

**COMBINATORIAL INTERPRETATION OF ROGERS-RAMANUJAN
IDENTITIES AND THEIR ANALOGUES**

*Dissertation submitted in partial fulfillment of the requirement for
the award of the degree of*

Master of Science

in

Mathematics and Computing

Submitted by

Rajni Bansal

Roll No. 300903012

Under the esteemed guidance of

Dr. Meenakshi Rana



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School of Mathematics and Computer Applications

Thapar University

Patiala – 147004 (Punjab)

INDIA

CERTIFICATE

I hereby certify that the work which is being presented in the thesis entitled "Combinatorial Interpretation of Rogers-Ramanujan Identities and their Analogues" in partial fulfillment of the requirements for the award of degree of Master of Science, School of Mathematics and Computer Applications (SMCA), Thapar University, Patiala is an authentic record of my own work carried out under the supervision of Dr. Meenakshi Rana.

The matter presented in this thesis has not been submitted for the award of any other degree of this or any other university.

Rajni Bansal
Rajni Bansal

Reg. No. 300903012

This is to certify that the above statement made by the candidate is correct and true to the best of my knowledge.

Meenakshi
14/7/11
Dr. Meenakshi Rana

Supervisor

Assistant Professor, SMCA

Thapar University, Patiala

Countersigned by:

S.S. Bhatia
Dr. S.S. Bhatia 15.7.11

Professor and Head

SMCA

Thapar University, Patiala

S.K. Mohapatra
Dr. S.K. Mohapatra
Dean of Academic Affairs
Thapar University
Patiala

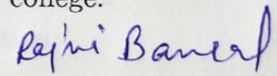
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July 14, 2011


Rajni Bansal.

Abstract

In this thesis, we study about partitions of positive integers. The study of partitions of positive integers has fascinated a number of great mathematicians: Euler, Legendre, Ramanujan, Hardy, Rademacher, Sylvester and Dyson. Here, we start from the basic q -series and then move quickly to one of the most magnificent and surprising results of the entire subject, the Rogers-Ramanujan identities.

In Chapter 1, we give a brief historical background of the subject matter and give some definitions, which we shall be using in this thesis.

In Chapter 2, we give analytic proof of Rogers-Ramanujan identities using Bailey's Lemma and a combinatorial proof of Rogers-Ramanujan identities due of MacMahon.

Finally in Chapter 3, we prove some more q -series identities combinatorially, analogues to Rogers-Ramanujan identities using ordinary partition function.

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CHAPTER I

Introduction and Basic Concepts

1.1 Basic Definitions and Notations

Definition 1.1.1 [4] A partition of a positive integer n is a finite non-increasing sequence of positive integers $\lambda_1, \lambda_2, \dots, \lambda_r$ such that $\sum_{i=1}^r \lambda_i = n$, where λ_i 's are called the parts of the partition.

For example: The partitions of 5 are

$$5, 4+1, 3+2, 3+1+1, 2+2+1, 2+1+1+1, 1+1+1+1+1.$$

Definition 1.1.2 The partition function $p(n)$ is the number of partitions of n .

For example: $p(5)=7$.

Remark. Obviously $p(n)=0$ when n is negative.

The partition function increases rapidly with n .

$$p(6)=11, p(7)=15, p(8)=22, p(9)=30, p(10)=42, \dots$$

Notation. Let $p(S, n)$ denote the number of partitions of n which belong to the set S .

For example: We might consider O the set of all partitions with odd parts.

$$p(O, 5) = 3$$

The relevant partitions are 5, 3+1+1, 1+1+1+1+1.

$$p(O, 7) = 5$$

The relevant partitions are 7, 5+1+1, 3+3+1, 3+1+1+1+1, 1+1+1+1+1+1+1.

1.2 Generating Functions

Definition 1.2.1 The generating function $f(q)$ for the sequence $a_0, a_1, a_2, a_3 \dots$ is

$$f(q) = \sum_{n=0}^{\infty} a_n q^n$$

Definition 1.2.2 [6] The generating function for $p(n)$ is given by

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{(q; q)_{\infty}}$$

where $|q| < 1$ and $(q; q)_{\infty}$ is q -rising factorial defined by

$$(a; q)_n = \prod_{i=0}^{\infty} \frac{(1 - aq^i)}{(1 - aq^{n+i})}$$

If n is a positive integer, then

$$(a; q)_n = (1 - a)(1 - aq)(1 - aq^2) \cdots (1 - aq^{n-1})$$

$$(a; q)_{\infty} = (1 - a)(1 - aq)(1 - aq^2) \cdots .$$

for any constant a .

The generating function for partition into odd parts is given by

$$\sum_{n=0}^{\infty} p(O, n)q^n = \prod_{n=0}^{\infty} \frac{1}{(1 - q^{2n+1})}$$

The generating function for partition into distinct parts is given by

$$\sum_{n=0}^{\infty} p(D, n)q^n = \prod_{n=0}^{\infty} (1 + q^n)$$

Theorem 1.2.1 The number of partitions of n into distinct parts is equal to the number of partitions of n into odd parts.

Theorem 1.2.2 The number of partitions of n in which each part appears two, three or five times equals the number of partitions of n into parts congruent to 2, 3, 6, 9 or 10 modulo 12.

Proof of Theorem 1.2.1

$$\begin{aligned}
\sum_{n=0}^{\infty} p(D,n)q^n &= \prod_{n=0}^{\infty} (1 + q^n) \\
&= \prod_{n=0}^{\infty} \frac{(1 - q^{2n})}{(1 - q^n)} \\
&= \prod_{n=0}^{\infty} \frac{1}{(1 - q^{2n+1})} \\
&= \sum_{n=0}^{\infty} p(O,n)q^n
\end{aligned}$$

comparing the coefficients of q^n on both sides we get;

$$p(O,n) = p(D,n) \text{ for all } n.$$

Proof of Theorem 1.2.2

The Generating function of number of partitions n which each part appears two, three or five times is

$$\begin{aligned}
\prod_{n=0}^{\infty} (1 + q^{2n} + q^{3n} + q^{5n}) &= \prod_{n=0}^{\infty} (1 + q^{2n} + q^{3n}(1 + q^{2n})) \\
&= \prod_{n=0}^{\infty} (1 + q^{2n})(1 + q^{3n}) \\
&= \prod_{n=0}^{\infty} \frac{(1 - q^{4n})(1 - q^{6n})}{(1 - q^{2n})(1 - q^{3n})} \\
&= \prod_{n=0}^{\infty} \frac{(1 - q^{4n})(1 - q^{6n})}{(1 - q^{4n})(1 - q^{4n+2})(1 - q^{6n})(1 - q^{6n+3})} \\
&= \prod_{n=0}^{\infty} \frac{1}{(1 - q^{4n+2})(1 - q^{6n+3})}
\end{aligned}$$

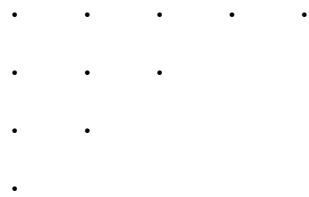
$$= \prod_{n=0}^{\infty} \frac{1}{(1 - q^{12n+2})(1 - q^{12n+3})(1 - q^{12n+6})(1 - q^{12n+9})(1 - q^{12n+10})}$$

is the Generating function of number of partitions of n into parts congruent to 2, 3, 6, 9 or 10 modulo 12.

1.3 Graphical Representation of Partitions and Some Theorems

Definition 1.3.1 [4] The Ferrers graph of a partition $\lambda_1, \lambda_2, \dots, \lambda_r$ of n is a set of rows of equi-spaced dots aligned on the left where the j th row has t_j dots.

For example: The Ferrer graph of the partition $5+3+2+1$ of 11 is,



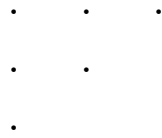
If the graph above is read vertically by columns then this represents the partition



$4+3+2+1+1$ of 11. This new partition is called the *conjugate* of given partition.

Definition 1.3.2 A partition is said to be self conjugate if it is identical with its conjugate.

For example:



The above graph represents the partition $3+2+1$, which is *self conjugate* partition.

