c_2}\right)}{e^{c_1}c_2\cosh\left(\frac{\log(\rho)-c_3}{2c_2}\right)}+c_4, \tag{5.4.6}$$

hence, after substituting the values of $\mu(\rho,t)$ and $\phi(\rho,t)$ in system (5.2.3)-(5.2.4), the value of metric function $\lambda(\rho,t)$ is

$$\lambda(\rho,t)=2\log(\rho)-2c_3+4\log\left(\cosh\left(\frac{\log(\rho)-c_3}{2c_2}\right)\right)+\frac{1}{2}\frac{\log(\rho)}{c_2^2}-\frac{1}{2}\frac{c_3}{c_2^2}. \tag{5.4.7}$$

Generator (ii)

case-(i): When $\alpha=1$

Similarity reductions corresponding to the symmetry generator $\Gamma_2+\alpha\Gamma_1$ is obtained by solving the characteristic equation are $\mu(\rho,t)=F(\zeta)$ and $\phi(\rho,t)=G(\zeta)$, $\zeta=\rho$, where $F(\zeta)$ and $G(\zeta)$ satisfies the system of ODEs:

$$F''(\zeta)+\frac{F'(\zeta)}{\zeta}-\frac{e^{F(\zeta)}(G'(\zeta)^2-1)}{\zeta^2}=0, \tag{5.4.8}$$

and

$$G''(\zeta)-\frac{G'(\zeta)}{\zeta}+F'(\zeta)G'(\zeta)=0. \tag{5.4.9}$$

From above ODE, $G'(\zeta)$ is

$$G'(\zeta)=c_1e^{-F(\zeta)}\zeta. \tag{5.4.10}$$

Here, it is difficult to solve further the system of ODEs.

case-(ii): When $\alpha=-1$

As above, system of ODEs can be found, when $\alpha=-1$.

case-(iii): When $\alpha=0$

The similarity variables, ρ and t are constants. Thus, $\mu(\rho, t)$ and $\phi(\rho, t)$ are also constants functions.

Generator (iii)

case-(i): When $\beta = 1$

For the generator $\Gamma_3 + \beta\Gamma_1$ under consideration, the associated similarity variable and similarity solutions are obtained as follows:

$\zeta = \rho, \quad \mu(\rho, t) = -2t + F(\zeta), \quad \phi(\rho, t) = e^t G(\zeta).$

In terms of these invariants the reduced system of ODEs is

$$F''(\zeta) + \frac{F'(\zeta)}{\zeta} - \frac{e^{F(\zeta)}(G'(\zeta)^2 - G(\zeta)^2)}{\zeta^2} = 0, \tag{5.4.11}$$

and

$$G''(\zeta) + G(\zeta) - \frac{G'(\zeta)}{\zeta} + F'(\zeta)G'(\zeta) = 0. \tag{5.4.12}$$

In this case also, it is difficult to solve further the system of ODEs.

case-(ii): When $\beta = -1$

As above, system of ODEs can be found, when $\beta = -1$.

case-(iii): When $\beta = 0$,

the invariants ρ and t are constants. Thus $\mu(\rho, t)$ and $\phi(\rho, t)$ are also constants functions.

Generator (iv)

case-(i): When $\gamma = 1$

The associated similarity variable and similarity solutions for the generator $\Gamma_4 + \gamma\Gamma_3$ are expressed as

$\zeta = \frac{\rho}{t}, \quad \mu(\rho, t) = F(\zeta), \quad \phi(\rho, t) = tG(\zeta).$

Substituting $\mu(\rho, t) = F(\zeta)$ and $\phi(\rho, t) = tG(\zeta)$ into system (5.2.1)-(5.2.2), the corresponding system of ODEs is

$$F''(\zeta)(1-\zeta^2)+\frac{F'(\zeta)}{\zeta}(1-2\zeta^2)-\frac{e^{F(\zeta)}(G'(\zeta)^2(1-\zeta^2)-G(\zeta)^2+2\zeta G(\zeta)G'(\zeta))}{\zeta^2}=0, \tag{5.4.13}$$

and

$$G''(\zeta)(1-\zeta^2)-\frac{G'(\zeta)}{\zeta}+F'(\zeta)G'(\zeta)(1-\zeta^2)+\zeta F'(\zeta)G(\zeta)=0. \tag{5.4.14}$$

Here, it is difficult to solve further the system of ODEs.

case-(ii): When $\gamma=-1$

As above, system of ODEs can be found, when $\gamma=-1$.

case-(iii): When $\gamma=0$

Similarly, from the equation (5.3.5), the invariants are

$\zeta=\frac{\rho}{t},\quad \mu(\rho,t)=\log(t^2F(\zeta)),\quad \phi(\rho,t)=G(\zeta).$

Then the corresponding reductions are

$$F''(\zeta)(1-\zeta^2)+\frac{F'(\zeta)}{\zeta}(1-2\zeta^2)-\frac{F'(\zeta)^2}{F(\zeta)}(1-\zeta^2)-\frac{F(\zeta)^2G'(\zeta)^2}{\zeta^2}(1-\zeta^2)+2F(\zeta)=0, \tag{5.4.15}$$

and

$$G''(\zeta)(1-\zeta^2)-\frac{G'(\zeta)}{\zeta}+\frac{F'(\zeta)G'(\zeta)}{F(\zeta)}(1-\zeta^2)=0. \tag{5.4.16}$$

Solution of this system of ODEs gives,

$$F(\zeta)=\frac{1}{2}c_1^2, \tag{5.4.17}$$

and

$$G(\zeta)=\pm\frac{2(\zeta^2-1)}{\sqrt{1-\zeta^2c_1}}+c_2. \tag{5.4.18}$$

Now, the solution of the system (5.2.1)-(5.2.2) is given as below:

$$\mu(\rho,t)=\log\left(\frac{t^2c_1^2}{2}\right), \tag{5.4.19}$$

and

$$\phi(\rho,t) = \pm\frac{2\left(\left(\frac{x}{t}\right)^2 - 1\right)}{\sqrt{1-\left(\frac{x}{t}\right)^2 c_1}} + c_2. \tag{5.4.20}$$

Substituting values of $\mu(\rho,t)$ and $\phi(\rho,t)$ in system (5.2.3)-(5.2.4), value of $\lambda(\rho,t)$ is

$$\lambda(\rho,t) = -2\log(\rho^2 - t^2) + 4\log(t) - 2\log(-\rho+t) - 2\log(\rho+t). \tag{5.4.21}$$

5.5 Pure Electric Fields

In this section for pure magnetic fields, i.e., when $\phi = 0$, the above system reduces to the following system of nonlinear PDEs:

$$\mu_{\rho\rho} - \mu_{tt} + \frac{\mu_\rho}{\rho} + e^{-\mu}(\psi_\rho^2 - \psi_t^2) = 0, \tag{5.5.1}$$

$$\psi_{\rho\rho} - \psi_{tt} + \frac{\psi_\rho}{\rho} - \mu_\rho\psi_\rho + \mu_t\phi_t = 0, \tag{5.5.2}$$

$$\lambda_\rho = \frac{\rho}{2}(\mu_\rho^2 + \mu_t^2) + \rho e^{-\mu}(\psi_\rho^2 + \psi_t^2), \tag{5.5.3}$$

$$\lambda_t = \rho\mu_\rho\mu_t + 2\rho e^{-\mu}\psi_\rho\psi_t. \tag{5.5.4}$$

Proceeding in the same way as mentioned earlier, in this section, Lie symmetries are found given as

$$\begin{aligned} \xi_2 &= a\rho, \\ \tau_2 &= at + b, \\ \eta^3 &= 2c, \\ \eta^4 &= c\psi + d. \end{aligned} \tag{5.5.5}$$

where ξ_2, τ_2, η^3 and η^4 are infinitesimals corresponding to x, t, μ and ψ respectively. The Lie algebra associated with the system of (5.5.1)-(5.5.2) consists of following four vector fields:

$$G_1 = \frac{\partial}{\partial t}, \quad G_2 = \frac{\partial}{\partial \psi}, \quad G_3 = 2\frac{\partial}{\partial \mu} + \phi\frac{\partial}{\partial \psi}, \quad G_4 = \rho\frac{\partial}{\partial \rho} + t\frac{\partial}{\partial t}.$$

The optimal system described by Olver [111] consists of the following four basic vector fields:

$$\langle G_1, \quad G_2+\alpha_2 G_1, \quad G_3+\beta_2 G_2, \quad V_4+\gamma_2 G_3,\rangle$$

where α_2, β_2 and γ_2 are arbitrary constants.

The similarity variables and forms can be obtained by solving characteristic equations (5.3.5). The general solution of these equations involves three variables; one becomes the new independent variable ζ and the other two, say F and G, play the role of new dependent variable. On substituting these solutions of (5.3.5) in system (5.5.1)-(5.5.2), the reduced ODEs are obtained.

Generator (i)

For the generator G_1, the associated equations are $\frac{d\rho}{0}=\frac{dt}{1}=\frac{d\psi}{0}$, which generates the invariants $\zeta=\rho, \quad \mu(\rho,t)=F(\zeta), \quad \psi(\rho,t)=G(\zeta)$.

The reduction correspond to case above to ODEs listed as follows:

$$F''(\zeta)+\frac{F'(\zeta)}{\zeta}+e^{-F(\zeta)}G'(\zeta)^2=0, \tag{5.5.6}$$

and

$$G''(\zeta)+\frac{G'(\zeta)}{\zeta}-F'(\zeta)G'(\zeta)=0. \tag{5.5.7}$$

On solving this above system of ODEs, the solution is given follows:

$$F(\zeta)=\log(\zeta)-\log(2)+\log\left(\frac{c_2}{\cosh\left(\sqrt{e^{c_1}c_2}(\log(\zeta)-c_3)e^{-c_1}\right)^2\zeta}\right)+c_1, \tag{5.5.8}$$

and

$$G(\zeta)=\frac{c_2 e^{c_1}\sinh\left(\frac{\sqrt{e^{c_1}c_2}(\log(\zeta)-c_3)}{2e^{c_1}}\right)}{\sqrt{e^{c_1}c_2}\cosh\left(\frac{\sqrt{e^{c_1}c_2}(\log(\zeta)-c_3)}{2e^{c_1}}\right)}+c_4. \tag{5.5.9}$$

Now, solution of the system (5.5.1)-(5.5.2) is:

$$\mu(\rho,t)=\log(\rho)-\log(2)+\log\left(\frac{c_2}{\cosh\left(\sqrt{e^{c_1}c_2}(\log(\rho)-c_3)e^{-c_1}\right)^2\rho}\right)+c_1, \tag{5.5.10}$$

and

$$\psi(\rho,t)=\frac{c_2e^{c_1}\sinh\left(\frac{\sqrt{e^{c_1}c_2}(\log(\rho)-c_3)}{2e^{c_1}}\right)}{\sqrt{e^{c_1}c_2}\cosh\left(\frac{\sqrt{e^{c_1}c_2}(\log(\rho)-c_3)}{2e^{c_1}}\right)}+c_4, \tag{5.5.11}$$

now, after substituting the values of $\mu(\rho,t)$ and $\psi(\rho,t)$ in system (5.5.3)-(5.5.4), the value of metric function $\lambda(\rho,t)$ is

$$\lambda(\rho,t)=\frac{1}{2}c_1e^{-c_1}\log(\rho). \tag{5.5.12}$$

Generator (ii)

case-(i): When $\alpha_2=1$

Also the characteristic equation corresponding to the generator $G_2+\alpha_2G_1$ is $\frac{d\rho}{0}=\frac{dt}{1}=\frac{d\psi}{\alpha_2}$, which give rise to the following functional forms

$\zeta=\rho,\quad \mu(\rho,t)=F(\zeta),\quad \psi(\rho,t)=t+G(\zeta).$

Substituting these similarity solutions into system (5.5.1)-(5.5.2), the resulting system of ODEs to determine $F(\zeta)$ and $G(\zeta)$ is as follows:

$$F''(\zeta)+\frac{F'(\zeta)}{\zeta}+e^{-F(\zeta)}(G'(\zeta)^2-1)=0, \tag{5.5.13}$$

and

$$G''(\zeta)+\frac{G'(\zeta)}{\zeta}-F'(\zeta)G'(\zeta)=0. \tag{5.5.14}$$

Solution of above ODE yields,

$$G'(\zeta)=c_1\frac{e^{F(\zeta)}}{\zeta}. \tag{5.5.15}$$

Here, it is difficult to solve further the system of ODEs.

case-(ii): When $\alpha_2=-1$

As above, system of ODEs can be found, when $\alpha_2=-1$.

case-(iii): When $\alpha_2=0$

We have ρ and t are constants. Thus $\mu(\rho,t)$ and $\psi(\rho,t)$ are also constants functions.

Generator (iii)

case-(i): When $\beta_2 = 1$

The generator $G_3 + \beta_2 G_2$ yields the invariants

$\zeta = \frac{\rho}{t}$ and $\mu(\rho, t) = F(\zeta), \quad \psi(\rho, t) = \log tG(\zeta)$.

Substituting these variables into (5.5.1)-(5.5.2) to determine system of ODEs:

$$F''(\zeta)(1-\zeta^2) + \frac{F'(\zeta)}{\zeta}(1-2\zeta^2) + e^{-F(\zeta)}\left(\frac{G'(\zeta)^2}{G(\zeta)^2}(1-\zeta^2) - 1 + \frac{2\zeta G'(\zeta)}{G(\zeta)}\right) = 0, \tag{5.5.16}$$

and

$$\begin{aligned} &G''(\zeta)(1-\zeta^2) + G(\zeta) + \tfrac{G'(\zeta)}{\zeta}(1-2\zeta^2) - F'(\zeta)G'(\zeta)(1-\zeta^2) - \tfrac{G'(\zeta)^2}{G(\zeta)}(1-\zeta^2) \\ &-\zeta G(\zeta)F'(\zeta) = 0. \end{aligned} \tag{5.5.17}$$

In this case also, it is difficult to solve further the system of ODEs.

case-(ii): When $\beta_2 = -1$

As above, system of ODEs can be derived, when $\beta_2 = -1$.

case-(iii): When $\beta_2 = 0$

The invariants associated with the symmetry operator Γ_3 are obtained as follows:

$\zeta = \frac{\rho}{t}$,$\mu(\rho, t) = F(\zeta)$,$\phi(\rho, t) = G(\zeta)$.

By using these similarity variable and similarity solution in system $(6.53) - (6.54)$, the reduced system of ODEs is as follows:

$$F''(\zeta)(1-\zeta^2) + \frac{F'(\zeta)}{\zeta}(1-2\zeta^2) + e^{-F(\zeta)}G'(\zeta)^2(1-\zeta^2) = 0, \tag{5.5.18}$$

and

$$G''(\zeta)(1-\zeta^2) + \frac{G'(\zeta)}{\zeta} - \frac{F'(\zeta)G'(\zeta)}{F(\zeta)}(1-\zeta^2) = 0. \tag{5.5.19}$$

Solution of system of these ODEs is,

$$\begin{aligned} F(\zeta) &= \tfrac{1}{2}\log(\zeta-1)+\log(\zeta)+\tfrac{1}{2}\log(\zeta+1) \\ &+\log\left(\frac{c_2}{\zeta\sqrt{\zeta-1}\sqrt{\zeta+1}\left(\cosh\left(\frac{\sqrt{e^{c_1}c_2}\arctan\left(\frac{1}{\sqrt{(\zeta^2-1)}}\right)}{e^{c_1}}+\frac{\sqrt{e^{c_1}c_2}c_3}{e^{c_1}}\right)+1\right)}\right)+c_1, \end{aligned} \tag{5.5.20}$$

and

$$G'(\zeta)=\int\frac{c_2}{\zeta\sqrt{\zeta-1}\sqrt{\zeta+1}\left(\cosh\left(\frac{\sqrt{e^{c_1}c_2}\arctan\left(\frac{1}{\sqrt{(\zeta^2-1)}}\right)}{e^{c_1}}+\frac{\sqrt{e^{c_1}c_2}c_3}{e^{c_1}}\right)+1\right)}d\zeta+c_1, \tag{5.5.21}$$

Generator *(iv)*

case-(i): When $\gamma_2 = 1$

By doing same to generator $G_4+\gamma_2 G_3$, the invariant is $\zeta=\frac{\rho}{t}$ then the similarity solutions are

$\mu(\rho,t)=\log(t^2F(\zeta)), \quad \psi(\rho,t)=tG(\zeta),$

and these similarity solutions satisfies the ODEs:

$$\begin{aligned} &F''(\zeta)(1-\zeta^2)+\tfrac{F'(\zeta)}{\zeta}(1-2\zeta^2)+\tfrac{F'(\zeta)^2}{F(\zeta)}(1-\zeta^2)+G'(\zeta)^2(1-\zeta^2) \\ &+2F(\zeta)-G(\zeta)^2+2\zeta G(\zeta)G'(\zeta)=0, \end{aligned} \tag{5.5.22}$$

and

$$G''(\zeta)(1-\zeta^2)+\frac{G'(\zeta)}{\zeta}-\frac{F'(\zeta)G'(\zeta)}{F(\zeta)}(1-\zeta^2)+2G(\zeta)-2\zeta G'(\zeta)-\zeta F'(\zeta)G(\zeta)=0. \tag{5.5.23}$$

Here, it is difficult to solve further the system of ODEs.

case-(ii): When $\gamma_2 = -1$

As above, system of ODEs can be obtained, when $\gamma = -1$.

case-(iii): When $\gamma_2 = 0$

The similarity variables ρ and t are constants. Thus $\mu(\rho,t)$ and $\psi(\rho,t)$ are also constants functions.

