ON PLANE PARTITIONS AND n–COLOR PARTITIONS

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Submitted by

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CERTIFICATE

I hereby certify that the work which is being presented in the thesis entitled **"On Plane Partitions and n–Color Partitions"** in partial fulfillment of the requirements for the award of degree of Master of Science, School Of Mathematics(SOM), Thapar University, Patiala is an authentic record of my own work carried out under the supervision of **Dr. Meenakshi Rana**.
The matter presented in this thesis has not been submitted in part or full to any other University or Institution for the award of any degree.

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This is to certify that the above statement made by the candidate is correct and true to the best of my knowledge.

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July, 2017 Amrinder Kaur

Abstract

The first chapter is devoted to preliminaries and provides introduction to plane partitions and n–color partitions.
The second chapter elaborates the bijection between plane partitions and n–color partitions through Bender and Knuth matrices using which some basic results are proved.
The third chapter includes some theorems which give a nice interaction between the plane partition and n–color partition. A relation between the rows of the plane partition and the subscripts of the n–color partition of any number ν is established in this chapter. Our results give a simpler way for finding the corresponding plane partition for a given n–color partition and vice–versa.

Contents

Chapter 1

Introduction

1.1 Partition

Definition 1.1.1 *[8] A Partition of a positive integer n is a finite non increasing sequence of positive integers*

$$a_1 \geq a_2 \geq .... \geq a_r$$

such that

$$\sum_{i=1}^{r} a_i = n.$$

The $a_i's$ are called the parts or summands of the partition. We denote by $p(n)$ the number of partitions of n. $p(0) = 1$ as zero has one partition which is the empty partition(It has no parts).
The Partitions of $n = 4$ are

$$4, \quad 3+1, \quad 2+2, \quad 2+1+1, \quad 1+1+1+1.$$

Hence $p(4) = 5$.
In the definition of partitions, the order does not matter. So 4+3 and 3+4 are the same partitions of 7. Thus a partition is an unordered collection of parts. An ordered collection is called a Composition. Thus 4+3 and 3+4 are two different compositions of 7.
The generating function of $p(n)$ is given by

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{(q;q)_\infty},$$

where $|q| < 1$ and $(q;q)_n$ is a rising q–factorial defined by

$$(a;q)_n = \prod_{i=0}^{\infty} \frac{(1-aq^i)}{(1-aq^{n+i})}$$

for any constant a. If n is a positive integer, then

$$\begin{aligned} (a;q)_n &= (1-a)(1-aq)\cdots(1-aq^{n-1}) \\ \text{and } (a;q)_\infty &= (1-a)(1-aq)(1-aq^2)\cdots \end{aligned}$$

An improved definition of partition regards the parts of the partition as being placed at points of a line

$$\bullet \geq \bullet \geq \bullet \geq \bullet \geq \bullet \geq \bullet$$

and the symbol $\geq$ as regulating the magnitude of the parts at any two adjacent points. It is important to realise that the partitions may be regarded as partitions on a line or in one dimension of space.

1.2 Plane Partitions

Definition 1.2.1 *[17] A Plane Partition of n is an array*

$$\begin{array}{|cccc} n_{1,1} & n_{1,2} & n_{1,3} & \cdots \\ n_{2,1} & n_{2,2} & n_{2,3} & \cdots \\ . & . & . & \\ . & . & . & \\ . & . & . & \end{array}$$

of non negative integers for which

$$\sum_{i,j} n_{ij} = n,$$

where the rows and columns are in non increasing order:

$$n_{ij} \geq n_{(i+1)j} \ , \ n_{ij} \geq n_{i(j+1)} \quad \forall\ i,j \geq 1.$$

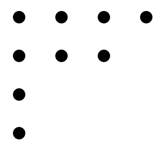
If $n_{ij} = 0$ for all $i > r$, it is an r–rowed partition and if $n_{ij} = 0$ for all $j > c$, it is a c–columned partition. If $n_{ij} \leq m \ \forall\ i,j \geq 1$, we say the parts do not exceed m.
The generating function for plane partitions is

$$\sum_{n=0}^{\infty} Pl(n)q^n = \prod_{n=1}^{\infty} \frac{1}{(1-q^n)^n},$$

where $Pl(n)$ denotes the number of plane partitions of n.
The six plane partitions of $n = 3$ are

$$\begin{array}{llllll} 3, & 2\ \ 1, & 2, & 1\ \ 1\ \ 1, & 1\ \ 1, & 1. \\ & & 1 & & 1 & 1 \\ & & & & & 1 \end{array}$$

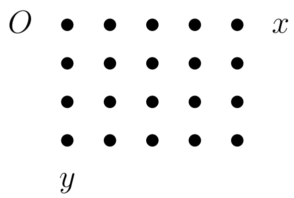
Plane partitions are the partitions in two dimension of space. It is interesting to observe that the Ferrers–Sylvester graph of a partition of a unipartite number is in reality a partition in two dimensions. Such a graph of the partition 4 3 1 1 is



For many purposes it is advantageous to replace the nodes by units, leading to the unit graph

$$\begin{array}{llll} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & \\ 1 & & & \\ 1 & & & \end{array}$$

This is a two–dimensional partition of the number 9 in agreement with the definition which is now given.
Consider the points of a two–dimensional lattice



and let the first row and first column be axes of x and y respectively. Suppose the parts of the partition to be placed at these points so that a descending order of magnitude is in evidence in each row in direction of the x–axis and also in each column in direction of the y–axis. The arrangement of numbers thus reached is defined to be a two–dimensional partition of the number partitioned. Clearly the unit graph of a partition is a two–dimensional partition in which the part magnitude is limited not to exceed unity. In the case of a partition in two dimensions we are concerned with three limiting numbers, for we may limit

1. the number of rows,

2. the number of columns,
3. the part magnitude.