Theorem 1.3.1 The number of partitions of n with distinct odd parts is equal to the number of self-conjugate partitions of n .

Theorem 1.3.2 The number of partitions of $a - c$ into exactly $b - 1$ parts, none exceeding c , equals the number of partitions of $a - b$ into $c - 1$ parts, none exceeding b .

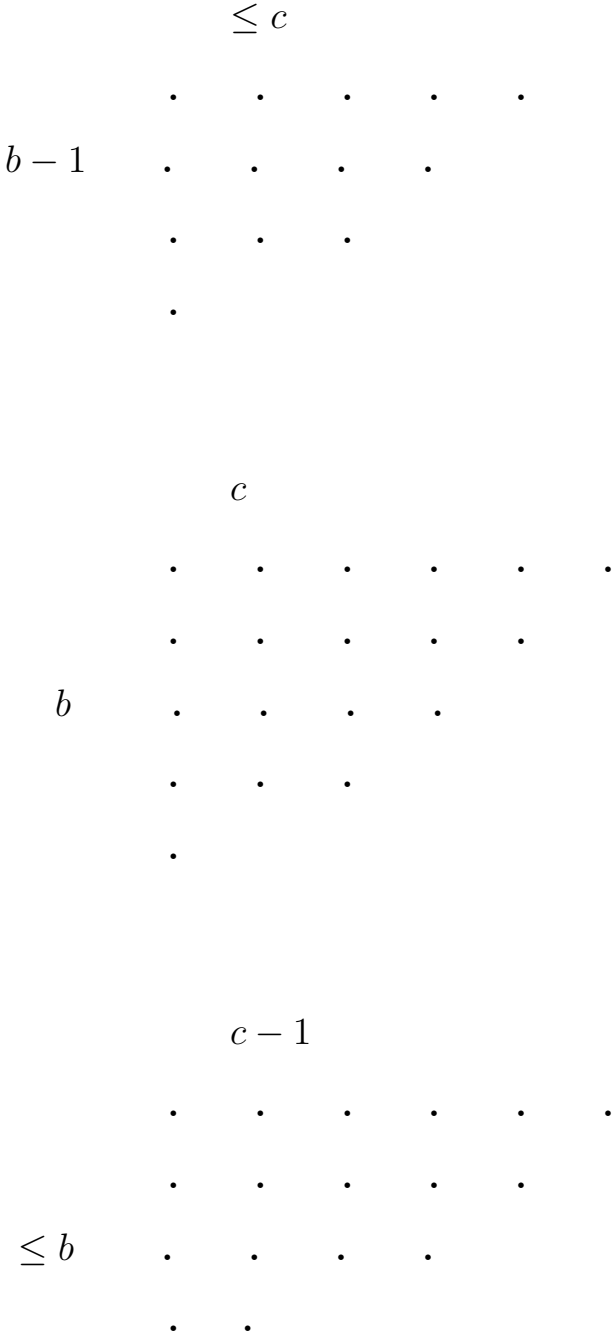
We give the proofs of Theorems 1.3.1 - 1.3.2 in the next section.

1.4 Proof of Theorems 1.3.1 - 1.3.2

Proof of Theorem 1.3.1 Take any partition of n into distinct odd parts. Draw its Ferrer's graph and bent it as right angle with number of nodes in row is equal to number of nodes in column.

The new graph represents self conjugate partition of n .

Proof of Theorem 1.3.2 Let us consider the graphical representation of partition $a-c$ into exactly $b-1$ parts, none exceeding c . We transform the partition as follows; first we adjoin a new top row of c nodes; then we delete the first column (which now has b nodes); and then we take the conjugate:



$$\begin{array}{cccc}
& & \leq b & \\
& \cdot & \cdot & \cdot & \cdot \\
& \cdot & \cdot & \cdot & \cdot \\
c - 1 & \cdot & \cdot & \cdot & \\
& \cdot & \cdot & \cdot & \\
& \cdot & \cdot & & \\
& \cdot & \cdot & & \\
& \cdot & & &
\end{array}$$

We see immediately that this composite transformation provides a one-to-one correspondence between the two types of partitions considered and consequently the theorem is established.

1.5 q-Binomial Theorem, q-Gauss Theorem

Definition 1.5.1 [5] The generalized basic hypergeometric series is defined as:

$${}_r\phi_s \left(\begin{matrix} a_1, a_2, \dots, a_r; z \\ b_1, b_2, \dots, b_s \end{matrix} \right) = \sum_{n=0}^{\infty} \frac{(a_1; q)_n (a_2; q)_n \dots (a_r; q)_n [(-1)^n q^{\frac{1}{2}n(n-1)}]^{1+s-r} z^n}{(b_1; q)_n (b_2; q)_n \dots (b_s; q)_n (q; q)_n}$$

where $q \neq 0$, $r > s + 1$.

Remark 1. The series is said to be *Balanced* if the product of denominator parameters is equal to q times the product of numerator parameters such that

$$b_1 b_2 \dots b_s = q a_1 a_2 \dots a_r.$$

Remark 2. The series is said to be *Well poised* if

$$a_1 q = a_2 b_1 = a_3 b_2 = \dots = a_r b_{s-1}.$$

It is said to be *Very well poised* if in the addition

$$b_1 = -b_2 = \sqrt{a_1}.$$

Theorem 1.5.1(q-binomial theorem) For $|q| < 1$, $|t| < 1$; Prove that

$$\sum_{n=0}^{\infty} \frac{(a; q)_n}{(q; q)_n} t^n = \prod_{n=0}^{\infty} \frac{(1 - atq^n)}{1 - tq^n}$$

Corollary 1.5.1 For $|q| < 1$, $|t| < 1$; Prove that

$$\sum_{n=0}^{\infty} \frac{t^n}{(q; q)_n} = \prod_{n=0}^{\infty} (1 - tq^n)^{-1} = \frac{1}{(t; q)_{\infty}}$$

Corollary 1.5.2 For $|q| < 1$, $|z| < 1$; Prove that

$$\sum_{n=0}^{\infty} \frac{z^n q^{\frac{n(n-1)}{2}}}{(q; q)_n} = \prod_{n=0}^{\infty} (1 + zq^n) = (-z; q)_{\infty}$$

Theorem 1.5.2(Heines Transformation) For $|q| < 1$, $|t| < 1$;

$${}_2\phi_1 \left(\begin{matrix} a, b; t \\ c \end{matrix} \right) = \frac{(b; q)_{\infty} (at; q)_{\infty}}{(c; q)_{\infty} (t; q)_{\infty}} {}_2\phi_1 \left(\begin{matrix} \frac{c}{b}, t; b \\ at \end{matrix} \right)$$

Theorem 1.5.3(q-Gauss Theorem) For $|q| < 1$, $|c| < |ab|$;

$${}_2\phi_1 \left(\begin{matrix} a, b; \frac{c}{ab} \\ c \end{matrix} \right) = \frac{(\frac{c}{a}; q)_{\infty} (\frac{c}{b}; q)_{\infty}}{(\frac{c}{ab}; q)_{\infty} (c; q)_{\infty}}$$

Proof of Theorem 1.5.1

$$F(t) = \prod_{n=0}^{\infty} \frac{(1 - atq^n)}{1 - tq^n} = \sum_{n=0}^{\infty} A_n t^n \tag{1.5.1}$$

We have to prove that;

$$A_n = \frac{(a; q)_n}{(q; q)_n} \tag{1.5.2}$$

$$F(t) = \prod_{n=0}^{\infty} \frac{(1 - atq^n)}{(1 - tq^n)} = \frac{(1 - at)}{(1 - t)} \prod_{n=1}^{\infty} \frac{(1 - atq^n)}{(1 - tq^n)}$$

put $n = n + 1$;

$$F(t) = \frac{(1 - at)}{(1 - t)} \prod_{n=0}^{\infty} \frac{(1 - atq^{(n+1)})}{(1 - tq^{(n+1)})}$$

$$F(t) = \frac{(1 - at)}{(1 - t)} F(tq)$$

$$(1 - t)F(t) = (1 - at)F(tq) \tag{1.5.3}$$

by (1.5.1),

$$F(t) = \sum_{n=0}^{\infty} A_n t^n$$

put this value in (1.5.3), it becomes;

$$\begin{aligned} (1 - t) \sum_{n=0}^{\infty} A_n t^n &= (1 - at) \sum_{n=0}^{\infty} A_n (tq)^n \\ &= (1 - at) \sum_{n=0}^{\infty} A_n q^n t^n \end{aligned}$$

now equating the powers of t^n , we get;

$$A_n - A_{n-1} = A_n q^n - a q^{n-1} A_{n-1}$$

$$A_n = \frac{(1 - a q^{n-1})}{(1 - q^n)} A_{n-1} \tag{1.5.4}$$

iterating (1.5.4) $n - times$, we get;

$$A_n = \frac{(1 - a q^{n-1})(1 - a q^{n-2}) \cdots (1 - a)}{(1 - q^n)(1 - q^{n-1}) \cdots (1 - q)} A_0.$$