5.6 Discussion and Concluding Remarks

In order to completely describe the pure magnetic fields and pure electric fields in general relativity, we need to find corresponding exact solutions. A system of non-linear PDEs derived from non static Einstein and Rosen metric has been studied for deriving the exact solutions. The Lie classical method, is used to obtain symmetries of Einstein field equations and utilized these symmetries for obtaining basic vector fields which are helpful in the reduction of system of PDEs to system of ODEs. After that by solving the reduced ODEs exact analytic solutions are obtained.

Chapter 6

Einstein Vacuum Equations For Axially Symmetric Gravitational Fields [1]

6.1 Introduction

Einstein field equations are at the core of General Relativity and hence their exact solutions play very important role in the discussion of this theory. In literature, exact solutions of Einstein field equations have been found by many researchers by using group-theoretic techniques, some important contributions are Ali [6], Asghar et al. [7], Attalaha et al. [8], Bhutani and Singh [10], Bhutani et al. [11] Stephani et al. [150] and Wilshire [173]. In this chapter, the main focus is on the exact solutions of system of nonlinear PDEs derived from Einstein vacuum field equations in general relativity.

In chapter 6, system of Einstein vacuum field equations has been analyzed using the Lie Classical Method. In Section 2, a system of PDEs from Weyl metric by using relation between Einstein tensor and metric tensor is derived. In section 3, the symmetries in the generalized form are derived, which are then used to obtain associated Lie algebra of vector fields. Exact solutions of system of PDEs has also

[1] The contents of this chapter are published in the *International Journal of Mathematical and Computational Sciences* 6 (2012) 104-107.

been obtained in this section. In section 4, conclusions are drawn.

6.2 Basic Field Equations

For axially symmetric vacuum static gravitational fields, the line element reduces to the Weyl [171] metric :

$$ds^2 = \frac{1}{f}[e^{2\gamma}(d\rho^2 + dz^2) + \rho^2 d\phi^2] - f dt^2. \tag{6.2.1}$$

Here ρ, ϕ, z and t are the canonical Weyl coordinates and time respectively; $f(\rho, z)$ and $\gamma(\rho, z)$ are two unknown functions to be determined from the field equations. For this metric the field equations take the form

$$G_{ik} = 0, \tag{6.2.2}$$

where G_{ik} is the Einstein tensor related to the Ricci tensor R_{ik} and the curvature scalar R by the relation

$$G_{ik} = R_{ik} - \frac{1}{2} g_{ik} R. \tag{6.2.3}$$

To calculate the Ricci tensor and the curvature scalar, one should use the formulae

$$\begin{aligned} R_{ik} &= -\tfrac{\partial \Gamma^l_{ik}}{\partial x_l} + \tfrac{\partial \Gamma^l_{il}}{\partial x^k} - \Gamma^l_{lm}\Gamma^m_{ik} + \Gamma^m_{il}\Gamma^l_{km}, \\ R &= R_{ik} g^{ik}, \end{aligned} \tag{6.2.4}$$

where the Christoffel symbols Γ^i_{kl} can be obtained from the relations:

$$\Gamma^i_{kl} = \frac{1}{2} g^{im} (\frac{\partial g_{mk}}{\partial x^l} + \frac{\partial g_{ml}}{\partial x^k} - \frac{\partial g_{kl}}{\partial x_m}), \tag{6.2.5}$$

where Γ^i_{kl} are Christoffel symbols of second kind.

From equation (6.2.3), the Einstein equations for an axially symmetric gravitational field outside the sources are derived:

$$f \Delta f = (\overrightarrow{\nabla} f)^2, \tag{6.2.6}$$

$$\begin{aligned} 4\tfrac{\partial\gamma}{\partial\rho} &= \rho\tfrac{1}{f^2}[\left(\tfrac{\partial f}{\partial\rho}\right)^2 + \left(\tfrac{\partial f}{\partial z}\right)^2], \\ 2\tfrac{\partial\gamma}{\partial z} &= \rho\tfrac{1}{f^2}\tfrac{\partial f}{\partial\rho}\tfrac{\partial f}{\partial z}, \end{aligned} \tag{6.2.7}$$

$$\frac{\partial^2\gamma}{\partial\rho^2} + \frac{\partial^2\gamma}{\partial z^2} = -\frac{1}{4}\frac{1}{f^2}(\overrightarrow{\nabla} f)^2, \tag{6.2.8}$$

The operator Δ and $\overrightarrow{\nabla}$ are defined by the formulae

$$\begin{aligned} \Delta &\equiv \tfrac{\partial^2}{\partial\rho^2} + \tfrac{1}{\rho}\tfrac{\partial}{\partial\rho} + \tfrac{\partial^2}{\partial z^2}, \\ \overrightarrow{\nabla} &\equiv \overrightarrow{\rho}_0\tfrac{\partial}{\partial\rho} + \overrightarrow{z}_0\tfrac{\partial}{\partial z}, \end{aligned} \tag{6.2.9}$$

where $\overrightarrow{\rho}_0$ and $\overrightarrow{z}_0$ being the unit vectors, they are similar to the ordinary Laplacian and gradient operators for flat space expressed in cylindrical coordinates provided that there is no angular coordinate dependence. Among the field equations (6.2.6)-(6.2.9), equation (6.2.6) is independent, whereas (6.2.8) is a consequence of both equations in (6.2.7).

Solutions of (6.2.6)-(6.2.8) define the Weyl class. With the substitution

$$f = e^{2\psi}, \tag{6.2.10}$$

equation (6.2.6) becomes

$$\Delta\psi \equiv \frac{\partial^2\psi}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial\psi}{\partial\rho} + \frac{\partial^2\psi}{\partial z^2} = 0. \tag{6.2.11}$$

Equations (6.2.7) can be rewritten in the form

$$\gamma(\rho, z) = \int \rho\left(\left(\frac{\partial\psi}{\partial\rho}\right)^2 + \left(\frac{\partial\psi}{\partial z}\right)^2\right)d\rho + \int 2\rho\frac{\partial\psi}{\partial\rho}\frac{\partial\psi}{\partial z}dz, \tag{6.2.12}$$

To determine the symmetry group and exact solutions for the pde (6.2.11), Lie classical method is utilized in next section. By utilizing this method, the basic fields of the optimal system which lead to the reductions are obtained. For each vector field, some exact solutions are examined for the system.

6.3 Group Invariant Solutions

To apply the classical method to equation (6.2.11), the one-parameter Lie group of infinitesimal transformations in (ρ, z, ψ) is considered. The associated Lie algebra of infinitesimal symmetries is the corresponding set of vector fields of the form

$$X = \xi(\rho, z, \psi)\frac{\partial}{\partial \rho} + \tau(\rho, z, \psi)\frac{\partial}{\partial z} + \phi(\rho, z, \psi)\frac{\partial}{\partial \psi}. \tag{6.3.1}$$

Then, it is required that this transformation leave the set of solutions of PDE (6.2.11) invariant. This yields an over determined linear system of equations for the infinitesimals $\xi(\rho, z, \psi), \tau(\rho, z, \psi)$ and $\phi(\rho, z, \psi)$. After the infinitesimals are determined, the symmetry variables are found by solving the invariant-surface conditions

$$\Phi \equiv \xi\frac{\partial \psi}{\partial \rho} + \tau\frac{\partial \psi}{\partial z} - \phi = 0. \tag{6.3.2}$$

Applying the classical method to pde (6.2.11) yields a system of equations that leads to a four-parameter Lie group. Associated with this Lie group, a Lie algebra that can be represented by the generators is given as

$$\begin{aligned} V_1 &= \tfrac{\partial}{\partial z}, \\ V_2 &= \rho\tfrac{\partial}{\partial \rho} + z\tfrac{\partial}{z}, \\ V_3 &= \psi\tfrac{\partial}{\partial \psi}, \\ V_4 &= z\rho\tfrac{\partial}{\partial \rho} + \tfrac{(\rho^2+z^2)}{2}\tfrac{\partial}{\partial z} - \tfrac{z\psi}{2}. \end{aligned} \tag{6.3.3}$$

The commutation relations between these generators is

$[V_1, V_2] = -[V_2, V_1] = V_1, [V_1, V_3] = -[V_3, V_1] = 0, [V_1, V_4] = -[V_4, V_1] = V_2,$

$[V_2, V_3] = -[V_3, V_2] = V_2, [V_2, V_4] = -[V_4, V_2] = V_4, [V_3, V_4] = -[V_4, V_3] = 0.$

The Adjoint table of these commutation relations is given by Table (6.1)

Adjoint Table (6.1)

Ad	V_1	V_2	V_3	V_4
V_1	V_1	$V_2 - \epsilon V_1$	V_3	$V_4 - \epsilon V_2 + \frac{\epsilon^2}{2} V_1$
V_2	$V_1 e^{\epsilon}$	V_2	V_3	$V_4 e^{-\epsilon}$
V_3	V_1	V_2	V_3	V_4
V_4	$V_1 + \epsilon V_2 - \frac{\epsilon^2}{2} V_4$	$V_2 + \epsilon V_4$	V_3	V_4

To find nonequivalent branches of solutions, the one-dimensional optimal system of subalgebras is constructed. The corresponding generators of the optimal system of subalgebras are

$$\langle V_1 \rangle, \qquad \langle V_2 \rangle, \qquad \langle V_3 + \alpha V_2 \rangle, \qquad \langle V_4 + \beta V_2 + \delta V_1 \rangle,$$

where α, β and δ are arbitrary constants.

The symmetry variables corresponding to each case are found by solving the auxiliary equation

$$\frac{d\rho}{\xi} = \frac{dz}{\tau} = \frac{d\psi}{\phi}. \tag{6.3.4}$$

Next, similarity variables and similarity solutions for all the four essential vector fields with optimal system are derived. The reduction of equation (6.2.11) into ODEs are obtained corresponds to each vector field in the optimal system. Some exact solutions of these ODEs and system (6.2.11)-(6.2.12) are also obtained.

(i) $\quad L_1 = \{V_1\} = \{\frac{\partial}{\partial \rho}\}$

The reduced equation for the similarity variables

$\zeta = \rho$, $\psi(\rho, z) = F(\zeta)$ and subalgebra $L_1 = V_1$ is

$$F''(\zeta) + \frac{F'(\zeta)}{\zeta} = 0, \tag{6.3.5}$$

On solving this ODE, $F(\zeta)$ is

$$F(\zeta) = c_1 + c_2 \log(\zeta). \tag{6.3.6}$$

Thus the solution of PDE (6.2.11) is

$$\psi(\rho, z) = c_1 + c_2 \log(\rho) \tag{6.3.7}$$

and after substituting value of ψ in (6.2.12), we have the metric function γ as

$$\gamma(\rho, z) = c_2^2 \log(\rho), \tag{6.3.8}$$

also the metric function $f(\rho, z)$, from (6.2.10) is

$$f(\rho, z) = e^{(2c_1 + c_2 \log(\rho))} \tag{6.3.9}$$

where c_1 and c_2 are arbitrary constants.

(ii) $L_2 = \{V_2\} = \{\frac{\partial}{\partial \rho} + \frac{\partial}{\partial z}\}$

The reduced equation for the similarity variables

$\zeta = \frac{\rho}{z}$, $\psi(\rho, z) = F(\zeta)$ and subalgebra $L_2 = V_2$ is

$$F''(\zeta) + \frac{F'(\zeta)}{\zeta} - \zeta^2 F''(\zeta) - 2\zeta F'(\zeta) = 0, \tag{6.3.10}$$

solution of this ODE is

$$F(\zeta) = c_1 + c_2 \arctan\left(\frac{1}{\sqrt{-1 + \zeta^2}}\right) \tag{6.3.11}$$

and solution of system (6.2.11)and (6.2.12) is

$$\psi(\rho, z) = c_1 + c_2 \arctan\left(\frac{1}{\sqrt{-1 + \left(\frac{\rho}{z}\right)^2}}\right) \tag{6.3.12}$$

$$\begin{aligned} \gamma(\rho, z) &= c_2^2((\log(\rho + z) - \log(\rho)) \\ &+ c_2^2 \left(\log(\rho - z) + \log((z - \rho)(z + \rho))\right). \end{aligned} \tag{6.3.13}$$

and the metric function $f(\rho, z)$ is

$$f(\rho, z) = e^{\left(2c_1 + c_2 \arctan\left(\frac{1}{\sqrt{-1 + \left(\frac{\rho}{z}\right)^2}}\right)\right)} \tag{6.3.14}$$

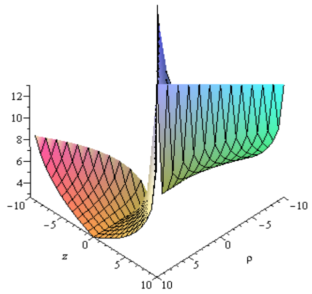

Figure 6.1: Behavior of Solution of (6.3.13) for constants $c_1 = C_2 = 1$

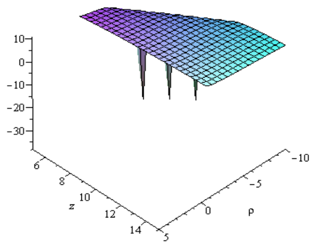

Figure 6.2: Behavior of Solution of (6.3.14) for constants $c_1 = C_2 = 1$

where c_1, c_2 and c_3 are arbitrary constants. The behavior of solutions (6.3.13) and (6.3.14) are represented by Fig1. and Fig2. respectively.

(iii) $\quad L_{2,3}(\alpha) = \{V_3 + \alpha V_2\} = \{\psi \frac{\partial}{\partial \psi} + \alpha(\frac{\partial}{\partial \rho} + \frac{\partial}{\partial z})\}$

The reduced ODE of PDE (6.2.11) for the subalgebra $L_{2,3}(\alpha)$ is

$$F''(\zeta) + \frac{F'(\zeta)}{\zeta} - \frac{F}{\alpha^2} + \frac{F}{\alpha} + \frac{2F(\zeta)F'(\zeta)}{\alpha} - \zeta^2 F''(\zeta) - 2\zeta F'(\zeta) = 0, \tag{6.3.15}$$

by solving above ODE, we get

$$\begin{aligned} F(\zeta) = c_1 &\left(\left(-\tfrac{1}{2\alpha}, \tfrac{1}{2}\tfrac{(\alpha-1)}{\alpha}\right), \left[\tfrac{1}{2}\tfrac{-2+\alpha}{\alpha}\right], -\zeta^2 + 1\right) \\ +c_2(\zeta^2 - 1)^{\frac{\alpha+2}{2\alpha}} &\left(\left[\tfrac{1+2\alpha}{2\alpha}, \tfrac{1+\alpha}{2\alpha}\right], \left[\tfrac{1}{2}\tfrac{3\alpha+2}{\alpha}\right], -\zeta^2 + 1\right), \end{aligned} \tag{6.3.16}$$

where c_1 and c_2 are arbitrary constants and is hypergeometric function.

