

Group Theoretic Techniques and Their Applications to Some Nonlinear Systems

Thesis

Submitted in fulfillment of the requirements of the degree of

Doctor of Philosophy
in
Mathematics

by
Rajeev

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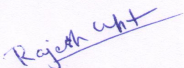


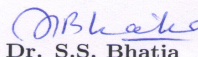
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Certificate

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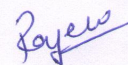
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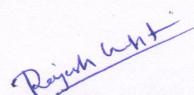
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Dedication

To the memory of my Father

and

To my Mother, my wife Amita and my daughter Varnika
for their love, support and encouragement.

Acknowledgements

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Abstract

The objective of this thesis is to exploit the group theoretic techniques for solving nonlinear partial differential equations (PDEs). Main aim of the present work is to find the symmetries of nonlinear PDEs in order to obtain their exact solutions by using different methods. Five nonlinear systems viz. generalized Kuramoto Sivashinsky equation, generalized Gardner equation, Benny equation, extended Ostrovsky equation and Gowdy equations are investigated for the symmetries and their solutions. Benny equation and extended Ostrovsky equation are also checked for their integrability by Painlevé test.

The present thesis comprises six chapters.

Chapter 1: Some mathematical background in the areas of symmetry analysis, particularly, the Lie point symmetry analysis has been discussed in this chapter. Basics definitions, methodologies utilized in later chapters as well as a brief description of the related studies made by various other researchers in the field are presented in this chapter.

Chapter 2: The variable coefficients form of generalized Kuramoto Sivashinsky (KS) equation

$$u_t + u^m u_x + a(t)u_{xx} + b(t)u_{xxx} + c(t)u_{xxxx} = 0,$$

where $a(t), b(t)$ and $c(t)$ are the arbitrary functions of independent variable t , has been investigated for symmetries by using Lie classical method. The optimal system of subalgebras are utilized to get the reduced ordinary differential equations. Some solutions have been obtained using tanh method.

Chapter 3: This chapter is devoted to the use of combination of Lie group method and $\frac{G'}{G}$ -expansion method to variable coefficients generalized Gardner equation (VCGE)

$$u_t + \alpha(t)u^m u_x + \beta(t)u^{2m}u_x + \theta(t)u_{xx} + \delta(t)u_{xxx} = 0,$$

where $\alpha(t), \beta(t), \theta(t)$ and $\delta(t)$ are arbitrary functions. The similarity reductions and exact solutions are derived by finding the complete sets of point symmetries of these equations. By using $\frac{G'}{G}$ - expansion method, more explicit traveling wave solutions involving arbitrary parameters are obtained. The solutions are expressed in terms of the trigonometric, hyperbolic and rational functions.

Chapter 4: The Benny equation (also called the KdV Burger–Kuramoto equation) with time dependent coefficients

$$u_t + uu_x + \alpha(t)u_{xx} + \beta(t)u_{xxx} + \delta(t)u_{xxxx} = 0,$$

where $\alpha(t), \beta(t)$ and $\delta(t)$ are arbitrary time functions, has been studied for symmetries and to obtained their solutions. Painlevé analysis has been applied to check the integrability of this equation.

Chapter 5: This chapter deals with the study of variable coefficients form of extended Ostrovsky equation (VCEOE):

$$(u_t + cu_x + \alpha(t)uu_x + \alpha_1(t)u^2u_x + \beta(t)u_{xxx})_x - \delta u = 0,$$

where $\alpha(t)$, $\alpha_1(t)$ and $\beta(t)$ are arbitrary time-dependent coefficients. Firstly, Painlevé analysis has been applied to check the integrability of the equation then, auto-Bäcklund transformation have been applied to find the solution of the equation. An investigation of similarity solutions of extended Ostrovsky equation has been undertaken by using Lie classical method and a number of explicit traveling wave solutions involving arbitrary parameters are obtained.

Chapter 6: This chapter is concerned with the Gowdy equation which have been investigated for Lie classical symmetries. Some new solutions of Gowdy equations have been obtained from reduced differential equations.

List of Research Papers

1. Lie Symmetry Analysis and Exact Solutions for a Variable Coefficients Generalised Kuramoto-Sivashinsky Equation,
Romanian Reports in Physics, Vol. 66, No. 4, 2014, 923-928,
SCI expanded, (Impact factor 1.23).
2. Symmetry Analysis and Some Solutions of Gowdy Equations,
Romanian Journal of Physics, Vol. 60, No. 1-2, 2015, 15-21,
SCI expanded, (Impact factor 0.74).
3. Painlevé Analysis and Some Solutions of Variable Coefficients Benny Equation,
Pramana Journal of Physics, Vol. 85, No. 6, 2015, 1111–1122 ,
SCI expanded, (Impact factor 0.72).
4. Invariant Solutions of Variable Coefficients Generalized Gardner Equation,
Nonlinear Dynamics, (in Press) ,
SCI expanded (Impact factor 2.849).
5. Painlevé, Auto-Bäcklund Transformation and Lie Symmetries for the Variable Coefficients Extended Ostrovsky Equation,
(communicated),
SCI expanded (Impact factor 0.726).

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Chapter 1

Introduction

1.1 Literature Survey and Motivation

Theory of a physical system, generally, involves a mathematical model defined by a certain set of differential equations which is supplemented by a set of rules for translating the mathematical results into meaningful descriptions. Physical systems occur in linear or nonlinear forms. Most of the linear system problems can be expressed by linear ordinary differential equations (ODEs) or linear partial differential equations (PDEs). Nonlinear systems are described by nonlinear ODEs or nonlinear PDEs. Solutions of these equations are of much interest from mathematical as well as applications point of view.

A number of mathematical theories have been developed from time to time to solve linear as well as nonlinear physical system problems. The field of PDEs enlightened much in the 18th century with the work of D'Álembert [97] followed by Laplace [164], Fourier [88] and many others. In 1752, D'Álembert introduced one-dimensional linear wave equation to model the problem on vibrating strings. In 1780, Laplace formulated the equation on gravitational fields. The heat equation

appeared in the work of Fourier on heat conduction in 1810. The development in basic sciences and engineering are always the major source of motivation for the growth and expansion of the theories of PDEs. The fundamental equations of physical sciences play significant roles in the advancement of the subject. PDEs are studied for several different prospects but mainly concerning with their solutions.

One of the most powerful general approach for constructing solutions for PDEs is to study their Lie group symmetries. Symmetry leaves the form of the differential equation invariant, but it need not always leave the solution invariant. Sophus Lie [190] initiated the concept of continuous one-parameter group of transformations. These transformations are essentially the flows associated with vector fields defined in the space of independent and dependent variables. These vector fields and their components are called infinitesimal generators and infinitesimals of the point transformations, respectively. The symmetry conditions for these one parameter groups of transformations are linear and these infinitesimals are much easier to analyse. The infinitesimals have been solved explicitly to yield the admitted group of point symmetries. It has become one of the most important tool for the geometric and algebraic studies of nonlinear PDEs and provides an approach based on infinitesimal transformations. An invariance under a one-parameter group of transformations can be utilized to reduce the number of independent variables for any given PDEs. The solutions obtained of PDEs are called similarity solutions or invariant solutions.

The Lie's approach to differential equations was not exploited till Birkhoff [55] reported its applications for the analysis of problems on fluid mechanics. Afterwards, Ovsiannikov and his co-workers [113] applied the Lie continuous group of transformations method to a wide range of problems. Bluman and Cole [70] determined the invariant solutions for boundary value problems. Connections between invariant solutions and separation of variables studied by Miller et al.[215, 216].

After 1960, there have been several generalizations of the Lie symmetry method mainly non classical symmetry method (also referred as the conditional method) by Bluman and Cole [71], direct method by Clarkson and Kruskal [156], differential constraint approach applied by Olver and Rosenau [161, 163], generalized conditional symmetry method given by Fokas and Liu [22]. In nonclassical symmetry method, the conditions for invariance of the system of differential equations under consideration are replaced by the weaker conditions for invariance of the combined system consisting of original differential equations along with the equations requiring the group-invariance solutions. The determining equations for infinitesimals appearing in nonclassical method are nonlinear and smaller in number than those of the classical method. It produces, generally, more solutions than classical method but it is difficult to solve the nonlinear determining equations. The direct method [156], represents a direct, algorithmic, and nongroup theoretic approach for finding symmetry reductions. Levi and Winternitz [41] established that by using a group theoretic explanation, the direct method solutions can also be determined by the nonclassical method. It has been shown that the similarity solutions corresponding to the nonclassical groups should, in general, constitute a larger family than as obtained by the direct method. A generalization of the non classical method is given by the differential constraint approach method. In this method, the original given system of PDEs can be enlarged by appending additional differential constraints, such that the outcome of over determined system of PDEs satisfies conditions of compatibility .

With the passage of time, a number of methods have been proposed to find higher-order symmetries of PDEs. The effective methods include Steinberg's symmetry reduction method [182], the modified direct method [196], Lie-Bäcklund symmetries [146, 179], generalized conditional symmetry method [35, 155], nonlo-

cal symmetry method [73], etc. Most of the methods determined the symmetry reductions of PDEs which fewer the number of independent variables and change PDEs equations to ODEs. The new applications of Lie group theory were being developed by a number of researchers. The monographs and books written by Bluman and Cole [72], Olver [161], Hydon [159], Cantwell [26] and Bluman and Anco [75] are all significant contributions to this field. Gandarias and Bruzón [128, 130, 131], Nucci [121], Anco and Dennis [181], Y.K. Gupta et al. [233, 234], Sharma et al. [207, 208], Hill [94], Singh [107], R. Gupta et al. [108, 148, 172, 173] and Biswas et al. [3, 10] have contributed a lot in this area.

The investigation of infinitesimal symmetries by using the software packages attracted considerable attention of the researchers. The symmetry analysis method is well-suited to computer algebra as it involves systematic algebraic manipulation and tedious calculations. The computer algebra implementation for symmetry analysis started in early 1980s, to determine the symmetries of DEs. The symmetry analysis packages have separated the process into three parts:

- (a) finding determining equations of DEs,
- (b) reducing these determining equations and
- (c) solving these reduced determining equations to get the symmetries.

The chances of solving determining equations increase significantly by the method of reducing the determining equations. Several Differential reduction and completion (DRC) methods [49, 76] such as the reduced involutive form algorithm [76], differential Gröbner basis [218] and Rosenfeld-Gröbner algorithm [52] have been implemented in computer algebra systems like Maple, etc. These computer algebra packages have been widely used for solving nonlinear PDEs.

A variety of powerful methods which are not based on the applications of

group theory, such as the tanh method [17], extended tanh method [116], *sech* method [37], sine-cosine method [15], Hirota method [168], homogeneous balance method [137], Jacobi elliptic function method [85], F-expansion method [138], homotopy perturbation method [38, 204], variational iteration method [90, 92], non-perturbative method [91] extended $\frac{G'}{G}$ -expansion method [184] and modified $\frac{G'}{G}$ -expansion method [1, 220], were developed to determine the exact solutions of PDEs.

In the last decade, the variable coefficients integrable nonlinear evolution equations attracted much attention. A variety of physical problems are explained by partial differential equations with variable coefficients [2, 107, 172]. The investigation of solutions to these partial differential equations are needed to explain the physical phenomena. Some recent analytical and numerical methods for finding the solutions for the variable coefficient of PDEs have been proposed or modified by Moussa et.al [11], Vaneeva et.al [153], Hong [27], Biswas et.al [122], etc.

The present work is devoted to the study of applications of Lie classical method to some specific physical systems, governed by nonlinear PDEs including variable and constant coefficients. Symmetries of all the nonlinear PDEs have been determined which have been applied for the reduction of PDEs into ODEs. The solutions are then deduced with the help of methods mainly $\frac{G'}{G}$ -expansion method [109, 184], tanh method [17, 194], etc. Two of the nonlinear PDEs are also been check by Painlevé test for their integrability.

1.2 Preliminaries

In this section, definitions and fundamentals concerning Lie group of transformations are presented.

1.2.1 Group

A group [75] G is a set of elements with a law of composition ψ between elements satisfying the following axioms:

(i) Closure Property : For any two elements a_1 and a_2 of G
 $\psi(a_1, a_2)$ is an element of G .

(ii) Associative Property: For any elements a_1, a_2, a_3 of G

$$\psi(a_1, \psi(a_2, a_3)) = \psi(\psi(a_1, a_2), a_3).$$

(iii) Identity Element: For any element a_1 of G there exists a unique identity e of G such that

$$\psi(a_1, e) = \psi(e, a_1) = a_1.$$

(iv) Inverse Element: For any element a_1 of G there exists a unique inverse element a_1^{-1} in G such that

$$\psi(a_1, a_1^{-1}) = \psi(a_1^{-1}, a_1) = e.$$

1.2.2 Abelian Group

A group G is Abelian [75] if $\psi(a_1, a_2) = \psi(a_2, a_1)$ holds for all elements a_1 and a_2 in G .

1.2.3 One Parameter Lie Group of Transformations

Let $x = (x_1, x_2, \dots, x_n)$ lies in a region $D \subset \mathfrak{R}^n$. The set of transformations

$$x^* = \Psi(x; \epsilon), \quad (1.2.1)$$

defined for each x in D , depending on parameter ϵ lying in set $S \subset \mathfrak{R}$, with $\psi(\epsilon, \delta)$ defining a law of composition of parameters ϵ and δ in S , forms a one parameter Lie group of transformations [75] on D if:

- (i) For every parameter ϵ in S , the transformations is one-to-one onto in D ; in particular x^* lies in D ,
- (ii) S with the law of composition ψ forms a group,
- (iii) $x^* = x$ if $\epsilon = e$, i.e. $\Psi(x; e) = x$,
- (iv) $x^* = \Psi(x; \epsilon)$, $x^{**} = \Psi(x^*; \delta)$, then $x^{**} = \Psi(x; \psi(\epsilon, \delta))$,
- (v) ϵ is a continuous parameter, i.e., S is an interval in \mathfrak{R} ; without loss of generality, $\epsilon = 0$ corresponds to the identity element e ,
- (vi) Ψ is infinitely differentiable with respect to x in D and an analytic function of ϵ in S and
- (vii) $\psi(\epsilon, \delta)$ is an analytic function of ϵ and δ , $\epsilon \in S$, $\delta \in S$.

1.2.4 Infinitesimal Generators

Consider a one parameter (ϵ) Lie group of transformation (1.2.1) with the identity $\epsilon = 0$ and law of composition ψ . Expanding x^* about $\epsilon = 0$, one gets,

$$x^* = x + \epsilon \left. \frac{\partial \Psi}{\partial \epsilon} \right|_{\epsilon=0} + O(\epsilon^2), \quad (1.2.2)$$

where $\frac{\partial \Psi}{\partial \epsilon} |_{\epsilon=0} = \xi(x)$. The transformation $x^* = x + \epsilon \xi(x)$ is known the infinitesimal transformation of the Lie group of transformations and the component $\xi(x)$ is called the infinitesimals of the transformation [75].

1.2.5 Invariant Surfaces

A surface $f(x)$ is said to be an invariant surface for a one-parameter Lie group of transformations [75] if and only if

$$f(x^*) = f(x) \quad (1.2.3)$$

or equivalently, if and only if $f(x)$ satisfies the condition

$$\sum_i^n \xi_i \frac{\partial f(x)}{\partial (x_i)} = 0. \quad (1.2.4)$$

The operator

$$X = X(x) = \sum_i^n \xi_i \frac{\partial}{\partial (x_i)} \quad (1.2.5)$$

is called the group operator or the infinitesimal generator and Xf is called the Lie derivative of f . The Taylor's series on expanding $f(x^*)$ around $\epsilon = 0$ gives

$$f(x^*) = f(x) + \epsilon Xf(x) + \frac{\epsilon^2}{2!} X^2 f(x) + \cdots + \frac{\epsilon^n}{n!} X^n f(x) + O(\epsilon^{n+1}) \quad (1.2.6)$$

If the series (1.2.6) converges, it is termed as Lie series and

$$f(x^*) = \sum_{n=0}^{\infty} \frac{\epsilon^n}{n!} X^n f(x) \quad (1.2.7)$$

1.2.6 Lie Algebra

If V_1 and V_2 are the infinitesimal generators, then the first order operator known as Commutator (Lie Bracket) of V_1, V_2 , is defined as

$$[V_1, V_2] = V_1V_2 - V_2V_1 \quad (1.2.8)$$

The commutator has the following properties:

(i) Bilinearity:

$$[a_1V_1 + a_2V_2, V_3] = a_1[V_1, V_3] + a_2[V_2, V_3], \quad (1.2.9)$$

$$[V_1, a_1V_2 + a_2V_3] = a_1[V_1, V_2] + a_2[V_1, V_3], \quad (1.2.10)$$

where $a_1, a_2, \in \mathfrak{R}$.

(ii) Skew-Symmetry:

$$[V_1, V_2] = -[V_2, V_1]. \quad (1.2.11)$$

(iii) Jacobi Identity:

$$[V_1, [V_2, V_3]] + [V_3, [V_1, V_2]] + [V_2, [V_3, V_1]] = 0. \quad (1.2.12)$$

The commutator of vector fields is again a vector field. Moreover, if V_i and V_j are two infinitesimal generators of a symmetry transformation, the commutator of both generators will again be a generator of a symmetry group. As a consequence, the set of all infinitesimal generators is closed under commutation of vector fields, thus possessing more structure than just that of vector space. This additional closure property endows the space of infinitesimal generators with an additional algebraic structure and well known as Lie algebra [75].

After finding some of the infinitesimal generators V_i of an r -parameter Lie group, it may be possible to construct new generators by computing the commutators of the known ones. A common way to visualise the structure of a Lie algebra is the commutator table [161]. Let V_1, V_2, \dots, V_r be basis of a Lie algebra, then its commutator table has (i, j) -th entry $[V_i, V_j]$. Because the commutator is antisymmetric it is sufficient to compute just the part above the diagonal, as $[V_i, V_j] = -[V_j, V_i]$. The commutator table therefore reads:

Table 1.1: Commutator Table

	V_1	V_2	...	V_r
V_1	0	$[V_1, V_2]$...	$[V_1, V_r]$
V_2	$-[V_1, V_2]$	0	...	$[V_2, V_r]$
...
V_r	$-[V_1, V_r]$	$-[V_2, V_r]$...	0

1.2.7 Classification of Subalgebras and Group

Invariant Solutions

Classification of subgroups [75] of Lie symmetry groups of differential equations is an important part in the study of the differential equations. This is because classification allows for an efficient computation of group-invariant solutions, without the possibility of an occurrence of equivalent solutions. Classifying subgroups further leads to the construction of simple ansätze for the corresponding equivalence classes of reduced differential equations. Thereby, the classification also provides an important step for further investigations of properties of these reduced equations.

The classification of subgroups of symmetry groups is usually done by the classification of the associated Lie subalgebras with respect to the adjoint representation

[113, 161] and the Lie series is use to compute the adjoint representation

$$Ad(\exp(\epsilon v))w_0 = w_0 - \epsilon[v, w_0] + \frac{\epsilon^2}{2}[v, [v, w_0]] + \dots \quad (1.2.13)$$

The classification of one-dimensional subalgebras of the whole symmetry algebra is done by an inductive approach. If V_1, V_2, \dots, V_r are elementary basis of Lie algebra, then we start with the most general infinitesimal generator,

$$V = a_1V_1 + a_2V_2 + a_3V_3 + \dots + a_rV_r, \quad (1.2.14)$$

and simplify it as much as possible by means of adjoint actions. Depending on the respective values of the coefficients $a_i, i = 1, \dots, r$, we will find the list of inequivalent one-dimensional subalgebras. On using the inequivalent one-dimensional subalgebras of the maximal Lie invariance algebra, group invariant reductions can easily carried out which corresponds to group invariant solutions of studied equations.

1.2.8 Extension of the One-Parameter Groups

Prolongation Formulas:

Consider a k^{th} order PDE by

$$F(x, u, \partial u, \dots, \partial^k u) = 0, \quad (1.2.15)$$

where $x = (x_1, x_2, \dots, x_n)$, u denotes the dependent variable and $\partial^j u$ denotes the set of coordinates corresponding to all j^{th} order partial derivatives of u with respect to independent variable x . The one-parameter Lie group of point transformations

$$x_i^* = \Psi_i(x, u; \epsilon) = x_i + \epsilon \xi_i(x, u) + O(\epsilon)^2, \quad (1.2.16)$$

$$u^* = U(x, u; \epsilon) = u + \epsilon \eta(x, u) + O(\epsilon)^2, \quad (1.2.17)$$

$i = 1, 2, \dots, n$ acting on (x, u) - space, has infinitesimal generator

$$X = \xi_i(x, u) \frac{\partial}{\partial x_i} + \eta(x, u) \frac{\partial}{\partial u}. \quad (1.2.18)$$

The k th extension [75] of (1.2.16) and (1.2.17) is given by

$$x_i^* = \Psi_i(x, u; \epsilon) = x_i + \epsilon \xi_i(x, u) + O(\epsilon^2), \quad (1.2.19)$$

$$u^* = U(x, u; \epsilon) = u + \epsilon \eta(x, u) + O(\epsilon^2), \quad (1.2.20)$$

$$u_i^* = U_i(x, u, \partial u; \epsilon) = u_i + \epsilon \eta_i^{(1)}(x, u, \partial u) + O(\epsilon^2), \quad (1.2.21)$$

\vdots

$$\begin{aligned} u_{i_1 i_2 \dots i_k}^* &= U_{i_1 i_2 \dots i_k}(x, u, \partial u, \dots, \partial^k u) \\ &= u_{i_1 i_2 \dots i_k} + \epsilon \eta_{i_1 i_2 \dots i_k}^{(k)}(x, u, \partial u, \dots, \partial^k u) + O(\epsilon^2), \end{aligned} \quad (1.2.22)$$

where $i = 1, 2, \dots, n$ and $i_l = 1, 2, \dots, n$ for $l = 1, 2, \dots, k$ with $k \geq 1$.

Its k th- extended infinitesimals

$$\xi(x, u), \eta(x, u), \eta_i^{(1)}(x, u, \partial u), \dots, \eta_{i_1 i_2 \dots i_k}^{(k)}(x, u, \partial u, \dots, \partial^k u), \dots \quad (1.2.23)$$

with the corresponding k th-extended infinitesimal generator is given as

$$X^k = \xi_i \frac{\partial}{\partial x_i} + \eta \frac{\partial}{\partial u} + \eta_i^1 \frac{\partial}{\partial u_i} + \dots + \eta_{i_1 i_2 \dots i_k}^{(k)} \frac{\partial}{\partial u_{i_1 i_2 \dots i_k}}. \quad k \geq 1 \quad (1.2.24)$$

Explicit formulas for the extended infinitesimals η^k satisfy the recursion relations:

$$\eta_i^{(1)} = D_i \eta - (D_i \xi_j) u_j, \quad i = 1, 2, \dots, n, \quad (1.2.25)$$

\vdots

$$\eta_{i_1 i_2 \dots i_k}^k = D_{i_k} \eta_{i_1 i_2 \dots i_{k-1}}^{(k-1)} - (D_{i_k} \xi_j) u_{i_1 i_2 \dots i_{k-1} j}, \quad (1.2.26)$$

$i_{l=1,2,\dots,n}$ for $l = 1, 2, \dots, k$ with $k \geq 2$.

Here

$$D_i = \frac{\partial}{\partial x_i} + u_i \frac{\partial}{\partial u_i} + \dots + u_{i_1 i_2 \dots i_n} \frac{\partial}{\partial u_{i_1 i_2 \dots i_n}} + \dots, \quad i = 1, 2, \dots, n. \quad (1.2.27)$$

The situation of n independent variables $x = (x_1, x_2, \dots, x_n)$ and m dependent variables $u = (u^1, u^2, \dots, u^m)$, with $m \geq 2$, arises in studying systems of differential equations. This leads to consideration of extended transformation from (x, u) space to $(x, u, \partial u, \partial^2 u, \dots, \partial^k u)$ -space where $\partial^k u$ denotes the components of all k^{th} -order partial derivatives of u with respect to x . Consider the k^{th} -extended transformation over the $(x, u, \partial u, \partial^2 u, \dots, \partial^k u)$ -space

$$\begin{aligned} (x^*)^i &= \Psi^i(x, u; \epsilon) = x^i + \epsilon \xi^i(x, u) + O(\epsilon^2), \quad i = 1, 2, \dots, n, \\ (u^*)^\mu &= U^\mu(x, u; \epsilon) = u^\mu + \epsilon \eta^\mu(x, u) + O(\epsilon^2), \quad \mu = 1, 2, \dots, m, \end{aligned} \quad (1.2.28)$$

$$(u^*)^\mu_i = u^\mu_i + \epsilon \eta_i^{(1)\mu}(x, u, \partial u) + O(\epsilon^2),$$

\vdots

$$(u^*)^\mu_{i_1 i_2 \dots i_k} = u^\mu_{i_1 i_2 \dots i_k} + \epsilon \eta_{i_1 i_2 \dots i_k}^{(k)\mu}(x, u, \partial u, \dots, \partial^k u) + O(\epsilon^2), \quad (1.2.29)$$

with the extended infinitesimals given by

$$\eta_i^{(1)\mu} = D_i \eta^\mu - (D_i \xi^j) u_j^\mu, \quad (1.2.30)$$

⋮

$$\eta_{i_1 i_2 \dots i_k}^{(k)\mu} = D_{i_k} \eta_{i_1 i_2 \dots, i_{k-1}}^{(k-1)\mu} - (D_{i_k} \xi^j) u_{i_1 i_2 \dots, i_{k-1} j}^\mu, \quad (1.2.31)$$

where $\mu = 1, 2, \dots, m$; $i, i_j = 1, 2, \dots, n$ for $j = 1, 2, \dots, k$ with $k \geq 2$. D_i is total derivative operator and given by

$$D_i = \frac{\partial}{\partial x^i} + u_i^\mu \frac{\partial}{\partial u^\mu} + u_{ii_1}^\mu \frac{\partial}{\partial u_{i_1}^\mu} + u_{ii_1 i_2}^\mu \frac{\partial}{\partial u_{i_1 i_2}^\mu} + \dots, \quad (1.2.32)$$

where $i = 1, 2, \dots, n$. Here, the k th extended infinitesimal generator is given by

$$\begin{aligned} X^{(k)} = & \xi^i(x, u) \frac{\partial}{\partial x^i} + \eta^\mu(x, u) \frac{\partial}{\partial u^\mu} + \eta_i^{(1)\mu}(x, u, \partial u) \frac{\partial}{\partial u_i^\mu} + \dots \\ & + \eta_{i_1 i_2, \dots, i_k}^{(k)\mu}(x, u, \partial u, \dots, \partial^k u) \frac{\partial}{\partial u_{i_1 i_2, \dots, i_k}^\mu}. \end{aligned} \quad (1.2.33)$$

1.2.9 Invariance Solutions of PDEs

Consider a system of N , PDEs with n independent variables $x = (x_1, x_2, x_3, \dots, x_n)$ and m dependent variables $u = (u_1, u_2, \dots, u_m)$ given by

$$F^\mu(x, u, \partial u, \partial^2 u, \dots, \partial^k u) = 0, \quad \mu = 1, 2, \dots, N, \quad (1.2.34)$$

where $\partial^j u$ denotes the coordinates with components $\frac{\partial^j u}{\partial x_{i_1} \partial x_{i_2} \dots \partial x_{i_j}} = u_{i_1 i_2 \dots, i_j}$, $i_j = 1, 2, \dots, n$, for $j = 1, 2, \dots, k$, corresponding to all j^{th} -order partial derivatives of u with respect to x . Let us write such a group of transformations in the form

$$\begin{aligned}x^* &= \Psi(x, u, \epsilon), \\u^* &= U(x, u, \epsilon),\end{aligned}\tag{1.2.35}$$

acting on the space R^{n+m} of the variables (x, u) .

Assume

$$X = \xi_i(x, u) \frac{\partial}{\partial x_i} + \eta^\alpha(x, u) \frac{\partial}{\partial u^\alpha}\tag{1.2.36}$$

be the infinitesimal generator of the Lie group of point transformations (1.2.35) and

$$\begin{aligned}X^k &= \xi_i(x, u) \frac{\partial}{\partial x_i} + \eta^\alpha(x, u) \frac{\partial}{\partial u^\alpha} + \eta_i^{(1)\alpha}(x, u, \partial u) \frac{\partial}{\partial u^\alpha} + \dots \\&+ \eta_{i_1 i_2 \dots i_k}^{(k)\alpha}(x, u, \partial u, \dots, \partial^k u) \frac{\partial}{\partial u_{i_1 i_2 \dots i_k}^\alpha}\end{aligned}\tag{1.2.37}$$

be the k th-extended infinitesimal generator of (1.2.36), where $\eta_i^{(1)\alpha}$ is given by (1.2.30) and $\eta_i^{(k)\alpha}$ is given by (1.2.31), $\alpha = 1, 2, \dots, m$ and $i_j = 1, 2, \dots, n$ for $j = 1, 2, \dots, k$.

The function $u = \Theta(x)$, with components $u^\alpha = \Theta^\alpha(x)$, ($\alpha = 1, 2, \dots, m$), is said to be invariant solution of (1.2.34) resulting from admitted point symmetry with infinitesimal generator (1.2.36) if and only if:

- (i) $u^\alpha = \Theta^\alpha(x)$ is an invariant surface of (1.2.35) for each ($\alpha = 1, 2, \dots, m$);
- (ii) $u = \Theta(x)$ solves (1.2.34).