We know that $A_0 = 1$

$$A_n = \frac{(1 - a q^{n-1})(1 - a q^{n-2}) \cdots (1 - a)}{(1 - q^n)(1 - q^{n-1}) \cdots (1 - q)} = \frac{(a; q)_n}{(q; q)_n}$$

put this value in (1.5.1);

$$\prod_{n=0}^{\infty} \frac{(1 - atq^n)}{1 - tq^n} = \sum_{n=0}^{\infty} \frac{(a; q)_n}{(q; q)_n} t^n.$$

Proof of Corollary 1.5.1

Put $a = 0$ in Theorem 1.5.1;

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{(0; q)_n}{(q; q)_n} t^n &= \prod_{n=0}^{\infty} \frac{1}{1 - tq^n} \\ \sum_{n=0}^{\infty} \frac{t^n}{(q; q)_n} &= \prod_{n=0}^{\infty} (1 - tq^n)^{-1} = \frac{1}{(t; q)_{\infty}} \end{aligned}$$

Proof of Corollary 1.5.2

Replace a by $\frac{a}{b}$, t by bz in Theorem 1.5.1;

$$\prod_{n=0}^{\infty} \frac{(\frac{a}{b}; q)_n}{(q; q)_n} (bz)^n = \sum_{n=0}^{\infty} \frac{(1 - azq^n)}{1 - bzq^n} = \frac{(az; q)_{\infty}}{(bz; q)_{\infty}} \quad (1.5.5)$$

Take left hand side of (1.5.5);

$$\begin{aligned} \prod_{n=0}^{\infty} \frac{(\frac{a}{b}; q)_n}{(q; q)_n} (bz)^n &= \prod_{n=0}^{\infty} \frac{(1 - \frac{a}{b})(1 - \frac{a}{b}q) \cdots (1 - \frac{a}{b}q^{n-1})}{(q; q)_n} (bz)^n \\ &= \prod_{n=0}^{\infty} \frac{(b - a)(b - aq) \cdots (b - aq^{n-1})}{(q; q)_n} z^n \end{aligned}$$

Take limit $a \rightarrow -1$ and $b \rightarrow 0$;

$$\begin{aligned} &= \prod_{n=0}^{\infty} \frac{q \cdot q^2 \cdots q^{n-1}}{(q; q)_n} z^n \\ &= \prod_{n=0}^{\infty} \frac{q^{\frac{n(n-1)}{2}}}{(q; q)_n} z^n \end{aligned} \quad (1.5.6)$$

take limit $a \rightarrow -1$ and $b \rightarrow 0$ in third series of (1.5.5);

It becomes

$$(-z; q)_\infty \tag{1.5.7}$$

by (1.5.5) and (1.5.7);

$$\sum_{n=0}^{\infty} \frac{z^n q^{\frac{n(n-1)}{2}}}{(q; q)_n} = \prod_{n=0}^{\infty} (1 + zq^n) = (-z; q)_\infty$$

Proof of Theorem 1.5.3

By theorem (1.5.2), we get;

$${}_2\phi_1 \left(\begin{matrix} a, b; \frac{c}{ab} \\ c \end{matrix} \right) = \frac{(b; q)_\infty (\frac{c}{b}; q)_\infty}{(c; q)_\infty (\frac{c}{ab}; q)_\infty} {}_2\phi_1 \left(\begin{matrix} \frac{c}{b}, \frac{c}{ab}; b \\ \frac{c}{b} \end{matrix} \right) \tag{1.5.8}$$

by definition (1.5.1) of generalized hyper geometric series;

$$\begin{aligned} {}_2\phi_1 \left(\begin{matrix} \frac{c}{b}, \frac{c}{ab}; b \\ \frac{c}{b} \end{matrix} \right) &= \sum_{n=0}^{\infty} \frac{(\frac{c}{b}; q)_n (\frac{c}{ab}; q)_n}{(\frac{c}{b}; q)_n (q; q)_n} b^n \\ &= \sum_{n=0}^{\infty} \frac{(\frac{c}{ab}; q)_n}{(q; q)_n} b^n \\ &= \frac{(\frac{c}{a}; q)_\infty}{(b; q)_\infty} \end{aligned}$$

Put this value in (1.5.8);

$$\begin{aligned} {}_2\phi_1 \left(\begin{matrix} a, b; \frac{c}{ab} \\ c \end{matrix} \right) &= \frac{(b; q)_\infty (\frac{c}{b}; q)_\infty (\frac{c}{a}; q)_\infty}{(c; q)_\infty (\frac{c}{ab}; q)_\infty (b; q)_\infty} \\ &= \frac{(\frac{c}{b}; q)_\infty (\frac{c}{a}; q)_\infty}{(c; q)_\infty (\frac{c}{ab}; q)_\infty}. \end{aligned}$$

CHAPTER II

Bailey Lemma and Rogers-Ramanujan Identities

2.1 Introduction

In this chapter, we prove Rogers-Ramanujan identities analytically using Bailey Lemma. These identities were first discovered by Rogers [16] in 1894 and were rediscovered by Ramanujan in 1913. Ramanujan [15] published a paper in 1919 which contains two proofs (one by Ramanujan and the other by Rogers) and a note by Hardy. After the publication of this paper these identities are known as Rogers-Ramanujan identities.

2.2 Jacobi Triple Product

Theorem 2.2.1(Jacobi triple product) For $|z| \neq 0$, $|q| < 1$;

$$\begin{aligned} \sum_{n=-\infty}^{\infty} z^n q^{n^2} &= \prod_{n=0}^{\infty} (1 - q^{2n+2})(1 + zq^{2n+1})(1 + z^{-1}q^{2n+1}) \\ &= (q^2; q^2)_{\infty} (-zq; q^2)_{\infty} (-z^{-1}q; q^2)_{\infty} \end{aligned}$$

Corollary 2.2.1(Gauss) $\sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} = \prod_{m=1}^{\infty} \frac{1 - q^m}{1 + q^m} = \frac{(q; q)_{\infty}}{(-q; q)_{\infty}}$

Corollary 2.2.2 $\sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}} = \prod_{m=1}^{\infty} \frac{1 - q^{2m}}{1 + q^{2m-1}}$

Corollary 2.2.3 $\sum_{n=-\infty}^{\infty} (-1)^n q^{(2k+1)\frac{n(n+1)}{2} - in} = \sum_{n=0}^{\infty} (-1)^n q^{(2k+1)n\frac{(n+1)}{2} - in} (1 - q^{i(2n+1)})$

$$= \prod_{n=0}^{\infty} (1 - q^{(2k+1)(n+1)})(1 - q^{(2k+1)n+i})(1 - q^{(2k+1)(n+1)-i})$$

2.3 Proof of Jacobi Triple Product

Proof of Theorem 2.2.1

Corollary (1.5.2) of q-binomial theorem is,

$$\sum_{n=0}^{\infty} \frac{t^n q^{\frac{n(n-1)}{2}}}{(q; q)_n} = \prod_{n=0}^{\infty} (1 + tq^n) = (-t; q)_{\infty} \quad (2.3.1)$$

Replace q by q^2 and t by zq ;

$$\sum_{n=0}^{\infty} \frac{z^n q^{n^2}}{(q^2; q^2)_n} = \prod_{n=0}^{\infty} (1 + zq^{2n+1}) = (-zq; q^2)_{\infty} \quad (2.3.2)$$