1.3 n–Color Partitions

Definition 1.3.1 *[2] An n–color partition (also called a partition with "n copies of n") of a positive integer ν is a partition in which a part of size n can come in n different colors denoted by subscripts $n_1, n_2, \cdots, n_n$ and the parts satisfy the order :*

$$1_1 < 2_1 < 2_2 < 3_1 < 3_2 < 3_3 < 4_1 < 4_2 < 4_3 < 4_4 < \cdots$$

If $P(\nu)$ denotes the number of n–color partitions of ν, then the generating function is

$$\sum_{\nu=0}^{\infty} P(\nu) q^{\nu} = \prod_{n=1}^{\infty} \frac{1}{(1-q^n)^n}.$$

The n–color partitions of $\nu = 3$ are

$$3_1,\ 3_2,\ 3_3,\ 2_1 1_1,\ 2_2 1_1,\ 1_1 1_1 1_1$$

1.4 BK_ν–matrices

Definition 1.4.1 *[6] For every non–negative integer ν, BK_ν–matrices(BK for Bender and Knuth) are defined as infinite matrices $[a_{i,j}]$ ($i,j \geq 1$) of non–negative integer entries which satisfy*

$$\sum_{r \geq 1} r \left(\sum_{i+j=r+1} a_{i,j} \right) = \nu$$

These are infinite matrices but will be represented in the sequel by the largest possible square matrices whose last row(column) is non–zero.

Thus, for example, the six relevant BK_3–matrices are given by

$$(3)\,, \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

1.5 $E_{i,j}$ matrices

Definition 1.5.1 *[1] We define a matrix $E_{i,j}$ as an infinite matrix whose $(i,j)^{th}$ entry is 1 and all the other entries are zeroes. We call $E_{i,j}$ distinct units of BK_ν–matrix.*

For example,

$$E_{23} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad \textit{and} \quad E_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

1.6 Rogers–Ramanujan Identities

A series involving factors like rising q–factorial $(a;q)_n$ defined by

$$(a;q)_n = \prod_{i=0}^{\infty} \frac{(1-aq^i)}{(1-aq^{n+i})}$$

is called a basic series (or q-series, or Eulerian series). The following two "sum–product" basic series identities are known as the Rogers–Ramanujan identities

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} = \prod_{n=1}^{\infty} \frac{1}{(1-q^{5n-1})(1-q^{5n-4})}, \tag{1.1}$$

$$\sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q;q)_n} = \prod_{n=1}^{\infty} \frac{1}{(1-q^{5n-2})(1-q^{5n-3})}. \tag{1.2}$$

They were first discovered by Rogers [12] and rediscovered by Ramanujan in 1913. MacMahon [17] gave the following partition theoretic interpretations of (1.1) and (1.2), respectively:

Theorem 1.6.1 *The number of partitions of n into parts with the minimal difference 2 equals the number of partitions of n into parts $\equiv \pm 1$ (mod 5).*

Theorem 1.6.2 *The number of partitions of n with minimal part 2 and minimal difference 2 equals the number of partitions of n into parts $\equiv \pm 2$ (mod 5).*

Partition theoretic interpretations of many more q–series identities like (1.1) and (1.2) have been given by several mathematicians. See, for instance, Göllnitz [10, 11], Gordon [5], Connor [19], Hirschhorn [13], Subbarao [15], Subbarao and Agarwal [16].

1.7 Remark

In the next chapter, we discuss the bijection between plane partitions and color partitions through a pair (S,T) of plane partitions. Each pair corresponds to a unique color partition and plane partition using BK_ν and $E_{i,j}$ matrices discussed in this chapter. Then we study some Rogers–Ramanujan type identities using BK_ν matrices.

Chapter 2

Bijection Between Plane Partition And n–Color Partition

2.1 Introduction

In [6], a one to one correspondence between plane partitions of ν and BK_ν–matrices is established(also elaborated in [4]). And there exists a bijection between BK_ν–matrices and n–color partitions of ν which is given in [1]. The results given in Chapter 3 use these bijections. Hence it is imperative to illustrate this bijection, given in section 2.2.

2.2 Bijection

Consider an n–color partition of ν

$$m_{1_{n_1}} + m_{2_{n_2}} + m_{3_{n_3}} + \cdots + m_{t_{n_t}},$$

where m_s is the part and n_s is the subscript. Each single part $m_{s_{n_s}}$ of an n–color partition of ν can be mapped to a single part $E_{p,q}$ as

$$\psi : m_i \to E_{i,m-i+1} \tag{2.1}$$

and the inverse is possible as

$$\psi^{-1} : E_{p,q} \to (p+q-1)_p$$

Using these $E_{p,q}'s$, we construct a matrix A as given below

$$A = a_{1,1}E_{1,1} + a_{1,2}E_{1,2} + \cdots + a_{2,1}E_{2,1} + a_{2,2}E_{2,2} + \cdots + a_{3,1}E_{3,1} + a_{3,2}E_{3,2} + \cdots$$

where $a_{p,q}'s$ are non–negative integers which denote the multiplicities of $E_{p,q}'s$.
From this matrix A, we first construct a two line array $\sigma_1(A)$ as given below:
Suppose $a_{p,q} = k > 0$. Then enter k copies of p in the first row of $\sigma_1(A)$ and k copies of q in the second row.
For example, if we start with

$$A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\sigma_1(A) = \begin{pmatrix} 1 & 1 & 1 & 2 & 2 & 3 \\ 1 & 3 & 3 & 2 & 2 & 1 \end{pmatrix}$$