It follows that $u = \Theta(x)$ a solution is invariant of PDEs (1.2.34), resulting from its invariance under the Lie group of point transformations (1.2.35) if and only if

$u = \Theta(x)$ satisfies:

$$(i) X(u^\alpha - \Theta^\alpha(x)) = 0, \text{ when } u = \Theta(x), \quad \alpha = 1, 2, \dots, m, \text{ i.e.,}$$

$$\xi_i(x, \Theta(x)) \frac{\partial \Theta(x)}{\partial x_i} = \eta^\alpha(x, \Theta(x)), \quad \alpha = 1, 2, \dots, m; \quad (1.2.38)$$

$$(ii) F^\mu(x, u, \partial u, \partial^2 u, \dots, \partial^k u) = 0, \quad \text{when } u = \Theta(x), \mu = 1, 2, \dots, N, \quad \text{i.e.};$$

$$F^\mu(x, \Theta(x), \partial \Theta(x), \partial^2 \Theta(x), \dots, \partial^k \Theta(x)) = 0 \quad , \mu = 1, 2, \dots, N. \quad (1.2.39)$$

Equations (1.2.38) are the invariant surface conditions for the invariant solutions of the system of PDEs (1.2.39) resulting from its invariance under the point symmetry (1.2.36). After finding Lie group of point transformations or point symmetry with infinitesimal generator X given by (1.2.36) through solving the linear system of determining equations (1.2.38), an invariant solution can be obtained by invariant form method for solving the system of equations (1.2.39). The procedure of Invariant form method is as follows:

Firstly the invariant surface conditions (1.2.38) are determined by explicitly solving the corresponding characteristic equations for $u = \Theta(x)$ given by

$$\frac{dx_1}{\xi_1(x, u)} = \frac{dx_2}{\xi_2(x, u)} = \dots = \frac{dx_n}{\xi_n(x, u)} = \dots = \frac{du^1}{\eta^1(x, u)} = \frac{du^2}{\eta^2(x, u)} = \dots = \frac{du^m}{\eta^m(x, u)}. \quad (1.2.40)$$

If $y_1(x, u), y_2(x, u), \dots, y_{n-1}(x, u), \nu^1(x, u), \nu^2(x, u), \dots, \nu^m(x, u)$ are $n+m-1$ functionally independent constants arise from solving the system of $n+m-1$ first order ODEs (1.2.40) with the Jacobian $\frac{\partial(\nu^1, \nu^2, \dots, \nu^m)}{\partial(u^1, u^2, \dots, u^m)} \neq 0$, then the general solution $u = \Theta(x)$ of the system of ODEs (1.2.38) is given implicitly by the invariant form

$$\nu^\alpha = \Phi^\alpha(y_1(x, u), y_2(x, u), \dots, y_{n-1}(x, u)), \quad (1.2.41)$$

where Φ^α is an arbitrary differentiable function of $y_1(x, u), y_2(x, u), \dots, y_{n-1}(x, u)$, for $\alpha = 1, 2, \dots, m$. Here $y_1(x, u), y_2(x, u), \dots, y_{n-1}(x, u), \nu^1(x, u), \nu^2(x, u), \dots, \nu^m(x, u)$ are $n + m - 1$ functionally independent group invariants of (1.2.41) and hence are $n + m - 1$ canonical coordinates for the Lie group of points transformations (1.2.35). If $y_n(x, u)$ be the $(n + m)$ th canonical coordinate satisfying $Xy_n = 1$, the system of PDEs (1.2.34) is transformed into a system of PDEs in terms of independent variables y_1, y_2, \dots, y_{n-1} and dependent variables $\nu^1, \nu^2, \dots, \nu^m$. The transformed system of PDEs admits the one-parameter Lie group of transformations

$$\begin{aligned} y_i^* &= y_i, \quad i = 1, 2, \dots, (n - 1) \\ y_n^* &= y_n + \epsilon, \\ \nu^{*\alpha} &= \nu^\alpha. \end{aligned} \tag{1.2.42}$$

Thus, the variable y_n does not appear explicitly in the transformed system of ODEs and, hence, the transformed system of PDEs has solutions of the form (1.2.41). Consequently, the system of PDEs (1.2.34) has invariant solutions given implicitly by the invariant form (1.2.41). Such solutions are found by solving a reduced system of differential equations with $n - 1$ independent variables y_1, y_2, \dots, y_{n-1} and m dependent variables $\nu^1, \nu^2, \dots, \nu^m$. The variables y_1, y_2, \dots, y_{n-1} are commonly called similarity variables. The reduced system of differential equations is found by substituting the invariant form (1.2.41) into the given system of PDEs (1.2.34). If $n = 2$ then the reduced system of differential equations is a system of ODEs.

1.2.10 Lie's Algorithm

The various steps involved in the execution of Lie's algorithm for calculating symmetries of system of PDEs [75] are as follows:

1. Consider the one parameter Lie group of point transformations (1.2.35) which makes the system of PDEs (1.2.34) invariant.
2. Write the generator of symmetry

$$X = \xi_i(x, u) \frac{\partial}{\partial x_i} + \eta^\alpha(x, u) \frac{\partial}{\partial u^\alpha} \quad (1.2.43)$$

and prolong the k^{th} symmetry generator of X given as

$$\begin{aligned} X^k = & \xi_i(x, u) \frac{\partial}{\partial x_i} + \eta^\alpha(x, u) \frac{\partial}{\partial u^\alpha} + \eta_i^{(1)\alpha}(x, u, \partial u) \frac{\partial}{\partial u^\alpha} + \dots \\ & + \eta_{i_1 i_2 \dots i_k}^{(k)\alpha}(x, u, \partial u, \dots, \partial^k u) \frac{\partial}{\partial u_{i_1 i_2 \dots i_k}^\alpha} \end{aligned} \quad (1.2.44)$$

3. Apply the prolonged generator X^k on (1.2.34), i.e.,

$$X^k(F^\mu(x, u, \partial u, \partial^2 u, \dots, \partial^k u)) = 0, \quad \mu = 1, 2, \dots, N, \quad (1.2.45)$$

4. From the invariance condition, a system of linear PDEs for ξ and η that constitutes a set of determining equations for the infinitesimal generator X admitted by the given system of PDEs (1.2.34) is obtained.
5. The explicit forms of ξ and η will be obtained from the solutions of the determining equations.
6. Construct the corresponding characteristics equations (1.2.40) and obtain u in terms of $(n - 1)$ new independent variables.
7. Rewrite the system (1.2.34) in these new coordinates to get the reduced form of the system.

1.3 Painlevé Analysis

From a qualitative point of view, integrability can be considered as a mathematical property that can be successfully used to obtain more predictive power and quantitative informations to understand the dynamics of the system locally and globally. Integrability nature of dynamical systems can be methodologically investigated employing Painlevé analysis. Singularity structure analysis admitting the Painlevé property advocated by Ablowitz et al. [124] for ordinary differential equations (ODEs) and extended by Weiss et al. [99, 100] to partial differential equations play a key role in the analysis of the integrable nonlinear dynamical systems. The applications of the Painlevé has been carried out by many researchers including Lakshmanan [127], Sachdev[160], Sahadevan [177], Chowdhury [20, 21], Kaliappan [126], Tamizhmani [105]. A brief outline of the Painlevé algorithm [136] is given below.

According to Weiss et al. [99, 100], a PDE has this property when the solutions of the PDE are single-valued about the movable, singular manifold $\phi(z_1, \dots, z_n) = 0$. If $u = u(z_1, \dots, z_n)$ is a solution to the PDE, then it is assumed that

$$u = \phi^\alpha \sum_{j=0}^{j=\infty} u_j \phi^j, \quad (1.3.1)$$

where $\phi(z_1, \dots, z_n)$, $u_j = u_j(z_1, \dots, z_n)$, $u_0 \neq 0$, are analytic functions of (z_j) in a neighbourhood of the manifold ϕ and α is a negative integer. The Painlevé analysis can be summarised as follows:

- (i) Substitute $u = u_0 \phi^\alpha$ into the PDE. Determine the possible leading order(s) α by matching two or more terms of the PDE such that for that α these terms dominate the others.

(ii) Discard the nondominating terms and solve the resulting equation for nonzero $u = u_0$. This may lead to several branches.

(iii) Find the resonances (values of j for which u_j is arbitrary) by inserting

$$u = u_0\phi^\alpha + \beta\phi^{\alpha+j},$$

into the PDE.

(iv) Substitute (1.3.1) into the PDE and equate like powers of ϕ to zero to obtain an overdetermined system of equations for ϕ , u_j , ($j = 0, 1, \dots$) and their derivatives.

(v) The PDE is said to possess the Painlevé property if the resonances are compatible, and this is often taken to be a definition of integrability of the differential equation.

1.4 Bäcklund Transformations

Bäcklund transformations [67, 84] which originated in the study of surfaces of constant mean curvature plays an important role in soliton theory. Bäcklund transformations method is an efficient method to obtain exact solutions of nonlinear PDEs. By the truncation of Painlevé expansion at the constant level term, one can obtain (auto) Bäcklund transformations [53, 68] of nonlinear PDEs. To be precise, if the singularity manifold is determined by $\phi(x, t) = 0$ and $u(x, t) = 0$ is a solution of the PDE

$$u_t = K(u, u_x, \dots). \tag{1.4.1}$$

Suppose that

$$u(x, t) = \frac{1}{\phi^\alpha} \sum_{j=0}^{\infty} u_j(x, t) \phi^j, \quad (1.4.2)$$

where α is a positive integer. $\phi(x, t), u_j(x, t)$ analytic functions in a neighbourhood of the manifold $\phi = 0$. Substituting (1.4.2) into PDE (1.4.1) determines the possible α and the recursion relations for $u_j, j = 0, 1, 2, \dots$.

Bäcklund transformations can be obtained by truncating expansion.

The main steps of Bäcklund transformations method are as follows

Assuming the solution for PDE (1.4.1) is of the form

$$u = \frac{\partial^\alpha}{\partial x^\alpha} f(\phi) + u_1, \quad (1.4.3)$$

where u_1 is also a solution of (1.4.1), f is determined later, α is a positive integer.

- (i) Substituting (1.4.3) into (1.4.1) determines the possible α through requiring the highest degree in nonlinear term and a highest order partial derivative term.
- (ii) Substituting (1.4.3) into (1.4.1) and collecting all terms with the highest degree of ϕ_x and setting its coefficient to zero, we obtain an ordinary equation then $f(\phi)$ can be determined.
- (iii) Collecting all terms with the same order derivatives and setting their coefficients to zero, respectively and then the compatibility conditions can be obtained.

1.5 Traveling Wave Solutions to Nonlinear Partial Differential Equations

Some of the important methods for finding traveling wave solutions are presented in this section:

1.5.1 Extended Tanh Method

Malfiet et al. [213] introduced the tanh method for the study of nonlinear wave equations. Later, extended tanh method has been formulated by the extension of tanh method given by Wazwaz [16]. It is a direct and effective algebraic method for handling nonlinear equations. The main steps of extended tanh Method are given as:

Step 1. Consider a given nonlinear evolution equation where u is dependent variable related with a physical field and x, t are two independent variables given as

$$F(u, u_t, u_x, \dots) = 0. \quad (1.5.1)$$

Then we seek the traveling wave solutions of (1.5.1) in the form

$u(x, t) = u(\xi)$, $\xi = (x - \beta t)$, which leads to a nonlinear ordinary differential equation (ODE)

$$G(U, U', U'', \dots) = 0, \quad (1.5.2)$$

where the prime $'$ denotes $\frac{d}{d\xi}$.

Step 2. Equation (1.5.2) is then integrated as long as all terms contain derivatives where integration constant is considered as zero, Introducing a new indepen-

dent variable

$$Y = \tanh(\xi) \quad (1.5.3)$$

leads to change of derivatives:

$$\begin{aligned} \frac{d}{d\xi} &= (1 - Y^2) \frac{d}{dY} \\ \frac{d^2}{d\xi^2} &= (1 - Y^2) \left(-2Y \frac{d}{dY} + (1 - Y^2) \frac{d^2}{dY^2} \right) \\ \frac{d^3}{d\xi^3} &= (1 - Y^2) \left((6Y^2 - 2) \frac{d}{dY} - 6Y(1 - Y^2) \frac{d^2}{dY^2} + (1 - Y^2)^2 \frac{d^3}{dY^3} \right) \end{aligned} \quad (1.5.4)$$

The extended tanh method admits the use of the finite expansion

$$U(\xi) = \sum_{k=0}^m a_k Y^k + \sum_{k=1}^m b_k Y^{-k}, \quad (1.5.5)$$

where m is a positive integer. The parameter m is usually obtained by balancing the highest order nonlinear terms and the linear terms of the highest order in the resulting equation.

Step 3. Substituting (1.5.5) into the ODE gives an algebraic system of equations in powers of Y which are utilized to find the parameters $a_k, (k = 0, 1, \dots, m)$, $b_k, (k = 1, \dots, m)$ and β .

1.5.2 $\frac{G'}{G}$ -Expansion Method

Wang et al. [139] introduced a new method called the $\frac{G'}{G}$ -expansion method to find traveling wave solutions of nonlinear differential equation. In this method, the travelling wave solutions of nonlinear equations can be expressed in terms of polynomial in $(\frac{G'}{G})$, where $G = G(\xi)$ satisfies the linear ordinary differential equation of second order given as $G''(\xi) + \lambda G'(\xi) + \mu G(\xi) = 0$, where $\xi = x + ct$ and λ, μ, c

are arbitrary unknown constants to be determined. The degree of this polynomial can be found out by considering the homogeneous balance between the nonlinear term and the highest order derivatives present in the given nonlinear equation. The coefficients of this polynomial can be obtained by solving a set of algebraic equations resulted from the process of using the proposed method.

Suppose that a nonlinear PDE in two independent variables x and t , is given by

$$P(u, u_t, u_x, u_{xt}, \dots) = 0, \quad (1.5.6)$$

where $u = u(x, t)$ is an unknown function, P is a polynomial in $u = u(x, t)$ and its various partial derivatives in which the nonlinear terms and highest order derivatives are involved. The main steps of the $(\frac{G'}{G})$ -expansion method are as follows:

Step 1. Combining the independent variables x and t into one variable $\xi = x - ct$ and

Assuming

$$u(x, t) = u(\xi), \quad (1.5.7)$$

the traveling wave variable (1.5.7) permits us reducing (1.5.6) to an ODE for $u = u(\xi)$

$$P(u, -cu', u', -cu'', \dots) = 0, \quad (1.5.8)$$

where prime $'$ denotes the derivative with respect to ξ .

Step 2. Supposing the solution of ordinary differential equation (1.5.8) can be expressed by a polynomial in $\frac{G'}{G}$ as:

$$u(\xi) = \alpha_m \left(\frac{G'}{G}\right)^m + \alpha_{m-1} \left(\frac{G'}{G}\right)^{m-1} + \dots, \quad (1.5.9)$$

where $G = G(\xi)$ satisfies the second order linear ODE in the following form:

$$G'' + \lambda G + \mu G = 0. \quad (1.5.10)$$

where $\alpha_m, \alpha_{m-1}, \dots, \lambda$ and μ are constants to be determined. The positive integer m can be determined by considering the homogeneous balance between the nonlinear terms and highest order derivatives present in (1.5.8). Exact solutions of Eq. (1.5.10) are obtained in three cases which are as follows:

(i) when $\Delta = \lambda^2 - 4\mu > 0$,

$$G(\xi) = (C_1 \sinh(\frac{\sqrt{\Delta}}{2}\xi) + C_2 \cosh(\frac{\sqrt{\Delta}}{2}\xi))e^{-\frac{\lambda}{2}\xi}. \quad (1.5.11)$$

(ii) when $\Delta = \lambda^2 - 4\mu < 0$,

$$G(\xi) = (C_1 \sin(\frac{\sqrt{-\Delta}}{2}\xi) + C_2 \cos(\frac{\sqrt{-\Delta}}{2}\xi))e^{-\frac{\lambda}{2}\xi} \quad (1.5.12)$$

(iii) when $\Delta = \lambda^2 - 4\mu = 0$,

$$G(\xi) = (C_1 + C_2\xi)e^{-\frac{\lambda}{2}\xi}, \quad (1.5.13)$$

where C_1 and C_2 are arbitrary constants.

Corresponding to above three cases, $\frac{G'}{G}$ expressed as follows:

(i) when $\Delta = \lambda^2 - 4\mu > 0$,

$$\frac{G'}{G} = \frac{\sqrt{\Delta} (C_1 \sinh(\frac{\sqrt{\Delta}}{2}\xi) + C_2 \cosh(\frac{\sqrt{\Delta}}{2}\xi))}{2 (C_1 \cosh(\frac{\sqrt{\Delta}}{2}\xi) + C_2 \sinh(\frac{\sqrt{\Delta}}{2}\xi))} - \frac{\lambda}{2} \quad (1.5.14)$$

(ii) when $\Delta = \lambda^2 - 4\mu < 0$,

$$\frac{G'}{G} = \frac{\sqrt{-\Delta} (C_1 \cos(\frac{\sqrt{-\Delta}}{2}\xi) - C_2 \sin(\frac{\sqrt{-\Delta}}{2}\xi))}{(C_1 \sin(\frac{\sqrt{-\Delta}}{2}\xi) + C_2 \cos(\frac{\sqrt{-\Delta}}{2}\xi))} - \frac{\lambda}{2} \quad (1.5.15)$$

(iii) when $\Delta = \lambda^2 - 4\mu = 0$,

$$\frac{G'}{G} = \frac{C_2}{(C_1 + C_2\xi)}, \quad (1.5.16)$$

where C_1 and C_2 are arbitrary constants.

Step 3. By substituting (1.5.9) into (1.5.8) and using second order linear ODE (1.5.10), collecting all terms with the same order of $(\frac{G'}{G})$ together, left-hand side of (1.5.8) is converted into a polynomial in $(\frac{G'}{G})$. A set of algebraic equations for $\alpha_m, \alpha_{m-1}, \dots, \lambda$ and μ by putting each coefficient of the polynomial to zero, are obtained .

Step 4. Determined the constants $\alpha_m, \alpha_{m-1}, \dots, \lambda$ and μ by solving the algebraic equations in Step 3. The general solutions of the second order linear ODE (1.5.10) have been well known for us. Using $\alpha_m, \alpha_{m-1}, \dots, \lambda, \mu$ and the general solutions of Eq.(1.5.10) into (1.5.9), we found more traveling wave solutions of the nonlinear equation (1.5.6).

1.5.3 Modified $\frac{G'}{G}$ -Expansion Method

Suppose that a nonlinear PDE is given by

$$P(u, u_t, u_x, u_{tt}, u_{xx}, \dots) = 0, \quad (1.5.17)$$

where $u = u(x, t)$ is an unknown function. The main steps of the modified $\frac{G'}{G}$ -expansion method [220] are as follows:

Step 1 . Consider the traveling wave variable

$$u(x, t) = u(\xi), \quad \xi = k(x - ct), \quad (1.5.18)$$

where k and c are constants, which permits us to reduce Eq. (1.5.17) into the following ODE:

$$F(u, u', u'', u''', \dots) = 0. \quad (1.5.19)$$

Step 2 . Suppose that the solution of Eq.(1.5.19) can be expressed by a polynomial $(\frac{G'}{G})$ as follows:

$$u(\xi) = \alpha_0 + \sum_{i=1}^m \left[\alpha_i \left(\frac{G'}{G} \right)^i + \alpha_{-i} \left(\frac{G'}{G} \right)^{-i} \right], \quad (1.5.20)$$

where $G = G(\xi)$ satisfies the second order linear ODE

$$G'' + \mu G = 0, \quad (1.5.21)$$

where $\alpha_0, \alpha_i, \alpha_{-i}, \mu$ are constants to be determined.

Step 3. The parameter m in (1.5.20) can be found by balancing the highest order derivative term and the highest nonlinear term in (1.5.19) and conclude the following:

- (a) If m is a positive integer then go to step 4,
- (b) If m is not positive integer, we put $u = v^m$ and then return to step 1.

Step 4. Substituting (1.5.20) into (1.5.19) and using (1.5.21), collecting all terms with the same powers of $(\frac{G'}{G})$ together and then equating each coefficient of the resulted polynomial to zero, yield a system of algebraic equations for $\alpha_0, \alpha_i, \alpha_{-i}, c, \mu$.

Step 5. Since the general solutions of (1.5.21) are well known to us, then substituting $\alpha_0, \alpha_i, \alpha_{-i}, \mu$ and the general solutions of (1.5.21) into (1.5.20) we have the traveling wave solutions of Eq.(1.5.17).

1.6 Objectives

The objective of the thesis is to exploit the group theoretic techniques for solving nonlinear partial differential equations (PDEs). Our main aim is to find the symmetries of nonlinear PDEs in order to obtain their exact solutions by using different methods. Lie's classical method algorithm leads to find the symmetries of constant and variable coefficients forms of nonlinear PDEs and their solutions will be determined. The symmetries and solutions of following nonlinear PDEs have been evaluated

1. The variable coefficients generalized Kuramoto Sivashinsky equation,
2. The variable coefficients generalized Gardner equation,
3. The Benny equation with its variable coefficients form,
4. The extended Ostrovsky equation with variable coefficients and
5. The Gowdy equations in cosmology.

The integrability of Benny equation and extended Ostrovsky equation are also checked by Painlevé test.

Chapter 2 deals with the study of generalized Kuramoto Sivashinsky (KS) equation which is fourth order nonlinear PDE:

$$u_t + u^m u_x + a u_{xx} + b u_{xxx} + c u_{xxxx} = 0, \quad (1.6.1)$$

where $u = u(x, t)$ and a, b and c are constant coefficients. The Kuramoto-Sivashinsky (KS) equation is a well-known model of one-dimensional turbulence, which was derived in various physical contexts. We consider its variable coefficients form as :

$$u_t + u^m u_x + a(t) u_{xx} + b(t) u_{xxx} + c(t) u_{xxxx} = 0. \quad (1.6.2)$$

The coefficients $a(t), b(t)$ and $c(t)$ are the variable coefficients. Lie symmetries of the KS equation, its Lie algebra and the corresponding optimal system has been studied. The reductions are obtained from the optimal system of subalgebras. Then, some classes of solutions of the obtained reduced equations are found by means of other techniques such as extended tanh method, etc.

Chapter 3 is devoted to the use of combination of Lie group method and $\frac{G'}{G}$ method to variable coefficients generalized Gardner equation:

$$u_t + \alpha(t) u^m u_x + \beta(t) u^{2m} u_x + \theta(t) u_{xx} + \delta(t) u_{xxx} = 0, \quad (1.6.3)$$

where $\alpha(t), \beta(t), \theta(t)$ and $\delta(t)$ are variable coefficients. Firstly, the similarity reductions are derived by determining the complete sets of point symmetries of these equations. Then, with the use of $\frac{G'}{G}$ method, more explicit traveling wave solutions involving arbitrary parameters are found out, which are expressed in terms of hyperbolic functions, trigonometric functions and rational functions.

Chapter 4 is devoted to the Benny equation also called the KdV Burger–Kuramoto equation with time dependent coefficients. Benny equation with variable coefficients (VCBE) is given as:

$$u_t + uu_x + \alpha(t)u_{xx} + \beta(t)u_{xxx} + \delta(t)u_{xxxx} = 0, \quad (1.6.4)$$

. We applied the Painlevé analysis only to check its integrability of Eq. (1.6.4). The efforts are concentrated only on finding the symmetries, reductions and exact solutions of (VCBE) equation by using various methods including modified $\frac{G'}{G}$ -expansion method and others.

In Chapter 5, Painlevé analysis of variable coefficients Extended Ostrovsky equation:

$$(u_t + cu_x + \alpha(t)uu_x + \alpha_1(t)u^2u_x + \beta(t)u_{xxx})_x - \delta u = 0, \quad (1.6.5)$$

where $\alpha(t), \alpha_1(t), \beta(t)$ are arbitrary time-dependent coefficients, has been applied to check the integrability. Some solutions are obtained by using auto-Bäcklund transformation. The symmetries and solutions, generated from group invariant reductions using other methods like $\frac{G'}{G}$ method are presented.

Chapter 6 is devoted to the study of following coupled system of nonlinear second order PDEs known as Gowdy equations:

$$\begin{aligned} P_{tt} + \frac{1}{t}P_t - P_{\theta\theta} - e^{2P}(Q_t^2 - Q_\theta^2) &= 0, \\ Q_{tt} + \frac{1}{t}Q_t - Q_{\theta\theta} + 2(Q_tP_t - Q_\theta P_\theta) &= 0. \end{aligned} \quad (1.6.6)$$

Lie classical method is applied to investigate symmetries of these Gowdy equations. Symmetries are applied to reduce the coupled system of PDEs into system of ODEs. Some new solutions of Gowdy equations are obtained after solving reduced ODEs.

Chapter 2

The Variable Coefficients

Generalized

Kuramoto-Sivashinsky Equation

2.1 Introduction

The Kuramoto-Sivashinsky (KS) equation in the form

$$u_t + uu_x + u_{xx} + \nu u_{xxxx} = 0, \tag{2.1.1}$$

where $u = u(x, t)$, was derived independently by Kuramoto [229] and Sivashinsky [64] in the late 70s, while working respectively, on turbulence phenomenon in chemistry and thermal diffusive instability in laminar flame fronts. The dissipative term u_{xxxx} of (2.1.1) provides damping at small scales. The nonlinear term uu_x stabilizes by transferring energy between large and small scales and u_{xx} is responsible for an instability at large scales.

Kuramoto-Sivashinsky equation is a nonlinear evolution equation. This equa-

tion has been derived in a variety of physical contexts driven far from equilibrium by intrinsic instabilities including instabilities of dissipative trapped ion modes in plasmas [25, 170], instabilities in laminar flame fronts [63], fluctuations in fluid films on inclines [65] and phase dynamics in reaction-diffusion systems [227]. Moreover, the KS equation is also considered to be a prototype of a system with self-generated chaos and the large class of generalized Burger's equations.

A large literature is available for the one-dimensional KS equation. A diligent study of the space-periodic KS equation in space dimension was initiated by Nicolaenko et al.[30]. In [51], the Cauchy problem for the KS equation has been studied. Guo [31] studied the solvability of the mixed problem in both one and multi-dimensional cases in bounded domains of the KS equation. In bounded domains global well-posedness of the mixed problem in classes of regular solutions for the KS equation with moving boundaries was proved by Cousin and Larkin [23].

Various forms of the KS equation has been in the literature describe the different physical processes in unstable systems. One of its form is known as KdV-KS equation [145] and is given as:

$$u_t + uu_x + \mu u_{xxx} + \nu (u_{xx} + u_{xxxx}) = 0, \quad (2.1.2)$$

where μ, ν are not negative constants, describes the modeling of long waves flowing down on an inclined plane in a viscous fluid. For $\nu = 0$ equation (2.1.2) becomes KdV equation [33, 200] and when $\mu = 0$ in equation (2.1.2), it gives Kuramoto-Sivashinsky equation (2.1.1). The Faedo-Galerkin method has been applied to solve the mixed problems for (2.1.2) in a bounded domains by Larken [145]. The Cauchy problem for (2.1.2) and the existence of a unique strong global solution and asymptotic behavior of solutions, as ν tends to zero has been studied by Biagioni et al. [77]. The periodic solutions of the KdV equation in approximating the KdV equation by

KS equations was studied by Ogawa in [201].

Kuramoto-Sivashinsky equation [143] also exists in following form:

$$u_t + u^m u_x + au_{xx} + bu_{xxx} + cu_{xxxx} = 0, \quad (2.1.3)$$

where $u = u(x, t)$ and the coefficients a, b and c are constants. This is a well known generalized form of the Kuramoto-Sivashinsky equation.

If $a = 1, b = 0$ and $m = 1$, Eq.(2.1.3) reduces to original KS equation [226]. This equation was derived by Kuramoto in the Belousov–Zhabotinsky reaction for the study of phase turbulence. For the propagation of a flame front in the case of mild combustion, an extension of this equation to two or more spatial dimensions was given by Sivashinsky [63].

The KS equation has some application areas including the representation of one class of pattern formation in model of bifurcation and chaos [34, 78]. If $b = 0, c = 1$ and $m = 1$, Eq.(2.1.3) represents various model of pattern formation on unstable flame fronts and thin hydrodynamic films [66]. If $m = 1$, Eq.(2.1.3) is also known as KdV–Burger–Kuramoto (KBK) equation [79]. The origin of persistent wave propagation of reaction-diffusion type through medium was explained in [228]. In plasma physics, mathematical model for consideration of dissipative waves by means of equation (2.1.3) was discussed in [25]. In [43], elementary particles as solutions of the Kuramoto-Sivashinsky equation was presented. Also at $m \neq 1$, Eq.(2.1.3) can be helpful for the explanation of nonlinear long waves in viscous-elastic tube [144].

Many authors [13, 56, 183, 231] have studied the KS equation numerically. Different methods like Runge-Kutta methods of different orders, Strang–split method, a split scheme of variable time method [231] and implicit-explicit methods [62, 205] have been applied to get the numerical solutions of KS equation.

Several classes of analytical solutions of the generalised Kuramoto-Sivashinsky equation has been constructed by means of inverse scattering transform method [142]. In [46, 143], solitary and periodic solutions of the generalized Kuramoto-Sivashinsky equation have been obtained. Ebadi et al.[58] applied $\frac{G'}{G}$ method to find the soliton solutions of KS equation.

Through, the nonlinear equations with constant coefficients describes different physical situations but to obtained more general solutions various authors attempted to study in variable coefficient forms [230, 235]. In this chapter, the generalized Kuramoto-Sivashinsky equation (GKS) [143] in variable coefficients form is considered as:

$$u_t + u^m u_x + a(t)u_{xx} + b(t)u_{xxx} + c(t)u_{xxxx} = 0. \quad (2.1.4)$$

Here, the coefficients $a(t)$, $b(t)$ and $c(t)$ are the variable coefficients. We have deduced the symmetries of variable coefficients GKS equation and utilized them to obtain new solutions.

The pattern of this chapter is as follows: Lie classical approach [75, 161], has been used to obtain the symmetries of the Eq.(2.1.4). These symmetries are applied to change the PDEs into ODEs as discussed in section 2.2. The solutions have been presented in section 2.3. Conclusions drawn are given in last section 2.4.