Now we know that;

$$(a; q)_n = \frac{(a; q)_{\infty}}{(aq^n; q)_{\infty}}.$$

Replace a by q ;

$$(q; q)_n = \frac{(q; q)_{\infty}}{(q^{n+1}; q)_{\infty}}.$$

Replace q by q^2 ;

$$(q^2; q^2)_n = \frac{(q^2; q^2)_{\infty}}{(q^{2n+2}; q^2)_{\infty}}.$$

Put the above value in (2.3.2);

$$\begin{aligned} \prod_{n=0}^{\infty} (1 + zq^{2n+1}) &= \sum_{n=0}^{\infty} \frac{z^n q^{n^2} (q^{2n+2}; q^2)_{\infty}}{(q^2; q^2)_{\infty}} = \frac{1}{(q^2; q^2)_{\infty}} \sum_{n=0}^{\infty} z^n q^{n^2} (q^{2n+2}; q^2)_{\infty} \\ \prod_{n=0}^{\infty} (1 + zq^{2n+1})(1 - q^{2n+2}) &= \sum_{n=-\infty}^{\infty} z^n q^{n^2} (q^{2n+2}; q^2)_{\infty} \end{aligned} \quad (2.3.3)$$

(since $(q^{2n+2}; q^2)_{\infty} = 0$; whenever $n < 0$)

replace t by $-q^{2n+2}$ and q by q^2 in (2.3.1);

$$\sum_{r=0}^{\infty} \frac{(-1)^r q^{r(2n+2)} q^{r(r-1)}}{(q^2; q^2)_r} = (q^{2n+2}; q^2)_{\infty}$$

put this value in (2.3.3);

We get;

$$\begin{aligned} \prod_{n=0}^{\infty} (1 + zq^{2n+1})(1 - q^{2n+2}) &= \sum_{n=-\infty}^{\infty} z^n q^{n^2} \sum_{r=0}^{\infty} \frac{(-1)^r q^{(r^2+2nr+r)}}{(q^2; q^2)_r} \\ &= \sum_{r=-\infty}^{\infty} \frac{(-1)^r (z^{-1}q)^r}{(q^2; q^2)_r} \sum_{n=-\infty}^{\infty} z^{n+r} q^{(n+r)^2}. \end{aligned}$$

Put $n + r = s$;

$$\begin{aligned} &= \sum_{r=-\infty}^{\infty} \frac{(-1)^r (z^{-1}q)^r}{(q^2; q^2)_r} \sum_{s=-\infty}^{\infty} z^s q^{s^2} \\ &= \sum_{r=-\infty}^{\infty} \frac{(-z^{-1}q)^r}{(q^2; q^2)_r} \sum_{s=-\infty}^{\infty} z^s q^{s^2} \end{aligned}$$

Replace t by $-z^{-1}q$, q by q^2 in corollary (1.5.1) of q-bionomial theorem;

$$= \frac{1}{(-z^{-1}q; q^2)_{\infty}} \sum_{s=-\infty}^{\infty} z^s q^{s^2}$$

$$\prod_{n=0}^{\infty} (1 - q^{2n+2})(1 + zq^{2n+1})(1 + z^{-1}q^{2n+1}) = \sum_{s=-\infty}^{\infty} z^s q^{s^2}$$

Now replace s by n ;

$$\begin{aligned} \sum_{n=-\infty}^{\infty} z^n q^{n^2} &= \prod_{n=0}^{\infty} (1 - q^{2n+2})(1 + zq^{2n+1})(1 + z^{-1}q^{2n+1}) \\ &= (q^2; q^2)_{\infty} (-zq; q^2)_{\infty} (-z^{-1}q; q^2)_{\infty} \end{aligned}$$

Proof of corollary 2.2.1

Replace z by -1 in theorem (2.2.1).

$$\sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} = \prod_{n=0}^{\infty} (1 - q^{2n+2})(1 - q^{2n+1})(1 - q^{2n+1})$$

$$\begin{aligned}
&= \prod_{n=1}^{\infty} (1 - q^{2n})(1 - q^{2n-1})(1 - q^{2n-1}) \\
&= \prod_{n=1}^{\infty} (1 - q^n)(1 - q^{2n-1}) \\
&= \prod_{n=1}^{\infty} \frac{(1 - q^n)}{(1 + q^n)}
\end{aligned}$$

Replace n by m ;

$$\sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} = \prod_{m=1}^{\infty} \frac{1 - q^m}{1 + q^m} = \frac{(q; q)_{\infty}}{(-q; q)_{\infty}}$$

Proof of Corollary 2.2.2

Replace q by $q^{\frac{1}{2}}$ then and z by $q^{\frac{1}{2}}$ in Theorem (2.2.1).

we see left hand side of theorem (2.2.1) becomes;

$$\begin{aligned}
\sum_{n=-\infty}^{\infty} q^{\frac{n}{2}} q^{\frac{n^2}{2}} &= \sum_{n=-\infty}^{\infty} q^{\frac{n(n+1)}{2}} \\
&= \sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}} + \sum_{n=-\infty}^{-1} q^{\frac{n(n+1)}{2}}
\end{aligned}$$

replace n by $-n$ in second series;

$$\begin{aligned}
&= \sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}} + \sum_{n=1}^{\infty} q^{\frac{-n(-n+1)}{2}} \\
&= \sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}} + \sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}} \\
&= 2 \sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}} \tag{2.3.4}
\end{aligned}$$

Now we see right hand side of theorem (2.2.1) becomes;

$$\begin{aligned}
\sum_{n=0}^{\infty} (1 - q^{n+1})(1 + q^{n+1})(1 + q^n) &= \sum_{n=0}^{\infty} (1 - q^{2n+2})(1 + q^n) \\
&= 2 \sum_{n=1}^{\infty} (1 - q^{2n}) \sum_{n=2}^{\infty} (1 + q^{n-1}) \\
&= 2 \sum_{n=1}^{\infty} (1 - q^{2n}) \sum_{n=1}^{\infty} (1 + q^n) \\
&= 2 \sum_{n=1}^{\infty} (1 - q^{2n})(1 + q^n) \\
&= 2 \sum_{n=1}^{\infty} \frac{(1 - q^{2n})}{(1 - q^{2n-1})}
\end{aligned} \tag{2.3.5}$$

by (2.3.4) and (2.3.5), we get;

$$2 \sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}} = 2 \sum_{n=1}^{\infty} \frac{(1 - q^{2n})}{(1 - q^{2n-1})}$$

replace n by m ;

$$\sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}} = \sum_{m=1}^{\infty} \frac{(1 - q^{2m})}{(1 - q^{2m-1})}$$

Proof of Corollary 2.2.3

Start with first series of corollary (2.2.3);

$$\begin{aligned}
&\sum_{n=-\infty}^{\infty} (-1)^n q^{(2k+1)n \frac{(n+1)}{2} - in} \\
&= \sum_{n=0}^{\infty} (-1)^n q^{(2k+1)n \frac{(n+1)}{2} - in} + \sum_{n=-\infty}^{-1} (-1)^n q^{(2k+1)n \frac{(n+1)}{2} - in} \\
&= \sum_{n=0}^{\infty} (-1)^n q^{(2k+1)n \frac{(n+1)}{2} - in} + \sum_{n=1}^{\infty} (-1)^n q^{(2k+1)n \frac{(n-1)}{2} + in} \\
&= \sum_{n=0}^{\infty} (-1)^n q^{(2k+1)n \frac{(n+1)}{2} - in} + \sum_{n=0}^{\infty} (-1)^{n+1} q^{(2k+1)n \frac{(n+1)}{2} + i(n+1)}
\end{aligned}$$

$$\begin{aligned}
&= \sum_{n=0}^{\infty} (-1)^n q^{(2k+1)n \frac{(n+1)}{2} - in} (1 + (-1)q^{in+i(n+1)}) \\
&= \sum_{n=0}^{\infty} (-1)^n q^{(2k+1)n \frac{(n+1)}{2} - in} (1 - q^{i(2n+1)})
\end{aligned} \tag{2.3.6}$$