Thus, the solution of equation (6.2.11) is

$$\begin{aligned} \psi(\rho, z) = c_1 &\left(\left[-\tfrac{1}{2\alpha}, \tfrac{1}{2}\tfrac{(\alpha-1)}{\alpha}\right], \left[\tfrac{1}{2}\tfrac{-2+\alpha}{\alpha}\right], -(\tfrac{\rho}{z})^2 + 1\right) \\ +c_2((\tfrac{\rho}{z})^2 - 1)^{\frac{\alpha+2}{2\alpha}} &\left(\left[\tfrac{1+2\alpha}{2\alpha}, \tfrac{1+\alpha}{2\alpha}\right], \left[\tfrac{1}{2}\tfrac{3\alpha+2}{\alpha}\right], -(\tfrac{\rho}{z})^2 + 1\right), \end{aligned} \tag{6.3.17}$$

solution of equation (6.2.12) will be in the form of integration and the metric function $f(\rho, z)$ is as follows

$$f(\rho, z) = e^{2\psi}, \tag{6.3.18}$$

where ψ is given by (6.3.17) and c_1 and c_2 are constants of integration.

(iv) $L_{4,3,2}(\beta, \delta) = \{V_4 + \beta V_3 + \delta V_2\} = \{(z\rho + \delta\rho)\frac{\partial}{\partial\rho} + \left(\delta z + \frac{(\rho^2+z^2)}{2}\right)\frac{\partial}{\partial z} + \left(\beta\psi + \frac{-z}{2}\right)\frac{\partial}{\psi}\}$

Corresponding to this subalgebra reduction is not possible.

6.4 Conclusion

In order to completely describe the gravitational field of a body, one must know the corresponding exact solutions. In this chapter, first a system of PDEs from Weyl metric is derived by using Einstein tensor. After that by applying Lie classical approach, the symmetries of Einstein vacuum equations are investigated and utilized these symmetries for obtaining group infinitesimals which are helpful in the reduction of PDE to ODE. Corresponding to these ODEs, some exact solutions are obtained that might be physically important in future work.

Chapter 7

Generalized Fifth Order Variable Coefficient KdV Equation [1]

7.1 Introduction

In this chapter, the generalized fifth order KdV equation with time dependent variable coefficient t is studied

$$u_t + \alpha(t)u^2u_x + \beta(t)u_xu_{xx} + \gamma(t)uu_{xxx} + \delta(t)u_{xxxxx} = 0, \tag{7.1.1}$$

where u is function of x and t. The generalized fifth-order KdV (fKdV) equation is a model equation for plasma waves, capillary-gravity water waves, and other dispersive phenomena when the cubic KdV-type dispersion is weak. The fKdV equation (7.1.1) describes the motion of long waves in shallow water under gravity and in one-dimensional nonlinear lattice. The nonlinear fifth-order KdV (7.1.1) is one of the important mathematical model with wide range of applications in quantum mechanics. Li et al. [84], studied the generalized fifth-order KdV model equation with constant coefficients and obtained solitary wave and soliton solutions. Wazwaz [168], found soliton solutions for several forms of fifth-order KdV with constant coefficients

[1] The contents of this chapter are communicated in *Journal of Differential Equations* .

by using tanh method. Wazwaz [169], derived soliton solutions of variable coefficients fifth order KdV by using wave ansätz method.

The physical phenomena in which many nonlinear integrable equations with constant coefficients arise tend to highly idealized. Therefore, equations with variable coefficients may provide various models for real physical phenomena, such as, in the propagation of small-amplitude surface waves, which runs on straits or large channels of slowly varying depth variable coefficients of nonlinear integrable evolution equations. As there are choices for parameters, the variable-coefficient nonlinear equations can be considered as generalization of the constant coefficients equations. The nonlinear wave equations with variable-coefficients are more realistic in various physical situations than their constant coefficients, since in realistic physical systems, no medium is homogenous due to the existence of inhomogeneities and non uniformities of boundaries. Here, if $\alpha(t), \beta(t), \gamma(t)$ are constants and $\delta(t) = 1$, then the above fifth order evolution equations include well-known Kaup-Kupershmidt (KK), Lax, Sawada-Kotera (SK), CaudreyDoddGibbon (CDG) and Ito equations. For example when:

(i) (α, β, γ) are $(30, 60, 270)$ and $(20, 40, 120)$, equation (7.1.1) is fifth order KdV equation, examined by [81].

(ii) (α, β, γ) are $(5, 5, 5)$ and $(-15, -15, 45)$, equation (7.1.1) is Sawada-Kotera equation, examined by [137].

(iii) (α, β, γ) are $(30, 75, 180)$ and $(10, 25, 20)$, equation (7.1.1) is Kaup-Kupershmidt equation, examined by [72, 79].

(iv) (α, β, γ) are $(30, 20, 10)$, equation (7.1.1) is Lax equation, examined by [81].

(v) (α, β, γ) are $(2, 6, 3)$, equation (7.1.1) is Ito equation, examined by [65].

(vi) (α, β, γ) are $(180, 30, 30)$, equation (7.1.1) is CaudreyDoddGibbon equation, examined by [24].

For instance, the Lax equation and the SK equation are completely integrable. These two equations have N-soliton solutions and an infinite set of conserved densities. Another example is the KK equation which is known to be integrable and has bilinear representations. A fourth equation in this class is the Ito equation which is not completely integrable, but has a limited number of special conserved densities. Obviously, for arbitrary values of the constants α, β and γ the equation (7.1.1) is not completely integrable, and therefore does not admit soliton solutions, which does not exclude the existence of solitary wave solutions. Note that with scales on u, x and t, the equations cannot be transformed into one another; they are fundamentally different.

In this chapter, exact analytic solutions of variable-coefficient generalized fKdV equation are derived. All the physical parameters in the solutions are evaluated as functions of varying coefficients. Moreover, by direct integration, it is difficult to solve fKdV equation with higher degree nonlinear terms with variable coefficients. Exact solutions of nonlinear evaluation equations are helpful for understanding the physical behavior of nonlinear phenomena and dynamical process modeled by these nonlinear models. Although, nonlinear models may posses constant or variable dependent coefficients which provide a rich variety of shape preserving waves and their interesting properties.

The detailed plan of the chapter is as follows. Section (7.2) is devoted to lie classical method, symmetries of generalized variable-coefficient fKdV equation and optimal system of basic vector fields. In Section (7.3), similarity variables and similarity solutions are derived and corresponding to these similarity solutions reduced ODEs and exact solutions have been obtained. Some more exact solutions of nonlinear equation (7.1.1) are obtained by using $(\frac{G'}{G})$ method in Section (7.4). In Section (7.5), some conclusions are drawn.

7.2 Lie algebra of VC fKdV Equation

In order to find the Lie point symmetry group of equation (7.1.1), all the possible invariance of equation (7.1.1) are found under the transformation

$$\{x, t, u\} \to \{x, t, u\} + \epsilon\{X, T, U\}, \tag{7.2.1}$$

where X, T and U are functions of x, t and u, and ϵ is an infinitesimal parameter. Substituting equation (7.2.1) into equation (7.1.1), expanding it to the first order of ϵ and removing the coefficients of the different derivatives of function u, the classical symmetries of the variable coefficient fifth order KdV equation are obtained. The final results read

$$\begin{aligned} X &= k_1 x + k_2, \\ T &= \tfrac{\int k_1\alpha(t)dt + k_3}{\alpha(t)}, \\ U &= 0, \end{aligned} \tag{7.2.2}$$

and $\alpha(t), \beta(t), \gamma(t), \delta(t)$ are obtained from following equations:

$$\begin{aligned} &T\alpha'(t) - \alpha(t)X_x + \alpha(t)T_t = 0, \\ &T\beta'(t) - 3\beta(t)X_x + \beta(t)T_t = 0, \\ &T\gamma'(t) - 3\gamma(t)X_x + \gamma(t)T_t = 0, \\ &T\delta'(t) - 5\delta(t)X_x + \delta(t)T_t = 0. \end{aligned} \tag{7.2.3}$$

with arbitrary constants k_1, k_2 and k_3.

Hence, the corresponding vector field can be written as

$$V = (k_1 x + k_2)\frac{\partial}{\partial x} + (\frac{\int k_1\alpha(t)dt + k_3}{\alpha(t)})\frac{\partial}{\partial t}. \tag{7.2.4}$$

Because of the three arbitrary constants in the vector V, three generators of the related Lie symmetry group of equation (7.1.1) are derived as:

$$\{X_1 = \frac{\partial}{\partial x}, X_2 = \frac{1}{\alpha(t)}\frac{\partial}{\partial t}, X_3 = x\frac{\partial}{\partial x} + \frac{\int \alpha(t)dt}{\alpha(t)}\frac{\partial}{\partial t}\}.$$

and $\beta(t), \gamma(t)$ and $\delta(t)$ are given as:

$$\begin{aligned} \beta(t) &= c_1(k_1 \textstyle\int \alpha(t)dt + k_3)^2 \alpha(t), \\ \gamma(t) &= c_2(k_1 \textstyle\int \alpha(t)dt + k_3)^2 \alpha(t), \\ \delta(t) &= c_3(k_1 \textstyle\int \alpha(t)dt + k_3)^4 \alpha(t), \end{aligned} \tag{7.2.5}$$

where c_1, c_2 and c_3 are arbitrary constants.

The commutation relations of the 3-dimensional Lie algebra spanned by the vector fields X_1, X_2 and X_3 have the form

$$[X_1, X_2] = -[X_2, X_1] = 0, [X_1, X_3] = -[X_3, X_1] = X_1, [X_2, X_3] = -[X_3, X_2] = X_2. \tag{7.2.6}$$

The Adjoint Table for the Lie algebra G, determined by X_1, X_2, X_3 are shown in Table (7.1).

Adjoint Table (7.1)

Ad	X_1	X_2	X_3
X_1	X_1	X_2	$X_3 - \epsilon X_1$
X_2	X_1	X_2	$X_3 - \epsilon X_2$
X_3	$X_1 e^{\epsilon}$	$X_2 e^{\epsilon}$	X_3

Following Ovsiannikov [120], an optimal system of sub algebras with their corresponding generators is derived as follows:

$$\langle X_3, \quad X_2 + X_1, \quad X_2 - X_1, \quad X_2, \quad X_1 \rangle$$

. Because of the symmetry $(x; t; u) \longrightarrow (-x; t; u), X_2 + X_1$ will map to $X_2 - X_1$, thus in the optimal system, reductions and exact solutions of remaining four essential vector fields of the optimal system are examined.

By using these basic vector fields by solving the characteristic equations

$$\frac{dx}{\xi_1} = \frac{dt}{\xi_2} = \frac{du}{\phi}. \tag{7.2.7}$$

reductions of equation (7.1.1) to ODEs have been obtained in next section.

7.3 Reduced ODEs and Exact Solutions

In the proceeding section, the symmetries for the equation (7.1.1) are derived. Now, symmetry reductions and exact solutions based on the Lie group analysis method are obtained.

Vector field (i) X_3

For the generator X_3, the similarity variables:

$\zeta = \frac{x}{\int \alpha(t)dt}$, $u(x,t) = F(\zeta)$

and coefficient functions $\beta(t), \gamma(t)$ and $\delta(t)$, for this vector field are given as:

$$\begin{aligned}\beta(t) &= c_1\left(\int \alpha(t)dt\right)^2\alpha(t),\\ \gamma(t) &= c_2\left(\int \alpha(t)dt\right)^2\alpha(t),\\ \delta(t) &= c_3\left(\int \alpha(t)dt\right)^4\alpha(t),\end{aligned} \tag{7.3.1}$$

with arbitrary constants c_1, c_2 and c_3.

Using these substitutions in equation (7.1.1), the following reduced ODE is obtained:

$$-\zeta F'(\zeta) + F(\zeta)^2 F'(\zeta) + c_1 F'(\zeta)F''(\zeta) + c_2 F(\zeta)F'''(\zeta) + c_3 F'''''(\zeta) = 0, \tag{7.3.2}$$

Next, solution of equation (7.3.2) in the form of power series is evaluated by taking

$$F(\zeta) = \sum_{n=0}^{\infty} a_n \zeta^n. \tag{7.3.3}$$

substituting (7.3.3) into (7.3.2) and by comparing coefficients, following coefficients of power series are evaluated

$$a_{n+5} = \frac{1}{c_3(n+1)(n+2)(n+3)(n+4)(n+5)}\Big(- na_n - \sum_{k=0}^{n}\sum_{j=0}^{k}(n+1-k)a_j a_{k-j} a_{n+1-k}$$
$$-c_1 \sum_{k=0}^{n}(n+2-k)(n+1-k)(k+1)a_{k+1}a_{n+2-k} - c_2 \sum_{k=0}^{n}(n+3-k)$$
$$(n+2-k)(n+1-k)a_{n+3-k}a_k\Big), \tag{7.3.4}$$

$n = 0, 1, 2, ...$. From (7.3.4), all coefficients $a_n, n \geq 1$ of power series (7.3.2) are obtained

$$a_5 = \frac{1}{120c_3}\big(- a_0^2 a_1 - 2c_1 a_1 a_2 - 6c_2 a_3 a_0\big),$$
$$a_6 = \frac{1}{720c_3}\big(- a_1 - 2\big(a_0^2 a_2 + a_0 a_1^2\big) - c_1\big(6a_1 a_3 + 4a_2^2\big) - c_2\big(24a_4 a_0 + 6a_3 a_1\big)\big),$$
$$a_7 = \frac{1}{2520c_3}\big(- 2a_2 - \big(3a_0^2 a_3 + 6a_0 a_1 a_2 + a_1^3\big) - c_1\big(12a_1 a_4 + 18a_2 a_3\big)$$
$$-c_2\big(60a_5 a_0 + 24a_4 a_1 + 6a_3 a_2\big)\big), \tag{7.3.5}$$

and so on. Hence, for arbitrary chosen constant numbers a_0, a_1, a_2, a_3 and a_4 the other terms of the sequence $\{a_n\}_{n=0}^{\infty}$ can be determined successively from (7.3.5) in a unique manner. This implies that for equation (7.3.2), there exists a power series solution (7.3.3) with the coefficients given by (7.3.4).

Now, the power series solution of (7.1.1) can be written as

$$u(x,t) = a_0 + a_1\zeta + a_2\zeta^2 + a_3\zeta^3 + a_4\zeta^4 + \frac{1}{120c_3}\big(- a_0^2 a_1 - 2c_1 a_1 a_2 - 6c_2 a_3 a_0\big)\zeta^5$$
$$+ \sum_{n=1}^{\infty} \frac{1}{c_3(n+1)(n+2)(n+3)(n+4)(n+5)}\Big(- na_n - \sum_{k=0}^{n}\sum_{j=0}^{k}(n+1-k)a_j a_{k-j} a_{n+1-k}$$
$$-c_1 \sum_{k=0}^{n}(n+2-k)(n+1-k)(k+1)a_{k+1}a_{n+2-k} - c_2 \sum_{k=0}^{n}(n+3-k)$$
$$(n+2-k)(n+1-k)a_{n+3-k}a_k\Big)\zeta^{n+5} \tag{7.3.6}$$

Hence, the exact power series solution of VC fKdV equation (7.1.1) is

$$u(x,t) = a_0 + a_1\Big(\tfrac{x}{\int \alpha(t)dt}\Big) + a_2\Big(\tfrac{x}{\int \alpha(t)dt}\Big)^2 + a_3\Big(\tfrac{x}{\int \alpha(t)dt}\Big)^3 + a_4\Big(\tfrac{x}{\int \alpha(t)dt}\Big)^4$$
$$+\tfrac{1}{120c_3}\Big(-a_0^2a_1 - 2c_1a_1a_2 - 6c_2a_3a_0\Big)\Big(\tfrac{x}{\int \alpha(t)dt}\Big)^5 + \sum_{n=1}^{\infty}\tfrac{1}{c_3(n+1)(n+2)(n+3)(n+4)(n+5)}$$
$$\Big(-na_n - \sum_{k=0}^{n}\sum_{j=0}^{k}(n+1-k)a_ja_{k-j}a_{n+1-k} - c_1\sum_{k=0}^{n}(n+2-k)(n+1-k)(k+1)a_{k+1}$$
$$a_{n+2-k} - c_2\sum_{k=0}^{n}(n+3-k)(n+2-k)(n+1-k)a_{n+3-k}a_k\Big)\Big(\tfrac{x}{\int \alpha(t)dt}\Big)^{n+5}, \tag{7.3.7}$$

where $a_0, a_1, a_2, a_3, a_4, c_1, c_2$ and c_3 are arbitrary constants.