Second, we permute the columns of $\sigma_1(A)$ so that
(a) the elements of the first row are in non–increasing order.
(b) within a block of constancy of the first row, the corresponding elements of the second row are in non–increasing order.
This yields $\sigma_2(A)$.
In the previous example,we have

$$\sigma_2(A) = \begin{pmatrix} 3 & 2 & 2 & 1 & 1 & 1 \\ 1 & 2 & 2 & 3 & 3 & 1 \end{pmatrix}$$

Third, from the two–line array $\sigma_2(A)$,

$$\sigma_2(A) = \begin{pmatrix} i_1 & i_2 & ,\cdots, & i_l \\ j_1 & j_2 & ,\cdots, & j_l \end{pmatrix}$$

we construct a pair (S,T) of plane partitions by an insertion and bumping procedure, as follows.
The plane partition S will be constructed from $i_1, i_2, ..., i_l$ and T from $j_1, j_2, ..., j_l$.
Recursively define $S^{(1)} = i_1$ and $T^{(1)} = j_1$. Suppose that $S^{(r)}$ and $T^{(r)}$ have been constructed, and these are plane partitions of the same shape, $S^{(r)}$ containing the parts $i_1, i_2, ..., i_r$ and $T^{(r)}$ containing $j_1, j_2, ..., j_r$.
We then insert j_{r+1} into the first row of $T^{(r)}$, immediately to the right of the rightmost entry which is $\geq j_{r+1}$. If this space is occupied by some element k, then by entering j_{r+1} into this space we bump k down to the second row, where it is then treated just as j_{r+1} was, so that another element may be

bumped to the third row, etc. If there is no entry that is $\geq j_{r+1}$ then j_{r+1} is inserted at the beginning of the row and bumps the former first element down.
In this way, $T^{(r+1)}$ is formed from $T^{(r)}$ and j_{r+1}. To construct $S^{(r+1)}$ just insert i_{r+1} into $S^{(r)}$ so that the resulting array has the same shape as $T^{(r+1)}$.

$$\begin{array}{lll} & S & \\ \end{array} \qquad \begin{array}{lll} & T & \\ \end{array}$$

$$\begin{array}{lll} 3 & & \end{array} \qquad \begin{array}{lll} 1 & & \end{array}$$

$$\begin{array}{lll} 3 & & \\ 2 & & \end{array} \qquad \begin{array}{lll} 2 & & \\ 1 & & \end{array}$$

$$\begin{array}{lll} 3 & 2 & \\ 2 & & \end{array} \qquad \begin{array}{lll} 2 & 2 & \\ 1 & & \end{array}$$

$$\begin{array}{lll} 3 & 2 & \\ 2 & & \\ 1 & & \end{array} \qquad \begin{array}{lll} 3 & 2 & \\ 2 & & \\ 1 & & \end{array}$$

$$\begin{array}{lll} 3 & 2 & 1 \\ 2 & 1 & \\ 1 & & \end{array} \qquad \begin{array}{lll} 3 & 3 & 1 \\ 2 & 2 & \\ 1 & & \end{array}$$

Interchange S and T. Thus the pair of plane partitions which correspond to A is

$$S = \begin{array}{lll} 3 & 3 & 1 \\ 2 & 2 & \\ 1 & & \end{array}$$

$$T = \begin{array}{lll} 3 & 2 & 1 \\ 2 & 1 & \\ 1 & & \end{array}$$

This completes the third phase of construction. Finally from ordered pair (S,T) of plane partitions, we construct a single plane partition by a method of Frobenius [9], as adapted by Bender and Knuth. From a column of S and a column of T we form a new column, as illustrated below.

				S
	•	•	•	3
	•	•	•	2
	•	•	•	1
T	2	1	0	

The labels at the right of the array is the first column of S, those at the bottom of the array is one less than the first column of T.
By counting the dots in each row, we find the first column of the plane partition, namely

$$\begin{array}{c}3\\3\\3\end{array}$$

Repeat this with the second column of S and T,

				S
	•	•	•	3
	•	•	•	2
T	1	0		

from which the second column of the plane partition is

$$\begin{array}{c}3\\3\end{array}$$

Finally from the third column of S and T, we get

		S
	•	1
T	0	

and the third column is

$$1$$

Thus the complete partition is

$$\begin{array}{ccc}3&3&1\\3&3&\\3&&\end{array}$$

2.3 Basic series and its Combinatorics

Recently in [14], the basic series

$$\sum_{n=0}^{\infty}\frac{q^{n(n+k-1)}(-q;q^2)_n}{(q^4;q^4)_n}, \tag{2.2}$$

where k is a positive integer, was interpreted as generating function of two different combinatorial objects, viz., an n–color partition function and a weighted lattice path function. The same series has been studied by Agarwal and Rana in [3] using Bender and Knuth matrices. Here in this section, we reproduce the results of [14] and [3].
The following theorem gives the combinatorial interpretation of (2.2) using n–color partition.