Now, replace q by $q^{k+\frac{1}{2}}$ and z by $-q^{k+\frac{1}{2}-i}$ in theorem (2.2.1),

$$\sum_{n=-\infty}^{\infty} z^n q^{n^2} = \prod_{n=0}^{\infty} (1 - q^{2n+2})(1 + zq^{2n+1})(1 + z^{-1}q^{2n+1})$$

we get;

$$\begin{aligned}
&\sum_{n=-\infty}^{\infty} (-q^{k+\frac{1}{2}-i})^n (q^{k+\frac{1}{2}})^{n^2} \\
&= \prod_{n=0}^{\infty} (1 - (q^{k+\frac{1}{2}})^{2n+2})(1 + (-q^{k+\frac{1}{2}-i})(q^{k+\frac{1}{2}})^{2n+1})(1 + (-q^{k+\frac{1}{2}-i})^{-1}(q^{k+\frac{1}{2}})^{2n+1}) \\
&\Rightarrow \sum_{n=-\infty}^{\infty} (-1)^n (q^{\frac{(2k+1)n}{2} - in + \frac{(2k+1)n^2}{2}}) \\
&= \prod_{n=0}^{\infty} (1 - (q^{(2k+1)(n+1)})(1 - q^{\frac{(2k+1)}{2} - i + \frac{(2k+1)(2n+1)}{2}}))(1 - q^{-\frac{(2k+1)}{2} + i + \frac{(2k+1)(2n+1)}{2}}) \\
&\Rightarrow \sum_{n=-\infty}^{\infty} (-1)^n q^{(2k+1) \frac{n(n+1)}{2} - in} \\
&= \prod_{n=0}^{\infty} (1 - (q^{(2k+1)(n+1)})(1 - q^{\frac{(2k+1)}{2}(2n+2) - i})(1 - q^{\frac{(2k+1)(2n)}{2} + i})) \\
&\Rightarrow \sum_{n=-\infty}^{\infty} (-1)^n q^{(2k+1) \frac{n(n+1)}{2} - in} \\
&= \prod_{n=0}^{\infty} (1 - (q^{(2k+1)(n+1)})(1 - q^{(2k+1)(n+1) - i})(1 - q^{(2k+1)(n) + i}))
\end{aligned} \tag{2.3.7}$$

by (2.3.6) and (2.3.7), we get the result;

$$\begin{aligned}
\sum_{n=-\infty}^{\infty} (-1)^n q^{(2k+1)\frac{n(n+1)}{2}-in} &= \sum_{n=0}^{\infty} (-1)^n q^{(2k+1)n\frac{(n+1)}{2}-in} (1 - q^{i(2n+1)}) \\
&= \prod_{n=0}^{\infty} (1 - q^{(2k+1)(n+1)})(1 - q^{(2k+1)n+i})(1 - q^{(2k+1)(n+1)-i})
\end{aligned}$$

2.4 Bailey's Lemma

Lemma 2.4.1 Let $\{\alpha_n\}$ and $\{\beta_n\}$ be two sequences; a is any constant. If, for $n \geq 0$;

$$\beta_n = \sum_{r=0}^n \frac{\alpha_r}{(q; q)_{n-r} (aq; q)_{n+r}}$$

then prove that

$$\sum_{n=0}^{\infty} \beta_n q^{n^2} a^n = \frac{1}{(aq; q)_{\infty}} \sum_{r=0}^{\infty} q^{r^2} \alpha_r a^r$$

and $\{\alpha_r, \beta_r\}$ is said to be *Bailey's pair*.

Proof of Bailey's lemma

$$\begin{aligned}
\sum_{n=0}^{\infty} \beta_n q^{n^2} a^n &= \sum_{n=0}^{\infty} \sum_{r=0}^n \frac{\alpha_r}{(q; q)_{n-r} (aq; q)_{n+r}} q^{r^2} a^n \\
&= \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} \frac{\alpha_r}{(q; q)_n (aq; q)_{n+2r}} q^{(n+r)^2} a^{n+r} \\
&= \sum_{r=0}^{\infty} \frac{\alpha_r}{(aq; q)_{2r}} q^{r^2} a^r \sum_{n=0}^{\infty} \frac{1}{(q; q)_n (aq^{2r+1}; q)_n} q^{n^2+2nr} a^n \quad (2.4.1)
\end{aligned}$$

We claim that: $\sum_{n=0}^{\infty} \frac{1}{(q; q)_n (aq^{2r+1}; q)_n} q^{n^2+2nr} a^n = \frac{1}{(aq^{2r+1}; q)_{\infty}}$.

By definition (1.5.1) of hypergeometric series;

$${}_2\phi_1 \left(\begin{matrix} \frac{1}{t}, \frac{1}{t} \\ at^2 q^{2r+1} \end{matrix}; at^2 q^{2r+1} \right) = \sum_{n=0}^{\infty} \frac{\left(\frac{1}{t}; q\right)_n \left(\frac{1}{t}; q\right)_n [(-1)^n q^{\frac{1}{2}n(n-1)}]^{1+1-2} (at^2 q^{2r+1})^n}{(aq^{2r+1}; q)_n (q; q)_n}$$

Take $t \rightarrow 0$;

$$\begin{aligned}
\lim_{t \rightarrow 0}^2 \phi_1 \left(\begin{array}{c} \frac{1}{t}, \frac{1}{t}; at^2 q^{2r+1} \\ aq^{2r+1} \end{array} \right) &= \lim_{t \rightarrow 0} \sum_{n=0}^{\infty} \frac{(\frac{1}{t}; q)_n (\frac{1}{t}; q)_n [(-1)^n q^{\frac{1}{2}n(n-1)}]^{1+1-2} (at^2 q^{2r+1})^n}{(aq^{2r+1}; q)_n (q; q)_n} \\
&= \lim_{t \rightarrow 0} \sum_{n=0}^{\infty} \frac{((\frac{1}{t}; q)_n)^2 (at^2 q^{2r+1})^n}{(aq^{2r+1}; q)_n (q; q)_n} \\
&= \lim_{t \rightarrow 0} \sum_{n=0}^{\infty} \frac{((1 - \frac{1}{t})(1 - \frac{1}{t}q) \dots (1 - \frac{1}{t}q^{n-1}))^2 (at^2 q^{2r+1})^n}{(aq^{2r+1}; q)_n (q; q)_n} \\
&= \lim_{t \rightarrow 0} \sum_{n=0}^{\infty} \frac{((t-1)(t-q) \dots (t-q^{n-1}))^2 (at^2 q^{2r+1})^n}{t^{2n} (aq^{2r+1}; q)_n (q; q)_n} \\
&= \sum_{n=0}^{\infty} \frac{(q^{1+2+\dots+n-1})^2 (aq^{2r+1})^n}{(aq^{2r+1}; q)_n (q; q)_n} \\
&= \sum_{n=0}^{\infty} \frac{q^{n(n-1)} (aq^{2r+1})^n}{(aq^{2r+1}; q)_n (q; q)_n} \\
&= \sum_{n=0}^{\infty} \frac{q^{n^2+2nr} a^n}{(aq^{2r+1}; q)_n (q; q)_n}. \tag{2.4.2}
\end{aligned}$$

In Theorem (1.5.3) put $a = \frac{1}{t}$, $b = \frac{1}{t}$, $c = aq^{2r+1}$;

$$\begin{aligned}
\lim_{t \rightarrow 0}^2 \phi_1 \left(\begin{array}{c} \frac{1}{t}, \frac{1}{t}; at^2 q^{2r+1} \\ aq^{2r+1} \end{array} \right) &= \lim_{t \rightarrow 0} \frac{(atq^{2r+1}; q)_{\infty} (atq^{2r+1}; q)_{\infty}}{(at^2 q^{2r+1}; q)_{\infty} (aq^{2r+1}; q)_{\infty}} \\
&= \frac{1}{(aq^{2r+1}; q)_{\infty}}. \tag{2.4.3}
\end{aligned}$$

By (2.4.2) and (2.4.3) claim is established.