Vector field *(ii)* $X_2 + X_1$

Similarly, from equation (7.2.7), the invariant $\zeta = x - \int \alpha(t)dt$ and the functional form of group invariant solution $u(x,t) = F(\zeta)$ are obtained. The coefficient functions for this vector field are

$$\begin{aligned}\beta(t) &= c_1\alpha(t),\\ \gamma(t) &= c_2\alpha(t),\\ \delta(t) &= c_3\alpha(t),\end{aligned} \tag{7.3.8}$$

and F satisfies the ordinary differential equation

$$-F'(\zeta) + F(\zeta)^2F'(\zeta) + c_1F'(\zeta)F''(\zeta) + c_2F(\zeta)F'''(\zeta) + c_3F'''''(\zeta) = 0 \tag{7.3.9}$$

The ordinary differential equation admits the family of solutions

$$F(\zeta) = \frac{24c_3c_4^4+1}{c_4^2c_1} - \frac{3}{2}\frac{(24c_3c_4^4+1)\tanh(c_5+c_4\zeta)^2}{c_4^2c_1}, \tag{7.3.10}$$

where $c_2 = \frac{64c_4^8c_3c_1^2+576c_4^8c_3^2-4c_1^2c_4^4+48c_3c_4^4+1}{8c_1c_4^2(24c_3c_4^4+1)}$

and hence, solution of VC fKdV equation (7.1.1) is

$$u(x,t) = \frac{24c_3c_4^4+1}{c_4^2c_1} - \frac{3}{2}\frac{(24c_3c_4^4+1)\tanh(c_5+c_4(x-\int \alpha(t)dt))^2}{c_4^2c_1}, \tag{7.3.11}$$

where c_2 is given as above and c_1, c_3, c_4 are arbitrary constants.

Vector field *(iii)* X_2

Corresponding to this vector field, only constant solution is obtained.

Vector field (iv) X_1

Similar computation gives similarity variable and similarity solution are as follows:

$\zeta = x$, $u(x,t) = F(\zeta)$,

and coefficient functions for vector field X_1 are

$$\begin{aligned} \beta(t) &= c_1\alpha(t), \\ \gamma(t) &= c_2\alpha(t), \\ \delta(t) &= c_3\alpha(t) \end{aligned} \tag{7.3.12}$$

Using these substitutions in equation (7.1.1), the reduced form of PDE is:

$$F(\zeta)^2F'(\zeta) + c_1F'(\zeta)F''(\zeta) + c_2F(\zeta)F'''(\zeta) + c_3F'''''(\zeta) = 0 \tag{7.3.13}$$

After solving this ODE, $F(\zeta)$ is

$$F(\zeta) = c_4^2\left(8c_2 - \frac{8}{3}c_1 + (-12c_2 + 4c_1)\tanh(c_3 + c_4(\zeta))^2\right), \tag{7.3.14}$$

where $c_2 = -\frac{1}{9}c_1^2 + \frac{1}{3}c_1c_2$.

Hence exact solution of VC fKdV is

$$u(x,t) = c_4^2\left(8c_2 - \frac{8}{3}c_1 + (-12c_2 + 4c_1)\tanh(c_3 + c_4(x))^2\right), \tag{7.3.15}$$

where c_2 is given as above and c_1, c_3, c_4 are arbitrary constants.

7.4 Some More Exact Solutions of VC fKdV Equation by $\left(\frac{G'}{G}\right)$-Expansion Method

In this section, some more exact solutions of VC fKdV equation via generalized $\left(\frac{G'}{G}\right)$-expansion method have been obtained. To study the traveling wave solutions of equation (7.1.1), a plane wave transformation is considered in the form

$$u(x,t) = u(\xi), \xi = kx + \int \tau(t)dt, \tag{7.4.1}$$

equation (7.1.1) is carried to an ODE, which takes the following form

$$\tau(t)u(\xi)+k\alpha(t)u(\xi)^2u'(\xi)+k^3\beta(t)u'(\xi)u''(\xi)+k^3\gamma(t)u\xi u'''(\xi)+\delta(t)u'''''(\xi)=0, \tag{7.4.2}$$

where the prime denotes the differential with respect to ξ.
In view of method $\left(\frac{G'}{G}\right)$, we introduce the ansätz

$$u(\xi)=\sum_{i=0}^{n}a_i\left(\frac{G'}{G}\right), \tag{7.4.3}$$

where a_i are constants to be determined later and $G=G(\xi)$ satisfies

$$G''(\xi)+\lambda G'(\xi)+\mu G(\xi)=0. \tag{7.4.4}$$

Considering the homogeneous balance between $u(\xi)^2u'(\xi)$ and $u'''''(\xi)$ in (7.4.2), yields $n=2$. The solution of equation (7.1.1) can be expressed by

$$u(\xi)=a_0+a_1\left(\frac{G'}{G}\right)+a_2\left(\frac{G'}{G}\right)^2, \tag{7.4.5}$$

where a_0, a_1 and a_2 are constants to be determined. Substituting (7.4.5) with (7.4.4) into (7.4.2) and by collecting the coefficients of $\left(\frac{G'}{G}\right)$, a set of algebraic equations for a_0, a_1, a_2 and $\tau(t)$ is obtained. After solving this system with the aid of Maple package the three sets of solutions are

Case – I

$$\begin{aligned}
a_0 &= \tfrac{2}{3}a_2\mu+\tfrac{1}{12}a_2\lambda^2, a_1=a_2\lambda,\\
\tau(t) &= \tfrac{k^3\left((576k^2\mu^2+36k^2\lambda^4-288k^2\lambda^2\mu)\delta(t)+(16a_2\mu^2+a_2\lambda^4-8a_2\lambda^2\mu)\beta(t)\right)}{24}\\
\alpha(t) &= \tfrac{-6k^2(60\delta(t)k^2+\beta(t)a_2+2a_2\gamma(t))}{a_2^2}.
\end{aligned} \tag{7.4.6}$$

Case – II

$$\begin{aligned}
&a_1=a_2\lambda, \alpha(t)=-\tfrac{6k^2\gamma(t)}{a_2}, \beta(t)=-\tfrac{60k^2\delta(t)+a_2\gamma(t)}{a_2}\\
&\tau(t)=-k^3\big((k^2\lambda^4a_2+16k^2a_2\mu^2-8k^2a_2\lambda^2\mu)\delta(t)+\big(-2a_2^2\mu^2-6a_0^2+a_0a_2\lambda^2\\
&-a_2^2\lambda^2\mu+8a_0a_2\mu\big)\gamma(t)\big).
\end{aligned} \tag{7.4.7}$$

Case – III

$$\begin{aligned} &a_0 = \tfrac{1}{12}a_2(8\mu + \lambda^2), a_1 = a_2\lambda, \alpha(t) = \tfrac{3k^2(48k^2\delta(t)-a_2\gamma(t))}{a_2^2}, \\ &\tau(t) = -\tfrac{k^3((512k^2\mu^2+32k^2\lambda^4-256k^2\lambda^2\mu)\delta(t)+(16a_2\mu^2+a_2\lambda^4-8a_2\lambda^2\mu)\gamma(t))}{16}, \\ &\beta(t) = -\tfrac{3}{2}\tfrac{56k^2\delta(t)+a_2\gamma(t)}{a_2}. \end{aligned} \tag{7.4.8}$$

From Case-I and equations (7.4.6) and (7.4.4), the following solutions of equation (7.1.1), are as follows.

When $\lambda^2 - 4\mu > 0$, solutions are obtained,

$$\begin{aligned} u_{11}(\xi) &= \tfrac{2}{3}a_2\mu + \tfrac{1}{12}a_2\lambda^2 + a_2\lambda\left(-\tfrac{\lambda}{2} + \tfrac{\sqrt{\lambda^2-4\mu}}{2}\tfrac{C_1 \sinh\frac{1}{2}(\sqrt{\lambda^2-4\mu}\xi)+C_2\cosh\frac{1}{2}(\sqrt{\lambda^2-4\mu}\xi)}{C_1\cosh\frac{1}{2}(\sqrt{\lambda^2-4\mu}\xi)+C_2\sinh\frac{1}{2}(\sqrt{\lambda^2-4\mu}\xi)}\right) \\ &+a_2\left(-\tfrac{\lambda}{2} + \tfrac{\sqrt{\lambda^2-4\mu}}{2}\tfrac{C_1 \sinh\frac{1}{2}(\sqrt{\lambda^2-4\mu}\xi)+C_2\cosh\frac{1}{2}(\sqrt{\lambda^2-4\mu}\xi)}{C_1\cosh\frac{1}{2}(\sqrt{\lambda^2-4\mu}\xi)+C_2\sinh\frac{1}{2}(\sqrt{\lambda^2-4\mu}\xi)}\right)^2, \end{aligned} \tag{7.4.9}$$

when $\lambda^2 - 4\mu < 0$, the solutions are,

$$\begin{aligned} u_{12}(\xi) &= \tfrac{2}{3}a_2\mu + \tfrac{1}{12}a_2\lambda^2 + a_2\lambda\left(-\tfrac{\lambda}{2} + \tfrac{\sqrt{-\lambda^2+4\mu}}{2}\tfrac{-C_1 \sin\frac{1}{2}(\sqrt{-\lambda^2+4\mu}\xi)+C_2\cos\frac{1}{2}(\sqrt{-\lambda^2+4\mu}\xi)}{C_1\cos\frac{1}{2}(\sqrt{-\lambda^2+4\mu}\xi)+C_2\sin\frac{1}{2}(\sqrt{-\lambda^2+4\mu}\xi)}\right) \\ &+a_2\left(-\tfrac{\lambda}{2} + \tfrac{\sqrt{-\lambda^2+4\mu}}{2}\tfrac{-C_1 \sin\frac{1}{2}(\sqrt{-\lambda^2+4\mu}\xi)+C_2\cos\frac{1}{2}(\sqrt{-\lambda^2+4\mu}\xi)}{C_1\cos\frac{1}{2}(\sqrt{-\lambda^2+4\mu}\xi)+C_2\sin\frac{1}{2}(\sqrt{-\lambda^2+4\mu}\xi)}\right)^2, \end{aligned} \tag{7.4.10}$$

when $\lambda^2 - 4\mu = 0$, the solutions are given as,

$$u_{13}(\xi) = \tfrac{2}{3}a_2\mu + \tfrac{1}{12}a_2\lambda^2 + a_2\lambda\left(-\tfrac{\lambda}{2} + \tfrac{C_2}{C_1+C_2\xi}\right) + a_2\left(-\tfrac{\lambda}{2} + \tfrac{C_2}{C_1+C_2\xi}\right)^2, \tag{7.4.11}$$

where

$\xi = kx$

$+\frac{1}{24}\left(k^3(576k^2\mu^2 + 36k^2\lambda^4 - 288k^2\lambda^2\mu)\int\delta(t)dt + (16a_2\mu^2 + a_2\lambda^4 - 8a_2\lambda^2\mu)\int\beta(t)dt\right).$

Next, from Case-II and equations (7.4.7) and (7.4.4), the corresponding solutions of equation (7.1.1), are as follows.

When $\lambda^2 - 4\mu > 0$, the solutions are in the form,

$$u_{21}(\xi) = a_0 + a_2\lambda\Bigg(-\tfrac{\lambda}{2} + \tfrac{\sqrt{\lambda^2-4\mu}}{2}\tfrac{C_1\sinh\frac{1}{2}(\sqrt{\lambda^2-4\mu}\xi)+C_2\cosh\frac{1}{2}(\sqrt{\lambda^2-4\mu}\xi)}{C_1\cosh\frac{1}{2}(\sqrt{\lambda^2-4\mu}\xi)+C_2\sinh\frac{1}{2}(\sqrt{\lambda^2-4\mu}\xi)}\Bigg) + a_2\left(-\tfrac{\lambda}{2} + \tfrac{\sqrt{\lambda^2-4\mu}}{2}\tfrac{C_1\sinh\frac{1}{2}(\sqrt{\lambda^2-4\mu}\xi)+C_2\cosh\frac{1}{2}(\sqrt{\lambda^2-4\mu}\xi)}{C_1\cosh\frac{1}{2}(\sqrt{\lambda^2-4\mu}\xi)+C_2\sinh\frac{1}{2}(\sqrt{\lambda^2-4\mu}\xi)}\right)^2, \tag{7.4.12}$$

when $\lambda^2 - 4\mu < 0$, the solutions become in the following form,

$$u_{22}(\xi) = a_0 + a_2\lambda\Bigg(-\tfrac{\lambda}{2} + \tfrac{\sqrt{-\lambda^2+4\mu}}{2}\tfrac{-C_1\sin\frac{1}{2}(\sqrt{-\lambda^2+4\mu}\xi)+C_2\cos\frac{1}{2}(\sqrt{-\lambda^2+4\mu}\xi)}{C_1\cos\frac{1}{2}(\sqrt{-\lambda^2+4\mu}\xi)+C_2\sin\frac{1}{2}(\sqrt{-\lambda^2+4\mu}\xi)}\Bigg) + a_2\left(-\tfrac{\lambda}{2} + \tfrac{\sqrt{-\lambda^2+4\mu}}{2}\tfrac{-C_1\sin\frac{1}{2}(\sqrt{-\lambda^2+4\mu}\xi)+C_2\cos\frac{1}{2}(\sqrt{-\lambda^2+4\mu}\xi)}{C_1\cos\frac{1}{2}(\sqrt{-\lambda^2+4\mu}\xi)+C_2\sin\frac{1}{2}(\sqrt{-\lambda^2+4\mu}\xi)}\right)^2, \tag{7.4.13}$$

when $\lambda^2 - 4\mu = 0$, the solutions are,

$$u_{23}(\xi) = a_0 + a_2\lambda\Big(-\tfrac{\lambda}{2} + \tfrac{C_2}{C_1+C_2\xi}\Big) + a_2\Big(-\tfrac{\lambda}{2} + \tfrac{C_2}{C_1+C_2\xi}\Big)^2, \tag{7.4.14}$$

where $\xi = kx - (m(x,t) + n(x,t))$ and

$m(x,t) = k^3(k^2\lambda^4 a_2 + 16k^2\mu^2 a_2 - 8k^2\lambda^2\mu a_2)\int \delta(t)dt$

$n(x,t) = (-2{a_2}^2\mu^2 - 6{a_0}^2 + a_0a_2\lambda^2 - {a_2}^2\lambda^2\mu + 8a_0a_2\mu)\int \gamma(t)dt.$

Now, from Case-III and equations (7.4.8) and (7.4.4), the following solutions of equation (7.1.1), are:

When $\lambda^2 - 4\mu > 0$, following solutions are derived,

$$u_{31}(\xi) = \tfrac{1}{2}a_2(8\mu+\lambda^2) + a_2\lambda\Bigg(-\tfrac{\lambda}{2} + \tfrac{\sqrt{\lambda^2-4\mu}}{2}\tfrac{C_1\sinh\frac{1}{2}(\sqrt{\lambda^2-4\mu}\xi)+C_2\cosh\frac{1}{2}(\sqrt{\lambda^2-4\mu}\xi)}{C_1\cosh\frac{1}{2}(\sqrt{\lambda^2-4\mu}\xi)+C_2\sinh\frac{1}{2}(\sqrt{\lambda^2-4\mu}\xi)}\Bigg) + a_2\left(-\tfrac{\lambda}{2} + \tfrac{\sqrt{\lambda^2-4\mu}}{2}\tfrac{C_1\sinh\frac{1}{2}(\sqrt{\lambda^2-4\mu}\xi)+C_2\cosh\frac{1}{2}(\sqrt{\lambda^2-4\mu}\xi)}{C_1\cosh\frac{1}{2}(\sqrt{\lambda^2-4\mu}\xi)+C_2\sinh\frac{1}{2}(\sqrt{\lambda^2-4\mu}\xi)}\right)^2, \tag{7.4.15}$$

when $\lambda^2 - 4\mu < 0$, obtained solutions are,

$$u_{22}(\xi) = \tfrac{1}{2}a_2(8\mu+\lambda^2) + a_2\lambda\Bigg(-\tfrac{\lambda}{2} + \tfrac{\sqrt{-\lambda^2+4\mu}}{2}\tfrac{-C_1\sin\frac{1}{2}(\sqrt{-\lambda^2+4\mu}\xi)+C_2\cos\frac{1}{2}(\sqrt{-\lambda^2+4\mu}\xi)}{C_1\cos\frac{1}{2}(\sqrt{-\lambda^2+4\mu}\xi)+C_2\sin\frac{1}{2}(\sqrt{-\lambda^2+4\mu}\xi)}\Bigg) + a_2\left(-\tfrac{\lambda}{2} + \tfrac{\sqrt{-\lambda^2+4\mu}}{2}\tfrac{-C_1\sin\frac{1}{2}(\sqrt{-\lambda^2+4\mu}\xi)+C_2\cos\frac{1}{2}(\sqrt{-\lambda^2+4\mu}\xi)}{C_1\cos\frac{1}{2}(\sqrt{-\lambda^2+4\mu}\xi)+C_2\sin\frac{1}{2}(\sqrt{-\lambda^2+4\mu}\xi)}\right)^2, \tag{7.4.16}$$

when $\lambda^2 - 4\mu = 0$, solutions are as follows,

$$u_{23}(\xi) = \frac{1}{2}a_2(8\mu + \lambda^2) + a_2\lambda\Big(-\frac{\lambda}{2} + \frac{C_2}{C_1+C_2\xi}\Big) + a_2\Big(-\frac{\lambda}{2} + \frac{C_2}{C_1+C_2\xi}\Big)^2, \qquad (7.4.17)$$

where

$\xi = kx + \frac{1}{16}$

$-\big(k^3(512k^2\mu^2 + 32k^2\lambda^4 - 256k^2\lambda^2\mu)\int\delta(t)dt + (16a_2\mu^2 + a_2\lambda^4 - 8a_2\lambda^2\mu)\int\gamma(t)dt\big)$.

7.5 Summary and Conclusion

In this chapter, VC fKdV equation is considered by using Lie symmetry analysis method. Similarity reductions and exact solutions based on the Lie group method are obtained by generating the group infinitesimals. Especially, all the similarity reductions and exact solutions are given for the first time in this chapter. The group invariant solutions to the reduced ODEs of VC fKdV are considered based on optimal system. These similarity solutions possess significant features in physical systems and cannot be derived from the method of dynamical systems.

Remark 1: As the nonlinear system (7.1.1) describes, physically important nonlinear equations (Kaup-Kupershmidt (KK), Lax, Sawada-Kotera (SK), CaudreyDoddGibbon (CDG) and Ito equations), so the similarity solutions of these equations can be obtained by substituting particular values of α, β, γ and δ.

Remark 2: Exact explicit solutions of highly nonlinear VC fKdV by using generalized $\big(\frac{G'}{G}\big)$-expansion method are evaluated. These solutions are expressed in terms of the hyperbolic functions, trigonometric functions and rational functions. Here, new exact solutions are found that might be useful for applications in mathematical physics and applied mathematics.

Chapter 8

Nano-Ionic Currents Along MTs In Biosciences [1]

8.1 Introduction

Microtubules (MTs) are important cytoskeleton structures, engaged in a number of specific cellular activities, including vesicular traffic, cell cyto-architecture and mobility, cell division and information processing within neuronal process. MTs have also been implicated in higher neuronal functions, including memory and emergence of "consciousness".

Recently, Wang et al. [166] proposed a new method called the $\left(\frac{G'}{G}\right)$-expansion method to construct traveling wave solutions for NLPDEs. The method is based on the homogeneous balance principle and linear ordinary differential equation (LODE) theory. In this work, the modified $\left(\frac{G'}{G}\right)$-expansion method is used to derive new traveling wave solutions of nonlinear partial differential equations of nano-ionic currents along MTs in biosciences.

The chapter is organized as follows. In section (8.2) and section (8.3), by utilizing the $\left(\frac{G'}{G}\right)$-expansion method to new equations of biosciences, the exact solutions are obtained. In section (8.4), discussion and concluding remarks are recorded.

[1]The contents of this chapter are communicated in *PRAMANA-Journal of Physics* .

8.2 New Solutions of Equation of Nano-Ionic Currents Along MTs

The new nonlinear partial differential equation is studied in this section which describes a model of MT as nonlinear transmission line described by Sekulić et al. [140]. In the contest of this model a MT is segmented into identical elementary rings (ERs) of nano-ionic currents along MTs:

$$\frac{l^3}{3}u_{xxx} + \frac{Z^{\frac{3}{2}}}{l}\left(\Psi G_0 - 2\lambda C_0\right)uu_t + 2u_x + \frac{ZC_0}{l}u_t + \frac{1}{l}\left(RZ^{-1} - G_0 Z\right)u = 0, \quad (8.2.1)$$

where $R = 0.34 \times 10^{19}\Omega$ is the resistance of the ER with length $l = 8 \times 10^{-9}m$, $C_0 = 1.8 \times 10^{-15}F$ is the total maximal capacitance of the ER, $G_0 = 1.1 \times 10^{13}Si$ is conductance of pertaining NPs and $Z = 5.56 \times 10^{10}\Omega$ is the characteristic impedance of our system. Parameters λ and Ψ describe nonlinearity of ER capacitor and conductance of NPs in ER, respectively.

According to the method discussed in chapter 1, using the traveling wave transformation $u(x,t) = u(\xi), \xi = x + kt$, equation (8.2.1) is reduced into an ODE

$$u'''(\xi) + k\beta u(\xi)u'(\xi) + 2\gamma u'(\xi) + \delta u'(\xi) + \nu u(\xi) = 0, \quad (8.2.2)$$

where $\beta = \frac{3Z^{\frac{3}{2}}(\Psi G_0 - 2\lambda C_0)}{l^4}, \gamma = \frac{3}{l^3}, \delta = \frac{3ZC_0}{l^4}$ and $\nu = \frac{\left(RZ^{-1} - G_0 Z\right)}{l^4}$.

From chapter 1, $m = 2$, by noticing the homogeneous balance between the highest order derivatives and nonlinear terms appearing in equation (8.2.2). The solution of (8.2.2) is in the form

$$u(\xi) = \alpha_0 + \alpha_1\left(\frac{G'}{G}\right) + \alpha_2\left(\frac{G'}{G}\right)^2 + \alpha_{-1}\left(\frac{G'}{G}\right)^{-1} + \alpha_{-2}\left(\frac{G'}{G}\right)^{-2}, \quad (8.2.3)$$

where $G = G(\xi)$ satisfies the second order LODE

$$G'' + \mu G = 0 \quad (8.2.4)$$

and $\alpha_0, \alpha_1, \alpha_2, \alpha_{-1}, \alpha_{-2}$ are arbitrary constants.

By substituting equation (8.2.3) into ODE (8.2.2), using the second order ODE (8.2.4) and by collecting the coefficients of all powers of $\left(\frac{G'}{G}\right)$ together, a set of over-determined algebraic equations for $\alpha_0, \alpha_2, \alpha_2, \alpha_{-1}, \alpha_{-2}, \beta, \gamma, \delta, \nu, \mu, k$ is obtained as follows:

$$
\begin{aligned}
&\left(\tfrac{G'}{G}\right)^5 : 24\alpha_2 + 2\beta k\alpha_2^2 = 0,\\
&\left(\tfrac{G'}{G}\right)^4 : 3\beta k\alpha_1\alpha_2 + 6\alpha_1 = 0,\\
&\left(\tfrac{G'}{G}\right)^3 : 40\alpha_2\mu + \beta k\alpha_1^2 + 2\delta k\alpha_2 + 2\beta k\alpha_2^2\mu + 4\gamma\alpha_2 + 2\beta k\alpha_0\alpha_2 = 0,\\
&\left(\tfrac{G'}{G}\right)^2 : 3\beta k\alpha_1\alpha_2\mu + 8\alpha_1\mu + \delta k\alpha_1 - \nu\alpha_2 + 2\gamma\alpha_1 + \beta k\alpha_0\alpha_1 + \beta k\alpha_2\alpha_{-1} = 0,\\
&\left(\tfrac{G'}{G}\right)^1 : 4\gamma\alpha_2\mu - \nu\alpha_1 + 16\alpha_2\mu^2 + 2\beta k\alpha_0\alpha_2\mu + \beta k\alpha_1^2\mu + 2\delta k\alpha_2\mu = 0,\\
&\left(\tfrac{G'}{G}\right)^0 : \beta k\alpha_0\alpha_1\mu + 2\gamma\alpha_1\mu - \beta k\alpha_1\alpha_{-2} + 2\alpha_1\mu^2 + \beta k\alpha_2\alpha_{-1}\mu - \beta k\alpha_0\alpha_{-1} - \nu\alpha_0\\
&\qquad +\delta k\alpha_1\mu - 2\alpha_{-1}\mu - \delta k\alpha_{-1} - 2\gamma\alpha_{-1} = 0,\\
&\left(\tfrac{G'}{G}\right)^{-1} : -2\beta k\alpha_0\alpha_{-2} - 16\alpha_{-2}\mu - \delta\alpha_{-1} - 4\gamma\alpha_{-2} - 2\delta k\alpha_{-2} - \beta k\alpha_{-1}^2 = 0,\\
&\left(\tfrac{G'}{G}\right)^{-2} : -\beta k\alpha_0\alpha_{-1}\mu - \beta k\alpha_1\alpha_{-2}\mu - \delta k\alpha_{-1}\mu - 3\beta k\alpha_{-1}\alpha_{-2} - 8\alpha_{-1}\mu^2 - 2\gamma\alpha_{-1}\mu -\\
&\qquad \delta\alpha_{-2} = 0,\\
&\left(\tfrac{G'}{G}\right)^{-3} : -2\delta k\alpha_{-2}\mu - 40\alpha_{-2}\mu^2 - \beta k\alpha_{-1}^2\mu - 2k\alpha_{-2}^2 - 2\beta k\alpha_0\alpha_{-2}\mu - 4\gamma\alpha_{-2}\mu = 0,\\
&\left(\tfrac{G'}{G}\right)^{-4} : -6\alpha_{-1}\mu^3 - 3\beta k\alpha_{-1}\alpha_{-2}\mu = 0,\\
&\left(\tfrac{G'}{G}\right)^{-5} : -24\alpha_{-2}\mu^3 - 2\beta k\alpha_{-2}^2\mu = 0.
\end{aligned}
\tag{8.2.5}
$$

Solving the system of over determined algebraic equations, the following cases arise:

Case 1. $\alpha_1 = 0, \alpha_2 = 0, \alpha_{-1} = 0, \gamma = \frac{2\mu(-2\beta\alpha_{-2}+3\beta\alpha_0\mu+3\delta\mu)}{\beta\alpha_{-2}}, k = \frac{-12\mu^2}{\beta\alpha_{-2}}, \nu = 0$ and $\beta, \mu, \delta, \alpha_0, \alpha_{-2}$ are arbitrary constants.

Case 2. $\alpha_1 = 0, \alpha_{-1} = 0, \nu = 0, \gamma = \frac{2\mu(-2\beta\alpha_{-2}+3\beta\alpha_0\mu+3\delta\mu)}{\beta\alpha_{-2}}, k = \frac{-12\mu^2}{\beta\alpha_{-2}}, \alpha_2 = \frac{\alpha_{-2}}{\mu^2}$ and $\beta, \alpha_0, \mu, \alpha_{-2}$ are arbitrary constants.

Case 3. $\alpha_1 = 0, \alpha_{-1} = 0, \alpha_{-2} = 0, \beta = \frac{-(8\alpha_2\mu+\delta k\alpha_2-12\alpha_0)}{2\alpha_2}, \nu = 0$ and $\delta, k, \mu, \alpha_0, \alpha_2$ are arbitrary constants.

Corresponding to above three cases, the following two subcases are, when $\mu > 0$ and $\mu < 0$.

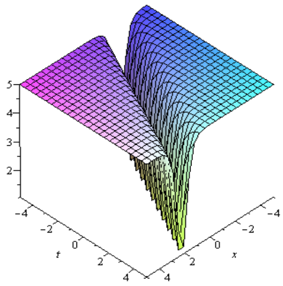

Figure 8.1: Traveling Wave Solution of (8.2.6)

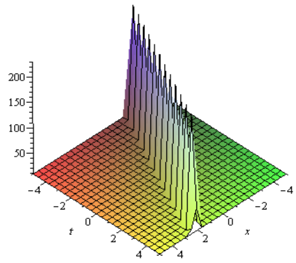

Figure 8.2: Traveling Wave Solution of (8.2.7)

Subcase 1. When $\mu > 0$, the trigonometric function solution of (8.2.2) are expressed as follows:

For Case 1, after substituting the solutions of second order LODE (8.2.4), the solutions of (8.2.2) are:

$$u(\xi) = \alpha_0 + \alpha_{-2}\left(\sqrt{\mu}\left(\frac{A\cos\sqrt{\mu}\xi - B\sin\sqrt{\mu}\xi}{A\sin\sqrt{\mu}\xi + B\cos\sqrt{\mu}\xi}\right)\right)^{-2}. \tag{8.2.6}$$

On choosing the constants $\alpha_0 = 1, \mu = 4, \alpha_{-2} = 16, \beta = 24, A = 1, B = 2$, the traveling waves corresponding to $u(x,t)$ is shown in Fig. 1.

Now for Case 2 and from equation(8.2.4), the obtained solutions are

$$u(\xi) = \alpha_0 + \frac{\alpha_{-2}}{\mu^2}\left(\sqrt{\mu}\left(\frac{A\cos\sqrt{\mu}\xi - B\sin\sqrt{\mu}\xi}{A\sin\sqrt{\mu}\xi + B\cos\sqrt{\mu}\xi}\right)\right)^{2} + \alpha_{-2}\left(\sqrt{\mu}\left(\frac{A\cos\sqrt{\mu}\xi - B\sin\sqrt{\mu}\xi}{A\sin\sqrt{\mu}\xi + B\cos\sqrt{\mu}\xi}\right)\right)^{-2}. \tag{8.2.7}$$

The traveling waves for constants $\alpha_0 = 1, \mu = 4, \alpha_{-2} = 16, \beta = 24, A = 1, B = 2$, corresponding to $u(x,t)$ is shown in Fig. 2.

Finally for Case 3 and by using solutions of(8.2.4), the corresponding solutions of (8.2.2) are:

$$u(\xi) = \alpha_0 + \alpha_2\left(\sqrt{\mu}\left(\frac{A\cos\sqrt{\mu}\xi - B\sin\sqrt{\mu}\xi}{A\sin\sqrt{\mu}\xi + B\cos\sqrt{\mu}\xi}\right)\right)^{2}. \tag{8.2.8}$$

Subcase 2. When $\mu < 0$, the hyperbolic function solutions of equation (8.2.2) are

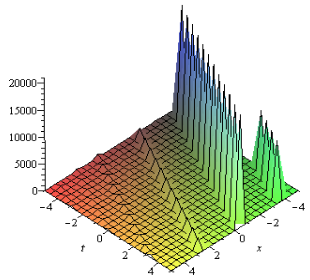

Figure 8.3: Traveling Wave Solution of (8.2.9)

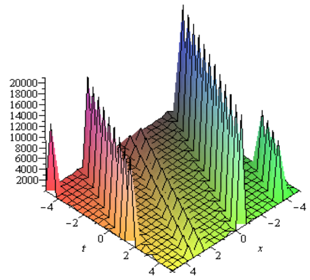

Figure 8.4: Traveling Wave Solution of (8.2.10)

as follows:

For Case 1 after substituting solutions of LODE (8.2.4), the solutions of (8.2.2) are:

$$u(\xi) = \alpha_0 + \alpha_{-2}\left(\sqrt{-\mu}\left(\frac{A\sinh\sqrt{-\mu}\xi + B\cosh\sqrt{-\mu}\xi}{A\cosh\sqrt{-\mu}\xi + B\sinh\sqrt{-\mu}\xi}\right)\right)^{-2}. \tag{8.2.9}$$

On choosing the constants $\alpha_0 = 1, \mu = -4, \alpha_{-2} = 16, \beta = 24, A = 1, B = 2$, the traveling waves corresponding to $u(x,t)$ is shown in Fig. 3.