Theorem 2.3.1 *For a positive integer k, let $A_k(\nu)$ denote the number of n–color partitions of ν such that*

2.3.1.1 *the parts are greater than or equal to k,*

2.3.1.2 *the parts are of the form $(2l-1)_1$ or $(2l)_2$, if k is an odd and of the form $(2l-1)_2$ or $(2l)_1$, if k is an even,*

2.3.1.3 *if the m_i is the smallest or the only part in the partition, then $m \equiv i+k-1 (mod\, 4)$ and*

2.3.1.4 *the weighted difference between any two consecutive parts is non negative and is $\equiv 0 (mod\, 4)$.*

Theorem 2.3.2 *For $k, \nu \geq 1$, let $B_k(\nu)$ denote the number of BK_ν–matrices Δ such that*

2.3.2.1 *if k is odd (resp. even), then even (resp. odd) columns are zero,*

2.3.2.2 *all rows after the second row are zero,*

2.3.2.3 *if $E_{i,j}$ is the $(i,j)^{th}$ entry in Δ such that either it is the only non–zero entry or $i+j$ is minimum, then $j \equiv k (mod\, 4)$,*

2.3.2.4 *the order difference of any two units of Δ is non negative and is $\equiv 0 (mod\, 4)$,*

2.3.2.5 *for odd $k > 1$, the first $\frac{(k-1)}{2}$ odd columns are zero and*

2.3.2.6 *for even $k > 2$, the first $\frac{(k-2)}{2}$ even columns are zero.*

Theorem 2.3.3 *For all k and ν,*

$$A_k(\nu) = B_k(\nu)$$

.

Proof of Theorem 2.3.1 We shall prove that

$$\sum_{\nu=0}^{\infty} A_k(\nu)q^{\nu} = \sum_{n=0}^{\infty} \frac{q^{n(n+k-1)}(-q;q^2)_n}{(q^4;q^4)_n}. \tag{2.3}$$

Let $A_k(m,\nu)$ denote the number of partitions enumerated by $A_k(\nu)$ into m parts. We shall prove the identity,

$$\begin{aligned} A_k(m,\nu) &= A_k(m-1,\nu-k-2m+2) + A_k(m-1,\nu-k-4m+3) \\ &+ A_k(m,\nu-4m). \end{aligned} \tag{2.4}$$

We give the proof of (2.4) for odd k as the proof for even k is similar and hence is omitted. To prove (2.4) for odd k, we split the partitions enumerated by $A_k(m,\nu)$ into three classes:

(i) those that have least part equal to k_1,

(ii) those that have least part equal to $(k+1)_2$, and

(iii) those that have least part greater than or equal to $(k+2)_1$.

We note that in class (iii) the parts are $\geq 5_1$ because if $k=1$ then 3_1 can not be a part in view of the condition (2.3.1.3) of the theorem.
We now transform the partitions in class (i) by deleting the least part k_1 and then subtracting 2 from all the remaining parts ignoring the subscripts. This produces a partition of $\nu-k-2(m-1)$ into exactly $(m-1)$ parts each of which is $\geq k_1$ (since originally the second smallest part was $\geq (k+2)_1$). Obviously, this transformation does not disturb the weighted difference condition (2.3.1.4) between the parts and so the transformed partition is of the type enumerated by $A_k(m-1,\nu-k-2m+2)$.
Next, we transform the partitions in class (ii) by deleting the least part $(k+1)_2$ and then subtracting 4 from all the remaining parts. This produces a partition of $\nu-(k+1)-4(m-1) = \nu-k-4m+3$ into $m-1$ parts, each of which is $\geq k_1$ (since originally the second smallest part was $\geq (k+4)_1$). Note that originally $(k+2)_1$ and $(k+3)_2$ could not be the second smallest part because of the weighted difference condition (2.3.1.4). Furthermore, since the weighted difference condition between the parts is not disturbed, we see that the transformed partition is of the type enumerated by $A_k(m-1,\nu-k-4m+3)$.
Finally, we transform the partitions in class (iii) by subtracting 4 from each part ignoring the subscripts. This produces a partition of $\nu-4m$ into m parts, each $\geq k_1$. Since the weighted difference condition (2.3.1.4) between the parts is again not disturbed, we see that the transformed partition is of

the type enumerated by $A_k(m, \nu - 4m)$.
The above transformations establish a bijection between the partitions enumerated by $A_k(m,\nu)$ and those enumerated by $A_k(m-1,\nu-k-2m+2)+A_k(m-1,\nu-k-4m+3)+A_k(m,\nu-4m)$. This proves the Identity (2.4).
For $|z|<|q|^{-1}$ and $|q|<1$, let

$$f_k(z;q)=\sum_{\nu=0}^{\infty}\sum_{m=0}^{\infty}A_k(m,\nu)z^m q^\nu. \tag{2.5}$$

Substituting for $A_k(m,\nu)$ from (2.4) in (2.5) and then simplifying, we get the following q–functional equation

$$f_k(z;q)=zq^k f_k(zq^2;q)+zq^{k+1}f_k(zq^4;q)+f_k(zq^4;q). \tag{2.6}$$

Since $f_k(0;q)=1$, we may easily check by coefficient comparison in (2.6) that

$$f_k(z;q)=\sum_{n=0}^{\infty}\frac{q^{n(n+k-1)}(-q;q^2)_n z^n}{(q^4;q^4)_n}$$

Now,

$$\begin{aligned}\sum_{\nu=0}^{\infty}A_k(\nu)q^\nu &= \sum_{\nu=0}^{\infty}(\sum_{m=0}^{\infty}A_k(m,\nu))q^\nu\\ &= f_k(1;q)\\ &= \sum_{n=0}^{\infty}\frac{q^{n(n+k-1)}(-q;q^2)_n}{(q^4;q^4)_n}\end{aligned}$$

This completes the proof of (2.3).