Put this value in (2.4.1),

$$\begin{aligned} \sum_{n=0}^{\infty} \beta_n q^{r^2} a^n &= \sum_{r=0}^{\infty} \frac{\alpha_r}{(aq; q)_{2r}} q^{r^2} a^r \frac{1}{(aq^{2r+1}; q)_{\infty}} \\ &= \frac{1}{(aq; q)_{\infty}} \sum_{r=0}^{\infty} q^{r^2} \alpha_r a^r \end{aligned}$$

2.5 Rogers-Ramanujan Identities (RRI)

The following two identities are known as Rogers-Ramanujan Identities:

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q; q)_n} = \prod_{n=1}^{\infty} (1 - q^{5n-1})^{-1} (1 - q^{5n-4})^{-1} \quad (2.5.1)$$

$$\sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q; q)_n} = \prod_{n=1}^{\infty} (1 - q^{5n-2})^{-1} (1 - q^{5n-3})^{-1} \quad (2.5.2)$$

Analytical proof of (2.5.1) and (2.5.2) is due to Rogers [16] and were rediscovered by Ramanujan in [15]. MacMohan [13] gave the following combinatorial interpretations of (2.5.1) and (2.5.2), respectively.

Theorem 2.5.1 The number of partitions of n into parts with minimum difference 2 equals the number of partitions of n into parts $\equiv 1$ or $4 \pmod{5}$.

Theorem 2.5.2 The number of partitions of n into parts greater than equals 2 with minimum difference 2 equals the number of partitions of n into parts $\equiv 2$ or $3 \pmod{5}$.

In the next sections we give complete proofs of Rogers-Ramanujan Identities analytically and combinatorially.

2.6 Analytical Proof of RRI

Proof of Identity (2.5.1)

We use (2.4.1) Bailey's lemma as;

$$a = 1, \beta_n = \frac{1}{(q; q)_n},$$

$$\alpha_n = \begin{pmatrix} 1 & \text{if } n = 0 \\ (-1)^n q^{\frac{n(3n-1)}{2}} (1 + q^n) & \text{if } n > 0 \end{pmatrix}$$

we get,

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q; q)_n} = \frac{1}{(aq; q)_{\infty}} \sum_{r=0}^{\infty} q^{r^2} \alpha_r \quad (2.6.1)$$

Take right hand side of (2.6.1),

$$\begin{aligned} \frac{1}{(q; q)_{\infty}} \sum_{r=0}^{\infty} q^{r^2} \alpha_r &= \frac{1}{(q; q)_{\infty}} \left(1 + \sum_{r=1}^{\infty} q^{r^2} \alpha_r \right) \\ &= \frac{1}{(q; q)_{\infty}} \left(1 + \sum_{r=1}^{\infty} q^{r^2} (-1)^r q^{\frac{r(3r-1)}{2}} (1 + q^r) \right) \\ &= \frac{1}{(q; q)_{\infty}} \left(1 + \sum_{r=1}^{\infty} q^{r^2} (-1)^r q^{\frac{r(5r-1)}{2}} + \sum_{r=1}^{\infty} (-1)^r q^{\frac{r(5r+1)}{2}} \right) \end{aligned}$$

we can replace r by $-r$ in second series;

$$\begin{aligned} &= \frac{1}{(q; q)_{\infty}} \left(1 + \sum_{r=1}^{\infty} q^{r^2} (-1)^r q^{\frac{r(5r-1)}{2}} + \sum_{r=-\infty}^{-1} (-1)^r q^{\frac{r(5r-1)}{2}} \right) \\ &= \frac{1}{(q; q)_{\infty}} \sum_{r=-\infty}^{\infty} (-1)^r q^{\frac{r(5r-1)}{2}} \quad (2.6.2) \end{aligned}$$

Now we will use corollary (2.2.3) of Jacobi triple product.

Put $k = 2$, $i = 2$ in corollary (2.2.3);

$$\sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{r(5r-1)}{2}} = \prod_{n=0}^{\infty} (1 - q^{5n+5})(1 - q^{5n+2})(1 - q^{5n+3})$$

put this value in (2.6.2), we get;

$$\begin{aligned} \frac{1}{(q; q)_{\infty}} \sum_{r=0}^{\infty} q^{r^2} \alpha_r &= \frac{1}{(q; q)_{\infty}} \prod_{n=0}^{\infty} (1 - q^{5n+5})(1 - q^{5n+2})(1 - q^{5n+3}) \\ &= \frac{1}{(q; q)_{\infty}} \prod_{n=1}^{\infty} (1 - q^{5n})(1 - q^{5n-3})(1 - q^{5n-2}) \\ &= \prod_{n=1}^{\infty} \frac{1}{(1 - q^{5n-1})(1 - q^{5n-4})} \\ &= \prod_{n=1}^{\infty} (1 - q^{5n-1})^{-1} (1 - q^{5n-4})^{-1} \\ \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q; q)_n} &= \prod_{n=1}^{\infty} (1 - q^{5n-1})^{-1} (1 - q^{5n-4})^{-1} \end{aligned}$$

Proof of Identity (2.5.2)

We use (2.4.1) Bailey's lemma as;

$$a = q, \beta_n = \frac{1}{(q; q)_n}, \alpha_n = \begin{pmatrix} \frac{1}{1-q} & \text{if } n = 0 \\ \frac{(-1)^n q^{\frac{n(3n-5)}{2}} (1 + q^{3n})}{1-q} & \text{if } n > 0 \end{pmatrix}$$

$$\sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q; q)_n} = \frac{1}{(q^2; q)_{\infty}} \sum_{r=0}^{\infty} q^{r^2+r} \alpha_r \quad (2.6.3)$$

take right hand side of (2.6.3) equation;

$$\frac{1}{(q^2; q)_{\infty}} \sum_{r=0}^{\infty} q^{r^2+r} \alpha_r = \frac{1}{(q^2; q)_{\infty}} \left(\frac{1}{1-q} + \sum_{r=1}^{\infty} q^{r^2+r} \alpha_r \right)$$

$$\begin{aligned}
&= \frac{1}{(q^2; q)_\infty} \left(\frac{1}{1-q} + \sum_{r=1}^{\infty} q^{r^2+r} \frac{(-1)^r q^{\frac{r(3r-5)}{2}} (1+q^{3r})}{1-q} \right) \\
&= \frac{1}{(q; q)_\infty} \left(1 + \sum_{r=1}^{\infty} q^{r^2+r} (-1)^r q^{\frac{r(3r-5)}{2}} (1+q^{3r}) \right) \\
&= \frac{1}{(q; q)_\infty} \left(1 + \sum_{r=1}^{\infty} (-1)^r q^{\frac{r(5r-3)}{2}} + \sum_{r=1}^{\infty} (-1)^r q^{\frac{r(5r+3)}{2}} \right)
\end{aligned}$$

Replace r by $-r$ in second series, we get

$$\begin{aligned}
&= \frac{1}{(q; q)_\infty} \left(1 + \sum_{r=1}^{\infty} (-1)^r q^{\frac{r(5r-3)}{2}} + \sum_{r=-\infty}^{-1} (-1)^r q^{\frac{r(5r-3)}{2}} \right) \\
&= \frac{1}{(q; q)_\infty} \sum_{r=0}^{\infty} (-1)^r q^{\frac{r(5r-3)}{2}} \tag{2.6.4}
\end{aligned}$$

Now we will use corollary (2.2.3) of Jacobi triple product.