2.2 Classical Lie Symmetries

The method of Lie symmetry groups is one of the important approach to obtain analytical solutions of nonlinear PDEs. According to the Lie Classical method [75], if a given system of PDEs is subjected to invariance under one-parameter Lie group of transformations, one arrives at an over determined linear system of differential

equations for the group infinitesimals. The obtained infinitesimals are utilized to do the reductions of the system. In view of the algorithmic steps of this method as described in section (1.2.10), following steps are followed:

We assumed the one-parameter Lie group of infinitesimal transformations in x, t and u , given as:

$$\begin{aligned}x^* &= x + \epsilon\xi(x, t, u) + O(\epsilon^2), \\t^* &= t + \epsilon\tau(x, t, u) + O(\epsilon^2), \\u^* &= u + \epsilon\eta(x, t, u) + O(\epsilon^2),\end{aligned}\tag{2.2.1}$$

where ϵ is the group parameter. These transformations leave the equation (2.1.4) invariant. It means, the transformations are to be considered in such a form that if u is a solution of Eq.(2.1.4) then u^* is also a solution. On applying the criteria of invariance, the relation from the coefficients of the first order of ϵ is deduced as follows:

$$\begin{aligned}\eta^t + \eta u_x u^{(m-1)} + \eta^x u^m + a(t)\eta^{xx} + \tau a(t)' u_{xx} + b(t)\eta^{xxx} \\+ \tau b(t)' u_{xxx} + c(t)\eta^{xxxx} + \tau c(t)' u_{xxxx} = 0,\end{aligned}\tag{2.2.2}$$

where prime($'$) indicates the derivative with respect to independent variable t and $\eta^t, \eta^x, \eta^{xx}, \eta^{xxx}$ and η^{xxxx} are extended (*prolonged*) infinitesimals acting on an enlarged space that include the derivatives of the dependent variables (more details in Appendix). The infinitesimals are then obtained from invariance condition Eq.(2.2.2) by putting the coefficients of different differentials equal to zero. The following system of determining equations are obtained:

$$\begin{aligned}
 \tau_x &= 0, \\
 \tau_u &= 0, \\
 \xi_u &= 0, \\
 \eta_{uu} &= 0, \\
 \tau_t c(t) + \tau c(t)' - 4c(t)\xi_x &= 0, \\
 \eta_t + a(t)\eta_{xx} + b(t)\eta_{xxx} + c(t)\eta_{xxxx} + \eta_x u^m &= 0, \\
 \tau_t b(t) + \tau b(t)' + 4c(t)\eta_{xu} - 3b(t)\xi_x - 6c(t)\xi_{xx} &= 0, \\
 \tau_t a(t) + \tau a(t)' + 6c(t)\eta_{xuu} - 3b(t)\xi_{xx} - 2a(t)\xi_x + 3b(t)\eta_{xu} - 4c(t)\xi_{xxx} &= 0 \\
 \xi_t - 2a(t)\eta_{xu} - m\eta u^{(m-1)} - 3b(t)\eta_{xuu} - 4c(t)\eta_{xxxu} - \tau_t u^m + b(t)\xi_{xxx} \\
 + c(t)\xi_{xxxx} + a(t)\xi_{xx} + \xi_x u^m &= 0.
 \end{aligned} \tag{2.2.3}$$

The solution of these determining equations have been obtained as:

$$\begin{aligned}
 \eta &= k_1 u, \\
 \xi &= k_2 x + k_3, \\
 \tau &= (k_2 - mk_1)t + k_4,
 \end{aligned} \tag{2.2.4}$$

where k_1, k_2, k_3 and k_4 are arbitrary constants. The coefficients functions $a(t), b(t)$ and $c(t)$ are given by the following conditions:

$$\begin{aligned}
 \tau_t a + a' \tau - 2a\xi_x &= 0, \\
 \tau_t b + b' \tau - 3b\xi_x &= 0, \\
 \tau_t c + c' \tau - 4c\xi_x &= 0.
 \end{aligned} \tag{2.2.5}$$

The associated infinitesimal generators or vector fields are represented by

$$\begin{aligned} V_1 &= u \frac{\partial}{\partial u} - mt \frac{\partial}{\partial t}, \\ V_2 &= x \frac{\partial}{\partial x} + t \frac{\partial}{\partial t}, \\ V_3 &= \frac{\partial}{\partial x}, \\ V_4 &= \frac{\partial}{\partial t}. \end{aligned} \tag{2.2.6}$$

Now, apply the method for finding the one-dimensional optimal system of subalgebras for the Eq.(2.1.4) as given in [113, 161]. This approach, in essence, is taking a general element from the Lie algebra and reducing it to its simplest equivalent form by applying carefully chosen adjoint transformations. The commutator table of the Lie symmetries of equation (2.1.4) and the adjoint representation of the symmetry group of (2.1.4) on its Lie algebra are shown in Tables (2.1) and (2.2) respectively.

Table 2.1: Commutator Table

	V_1	V_2	V_3	V_4
V_1	0	0	$-V_3$	mV_4
V_2	0	0	$-V_3$	$-V_4$
V_3	V_3	V_3	0	0
V_4	$-mV_1$	V_4	0	0

Table 2.2: Adjoint Table

	V_1	V_2	V_3	V_4
V_1	V_1	V_2	$V_3 e^\epsilon$	$V_4 e^{m\epsilon}$
V_2	V_1	V_2	$V_3 e^\epsilon$	$V_4 e^\epsilon$
V_3	$V_1 - \epsilon V_3$	$V_2 - \epsilon V_3$	V_3	V_4
V_4	$V_1 + m\epsilon V_4$	$V_2 - m\epsilon V_4$	V_3	V_4

The optimal system for the equation (2.1.4) is given as

(i) $V_1 + \lambda V_2 + \mu V_4$

(ii) $V_1 + \nu V_2$

(iii) $V_2 + \delta V_4$

(iv) $V_3 + \delta V_4$

(v) V_4

For obtaining the similarity variable and the form, the following characteristic equation

$$\frac{dt}{\tau} = \frac{dx}{\xi} = \frac{du}{\eta} \quad (2.2.7)$$

is utilized. Two constants are involved in the general solution of these equations. One becomes the new independent variable ζ and the other, say f , plays the role of new dependent variable. After substituting these solutions of (2.2.7) in Eq.(2.1.4), we get the reduced ODE as discussed in following subsection.

2.2.1 Symmetry Reductions

(i) **Vector field** $V_1 + \lambda V_2 + \mu V_4$

On using the characteristic Eq.(2.2.7) for the vector field $V_1 + \lambda V_2 + \mu V_4$, the similarity variable, the form of the similarity solution and coefficients functions are obtained as :

$$\begin{aligned} u &= x^{\frac{1}{\lambda}} f(\zeta), \\ \zeta &= x [(\lambda - m)t + \mu]^{\frac{\lambda}{m-\lambda}} \\ a(t) &= q_3, \quad b(t) = q_2, \quad c(t) = q_1, \end{aligned} \quad (2.2.8)$$

where q_1, q_2 and q_3 are arbitrary constants. Substituting Eq.(2.2.8) into Eq.(2.1.4), the following ODE is obtained:

$$f^m(\zeta) f'(\zeta) - f(\zeta) - \lambda \zeta f'(\zeta) + q_3 f''(\zeta) + q_2 f'''(\zeta) + q_1 f''''(\zeta) = 0, \quad (2.2.9)$$

where prime (') means the derivative with respect to the new independent variable ζ .

(ii) **Vector field** $V_1 + \nu V_2$

The similarity variable, the form of the similarity solutions and coefficients functions for the vector field $V_1 + \nu V_2$ is as follows:

$$\begin{aligned} u &= t^{\frac{1}{(\nu-m)}} f(\zeta), \\ \zeta &= t^{\frac{1}{(\nu-m)}} x^{(-1)}, \\ a(t) &= q_4 t^{\frac{(\nu+m)}{(\nu-m)}}, \quad b(t) = q_5 t^{\frac{(2\nu+m)}{(\nu-m)}}, \quad c(t) = q_6 t^{\frac{(3\nu+m)}{(\nu-m)}}, \end{aligned} \quad (2.2.10)$$

where q_4, q_5, q_6 are arbitrary constants. Substituting Eq.(2.2.10) into Eq.(2.1.4), the function $f(\zeta)$, satisfying the following ODE is obtained:

$$\begin{aligned} q_6 f''''(\zeta) + q_5 f'''(\zeta) + q_4 f''(\zeta) + f^m(\zeta) f'(\zeta) - \left(\frac{\nu}{\nu-m}\right) \zeta f'(\zeta) \\ + \left(\frac{1}{\nu-m}\right) f(\zeta) = 0, \end{aligned} \quad (2.2.11)$$

where prime (') means the differentiation with respect to the new independent variable ζ .

(iii) **Vector field** $V_2 + \delta V_4$

The similarity variable, similarity solution and coefficient functions for the

vector field $V_2 + \delta V_4$ are deduced as:

$$\begin{aligned} u &= f(\zeta), \quad \zeta = \frac{x}{t + \delta} \\ a(t) &= q_7(t + \delta), \quad b(t) = q_8(t + \delta)^2 \quad c(t) = q_9(t + \delta)^3. \end{aligned} \quad (2.2.12)$$

The reduced ODE after substituting Eq.(2.2.12) into Eq.(2.1.4) is given as:

$$q_9 f''''(\zeta) + q_8 f'''(\zeta) + q_7 f''(\zeta) + f^m(\zeta) f'(\zeta) - \zeta f'(\zeta) = 0, \quad (2.2.13)$$

where q_7, q_8, q_9 are arbitrary constants.

(iv) **Vector field** $V_3 + \omega V_4$

For the vector field $V_3 + \omega V_4$, the similarity variable, similarity solution and coefficient functions are deduced as:

$$\begin{aligned} u &= f(\zeta), \quad \zeta = (x - \omega t) \\ a(t) &= q_{12}, \quad b(t) = q_{11}, \quad c(t) = q_{10}. \end{aligned} \quad (2.2.14)$$

Substituting Eq.(2.2.14) into Eq.(2.1.4), the reduced ODE:

$$q_{10} f''''(\zeta) + q_{11} f'''(\zeta) + q_{12} f''(\zeta) + f'(\zeta) f^m(\zeta) - \omega f'(\zeta) = 0, \quad (2.2.15)$$

is obtained. Here q_{10}, q_{11} and q_{12} are arbitrary constants.

(v) **Vector field** V_4

For the Vector field V_4 , the similarity variable, similarity solution and coefficient functions are constants. So, the Eq.(2.1.4) reduces to:

$$u = K_1, \quad (2.2.16)$$

where K_1 is constant.

2.3 Exact Solutions

The solutions of generalized Kuramoto-Sivashinsky equation are useful for both mathematical and physical point of view. In this section, the solutions of Eq.(2.1.4) for the reduced ODEs equations of section 2.2.1 has been evaluated.

1. Vector field $V_1 + \lambda V_2 + \mu V_4$

The reduced ODE corresponding to this vector field as given by Eq.(2.2.9) has the solutions in the following form

$$f(\zeta) = A\zeta^p, \quad (2.3.1)$$

where A and p are constants to be found out. Equate the exponents of ζ suitably and have put their respective coefficients equal to zero.

By putting $\lambda = 1$ in Eq.(2.2.9) and integrating, we get the equation

$$q_1 f'''(\zeta) + q_2 f''(\zeta) + q_3 f'(\zeta) + \frac{f^{m+1}(\zeta)}{(m+1)} - \zeta f(\zeta) = 0, \quad (2.3.2)$$

where constant of integration is taken as zero.

On equating the exponents $(p-3)$ and $p(m+1)$, we get $p = \frac{-3}{m}$. The solution of Eq.(2.1.4) is obtained after substituting Eq.(2.3.1) into Eq.(2.3.2) as :

$$u = \left[\frac{m^3}{q_1(1+m)(6m^2+27m+27)} \right]^{\frac{-1}{m}} x^{\frac{(m-3)}{m}} [(1-m)t + \mu]^{\frac{(-3)}{m(m-1)}} \quad (2.3.3)$$

2. Vector field $V_1 + \nu V_2$

For $\nu = 1$, we are able to obtain the solution for the reduced ODE corresponding

to vector field $V_1 + \nu V_2$ given by Eq.(2.2.11) in the following form

$$f(\zeta) = A\zeta^p. \quad (2.3.4)$$

where A and p are constants to be found out. We equated the exponents of ζ and put their respective coefficients equal to zero. By equating the exponents $(p-4)$ and $(pm+p-1)$, we got $p = \frac{-3}{m}$. On substituting (2.3.4) into Eq.(2.2.11), the following solution of Eq.(2.1.4) is obtained:

$$u(x, t) = \left[\frac{6q_6m^3 + 33q_6m^2 + 54q_6m + 27q_6}{m^3} \right]^{\frac{1}{m}} \left(\frac{t^{(1-m)}}{x} \right)^{\frac{-3}{m}}. \quad (2.3.5)$$

3. Vector field $V_2 + \delta V_4$

For the solution of reduced ODE given in Eq.(2.2.13), the transformation

$$f(\zeta) = H(\zeta)^{\frac{1}{m}}, \quad (2.3.6)$$

has been used. By substituting the transformation (2.3.6) into Eq.(2.2.13), the following nonlinear ODE is obtained:

$$\begin{aligned} & q_9(H')^4 + 6mq_8(H')^2H''H - 6mq_9(H')^4 + 3m^2q_9(H'')^2H^2 \\ & + 4m^2q_9H'H'''H^2 + 11m^2q_9(H')^4 + q_9m^3H^3H'''' - 4q_9m^3H'H'''H^2 \\ & + 12q_9m^3H(H')^2H'' - 3q_9m^3H^2(H'')^2 - 6q_9m^3(H')^4 + q_8mH(H')^3 \\ & + 3q_8m^2H^2H'H'' - 3q_8m^2H(H')^3 + q_8m^3H^3H''' - 3q_8m^3H^2H'H'' \\ & + q_7m^2H^2(H')^2 + q_7m^3(H^3)H'' - 18q_9m^2(H')^2H''H + 2q_8m^3H(H')^3 \\ & - q_7m^3H^2(H')^2 + m^3HH' - \zeta m^3HH' = 0. \end{aligned} \quad (2.3.7)$$

Considering the solution for Eq.(2.3.7) in the form

$$H(\zeta) = a_0 + a_1\zeta + a_2\zeta^2, \quad (2.3.8)$$

and substituting Eq.(2.3.8) in Eq.(2.3.7) gave the coefficient of powers of ζ which is a system of algebraic equations in terms of a_0 , a_1 and a_2 . On solving these algebraic equations for a_0 , a_1 and a_2 by a suitable software, following set of results has been obtained.

Set I

$$\begin{aligned} a_0 &= 0, & a_1 &= \frac{7}{16}, \\ a_2 &= \frac{7}{64} \frac{q_8 m}{q_9 (3m - 2)}. \end{aligned}$$

Using these values, the solution of Eq.(2.1.4) is obtained as

$$u(x, t) = \left[\frac{7}{16} \left(\frac{x}{t + \delta} \right) + \frac{7}{64} \frac{q_8 m}{q_9 (3m - 2)} \left(\frac{x}{t + \delta} \right)^2 \right]^{\frac{1}{m}}. \quad (2.3.9)$$

Set II

$$\begin{aligned} a_0 &= a_0, & a_1 &= 0, \\ a_2 &= -\frac{1}{3} \frac{m a_0}{q_7 (m - 2)}. \end{aligned}$$

For this set, the solution of Eq.(2.1.4) is

$$u(x, t) = \left[a_0 - \frac{1}{3} \frac{m a_0}{q_7 (m - 2)} \left(\frac{x}{t + \delta} \right)^2 \right]^{\frac{1}{m}}. \quad (2.3.10)$$

Set III

$$a_0 = \frac{1}{3} \frac{\sqrt{3}\sqrt{mq_7(-1+m)}}{m}, \quad a_1 = 1,$$

$$a_2 = 0.$$

These values gave the solution of Eq.(2.1.4) as

$$u(x, t) = \left[\frac{1}{3} \frac{\sqrt{3}\sqrt{mq_7(-1+m)}}{m} - \left(\frac{x}{t+\delta} \right) \right]^{\frac{1}{m}}. \quad (2.3.11)$$

4. Vector field $V_3 + \omega V_4$

The reduced ODE Eq.(2.2.15) on integration gives the following ODE;

$$q_{10}f'''(\zeta) + q_{11}f''(\zeta) + q_{12}f'(\zeta) + \frac{f^{(m+1)}(\zeta)}{(m+1)} - \omega f(\zeta) = 0, \quad (2.3.12)$$

where constant of integration is considered to be zero.

Solutions of Eq.(2.1.4) are obtained for the cases $m = 3$ and $m = 1$ in (2.3.12).

Case I: For $m = 3$

Putting $m = 3$ in Eq.(2.3.12), we get

$$q_{10}f'''(\zeta) + q_{11}f''(\zeta) + q_{12}f'(\zeta) + \frac{f^{(4)}(\zeta)}{(4)} - \omega f(\zeta) = 0. \quad (2.3.13)$$

For solving the above nonlinear ODE, using extended tanh method as explained in section (1.5.1). Balancing higher order term of Eq.(2.3.13) f''' with nonlinear term f^4 , we get $n = 1$. On introducing new independent variable $y = \tanh \zeta$ we obtained,

$$f(\zeta) = a_0 + a_1 y + a_2 y^{-1}. \quad (2.3.14)$$

Substituting Eq.(2.3.14) in Eq.(2.3.13) and by equating the coefficient of powers of y gives the system of algebraic equations in terms of coefficients a_0, a_1, a_2 and ω . After simplifying the algebraic equations, the following set of results are obtained:

Set I

The values of coefficients are obtained as

$$a_0 = 0, \quad a_1 = (24q_{10})^{\frac{1}{3}}, \quad \omega = -2q_{11},$$

$$a_2 = \frac{(q_{12} - 8q_{10})}{(24q_{10})^{\frac{2}{3}}}$$

and the corresponding solution of Eq.(2.1.4) is:

$$u = (24q_{10})^{\frac{1}{3}} \tanh(x - \omega t) + \frac{(q_{12} - 8q_{10})}{(24q_{10})^{\frac{2}{3}}} \coth(x - \omega t). \quad (2.3.15)$$

Set II

$$a_0 = \frac{-1}{12} \frac{q_{11} \sqrt[3]{24}}{q_{10}^{2/3}}, \quad a_1 = 0, \quad a_2 = \sqrt[3]{24} \sqrt[3]{q_{10}},$$

$$\omega = -\frac{1}{288} \frac{3456 q_{12} q_{10}^3 - 6912 q_{10}^4 + q_{11}^4}{q_{11} c_4^2}.$$

For this set, the solution of Eq.(2.1.4):

$$u = \frac{-1}{12} \frac{q_{11} \sqrt[3]{24}}{q_{10}^{2/3}} + \left(\sqrt[3]{24} \sqrt[3]{q_{10}} \right) \coth(x - \omega t). \quad (2.3.16)$$

Set III

$$a_0 = \frac{-1}{6} \frac{q_{11} \sqrt[3]{3}}{q_{10}^{2/3}}, \quad a_1 = 2 \sqrt[3]{3} \sqrt[3]{q_{10}},$$

$$a_2 = 0, \quad \omega = -\frac{1}{216} \frac{q_{11} (q_{11}^2 \sqrt[3]{3} + 216 a_2 q_{10}^{5/3} + 144 \sqrt[3]{3} q_{10}^2) 3^{2/3}}{q_{10}^2}. \quad (2.3.17)$$

The solution for Eq.(2.1.4)by using this set is

$$u = \frac{-1}{6} \frac{q_{11} \sqrt[3]{3}}{q_{10}^{2/3}} + \left(2 \sqrt[3]{3} \sqrt[3]{q_{10}} \right) \tanh(x - \omega t). \quad (2.3.18)$$

. **Case II:** For $m = 1$

Substituting $m = 1$ in Eq.(2.3.12), we get

$$q_{10} f''''(\zeta) + q_{11} f''(\zeta) + q_{12} f'(\zeta) + \frac{f^{(2)}(\zeta)}{2} - \omega f(\zeta) = 0. \quad (2.3.19)$$

Balancing f'''' with f^2 in Eq.(2.3.19), we get $n=3$. By using extended tanh method, we get

$$f(\zeta) = a_0 + a_1 y + a_2 y^2 + a_3 y^3 + a_4 y^{-1} + a_5 y^{-2} + a_6 y^{-3}. \quad (2.3.20)$$

Substituting Eq.(2.3.20) in Eq.(2.3.19) and by equating the coefficients of powers of y , we get the system of algebraic equations for $a_0, a_1, a_2, a_3, a_4, a_5, a_6$ and ω . After solving the algebraic equations by a software, we get following set of solutions.

Set I

The values of the coefficients are

$$\begin{aligned} a_0 &= q_{12}, & a_1 &= 2q_{12}, & a_2 &= 0, & a_3 &= 0, \\ a_4 &= 0, & a_5 &= -12q_{11}, & a_6 &= 0, & \omega &= q_{12}. \end{aligned} \quad (2.3.21)$$

Corresponding to this set, solution of Eq.(2.1.4) is obtained as

$$u = q_{12} + 2q_{12} \tanh(x - \omega t) - 12q_{11} \coth^2(x - \omega t). \quad (2.3.22)$$

Set II

$$\begin{aligned}
a_0 &= 0, & a_1 &= 0, & a_2 &= 0, & a_3 &= 0, \\
a_4 &= \left(\frac{60}{19}q_{12} - \frac{1200}{19}q_{10} \right), & a_5 &= 0, \\
a_6 &= 120q_{10}, & \omega &= -2q_{12}.
\end{aligned} \tag{2.3.23}$$

Using these values the solution of Eq.(2.1.4) is obtained as

$$u = \left(\frac{60}{19}q_{12} - \frac{1200}{19}q_{10} \right) \coth(x - \omega t) + 120q_{10} \coth^3(x - \omega t). \tag{2.3.24}$$

Set III

$$\begin{aligned}
a_0 &= 2c_{12}, & a_1 &= 2q_{12}, & a_2 &= 0, & a_3 &= 0, \\
a_4 &= 0, & a_5 &= -12q_{11}, & a_6 &= 120q_{10}, & \omega &= 2q_{12}.
\end{aligned} \tag{2.3.25}$$

Using these set of values, the corresponding solution of Eq.(2.1.4) is obtained as

$$u = 2q_{12} + 2q_{12} \tanh(x - \omega t) - 12q_{11} \coth^2(x - \omega t) + 120q_{10} \coth^3(x - \omega t). \tag{2.3.26}$$

Set IV

$$\begin{aligned}
a_0 &= 0, & a_1 &= 0, & a_2 &= \frac{-1}{4} \frac{q_{11}q_{12}}{q_{10}}, \\
a_3 &= 0, & a_4 &= 0, & a_5 &= 15q_{11}, \\
a_6 &= 120q_{10}, & \omega &= -2q_{11}.
\end{aligned} \tag{2.3.27}$$

Under this set of values, we get the solution of Eq.(2.1.4) as:

$$u = \frac{-1}{4} \frac{q_{11}q_{12}}{q_{10}} \tanh^2(x - \omega t) + 15q_{11} \coth^2(x - \omega t) + 120q_{10} \coth^3(x - \omega t). \quad (2.3.28)$$

Set V

$$\begin{aligned} a_0 &= 0, & a_1 &= 2q_{12}, & a_2 &= \frac{-4q_{12}q_{11}}{(4q_{12} - 12q_{10})}, \\ a_3 &= 0, & a_4 &= (2c_6 - 12q_{10}), \\ a_5 &= 0, & a_6 &= 0, & \omega &= -2q_{11}. \end{aligned} \quad (2.3.29)$$

For this set, the solution of Eq.(2.1.4) is obtained in the following form

$$\begin{aligned} u &= 2q_{12} \tanh(x - \omega t) + \frac{-4q_{12}q_{11}}{(4q_{12} - 12q_{10})} \tanh^2(x - \omega t) \\ &+ (2q_{12} - 12q_{10}) \coth(x - \omega t). \end{aligned} \quad (2.3.30)$$

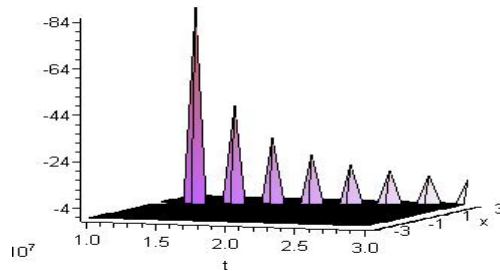


Figure 2.1: The solution (2.3.15) for the values of $q_{10} = 1, q_{11} = 1,$ and $q_{12} = 1,$ within the interval $-3 \leq x, t \leq 3.$

2.4 Conclusion

Lie classical method has been successfully applied to find the symmetries of variable coefficients form of generalized Kuramoto-Sivashinsky equation. The symmetries obtained are further utilized to reduce the partial differential equations into ordinary differential equation corresponding to optimal vector fields. Various methods like power series and extended tanh method has been applied to obtain the solutions of KS equation. The graphical representation of solution (2.3.15) for the values of $q_{10} = 1, q_{11} = 1,$ and $q_{12} = 1,$ within the interval $-3 \leq x, t \leq 3$ gives singular solution (peaks) in Fig. 2.1.

Chapter 3

Invariant Solutions of Variable Coefficients Generalized Gardner Equation

3.1 Introduction

A generic evolutionary partial differential equation (PDE) known as Korteweg-de Vries (KdV) equation is a well-known nonlinear PDE. It is frequently used to model weakly nonlinear long waves, incorporating a certain balance of leading-order nonlinearity and dispersion. In different branches of science and engineering like modulation in pulse-width [36], mass transports in a chemical response theory [240], dust acoustic solitary structures in magnetized dusty plasmas [115] and nonlinear long dynamo waves observed in the Sun [93], the Korteweg-de Vries (KdV)-type equations have become important nonlinear model due to its nonlinearity [32]. Expansion of KdV equation to a higher order nonlinearity and also including more nonlinear terms give extension of this equation. The extension is a combination of

KdV and modified KdV equation (KdV-mKdV), collectively called as the extended KdV (eKdV) or Gardner equation. This extended KdV (eKdV) is given by

$$u_t + auu_x + bu^2u_x + \mu u_{xxx} = 0, \quad (3.1.1)$$

where u is a function of both x and t . The properties of extended KdV (eKdV) or gardner equation are same as that of classical KdV. In most of the applications, $u = u(x, t)$ represents the wave mode amplitude, the terms uu_x and u^2u_x represent nonlinear wave steepening where as the term u_{xxx} , represents dispersive wave effects. The coefficients of the nonlinear terms a and b and the dispersive term μ regulate the steady oceanic background density and flow stratification through the linear eigen mode of the internal waves.

In literature [154, 185], it has been shown that different forms of Gardner equation describe various interesting physical phenomena such as stratified ocean internal waves, the long wave propagation in a two-layer inhomogeneous shallow liquid and ion acoustic waves in plasma with non positive ion interfacial solitary waves over, slowly varying topographies [110] and a nonlinear elastic structural element with large deflection wave motion [165]. The generalized form of Gardner equation [18, 103] is given as

$$u_t + \alpha u^m u_x + \beta u^{2m} u_x + \theta u_{xx} + \delta u_{xxx} = 0, \quad (3.1.2)$$

where α, β, θ and δ are constants. From this generalized Gardner equation (3.1.2), the following important cases can be distinguished depending on values of these parameters:

- (i) If $\beta = 1$ and $m = 1$, Eq.(3.1.2) becomes the Burgers-Korteweg-de Vries equation [186].

- (ii) If $\alpha = 0$ and $m = 1$, Eq.(3.1.2) becomes the modified Burgers-Korteweg-de Vries equation [237].
- (iii) if $m = 1$, $\theta = 0$, $\alpha \neq 0$ and $\beta \neq 0$, Eq.(3.1.2) turns into the combined KdV and modified KdV equation (KdV-mKdV) [42].
- (iv) If $m = 1$, $\theta = 0$, $\alpha \neq 0$ and $\beta = 0$, Eq.(3.1.2) further changes to the classical KdV equation [132].

Various techniques and methods [29] have been applied to find the solutions of generalized Gardner equation (3.1.2). Nonlinear evolution equations (NLEEs) with variable coefficients forms can provide more powerful and realistic models as compare with constant coefficients forms. A lot of important and significant phenomena with the time-dependent coefficients of inhomogeneous Gardner models have been derived to describe in ocean dynamics, fluid mechanics and plasma physics. Some of them are as follows:

- (i) The variable coefficient mKdV model [101, 158, 199, 202, 206] given as:

$$u_t + a(t)uu_x + \kappa u^2 u_x + u_{xxx} = 0, \quad (3.1.3)$$

where pycnocline location depends on the coefficient $a(t)$ and coefficient κ is a constant, explains the stratified ocean internal waves if the pycnocline lies midway between the surface and sea bed .

- (ii) The modelling of the governing equation for the propagation of long wave in a two layer inhomogeneous shallow liquid, is the variable coefficients extended KdV equation given as [80, 87, 217]

$$u_t - 6\alpha(t)uu_x - 6\gamma u^2 u_x + \theta(t)u_{xxx} = 0. \quad (3.1.4)$$

Here ($u = u(x, t)$) is proportional to the elevation of the interface between two layers, coefficients $\alpha(t)$ and $\theta(t)$ depend on coordinate t imply the ratio of the depths of two layers.

(iii) A perturbed mKdV model [80] is proposed during the investigation on the dynamics hidden in the plasma sheath transition layer and inner sheath layer.

The perturbed mKdV is given as

$$\psi_\tau + 6\psi\psi_\eta + \sigma\psi^2\psi_\eta - \psi_{\eta\eta\eta} + h(\tau)\psi_\eta = 0, \quad (3.1.5)$$

where σ is a constant, $h(\tau)$ is an analytic function.

This chapter deals with the variable coefficients form of generalized Gardner equation gives as:

$$u_t + \alpha(t)u^m u_x + \beta(t)u^{2m}u_x + \theta(t)u_{xx} + \delta(t)u_{xxx} = 0, \quad (3.1.6)$$

where $\alpha(t)$, $\beta(t)$, $\theta(t)$ and $\delta(t)$ are arbitrary time-dependent coefficients.

In section 3.2 , the symmetries of this generalized Gardner equation (3.1.6) has been obtained using the Lie classical method. Lie algebra of these symmetries are presented. In Section 3.3, the corresponding generators have been used to reduce equation (3.1.6) to lower dimensional differential equations. Some exact solutions have been obtained by using $\frac{G'}{G}$ method. In Section 3.4, explanation of the solutions has been given. Conclusion are presented in last section.