Proof of Theorem 2.3.2 First we define the mapping denoted by f as

$$f: E_{p,q}\to(p+q-1)_p, \tag{2.7}$$

and the inverse mapping f^{-1} is easily seen to be

$$f^{-1}: m_i\to E_{i,m-i+1}. \tag{2.8}$$

We shall prove that if Δ is a matrix enumerated by $B_k(\nu)$ then the n–color partition $f(\Delta)$ is enumerated by $A_k(\nu)$, and conversely, if π is an n–color partition enumerated by $A_k(\nu)$, then the BK_ν–matrix $f^{-1}(\pi)$ is enumerated by $B_k(\nu)$.
Let the matrix enumerated by $D_k(\nu)$ has the following representation.

$$\Delta=a_{1,1}E_{1,1}+a_{1,2}E_{1,2}+\cdots+a_{2,1}E_{2,1}+a_{2,2}E_{2,2}+\cdots+a_{3,1}E_{3,1}+a_{3,2}E_{3,2}+\cdots$$

Clearly, in view of the condition (2.3.2.4), the entries in Δ cannot exceed one. Hence, each $a_{i,j} = 1$ or 0. Let $E_{p,q}, E_{r,s}(p+q \geq r+s)$ be two units of Δ which correspond to two n–color parts m_i, n_j of $f(\Delta)$. Then $m_i = (p+q-1)_p$ and $n_j = (r+s-1)_r$ by (2.7). Since $(p+q \geq r+s)$, therefore $m \geq n$ and

$$((m_i - n_j)) = (p+q-1) - p - (r+s-1) - r = q - s - 2r = \{\{E_{p,q} - E_{r,s}\}\}$$

which is non negative and $\equiv 0(mod\ 4)$. This shows that (2.3.2.4) implies (2.3.1.4). Since $f(E_{i,j}) = (i+j-1)_i = m_i$ (say), by (2.7), so if $E_{i,j}$ is the only nonzero entry in Δ or $i+j$ is minimum it means that in $f(\Delta)$ either m_i is the only part or the least part. Thus (2.3.2.3) implies (2.3.1.3).
Next, we see that if k is odd, then by (2.3.2.1) even columns in Δ are zero which means that in $E_{p,q}$, q is odd. Further since $p \leq 2$ by (2.3.2.2), we conclude that

$$f(E_{p,q}) = \begin{cases} q_1 & \text{if } p = 1 \\ (q+1)_2 & \text{if } p = 2 \end{cases}$$

This shows that in $f(\Delta)$ the parts are of the form $(2l-1)_1$ or $(2l)_2$. Similarly, we can show that if k is even, then in $f(\Delta)$ the parts are of the form $(2l-1)_2$ or $(2l)_1$. Thus (2.3.2.1) and (2.3.2.2) imply (2.3.1.2). Finally, when k is odd, say, $(2l-1)$, the first $(l-1)$ odd columns, that is, $1st, 3rd, \cdots, (2l-3)th$ are zero by (2.3.2.5) and since $E_{1,2l-3} = (2l-3)_1$ and $E_{2,2l-3} = (2l-2)_2$, we see that in $f(\Delta)$ the parts are $\geq k$. Thus (2.3.2.5) implies (2.3.1.1) when k is odd. Similarly, we can show that (2.3.2.6) implies (2.3.1.1) when k is even. Thus $f(\Delta)$ is enumerated by $A_k(\nu)$.

To see the reverse implication, let π be an n–color partition of ν enumerated by $A_k(\nu)$. We shall prove that the BK_ν–matrix $f^{-1}(\pi)$ is enumerated by $B_k(\nu)$. Let m_i, n_j ($m \geq$ n) be two parts of π such that $f^{-1}(m_i) = E_{p,q}$ and $f^{-1}(n_j) = E_{r,s}$. Then $E_{p,q} = E_{i,m-i+1}$ and $E_{r,s} = E_{j,n-j+1}$ by (2.8). Since $(m \geq n)$, we have $p+q = m+1 \geq n+1 = r+s$, and

$$\begin{aligned} \{\{E_{p,q} - E_{r,s}\}\} &= \{\{E_{i,m-i+1} - E_{j,n-j+1}\}\} \\ &= (m-i+1) - (n-j+1) - 2j \\ &= m - n - i - j \\ &= ((m_i - n_j)) \end{aligned}$$

Thus (2.3.1.4) implies (2.3.2.4) since $f^{-1}(m_i) = E_{i,m-i+1} = E_{i,j}$ (say)(by (2.8)). So if m_i is the only part or the least part of π it means that in $f^{-1}(\pi)$ either $E_{i,j}$ is the only non zero entry or $i+j$ is minimum. Thus (2.3.1.3) implies (2.3.2.3).
To prove (2.3.2.1), (2.3.2.2), (2.3.2.5) and (2.3.2.6), we first consider the case

when k is odd. Since $f^{-1}((2l-1)_1) = E_{1,2l-1}$ and $f^{-1}((2l)_2) = E_{2,2l-1}$, we see that in $f^{-1}(\pi)$ even columns are zero and all rows after the second row are zero. This proves (2.3.2.1) and (2.3.2.2). Furthermore, by (2.3.1.1) we see that in $f^{-1}((2l-1)_1) = E_{1,2l-1}$, $(2l-1) \geq k$ and in $f^{-1}((2l)_2) = E_{2,2l-1}$, $(2l) \geq k$, that is (2.3.2.5) and (2.3.2.6) are satisfied. Similarly, we can prove the case when k is even. This completes the proof of Theorem 2.3.2.