Put $k = 2$, $i = 1$ in corollary (2.2.3);

$$\sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{n(5n-3)}{2}} = \prod_{n=0}^{\infty} (1 - q^{5n+5})(1 - q^{5n+1})(1 - q^{5n+4})$$

Put this value in (2.6.4) and by (2.6.3), we get;

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q; q)_n} &= \frac{1}{(q; q)_\infty} \prod_{n=0}^{\infty} (1 - q^{5n+5})(1 - q^{5n+1})(1 - q^{5n+4}) \\
&= \frac{1}{(q; q)_\infty} \prod_{n=1}^{\infty} (1 - q^{5n})(1 - q^{5n-4})(1 - q^{5n-1}) \\
&= \prod_{n=1}^{\infty} (1 - q^{5n-2})^{-1} (1 - q^{5n-3})^{-1} \\
\sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q; q)_n} &= \prod_{n=1}^{\infty} (1 - q^{5n-2})^{-1} (1 - q^{5n-3})^{-1}
\end{aligned}$$

2.7 Combinatorial Proof of RRI

Proof of Theorem 2.5.1

$\frac{1}{(q; q)_\infty}$ generates partitions in atmost n parts

or

$\frac{1}{(q; q)_\infty}$ generates partitions into exactly n non-negative parts

Let $a_1 + a_2 + a_3 + \dots + a_n$ be a partition of N generated by $\frac{1}{(q; q)_\infty}$

$$0 \leq a_1 \leq a_2 \leq \dots \leq a_n$$

$$N = a_1 + a_2 + a_3 + \dots + a_n$$

$$b_1 = a_1 + 1, b_2 = a_2 + 3, \dots, b_n = a_n + (2n-1)$$

$$b_1 + b_2 + \dots + b_n = N + (1 + 3 + \dots + (2n - 1))$$

$$b_1 + b_2 + \dots + b_n = N + n^2$$

The number of partitions of $N + n^2$ into n parts with minimum difference 2 between the parts equals the number of partitions of N into atmost n parts

$$= \text{the coefficient of } q^N \text{ in } \frac{1}{(q; q)_\infty}.$$

Replace N by $N - n^2$,

The number of partitions of N into n parts with minimum difference 2 between the parts = the coefficient of q^{N-n^2} in $\frac{1}{(q; q)_\infty}$

$$= \text{the coefficient of } q^N \text{ in } \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q; q)_n}$$

The required generating function for partition with minimum difference 2 =

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q; q)_n}.$$

Proof of Theorem 2.5.2

$\frac{1}{(q; q)_\infty}$ generates partitions in atmost n parts

or

$\frac{1}{(q; q)_\infty}$ generates partitions into exactly n non-negative parts

Let $a_1 + a_2 + a_3 + \dots + a_n$ be a partition of N generated by $\frac{1}{(q; q)_\infty}$

$$0 \leq a_1 \leq a_2 \leq \dots \leq a_n$$

$$N = a_1 + a_2 + a_3 + \dots + a_n$$

$$b_1 = a_1 + 2, b_2 = a_2 + 4, \dots, b_n = a_n + 2n$$

$$b_1 + b_2 + \dots + b_n = N + (2 + 4 + \dots + 2n)$$

$$b_1 + b_2 + \dots + b_n = N + n^2 + n$$

The number of partitions of $N + n^2 + n$ into n parts with minimum difference 2 between the parts and parts ≥ 2 equals to the partition of N into atmost n parts.

$$= \text{the coefficient of } q^N \text{ in } \frac{1}{(q; q)_\infty}.$$

Replace N by $N - n^2 - n$,

The number of partitions of N into n parts with minimum difference 2 between the parts and parts $\geq 2 =$ the coefficient of q^{N-n^2-n} in $\frac{1}{(q; q)_\infty}$

$$= \text{the coefficient of } q^N \text{ in } \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q; q)_n}$$

The required generating function for partition with minimum difference 2 =

$$\sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q; q)_n}.$$

CHAPTER III

Some Results Analogues to Rogers-Ramanujan Identities

3.1 Introduction

Our objective in this chapter is to interpret some q -series identities [14] as generating function for certain ordinary partition function. In the next section we give our main results.

3.2 Main Results

Theorem 3.2.1 The number of partitions of k ,

$$k = a_1 + a_2 + a_3 + \cdots$$

with

$$a_1 > a_2 \geq a_3 > a_4 \geq \cdots$$

is equal to the number of partitions of k into parts which are 1, 3, 4, 5, 7, 9, 11, 13, 15, 16, 17 or 19 mod 20.

Theorem 3.2.2 The number of partitions of k ,

$$k = a_1 + a_2 + a_3 + \cdots$$

with

$$a_1 \geq a_2 > a_3 \geq a_4 > \cdots$$

is equal to the number of partitions of k into parts which are 1, 2, 5, 6, 8, 9, 11, 12, 14, 15, 18 or 19 mod 20.

Theorem 3.2.3 The number of partitions of k ,

$$k = a_1 + a_2 + a_3 + \cdots$$

with

$$a_2 - a_1 \geq 2, a_3 \geq a_2, a_4 - a_3 \geq 2, \cdots$$

is equal to the number of partitions of k into parts which are 1, 4, 6, 7, 9, 10, 12, 15 mod 16.

Theorem 3.2.4 The number of partitions of k ,

$$k = a_1 + a_2 + a_3 + \cdots$$

with

$$a_1 \geq 2, a_2 \geq a_1, a_3 - a_2 \geq 2, a_4 \geq a_3, \cdots$$

is equal to the number of partitions of k into parts which are 2, 3, 4, 5, 11, 12, 13, 14 mod 16.

Our proofs of theorems 3.2.1 – 3.2.4 depend upon the L.J. Slater identities [16].

3.3 Proof of Main Results

Proof of Theorem 3.2.1 The generating function for partition function defined in theorem 3.2.1 is;

$$\sum_{r=0}^{\infty} \frac{q^{r^2}}{(q; q)_{2r}} =$$

$$\prod_{r=0}^{\infty} \frac{1}{(1 - q^{20r+1})(1 - q^{20r+3})(1 - q^{20r+4})(1 - q^{20r+5})(1 - q^{20r+7})(1 - q^{20r+9})(1 - q^{20r+11})(1 - q^{20r+13})(1 - q^{20r+15})(1 - q^{20r+16})(1 - q^{20r+17})(1 - q^{20r+19})}$$

(3.3.1)

Let $p_1(k)$ denote the number of partitions of k ,

$$k = a_1 + a_2 + a_3 + \cdots$$

with

$$a_1 > a_2 \geq a_3 > a_4 \geq \cdots$$

Any such partition of k has, for some $s \geq 1$, $2s - 1$ or $2s$ parts.

Suppose

$$k = a_1 + a_2 + a_3 + \cdots + a_{2s-1}$$

with

$$a_1 > a_2 \geq a_3 > a_4 \geq \cdots \geq a_{2s-1}$$

then

$$a_{2s-1} \geq 1, a_{2s-2} \geq 1, a_{2s-3} \geq 2, \cdots, a_2 \geq s - 1, a_1 \geq s;$$

and if we subtract

$$1 \text{ from } a_{2s-1}, 1 \text{ from } a_{2s-2}, \cdots, s \text{ from } a_1,$$

there remains a partition of $k - s^2$ into at most $2s - 1$ parts.

Similarly if,

$$k = a_1 + a_2 + a_3 + \cdots + a_{2s}$$

with

$$a_1 > a_2 \geq a_3 > a_4 \geq \cdots \geq a_{2s-1} > a_{2s}$$

then

$$a_{2s} \geq 1, a_{2s-1} \geq 2, a_{2s-2} \geq 2, \dots, a_2 \geq s, a_1 \geq s+1;$$

and if we subtract

$$1 \text{ from } a_{2s}, 2 \text{ from } a_{2s-1}, \dots, s+1 \text{ from } a_1.$$

there remains a partition of $k - (s^2 + 2s)$ into at most $2s$ parts. Thus,

$$\begin{aligned} 1 + \sum_{k=1}^{\infty} p_1(k)q^k &= 1 + \sum_{s=1}^{\infty} \frac{q^{s^2}}{(q; q)_{2s-1}} + \sum_{s=1}^{\infty} \frac{q^{s^2+2s}}{(q; q)_{2s}} \\ &= \sum_{r=0}^{\infty} \frac{q^{r^2}}{(q; q)_{2r}} \end{aligned}$$

by (3.3.1) establishes the theorem.

Proof of Theorem 3.2.2 The generating function for partition function defined in theorem 3.2.2 is;

$$\begin{aligned} \sum_{r=0}^{\infty} \frac{q^{r^2+r}}{(q; q)_{2r+1}} &= \\ &= \prod_{r=0}^{\infty} \frac{1}{(1-q^{20r+1})(1-q^{20r+2})(1-q^{20r+5})(1-q^{20r+6})(1-q^{20r+8})(1-q^{20r+9})} \\ &\quad (1-q^{20r+11})(1-q^{20r+12})(1-q^{20r+14})(1-q^{20r+15})(1-q^{20r+18})(1-q^{20r+19}) \end{aligned} \tag{3.3.2}$$

Let $p_2(k)$ denote the number of partitions of k ,

$$k = a_1 + a_2 + a_3 + \dots +$$

with

$$a_1 \geq a_2 > a_3 \geq a_4 > \dots$$

Any such partition of k has, for some $s \geq 1$, $2s - 1$ or $2s$ parts.