3.2 Classical Lie Symmetry Analysis

The symmetries of Eq.(3.1.6) are calculated using Lie classical method. The one-parameter of infinitesimal Lie group of point transformations leaving (3.1.6) invari-

ant is defined as :

$$\begin{aligned} u^* &= u + \epsilon\eta(x, t, u) + O(\epsilon^2) \\ x^* &= x + \epsilon\xi(x, t, u) + O(\epsilon^2) \\ t^* &= t + \epsilon\tau(x, t, u) + O(\epsilon^2). \end{aligned} \quad (3.2.1)$$

The invariance condition of (3.1.6) under the transformations (3.2.1) is obtained as :

$$\begin{aligned} \eta^t + m\alpha(t)u^{m-1}\eta u_x + \alpha(t)\eta^x u^m + \tau\alpha' u_x u^m + \beta(t)\eta^x u^{2m} + 2m\beta(t)\eta u^{2m-1}u_x \\ + \tau\beta(t)' u_x u^{2m} + \eta^{xx}\theta(t) + \tau\theta(t)' u_{xx} + \eta^{xxx}\delta(t) + \tau\delta(t)' u_{xxx} = 0, \end{aligned} \quad (3.2.2)$$

where prime (') means the derivatives with respect to independent variable t and η^t , η^x , η^{xx} and η^{xxx} are extended (prolonged) infinitesimals acting on an enlarged space which have all derivatives of the dependent variables (details been discussed in Appendix). The infinitesimals are determined from the determining equations which are obtained from invariance condition (3.2.2) by putting the coefficients of different differentials equal to zero. The determining equations of Eq. (3.1.6) are as follows:

$$\begin{aligned} \tau_x = 0, \tau_u = 0, \xi_u = 0, \eta_{uu} = 0, \\ \tau\delta(t)' + \tau_t\delta(t) - 3\delta(t)\xi_x = 0, \\ \tau\theta(t)' + \tau_t\theta(t) - 2\theta(t)\xi_x + 3\delta(t)\eta_{xu} - 3\delta(t)\xi_{xx} = 0, \\ \eta_t + \alpha(t)u^m\eta_x + \theta(t)\eta_{xx} + \beta(t)u^{2m}\eta_x + \delta(t)\eta_{xxx} = 0, \\ \tau\alpha(t)' u^m - \theta(t)\xi_{xx} + 2\theta(t)\eta_{xu} - \xi_t + 3\delta(t)\eta_{xxu} - \delta(t)\xi_{xxx} \end{aligned}$$

$$\begin{aligned}
 & -\alpha(t)u^m\xi_x - \beta(t)u^{2m}\xi_x + \tau_t\alpha(t)u^m + \tau_t\beta(t)u^{2m} + 2m\beta(t)\eta u^{(2m-1)} \\
 & + \tau_t\beta(t)'u^{2m} + m\alpha(t)\eta u^{(m-1)} = 0.
 \end{aligned} \tag{3.2.3}$$

On solving these set of equations (3.2.3), the infinitesimals ξ , τ and η are obtained as:

$$\begin{aligned}
 \eta &= uk_1, \\
 \xi &= xk_2 + k_3, \\
 \tau &= \frac{1}{\delta(t)} \left[\left(3 \int \delta(t) dt \right) k_2 + k_4 \right]
 \end{aligned} \tag{3.2.4}$$

where k_i , ($i = 1, 2, 3, 4$) are arbitrary constants and the equations which govern $\alpha(t)$, $\beta(t)$, $\theta(t)$ and $\delta(t)$ are given as:

$$\begin{aligned}
 \theta(t)'\tau + \theta(t)\tau_t - 2\theta(t)\xi_x &= 0, \\
 \beta(t)'\tau + \beta(t)\tau_t + 2m\beta(t)\eta_u - \xi_x\beta(t) &= 0, \\
 \alpha(t)'\tau + \alpha(t)\tau_t + m\alpha(t)\eta_u - \xi_x\alpha(t) &= 0.
 \end{aligned} \tag{3.2.5}$$

The corresponding infinitesimal generators of the Lie algebra are given by

$$\begin{aligned}
 V_1 &= \frac{1}{\delta(t)} \left(3 \int \delta(t) dt \right) \frac{\partial}{\partial t} + x \frac{\partial}{\partial x}, \\
 V_2 &= \frac{1}{\delta(t)} \frac{\partial}{\partial t}, \\
 V_3 &= \frac{\partial}{\partial x}, \\
 V_4 &= u \frac{\partial}{\partial u}.
 \end{aligned} \tag{3.2.6}$$

The adjoint action is obtained by the use of Lie series

$$Ad(\exp(\epsilon V_i))V_j = V_j - \epsilon[V_i, V_j] + \frac{\epsilon^2}{2}[V_i, [V_i, V_j]] - \dots, \quad (3.2.7)$$

where $[V_i, V_j] = V_i V_j - V_j V_i$ are the Lie algebra commutator and ϵ is a small parameter. With the help of Lie series (3.2.7), the commutator table and adjoint table of Lie algebra (3.2.6) can easily be formed shown in Table 3.1 and 3.2:

Table 3.1: Commutator Table

	V_1	V_2	V_3	V_4
V_1	0	$-3V_2$	$-V_3$	0
V_2	$3V_2$	0	0	0
V_3	V_3	0	0	0
V_4	0	0	0	0

Table 3.2: Adjoint Table

	V_1	V_2	V_3	V_4
V_1	V_1	$V_2 e^{3\epsilon}$	$V_3 e^\epsilon$	V_4
V_2	$V_1 - \epsilon V_2$	V_2	V_3	V_4
V_3	$V_1 - \epsilon V_3$	V_2	V_3	V_4
V_4	V_1	V_2	V_3	V_4

Since linear combination of generators is also a generator, a large number of vector fields of the Lie algebra form with the linear combination of the generators V_j ; $j = 1,2,3,4$ and the reduction can be obtained for any vector field by using characteristic equations:

$$\frac{dx}{\xi} = \frac{dt}{\tau} = \frac{du}{\eta}. \quad (3.2.8)$$

The basic fields which give the optimal system [161] for Eq.(3.1.6) are

- (i) $V_1 + a_1 V_4$

$$(ii) V_2 + a_2 V_3$$

$$(iii) V_2 + a_3 V_4$$

$$(iv) V_2 + a_4 V_3 + a_5 V_4$$

By using the generators of the optimal system, various reductions of the Eq.(3.1.6) to ODE's are obtained as follows:

3.3 Reductions and Exact Solutions

1. Vector field $V_1 + a_1 V_4$

Assuming $a_1 = 1$, then the corresponding generator is $V_1 + V_4$. By using the characteristic equation (3.2.8), the similarity variable and similarity solution of the Eq.(3.1.6) are obtained as

$$\begin{aligned} \zeta &= \frac{x^3}{\left(\int \delta(t) dt\right)}, \\ u &= \left[\left(\int \delta(t) dt\right)\right]^{\frac{1}{3}} F(\zeta), \end{aligned} \quad (3.3.1)$$

and equation (3.2.5) gives the coefficient functions as

$$\begin{aligned} \theta(t) &= \frac{p_1 \delta(t)}{\left(\int \delta(t) dt\right)^{\frac{1}{3}}}, \\ \beta(t) &= \frac{p_2 \delta(t)}{\left(\int \delta(t) dt\right)^{\frac{2m+2}{3}}}, \\ \alpha(t) &= \frac{p_3 \delta(t)}{\left(\int \delta(t) dt\right)^{\frac{2+m}{3}}}. \end{aligned} \quad (3.3.2)$$

Substitution of Eq.(3.3.1) and Eq.(3.3.2) into Eq.(3.1.6) yields a reduced ODE with variable coefficients:

$$27\zeta^2 F(\zeta)''' + 54\zeta F(\zeta)'' + 6F(\zeta)' + 9p_1\zeta^{\frac{4}{3}} F(\zeta)'' + 6p_1\zeta^{\frac{1}{3}} F(\zeta)' + 3p_2\zeta^{\frac{2}{3}} F(\zeta)^{2m} F(\zeta)' + 3p_3\zeta^{\frac{2}{3}} F(\zeta)^m F(\zeta)' + \frac{1}{3} F(\zeta) - \zeta F(\zeta)' = 0, \quad (3.3.3)$$

where the prime (') means the differentiation with respect to the independent variable ζ . Solving the Eq.(3.3.3), the trival solution of Eq.(3.1.6) is obtained.

2. Vector field $V_2 + a_2 V_3$

Let $a_2 = 1$, then by using generator $V_2 + V_3$; the similarity variable, similarity solution and coefficient functions are obtained as:

$$\begin{aligned} \zeta &= x - \left(\int \delta(t) dt \right), \\ u &= F(\zeta), \\ \theta &= p_7\delta(t), \quad \beta = p_5\delta(t), \quad \alpha = p_6\delta(t), \end{aligned} \quad (3.3.4)$$

and the corresponding ODE as

$$F(\zeta)''' + p_7 F(\zeta)'' + p_5 F(\zeta)' F(\zeta)^{2m} + p_6 F(\zeta)' F(\zeta)^m + F(\zeta)' = 0. \quad (3.3.5)$$

We evaluated traveling wave solutions for this equation(3.3.5) by using the extended $\frac{G'}{G}$ -expansion method.

On integrating Eq.(3.3.5), we get,

$$F(\zeta)'' + p_7 F(\zeta)' + p_5 \frac{F(\zeta)^{2m+1}}{2m+1} + p_6 \frac{F(\zeta)^{m+1}}{m+1} + F(\zeta) = 0. \quad (3.3.6)$$

The following steps are used to solve the equation (3.3.6).

- (i) Assuming that Eq.(3.3.6) solution can be represented in $\frac{G'}{G}$ as follows:

$$F(\zeta) = \sum_{i=0}^n A_i \left(\frac{G'}{G} \right)^i, \quad (3.3.7)$$

where $G = G(\zeta)$ is the solution of second order ODE in the form

$$G'' + \lambda G' + \mu G = 0, \quad (3.3.8)$$

where A_i, λ, μ are constants to be found out and n is called the balance number.

- (ii) The homogeneous balancing between the derivative of highest order and the nonlinear terms present in equation (3.3.6) gives the positive integer n .
- (iii) Substituting (3.3.7) into (3.3.6) and using the ODE (3.3.8), we get a polynomial in $\frac{G'}{G}$ after collecting all terms of $\frac{G'}{G}$ with the same order together. Then, a set of algebraic equations will be obtained by equating each coefficient of this polynomial to zero, which inturn is to be used to get value of A_i, λ and μ .
- (iv) The general solution of Eq.(3.3.8) will be easily find out and using these solutions into (3.3.7) and values of A_i , the traveling wave solutions of Eq.(3.1.6) is obtained.

In equation (3.3.6), applying the method of homogeneous balance between the terms u'' and u^{2m+1} , we determined $n = \frac{1}{m}$. Heren is not a positive integer. To solve

the solution of equation (3.3.6), we have consider the solution in the form [50, 86],

$$F(\zeta) = A \left(\frac{G'}{G} \right)^{\frac{1}{m}} \quad (3.3.9)$$

where constant value of A is to be found out and G satisfies the Eq.(3.3.8). Substituting (3.3.9) into (3.3.6) get the terms in powers of $\frac{G'}{G}$. A set of algebraic equations are obtained after collecting all terms of same powers of $\frac{G'}{G}$ and equating the coefficient equal to zero in order to determine the value of A, μ and λ ,e.g.,

$$\begin{aligned} \frac{A}{m^2} + \frac{p_5 A^{2m+1}}{2m+1} + \frac{A}{m} &= 0, \\ \frac{p_6 A^{m+1}}{m+1} - \frac{p_7 A}{m} + \frac{A\lambda}{m} + 2 \frac{A\lambda}{m^2} &= 0, \\ -\frac{p_7 A\mu}{m} + 2 \frac{A\lambda\mu}{m^2} - \frac{A\lambda\mu}{m} &= 0, \\ \frac{A\lambda^2}{m^2} + A + 2 \frac{A\mu}{m^2} - \frac{p_7 A\lambda}{m} &= 0, \\ -\frac{A\mu^2}{m} + \frac{A\mu^2}{m^2} &= 0. \end{aligned} \quad (3.3.10)$$

The result obtained by solving the above algebraic equations by using software is given as:

$$\begin{aligned} \lambda &= \frac{p_7 m}{2-m}, \\ \mu &= -\frac{1}{2} \frac{p_7^2 m^2}{(-2+m)^2} - \frac{1}{2} m^2 - \frac{1}{2} \frac{p_7^2 m^2}{(-2+m)}, \\ A &= \left(-\frac{(m+1)(2m+1)}{p_5 m^2} \right)^{\frac{1}{2m}}. \end{aligned} \quad (3.3.11)$$

When $\lambda^2 - 4\mu > 0$, then solution set represents the hyperbolic exact traveling wave solution:

$$u(x, t) = \left(-\frac{(m+1)(2m+1)}{p_5 m^2} \right)^{\frac{1}{2m}} \times \left(\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2} \left(\frac{A_1 \sinh\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2}\zeta\right) + B_1 \cosh\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2}\zeta\right)}{A_1 \cosh\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2}\zeta\right) + B_1 \sinh\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2}\zeta\right)} \right) - \frac{\lambda}{2} \right)^{\frac{1}{m}} \right), \quad (3.3.12)$$

where $\zeta = x - \left(\int \delta(t) dt\right)$, $\lambda = \frac{p_7 m}{2-m}$, $\mu = -\frac{1}{2} \frac{p_7^2 m^2}{(-2+m)^2} - \frac{1}{2} m^2 - \frac{1}{2} \frac{p_7^2 m^2}{(-2+m)}$.

When $\lambda^2 - 4\mu < 0$, then solution set gives the trigonometric exact traveling wave solution:

$$u(x, t) = \left(-\frac{(m+1)(2m+1)}{p_5 m^2} \right)^{\frac{1}{2m}} \times \left(\left(\frac{\sqrt{4\mu - \lambda^2}}{2} \left(\frac{-A_1 \sin\left(\frac{\sqrt{4\mu - \lambda^2}}{2}\zeta\right) + B_1 \cos\left(\frac{\sqrt{4\mu - \lambda^2}}{2}\zeta\right)}{A_1 \cos\left(\frac{\sqrt{4\mu - \lambda^2}}{2}\zeta\right) + B_1 \sin\left(\frac{\sqrt{4\mu - \lambda^2}}{2}\zeta\right)} \right) - \frac{\lambda}{2} \right)^{\frac{1}{m}} \right), \quad (3.3.13)$$

where $\zeta = x - \left(\int \delta(t) dt\right)$, $\lambda = \frac{p_7 m}{2-m}$, $\mu = -\frac{1}{2} \frac{p_7^2 m^2}{(-2+m)^2} - \frac{1}{2} m^2 - \frac{1}{2} \frac{p_7^2 m^2}{(-2+m)}$.

3. Vector field $V_2 + a_3 V_4$

Assuming the scalar $a_3 = 1$, the vector field $V_2 + V_4$ gives the similarity variable, similarity solution and coefficient functions as:

$$\begin{aligned} \zeta &= x, \\ u &= e^{\int \delta(t) dt} F(\zeta) \\ \theta(t) &= p_8 \delta(t), \quad \beta(t) = p_9 \delta(t) e^{-2m \int \delta(t) dt}, \quad \alpha(t) = p_{10} \delta(t) e^{-m \int \delta(t) dt}. \end{aligned} \quad (3.3.14)$$

Substitution of Eq.(3.3.14) in Eq.(3.1.6) gives the ODE as

$$F(\zeta) + p_{10} F(\zeta)' F(\zeta)^m + p_9 F(\zeta)' F(\zeta)^{2m} + p_8 F(\zeta)'' + F(\zeta)''' = 0. \quad (3.3.15)$$

Assuming in Eq.(3.3.15)

$$F(\zeta) = H(\zeta)^{\frac{1}{m}}, \quad (3.3.16)$$

Eq.(3.3.15) takes the form

$$\begin{aligned} & m^2 H(\zeta)''' H(\zeta)^2 - 3m^2 H(\zeta)' H(\zeta)'' H(\zeta) + 2m^2 H(\zeta)^3 - 3m \left(H(\zeta)' \right)^3 \\ & + 3m H(\zeta)' H(\zeta)'' H(\zeta) - p_8 m^2 H(\zeta) \left(H(\zeta)' \right)^2 + p_8 m^2 H(\zeta)^2 H(\zeta)'' \\ & + p_8 m H(\zeta) \left(H(\zeta)' \right)^2 + p_9 m^2 H(\zeta)^4 H(\zeta)' + p_{10} m^2 H(\zeta)^3 H(\zeta)' \\ & + m^3 H(\zeta)^3 = 0. \end{aligned} \quad (3.3.17)$$

Applying $\frac{G'}{G}$ method [184] for the Eq.(3.3.17) and by using homogeneous balancing method, we get $n = 1$. Assuming the solution of Eq.(3.3.17) in $\frac{G'}{G}$ as follows

$$H(\zeta) = a_0 + a_1 \frac{G'}{G}, \quad (3.3.18)$$

where $G = G(\zeta)$. Substituting Eq.(3.3.18) in Eq.(3.3.17) and using (3.3.8), collecting all the terms of similar powers of $\frac{G'}{G}$ and putting each coefficient of them equal to zero, gives a set of following algebraic equations for a_0 , a_1 , λ and μ .

$$\begin{aligned} & -p_9 m^2 a_1^5 - a_1^3 - 3ma_1^3 - 2m^2 a_1^3 = 0, \\ & p_8 m a_1^3 - 3m^2 a_1^3 \lambda - 6m^2 a_1^2 a_0 - 6ma_1^2 a_0 \\ & - 4p_9 m^2 a_1^4 a_0 - 6ma_1^3 \lambda + p_8 m^2 a_1^3 - 3a_1^3 \lambda \\ & - p_9 m^2 a_1^5 \lambda - p_{10} m^2 a_1^4 = 0, \\ & p_8 m^2 a_1^3 \lambda + p_8 m a_1^2 a_0 - 6m^2 a_1 a_0^2 - 3a_1^3 \mu - 3ma_1^3 \lambda^2 \\ & - 6p_9 m^2 a_1^3 a_0^2 - m^2 a_1^3 \lambda^2 - p_{10} m^2 a_1^4 \lambda - 2m^2 a_1^3 \mu - 15ma_1^2 \lambda a_0 \end{aligned}$$

$$\begin{aligned}
& + 2p_8ma_1^3\lambda - 9m^2a_1^2\lambda a_0 - 4p_9m^2a_1^4\lambda a_0 - 3p_{10}m^2a_1^3a_0 \\
& - 3ma_1^3\mu - 3a_1^3\lambda^2 + 3p_8m^2a_1^2a_0 - p_9m^2a_1^5\mu = 0, \\
& - 2m^2a_1^3\lambda\mu - 6a_1^3\lambda\mu - 12ma_1^2\mu a_0 + 2p_8m^2a_1a_0^2 \\
& - 2m^2a_1^2\lambda^2a_0 + 2p_8ma_1^3\mu + m^3a_1^3 + p_8ma_1^3\lambda^2 \\
& - p_{10}m^2a_1^4\mu + 2p_8ma_1^2\lambda a_0 - 3p_{10}m^2a_1^2a_0^2 - 12m^2a_1\lambda a_0^2 \\
& + 4p_8m^2a_1^2\lambda a_0 - 3p_{10}m^2a_1^3\lambda a_0 - 4p_9m^2a_1^4\mu a_0 - 4m^2a_1^2\mu a_0 \\
& - 6p_9m^2a_1^3\lambda a_0^2 - 4p_9m^2a_1^2a_0^3 - 12ma_1^2\lambda^2a_0 - a_1^3\lambda^3 = 0, \\
& 2m^2a_1^2\mu^2a_0 - 3a_1^3\lambda\mu^2 - 6ma_1^2\lambda^2\mu a_0 - p_9m^2a_1^3\lambda\mu^2\lambda a_0^4 \\
& - 6ma_1^2\mu^2a_0 + 6ma_1 + p_8m^2a_1\lambda^2a_0^2 + 2p_8m^2a_1\mu a_0^2 + 2p_8ma_1^2\lambda\mu a_0 \\
& + p_8ma_1^3\mu^2 - 8m^2a_1\lambda\mu a_0^2 - p_{10}m^2a_1\lambda a_0^3 + 4m^2a_1^2\lambda^2\mu a_0 \\
& - 3m^2a_1^3\lambda\mu^2 - 4p_9m^2a_1^2\mu a_0^3 - 3p_{10}m^2a_1^2\mu a_0^2 - p_8m^2a_1^3\mu^2 \\
& + 3m^3a_0^2a_1 - m^2a_1\lambda^3a_0^2 = 0, \\
& 3ma_1^3\mu^3 - p_9m^2a_1\mu a_0^4 - a_1^3\mu^3 - 2m^2a_1^3\mu^3 - 2m^2a_1\mu^2a_0^2 \\
& - 3ma_1^2\mu^2\lambda a_0 + 3m^2a_1^2\mu^2\lambda a_0 + p_8m^2a_1\lambda\mu a_0^2 - p_8m^2a_1^2\mu^2a_0 \\
& - m^2a_1\lambda^2\mu a_0^2 + m^3a_0^3 - p_{10}m^2a_1\mu a_0^3 + p_8ma_1^2\mu^2a_0 = 0. \tag{3.3.19}
\end{aligned}$$

Solving these algebraic equations simultaneously we get:

$$\begin{aligned}
a_0 &= 0, \quad a_1 = \frac{\sqrt{-p_9(1+3m+2m^2)}}{p_9m}, \\
\lambda &= -\frac{1}{3} \left(\frac{p_8m}{m-1} \right), \\
\mu &= \frac{1}{9} \left(\frac{p_8^2m^2(2m^2-6m+3)}{-9m+11m^2+3-7m^3+2m^4} \right). \tag{3.3.20}
\end{aligned}$$

When $\lambda^2 - 4\mu > 0$, then the above solution set represents the hyperbolic exact

traveling wave solution which is given as:

$$u(x, t) = e^{\int \delta(t) dt} \left(\frac{\sqrt{-p_9(1+3m+2m^2)}}{p_9m} \right)^{\frac{1}{m}} \times \left(\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2} \left(\frac{A_1 \sinh\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2}x\right) + B_1 \cosh\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2}x\right)}{A_1 \cosh\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2}x\right) + B_1 \sinh\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2}x\right)} \right) \right) - \frac{\lambda}{2} \right)^{\frac{1}{m}}, \quad (3.3.21)$$

where $\lambda = -\frac{1}{3} \left(\frac{p_8m}{m-1} \right)$, $\mu = \frac{1}{9} \left(\frac{p_8^2m^2(2m^2-6m+3)}{-9m+11m^2+3-7m^3+2m^4} \right)$.

When $\lambda^2 - 4\mu < 0$, then the solution set (3.3.20) gives the trigonometric exact traveling wave solution:

$$u(x, t) = e^{\int \delta(t) dt} \left(\frac{\sqrt{-p_9(1+3m+2m^2)}}{p_9m} \right)^{\frac{1}{m}} \times \left(\left(\frac{\sqrt{4\mu - \lambda^2}}{2} \left(\frac{-A_1 \sin\left(\frac{\sqrt{4\mu - \lambda^2}}{2}x\right) + B_1 \cos\left(\frac{\sqrt{4\mu - \lambda^2}}{2}x\right)}{A_1 \cos\left(\frac{\sqrt{4\mu - \lambda^2}}{2}x\right) + B_1 \sin\left(\frac{\sqrt{4\mu - \lambda^2}}{2}x\right)} \right) \right) - \frac{\lambda}{2} \right)^{\frac{1}{m}}, \quad (3.3.22)$$

where $\lambda = -\frac{1}{3} \left(\frac{p_8m}{m-1} \right)$, $\mu = \frac{1}{9} \left(\frac{p_8^2m^2(2m^2-6m+3)}{-9m+11m^2+3-7m^3+2m^4} \right)$.

4. Vector field $V_2 + a_4V_3 + a_5V_4$

Assuming the scalars $a_4 = a_5 = 1$, the corresponding similarity variable, similarity solution, and coefficient functions are given as:

$$\begin{aligned} \zeta &= x - \int \delta(t) dt, \\ u &= e^{\int \delta(t) dt} F(\zeta) \\ \theta(t) &= p_{11}\delta(t), \quad \beta(t) = p_{12}\delta(t)e^{-2m \int \delta(t) dt}, \quad \alpha(t) = p_{13}\delta(t)e^{-m \int \delta(t) dt}. \end{aligned} \quad (3.3.23)$$

The reduced ODE is obtained by substituting Eq.(3.3.23) into Eq.(3.1.6) and is

given as:

$$F(\zeta) - F(\zeta)' + p_{13}F(\zeta)'F(\zeta)^m + p_{12}F(\zeta)'F(\zeta)^{2m} + p_{11}F(\zeta)'' + F(\zeta)''' = 0. \quad (3.3.24)$$

To obtain the solution of Eq.(3.3.24), we apply the same procedure as for solution of Eq.(3.3.6) and the method of homogeneous balance for the term u''' and u^{2m} gives $n = \frac{1}{m}$. It should be noted that n is not a positive integer. We assume the solution of Eq.(3.3.24) in the form [50, 86]

$$F(\zeta) = A \left(\frac{G'}{G} \right)^{\frac{1}{m}} \quad (3.3.25)$$

where A is a constant to be found out and G satisfies Eq.(3.3.8). Substituting (3.3.25) into (3.3.24), collecting all the terms of similar powers of $\frac{G'}{G}$ and putting each coefficient of them equal to zero, gives a set of following algebraic equations for A, μ and λ .

$$\begin{aligned} p_{12}A^{2m+1}m^2 + 3Am + A + 2Am^2 &= 0, \\ p_{13}A^{m+1}\lambda m^2 - p_{11}A\lambda m^2 + 3A\lambda^2 + 3A\mu m + 3A\lambda^2m + A\lambda^2m^2 \\ + p_{12}A^{2m+1}\mu m^2 - Am^2 + 2A\mu m^2 - 2p_{11}A\lambda m + 3A\mu &= 0, \\ 6A\lambda\mu - A\lambda m^2 + 2A\lambda\mu m^2 - 2p_{11}A\mu m - p_{11}A\lambda^2m + A\lambda^3 \\ - Am^3 + p_{13}A^{m+1}\mu m^2 &= 0, \\ p_{11}A\lambda\mu m^2 + A\lambda^2\mu m^2 - 2p_{11}A\lambda\mu m + 2A\mu^2m^2 + 3A\mu^2 \\ - A\mu m^2 + 3A\lambda^2\mu - 3A\mu^2m - 3A\lambda^2\mu m &= 0, \\ p_{11}A\mu^2m^2 + 3A\lambda\mu^2m^2 - p_{11}A\mu^2m + 3A\lambda\mu^2 - 6A\lambda\mu^2m &= 0, \end{aligned}$$

$$-p_{11}Am^2 + 6A\lambda m - p_{11}Am + 3A\lambda m^2 + 3A\lambda + p_{13}A^{m+1}m^2 + p_{12}A^{2m+1}\lambda m^2 = 0. \quad (3.3.26)$$

The result by solving the above algebraic equations by software gives the following set of solution

$$\begin{aligned} \lambda &= -\frac{1}{3} \frac{p_{11}m}{m-1}, \\ \mu &= \frac{1}{9} \frac{m^2(2p_{11}^2m^2 - 6p_{11}^2m + 3p_{11}^2 - 18m + 9m^2 + 9)}{-7m^3 + 2m^4 + 11m^2 - 9m + 3}, \\ A &= \left[-\frac{(m+1)(2m+1)}{m^2} \right]^{\frac{1}{2m}}. \end{aligned} \quad (3.3.27)$$

When $\lambda^2 - 4\mu > 0$, then solution set gives the hyperbolic exact traveling wave solution:

$$\begin{aligned} u(x, t) &= \left[-\frac{(m+1)(2m+1)}{m^2} \right]^{\frac{1}{2m}} \\ &\times \left(\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2} \left(\frac{A_1 \sinh\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2}\zeta\right) + B_1 \cosh\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2}\zeta\right)}{A_1 \cosh\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2}\zeta\right) + B_1 \sinh\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2}\zeta\right)} \right) - \frac{\lambda}{2} \right) \right)^{\frac{1}{m}}, \end{aligned} \quad (3.3.28)$$

where $\zeta = x - (\int \delta(t) dt)$, $\lambda = -\frac{1}{3} \frac{p_{11}m}{m-1}$, and $\mu = \frac{1}{9} \frac{m^2(2p_{11}^2m^2 - 6p_{11}^2m + 3p_{11}^2 - 18m + 9m^2 + 9)}{-7m^3 + 2m^4 + 11m^2 - 9m + 3}$.

When $\lambda^2 - 4\mu < 0$, then solution set gives the trigonometric exact traveling wave solution:

$$\begin{aligned} u(x, t) &= \left[-\frac{(m+1)(2m+1)}{m^2} \right]^{\frac{1}{2m}} \\ &\times \left(\left(\frac{\sqrt{4\mu - \lambda^2}}{2} \left(\frac{-A_1 \sin\left(\frac{\sqrt{4\mu - \lambda^2}}{2}\zeta\right) + B_1 \cos\left(\frac{\sqrt{4\mu - \lambda^2}}{2}\zeta\right)}{A_1 \cos\left(\frac{\sqrt{4\mu - \lambda^2}}{2}\zeta\right) + B_1 \sin\left(\frac{\sqrt{4\mu - \lambda^2}}{2}\zeta\right)} \right) - \frac{\lambda}{2} \right) \right)^{\frac{1}{m}}, \end{aligned} \quad (3.3.29)$$

where $\zeta = x - (\int \delta(t) dt)$, $\lambda = -\frac{1}{3} \frac{p_{11}m}{m-1}$, and $\mu = \frac{1}{9} \frac{m^2(2p_{11}^2m^2 - 6p_{11}^2m + 3p_{11}^2 - 18m + 9m^2 + 9)}{-7m^3 + 2m^4 + 11m^2 - 9m + 3}$.

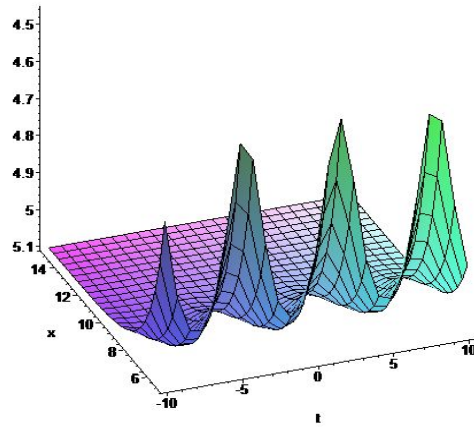


Figure 3.1: The solution (3.3.12) for $m = \frac{1}{2}$, $A_1 = 1$, $B_1 = 0$, $p_5 = -1$, $p_7 = 1$, $\delta(t) = \cos(t)$.

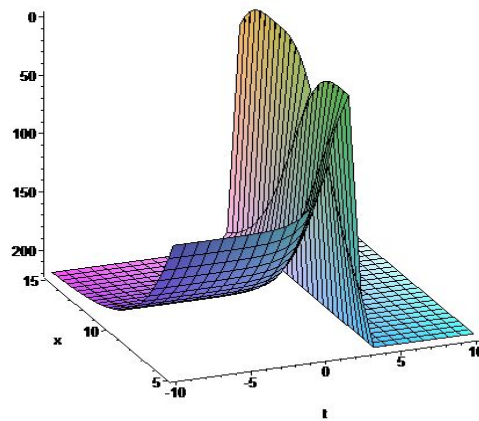


Figure 3.2: The solution (3.3.12) for $m = \frac{1}{4}$, $A_1 = 1$, $B_1 = 0$, $p_5 = -1$, $p_7 = 1$, $\delta(t) = e^t$.

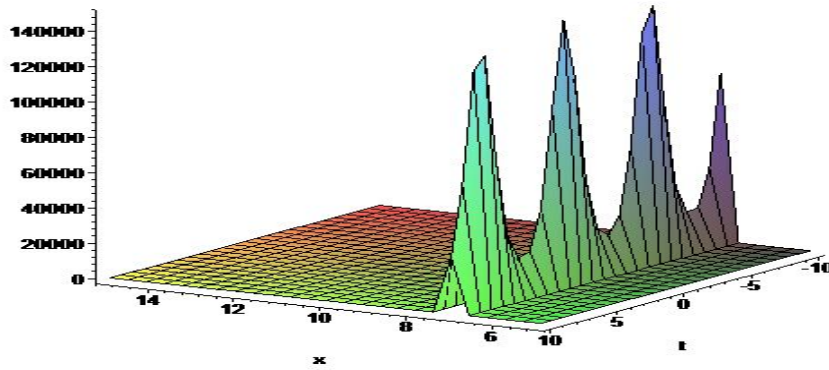


Figure 3.3: The solution (3.3.21) for $m = \frac{1}{2}, p_8 = 1, p_9 = -1, A_1 = 1, B_1 = 0, \delta(t) = \cos(t)$.