Theorem 2.3.3 leads to a 2–way infinite identity. In [3], for $k = 1$ and $k = 3$, the following Rogers–Ramanujan Identities arise as a particular case

$$\prod_{n=1, n\equiv\pm1,\pm2(mod\ 6)}^{\infty} \frac{1}{(1-q^n)} = \left(\sum_{n=0}^{\infty} \frac{q^{n^2}(-q;q^2)}{(q^4;q^4)_n}\right)\left(\prod_{n=1, n\equiv\pm2,\pm3,6(mod\ 12)}^{\infty} \frac{1}{1-q^n}\right),$$
$$\prod_{n=1, n\equiv\pm2,3(mod\ 6)}^{\infty} \frac{1}{(1-q^n)} = \left(\sum_{n=0}^{\infty} \frac{q^{n^2+2n}(-q;q^2)}{(q^4;q^4)_n}\right)\left(\prod_{n=1}^{\infty} \frac{1}{1-q^{4n-2}}\right).$$

The above identities have the following combinatorial interpretations respectively:

Theorem 2.3.4 *Let $C_1(\nu)$ denote the number of partitions of ν into parts $\equiv \pm2, \pm3, 6(mod\ 12)$. And let $D_1(\nu)$ denote the number of partitions into parts $\equiv \pm1, \pm2(mod\ 6)$. Then*

$$D_1(\nu) = \sum_{k=0}^{\nu} A_1(k)C_1(\nu-k) = \sum_{k=0}^{\nu} B_1(k)C_1(\nu-k)$$

Theorem 2.3.5 *Let $C_3(\nu)$ denote the number of partitions of ν into parts $\equiv 2(mod\ 4)$. And let $D_3(\nu)$ denote the number of partitions of ν into parts $\equiv \pm2, 3(mod\ 6)$. Then*

$$D_3(\nu) = \sum_{k=0}^{\nu} A_3(k)C_3(\nu-k) = \sum_{k=0}^{\nu} B_3(k)C_3(\nu-k)$$

Theorem 2.3.4 and Theorem 2.3.5 lead to a 3–way combinatorial identity

$$A_k(\nu) = B_k(\nu) = C_k(\nu), \quad k = 1, 3.$$

2.4 Conclusion

In this chapter, we have explored the bijection between plane partitions and n–color partitions of a number ν. This mapping leads to interesting results as discussed in previous section. The bijection using Bender and Knuth matrices is also used in [18] to prove results on symmetric functions.

Chapter 3

On Extensions Of Relations Between Plane Partitions and n–Color Partitions

3.1 Introduction

The purpose of this chapter is to further explore the bijection and find some relations between plane partitions and color partitions through Bender and Knuth matrices to establish a simple connection between them. Here, we establish a relation between the number of rows of plane partition and the subscripts of the corresponding n–color partition.

3.2 Main Results

Theorem 3.2.1 *Consider an r–rowed plane partition of ν. Then*

$$r = Max[n_s] \quad ,for\ s = 1, 2, \cdots, t$$

where

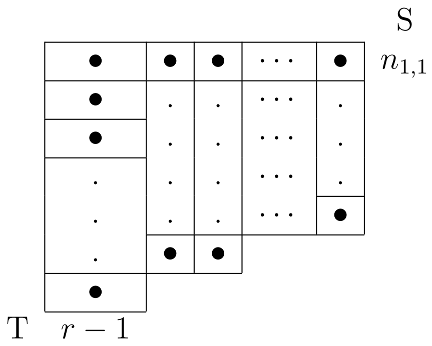
$$m_{1_{n_1}} + m_{2_{n_2}} + m_{3_{n_3}} + \cdots + m_{t_{n_t}}$$

is an n–color partition of ν.

Proof. Consider an r–rowed plane partition with l columns.

$$\begin{array}{ccccc} n_{1,1} & n_{1,2} & n_{1,3} & \cdots & n_{1,l} \\ n_{2,1} & n_{2,2} & n_{2,3} & \cdots & n_{2,l} \\ \vdots & \vdots & \vdots & & \vdots \\ n_{r,1} & n_{r,2} & n_{r,3} & \cdots & n_{r,l} \end{array}$$

During the first phase of conversion, each column of the plane partition is converted to a column of S and T. The entries on the right becomes the column of S and each entry on the bottom is added with one and then becomes the column of T. Converting every column of the plane partition, we get an ordered pair of plane partitions S and T.



Above the Frobenius construction is shown by taking the first column of the plane partition. The first row has $n_{1,1}$ dots. The first column has r dots, but the first dot is included in the $n_{1,1}$ dots of the first row. Hence, we are left with $r-1$ dots. So, the first entry of first column of T is equal to r.
This entry is the maximum entry of T. It can be easily seen that

$$r = Max\ [entries\ of\ T].$$

S and T are interchanged. Hence r becomes,

$$r = Max\ [entries\ of\ S].$$

In the second phase of construction, a two–line array is generated from the ordered pair (S,T) such that the entries in the first row are taken from S and the entries in the second row are taken from T.

$$r = Max\ [Entries\ of\ top\ row\ of\ the\ two\text{–}line\ array].$$

In the third phase, we obtain $E_{p,q}$ matrices which give the matrix A. For $k > 0$, if there are k copies of p in the first row and k copies of q in the second row, then $a_{p,q} = k$.

$$r = Max\ [p : a_{p,q} = k > 0].$$
$$r = Max\ [p : E_{p,q}\ appears\ in\ the\ representation\ of\ matrix\ A].$$

The row number p in $E_{p,q}$ is nothing but the subscript n_s if $m_{s_{n_s}}$ is the corresponding part of $E_{p,q}$ matrix in the n–colour partition of ν. Hence,

$$r = Max\ [n_s]\ where\ 1 \le s \le t.$$

Theorem 3.2.2 *If m_i is the only part in an n–color partition of ν, then the correponding plane partition has i rows, 1 column, $m-i+1$ occurs as an entry in the first row and the rest $i-1$ entries are all 1's.*