Now for Case 2 and by using ODE (8.2.4), the solutions of (8.2.2) are derived as:

$$\begin{aligned} u(\xi) = \alpha_0 + \tfrac{\alpha_{-2}}{\mu^2}\left(\sqrt{-\mu}\left(\tfrac{A\sinh\sqrt{-\mu}\xi+B\cosh\sqrt{-\mu}\xi}{A\cosh\sqrt{-\mu}\xi+B\sinh\sqrt{-\mu}\xi}\right)\right)^2 \\ +\alpha_{-2}\left(\sqrt{-\mu}\left(\tfrac{A\sinh\sqrt{-\mu}\xi+B\cosh\sqrt{-\mu}\xi}{A\cosh\sqrt{-\mu}\xi+B\sinh\sqrt{-\mu}\xi}\right)\right)^{-2}. \end{aligned} \tag{8.2.10}$$

The traveling waves for constants $\alpha_0 = 1, \mu = -4, \alpha_{-2} = 16, \beta = 24, A = 1, B = 2$, corresponding to $u(x,t)$ is shown in Fig. 4.

Again for Case 3 and from equation (8.2.4), corresponding solutions of (8.2.2) are:

$$u(\xi) = \alpha_0 + \alpha_2\left(\sqrt{-\mu}\left(\frac{A\sinh\sqrt{-\mu}\xi + B\cosh\sqrt{-\mu}\xi}{A\cosh\sqrt{-\mu}\xi + B\sinh\sqrt{-\mu}\xi}\right)\right)^{2}, \tag{8.2.11}$$

where A and B are arbitrary constants and $\xi = x + kt$, k is arbitrary constant.

8.3 New Solutions of Equation of MTs as Nonlinear RLC Transmission Line

Second equation of MTs as nonlinear transmission line, derived by Satarić et al. [136] is

$$R_2C_0l^2u_{xxt} + l^2u_{xx} + 2R_1C_0\lambda uu_t - R_1C_0u_t = 0 \qquad (8.3.1)$$

Here $l = 8\times10^{-9}m$ and $C_0 = 1.32\times10^{-15}F$ have the same meanings as in equation (8.2.1), while $R_1 = 10^9\Omega$ and $R_2 = 7\times10^6\Omega$ stand for longitudinal and transversal component of resistance of an ER. Again, parameter $\lambda(\lambda < 1)$ describes nonlinearity of ER capacitor in MT.

Now, by using the traveling wave transformation $u(x,t) = u(\xi), \xi = x + kt$, the reduced equation of (8.3.1) into ODE is as follows:

$$ku'''(\xi) + \beta u'' + 2\gamma kuu' - \delta ku' = 0, \qquad (8.3.2)$$

where $\beta = \frac{1}{R_2C_0}, \gamma = \frac{R_1\lambda}{R_2l^2}, \delta = \frac{R_1}{R_2l^2}$.

On balancing highest order derivatives and nonlinear terms appearing in equation (8.3.2), $m = 2$, is obtained. In the same way as in section (8.2), solution of equation (8.3.2) is of the form

$$u(\xi) = \alpha_0 + \alpha_1\left(\frac{G'}{G}\right) + \alpha_2\left(\frac{G'}{G}\right)^2 + \alpha_{-1}\left(\frac{G'}{G}\right)^{-1} + \alpha_{-2}\left(\frac{G'}{G}\right)^{-2}, \qquad (8.3.3)$$

where $G = G(\xi)$ satisfies the second order LODE (8.2.4) and $\alpha_0, \alpha_1, \alpha_2, \alpha_{-1}, \alpha_{-2}$ are arbitrary constants.

By substituting equation (8.3.3) into ODE (8.3.2) and using the ODE (8.2.4), collecting all coefficients of $\left(\frac{G'}{G}\right)$ together, a set of over-determined algebraic equations for $\alpha_0, \alpha_2, \alpha_2, \alpha_{-1}, \alpha_{-2}, \beta, \gamma, \delta, \mu, k$ is as follows:

$$\begin{aligned}
&\left(\tfrac{G'}{G}\right)^5 : 4\gamma k\alpha_2^2 + 24k\alpha_2 = 0,\\
&\left(\tfrac{G'}{G}\right)^4 : 6\gamma k\alpha_1\alpha_2 - 6\beta\alpha_2 + 6k\alpha_1 = 0,\\
&\left(\tfrac{G'}{G}\right)^3 : 40k\alpha_2\mu - 2k\delta\alpha_2 + 2\gamma k\alpha_1^2 + 4\gamma k\alpha_2^2\mu - 2\beta\alpha_1 + 4\gamma k\alpha_0\alpha_2 = 0,\\
&\left(\tfrac{G'}{G}\right)^2 : 2\gamma k\alpha_2\alpha_{-1} - k\delta\alpha_1 - 8\beta\alpha_2\mu + 8k\alpha_1\mu + 6\gamma k\alpha_1\alpha_2\mu + 2\gamma k\alpha_0\alpha_1 = 0,\\
&\left(\tfrac{G'}{G}\right)^1 : -2k\delta\alpha_2\mu + 2\gamma k\alpha_1^2\mu + 4\gamma k\alpha_0\alpha_2\mu - 2\beta\alpha_1\mu + 16k\alpha_2\mu^2 = 0,\\
&\left(\tfrac{G'}{G}\right)^0 : -2\gamma k\alpha_1\alpha_{-2} + k\delta\alpha_{-1} + 2\gamma k\alpha_{-2}\alpha_{-1}\mu - 2\beta\alpha_{-2} - k\delta\alpha_1\mu + 2\gamma k\alpha_0\alpha_1\mu - 2\gamma k\alpha_0\alpha_{-1}\\
&-2\gamma\alpha_2\mu^2 + 2k\alpha_1\mu^2 - 2k\alpha_{-1}\mu = 0,\\
&\left(\tfrac{G'}{G}\right)^{-1} : -16k\alpha_{-2}\mu + 2k\delta\alpha_{-2} - 2\gamma k\alpha_{-1}^2 - 4\gamma k\alpha_0\alpha_{-2} - 2\beta\alpha_{-1}\mu = 0,\\
&\left(\tfrac{G'}{G}\right)^{-2} - 2\gamma k\alpha_1\alpha_{-2}\mu - 2\gamma k\alpha_0\alpha_{-1}\mu + k\delta\alpha_{-1}\mu - 6\gamma k\alpha_{-1}\alpha_{-2} - 8\beta\alpha_{-2}\mu - 8k\alpha_{-1}\mu^2 = 0,\\
&\left(\tfrac{G'}{G}\right)^{-3} : -2\beta\alpha_{-1}\mu^2 - 4\gamma k\alpha_{-2}^2 - 4\gamma k\alpha_0\alpha_{-2}\mu - 2\gamma k\alpha_{-1}^2\mu + 2k\delta\alpha_{-2}\mu - 40k\alpha_{-2}\mu^2 = 0,\\
&\left(\tfrac{G'}{G}\right)^{-4} : -6k\alpha_{-1}\mu^3 - 6\beta\alpha_{-2}\mu^2 - 6\gamma k\alpha_{-1}\alpha_{-2}\mu = 0,\\
&\left(\tfrac{G'}{G}\right)^{-5} : -4\gamma k\alpha_{-2}^2\mu - 24k\alpha_{-2}\mu^3 = 0.
\end{aligned} \tag{8.3.4}$$

Solving the system of over determined algebraic equations (8.3.4), the following cases arise:

Case 1. $\beta = \frac{-5k\alpha_1}{\alpha_2}, \gamma = \frac{-6}{\alpha_2}, \mu = \frac{-\alpha_1^2}{4\alpha_2^2}, \delta = \frac{-3(\alpha_1^2+4\alpha_0\alpha_2)}{\alpha_2^2}, \alpha_{-1} = 0, \alpha_{-2} = 0$ and $k, \alpha_0, \alpha_1, \alpha_2$ are arbitrary constants.

Case 2. $\beta = \frac{-20k\alpha_{-2}}{\alpha_{-1}}, \gamma = \frac{-96\alpha_{-2}^3}{\alpha_{-1}^4}, \mu = \frac{-4\alpha_{-2}^2}{\alpha_{-1}^2}, \delta = \frac{-48\alpha_{-2}^2(\alpha_{-1}^2+4\alpha_0\alpha_{-2})}{\alpha_{-1}^4}, \alpha_1 = 0, \alpha_2 = 0$ and $k, \alpha_0, \alpha_{-1}, \alpha_{-2}$ are arbitrary constants.

Case 3. $\beta = \frac{-80k\alpha_{-2}}{\alpha_{-1}}, \gamma = \frac{-1536\alpha_{-2}^3}{\alpha_{-1}^4}, \mu = \frac{-16\alpha_{-2}^2}{\alpha_{-1}^2}, \delta = \frac{-384\alpha_{-2}^2(\alpha_{-1}^2+8\alpha_0\alpha_{-2})}{\alpha_{-1}^4}, \alpha_1 = \frac{\alpha_{-1}^3}{16\alpha_{-2}^2}, \alpha_2 = \frac{\alpha_{-1}^4}{256\alpha_{-2}^3}$ and $k, \alpha_0, \alpha_{-1}, \alpha_{-2}$ are arbitrary constants.

Corresponding to above six cases, the following two subcases are possible, when $\mu > 0$ and $\mu < 0$.

Subcase 1. When $\mu > 0$,

Substituting the solution set from Case 1 and corresponding solution sets of (8.2.4) into (8.3.3), the solutions of (8.3.2) are as follows:

$$u(\xi) = \alpha_0 + \alpha_1 \left(\sqrt{\mu} \left(\frac{A \cos \sqrt{\mu}\xi - B \sin \sqrt{\mu}\xi}{A \sin \sqrt{\mu}\xi + B \cos \sqrt{\mu}\xi} \right) \right) + \alpha_2 \left(\sqrt{\mu} \left(\frac{A \cos \sqrt{\mu}\xi - B \sin \sqrt{\mu}\xi}{A \sin \sqrt{\mu}\xi + B \cos \sqrt{\mu}\xi} \right) \right)^2 . \tag{8.3.5}$$

Using Case 2 and (8.4.2), the solutions of (8.3.2) are:

$$u(\xi) = \alpha_0 + \alpha_{-1} \left(\sqrt{\mu} \left(\frac{A \cos \sqrt{\mu}\xi - B \sin \sqrt{\mu}\xi}{A \sin \sqrt{\mu}\xi + B \cos \sqrt{\mu}\xi} \right) \right)^{-1} + \alpha_{-2} \left(\sqrt{\mu} \left(\frac{A \cos \sqrt{\mu}\xi - B \sin \sqrt{\mu}\xi}{A \sin \sqrt{\mu}\xi + B \cos \sqrt{\mu}\xi} \right) \right)^{-2} . \tag{8.3.6}$$

Again, for Case 3 and (8.4.2), the corresponding solutions of (8.3.2) are:

$$\begin{aligned} u(\xi) &= \alpha_0 + \alpha_1 \left(\sqrt{\mu} \left(\tfrac{A \cos \sqrt{\mu}\xi - B \sin \sqrt{\mu}\xi}{A \sin \sqrt{\mu}\xi + B \cos \sqrt{\mu}\xi} \right) \right) + \alpha_2 \left(\sqrt{\mu} \left(\tfrac{A \cos \sqrt{\mu}\xi - B \sin \sqrt{\mu}\xi}{A \sin \sqrt{\mu}\xi + B \cos \sqrt{\mu}\xi} \right) \right)^2 \\ &+ \alpha_{-1} \left(\sqrt{\mu} \left(\tfrac{A \cos \sqrt{\mu}\xi - B \sin \sqrt{\mu}\xi}{A \sin \sqrt{\mu}\xi + B \cos \sqrt{\mu}\xi} \right) \right)^{-1} + \alpha_{-2} \left(\sqrt{\mu} \left(\tfrac{A \cos \sqrt{\mu}\xi - B \sin \sqrt{\mu}\xi}{A \sin \sqrt{\mu}\xi + B \cos \sqrt{\mu}\xi} \right) \right)^{-2} . \end{aligned} \tag{8.3.7}$$

Subcase 2. When $\mu < 0$,

corresponding to the Case 1 and from equation (8.2.4), the solutions of (8.3.2) are shown as below:

$$\begin{aligned} u(\xi) &= \alpha_0 + \alpha_1 \left(\sqrt{-\mu} \left(\tfrac{A \sinh \sqrt{-\mu}\xi + B \cosh \sqrt{-\mu}\xi}{A \cosh \sqrt{-\mu}\xi + B \sinh \sqrt{-\mu}\xi} \right) \right) \\ &+ \alpha_2 \left(\sqrt{-\mu} \left(\tfrac{A \sinh \sqrt{-\mu}\xi + B \cosh \sqrt{-\mu}\xi}{A \cosh \sqrt{-\mu}\xi + B \sinh \sqrt{-\mu}\xi} \right) \right)^2 . \end{aligned} \tag{8.3.8}$$

For Case 2 and from second order LODE (8.4.2), the solutions of (8.3.2) are:

$$\begin{aligned} u(\xi) &= \alpha_0 + \alpha_{-1} \left(\sqrt{-\mu} \left(\tfrac{A \sinh \sqrt{-\mu}\xi + B \cosh \sqrt{-\mu}\xi}{A \cosh \sqrt{-\mu}\xi + B \sinh \sqrt{-\mu}\xi} \right) \right)^{-1} \\ &+ \alpha_{-2} \left(\sqrt{-\mu} \left(\tfrac{A \sinh \sqrt{-\mu}\xi + B \cosh \sqrt{-\mu}\xi}{A \cosh \sqrt{-\mu}\xi + B \sinh \sqrt{-\mu}\xi} \right) \right)^{-2} . \end{aligned} \tag{8.3.9}$$

Finally for Case 3 and from solution sets of (8.4.2), the obtained solutions of (8.3.2) are:

$$\begin{aligned} u(\xi) &= \alpha_0 + \alpha_1 \left(\sqrt{-\mu} \left(\tfrac{A \sinh \sqrt{-\mu}\xi + B \cosh \sqrt{-\mu}\xi}{A \cosh \sqrt{-\mu}\xi + B \sinh \sqrt{-\mu}\xi} \right) \right) + \alpha_2 \left(\sqrt{-\mu} \left(\tfrac{A \sinh \sqrt{-\mu}\xi + B \cosh \sqrt{-\mu}\xi}{A \cosh \sqrt{-\mu}\xi + B \sinh \sqrt{-\mu}\xi} \right) \right)^2 \\ &+ \alpha_{-1} \left(\sqrt{-\mu} \left(\tfrac{A \sinh \sqrt{-\mu}\xi + B \cosh \sqrt{-\mu}\xi}{A \cosh \sqrt{-\mu}\xi + B \sinh \sqrt{-\mu}\xi} \right) \right)^{-1} + \alpha_{-2} \left(\sqrt{-\mu} \left(\tfrac{A \sinh \sqrt{-\mu}\xi + B \cosh \sqrt{-\mu}\xi}{A \cosh \sqrt{-\mu}\xi + B \sinh \sqrt{-\mu}\xi} \right) \right)^{-2} , \end{aligned} \tag{8.3.10}$$

where A and B are arbitrary constants and $\xi = x + kt$.