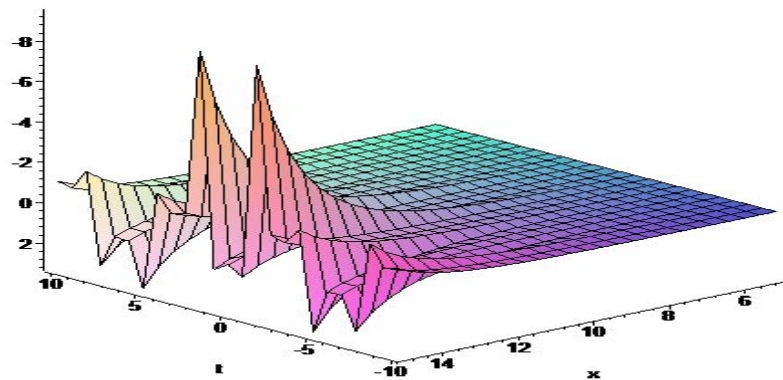


Figure 3.4: The solution (3.3.21) for $m = \frac{1}{4}, p_8 = 1, p_9 = -1, A_1 = 1, B_1 = 0, \delta(t) = \tan(t)$.

3.4 Results and Discussions

- (i) The determined solutions of equation (3.3.12) represents hyperbolic function solutions and are said to be traveling wave solutions. For $\lambda^2 - 4\mu > 0$, the solutions depend upon p_5, p_7, m, A_1, B_1 and $\delta(t)$. By considering $p_5 = -1, p_7 = 1, A_1 = 1, B_1 = 0, m = \frac{1}{2}$ and $\delta(t) = \cos(t)$ represents kinky periodic solutions and $p_5 = -1, p_7 = 1, A_1 = 1, B_1 = 0, m = \frac{1}{4}$ and $\delta(t) = e^t$ solutions which are special kind of solitary waves. These solutions are shown graphically in Fig. 3.1 and Fig. 3.2. respectively.
- (ii) It is also analysed that the solutions (3.3.21) behaves as periodic solution for $p_8 = 1, p_9 = -1, m = \frac{1}{2}$ and $\delta(t) = \cos(t)$ as well as for $p_8 = 1, p_9 = -1, m = \frac{1}{4}, A_1 = 1, B_1 = 0$ and $\delta(t) = \tan(t)$. The graphical representations of these solutions are shown in Fig. 3.3 and Fig. 3.4.

3.5 Conclusion

Symmetries of the variable coefficients generalized Gardner equation have been obtained by Lie classical method. Using these symmetries the generalized Gardner equation have been transformed into ordinary differential equations corresponding to the respective optimal vector fields. The traveling wave solutions are successfully obtained by using $\frac{G'}{G}$ -Expansion method. The solutions involve hyperbolic and trigonometric functions. The behaviour of all the obtained solutions can be controlled by the choice of $\delta(t)$ and other constants occurring in the solution.

Chapter 4

Painlevé Analysis and Some Solutions of Variable Coefficients Benny Equation

4.1 Introduction

The physical phenomena of a nonlinear partial differential equations (PDEs) can be well describe by their solutions. In the last decade various approaches have been applied to obtain the analytical solutions of nonlinear PDEs, like homogeneous balance method [44], the Jacobi elliptical function expansion method [28] and some numerical approaches like variational derivative method [198], have been applied to get the approximate solutions. Zhang et al. [235] investigated solitary wave solutions of nonlinear PDEs by introducing an auxiliary equation as a source equation. Khater et al.[12] found the traveling wave solutions by applying the improved sine–cosine method and Wu’s elimination method. Chen and Ding [236] investigated traveling wave solutions for $(1 + 1)$ and $(2 + 1)$ -dimensional nonlinear evolution

equations (NLEE) by employing an improved projective Riccati equation method and symbolic computation system Maple. Yong and Chen [219] gave extension to this method to get the exact solutions of the coupled KdV systems.

The partial differential equations with variable coefficients can describe the system more realistically than those with constant coefficients. Many innovative techniques present for determining the exact solutions of nonlinear partial differential equations but most of them work only for limited problems because of their complexity.

In nineteenth century, Sophus Lie [190, 191] developed Lie point symmetry method [74, 190]. In Lie's symmetry method [171], the invariance, symmetry properties and similarity reductions of nonlinear PDEs equations are obtained. Weiss et al. (WTC) [99, 111] presented the Painlevé test for nonlinear partial differential equations to check its integrability. According to the WTC method, a nonlinear partial differential equation has Painlevé property if its solutions are single-valued about a movable singularity manifold. The further developments to this method include Kruskal's simplified method [123] and Conte's invariant method [166]. Painlevé applications has been carried out by Lakshmanan [127], Sahadevan [177], Chowdhury [20, 21], Kaliappan [126], Tamizhmani [105], etc.

Benny equation, also known as KdV Burger–Kuramoto equation, is the fourth-order nonlinear partial differential equation:

$$u_t + uu_x + \alpha u_{xx} + \beta u_{xxx} + \delta u_{xxxx} = 0, \quad (4.1.1)$$

which describes physical process in motion of turbulence and other unstable systems. Depending on coefficients α , β and δ , there are several special cases of Eq.(4.1.1),

- (i) if $\beta = \delta = 0$, Eq.(4.1.1) becomes the Burgers equation [125]:

$$u_t + uu_x + \alpha u_{xx} = 0,$$

(ii) if $\alpha = \delta = 0$, Eq.(4.1.1) represents the KdV equation [40]:

$$u_t + uu_x + \beta u_{xxx} = 0,$$

(iii) if $\delta = 0$, Eq.(4.1.1) represents the KdV–Burgers equation [178, 239]:

$$u_t + uu_x + \alpha u_{xx} + \beta(t)u_{xxx} = 0$$

(iv) and if $\beta = 0$, Eq.(4.1.1) becomes the Kuramoto–Sivashinsky equation [227]:

$$u_t + uu_x + \alpha u_{xx} + \delta u_{xxxx} = 0.$$

The Benny equation is a composition of the KdV, Burgers, KdV–Burgers and the KS equations. It involves nonlinearity, dispersion, dissipation and instability simultaneously. This equation is an important mathematical model arising in many different physical phenomena especially turbulence processes. Different effective methods have been applied to construct the exact solutions of the Benny equation. These methods include tanh function method, homogeneous balance method and the generalized F-expansion method. New exact traveling wave solutions are obtained including trigonometric function solutions, combined non-degenerate Jacobi elliptic function solutions and solitary wave solutions ,etc., [96].

In this chapter, the Benny equation with variable coefficients

$$u_t + uu_x + \alpha(t)u_{xx} + \beta(t)u_{xxx} + \delta(t)u_{xxxx} = 0. \tag{4.1.2}$$

has been considered for checking the integrability and also finding its exact solutions. The solutions of equation (4.1.2) have been obtained by finding the symmetries

using Lie's classical method and solving the reduced ODEs. The integrability of this equation has been checked by using Painlevé test before finding its solutions.

4.2 Painlevé Analysis

A partial differential equation which possess the Painlevé property must be single valued about the movable singularity manifold [99, 100, 127, 136]. To check the integrability of equation (4.1.2) by Painlevé test, let the solutions of this equation in the form of Laurent expansion given as

$$u(x, t) = g(x, t)^\gamma \sum_{j=0}^{\infty} u_j g^j(x, t), \quad (4.2.1)$$

where $g = g(x, t)$, $u_j = u_j(x, t)$ and $u_0 \neq 0$, are the analytic functions of (x, t) in the neighbourhood of $g = 0$ and γ is a negative integer. Substituting Eq.(4.2.1) in Benny equation (4.1.2), we obtained the value of leading order is $\gamma = -3$ and

$$u_0 = 120\delta(t)g_x^3, \quad (4.2.2)$$

where g_x represent the partial derivative of $g(x, t)$ with respect to independent variable x . The resonances are determined by substituting the laurent series expansion (4.2.1) into Eq.(4.1.2) and then by collecting the coefficients of like terms, we get the polynomial equation given as

$$(j - 6)(j + 1)(j^2 - 13j + 60)\delta(t)g^{(j-3)}u_j = F(u_{j-1}, \dots, u_0, g_x, g_{xx} \dots). \quad (4.2.3)$$

Using Eq.(4.2.3) the resonance are found to be $j = -1, 6, \frac{13+i\sqrt{71}}{2}, \frac{13-i\sqrt{71}}{2}$. The arbitrariness of the function $g(x, t) = 0$ is given at $j = -1$ but for the resonances

at $j = \frac{13+i\sqrt{71}}{2}$ and $j = \frac{13-i\sqrt{71}}{2}$, the real parts of two resonances are larger than zero which indicates the equation (4.1.2) has a movable branch point at $g(x, t) = 0$. Thus Eq.(4.1.2) fails to pass Painleve test [193].

4.3 Symmetry Analysis and Exact Solutions

Lie classical method have been applied for finding the symmetries of Benny equation (4.1.2) which, in turn, has been used to obtain reduced ODEs corresponding to each vector field in the optimal system. Solutions of the Eq.(4.1.2) have been found out by using $\frac{G'}{G}$ -method , tanh method, etc.

Let $\xi(x, t, u), \tau(x, t, u), \eta(x, t, u)$ be the infinitesimals corresponding to x, t, u respectively and impose the condition of invariance on (4.1.2). On using the invariance criteria as mentioned in [161], the following relation from the coefficients of the first order of ϵ is obtained:

$$\begin{aligned} \eta^t + \eta^x u + \eta u_x + \alpha(t)\eta^{xx} + \tau\alpha' u_{xx} + \beta(t)\eta^{xxx} + \tau\beta(t)' u_{xxx} \\ + \delta(t)\eta^{xxxx} + \tau\delta(t)' u_{xxxx} = 0, \end{aligned} \tag{4.3.1}$$

where (') indicates the derivative with respect to t and $\eta^t, \eta^x, \eta^{xx}, \eta^{xxx}$ and η^{xxxx} are extended (prolonged) infinitesimals acting on an enlarged space (jet space) which include all derivatives of the dependent variables. The set of determining equations for the group infinitesimals ξ, τ and η which is obtained from 4.3.1, after equating the coefficients of various derivative terms to zero, are as follows:

$$\begin{aligned}
 \tau_x &= 0, \tau_u = 0, \\
 \xi_u &= 0, \eta_{uu} = 0, \\
 \tau_t \delta(t) - 4\delta(t)\xi_x + \tau \delta(t)' &= 0, \\
 \tau \beta(t)' - 3\beta(t)\xi_x + \tau_t \beta(t) + 4\delta(t)\eta_{xu} - 6\delta(t)\xi_{xx} &= 0, \\
 \eta_t + \alpha(t)\eta_{xx} + \beta\eta_{xxx} + \delta(t)\eta_{xxxx} + u\eta_x &= 0, \\
 \tau \alpha(t)' - 2\alpha(t)\xi_x + \tau_t \alpha(t) - 4\delta(t)\xi_{xxx} + 6\delta(t)\eta_{xxu} - 3\beta(t)\xi_{xx} + 3\beta(t)\eta_{xu} &= 0, \\
 2\alpha(t)\eta_{xu} - \beta(t)\xi_{xxx} - \alpha(t)\xi_{xx} - \delta(t)\xi_{xxx} - u\xi_x + 3\beta(t)\eta_{xxu} \\
 - \xi_t + \tau_t u + \eta + 4\delta(t)\eta_{xxxu} &= 0.
 \end{aligned} \tag{4.3.2}$$

The general solution of the above system provides the infinitesimal elements ξ , τ , η

$$\begin{aligned}
 \xi &= k_1 xt + k_2 t + k_3 x + k_4, \\
 \tau &= k_1 t^2 + (k_3 - k_5)t + k_6, \\
 \eta &= k_1(x - ut) + uk_5 + k_2,
 \end{aligned} \tag{4.3.3}$$

where k_i , $i = 1, 2, \dots, 6$ are arbitrary constants and other admissible forms of coefficients functions $\alpha(t)$, $\beta(t)$ and $\delta(t)$ as follows:

$$\begin{aligned}
 \tau \delta(t)' - 4\delta(t)\xi_x + \tau_t \delta(t) &= 0, \\
 \tau \beta(t)' + \tau_t \beta(t) - 3\beta(t)\xi_x + 4\delta(t)\eta_{xu} - 6\delta(t)\xi_{xx} &= 0, \\
 \tau \alpha(t)' - 2\alpha(t)\xi_x + \tau_t \alpha(t) + 6\delta t \eta_{xxu} - 3\beta(t)\xi_{xx} + 3\beta(t)\eta_{xu} - 4\delta t \xi_{xxx} &= 0.
 \end{aligned} \tag{4.3.4}$$

The corresponding infinitesimal generators or vector fields are given by

$$\begin{aligned}
 X_1 &= t^2 \frac{\partial}{\partial t} + xt \frac{\partial}{\partial x} + (x - ut) \frac{\partial}{\partial u}, \\
 X_2 &= -t \frac{\partial}{\partial t} + u \frac{\partial}{\partial u}, \\
 X_3 &= t \frac{\partial}{\partial t} + x \frac{\partial}{\partial x}, \\
 X_4 &= t \frac{\partial}{\partial x} + \frac{\partial}{\partial u}, \\
 X_5 &= \frac{\partial}{\partial t}, \\
 X_6 &= \frac{\partial}{\partial x}.
 \end{aligned} \tag{4.3.5}$$

The nonzero commutation relations of these vector fields are

$$\begin{aligned}
 [X_1, X_2] &= -[X_2, X_1] = X_1, & [X_1, X_3] &= -[X_3, X_1] = -X_1, \\
 [X_1, X_5] &= -[X_5, X_1] = X_2 - X_3, & [X_1, X_6] &= -[X_6, X_1] = -X_4, \\
 [X_2, X_4] &= -[X_4, X_2] = -X_4, & [X_2, X_5] &= -[X_5, X_2] = X_5, \\
 [X_3, X_5] &= -[X_5, X_3] = -X_5, & [X_3, X_6] &= -[X_6, X_3] = -X_6, \\
 [X_4, X_5] &= -[X_5, X_4] = -X_6.
 \end{aligned} \tag{4.3.6}$$

Linear combination of generators $X_j; j = 1, \dots, 6$ is also a generator which is useful in constructing the optimal system. The formation of the optimal system of subalgebras can be carried out by using the adjoint transformations as suggested by Ovsianikov [113]. The basic optimal vector fields has been obtained as:

$$\langle X_6 + q_1 X_5, X_2 + q_2 X_3 + q_3 X_5, X_1, X_3, X_4 \rangle.$$

The similarity variable can be determined by simplifying the characteristic equation:

$$\frac{dx}{\xi} = \frac{dt}{\tau} = \frac{du}{\eta}. \tag{4.3.7}$$

Two constants are involved in the general solution of these equations. First is treated as new independent variable ζ and second one as the new dependent variable $f(\zeta)$. The reduced ODEs are obtained for optimal vector fields after by using the solution of (4.3.7) in Eq.(4.1.2):

(i) For the generator $X_6 + q_1 X_5$, the similarity variables are:

$\zeta = x - q_1 t$, $u = f(\zeta)$ and the group invariant solutions is

$$u = f(x - q_1 t). \tag{4.3.8}$$

By substituting (4.3.8) in Eq.(4.1.2), the reduced ordinary differential equation(ODE) is obtained as:

$$c_1 f'''' + c_2 f''' + c_3 f'' + f' f - q_1 f' = 0, \tag{4.3.9}$$

on integrating, we get

$$c_1 f'''' + c_2 f''' + c_3 f'' + \frac{f^2}{2} - q_1 f + p_1 = 0, \tag{4.3.10}$$

where $f' = \frac{df}{d\zeta}$ and $p_1, c_i, i = 1, 2, 3$ are constants.

Apply modified $\frac{G'}{G}$ -expansion method [1, 139] for the equation (4.3.10) by taking integration constant as zero. Assume the equation(4.3.10) have the solution in the form of a polynomial in $\frac{G'}{G}$ as:

$$f(\zeta) = a_0 + \sum_{i=1}^n \left\{ a_i \left(\frac{G'}{G} \right)^i + b_i \left(\frac{G'}{G} \right)^{-i} \right\}, \tag{4.3.11}$$

where a_0, a_i and b_i are constants and n is non negative value which is to be find out by considering the homogeneous balance of the highest-order nonlinear and highest-order derivatives present in ODEs equation (4.3.10). The function $G(\zeta)$ is

the auxiliary linear ODE solution given as

$$G''(\zeta) + \mu G(\zeta) = 0, \quad (4.3.12)$$

where constant μ is to be find out. Following types of solutions depending upon μ are obtained as:

1. If $\mu < 0$, the hyperbolic-type solution

$$\frac{G'}{G} = \sqrt{-\mu} \left(\frac{A \sinh \sqrt{-\mu}\zeta + B \cosh \sqrt{-\mu}\zeta}{A \cosh \sqrt{-\mu}\zeta + B \sinh \sqrt{-\mu}\zeta} \right), \quad (4.3.13)$$

2. If $\mu > 0$, the trigonometric-type solution

$$\frac{G'}{G} = \sqrt{\mu} \left(\frac{A \cos \sqrt{\mu}\zeta - B \sin \sqrt{\mu}\zeta}{A \sin \sqrt{\mu}\zeta + B \cos \sqrt{\mu}\zeta} \right), \quad (4.3.14)$$

where A, B are arbitrary constants.

To solve the ODE (4.3.10), apply the homogeneous balancing method for highest order derivatives and the nonlinear terms. We get the value of $n = 3$, using it in Eq.(4.3.11) gives

$$\begin{aligned} f(\zeta) = & a_0 + a_1 \left(\frac{G'}{G} \right) + a_2 \left(\frac{G'}{G} \right)^2 + a_3 \left(\frac{G'}{G} \right)^3 \\ & + b_1 \left(\frac{G'}{G} \right)^{-1} + b_2 \left(\frac{G'}{G} \right)^{-2} + b_3 \left(\frac{G'}{G} \right)^{-3}. \end{aligned} \quad (4.3.15)$$

After substituting Eq.(4.3.15) into the ODE Eq.(4.3.10), collect all the terms of same powers of $\left(\frac{G'}{G} \right)$ together and on equating their coefficient to zero gives a system of algebraic equations for $a_0, a_1, a_2, a_3, b_1, b_2, b_3$ and μ . The results obtained after solving the set of equations are given as:

Set I

$$\begin{aligned}
 a_0 = 0, b_1 = 0, b_3 = 0, a_2 = -15c_2, a_3 = 120c_1, \\
 b_2 = -12c_2\mu^2, a_1 = \frac{15}{76} \frac{608 c_1^2 \mu - c_2^2 + 16 c_3 c_1}{c_1}, \mu = \frac{q_1}{8c_2},
 \end{aligned} \tag{4.3.16}$$

Set II

$$\begin{aligned}
 a_0 = 0, b_2 = 0, a_2 = -15c_2, a_3 = 120c_1, b_3 = -120c_1\mu^3, \\
 b_1 = -\frac{60}{19} \mu (c_3 + 38 c_1 \mu), a_1 = \frac{15}{76} \frac{608 c_1^2 \mu - c_2^2 + 16 c_3 c_1}{c_1}, \\
 \mu = \frac{1}{6080} \frac{56 c_2 c_3 c_1 - 13 c_2^3 + 608 q_1 c_1^2}{c_2 c_1^2},
 \end{aligned} \tag{4.3.17}$$

Set III

$$\begin{aligned}
 a_0 = 0, b_1 = 0, a_1 = 0, a_2 = -12c_2, a_3 = 0, , \\
 b_2 = -15c_2\mu^2, b_3 = -120c_1\mu^3, \mu = \frac{q_1}{8c_2}.
 \end{aligned} \tag{4.3.18}$$

From Eq.(4.3.11) and the general solution of Eq.(4.3.12), the traveling wave solutions of Eq. (4.1.2) is obtained as follows:

If $\mu < 0$

Set I gives solution of $u(x, t)$ as :

$$\begin{aligned}
 u_1(x, t) = & \frac{15}{76} \frac{608 c_1^2 \frac{q_1}{8c_2} - c_2^2 + 16 c_3 c_1}{c_1} \\
 & \left(\sqrt{-\frac{q_1}{8c_2}} \left(\frac{A \sinh \sqrt{-\frac{q_1}{8c_2}}(x - q_1 t) + B \cosh \sqrt{-\frac{q_1}{8c_2}}(x - q_1 t)}{A \cosh \sqrt{-\frac{q_1}{8c_2}}(x - q_1 t) + B \sinh \sqrt{-\frac{q_1}{8c_2}}(x - q_1 t)} \right) \right) \\
 & - 15c_2 \left(\sqrt{-\frac{q_1}{8c_2}} \left(\frac{A \sinh \sqrt{-\frac{q_1}{8c_2}}(x - q_1 t) + B \cosh \sqrt{-\frac{q_1}{8c_2}}\zeta}{A \cosh \sqrt{-\frac{q_1}{8c_2}}(x - q_1 t) + B \sinh \sqrt{-\frac{q_1}{8c_2}}(x - q_1 t)} \right) \right)^2 \\
 & + 120c_1 \left(\sqrt{-\frac{q_1}{8c_2}} \left(\frac{A \sinh \sqrt{-\frac{q_1}{8c_2}}\zeta + B \cosh \sqrt{-\frac{q_1}{8c_2}}(x - q_1 t)}{A \cosh \sqrt{-\frac{q_1}{8c_2}}(x - q_1 t) + B \sinh \sqrt{-\frac{q_1}{8c_2}}(x - q_1 t)} \right) \right)^3 \\
 & - 12c_2 \left(\frac{q_1}{8c_2} \right)^2 \left(\sqrt{-\frac{q_1}{8c_2}} \left(\frac{A \sinh \sqrt{-\frac{q_1}{8c_2}}(x - q_1 t) + B \cosh \sqrt{-\frac{q_1}{8c_2}}\zeta}{A \cosh \sqrt{-\frac{q_1}{8c_2}}(x - q_1 t) + B \sinh \sqrt{-\frac{q_1}{8c_2}}(x - q_1 t)} \right) \right)^{-2}.
 \end{aligned} \tag{4.3.19}$$

Set II gives the solution

$$\begin{aligned}
 u_2(x, t) = & \frac{15}{76} \frac{608 c_1^2 \mu - c_2^2 + 16 c_3 c_1}{c_1} \\
 & \left(\sqrt{-\mu} \left(\frac{A \sinh \sqrt{-\mu}(x - q_1 t) + B \cosh \sqrt{-\mu}(x - q_1 t)}{A \cosh \sqrt{-\mu}(x - q_1 t) + B \sinh \sqrt{-\mu}(x - q_1 t)} \right) \right) \\
 & - 15c_2 \sqrt{-\mu} \left(\frac{A \sinh \sqrt{-\mu}(x - q_1 t) + B \cosh \sqrt{-\mu}(x - q_1 t)}{A \cosh \sqrt{-\mu}(x - q_1 t) + B \sinh \sqrt{-\mu}(x - q_1 t)} \right)^2 \\
 & + 120c_1 \sqrt{-\mu} \left(\frac{A \sinh \sqrt{-\mu}(x - q_1 t) + B \cosh \sqrt{-\mu}(x - q_1 t)}{A \cosh \sqrt{-\mu}(x - q_1 t) + B \sinh \sqrt{-\mu}(x - q_1 t)} \right)^3 \\
 & - \frac{60}{19} \mu (c_3 + 38 c_1 \mu) \sqrt{-\mu} \left(\frac{A \sinh \sqrt{-\mu}(x - q_1 t) + B \cosh \sqrt{-\mu}(x - q_1 t)}{A \cosh \sqrt{-\mu}(x - q_1 t) + B \sinh \sqrt{-\mu}(x - q_1 t)} \right)^{-1} \\
 & - 120(c_1 \mu^3) \sqrt{-\mu} \left(\frac{A \sinh \sqrt{-\mu}(x - q_1 t) + B \cosh \sqrt{-\mu}(x - q_1 t)}{A \cosh \sqrt{-\mu}(x - q_1 t) + B \sinh \sqrt{-\mu}(x - q_1 t)} \right)^{-3},
 \end{aligned} \tag{4.3.20}$$

where $\mu = \frac{1}{6080} \frac{56 c_2 c_3 c_1 - 13 c_2^3 + 608 q_1 c_1^2}{c_2 c_1^2}$.

Set III gives solution as

$$\begin{aligned}
 u_3(x, t) = & -12c_2 \sqrt{-\left(\frac{q_1}{8c_2}\right)} \left(\frac{A \sinh \sqrt{-\left(\frac{q_1}{8c_2}\right)}(x - q_1t) + B \cosh \sqrt{-\left(\frac{q_1}{8c_2}\right)}(x - q_1t)}{A \cosh \sqrt{-\left(\frac{q_1}{8c_2}\right)}(x - q_1t) + B \sinh \sqrt{-\left(\frac{q_1}{8c_2}\right)}(x - q_1t)} \right)^2 \\
 & - 15c_2 \left(\frac{q_1}{8c_2}\right)^2 \sqrt{-\left(\frac{q_1}{8c_2}\right)} \left(\frac{A \sinh \sqrt{-\left(\frac{q_1}{8c_2}\right)}(x - q_1t) + B \cosh \sqrt{-\left(\frac{q_1}{8c_2}\right)}(x - q_1t)}{A \cosh \sqrt{-\left(\frac{q_1}{8c_2}\right)}(x - q_1t) + B \sinh \sqrt{-\left(\frac{q_1}{8c_2}\right)}(x - q_1t)} \right)^{-2} \\
 & - 120c_1 \left(\frac{q_1}{8c_2}\right)^3 \sqrt{-\left(\frac{q_1}{8c_2}\right)} \left(\frac{A \sinh \sqrt{-\left(\frac{q_1}{8c_2}\right)}(x - q_1t) + B \cosh \sqrt{-\left(\frac{q_1}{8c_2}\right)}(x - q_1t)}{A \cosh \sqrt{-\left(\frac{q_1}{8c_2}\right)}(x - q_1t) + B \sinh \sqrt{-\left(\frac{q_1}{8c_2}\right)}(x - q_1t)} \right)^{-3}.
 \end{aligned} \tag{4.3.21}$$

If $\mu \succ 0$

For Set I, the solution is

$$\begin{aligned}
 u_4(x, t) = & \frac{15}{76} \frac{608 c_1^2 \frac{q_1}{8c_2} - c_2^2 + 16 c_3 c_1}{c_1} \\
 & \left(\sqrt{\frac{q_1}{8c_2}} \left(\frac{A \cos \sqrt{\frac{q_1}{8c_2}}(x - q_1t) - B \sin \sqrt{\frac{q_1}{8c_2}}(x - q_1t)}{A \sin \sqrt{\frac{q_1}{8c_2}}(x - q_1t) + B \cos \sqrt{\frac{q_1}{8c_2}}(x - q_1t)} \right) \right) \\
 & - 15c_2 \left(\sqrt{\frac{q_1}{8c_2}} \left(\frac{A \cos \sqrt{\frac{q_1}{8c_2}}(x - q_1t) - B \sin \sqrt{\frac{q_1}{8c_2}}\zeta}{A \sin \sqrt{\frac{q_1}{8c_2}}(x - q_1t) + B \cos \sqrt{\frac{q_1}{8c_2}}(x - q_1t)} \right) \right)^2 \\
 & + 120c_1 \left(\frac{q_1}{8c_2}\right)^2 \left(\sqrt{\frac{q_1}{8c_2}} \left(\frac{A \cos \sqrt{\frac{q_1}{8c_2}}\zeta - B \sin \sqrt{\frac{q_1}{8c_2}}(x - q_1t)}{A \sin \sqrt{\frac{q_1}{8c_2}}\zeta + B \cos \sqrt{\frac{q_1}{8c_2}}(x - q_1t)} \right) \right)^3 \\
 & - 12c_2 \left(\frac{q_1}{8c_2}\right)^2 \left(\sqrt{\frac{q_1}{8c_2}} \left(\frac{A \cos \sqrt{\frac{q_1}{8c_2}}(x - q_1t) - B \sin \sqrt{\frac{q_1}{8c_2}}(x - q_1t)}{A \sin \sqrt{\frac{q_1}{8c_2}}(x - q_1t) + B \cos \sqrt{\frac{q_1}{8c_2}}(x - q_1t)} \right) \right)^{-2}.
 \end{aligned} \tag{4.3.22}$$

Set II gives the solution

$$\begin{aligned}
 u_5(x, t) = & \frac{15}{76} \frac{608 c_1^2 \mu - c_2^2 + 16 c_3 c_1}{c_1} \\
 & \left(\sqrt{\mu} \left(\frac{A \cos \sqrt{\mu}(x - q_1 t) - B \sin \sqrt{\mu}(x - q_1 t)}{A \sin \sqrt{\mu}(x - q_1 t) + B \cos \sqrt{\mu}(x - q_1 t)} \right) \right) \\
 & - 15 c_2 \sqrt{\mu} \left(\frac{A \cos \sqrt{\mu}(x - q_1 t) - B \sin \sqrt{\mu}(x - q_1 t)}{A \sin \sqrt{\mu}(x - q_1 t) + B \cos \sqrt{\mu}(x - q_1 t)} \right)^2 \\
 & + 120 c_1 \sqrt{\mu} \left(\frac{A \cos \sqrt{\mu}(x - q_1 t) - B \sin \sqrt{\mu}(x - q_1 t)}{A \sin \sqrt{\mu}(x - q_1 t) + B \cos \sqrt{\mu}(x - q_1 t)} \right)^3 \\
 & - \frac{60}{19} \mu (c_3 + 38 c_1 \mu) \sqrt{\mu} \left(\frac{A \cos \sqrt{\mu}(x - q_1 t) - B \sin \sqrt{\mu}(x - q_1 t)}{A \sin \sqrt{\mu}(x - q_1 t) + B \cos \sqrt{\mu}(x - q_1 t)} \right)^{-1} \\
 & - 120 (c_1 \mu^3) \sqrt{\mu} \left(\frac{A \cos \sqrt{\mu}(x - q_1 t) - B \sin \sqrt{\mu}(x - q_1 t)}{A \sin \sqrt{\mu}(x - q_1 t) + B \cos \sqrt{\mu}(x - q_1 t)} \right)^{-3}, \quad (4.3.23)
 \end{aligned}$$

where $\mu = \frac{1}{6080} \frac{56 c_2 c_3 c_1 - 13 c_2^3 + 608 q_1 c_1^2}{c_2 c_1^2}$.