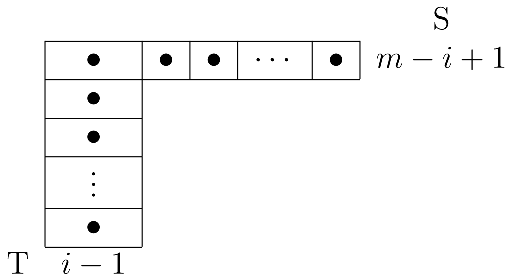
Proof. Since m_i is the only part, there is only one $E_{p,q}$ matrix which is $E_{i,m-i+1}$. As $a_{i,m-i+1} = 1 > 0$, there is one copy of i in the first row and one copy of $m-i+1$ in the second row of the two–line array. The two–line array is

$$\begin{pmatrix} i \\ m-i+1 \end{pmatrix}$$

The ordered pair of plane partitions (S,T) after interchanging becomes

$$\begin{array}{cc} S & T \\ m-i+1 & i \end{array}$$

By the method of Frobenius, taking entries of S on the right and one less than entries of T on the bottom, we get



The first entry of S is $m-i+1$, hence adding $m-i+1$ dots in the first row. The first entry of T is $i-1$, so creating $i-1$ rows with one dot in each row. The corresponding plane partition can be obtained by counting the number of dots in each row. Hence, we obtain

$$\begin{array}{c} m-i+1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{array}$$

It has i rows, 1 column, $m-i+1$ occurs as an entry in the first row and the rest $i-1$ entries are all 1's.

Remark 3.2.1 *If we have t parts in an n–color partition of* ν*,*

$$m_{1_{n_1}} + m_{2_{n_2}} + m_{3_{n_3}} + \cdots + m_{t_{n_t}}$$

where

$$m_s \geq m_{s+1} \;\; , \;\; n_s \geq n_{s+1} \;\; , \;\; m_s - n_s + 1 \geq m_{s+1} - n_{s+1} + 1 \qquad \forall \;\; 1 \leq s \leq t-1$$

then converting each part using Theorem 3.2.2 to a column of plane partition we get the corresponding plane partition.

$$\begin{array}{cccccc}
m_1 - n_1 + 1 & m_2 - n_2 + 1 & . & . & . & m_t - n_t + 1 \\
1 & 1 & . & . & . & 1 \\
. & . & . & . & 1 & \\
. & . & 1 & 1 & & \\
1 & 1 & & & & \\
1 & & & & &
\end{array}$$

Since n_1 *is the largest subscript, there are* n_1 *rows in the plane partition. Each part of the n–color partition corresponds to one column in this case, hence, there are t columns. The first column has* $n_1 - 1$ *entries equal to one, second column has* $n_2 - 1$ *entries equal to one, and so on.*

Corollary 3.2.3 *If*

$$m_{1_1} + m_{2_1} + m_{3_1} + \cdots + m_{t_1}$$

where $m_s \geq m_{s+1} \;\; \forall \; s \geq 1$ *is an n–color partition of* ν*, then the corresponding plane partition is*

$$m_1 \quad m_2 \quad m_3 \quad \cdots \quad m_t$$

in 1 row and t columns.

Proof. Since $n_s = 1 \; \forall \; s \geq 1, \;\; n_s \geq n_{s+1}$ is satisfied. Also $m_s \geq m_{s+1} \; \forall s$. Since $m_s - 1 + 1 = m_s$, $m_s - n_s + 1 \geq m_{s+1} - n_{s+1} + 1$ is also satisfied. Hence using Theorem 3.2.2 and Remark 3.2.1, the corresponding plane partition is

$$m_1 \quad m_2 \quad m_3 \quad \cdots \quad m_t$$

Corollary 3.2.4 *If*

$$m_{1_{m_1}} + m_{2_{m_2}} + m_{3_{m_3}} + \cdots + m_{t_{m_t}}$$

with $m_s \geq m_{s+1}$ *is an n–color partition of* ν*, then the corresponding plane partition has* m_1 *rows, t columns with all entries equal to one.*

Proof. Since $m_s \geq m_{s+1}\ \forall\ s \geq 1$, all the three conditions stated in Remark 3.2.1 are satisified. Hence, by Theorem 3.2.2 and Remark 3.2.1, the corresponding plane partition is

$$\begin{array}{cccccc} 1 & 1 & . & . & . & 1 \\ 1 & 1 & . & . & . & 1 \\ . & . & . & . & 1 & \\ . & . & . & 1 & & \\ 1 & 1 & & & & \\ 1 & & & & & \end{array}$$

Since m_1 is the largest subscript of all the parts of the n–color partition, the number of rows in the plane partition is m_1.
As $m_s - n_s + 1 = 1\ \forall s = 1, 2, \cdots, t$, all the entries of the first row are equivalent to one. The first column has m_1 number of 1's, the second column has m_2 number of 1's, $\cdots$, the t^{th} column has m_t number of 1's.