Suppose

$$k = a_1 + a_2 + a_3 + \cdots + a_{2s-1}$$

with

$$a_1 \geq a_2 > a_3 \geq \cdots \geq a_{2s-2} > a_{2s-1}$$

then

$$a_{2s-1} \geq 1, a_{2s-2} \geq 2, a_{2s-3} \geq 2, \cdots, a_2 \geq s, a_1 \geq s;$$

and if we subtract

$$1 \text{ from } a_{2s-1}, 2 \text{ from } a_{2s-2}, \cdots, s \text{ from } a_2, s \text{ from } a_1,$$

there remains a partition of $k - (s^2 + s - 1)$ into at most $2s - 1$ parts.

Similarly if,

$$k = a_1 + a_2 + a_3 + \cdots + a_{2s}$$

with

$$a_1 \geq a_2 > a_3 \geq \cdots > a_{2s-1} \geq a_{2s}$$

then

$$a_{2s} \geq 1, a_{2s-1} \geq 1, a_{2s-2} \geq 2, a_{2s-3} \geq 2, \cdots, a_2 \geq s, a_1 \geq s$$

and if we subtract

$$1 \text{ from } a_{2s}, 1 \text{ from } a_{2s-1}, 2 \text{ from } a_{2s-2}, \cdots, s \text{ from } a_2, s \text{ from } a_1;$$

there remains a partition of $k - (s^2 + s)$ into at most $2s$ parts. Thus,

$$1 + \sum_{k=1}^{\infty} p_2(k)q^k = 1 + \sum_{s=1}^{\infty} \frac{q^{s^2+s-1}}{(q; q)_{2s-1}} + \sum_{s=1}^{\infty} \frac{q^{s^2+s}}{(q; q)_{2s}}$$

$$= \sum_{r=0}^{\infty} \frac{q^{r^2+r}}{(q; q)_{2r+1}}$$

by (3.3.2) establishes the theorem.

Proof of Theorem 3.2.3 The generating function for partition function defined in theorem 3.2.3 is;

$$\sum_{r=0}^{\infty} \frac{q^{2r^2+2r}}{(q; q)_{2r+1}} = \prod_{r=0}^{\infty} \frac{1}{(1 - q^{16r+1})(1 - q^{16r+4})(1 - q^{16r+6})(1 - q^{16r+7})(1 - q^{16r+9})(1 - q^{16r+10})(1 - q^{16r+12})(1 - q^{16r+15})}$$

(3.3.3)

Let $p_3(k)$ denote the number of partitions of k ,

$$k = a_1 + a_2 + a_3 + \cdots +$$

with

$$a_2 - a_1 \geq 2, a_3 \geq a_2, a_4 - a_3 \geq 2, \cdots$$

$$a_1 \geq 1, a_2 \geq 3, a_3 \geq 3, a_4 \geq 5, a_5 \geq 5, \cdots$$

Any such partition of k has, for some $s \geq 1$, $2s - 1$ or $2s$ parts.

Suppose

$$k = a_1 + a_2 + a_3 + \cdots + a_{2s-1}$$

with

$$a_1 \geq 1, a_2 \geq 3, a_3 \geq 3, \cdots, a_{2s-2} \geq 2s - 1, a_{2s-1} \geq 2s - 1$$

and if we subtract

1 from a_1 , 3 from a_2, \dots , $2s - 1$ from a_{2s-2} , $2s - 1$ from a_{2s-1} ,

there remains a partition of $k - (2s^2 - 1)$ into at most $2s - 1$ parts.

Similarly, suppose

$$k = a_1 + a_2 + a_3 + \dots + a_{2s}$$

with

$$a_1 \geq 1, a_2 \geq 3, a_3 \geq 3, \dots, a_{2s-2} \geq 2s - 1, a_{2s-1} \geq 2s - 1, a_{2s} \geq 2s + 1;$$

and if we subtract

$$1 \text{ from } a_1, 3 \text{ from } a_2, \dots, 2s-1 \text{ from } a_{2s-2}, 2s-1 \text{ from } a_{2s-1}, 2s+1 \text{ from } a_{2s}$$

there remains a partition of $k - (2s^2 + 2s)$ into at most $2s$ parts. Thus,

$$\begin{aligned} 1 + \sum_{k=1}^{\infty} p_3(k)q^k &= 1 + \sum_{s=1}^{\infty} \frac{q^{2s^2-1}}{(q; q)_{2s-1}} + \sum_{s=1}^{\infty} \frac{q^{2s^2+2s}}{(q; q)_{2s}} \\ &= \sum_{r=0}^{\infty} \frac{q^{2r^2+2r}}{(q; q)_{2r+1}} \end{aligned}$$

by (3.3.3) establishes the theorem.

Proof of Theorem 3.2.4 The generating function for partition function defined in theorem 3.2.4 is;

$$\begin{aligned} \sum_{r=0}^{\infty} \frac{q^{2r^2}}{(q; q)_{2r}} &= \prod_{r=0}^{\infty} \frac{1}{(1 - q^{16r+2})(1 - q^{16r+3})(1 - q^{16r+4})(1 - q^{16r+5})(1 - q^{16r+11})} \\ &\quad (1 - q^{16r+12})(1 - q^{16r+13})(1 - q^{16r+14}) \end{aligned} \tag{3.3.4}$$

Let $p_4(k)$ denote the number of partitions of k ,

$$k = a_1 + a_2 + a_3 + \cdots$$

with

$$a_1 \geq 2, a_2 \geq a_1, a_3 - a_2 \geq 2, a_4 \geq a_3, \cdots$$

$$a_1 \geq 2, a_2 \geq 2, a_3 \geq 4, a_4 \geq 4, a_5 \geq 6, \cdots$$

Any such partition of k has, for some $s \geq 1$, $2s - 1$ or $2s$ parts.

Suppose

$$k = a_1 + a_2 + a_3 + \cdots + a_{2s-1}$$

with

$$a_1 \geq 2, a_2 \geq 2, a_3 \geq 4, \cdots, a_{2s-2} \geq 2s - 2, a_{2s-1} \geq 2s$$

and if we subtract

$$2 \text{ from } a_1, 2 \text{ from } a_2, \cdots, 2s - 2 \text{ from } a_{2s-2}, 2s \text{ from } a_{2s-1},$$

there remains a partition of $k - (2s^2)$ into at most $2s - 1$ parts.

Similarly, Suppose

$$k = a_1 + a_2 + a_3 + \cdots + a_{2s}$$

with

$$a_1 \geq 2, a_2 \geq 2, a_3 \geq 4, \cdots, a_{2s-2} \geq 2s - 2, a_{2s-1} \geq 2s, a_{2s} \geq 2s$$

if we subtract,

2 from a_1 , 2 from a_2, \dots , $2s - 2$ from a_{2s-2} , $2s$ from a_{2s-1} , $2s$ from a_{2s}

there remains a partition of $k - (2s^2 + 2s)$ into at most $2s$ parts. Thus,

$$\begin{aligned}
 1 + \sum_{k=1}^{\infty} p_4(k)q^k &= 1 + \sum_{s=1}^{\infty} \frac{q^{2s^2}}{(q; q)_{2s-1}} + \sum_{s=1}^{\infty} \frac{q^{2s^2+2s}}{(q; q)_{2s}} \\
 &= \sum_{r=0}^{\infty} \frac{q^{2r^2}}{(q; q)_{2r}}
 \end{aligned}$$

by (3.3.4) establishes the theorem.

3.4 Conclusion

Many more q-series identities similar to Roger-Ramanujan identities may be found in [1], [12], [11], [7], [9], [16], [8]. In this thesis, we have used only ordinary partition theoretic interpretations, but in literature we may find many more combinatorial objects such as coloured partition [2] lattice paths [3] and Frobenius Partitions [10] for studying combinatorial interpretations of some q-series identities.

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