Set III gives solution as

$$\begin{aligned}
 u_6(x, t) = & -12 c_2 \sqrt{\left(\frac{q_1}{8 c_2}\right)} \left(\frac{A \cos \sqrt{\left(\frac{q_1}{8 c_2}\right)}(x - q_1 t) - B \sin \sqrt{\left(\frac{q_1}{8 c_2}\right)}(x - q_1 t)}{A \sin \sqrt{\left(\frac{q_1}{8 c_2}\right)}(x - q_1 t) + B \cos \sqrt{\left(\frac{q_1}{8 c_2}\right)}(x - q_1 t)} \right)^2 \\
 & - 15 c_2 \left(\frac{q_1}{8 c_2}\right)^2 \sqrt{\left(\frac{q_1}{8 c_2}\right)} \left(\frac{A \cos \sqrt{\left(\frac{q_1}{8 c_2}\right)}(x - q_1 t) - B \sin \sqrt{\left(\frac{q_1}{8 c_2}\right)}(x - q_1 t)}{A \sin \sqrt{\left(\frac{q_1}{8 c_2}\right)}(x - q_1 t) + B \cos \sqrt{\left(\frac{q_1}{8 c_2}\right)}(x - q_1 t)} \right)^{-2} \\
 & - 120 c_1 \left(\frac{q_1}{8 c_2}\right)^3 \sqrt{\left(\frac{q_1}{8 c_2}\right)} \left(\frac{A \cos \sqrt{-\left(\frac{q_1}{8 c_2}\right)}(x - q_1 t) - B \sin \sqrt{\left(\frac{q_1}{8 c_2}\right)}(x - q_1 t)}{A \sin \sqrt{-\left(\frac{q_1}{8 c_2}\right)}(x - q_1 t) + B \cos \sqrt{\left(\frac{q_1}{8 c_2}\right)}(x - q_1 t)} \right)^{-3}. \quad (4.3.24)
 \end{aligned}$$

(ii) For the generator $X_2 + q_2 X_3 + q_3 X_5$

By consider the scalars $q_2 = q_3 = 1$, the similarity variable, similarity solution and coefficient functions are obtained as:

$$\zeta = xe^{-t}, u = e^t f(\zeta), \delta(t) = c_4 e^{4t}, \beta(t) = c_5 e^{3t}, \alpha(t) = c_6 e^{2t}. \quad (4.3.25)$$

Substituting (4.3.25) into Eq.(4.1.2), we get

$$c_4 f'''' + c_5 f''' + c_6 f'' + ff' - \zeta f' + f = 0, \quad (4.3.26)$$

where $c_i, i = 4, 5, 6$ are constants and prime (') is the derivative with respect to independent variable ζ .

To find the solution of Eq.(4.3.26), assume the solution in the form

$$f = a_0 + \sum_{i=1}^n \frac{a_i}{\zeta^i} + \sum_{i=1}^n b_i \zeta^i. \quad (4.3.27)$$

By balancing the f'''' and ff' in equation 4.3.26, we get $n = 3$. Putting (4.3.27) in (4.3.26), a set of algebraic equations is obtained in terms of $a_0, a_1, a_2, a_3, b_1, b_2, b_3$ as:

- (i) $2b_2^2 + 4b_1b_3 - 2b_3 = 0,$
- (ii) $3a_0b_3 + 3b_1b_2 - b_2 = 0,$
- (iii) $6c_6b_3 + 2a_1b_3 + 2a_0b_2 + b_1^2 = 0,$
- (iv) $a_2b_3 + 2c_6b_2 + a_0b_1 + a_0 + a_1b_2 + 6c_5b_3 = 0,$
- (v) $3a_2 - a_0a_1 - a_3b_2 - a_2b_1 = 0,$
- (vi) $4a_3 + 2c_6a_1 - a_1^2 - 2a_3b_1 - 2a_0a_2 = 0,$
- (vii) $6c_6a_2 - 3a_1a_2 - 3a_0a_3 - 6c_4a_1 = 0,$
- (viii) $24c_4a_1 + 12c_6a_3 - 4a_1a_3 - 24c_5a_2 - 2a_2^2 = 0,$

$$\begin{aligned}
 (ix) \quad & 120c_4a_2 - 5a_2a_3 - 60c_5a_3 = 0, \\
 (x) \quad & 360c_4a_3 - 3a_3^2 = 0.
 \end{aligned} \tag{4.3.28}$$

On simplifying these equations by a software, we obtain the following results:

Set I

$$\begin{aligned}
 a_0 = 0, \quad a_1 = -2c_6, \quad a_2 = c_5, \quad a_3 = 0, \\
 b_1 = -3, \quad b_2 = 0, \quad b_3 = \frac{-9}{2c_6}.
 \end{aligned} \tag{4.3.29}$$

Set II

$$\begin{aligned}
 a_0 = 0, \quad a_1 = \frac{(180c_{13}c_4 + 55c_5^2)}{63c_4}, \quad a_2 = 10c_5, \quad a_3 = -120c_4, \\
 b_1 = 0, \quad b_2 = \frac{-c_{12}}{4c_4}, \quad b_3 = \frac{c_5}{6c_4^2}.
 \end{aligned} \tag{4.3.30}$$

The solution of Eq.(4.3.26) corresponding to Set I is found as

$$f(\zeta) = \left[\frac{-2c_6}{\zeta} + \frac{c_5}{\zeta^2} - 3\zeta + \frac{-9\zeta^3}{2c_6} \right], \tag{4.3.31}$$

by back substitution, the solution of Benny equation (4.1.2) is

$$u_7(x, t) = e^t \left[\frac{-2c_6e^t}{x} + \frac{c_5e^{2t}}{x^2} - 3\frac{x}{e^t} - \frac{9x^3}{2c_6e^{3t}} \right]. \tag{4.3.32}$$

Set II gives the solution of Eq.(4.3.37) as

$$f(\zeta) = \frac{(180c_{13}c_4 + 55c_5^2)}{63c_4} \frac{1}{\zeta} + \frac{10c_5}{\zeta^2} - 120\frac{c_4}{\zeta^3} - \frac{c_5}{4c_4}\zeta^2 + \frac{c_5}{6c_4^2}\zeta^3, \tag{4.3.33}$$

using (4.3.33), the solution of Eq.(4.1.2) is obtained as:

$$u_8(x, t) = e^t \left[\frac{(180c_6c_4 + 55c_5^2) e^t}{63c_4} \frac{1}{x} + 10c_5 \frac{e^{2t}}{x^2} - 120c_4 \frac{e^{3t}}{x^3} - \frac{c_5}{4c_4} \frac{x^2}{e^{2t}} + \frac{c_5}{6c_4^2} \frac{x^3}{e^{3t}} \right]. \quad (4.3.34)$$

(iii) For the generator X_1

We get $\zeta = \frac{x}{t}$ and $u = \frac{x}{t} + \frac{f(\zeta)}{t}$ and the group invariant solutions as

$$u = \zeta + \frac{f(\zeta)}{t}. \quad (4.3.35)$$

Substituting Eq.(4.3.35) in Eq.(4.1.2), we get following reduce ODE:

$$c_7 f'''' + c_8 f''' + c_9 f'' + f' f = 0, \quad (4.3.36)$$

on integrating this equation w.r.t. ζ , we get

$$c_7 f''' + c_8 f'' + c_9 f' + \frac{f^2}{2} + p_2 = 0, \quad (4.3.37)$$

where $f' = \frac{df}{d\zeta}$ and $p_2, c_i, i = 7, 8, 9$ are constants. Apply the tanh method [16, 180] for the ODE Eq.(4.3.37) by considering the constant of integration as zero.

To find the solution of Eq.(4.3.37), assume

$$f(\zeta) = \sum_{i=0}^n a_i (\tanh(\mu\zeta))^i, \quad (4.3.38)$$

where $a_i, i = 1, 2, 3, \dots$ and μ are all constants to be found out. The value of $n = 3$ is obtained by balancing the the higher derivative of the equation and nonlinear terms in Eq.(4.3.37). Substituting $n = 3$ in Eq.(4.3.38) and using the resulting equation in Eq.(4.3.37) gives several possibilities for the different values of the constants.

Depending upon these possibilities, the following sets solutions of Benny equation (4.1.2) are obtained.

Set I

$$\begin{aligned}
 a_0 &= 2c_8\mu^2, \quad a_1 = (-16c_7\mu^3 + 2c_9\mu), \quad a_2 = 0, \quad a_3 = 0, \\
 \mu &= \frac{1}{8} \left(\frac{\sqrt{2}\sqrt{10c_7c_9 - c_8^2} + \sqrt{36c_7^2c_9^2 - 20c_7c_9c_8^2 + c_8^4}}{c_7} \right),
 \end{aligned}
 \tag{4.3.39}$$

gives the solution

$$u_9 = 2c_8\mu^2 + (-16c_7\mu^3 + 2c_9\mu) \left(\tanh\left(\mu \frac{x}{t}\right) \right).
 \tag{4.3.40}$$

Set II

$$\begin{aligned}
 a_0 &= 0, \quad a_1 = 0, \\
 a_2 &= -\frac{1}{20} \frac{(-15c_8^2 + 45c_7c_9)(-c_9 + \frac{1}{45} \frac{15c_8^2 + 45c_7c_9}{c_7})}{c_7c_8} \\
 a_3 &= \frac{1}{225} \frac{(15c_8^2 + 45c_7c_9)}{(c_7^2)} \\
 \mu &= \frac{1}{30} \frac{\sqrt{15c_8^2 + 45c_7c_9}}{c_7},
 \end{aligned}
 \tag{4.3.41}$$

the corresponds solution is

$$\begin{aligned}
 u_{10} &= -\frac{1}{20} \frac{(-15c_8^2 + 45c_7c_9)(-c_9 + \frac{1}{45} \frac{15c_8^2 + 45c_7c_9}{c_7})}{c_7c_8} \left(\tanh\left(\mu \frac{x}{t}\right) \right)^2 \\
 &+ \frac{1}{225} \frac{(15c_8^2 + 45c_7c_9)}{(c_7^2)} \left(\tanh\left(\mu \frac{x}{t}\right) \right)^3.
 \end{aligned}
 \tag{4.3.42}$$

Set III

$$\begin{aligned} a_0 = 0, a_1 &= \frac{45}{361} \frac{\sqrt{-19c_7c_8c_9}}{c_7}, \\ a_2 = 0, a_3 &= \frac{15}{6859} \frac{(-19c_7c_9)}{c_7^2} \mu = \frac{1}{38} \frac{\sqrt{-19c_7c_9}}{c_7}, \end{aligned} \quad (4.3.43)$$

results the solution

$$u_{11} = \frac{45}{361} \frac{\sqrt{-19c_7c_8c_9}}{c_7} (\tanh(\mu \frac{x}{t})) + \frac{15}{6859} \frac{(-19c_7c_8)}{c_7^2} (\tanh(\mu \frac{x}{t}))^3. \quad (4.3.44)$$

Set IV

$$\begin{aligned} a_0 &= \frac{2}{5} \frac{(c_8c_9)}{c_7}, a_1 = 0, a_2 = \frac{-4}{5} \frac{(c_8c_9)}{c_7}, a_3 = 0 \\ \mu &= \frac{1}{10} \frac{(\sqrt{5})(\sqrt{(c_7c_9)})}{c_7}, \end{aligned} \quad (4.3.45)$$

gives the solution as

$$u_{12} = \frac{2}{5} \frac{(c_8c_9)}{c_7} + \frac{-4}{5} \frac{(c_8c_9)}{c_7} (\tanh(\mu \frac{x}{t}))^2. \quad (4.3.46)$$

(iv) For the generator X_3

We get $\zeta = \frac{x}{t}$ and $u = f(\zeta)$, the group invariant solutions as

$$u = f(\zeta). \quad (4.3.47)$$

Substituting (4.3.47) in Eq.(4.1.2), the following reduce ODE is obtained:

$$c_{10}f'''' + c_{11}f''' + c_{12}f'' + f'f - \zeta f' = 0, \quad (4.3.48)$$

where $f' = \frac{df}{d\zeta}$ and $c_i, i = 10, 11, 12$ are constants. For the solution of Eq.(4.3.48), consider the solution in the form

$$f = a_0 + \sum_{i=1}^n \frac{a_i}{\zeta^i} + \sum_{i=1}^n b_i \zeta^i. \quad (4.3.49)$$

By balancing the f'''' and ff' in Eq.4.3.49, we get $n = 3$. Using $n = 3$ in Eq. (4.3.49) then substituting the resulting equation in Eq.(4.3.48), a set of algebraic equations is obtained for $a_0, a_1, a_2, a_3, b_1, b_2, b_3$ given as

$$\begin{aligned} (i) \quad & -3b_3 + 2b_2^2 + 4b_1b_3 = 0, \\ (ii) \quad & 3a_0b_3 - 2b_2 + 3b_1b_2 = 0, \\ (iii) \quad & 2a_1b_3 + 2a_0b_2 - b_1 + 6c_{12}b_3 + b_1^2 = 0, \\ (iv) \quad & a_0b_1 + a_1b_2 + a_2b_3 + 6c_{11}b_3 + 2c_{12}b_2 = 0, \\ (v) \quad & 2a_2 - a_3b_2 - a_0a_1 - a_2b_1 = 0, \\ (vi) \quad & -2a_0a_2 + 3a_3 - 2a_3b_1 + 2c_{12}a_1 - a_1^2 = 0, \\ (vii) \quad & 6c_{12}a_2 - 3a_0a_3 - 6c_{11}a_1 - 3a_1a_2 = 0, \\ (viii) \quad & -4a_1a_3 - 24c_{11}a_2 - 2a_2^2 + 24c_{10}a_1 + 12c_{12}a_3 = 0, \\ (ix) \quad & 120c_{10}a_2 - 5a_2a_3 - 60c_{11}a_3 = 0, \\ (x) \quad & -3a_3^2 + 360c_{10}a_3 = 0. \end{aligned} \quad (4.3.50)$$

The result obtained after solving the above equations by software is

$$\begin{aligned} a_0 = 0, \quad a_1 = \frac{5c_{11}}{2}, \quad a_2 = -10c_{11}, \quad a_3 = -120c_{10}, \\ b_1 = 0, \quad b_2 = \frac{16c_{11}}{891c_{12}^2}, \quad b_3 = \frac{2}{99c_{12}}. \end{aligned} \quad (4.3.51)$$

The corresponding solution of Eq.(4.1.2) is obtained as:

$$u_{13}(x, t) = \frac{5c_{11}}{2} \frac{t}{x} - 10c_{11} \frac{t^2}{x^2} - 120c_{10} \frac{t^3}{x^3} + \frac{16c_{11}}{891c_{12}^2} \frac{x^2}{t^2} + \frac{2}{99c_{12}} \frac{x^3}{t^3}. \quad (4.3.52)$$

(v) For the generator X_4

This equation has a trivial solution $u_{14} = c$, where c is an arbitrary constant.

4.4 Conculsion

In the present chapter, Painlevé test has been used for the Benny equation in three time-dependent variable coefficients. It has been found that Benny equation does not pass the Painlevé test and, therefore, it is not integrable. The symmetries of Benny equation are determined by the Lie's classical method. The symmetries are applied to get the optimal vector fields which are further used to reduce the number of independent variables. The reduced equation corresponding to respective vector field is in the form of ordinary differential equation. The solutions of these ODEs are found by applying $\frac{G'}{G}$ -expansion method, tanh method. The solutions of Benny equations are hyperbolic functions and trigonometric functions solutions have been obtained. In $\frac{G'}{G}$ -expansion method, for $\mu < 0$, hyperbolic solutions have been obtained whereas for $\mu > 0$ the trigonometric solutions are obtained. The tanh hyperbolic method gave the solutions in the form of tanh function. Power series solutions of Benny equation are also obtained.

Chapter 5

Painlevé Analysis, Auto-Bäcklund Transformation and Lie Symmetries for the Variable Coefficients Extended Ostrovsky Equation

5.1 Introduction

The extended Ostrovsky equation (EOE),

$$(u_t + cu_x + \alpha uu_x + \alpha_1 u^2 u_x + \beta u_{xxx})_x - \delta u = 0, \quad (5.1.1)$$

where $u = u(x, t)$, is used as a model of large-amplitude internal oceanic waves affected by earth's rotation [119, 134]. It contains two types of dispersion, the Boussinesq dispersion (β) for non-hydrostatic effects and Coriolis dispersion (δ) for

the earth's rotation. The equation also contain two nonlinear terms α_1 and α which are the coefficients of cubic and quadratic nonlinearities, respectively. Equation (5.1.1) in the particular cases reduces to different well-known model equations:

- (i) If $\delta \neq 0$, but $\alpha_1 = 0$, Eq.(5.1.1) is known as the Ostrovsky equation which is used for explaining the internal and surface waves in rotating oceans [114].
- (ii) If $\delta = 0$, Eq.(5.1.1)) represents the Gardner equation [81].
- (iii) If $\delta = 0$ and $\alpha_1 = 0$, Eq.(5.1.1) represents the KdV equation [40].
- (iv) If $c = 0$, $\alpha_1 = 0$, $\alpha = 1$, $\beta = 1$ and $\delta = 1$, Eq.(5.1.1)) represents the short-wave equation. [118]
- (v) If $\beta = 0$, Eq.(5.1.1) represents the reduced form of Ostrovsky equation (ROE) [47, 210].

Mathematical properties of the EOE and their various forms were studied the global and local well-posedness in energy space [54], solitary waves stability [224] and solutions convergence in the limit of the Kortewegde Vries equation [188]. Varlamov and Liu [211, 223] explained the initial-value problems in unidirectional long waves propagation in a rotating homogeneous incompressible fluid. Various Numerical scheme like finite-difference have been developed for the solution of Ostrovsky equations [151, 203].

Some methods like F-expansion method, exponential rational function method, hyperbolic tangent method, have been applied for finding solitary solutions of the various forms of the nonlinear EOE. Liu et al. [225] used Jacobi elliptic function method to obtain exact solitary and periodic wave solutions. Parkes [48] gave the categorization of the wave solutions. Xie and Cai [195] applied the method of bifurcation for dynamic systems and method of simulation of differential equations

to get exact compacton and generalized kink wave solutions. Stepanyants [222] applied the qualitative theory of differential equations to give the classification of solutions. Biswas et al.[4] applied the mapping and ansatz method to solve the Ostrovsky equation. Gandarias and Bruzón [129] find the classical Lie symmetries of some particular form of Ostrovsky equation.

Many physical situations can be explained by nonlinear equations in constant coefficients form. Now, a lot of attention has been given to the variable coefficient form of these nonlinear equations. Solution of the variable coefficients also explain the nonlinear phenomena and describe the various aspects of the physical phenomena.

In this chapter, the extended Ostrovsky equation (EOE) [134] with time dependent variable coefficients form is considered as:

$$(u_t + cu_x + \alpha(t)uu_x + \alpha_1(t)u^2u_x + \beta(t)u_{xxx})_x - \delta u = 0. \quad (5.1.2)$$

For a variable coefficients extended Ostrovsky equation (EOE), it is completely integrable if the variable coefficients must satisfy some specific constraint conditions. Painlevé test has been applied to get the constraint conditions in section 5.2. Auto-Bäcklund transformation [68] using the truncated Painlevé expansion analysis are applied to the solutions in section 5.3.

Lie classical method is used to obtain the various symmetries of the variable coefficients extended Ostrovsky equation (EOE) and an optimal system using basic vector fields is identified and the reduced ordinary differential equations (ODEs) and their exact solutions are obtained as discussed in section 5.4 and 5.5. Conclusion drawn are presented in the last section.

5.2 Painlevé Analysis

In 1983, the Painlevé property has been defined by Weiss et.al [99, 100] for partial differential equation. They developed a method for testing a particular type of movable singularity, without performing any similarity reductions. Painlevé test consisted by three steps as discuss in section (1.3).

- (i) The leading order analysis,
- (ii) The resonances determination,
- (iii) The resonances condition veification.

Applying the approach of Painlevé PDE test, the solutions of Eq.(5.1.2) is consider in the form of Laurent expansion given as:

$$u(x, t) = g(x, t)^\gamma \sum_{i=0}^{\infty} u_i g^i(x, t), \quad (5.2.1)$$

where $u_i = u_i(x, t)$, $u_0 \neq 0$ and $g = g(x, t)$ is the analytic functions in the neighbourhood of $g = 0$. The leading order for this case is $\gamma = -1$ and following the procedure in section (1.3), we get

$$u_0 = \frac{\sqrt{-6 \alpha_1(t) \beta(t)} g_x(x, t)}{\alpha_1(t)}, \quad (5.2.2)$$

where g_x means the partial differentiation of $g(x, t)$ with respect to independent variable x . The resonances is foundd out by subsituting the Laurent series (5.2.1) into Eq.(5.1.2). Equating the coefficients of same terms, the polynomial equation is given as

$$\alpha_1(t)(r+1)(r-3)(r-4)^2 u_r(x, t) (g(x, t))^{r-1} = F(u_{i-1}, \dots, u_0, g_x, g_{xx} \dots). \quad (5.2.3)$$

Using Eq.(5.2.3) the resonances are found to be $i = -1, 3, 4, 4$. As usual, the resonance at $i = -1$ corresponds to the arbitrariness of the singular manifold $g(x, t) = 0$. There are two compatibility tests at $i = 3$ and 4 . When $i = 3$, u_3 is arbitrary and when $i = 4$, we can get the compatibility condition of variable coefficients of equation (5.1.2) which is obtained as:

$$\begin{aligned} (\alpha(t))^2 &= 4c\alpha_1(t), \\ \left(\frac{d}{dt}\alpha_1(t)\right)\beta(t) - \alpha_1(t)\frac{d}{dt}\beta(t) &= 0. \end{aligned} \quad (5.2.4)$$

The explicit conditions after solving the Eq.(5.2.4) for the variable coefficients $\alpha(t)$, $\alpha_1(t)$ and $\beta(t)$ of Eq.(5.1.2) to pass the Painlevé test is given as:

$$\begin{aligned} \alpha(t) &= \sqrt{4c\alpha_1(t)}, \\ \beta(t) &= c_1\alpha_1(t), \end{aligned} \quad (5.2.5)$$

where c_1 is an arbitrary constant.

5.3 Auto-Bäcklund Transformations

For the derivation of auto-Bäcklund transformation [67, 84] of Eq.(5.1.2), we consider the truncated Painlevé expansion as

$$u(x, t) = \frac{u_0(x, t)}{g(x, t)} + u_1(x, t). \quad (5.3.1)$$

Substituting (5.3.1) into Eq.(5.1.2) and collecting the coefficients of same powers of $g(x, t)$ gives :

$$g^{-5} : 24\beta(t)u_0g_x^4 + 4\alpha_1(t)u_0^3g_x^3 = 0, \quad (5.3.2)$$

$$\begin{aligned} g^{-4} : 3\alpha(t)u_0g_x^2 - 24\beta(t)u_{0,x}g_x^3 + 6\alpha_1(t)u_0^2u_1g_x^2 - 36\beta(t)u_0g_x^2g_{xx} \\ - 6\alpha_1(t)u_0^2u_{0,x}g_x - \alpha_1(t)u_0^3g_{xx} = 0, \end{aligned} \quad (5.3.3)$$

$$\begin{aligned} g^{-3} : 24\beta(t)u_{0,x}g_xg_{xx} + \alpha_1(t)u_0^2u_{0,xx} - 4\alpha(t)u_{0,x}g_xu_0 - \alpha(t)u_0^2g_{xx} \\ + 2\alpha(t)u_0g_x^2u_1 + 8\beta(t)u_0g_xg_{xxx} - 2\alpha_1(t)u_0^2u_1g_{xx} + 6\beta(t)u_0g_{xx}^2 \\ - 4\alpha_1(t)u_0^2g_xu_{1,x} + 2\alpha_1(t)u_1^2u_0g_x^2 + 2u_0g_tg_x + 2cu_0g_x^2 \\ - 8\alpha_1u_0u_1u_{0,x}g_x + 2\alpha_1(t)u_{0,x}u_{0,x}^2 + 12\beta(t)u_{0,xx}u_{0,x}^2 = 0, \end{aligned} \quad (5.3.4)$$

$$\begin{aligned} g^{-2} : \alpha(t)u_{0,xx}u_0 - u_0g_{xt} - 2\alpha_1(t)u_1^2u_{0,x}g_x - \alpha(t)u_0g_{xx}u_1 - 2\alpha(t)u_0g_xu_{1,x} \\ + 2\alpha_1(t)u_1u_{0,x}^2 - 4\beta(t)u_{0,xxx}g_x - \alpha_1u_1^2u_0g_{xx} - 2cu_{0,x} + 4\alpha_1(t)u_0u_{0,x}u_{1,x} \\ - 4\beta(t)u_{0,x}g_{xxx} - cu_0g_{xx} - 6\beta(t)u_{0,xx}g_{xx} - \beta(t)u_0g_{xxxx} + \alpha(t)u_{0,x}^2 \\ + \alpha_1(t)u_0^2u_{1,xx}g_x - u_{0,t} - 2\alpha_1(t)u_{0,x}g_xu_1 - 4\alpha_1(t)u_0u_1g_xu_{1,x} \\ + 2\alpha_1(t)u_0u_1u_{0,xx} = 0, \end{aligned} \quad (5.3.5)$$

$$\begin{aligned} g^{-1} : 4\alpha_1(t)u_1u_{0,x}u_{1,x} + \alpha(t)u_{1,xx}u_{0,x} + 2\alpha_1(t)u_0u_{1,x}^2 + \beta(t)u_{0,xxxx} \\ + \alpha(t)u_{0,xx}u_1 + u_{0,xt} + 2\alpha(t)u_{0,x}u_{1,x} + 2\alpha_1(t)u_0u_1u_{1,xx} - \delta u_0 \\ + cu_{0,xx} + \alpha_1(t)u_1^2u_{0,xx} = 0, \end{aligned} \quad (5.3.6)$$

$$\begin{aligned}
 g^0 : \beta(t)u_{1,xxxx} + \alpha(t)u_{1,x}^2 + \alpha(t)u_{1,xx}u_1 + \alpha_1(t)u_1^2u_{1,xx} + cu_{1,xx} \\
 + 2\alpha_1(t)u_1u_{1,x}^2 + u_{1,xt} - \delta u_1 = 0. \tag{5.3.7}
 \end{aligned}$$

From Eq.(5.3.2), we get

$$u_0(x, t) = \frac{\sqrt{-6\alpha_1(t)\beta(t)}g_x}{\alpha_1(t)}. \tag{5.3.8}$$

Substituting Eq.(5.3.8) in equations (5.3.3) to (5.3.6), we get, respectively,

$$\alpha(t)g_x + \sqrt{6}\sqrt{-\alpha_1(t)\beta(t)}g_{xx} + 2g_xu_1\alpha_1(t) = 0, \tag{5.3.9}$$

$$\begin{aligned}
 9\beta(t)\sqrt{6}\sqrt{-\alpha_1(t)\beta(t)}g_{xx}^2 + 7\beta(t)g_x\sqrt{6}\sqrt{-\alpha_1(t)\beta(t)}g_{xxx} + 15\alpha(t)\beta(t)g_{xx}g_x \\
 \alpha(t)\sqrt{6}\sqrt{-\alpha_1(t)\beta(t)}g_x^2u_1 + 30\beta(t)g_xg_{xx}u_1\alpha_1(t) + 12\beta(t)g_x^2u_{1,x}\alpha_1(t) \\
 + u_1^2\sqrt{6}\sqrt{-\alpha_1(t)\beta(t)}g_x^2\alpha_1(t) + \sqrt{6}\sqrt{-\alpha_1(t)\beta(t)}g_xg_t + c\sqrt{6}\sqrt{-\alpha_1(t)\beta(t)}g_x^2 = 0, \tag{5.3.10}
 \end{aligned}$$

$$\begin{aligned}
 - 2\alpha(t)\beta(t)g_{xxx}g_x\sqrt{6}\sqrt{-\alpha_1(t)\beta(t)} + 4\alpha_1(t)\beta(t)g_xg_{xt} + 6u_1^2\alpha_1(t)\beta(t)g_xg_{xx} \\
 6\alpha(t)\alpha_1(t)\beta(t)g_xg_{xx}u_1 + 4\alpha(t)\alpha_1(t)\beta(t)g_xg_x^2u_{1,x} - 4u_1\beta(t)g_{xx}^2\alpha_1(t)\sqrt{6}\sqrt{-\alpha_1(t)\beta(t)} \\
 10\beta(t)^2\alpha_1(t)g_{xxxx}g_x + 6c\beta(t)\alpha_1(t)g_{xx}g_x - 8\beta(t)g_xg_{xx}u_{1,x}\alpha_1(t)\sqrt{6}\sqrt{-\alpha_1(t)\beta(t)} \\
 - 2\alpha(t)\beta(t)g_{xx}^2\sqrt{6}\sqrt{-\alpha_1(t)\beta(t)} - 2\beta(t)g_x^2g_{xx}\alpha_1(t)\sqrt{6}\sqrt{-\alpha_1(t)\beta(t)} \\
 - g_x^2\alpha_1(t)'\beta(t) + g_x^2\alpha_1(t)\beta(t)' + 2\alpha_1(t)\beta(t)g_{xx}g_t + 8u_1\alpha_1(t)^2\beta(t)g_x^2u_{1,x} \\
 + 20\beta(t)^2\alpha_1(t)g_{xx}g_{xxx} - 4g_xu_1g_{xxx}\alpha_1(t)\sqrt{6}\sqrt{-\alpha_1(t)\beta(t)} = 0, \tag{5.3.11}
 \end{aligned}$$

$$\begin{aligned}
 & 8u_1\alpha_1(t)^2\beta(t)g_{xx}u_{1,x} + 2\alpha(t)u_{1,xx}\alpha_1(t)\beta(t)g_x + 4\alpha_1(t)^2\beta(t)g_xu_{1,x}^2 + 2\beta(t)^2\alpha_1(t)g_{xxxx} \\
 & + 2\alpha_1(t)\alpha(t)\beta(t)u_1g_{xxx} - \alpha(t)'\beta(t)g_{xx} + 4\alpha_1(t)^2\beta(t)u_1u_{1,xx}g_x \\
 & - 2\delta\alpha_1(t)\beta(t)g_x + 2c\alpha_1(t)\beta(t)g_{xxx} + 2u_1^2\alpha_1(t)^2\beta(t)g_{xxx} = 0.
 \end{aligned} \tag{5.3.12}$$