8.4 Discussion

In this chapter, hyperbolic and trigonometric traveling wave solutions of two nonlinear new partial differential equations recently modeled in biosciences, are obtained. The solutions are derived by using modified $\left(\frac{G'}{G}\right)$-expansion method. In order to illustrate the validity and advantages of this algorithm, it is applied to NLPDEs derived in biosciences. The performance of the modified $\left(\frac{G'}{G}\right)$-expansion method is reliable, simple, direct, concise and gives more new exact solutions compared to the other method. This method allow us to solve more complicated NLPDEs. The derived solutions contain free parameters by which the behavior of these solutions are graphically represented in figures by changing the values of these parameters.

Bibliography

[1] Ablowitz M. J., Ramani A. and Segur H., A connection between nonlinear evolution equations and ordinary differential equtions of P-type. 1, Journal of Mathematical Physics 21 (1980) 715-721.

[2] Ablowitz M. J. and Clarkson P. A., Solitons, Nonlinear Evolution Equations and Inverse Scattering, Cambridge University Press, Cambridge, 1991.

[3] Aguilera D. N., Pons J. A. and Miralles J. A., The impact of magnetic field on the thermal evolution of neutron stars, The Astrophysical Journal 673 (2008) L167-L170.

[4] Aharonian F. A., Proton-Synchrotron radiation of large-scale jets in active nuclei, Monthly Notices of The Royal Astronomical Society 332 (2002) 215-230.

[5] Akhatov I. S., Gazizov R. K. and Ibragimov N. K., Nonlocal symmetries. Heuristic approach (in Russian), Sovremennye Problemy Matematiki. Novejshie Dostizheniya 34 (1989) 3-89.

[6] Ali A. T., New exact solutions of the Einstein vacuum equations for the rotating axially symmetric fields, Physica Scripta 79 (2009) 035006-035014.

[7] Asghar S., Mustaq M. and Kara A. H., Exact solutions using symmetry methods and conservation laws for the viscous flow through expanding-contracting channels, Applied Mathematical Modelling 32 (2008) 2936-2940.

[8] Attallah S. K., El-Sabbagh M. F. and Ali A. T., Isovector fields and similarity solutions of Einstein vacuum equations for the rotating fields, Communication in Nonlinear Science and Numerical Simulation 12 (2007) 1153-1161.

[9] Bai C. L. and Zhao H., Complex hyperbolic-function method and its applications to nonlinear equations, Physics Letters A 355 (2006) 32-38.

[10] Bhutani O. P. and Singh K., Generalized similarity solutions for the type D fluid in five-dimensional flat space, Journal of Mathematical Physics 39 (1998) 3203-3212.

[11] Bhutani O. P., Singh K. and Kalra D. K., On certain classes of exact solutions of Einstein equations for rotating fields in conventional and nonconventional form, International Journal of Engineering Science 41 (2003) 769-786.

[12] Bihlo A. and Popovych R. O., Lie symmetry analysis and exact solutions of the quasigeostropic two layer problem, Journal of Mathematical Physics 52 (2011) 033103-1-033103-24.

[13] Biswas A. 1-Soliton solution of the generalized Zakharov Kuznetsov modified equal width equation, Applied Mathematics Letters 22 (2009) 1175-1777.

[14] Biswas A., 1-Soliton solutions of Benjamin-Bona-Mahony equation with dual-power law nonlinearity, Communication in Nonlinear Science and Numerical Simulation 15 (2010) 2744-2746.

[15] Biswas A., Moran A., Milovic D., Majid F. and Biswas K. C., An exact solution for the modified nonlinear Schrödinger's equation for Davydov solitons in alpha-helix proteins, Mathematical Biosciences 227 (2010) 68-71.

[16] Bluman G. W. and Cole J. D., Similarity Methods for Differential Equations, Springer, New York, 1974.

[17] Bluman G. W. and Kumei S., Symmetries and Differential Equations, Springer, New York, 1989.

[18] Bluman G. W. and Anco S. C., Symmetry and Integration Methods for Differential Equations, Springer, New York, 2002.

[19] Bondi H., Pirani F. A. E. and Robinson I., Gravitational waves in general relativity. III. Exact plane waves, Proceedings of the Royal Society A 251 (1959) 519-533.

[20] Bonnor W. B., Certain exact solutions of the equations of general relativity with an electrostatic field, Proceedings of the Physical Society Section A 66 (1953) 145-152.

[21] Bruzon M. S., Gandarias M .L and Camacho J. C., Symmetry analysis and solutions for a generalization of a family of BBM equations, Journal of Nonlinear Mathematical Physics 15 (2008) 81-90.

[22] Butler S. and Joshi N., An inverse scattering transform for the lattice potential KdV equation, Inverse Problems 26 (2010) 115012-115040.

[23] Cai G. and Wang Q., A modified F-expansion method for solving nonlinear PDEs, Journal of Information and Computing Science 2 (2007) 03-16.

[24] Caudrey P. J., Dodd R. K. and Gibbon J. D., A new hierachy of Korteweg-de Vries equations, Proceedings of the Royal Society A 351 (1976) 407-422.

[25] Chevalley C., The Theory of Lie Groups, Princeton University Press, Princeton, 1946.

[26] Chowdhury A. R.. Painlevé Analysis and its Applications, volume 105 of Monographs and Surveys in Pure and Applied Mathematics, Chapman and Hall/CRC, Boca Raton, 2000.

[27] Cicogna G., Cecherini F. and Pegoraro F., Applications of symmetry methods to the theory of plasma physics, Symmetry, Integrability and Geometry: Methods and Applications 2 (2006) 017-034.

[28] Clarkson P. A. and Mansfield E. L., Algorithms for the nonclassical method of symmetry reductions, SIAM Journal of Applied Mathematics 54 (1994) 1693-1719.

[29] Conte R., The Painlevé Property: One Century Later, CRM Series in Mathematical Physics. Springer, New York, 1999.

[30] de Felice F. and Clark C. J. S., Relativity on Manifolds, Cambridge University Press, Cambridge, 1990.

[31] de Felice F., Siming L. and Yunqiang Y., Relativistic charged spheres: II. Regularity and stability, Classical Quantum Gravity 16 (1999) 2669-2680.

[32] Delgaty M. S. R. and Lake K., Physical acceeptability of isolated, static, spherically symmetric, perfect fluid solutions of Einstein's equation, Computer Physics Communications 115 (1998) 395-415.

[33] do Carmo M. P., Riemannian Geometry, Boston: Birkhäuser, 1992.

[34] DoranWu P. R., Extension of Lie's Algorithm: A Potential Symmetries Classification of PDEs, Ph.D thesis: University of British Columbia, Canada, 1996.

[35] Durgapal M. C. and Bannerji R., New analytical stellar model in general relativity, Physics Review D 27 (1983) 328-331.

[36] Durgapal M. C. and Fuloria R. S., Analytic relativistic model for a superdense star, General Relativity and Gravitation 17 (1985) 671-681.

[37] Eddington A. S., The Mathematical Theory of Relativity, Cambridge University Press, Cambridge, 1924.

[38] Einstein A. and Rosen N., On gravitational waves, Journal of Franklin Institute 223 (1937) 43-54.

[39] Finch M. R. and Skea J. E. F., A realistic stellar model based on an ansatz of Duorah and Ray, Classical Quantum Gravity 6 (1989) 467-476.

[40] Gandarias M. L. and Bruzon M. S., Classical and noncalssical symmetries of a generalized Boussinesq equation, Journal of Nonlinear Mathematical Physics 5 (1998) 8-12.

[41] Gandarias M.L. and Bruzon M. S., Nonlocal potential symmetries for the Burger equation, Nonlinear Analysis 71 (2009) 1826-1834.

[42] Garćia-Reyes G. and González G. A., Rotating and counter rotating relativistic thin disks as sources of stationary electrovacuum spacetimes, Brazilian Journal of Physics 37 (2007) 1094-1105.

[43] Gepreel K. A., A generalized $\left(\frac{G'}{G}\right)$-expansion method to find the traveling wave solutions of nonlinear evalution equations, Journal of Partial Differential Equations 24 (2011) 55-69.

[44] Geroch R., A method for generating new solutions of Einstein's equation. II, Journal of Mathematical Physics 13 (1972) 394-404.

[45] Göktas Ü. and Hereman W., Symbolic computation of conserved densities for the system of nonlinear evolution equations, Journal of Symbolic Computation 24 (1997) 591-621.

[46] Goyal N. and Gupta R. K., Symmetries and exact solutions of the nondiagonal Einstein-Rosen metrics, Physica Scripta 85 (2012) 015004-015010.

[47] Goyal N. and Gupta R. K., A class of exact solutions of Einstein field equations, Physica Scripta 85 (2012) 055011-055017.

[48] Goyal N. and Gupta R. K., On symmetries and exact solutions of Einstein vacuum equations for axially symmetric gravitational fields, International Journal of Mathematical and Computational Sciences, 6 (2012) 104-107.

[49] Gupta Y. K. and Jasim M. K., On most general exact solution for Vaidya-Tikekar isentropic superdense star, Astrophysics and Space Science 27 (2004) 403-415.

[50] Gupta Y. K. and Kumar M., On the general solution for a class of charged fluid spheres, General Relativity and Gravitation 37 (2005) 233-236.

[51] Gupta Y. K., Pratibha and Jasim M. K., Similarity solutions for relativistic accelrating fluid plates of embedding class one using symbolic computation, Advanced Studies in Theoretical Physics 4 (2010) 449-466.

[52] Gupta R. K. and Singh K., Similarity analysis and some exact solutions of cylindrically symmetric null fields in general relativity, Communication in Nonlinear Science and Numerical Simulation 16 (2011) 4189-4196.

[53] Harris S. E. and Clarkson P. A., Painlevé analysis and similarity reductions for the Magma equation, Symmetry, Integrability and Geometry: Methods and Applications 2 (2006) 068-085.

[54] Harrison B. K. and Estabrook F. B., Geometric approach to invariance groups and solution of partial differential systems, Journal of Mathematical Physics 12 (1971) 653-666.

[55] Hawking S. and Wand Ellis G. F. R., The Large Scale Structure of Spacetime, Cambridge University Press, Cambridge, 1973.

[56] Hennebelle P. and Teyssier R., Magnetic processes in a collasping dense case. II-fragmentation. Is the fragmentation crisis?, Astronomy and Astrophysics 477 (2008) 25-34.

[57] Hill J. M., Avagiliano A. J. and Edwards M. P., Some exact results for nonlinear diffusions with absorption, IMA Journal of Applied Mathematics 48 (1992) 283-304.

[58] Hill J. M. and Hill D. L., On the derivation of first integrals for similarity solutions, Journal of Engineering Mathematics 25 (1992) 287-299.

[59] Hu H. C., Lou S. Y. and Liu Q. P., Darboux transformation and variable separation approach: the Nizhnik-Novikov-Vesselov equation, Chinese Physics Letters 20 (2003) 1413.

[60] Ibragimov N. H. (Ed.), CRC Handbook of Lie Group Analysis of Differential Equations, CRC Press, Boca Raton, Vol.1, 1994; Vol.2, 1995; Vol.3, 1996.

[61] Ibragimov N .H., Elementary Lie Group Analysis and Ordinary Differential Equations, John Wiley, Chichester, 1999.

[62] Ibrahim A. I., Swank J. H. and Parke W., New evidence of proton-cyclotron resonance in a magnetar strength field from SGR 1806-20, The Astrophysical Journal 584 (2003) L17-L21.

[63] Ibrahim R. S., Truncated Painlevé expansion and reduction of the sine-Poission equation to a quadrature in collisionless old plasma, IMA Journal of Applied Mathematics 68 (2003) 523-535.

[64] Islam J. N., Rotating Fields in General Relativity, Cambridge University Press, Cambridge, 1985.

[65] Ito M., An extension of nonlinear evolution equation of the KdV (mKdV) type of higher orders, Journal of Physical Society of Japan 49 (1980) 771-778.

[66] Ivanov B. V., Static charged perfect fluid spheres in general relativity, Physics Review D 65 (2002) 104001-104018.

[67] Ji A. C., Liu W. M., Song J. L. and Zhou F., Dynamical creation of fractionalized vortices and vortex lattices, Physical Review Letters 101 (2008) 010402.

[68] Jianming L., Jie D. and Wenjum Y., Bäcklund transformation and new exact solutions of the Sharma-Tasso-Olver equation, Abstract and Applied Analysis 2011 (2011) 935710-935718.

[69] Joshi P. S., Global Aspects in Gravitation and Cosmology, Clarendon Press, Oxford, 1993.

[70] Kara A. H., Potential symmetry generators of some perturbe equations, Applied Mathematics and Computations 8 (2003) 89-94.

[71] Kara A. H. and Mahomed F. M., Noether-type symmetries and conservation laws via partial Langrangians, Nonlinear Dynamics 45 (2006) 367-383.

[72] Kaup D. J., On the inverse scattering problem for the cubic eigenvalue problems of the class $\psi_{xxx} + \psi_x + 6R\psi = \lambda\psi$, Studies in Applied Mathematics 62 (1980) 189-216.

[73] Kerr R. P., Gravitational field of a spinning mass as an example of algebraically special metrics, Physical Review Letters 11 (1963) 237-238.

[74] Kheiri H., Moghaddam M. R. and Vafaei V., Application of the $\left(\frac{G'}{G}\right)$-expansion method for the Burgers, BurgersHuxley and modified BurgersKdV equations, PRAMANA- Journal of Physics 76 (2011) 831-842.

[75] Knutsen H., On the Vaidya-Tikekar model for a neutron star, Astrophysics and Space Science 98 (1984) 207-209.

[76] Knutsen H., On the stability and physical properties of an exact relativistic model for a superdense star, Monthly Notices of the Royal Astronomical Society 232 (1988) 163-174.

[77] Kompaneets A. S., The estalishment of thermal equilbrium between quanta and electrons, Soviet Physics-JETP 4 (1957) 730-737.

[78] Krasinski A., Inhomogeneous Cosmological Models, Cambridge University Press, Cambridge, 1997.

[79] Kuperschmidt B. A., A super Korteweg-de Vries equation: An integrable system, Physics letters A 102 (1984) 213-215.

[80] Lake K., All static spherically symmetric perfect fluid solutions of Einstein's equations, Physics Review D 67 (2003) 104015-104019.

[81] Lax P. D., Integrals of nonlinear equations of evalution and solitary waves, Communications on Pure and Applied Mathematics 21 (1968) 467-490.

[82] Lee J. and Sakthivel R., New exact traveling wave solutions of bidirectional wave equations, PRAMANA-Journal of Physics 76 (2011) 819-829.

[83] Lewis T., Some special solutions of the equation of axially symmetric gravitational fields, Proceeding of the Royal Society A 136 (1932) 176-192.

[84] Li H., Wan X., Fu Z. and Liu S., New special structures to the (2+1)-dimensional breaking soliton solutions, Physica Scripta 84 (2011) 035005-035010.

[85] Li X. and Wang M., A sub-ODE method for finding exact solutions of a generalized KdV-mKdV equations with high order nonlinear terms, Physics Letters A 361 (2007) 115-118.

[86] Li Y., Fang J. H. and Zhang K. J., Conformal invariance and a kind of Hojman conserved quantity of the Nambu system, Chinese Physics B 20 (2011) 030201-030207.

[87] Li Z., Liu Y. and Wang M., Exact solitary wave and soliton solutions of the fifth order model equation, Acta Mathematica Scientia 22B (2002) 138-144.

[88] Liang C., A family of cylindrically symmetric solutions to Einstein-Maxwell equations, General Relativity and Gravitation 27 (1971) 669-677.

[89] Liang Z. X., Zhang Z. D. and Liu W. M., Dynamics of a bright soliton in Bose-Einstein condensates with time-dependent atomic scattering length in an explusive parabolic potential, Physical Review Letters (PRL) 94 (2005) 050402-050406.

[90] Lie S., Über die Integration durch bestimmte Integrale von einer Klasse linear partieller Differential gleichung, Archieve der Mathematik 6 (1881) 328-368

[91] Lisle I. G., Equivalence Transformations for a Class of Differential Equations, Ph.D thesis, University of British Columbia, Canada, 1992.