Assuming the explicit constraints on the variable coefficients of Eq.(5.1.2) as

$$\begin{aligned}
 \alpha_1(t) &= c_1\beta(t), \\
 \alpha_1(t) &= 2\sqrt{cc_1}\beta(t),
 \end{aligned} \tag{5.3.13}$$

and considering the solution by auto-Bäcklund transformation as

$$g(x, t) = 1 + e^{p(t)x+q(t)}, \tag{5.3.14}$$

$$u_1(x, t) = r(t)x + s(t), \tag{5.3.15}$$

where c_1 is constant, the differentiable functions $p(t) \neq 0, q(t), r(t)$ and $s(t)$ are to be found out. Substituting Eq.(5.3.13) to Eq.(5.3.15) into Eq.(5.3.9) to Eq.(5.3.12) give rise to a set of following equations in the order:

$$2\sqrt{cc_1}\beta(t)p(t) + \sqrt{6}\sqrt{-c_1(\beta(t))^2(p(t))^2 + 2p(t)(r(t)x + s(t))c_1}\beta(t) = 0, \tag{5.3.16}$$

$$\begin{aligned}
 & 16\beta(t)\sqrt{6}\sqrt{-c_1(\beta(t))^2(p(t))^4 + 30\sqrt{cc_1}\beta(t)}\beta(t)(p(t))^3 \\
 & + 2\sqrt{cc_1}\beta(t)\sqrt{6}\sqrt{-c_1(\beta(t))^2(p(t))^2(r(t)x + s(t))c_1 + 30(\beta(t))^2(p(t))^3} \\
 & (r(t)x + s(t))c_1 + 12(\beta(t))^2(p(t))^2r(t)c_1 + (r(t)x + s(t))^2\sqrt{6}\sqrt{-c_1(\beta(t))^2}
 \end{aligned}$$

$$\begin{aligned} & (p(t))^2 c_1 \beta(t) + \sqrt{6} \sqrt{-c_1 (\beta(t))^2} p(t) \left(\left(\frac{d}{dt} p(t) \right) x + \frac{d}{dt} q(t) \right) \\ & + c \sqrt{6} \sqrt{-c_1 (\beta(t))^2} (p(t))^2 = 0, \end{aligned} \quad (5.3.17)$$

$$\begin{aligned} & - 8 \sqrt{cc_1 \beta(t)} \beta(t) (p(t))^4 \sqrt{6} \sqrt{-c_1 (\beta(t))^2} + 6 (r(t)x + s(t))^2 c_1^2 (\beta(t))^3 (p(t))^3 \\ & + 12 \sqrt{cc_1 \beta(t)} c_1 (\beta(t))^2 (p(t))^3 (r(t)x + s(t)) + 8 \sqrt{cc_1 \beta(t)} c_1 (\beta(t))^2 (p(t))^2 r(t) \\ & - 8 (r(t)x + s(t)) (\beta(t))^2 (p(t))^4 c_1 \sqrt{6} \sqrt{-c_1 (\beta(t))^2} 30 (\beta(t))^3 c_1 (p(t))^5 + 6 cc_1 (\beta(t))^2 (p(t))^3 \\ & - 8 (\beta(t))^2 (p(t))^3 r(t) c_1 \sqrt{6} \sqrt{-c_1 (\beta(t))^2} + 2 c_1 (\beta(t))^2 (p(t))^2 \left(\left(\frac{d}{dt} p(t) \right) x + \frac{d}{dt} q(t) \right) \\ & + 8 (r(t)x + s(t)) c_1^2 (\beta(t))^3 (p(t))^2 r(t) \\ & + 4 c_1 (\beta(t))^2 p(t) \left(\frac{d}{dt} p(t) + \left(\left(\frac{d}{dt} p(t) \right) x + \frac{d}{dt} q(t) \right) p(t) \right) = 0, \end{aligned} \quad (5.3.18)$$

$$\begin{aligned} & 8 (r(t)x + s(t)) c_1^2 (\beta(t))^3 (p(t))^2 r(t) + 4 c_1^2 (\beta(t))^3 p(t) (r(t))^2 + 2 (\beta(t))^3 c_1 (p(t))^5 \\ & 4 \sqrt{cc_1 \beta(t)} c_1 (\beta(t))^2 (p(t))^3 (r(t)x + s(t)) - 2 \delta c_1 (\beta(t))^2 p(t) + 2 cc_1 (\beta(t))^2 (p(t))^3 \\ & + 2 c_1 (\beta(t))^2 \left(2 \left(\frac{d}{dt} p(t) \right) p(t) + \left(\left(\frac{d}{dt} p(t) \right) x + \frac{d}{dt} q(t) \right) (p(t))^2 \right) \\ & + 8 \sqrt{cc_1 \beta(t)} c_1 (\beta(t))^2 (p(t))^2 r(t) + 2 (r(t)x + s(t))^2 c_1^2 (\beta(t))^3 (p(t))^3 = 0. \end{aligned} \quad (5.3.19)$$

Solving these from Eq.(5.3.16) to (5.3.19) by considering $r(t) = 0$, we get

$$\begin{aligned} p(t) &= k_1, \\ q(t) &= \int \frac{1}{2} \beta(t) k_1^3 dt + c_1, \end{aligned}$$

$$s(t) = -\frac{1}{2} \frac{2\sqrt{cc_1\beta(t)} + \sqrt{6}\sqrt{-c_1(\beta(t))^2}k_1}{c_1\beta(t)}, \quad (5.3.20)$$

where k_1 is constant.

For these values, the auto-Bäcklund transformation give the solution of Eq.(5.1.2) as

$$u = \frac{\sqrt{6}\sqrt{-c_1(\beta(t))^2}}{c_1\beta(t)} \log g(x) - \frac{1}{2} \frac{2\sqrt{cc_1\beta(t)} + \sqrt{6}\sqrt{-c_1(\beta(t))^2}k_1}{c_1\beta(t)}. \quad (5.3.21)$$

5.4 Lie Symmetries

In this section, the symmetries of Eq.(5.1.2) is found out by using Lie classical method. Assuming the one-parameter Lie group of infinitesimal transformation as:

$$\begin{aligned} x &\rightarrow x + \epsilon\xi(x, t, u), \\ t &\rightarrow t + \epsilon\tau(x, t, u), \\ u &\rightarrow u + \epsilon\eta(x, t, u), \end{aligned} \quad (5.4.1)$$

with a small parameter $\epsilon \ll 1$.

The vector field corresponding to the group of transformations can be written as

$$X = \xi(x, t, u) \frac{\partial}{\partial x} + \tau(x, t, u) \frac{\partial}{\partial t} + \eta(x, t, u) \frac{\partial}{\partial u}. \quad (5.4.2)$$

The symmetry group of Eq.(5.1.2) will be generated by the vector field of the form (5.4.2). Applying the fourth prolongation $\text{pr}^4 X$ of X to Eq.(5.1.2), we find the symmetry condition that satisfy by the coefficients functions ξ, τ and η :

$$\begin{aligned}
 & \eta_{xt} + c\eta_{xx} + \alpha(t)\eta^{xx}u + \alpha(t)\eta u_{xx} + \tau\alpha(t)'uu_{xx} + 2\alpha(t)\eta^x u_x \\
 & + \tau\alpha(t)'u_x^2 + \beta(t)\eta^{xxxx} + \tau\beta(t)'u_{xxxx} - \delta\eta + \alpha_1(t)\eta^{xx}u^2 \\
 & + 2\alpha_1(t)\eta uu_{xx} + \tau\alpha_1(t)'u_{xx}u^2 + 4\alpha_1(t)\eta^x uu_x + 2\eta\alpha_1(t)u_x^2 \\
 & + 2\tau\alpha_1(t)'uu_x^2 = 0,
 \end{aligned} \tag{5.4.3}$$

where $\eta^x, \eta^t, \eta^{xx}, \eta^{xt}, \eta^{xxxx}$ are coefficients of pr^4X . Using the expressions for $\eta^x, \eta^t, \eta^{xx}, \eta^{xt}, \eta^{xxxx}$ in Eq.(5.4.3) and u_{xt} is replaced by Eq.(5.1.2). The system of determining equations is obtained by putting the coefficients of different differentials equal to zero. On solving the system of determining equations, we obtained the following forms for the infinitesimal elements ξ, τ, η of Eq.(5.1.2) as:

$$\begin{aligned}
 \xi &= k_1x - 2ck_1t + k_4, \\
 \tau &= k_3 - k_1t, \\
 \eta &= k_2u,
 \end{aligned} \tag{5.4.4}$$

and their admissible forms of various coefficients $\beta(t), \alpha(t)$ and $\alpha_1(t)$ satisfy the following equation:

$$\begin{aligned}
 & \tau\beta(t)' - 3\beta(t)\xi_x + \tau_t\beta(t) = 0, \\
 & \alpha(t)\eta_u - \alpha(t)\xi_x - 2\alpha_1(t)\xi_x u + 2\eta\alpha_1(t) + \tau_t\alpha(t) + 2\tau_t\alpha_1(t)u \\
 & + \tau\alpha'(t) + 2\alpha_1'(t)\tau u + 2\alpha_1(t)u\eta_u = 0.
 \end{aligned} \tag{5.4.5}$$

The corresponding infinitesimal generators are as follows:

$$\begin{aligned}
 X_1 &= (x - 2ct) \frac{\partial}{\partial x} - t \frac{\partial}{\partial t}, \\
 X_2 &= u \frac{\partial}{\partial u}, \\
 X_3 &= \frac{\partial}{\partial x}, \\
 X_4 &= \frac{\partial}{\partial t}.
 \end{aligned} \tag{5.4.6}$$

The non-zero Lie bracket relationships are given as :

$$[X_1, X_3] = -X_3, [X_1, X_4] = 2cX_3 + X_4, [X_3, X_1] = X_3, [X_4, X_1] = -(2cX_3 + X_4), \tag{5.4.7}$$

for the vector fields. The basic vector fields which form an optimal system is given as :

$$\begin{aligned}
 &(i)X_1 + \lambda_1 X_2, (ii)X_3 + \lambda_2 X_2, (iii)X_2 + \lambda_3 X_3 + \lambda_4 X_4, \\
 &(iv)X_2 + \lambda_5 X_4, (v)X_1.
 \end{aligned} \tag{5.4.8}$$

5.4.1 Symmetry Reductions and Exact Solutions

In this subsection, the basic optimal vector field are used to reduce variable coefficients extended Ostrovsky equation into the ordinary differential equations. The similarity variables and the similarity solutions of the variable coefficients extended Ostrovsky equation can be determined by simplifying the characteristic equation :

$$\frac{\partial x}{\xi} = \frac{\partial t}{\tau} = \frac{\partial u}{\eta}, \tag{5.4.9}$$

and the coefficients functions given by Eq.(5.4.5). The general solution of characteristic equation involves two constants, first constant becomes independent variable ζ and second one plays the role of new dependent variable $F(\zeta)$.

1. Vector Field $X_1 + \lambda_1 X_2$

For the vector field $X_1 + \lambda_1 X_2$, assuming $\lambda_1 = 1$ and on solving the Eq.(5.4.5) and Eq.(5.4.9), we obtain

$$\begin{aligned} u(x, t) &= \frac{F(\zeta)}{t}, \quad \zeta = t(x - ct), \\ \beta(t) &= \frac{b_1}{t^4}, \quad \alpha_1(t) = b_2, \quad \alpha(t) = \frac{b_3}{t}, \end{aligned} \quad (5.4.10)$$

where b_1, b_2 and b_3 are arbitrary constants. Substituting (5.4.10) into Eq.(5.1.2), we get the function $F(\zeta)$ which satisfy ordinary differential equation given as:

$$F''\zeta + b_3 F''F + b_3 (F')^2 + b_1 F'''' - \delta F + b_2 F''F^2 + 2b_2 F(F')^2 = 0, \quad (5.4.11)$$

where prime (') indicates the differentiation with respect to the independent variable ζ . To obtain the solution for Eq.(5.4.11), we assume the solution in the form

$$F(\zeta) = \frac{s_0}{\zeta} + s_1 + s_2 \zeta. \quad (5.4.12)$$

Substituting (5.4.12) into Eq.(5.4.11), we obtain $s_0 = s_1 = 0$ and $s_2 = \sqrt{\frac{\delta}{2b_2}}$.

Hence, the solution of Eq.(5.1.2) is given as:

$$u = \sqrt{\frac{\delta}{2b_2}}(x - ct). \quad (5.4.13)$$

2. Vector Field $X_3 + \lambda_2 X_1$

Applying the same procedure as given above, we get

$$\begin{aligned} u(x, t) &= F(\zeta), \quad \zeta = (x - \lambda_2 t), \\ \beta(t) &= b_5, \quad \alpha(t) = b_6, \quad \alpha_1(t) = b_7, \end{aligned} \quad (5.4.14)$$

where b_5, b_6 and b_7 are arbitrary constants. Substituting (5.4.14) into Eq.(5.1.2), we have the function $F(\zeta)$ which must satisfy the following ordinary differential equation;

$$F'' (c - \lambda_2) + b_6 F'' F + b_6 (F')^2 + b_5 F'''' - \delta F + b_7 F'' F^2 + 2b_7 F (F')^2 = 0. \quad (5.4.15)$$

For the solutions of Eq.(5.4.15), we apply $\frac{G'}{G}$ -expansion method . We considered

$$F(\zeta) = p_0 + \sum_{i=1}^n \left\{ p_i \left(\frac{G'}{G} \right) + q_i \left(\frac{G'}{G} \right)^{-i} \right\} \quad (5.4.16)$$

where p_0, p_i, q_i are constants and the value of non negative integer n can be determined by considering the homogeneous balance of the highest-order nonlinear and highest-order derivatives present in ordinary differential equations (5.4.15). The function $G(\zeta)$ is the auxiliary linearly ODE solution given as :

$$G''(\zeta) + \mu G(\zeta) = 0, \quad (5.4.17)$$

where μ is a unknown constant to be found out. We obtain $n = 1$, by balancing the highest-order derivatives with the nonlinear terms appearing in Eq.(5.4.15). Substituting Eq.(5.4.16) in (5.4.15) and using Eq.(5.4.17), a set of algebraic equations for p_0, p_1, q_1 and μ are obtained after collecting all the terms same powers of $\left(\frac{G'}{G} \right)^k$.

On Solving the algebraic equations by using a software, the following results are obtained as:

Set I

$$\begin{aligned} q_1 = 0, p_0 = -\frac{b_6}{2b_7}, p_1 = \frac{\sqrt{-6b_7b_5}}{b_7} \\ \mu = -\frac{1}{8} \frac{-4\lambda_2 b_7 - b_6^2 + 4cb_7}{b_7b_5}. \end{aligned} \quad (5.4.18)$$

Set II

$$\begin{aligned} p_0 = 0, p_1 = \frac{\sqrt{-6b_7b_5}}{b_7}, q_1 = \frac{1}{6} \frac{\sqrt{6}\sqrt{-b_7b_5}(-\lambda_2 + c)}{b_7b_5} \\ \mu = -\frac{1}{12} \frac{(-\lambda_2 \sqrt{6}\sqrt{-b_7b_5} + c\sqrt{6}\sqrt{-b_7b_5} - 6b_5q_1b_7)\sqrt{6}}{b_5\sqrt{-b_7b_5}}. \end{aligned} \quad (5.4.19)$$

Set III

$$\begin{aligned} p_1 = 0, p_0 = -\frac{b_6}{b_2b_7} \\ q_1 = \frac{\sqrt{-6b_7b_5}\mu}{b_7} \\ \mu = \frac{1}{16} \frac{4\lambda b_7 - 4cb_7 + b_6^2 + \sqrt{16\lambda^2b_7^2 - 32\lambda b_7^2c + 8\lambda b_7b_6^2 + 16c^2b_7^2 - 8cb_7b_6^2 + b_6^4 + 64b_7^2b_5\delta}}{b_7b_5}. \end{aligned} \quad (5.4.20)$$

Depending upon μ , there are two cases as follows:

Case I For $\mu < 0$, the hyperbolic-type solutions of $u(x,t)$ for the sets I, II, III are obtained and are given below, respectively:

$$u_1(x, t) = -\frac{b_6}{2b_7} + \frac{\sqrt{-6b_7b_5}}{b_7} \left(\sqrt{-\mu} \left(\frac{A \sinh \sqrt{-\mu}(x - \lambda_2 t) + B \cosh \sqrt{-\mu}(x - \lambda_2 t)}{A \cosh \sqrt{-\mu}(x - \lambda_2 t) + B \sinh \sqrt{-\mu}(x - \lambda_2 t)} \right) \right), \quad (5.4.21)$$

where $\mu = -\frac{1}{8} \frac{-4\lambda_2 b_7 - b_6^2 + 4cb_7}{b_7 b_5}$,

$$u_2(x, t) = \frac{\sqrt{-6b_7b_5}}{b_7} \left(\sqrt{-\mu} \left(\frac{A \sinh \sqrt{-\mu}(x - \lambda_2 t) + B \cosh \sqrt{-\mu}(x - \lambda_2 t)}{A \cosh \sqrt{-\mu}(x - \lambda_2 t) + B \sinh \sqrt{-\mu}(x - \lambda_2 t)} \right) \right) + \frac{1}{6} \frac{\sqrt{6}\sqrt{-b_7b_5}(-\lambda_2 + c)}{b_7 b_5} \left(\sqrt{-\mu} \left(\frac{A \sinh \sqrt{-\mu}(x - \lambda_2 t) + B \cosh \sqrt{-\mu}(x - \lambda_2 t)}{A \cosh \sqrt{-\mu}(x - \lambda_2 t) + B \sinh \sqrt{-\mu}(x - \lambda_2 t)} \right) \right)^{-1}, \quad (5.4.22)$$

where $\mu = -\frac{1}{12} \frac{(-\lambda_2 \sqrt{6}\sqrt{-b_7b_5} + c\sqrt{6}\sqrt{-b_7b_5} - 6b_5 q_1 b_7)\sqrt{6}}{b_5 \sqrt{-b_7b_5}}$,

and

$$u_3(x, t) = -\frac{b_6}{b_2 b_7} + \frac{\sqrt{-6b_7b_5}\mu}{b_7} \left(\sqrt{-\mu} \left(\frac{A \sinh \sqrt{-\mu}(x - \lambda_2 t) + B \cosh \sqrt{-\mu}(x - \lambda_2 t)}{A \cosh \sqrt{-\mu}(x - \lambda_2 t) + B \sinh \sqrt{-\mu}(x - \lambda_2 t)} \right) \right)^{-1}, \quad (5.4.23)$$

where $\mu = \frac{1}{16} \frac{4\lambda_2 b_7 - 4cb_7 + b_6^2 + \sqrt{16\lambda_2^2 b_7^2 - 32\lambda_2 b_7^2 c + 8\lambda_2 b_7 b_6^2 + 16c^2 b_7^2 - 8cb_7 b_6^2 + b_6^4 + 64b_7^2 b_5 \delta}}{b_7 b_5}$.

Case II . If $\mu > 0$, the trigonometric-type solutions of $u(x,t)$ for the sets I, II, III are obtained and are given below, respectively:

$$u_4(x, t) = -\frac{b_6}{2b_7} + \frac{\sqrt{-6b_7b_5}}{b_7} \left(\sqrt{\mu} \left(\frac{A \cos \sqrt{\mu}(x - \lambda_2 t) - B \sin \sqrt{\mu}(x - \lambda_2 t)}{A \sin \sqrt{\mu}(x - \lambda_2 t) + B \cos \sqrt{\mu}(x - \lambda_2 t)} \right) \right), \quad (5.4.24)$$

where $\mu = -\frac{1}{8} \frac{-4\lambda_2 b_7 - b_6^2 + 4cb_7}{b_7 b_5}$,

$$\begin{aligned}
 u_5(x, t) = & \frac{\sqrt{-6b_7b_5}}{b_7} \left(\sqrt{\mu} \left(\frac{A \cos \sqrt{\mu}(x - \lambda_2 t) - B \sin \sqrt{\mu}(x - \lambda_2 t)}{A \sin \sqrt{\mu}(x - \lambda_2 t) + B \cos \sqrt{\mu}(x - \lambda_2 t)} \right) \right) \\
 & + \frac{1}{6} \frac{\sqrt{6}\sqrt{-b_7b_5}(-\lambda_2 + c)}{b_7b_5} \left(\sqrt{\mu} \left(\frac{A \cos \sqrt{\mu}(x - \lambda_2 t) - B \sin \sqrt{\mu}(x - \lambda_2 t)}{A \sin \sqrt{\mu}(x - \lambda_2 t) + B \cos \sqrt{\mu}(x - \lambda_2 t)} \right) \right)^{-1},
 \end{aligned} \tag{5.4.25}$$

where $\mu = -\frac{1}{12} \frac{(-\lambda_2 \sqrt{6}\sqrt{-b_7b_5} + c\sqrt{6}\sqrt{-b_7b_5} - 6b_5q_1b_7)\sqrt{6}}{b_5\sqrt{-b_7b_5}}$,

and

$$u_6(x, t) = -\frac{b_6}{b_{2b_7}} + \frac{\sqrt{-6b_7b_5}\mu}{b_7} \left(\sqrt{\mu} \left(\frac{A \cos \sqrt{\mu}(x - \lambda_2 t) - B \sin \sqrt{\mu}(x - \lambda_2 t)}{A \sin \sqrt{\mu}(x - \lambda_2 t) + B \cos \sqrt{\mu}(x - \lambda_2 t)} \right) \right)^{-1}, \tag{5.4.26}$$

where $\mu = \frac{1}{16} \frac{4\lambda_2 b_7 - 4cb_7 + b_6^2 + \sqrt{16\lambda_2^2 b_7^2 - 32\lambda_2 b_7^2 c + 8\lambda_2 b_7 b_6^2 + 16c^2 b_7^2 - 8cb_7 b_6^2 + b_6^4 + 64b_7^2 b_5 \delta}}{b_7 b_5}$,

and A, B are arbitrary constants.

3. Vector Field $X_2 + \lambda_3 X_3 + \lambda_4 X_4$

For this vector field, we consider the scalar $\lambda_3 = \lambda_4 = 1$ and on solving the Eq.(5.4.5)

and Eq.(5.4.9), we obtained

$$u(x, t) = e^t F(\zeta), \zeta = (x - t),$$

$$\beta(t) = b_8, \alpha(t) = b_9 e^{-t}, \alpha_1(t) = b_{10} e^{-2t}, \tag{5.4.27}$$

where b_8, b_9 and b_{10} are arbitrary constants. Substituting (5.4.27) into Eq.(5.1.2), we have the function $F(\zeta)$ which satisfy the following ordinary differential equation:

$$F' - F'' + cF'' + b_9 F'' F + b_9 (F')^2 + b_8 F'''' - \delta F + b_{10} F'' F^2 + 2b_{10} F (F')^2 = 0, \tag{5.4.28}$$

We applied extended tanh method [46, 180] for solving the Eq.(5.4.28). $F(\zeta)$ is

assumed in the following form:

$$F(\zeta) = \sum_{k=0}^n r_k Y^k + \sum_{k=1}^n s_k Y^{-k}, \quad (5.4.29)$$

where $Y = \tanh(\zeta)$ and n is positive integer which is to be obtained by balancing the highest order nonlinear term with linear terms of the highest order in the equation. On balancing Eq.5.4.28, we get $n = 1$ and putting in Eq.(5.4.29) the finite expansion in tanh function is obtained as:

$$F(\zeta) = r_0 + r_1 \tanh(\zeta) + \frac{s_1}{\tanh(\zeta)}. \quad (5.4.30)$$

Substituting Eq.(5.4.30) into Eq.(5.4.28) and equating the coefficients of the same powers of $\tanh(\zeta)$, a system of algebraic equations for r_0, r_1 and s_1 is obtained. By solving these algebraic equations three sets of solutions are obtained which are as follows:

Set I

$$\begin{aligned} r_1 &= 0, s_1 = \frac{\sqrt{-6 b_{10} b_8}}{b_{10}}, \\ r_0 &= \frac{-6 b_9 b_8 + \sqrt{-6 b_{10} b_8}}{b_{10} (12 b_8 + \delta)}. \end{aligned} \quad (5.4.31)$$

Set II

$$\begin{aligned} s_1 &= 0, r_1 = \frac{\sqrt{-6 b_{10} b_8}}{b_{10}}, \\ r_0 &= \frac{-6 b_9 b_8 + \sqrt{-6 b_{10} b_8}}{b_{10} (12 b_8 + \delta)}. \end{aligned} \quad (5.4.32)$$

Set III

$$r_0 = 0, r_1 = -\frac{1}{4b_9},$$

$$s_1 = -\frac{1}{4b_9}. \quad (5.4.33)$$

Corresponding to these sets the solutions of equation (5.1.2) have been found as

$$u_1(x, t) = e^t \left(\frac{-6 b_9 b_8 + \sqrt{-6 b_{10} b_8}}{b_{10} (12 b_8 + \delta)} + \frac{\sqrt{-6 b_{10} b_8}}{b_{10}} \coth(x - t) \right), \quad (5.4.34)$$

$$u_2(x, t) = e^t \left(\frac{-6 b_9 b_8 + \sqrt{-6 b_{10} b_8}}{b_{10} (12 b_8 + \delta)} + \frac{\sqrt{-6 b_{10} b_8}}{b_{10}} \tanh(x - t) \right), \quad (5.4.35)$$

$$u_3(x, t) = e^t \left(-\frac{1}{4b_9} (\tanh(x - t) + \coth(x - t)) \right). \quad (5.4.36)$$

4. Vector Field $X_2 + \lambda_5 X_4$

Assuming $\lambda_5 = 1$, following the same procedure as above, we obtained

$$u(x, t) = e^t F(\zeta), \zeta = x,$$

$$\beta(t) = b_{11}, \alpha(t) = b_{12}e^{-t}, \alpha_1(t) = b_{13}e^{-2t}, \quad (5.4.37)$$

where b_{11}, b_{12} and b_{13} are arbitrary constants. Substituting (5.4.37) into Eq.(5.1.2), we get the function $F(\zeta)$ which satisfy the following ODE:

$$F' + cF'' + b_{12}F''F + b_{12}(F')^2 + b_{11}F'''' - \delta F + b_{13}F''F^2 + 2b_{13}F(F')^2 = 0. \quad (5.4.38)$$

Since the Eq.(5.4.38) is similar type of Eq.(5.4.28) similar solutions will be obtained.

5. Vector Field X_1

Applying the same procedure, we get

$$u(x, t) = F(\zeta), \zeta = t(x - ct),$$

$$\beta(t) = \frac{b_{14}}{t^4}, \alpha(t) = \frac{b_{15}}{t^2}, \alpha_1(t) = \frac{b_{16}}{t^4}, \quad (5.4.39)$$

where b_{14}, b_{15} and b_{16} are arbitrary constants. Substituting (5.4.39) into Eq.(5.1.2), we have the function $F(\zeta)$ which satisfy the following ODE:

$$F' + F'' \zeta + b_{15} F F'' + b_{15} (F')^2 + b_{14} F'''' - \delta F + b_{16} F'' F^2 + 2b_{16} F (F')^2 = 0. \quad (5.4.40)$$

Assuming the Eq.(5.4.40) has the solutions in the form of

$$F(\zeta) = s_0 + s_1 \zeta + \frac{s_2}{\zeta}, \quad (5.4.41)$$

where s_0, s_1 and s_2 are constants. On substituting Eq.(5.4.41) into Eq.(5.4.42), we get $s_1 = 0, s_0 = -\frac{b_{15}}{b_{16}}$ and $s_2 = \frac{\sqrt{-6 b_{16} b_{14}}}{b_{16}}$.

Hence, the solution of Eq.(5.1.2) is given as:

$$u(x, t) = -\frac{b_{15}}{b_{16}} + \frac{\sqrt{-6 b_{16} b_{14}}}{b_{16} t (x - ct)}. \quad (5.4.42)$$

5.5 Conclusion

Painlevé analysis is used for the extended Ostrovsky equation with time dependent variable coefficients. We found that it passes Painlevé test if and only if the variable coefficients satisfy the particular constraints. Auto-Bäcklund transformations are found by using the truncated Painlevé expansion analysis and solution is ob-

tained. Finally, Lie symmetry reduction procedure have been applied to reduce the variable coefficients extended Ostrovsky equation to ordinary differential equation. Different methods have been applied to obtained the wide family of explicit solutions including traveling waves solutions and singular solutions of the reduced ordinary differential equations. The solutions obtained are hyperbolic, trigonometric and series solutions.

Chapter 6

Symmetry Analysis and Some Solutions of Gowdy Equations

6.1 Introduction

Einstein's field equations are inherently nonlinear and their exact solutions are not easy to obtain. Because of their nonlinearity, they are expected to have solutions, at least for some spacetimes. Belinski and Zakharov modified the inverse scattering transform (IST) for application to Einstein's equation in vacuum and since then there has been a consistent search for soliton solutions of the vacuum equations based on their techniques. The problem of obtaining solutions for the field equations without linearization is however a more difficult task as no general solution of these equations is known.

The several methods have been used to study the spatially inhomogeneous cosmologies and the search for solutions of Einstein field equations plays a specially important role. Due to the highly nonlinear character of the field equations, the knowledge of solutions is crucial for the understanding of specific qualitative as-

pects that can be used as a guide for the study of more general models. The Gowdy spacetimes, studied by Robert H. Gowdy in 1974, [24] are one of the model systems for obtaining a mathematical understanding of the dynamics of inhomogeneous solutions of the Einstein vacuum equations. In the Gowdy class the spacetime metric can be written in the form

$$ds^2 = t^{-\frac{1}{2}} e^{\frac{\lambda}{2}} \left(-dt^2 + d\theta^2 \right) + t \left(e^P (dx + Qdy)^2 + e^{-P} dy^2 \right). \quad (6.1.1)$$

Here the functions P , Q and λ are function of t and θ . The essential equations describing these spacetimes are a system of semilinear wave equations for two functions P and Q which are assumed to be periodic with period 2π in the spatial coordinate t . The vacuum Einstein field equations [9] split into two systems, in which first is evolution equations for P and Q :

$$\begin{aligned} P_{tt} + \frac{1}{t} P_t - P_{\theta\theta} - e^{2P} (Q_t^2 - Q_\theta^2) &= 0 \\ Q_{tt} + \frac{1}{t} Q_t - Q_{\theta\theta} + 2(Q_t P_t - Q_\theta P_\theta) &= 0, \end{aligned} \quad (6.1.2)$$

and the other is constraint equations for λ :

$$\begin{aligned} \lambda_\theta &= t \left[P_\theta^2 + P_t^2 + e^{2P} (Q_t^2 + Q_\theta^2) \right], \\ \lambda_t &= 2t \left(P_t P_\theta + e^{2P} Q_t Q_\theta \right). \end{aligned} \quad (6.1.3)$$

The constraint equations determine λ , if P and Q are known. The integrability conditions for the constraint equations are satisfied as a consequence of the evolution equations. Since the evolution equations do not depend on λ there is essentially a complete decoupling of constraints from evolution equations.

In this chapter, we will deal with Eq.(6.1.2), which we call the Gowdy equations

[8, 24].

In Section 6.2, the outline of the Lie classical method [75, 191] is presented for generating the symmetries of the Gowdy equations. In Section 6.3, the determination of system of ordinary differential equations (ODEs) by the transformation group for the reduction of system (6.1.2) is presented and exact solutions have been found out. Conclusion are given in last Section.

6.2 Lie Symmetry Analysis

For the partial differential equation (PDE), a Lie point symmetry is an invertible transformation of the independent and dependent variables that makes the equation invariant. Lie find out that if we restrict ourself to symmetries that depend continuously on a small parameter and that form a continuous one-parameter group of transformations, one can linearize the symmetry conditions and end up with an algorithm for calculating continuous symmetries. In view of the algorithmic steps of Lie classical method , we proceed as follows:

Assuming the Lie group of point transformations as

$$\begin{aligned} t^* &= t + \xi(\bar{X}, \bar{\sigma}) + O(\epsilon^2), \\ \theta^* &= \theta + \tau(\bar{X}, \bar{\sigma}) + O(\epsilon^2), \\ \bar{\sigma}^* &= \bar{\sigma} + \bar{\phi}(\bar{X}, \bar{\sigma}) + O(\epsilon^2), \end{aligned} \tag{6.2.1}$$

where $\bar{X} = (t, \theta)$, $\bar{\sigma} = (P, Q)$ and $\bar{\phi} = (\eta, \phi)$ make the system (6.1.2) invariant. In other words, the transformations are such that $\bar{\sigma}^*$ is also a solution if $\bar{\sigma}$ is a solution of system (6.1.2). The method for finding the symmetry group of system (6.1.2) mainly consists of determining the infinitesimals ξ , τ , η and ϕ , which are functions

of t, θ, P and Q . The relations from the coefficients of the first order of ϵ is given as

$$\begin{aligned}\eta^{tt} + t\eta^t - \xi P_t - \eta^{\theta\theta} - 2e^{2P} (\phi^t Q_t - \phi^\theta Q_\theta) - 2e^{2P} \eta (Q_t^2 - Q_\theta^2) &= 0, \\ \phi^{tt} + \phi^t t - \xi Q_t - \phi^{\theta\theta} + 2\phi^t P_t + 2\eta^t Q_t - 2\phi^\theta P_\theta - 2\eta^\theta Q_\theta &= 0,\end{aligned}\quad (6.2.2)$$

where $\eta^t, \eta^\theta, \phi^t, \phi^\theta, \eta^{tt}, \eta^{\theta\theta}, \phi^{\theta\theta}, \phi^{\theta t}$ are extended (prolonged) infinitesimals acting on an enlarged space that include dependent variables of $P_t, P_\theta, Q_t, Q_\theta, P_{tt}, P_{\theta\theta}, Q_{tt}$ and $Q_{\theta\theta}$. The infinitesimals are obtained from the invariance conditions (6.2.2) by putting the different differentials coefficients equal to zero. A number of PDEs in ξ, τ, η and ϕ are obtained and are known as determining equations. The general solution of this system gives the infinitesimal elements ξ, τ, η and ϕ as follows :

$$\begin{aligned}\tau &= k_1\theta + k_2, \\ \xi &= k_1t, \\ \eta &= k_3, \\ \phi &= k_4 - k_3Q,\end{aligned}\quad (6.2.3)$$

where $k_i, i=1,2,3,4$ are arbitrary constants.

The associated Lie algebra of the system (6.1.2) consists of the following four vector fields:

$$\begin{aligned}X_4 &= \frac{\partial}{\partial\theta}, \\ X_3 &= \frac{\partial}{\partial Q}, \\ X_2 &= \theta \frac{\partial}{\partial\theta} + t \frac{\partial}{\partial t}, \\ X_1 &= \frac{\partial}{\partial P} - Q \frac{\partial}{\partial Q}.\end{aligned}\quad (6.2.4)$$

These vector fields are utilized to determine solutions of Eq.(6.1.2) as discussed in next section.

6.3 Optimal System

In this section, we deduced the one-dimensional optimal system of Eq.(6.1.2). Following the earlier described procedure, the symmetries of Eq.(6.1.2) are obtained. The symmetries form a four-dimensional Lie algebra generated by vector fields (6.2.4). To find an optimal system, we found the nonzero Lie brackets and adjoint representations for the Lie algebra which are given as follows:

$$\begin{aligned} [X_1, X_3] &= -X_3, [X_2, X_4] = -X_4, \\ [X_3, X_1] &= X_3, [X_4, X_2] = X_4. \end{aligned} \quad (6.3.1)$$

where the expression $[X_i, X_j] = -[X_j, X_i]$ is the Lie brackets and the adjoint relations are obtained by using the Lie series

$$Ad(exp(\epsilon X_i))X_j = X_j - \epsilon[X_i, X_j] + \frac{\epsilon^2}{2}[X_i, [X_3, X_1]] - \dots \quad (6.3.2)$$

Here, we are showing the only cases where $Ad(exp(\epsilon X_i))X_j \neq X_j$. Thus the adjoint relations for the Lie algebra for Eq.(6.1.2) are calculated as follows:

$$\begin{aligned} Ad(exp(\epsilon X_1))X_2 &= e^{-\epsilon} X_3, & Ad(exp(\epsilon X_2))X_4 &= e^{\epsilon} X_4 \\ Ad(exp(\epsilon X_3))X_1 &= X_1 - \epsilon X_3, & Ad(exp(\epsilon X_4))X_2 &= X_2 - \epsilon X_3. \end{aligned} \quad (6.3.3)$$

Optimal System for Eq.(6.1.2) given as:

$$\begin{aligned} X_2, \quad X_4, \\ X_1 + aX_2, \quad X_1 + bX_4, \\ X_2 + cX_3, \quad X_3 + dX_4 \end{aligned} \tag{6.3.4}$$

6.4 Symmetry Reductions and Solutions

The main aim for obtaining the symmetries of a differential equation is to utilize it for doing the symmetry reductions and finding exact solutions of it. In this section, the linear combination of system of vector fields (6.2.4) is considered which is being used to reduce the system equations (6.1.2) to systems of ODEs. The similarity variables and the similarity solutions of the Gowdy equation are obtained by simplifying the characteristic equation

$$\frac{dt}{\xi} = \frac{d\theta}{\tau} = \frac{dP}{\eta} = \frac{dQ}{\phi}. \tag{6.4.1}$$

The general solution of characteristic equation contain three variables; one represents the new independent variable ζ and the two other F and G are considered as dependent variables.

1. Vector field X_2

For this vector field, solving the system of equations (6.1.2) and (6.4.1) we get

$$\zeta = \frac{t}{\theta}, P = F(\zeta), Q = G(\zeta). \tag{6.4.2}$$

These similarity variables and similarity solutions are used for reducing the system

(6.1.2) into new reduced system of ordinary differential equation given as:

$$\begin{aligned} F'' + \left(\frac{1}{\zeta} - 2\zeta\right) F' - \zeta^2 F'' - e^{2F} (1 - \zeta^2) G'^2 &= 0, \\ G'' + \left(\frac{1}{\zeta} - 2\zeta\right) G' - \zeta^2 G'' + 2(1 - \zeta^2) F' G' &= 0. \end{aligned} \quad (6.4.3)$$

Solution of the system (6.4.3) of ODEs is as follows:

We considered $G(\zeta) = \exp(-F(\zeta)) H(\zeta)$ and $F'(\zeta) = N(\zeta)$.

Using these in Eq.equa.6.10 the system reduce to

$$\begin{aligned} N''\zeta + N - 2N\zeta^2 - N'\zeta^3 - \zeta N^2 H^2 + 2\zeta N H H' - \zeta H'^2 + \zeta^3 N^2 H^2 \\ - 2\zeta^3 N H H' + \zeta^3 H'^2 &= 0, \\ H''\zeta - N' H \zeta - \zeta H N^2 - N H + 2N H \zeta^2 + H' - 2\zeta^2 H' + \zeta^3 N' H + \zeta^3 N^2 \\ H - \zeta^3 H'' &= 0. \end{aligned} \quad (6.4.4)$$

Two following cases are consider:

Case 1 $H(\zeta) = 0$, it is not a physically existing case.

Case 2 $H(\zeta) = \iota$, where ι denotes the complex number iota.

Using case 2, our system (6.4.4) reduced to single ordinary differential equation:

$$\begin{aligned} (\zeta + 1)^2 \zeta (1 - \zeta) N' + \zeta^2 (1 - \zeta) (\zeta + 1)^2 N^2 + (\zeta + 1) (2\zeta^3 - 4\zeta^2 + 1) N \\ - \zeta^3 + 2\zeta^2 + \zeta - 1 &= 0, \end{aligned} \quad (6.4.5)$$

which is further solve and gave the solution

$$N(\zeta) = \frac{\sqrt{\zeta^2 - 1}\zeta + c_1 \ln\left(\zeta + \sqrt{\zeta^2 - 1}\right) \sqrt{\zeta^2 - 1}\zeta + c_1 \zeta + c_2}{\left(c_1 \ln\left(\zeta + \sqrt{\zeta^2 - 1}\right) + 1\right) (\zeta + 1) \sqrt{\zeta^2 - 1}\zeta}, \quad (6.4.6)$$

where c_1, c_2, d_1 are constants.

Using above we get final solution of system (6.1.2),

$$P = \int \frac{\sqrt{\zeta^2 - 1}\zeta + c_1 \ln(\zeta + \sqrt{\zeta^2 - 1}) \sqrt{\zeta^2 - 1}\zeta + c_1 \zeta + c_2}{(c_1 \ln(\zeta + \sqrt{\zeta^2 - 1}) + 1)(\zeta + 1)\sqrt{\zeta^2 - 1}\zeta} d\zeta + d_1.$$

$$Q = \iota \exp \left[- \left(\int \frac{\sqrt{\zeta^2 - 1}\zeta + c_1 \ln(\zeta + \sqrt{\zeta^2 - 1}) \sqrt{\zeta^2 - 1}\zeta + c_1 \zeta + c_2}{(c_1 \ln(\zeta + \sqrt{\zeta^2 - 1}) + 1)(\zeta + 1)\sqrt{\zeta^2 - 1}\zeta} d\zeta + d_1 \right) \right],$$
(6.4.7)

where $\zeta = \frac{t}{\theta}$ and d_1 is constant. On substituting (6.4.7) in (6.1.3), we get $\lambda = \text{constant}$.

2. Vector field X_4

Following the same way as above, we get

$$\zeta = t, P = F(\zeta), Q = G(\zeta). \quad (6.4.8)$$

Substituting Eq. (6.4.8) into system of Eq.(6.1.2) yields a reduced ODEs as :

$$F'' + \frac{F'}{\zeta} - e^{2F} G'^2 = 0,$$

$$G'' + \frac{G'}{\zeta} + 2F' G' = 0. \quad (6.4.9)$$

Solution of above ODEs system (6.4.9) is obtained as:

Assuming $G(\zeta) = \exp(-F(\zeta)) H(\zeta)$ and $F'(\zeta) = N(\zeta)$

using these substitutions the system Eq.6.4.9 reduce to

$$N' + \frac{N}{\zeta} - N^2 H^2 + 2N H H' - H'^2 = 0,$$

$$\zeta H'' - \zeta N' H + H' - N H - \zeta N^2 H = 0. \quad (6.4.10)$$

We consider the following cases:

Case 1 $H(\zeta) = 0$, it is not a physically existing case.

Case 2 $H(\zeta) = -\iota$, where ι denotes the complex number *iota*.

By using the case 2 the system (6.4.10) reduced to single ordinary differential equation:

$$(1 - \zeta^2) N' + \frac{N}{\zeta} - 3\zeta N = 0. \quad (6.4.11)$$

The solution of this ODE is found as:

$$N(\zeta) = \frac{\zeta^2 + d_2}{(\zeta)(\zeta^2 - 1)}, \quad (6.4.12)$$

where d_2 is constant. Using the above results, the final solution of system (6.1.2) is obtained as:

$$\begin{aligned} P &= \log \left[\frac{\sqrt{(t^2 - 1)}}{t} d_2 \right] + \log \left[\sqrt{(t^2 - 1)} \right], \\ Q &= -\iota \exp(-(\log(\frac{\sqrt{(t^2 - 1)}}{t} d_2) + \log \left[\sqrt{(t^2 - 1)} \right])). \end{aligned} \quad (6.4.13)$$

On substituting (6.4.13) in (6.1.3), we get $\lambda = \text{constant}$.

3. Vector field $X_1 + aX_2$

Assuming the scalar $a = 1$, the similarity variable and similarity solutions for the Eq.(6.1.2) are obtained as:

$$\zeta = \frac{\theta}{t}, P = \log(tF(\zeta)), Q = \frac{G(\zeta)}{t}. \quad (6.4.14)$$

Using the above, the reduced ODEs of the system equation (6.1.2) are as follows:

$$\begin{aligned}
(1 - \zeta)FF'' + (1 - \zeta^2)F^4(G')^2 - 2\zeta F^4GG' + (1 - \zeta^2)(F')^2 + \zeta FF' - F^4G^2 &= 0, \\
(1 - \zeta)FG'' + \zeta FG' + 2(\zeta^2 - 1)F'G' + 2\zeta GF' - GF &= 0, .
\end{aligned} \tag{6.4.15}$$

where prime (') means the differentiation with respect to the independent variable ζ .

Assuming the solution of Eq.(6.4.15) of the form

$$\begin{aligned}
F(\zeta) &= a_0 + \frac{a_1}{\zeta} + a_2\zeta, \\
G(\zeta) &= b_0 + \frac{b_1}{\zeta} + b_2\zeta,
\end{aligned} \tag{6.4.16}$$

where $a_i, b_i, i = 0, 1, 2$ are arbitrary constants. On substituting (6.4.16) in Eq.(6.4.15), we get $a_0 = 0, a_1 = a_1, a_2 = -16a_1, b_0 = 0, b_1 = b_1, b_2 = -32b_1$.

The corresponding solution of (6.1.2) is obtained as :

$$\begin{aligned}
P &= \log\left[a_1 \frac{(t^2 - 16\theta^2)}{\theta}\right], \\
Q &= b_1 \left[\frac{1}{\theta} - \frac{32\theta}{t^2}\right].
\end{aligned} \tag{6.4.17}$$

4. Vector field $X_1 + bX_4$

Assuming $b = 1$ then the similarity variable and similarity solution of Eq.(6.1.2) corresponding to this vector field are given as:

$$\zeta = \theta, P = F(\zeta), Q = G(\zeta). \tag{6.4.18}$$

Using above the system equation (6.1.2), reduces to the ODEs given as:

$$\begin{aligned} F'' - e^{2F} G'^2 &= 0, \\ G'' - 2F' G' &= 0. \end{aligned} \tag{6.4.19}$$

For the solution of the system (6.4.19) of ODEs we consider

$$G(\zeta) = \exp(-F(\zeta)) H(\zeta) \text{ and } F'(\zeta) = N(\zeta).$$

Using the substitutions, the system 6.4.19 reduce to

$$\begin{aligned} N' - N^2 H^2 + 2NHH' - H'^2 &= 0, \\ H'' - N'H - 4NH' + 3N^2 H &= 0. \end{aligned} \tag{6.4.20}$$

Two following cases are consider:

Case 1. $H(\zeta) = 0$, it is not a physically existing case.

Case 2. $H(\zeta) = -\iota$, where ι denotes the complex number *iota*.

By using case2, our system (6.4.20) reduces to single ordinary differential equation:

$$(1 - \zeta)N' + N^2\zeta^2 - (2\zeta + 4)N + 3N^2\zeta + 1 = 0, \tag{6.4.21}$$

which is further simplify and gave the solution as:

$$N(\zeta) = \frac{(\zeta - 1)^{2\sqrt{5}} - d_3}{d_3\sqrt{5} - 2d_3 - d_3\zeta + 2(\zeta - 1)^{2\sqrt{5}} + (\zeta - 1)^{2\sqrt{5}}\zeta + (\zeta - 1)^{2\sqrt{5}}\sqrt{5}}, \tag{6.4.22}$$

where d_3 is constant. The final solution of system (6.1.2) by using above results is

obtained as

$$\begin{aligned}
 P &= \int \left(\frac{(\theta - 1)^{2\sqrt{5}} - d_{13}}{d_3\sqrt{5} - 2d_3 - d_3\theta + 2(\theta - 1)^{2\sqrt{5}} + (\theta - 1)^{2\sqrt{5}}\theta + (\theta - 1)^{2\sqrt{5}}\sqrt{5}} \right) d\theta, \\
 Q &= -i \exp \left[- \int \left(\frac{(\theta - 1)^{2\sqrt{5}} - d_4}{d_3\sqrt{5} - 2d_3 - d_3\theta + 2(\theta - 1)^{2\sqrt{5}} + (\theta - 1)^{2\sqrt{5}}\theta + (\theta - 1)^{2\sqrt{5}}\sqrt{5}} \right) d\theta \right].
 \end{aligned} \tag{6.4.23}$$

On substituting (6.4.23) in (6.1.3), we get $\lambda = \text{constant}$.

5. Vector field $X_2 + cX_3$

For this vector field, the similarity variable and similarity solutions after assuming $c = 1$ are as follows:

$$\zeta = \frac{\theta}{t}, P = F(\zeta), Q = \log(tG(\zeta)). \tag{6.4.24}$$

Substituting (6.4.24) into (6.1.2), we get the following reduced ODEs:

$$\begin{aligned}
 (\zeta - 1)G^2F'' + \zeta G^2F' - e^{2F}G^2 + 2\zeta e^{2F}GG' + (1 - \zeta^2)e^{2F}(G')^2 &= 0, \\
 (\zeta - 1)GG'' + (1 - \zeta^2)(G')^2 + 2(\zeta^2 - 1)GG'F' + \zeta GG' - 2\zeta G^2F' &= 0,
 \end{aligned} \tag{6.4.25}$$

Assuming $F(\zeta) = \log(G(\zeta))$ to obtain the solution of ODEs (6.4.25). The reduced ODEs are found as:

$$\begin{aligned}
 (\zeta^2 - 1)GG'' + (1 - \zeta^2)(G')^2G^2 - G^4 + 2\zeta G^3G' + \zeta GG' + (1 - \zeta^2)(G')^2 &= 0, \\
 (\zeta - 1)GG'' + (\zeta^2 - 1)(G')^2 - \zeta^2 GG' &= 0.
 \end{aligned} \tag{6.4.26}$$

On solving the ODEs (6.4.26), we get

$$\zeta(\zeta - 1)G'' + (\zeta^2 - 1)(G')^2 + 2(1 - \zeta^2)(G')^2 + 2\zeta GG' + 2\zeta G^3G' - G^4 = 0. \tag{6.4.27}$$

To obtain the solution of Eq.(6.4.27), we consider the solution in the form as:

$$G(\zeta) = b_0 + \frac{b_1}{\zeta} + b_2\zeta. \quad (6.4.28)$$

Substituting this in Eq.(6.4.27), we get $b_0 = 0, b_1 = \frac{1}{\sqrt{2}}, b_2 = (-2\sqrt{2} + \sqrt{7})$. Hence the solution of Eq.(6.1.2) is obtained as:

$$\begin{aligned} P &= \log\left[\frac{t}{\sqrt{2}\theta} + (-2\sqrt{2} + \sqrt{7})\frac{\theta}{t}\right], \\ Q &= \log\left[\frac{t^2}{\sqrt{2}\theta} + (-2\sqrt{2} + \sqrt{7})\theta\right]. \end{aligned} \quad (6.4.29)$$

6. Vector field $X_3 + dX_4$

Following the same procedure as above, the similarity variable and similarity solution are obtained as:

$$\zeta = t, \quad P = f(\zeta) \quad Q = G(\zeta). \quad (6.4.30)$$

On substituting (6.4.30) in Eq.(6.1.2) we get the ODEs as:

$$\begin{aligned} F'' + \frac{F'}{\zeta} - e^{2F}(G')^2 &= 0, \\ G'' + \frac{G'}{\zeta} + 2G'F' &= 0, \end{aligned} \quad (6.4.31)$$

which is similar to Eq.(6.4.9). The solutions is similar to solutions of Eq.(6.4.9).

6.5 Conclusion

This chapter intended to give some key aspects on a vacuum Einstein field equations whose evolution is described by a nonlinear differential equation known as Gowdy equation. The symmetries are obtained by using the Lie classical symmetry method.

The main steps which have to be performed in order to determine the complete set of exact solutions are:

- (i) Determination of the general form for the symmetry operator;
- (ii) Based on the different set of operators and using the similarity reduction procedure, invariant solutions can be generated;
- (iii) Last but not least, a special technique has been applied to obtain the solution.

This is a new solution of the nonlinear system of vacuum Einstein field equations that are not found in the literature.

Summary

The present research work is a contribution to the area of applications of group theoretic techniques to systems of nonlinear partial differential equations. We have used Lie classical method as our main tool for finding the symmetries and solutions to some nonlinear PDEs. We have evaluated the symmetries and exact solutions of certain nonlinear PDEs mainly with variable coefficients forms.

Lie's classical point symmetry method is applied successfully in obtaining new symmetry reductions to both variable as well as constant coefficients nonlinear partial differential equation. The reduced equations have then been solved by several methods including extended $\frac{G'}{G}$ -expansion method, tanh method, power series method, etc. Solutions of nonlinear PDEs under considerations are obtained in terms of the hyperbolic functions, trigonometric functions and rational functions.

In chapter 1, we presented a brief literature survey related to symmetries. Some basics definitions and methodologies utilized in the thesis are described. The objectives of present work is outlined. In chapter 2, 3, 4 & 5, the variable coefficients forms of nonlinear PDEs have been considered for finding the symmetries and their solutions. The PDEs studied are generalized Kuramoto Sivashinsky equation, generalized Gardner equation, Benny equation and extended Ostrovsky equation, respectively. Lie classical method has been applied to find the symmetries of the PDEs. The symmetries are then used to reduce the PDEs into ODEs. The solu-

tions of ODEs are determined by tanh method, $\frac{G'}{G}$ -expansion method, etc. Some of the solutions involve an arbitrary coefficient function and it enables us to control and discuss the behaviour of solutions as governed by the choice of this arbitrary function. Apart from finding the symmetries of PDEs in chapter 4 and chapter 5, Painlevé test has also been applied to check their integrability. Solution of extended Ostrovsky equation has been obtained by using auto-Bäcklund transformation after passing the Painlevé test. In chapter 6, Lie classical method has been applied to obtain the symmetries of Gowdy equation. We have determined the complete Lie group classification and obtained the solutions.

The group classification and the exact solutions of the nonlinear systems under examination are not only interesting from mathematical point of view, but may be important for applications.

Future Scope

We have studied some of the nonlinear PDEs in variable coefficients forms. In all cases, the symmetries have been obtained by Lie classical method which are helpful to find the new solutions. To study the non-classical symmetries of variable coefficients equations and finding the exact solutions and their physical interpretation brings forth tremendous scope for future work.

We have checked the integrability of only for two nonlinear PDEs by Painlevé test. Painlevé properties of remaining nonlinear PDEs and how are they related to the symmetries is planned to be investigated in future.

Appendix

A.1 Extended infinitesimals for (1+1)-dimensional PDEs

On using the equations (1.2.31), the following extended infinitesimals for $\eta^t, \eta^x, \eta^{xx}, \eta^{tt}, \eta^{xxx}, \eta^{xxt}, \eta^{xxx}$ can be obtained for dependent variable $u(x, t)$ and ξ, τ, η are infinitesimals corresponding to x, t, u :

$$\eta^t = \eta_t + (\eta_u - \tau_t) u_t - \xi_t u_x - \tau_u u_t^2 - \xi_u u_x u_t \quad (A.1.1)$$

$$\eta^x = \eta_x + (\eta_u - \xi_x) u_x - \tau_x u_t - \xi_u u_x^2 - \tau_u u_x u_t \quad (A.1.2)$$

$$\begin{aligned} \eta^{xx} = & \eta_{xx} + (2\eta_{xu} - \xi_{xx}) u_x - \tau_{xx} u_t + (\eta_{uu} - 2\xi_{xu}) u_x^2 - 2\tau_{xu} u_x u_t + (\eta_u - 2\xi_x) u_{xx} \\ & - 2\tau_x u_{tx} - 3\xi_u u_x u_{xx} - \tau_u u_t u_{xx} - 2\tau_u u_x u_{tx} - \tau_{uu} u_x^2 u_t - \xi_{uu} u_x^3 \quad (A.1.3) \end{aligned}$$

$$\begin{aligned} \eta^{tt} = & \eta_u u_{tt} - 2\tau_{tu} u_t^2 - 2u_{tt} \tau_t - u_x \xi_{tt} + 2\eta_{tu} u_t - u_t \tau_{tt} - 3\tau_u u_t u_{tt} + \eta_{uu} u_t^2 - u_t^2 u_x \xi_{uu} \\ & - u_t^3 \tau_{uu} - 2u_{xt} \xi_u u_t - 2u_x \xi_{tu} u_t - 2u_{xt} \xi_t - u_x \xi_u u_{tt} + \eta_{tt} \quad (A.1.4) \end{aligned}$$

$$\begin{aligned} \eta^{xxx} = & \eta_{xxx} + (3\eta_{xuu} - \xi_{xxx}) u_x - \tau_{xxx} u_t - 3\tau_{xuu} u_x u_t + 3(\eta_{xuu} - \xi_{xuu}) u_x^2 - 3\tau_{xuu} u_t u_x^2 \\ & + (\eta_{uuu} - 3\xi_{xuu}) u_x^3 - \tau_{uuu} u_x^3 u_t - \xi_{uuu} u_x^4 - 3\tau_{xx} u_{tx} + 3(\eta_{xu} - \xi_{xx}) u_{xx} - 3\tau_{xu} u_t u_{xx} \\ & + 3(\eta_{uu} - 3\xi_{xu}) u_x u_{xx} - 6\tau_{xu} u_x u_{tx} - 6\xi_{uu} u_x^2 u_{xx} - 3\tau_{uu} u_{xt} u_x^2 - 3\tau_{uu} u_{xx} u_x u_t \\ & - 3\xi_u u_{xx}^2 - 3\tau_u u_{xx} u_{tx} - 3\tau_x u_{txx} + (\eta_u - 3\xi_x) u_{xxx} - 4\xi_u u_x u_{xxx} - \tau_u u_{xxx} u_t - 3\tau_u u_{xxt} u_x \quad (A.1.5) \end{aligned}$$

$$\begin{aligned}
\eta^{xxt} = & -3\xi_u u_{tx} u_{xx} - 2\tau_{xu} u_{tt} u_x - \tau_{uu} u_{xx} u_t^2 - 3\xi_{tu} u_x u_{xx} + (\eta_{uu} - 2\xi_{xu} - \tau_{tu}) u_{xx} u_t \\
& - \tau_{xx} \phi_{tt} - 3\xi_u u_x u_{txx} - 2\tau_x u_{ttx} + (2\eta_{xtu} - \xi_{xxt}) u_x - \xi_t u_{xxx} - 3\xi_{uu} u_t u_x u_{xx} - 4\tau_{uu} u_x u_t u_{tx} \\
& - \tau_{xxu} u_t^2 - \tau_{xuu} u_x u_t^2 + (\eta_u - 2\xi_x - \tau_t) u_{txx} - 2\tau_u u_{xt}^2 + \eta_{xxt} + (2\eta_{xuu} - \xi_{xxu} - 2\tau_{xtu}) u_x u_t \\
& - \tau_{uuu} u_x^2 u_t^2 - 2\tau_u u_t u_{txx} - \xi_u u_t u_{xxx} - 2\tau_u u_x u_{ttx} - \tau_{uu} u_x^2 u_{tt} + (2\eta_{uu} - \xi_{xx} - 2\tau_{tx}) u_{tx} \\
& - \tau_u u_{xx} u_{tt} + (\eta_{xru} - \tau_{xxt}) u_t + (\eta_{uuu} - 2\xi_{xuu} - \tau_{tuu}) u_x^2 u_t - \xi_{uuu} u_x^3 u_t - 4\tau_{xu} u_t u_{xt} \\
& + (\eta_{uut} - 2\xi_{xtu}) u_x^2 - \xi_{uut} u_x^3 + (\eta_{tu} - 2\xi_{tx}) u_{xx} + 2(\eta_{uu} - 2\xi_{xu} - \tau_{tu}) u_x u_{tx} \quad (A.1.6)
\end{aligned}$$

$$\begin{aligned}
\eta^{xxxx} = & 4\eta_{uu} u_x u_{xxx} - 4u_t u_x^3 \tau_{uuu} - 4u_{xxx} \tau_u u_x - 5\xi_u u_x u_{xxxx} + 6u_x^2 \eta_{uuu} u_{xx} \\
& - 10\xi_u u_{xx} u_{xxx} - 4u_{xt} u_x^3 \tau_{uuu} - 6u_t u_{xx} \tau_{xru} - 24u_x^2 \xi_{xuu} u_{xx} - 4u_t u_x \tau_{xxru} - 6u_t u_x^2 \tau_{xru} \\
& - 4u_{xt} \tau_u u_{xxx} + \eta_{xxxx} - 12u_{xt} u_x \tau_{xru} + 12u_x \eta_{xru} u_{xx} - 12u_t u_x \tau_{xru} u_{xx} - 12u_{xt} u_x \tau_{uu} u_{xx} \\
& - 4u_t u_x \tau_{uu} u_{xxx} - 3u_t u_{xx}^2 \tau_{uu} - 10u_x^2 \xi_{uu} u_{xxx} - 10u_x^3 \xi_{uuu} u_{xx} - u_t \tau_u u_{xxxx} - 6u_{xt} u_x^2 \tau_{uu} \\
& - 12u_{xt} \tau_{ru} u_{xx} - 18u_x \xi_{xru} u_{xx} - 6u_t u_x^2 \tau_{uuu} u_{xx} - 12u_{xt} u_x^2 \tau_{xru} - 12\xi_{xu} u_{xx}^2 - 4u_x^4 \xi_{xuuu} \\
& - u_x \xi_{xxxx} + 4\eta_{xru} u_x - 4u_{xt} \tau_{xxx} + 4u_x^3 \eta_{xuuu} - 4\xi_{xru} u_x^2 - 6u_x^3 \xi_{xruu} + 6u_x^2 \eta_{xruu} \\
& - 4u_{xx} \xi_{xxx} - 6u_{xt} \tau_{xx} - 4u_{xxx} \tau_x - 6u_{xxt} \tau_u u_{xx} + u_x^4 \eta_{uuuu} - u_x^5 \xi_{uuuu} - u_t \tau_{xxxx} \\
& - 6u_{xx} \xi_{xx} - 4u_{xxx} \xi_x + 6\eta_{xru} u_{xx} + \eta_u u_{xxx} + 4\eta_{xu} u_{xxx} + 3\eta_{uu} u_{xx}^2 - 12u_{xt} \tau_{ru} u_x \\
& - 15u_x \xi_{uu} u_{xx}^2 - 16\xi_{ru} u_x u_{xxx} - u_t u_x^4 \tau_{uuuu} - 4u_t \tau_{ru} u_{xxx} \quad (A.1.7)
\end{aligned}$$

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