[92] Liu H., Li J. and Liu L., Lie symmetry analysis, optimal systems and exact solutions to the fifth order KdV types of equations, Journal of Mathematical Analysis and Applications 368 (2010) 551-558.

[93] Liu H. and Liu L., Symmetry reductions, variable transformations and exact solutions to the second-order PDEs, Journal of Applied Mathematics and Informatics 29 (2011) 536-572.

[94] Liu W. M., Wu B. and Niu Q., Nonlinear effects in interference of Bose-Einstein condensates, Physical Review Letters 84 (2000) 2294-2297.

[95] Liu Y. T., Shapiro S. L. and Stephnes B. C., Magnetorotational collapse of very massive stars to black holes in full general relativity, Physical Review D 76 (2007) 084017-084034.

[96] Liu X., Zhang W. and Li Z., Applications of improved $\left(\frac{G'}{G}\right)$-expansion method to traveling wave solutions of two nonlinear evolution equations, Advances in Applied Mathematics and Mechanics 4 (2010) 122-130.

[97] Lobanov A. P., Ultracompact jets in active galactic nuclei, Astronomy and Astrophysics 330 (1998) 79-89.

[98] Ma W. X. and Chen M., Direct search for exact solutions to the nonlinear Schrödinger equation, Applied Mathematics and Computation 215 (2009) 2835-2842.

[99] Maharaj S. D. and Leach P. G. L., Exact solutions for the Tikekar superdense star, Journal of Mathematical Physics 37 (1996) 430-437.

[100] Maharaj S. D. and Chaisi M., New anisotropic models from isotropic solutions, Mathematical Methods in Applied Sciences 29 (2006) 67-83.

[101] Mahomed F. M. and Leach P. G. L., The linear symmetries of a nonlinear differential equation, Quaestiones Mathematicae 8 (1985) 241-274.

[102] Mahomed F. M., Symmetry Lie Algebras of nth Order Ordinary Differential Equations, Ph.D thesis, University of the Witwatersrand, Johannesburg, 1989.

[103] Majumdar S. D., A class of exact solutions of Einstein field equations, Physical Review 72 No. 5 (1947) 390-398.

[104] McVittie G. C., Solution with axial symmetry of Einstein equations of telleparallelism, Proceedings of the Edinburg Mathematical Society (Series 2) 2 (1931) 140-150.

[105] Mehra A. L. and Bohra M. L., A solution for a charged sphere in general relativity, General Relativity and Gravitation 11 (1979) 333-336.

[106] Misner C. W., Thorne K. S. and Wheeler J. A., Gravitation, Freeman, San Fransisco, 1973.

[107] Misra M. and Radhakrishna L., Some electromagnetic fields of cylindrically symmetry, Proceedings of National Institute of Sciences of India, Part A 28 (1962) 632-645.

[108] Morozov V. V., Classification of six-dimensional nilpotent Lie algebras (in Russian), Izvestiya Vyssh, Ucheb, Zaved 5 (1958) 161-171.

[109] Mubarakzyanov G. M., On solvable Lie algebras (in Russian), Izvestiya Vyssh, Ucheb, Zaved 32 (1963) 114-123.

[110] Mubarakzyanov G. M., The classification of the real structure of five-dimensional Lie algebras (in Russian), Izvestiya Vyssh, Ucheb, Zaved 34 (1963) 99-105.

[111] Mubarakzyanov G. M., The classification of six-dimensional Lie algebras with nilpotent basis element (in Russian), Izvestiya Vyssh, Ucheb, Zaved 35 (1963) 104-116.

[112] Mubarakzyanov G. M., Some theorems on solvable Lie algebras (in Russian), Izvestiya Vyssh, Ucheb, Zaved 55 (1966) 95-98.

[113] Mukherjee S., Paul B. C. and Dadhich N. K., General solution for a relativistic star, Classical Quantum Gravity 14 (1997) 3475-3480.

[114] Murata S., Non-classical symmetry and Riemann invariants, International Journal of Nonlinear Mechanics 41 (2006) 242-246.

[115] Naher H., Abdullah F.A. and Akber M.A., The Exp-function method for new exact solitions of the nonlinear partial differential equations, International Journal of Physical Sciences 6 (2011) 6706-6716.

[116] Naicker V., Andriopoulos K. and Leach P. G. L., Symmetry reductions of a Hamiltonian-Jacobi-Bellman equation arising in financial mathematics, Journal of Nonlinear Mathematical Physics 12 (2005) 268-283.

[117] Noether E., Invariante Variationsprobleme, Nachr. König. Gesell. Wissen. Göttingen, Math-Phys. Kl. 2 (1918) 235-257.

[118] Nordstrom G., On the energy of the gravitational field in Einstein's theory, Verhandlingen der Koninklijke Akademie van Wetenschpappen Afdeeling Natuurkunde 26 (1918) 1201-1208.

[119] Olver P. J., Applications of Lie Groups to Differential Equations, Springer, New York, 1986.

[120] Ovsiannikov L. V., Group Properties of Differential Equations (in Russian), English translation by Bluman G. W., Novosibirsk: USSR Academy of Science, Siberian Section, 1962.

[121] Ovsiannikov L. V., Group Analysis of Differential Equations, Academic Press, New York, 1982.

[122] Parker A. and Dye J. M., Bossinesq-type equations and "Switching" solutions, Proceedings of Institute of Mathematics of NAS of Ukraine 43 (2002) 344-351.

[123] Patel L. K. and Koppar S. S., A charged analogue of the Vaidya-Tikekar solution, Australian Journal of Physics 40 (1987) 441-447.

[124] Patel L. K., Tikekar R. and Sabu M. C., Exact interior solutions for charged fluid spheres, General Relativity and Gravitation 29 (1997) 489-497.

[125] Patera J. and Winternitz P., Subalgebras of real three- and four-dimensional Lie algebras, Journal of Mathematical Physics 18 (1977) 1449-1455.

[126] Pavlov G. G. and Bezchastnov V. G., Once-ionized helium in superstrong magnetic fields, The Astronomy Journal 635 (2005) L61-L64.

[127] Pirani F. A. E., Invariant formulation of gravitational radiation theory, Physical Review 105 (1957) 1089-1099.

[128] Prinari B., Biondini G. and Trubatch A.D., Inverse scattering transform for the multicomponent nonlinear Schrödinger equation with nonzero boundary conditions, Studies in Applied Mathematics 126 (2011) 245-302.

[129] Qian S., Painlevé analysis and symmetry reductions to the strong dispersive DGH equation, International Journal of Nonlinear Science 1 (2006) 119-123.

[130] Qiu Y. and Tian B., Generalized $\left(\frac{G'}{G}\right)$-expansion method and its applications, International Mathematical Forum 6 (2011) 147-157.

[131] Quevedo H., General static axisymmetric solution of Einstein's vacuum field equations in prolate spheroidal coordinates, Physical Review D 39 (1989) 2904-2911.

[132] Rao J. K., Radiating Levi-civita metric, Journal of Physics A: Mathematical and General 4 (1971) 17-20.

[133] Reissner H., Über die eigengravitation des elektrischen fields nach der Einsteinschen theorie, Annalen der Physik 355 (1916) 106-120.

[134] Rogers C. and Ames W. F., Nonlinear Boundary Value Problems in Sciences and Engineering, Academic Press, Boston 1992.

[135] Rosu H. C. and Ojeda-May P., Supersymmetry of FRW barotropic cosmologies, International Journal of Theoretical Physics 45 (2006) 1191-1996.

[136] Satarić M. V., Ilić D. I., Ralević N. and Tuszynski J. A., A nonlinear model of ionic wave propagation along microtubules, European Biophysics Journal 38 (2009) 637-647.

[137] Sawada K. and Kotera T., A method for finding N-soliton solutions of the K.d.V equation and K.d.V-like equations, Progress in Theoretical Physics 51 (1974) 1355-1367.

[138] Schwarzschild K., On the gravitational field of a point mass in Einstein's theory, Sitzungsberichte der Königlich-Preussischen akademie der Wissenschaften (1916a) 189-196.

[139] Schwarzschild K., Über das Gravitationsfeld einer Kugel aus inkompressibler Flüssigkeit nach der Einstein Theorie, Sitzungsberichte der Königlich-Preussischen Akademie der Wissenschaften 24 (1916b) 424-434.

[140] Sekulić D. L., Satarić B. M., Tuszynski J.A. and Satarić M. V., Nonlinear ionic pulses along microtubules, European Physical Journal E: Soft Matter 34 (2011) 49-60.

[141] Sharma R. and Mukherjee S., Her X-1: a quark-diquark star?, Modren Physics Letters A 16 (2001) 1049-1059.

[142] Sharma R. and Mukherjee S., Compact stars: a core-envelope model, Modren Physics Letters A 17 (2002) 2535-2544.

[143] Sharma R., Karmakar S. and Mukherjee S., Maximum mass of a cold compact star, International Journal of Modren Physics D 15 (2006) 405-418.

[144] Shen S., Lie symmetry analysis and Painlevé analysis of the new (2+1)-diamensional KdV equation, Applied Mathematics-A Journal of Chinese Universities B 22 (2007) 207-212.

[145] Singh K. and Gupta R. K., On symmetries and invariance solutions of a coupled KdV system with variable coefficients, International Journal of Mathematics and Mathematical Sciences 23 (2005) 3711-3725.

[146] Singh K. and Gupta R. K., Lie symmetries and exact solutions of a new generalized Hirota-Satsuma coupled KdV system with variable coefficients, International Journal of Engineering Science 44 (2006) 241-255.

[147] Singh K. and Gupta R. K., Exact solutions of a variant Boussinesq system, International Journal of Engineering Science 44 (2006) 1256-1268.

[148] Steeb W. H. and Euler N., Nonlinear Evolution Equations and Painlevé Test, World Scientific, Singapore, 1988.

[149] Steinberg S., Symmetry Methods in Differential Equations, Technical Report No. 367, The University of New Mexico, 1979.

[150] Stephani H., Kramer D., MacCallum M., Hoenselares C. and Herlt E., Exact Solutions of Einstein's Field Equations, Cambridge University Press, Cambridge, 2003.

[151] Stubhaug A., The mathematician Sophus Lie-It was the audacity of my thinking, English translation by Daly R. H., Springer, Berlin, 2002.

[152] Taub A. H., Empty space-times admitting a three parameter group of motions, Annals of Mathematics 53 (1951) 472-490.

[153] Thomas V. O., Ratanpal B. S. and Vinodkumar P. C., Core-envelope models of superdense star with anisotropic envelope, International Journal of Modren Physics D 14 (2005) 85-96.

[154] Tikekar R., A class of static charged dust spheres in general relativity, General Relativity and Gravitation 16 (1984) 445-451.

[155] Tikekar R., Exact model for a relativistic star, Journal of Mathematical Physics 31 (1990) 2454-2458.

[156] Tikekar R. and Thomas V. O., Relativistic fluid sphere on pseudo-spheroidal space- time, Pramana - Journal of Physics 50 (1998) 95-103.

[157] Tikekar R. and Jotania K., Relativistic superdense star models of pseudo spheroidal space-time, International Journal of Modren Physics D 14 (2005) 1037-1048.

[158] Turkowski P., Low-dimensional real Lie algebras, Journal of Mathematical Physics 29 (1988) 2139-2144.

[159] Vaidya P. C. and Tikekar R., Exact relativistic model for a superdense star, Journal of Astrophysics and Astronomy 3 (1982) 325-334.

[160] Verdaguer E., Soliton solutions in spacetimes with two spacelike killing fields, Physics Reports 229 (1993) 1-80.

[161] Vlemmings W. H. T., Diamond P. J. and Imai H., A magnetically collimated jet form an evolved star, Nature 440 (2006) 58-60.

[162] Wafo Soh C., Lie Algebraic Properties of Systems of Differential Equations, Ph.D thesis, University of the Witwatersrand, Johannesburg, 1999.

[163] Wang D., Zhang R. B. and Zhang X., Exact Solutions of non-commutative vacuum field equations and plane gravitational waves, European Physical Journal C 64 (2009) 439-444.

[164] Wang D. S., Ren Y. J. and Zhang H. Q., Further extended sinh-cosh and sine-cosine methods and new travelling wave solutions of the (2+1) dimensional dispersive long wave equations, Applied Mathematics E-Notes, 5 (2005) 157-163.

[165] Wang M., Zhou Y. and Li Z., Applications of a homogenous balance method to exact solutions of nonlinear equations in mathematical physics, Physics Letters A, 216 (1996) 67-75.

[166] Wang M. L., Zhang Z. L. and Li X. Z., The $\left(\frac{G'}{G}\right)$-expansion method and traveling wave solutions of nonlinear evolution equations in mathematical physics, Phys. Lett. A 372 (2008) 417-423.

[167] Wazwaz A. M., Abundant soliton solutions for several forms of the fifth order KdV equation by using the tanh method, Applied Mathematics and Computation 182 (2006) 283-300.

[168] Wazwaz A. M., Soliton solutions for the fifth order KdV equation and the Kawahara equation with the time-dependent coefficients, Physica Scripta 82 (2010) 035009-035013.

[169] Wazwaz A. M, Soliton solutions of the dispersive Sin-Gordon and the dispersive sinh- Gordon equations with fourth spatial or spatio-temporal derivatives, Physica Scripta, 84 (2011) 065007-065011.

[170] Weber J. and Wheeler J. A., Reality of the cylindrical gravitational waves of the Einstein and Rosen, Reviews of Modren Physics 29 (1957) 509-515.

[171] Weyl H., Zur gravitationstheorie, Annalen der Physik 359 (1917) 117-145.

[172] Wilson S. J., Exact solution of a static charged sphere in general relativity, Canadian Journal of Physics 47 (1969) 2401-2404.

[173] Wiltshire R., Isotropy, shear, symmetry and exact solutions for relativistic fluid spheres, Classical Quantum Gravity 23 (2006) 1365-1380.

[174] Wu J., Bilinear transformation for a variable coefficient Kadomtsev Petviashvili equation, Chinese Physics Letters 28 (2011) 060207-060211.

[175] Xu S., He J. and Wang L., The Darboux transformation of the derivative nonlinear Schrödinger equation, Journal of Physics A: Mathematical and Theoretical 44 (2011) 305203-305225.

[176] Zayed E. M. E. and Abdelaziz M. A. M., The modified sine-cosine method and its applications to the generalized $k(n, m)$ and BBM equations with variable coefficients, International Journal of Nonlinear Science 12 (2011) 95-99.

[177] Zedan H. A., Blow-up the symmetry analysis for the Hirota-Satsuma equations, Journal of Contemporary Mathematical Analysis 43 (2008) 108-117.

[178] Zhang S., Dong L., Ba J. M. and Sun Y. N., The $\left(\frac{G'}{G}\right)$-expansion method for for a discrete nonlinear Schrödinger equation, PRAMANA-Journal of Physics 74 (2010) 207-218.

[179] Zhang W., The extended tanh method and exp-function method to solve a kind of nonlinear heat equation, Mathematical Problems in Engineering (2010) 935873-935885.

[180] Zhdanov R., Nonlocal symmetries of systems of evalution equations, Advances in Mathematical Physics 2011 (2011) 456784-456798.

[181] Zhou Y. B. and Li C., Application of modified $\left(\frac{G'}{G}\right)$-expansion method to traveling wave solutions for Whitham Broer Kaup-like equations, Communication in Theoretical Physics 51 (2009) 664-670.

[182] Zhou L., Fu J. L., and Chen B. Y., Lie symmetries and conserved quantities for a two dimensional nonlinear diffusion equation of concentration, Chinese Physics B 19 (2010) 010301-010306.

[183] Zhu S., The extended $\left(\frac{G'}{G}\right)$-expansion method and traveling wave solutions of nonlinear evolution equations, Mathematical and Computational Applications 15 (2010) 924-929.

[184] Zuo J. M. and Zhang Y. M., The Hirota bilinear method for the coupled Burgers equation and the higher-order Boussinesq-Burger equation, Chinese Physics B 20 (2011) 010205-010213.