Theorem 3.2.5 *If $m_1 + m_2 + m_3 + \cdots + m_{t-1} + m_t$ with $t \leq m$ is an n–color partition of ν, then the corresponding plane partition has t rows and one column, with each entry equal to m.*

Proof. Let $m_1 + m_2 + m_3 + \cdots + m_{t-1} + m_t$ be an n–color partition of ν, then the matrix A can be written as sum of $E_{p,q}$ matrices as

$$A = E_{1,m} + E_{2,m-1} + E_{3,m-2} + \cdots + E_{t-1,m-t+2} + E_{t,m-t+1}$$

The two–rowed array becomes

$$\sigma_1(A) = \begin{pmatrix} 1 & 2 & 3 & \cdots & t-1 & t \\ m & m-1 & m-2 & \cdots & m-t+2 & m-t+1 \end{pmatrix}$$

The entries of the top row should be in non-increasing order.

$$\sigma_2(A) = \begin{pmatrix} t & t-1 & \cdots & 3 & 2 & 1 \\ m-t+1 & m-t+2 & \cdots & m-2 & m-1 & m \end{pmatrix}$$

The ordered pair (S,T) of plane partitions becomes

$$\begin{array}{cc} T & S \\ & \\ t & m \\ t-1 & m-1 \\ \vdots & \vdots \\ 3 & m-t+3 \\ 2 & m-t+2 \\ 1 & m-t+1 \end{array}$$

By Frobenius construction,

						S
	•	•	•	$\cdots$	•	m
	•	•	•	$\cdots$	•	$m-1$
	•	•	•	$\cdots$	•	$m-2$
	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$
	•	•	•	$\cdots$	•	$m-t+1$
T	$t-1$	$t-2$	$t-3$	$\cdots$	0	

Each row has m dots. The plane partition is

$$\begin{matrix} m \\ m \\ m \\ \vdots \\ m \end{matrix}$$

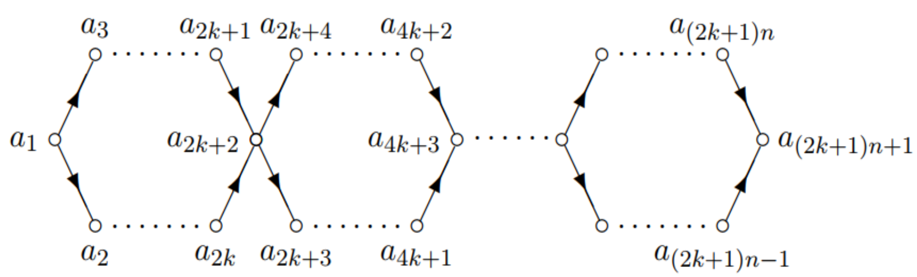
having t rows and a single column.

3.3 Applications of Plane partitions

3.3.1 Diamond Partitions

Diamond Partitions were introduced by G.E. Andrews and P. Paule in [7] as new variations of plane partitions in 2001. The culmination of their study leads to an infinite family of modular forms. These, in turn, lead to interesting arithmetic theorems and conjectures for the related partition functions.

k–Elongated Partition Diamond of length n



There are $(2k+1)n+1$ parts where n is the length and k is the elongation in diamond.

Its generating function is

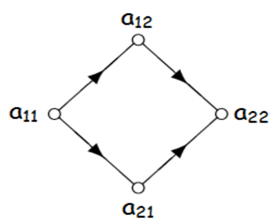
$$h_{n,k} = \frac{\prod_{j=0}^{n-1}(1+q^{(2k+1)j+2})(1+q^{(2k+1)j+4})......(1+q^{(2k+1)j+2k})}{\prod_{j=1}^{(2k+1)n+1}(1-q^j)}$$

A diamond partition can be obtained from a plane partition only when the plane partition is of the following forms:

1. ***2*** $\times$ ***2***
   A 2×2 plane partition

$$\begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{matrix}$$

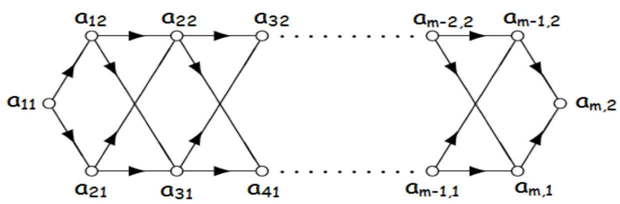
   can be written as a 1–elongated diamond partition of length one.



2. ***m*** $\times$ ***2***
   A $m \times 2$ plane partition

$$\begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \\ \vdots & \vdots \\ a_{m1} & a_{m2} \end{matrix}$$

   can be written as a $m-1$ elongated diamond partition



if the plane partition satisfies some additional conditions:

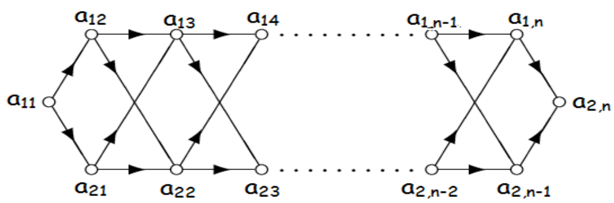
$$\begin{aligned} a_{12} &\geq a_{31} \\ a_{22} &\geq a_{41} \\ a_{32} &\geq a_{51} \\ \vdots \quad & \quad \vdots \\ a_{(m-2)2} &\geq a_{m1} \end{aligned}$$

3. ***2*** $\times$ ***n***

   A $2 \times n$ plane partition

$$\begin{matrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \end{matrix}$$

   can be written as a $n-1$ elongated diamond partition



   if the plane partition satisifies some additional conditions:

$$\begin{aligned} a_{21} &\geq a_{13} \\ a_{22} &\geq a_{14} \\ a_{23} &\geq a_{15} \\ \vdots \quad & \quad \vdots \\ a_{2(n-2)} &\geq a_{1n} \end{aligned}$$

3.3.2 Solid Partitions

In mathematics, solid partitions are natural generalizations of partitions and plane partitions defined by Percy Alexander MacMahon in [17]. A solid partition of n is a three–dimensional array, $n_{i,j,k}$, of non–negative integers (the indices $i, j, k \geq 1$) such that

$$\sum_{i,j,k} n_{i,j,k} = n$$

and

$$n_{i,j,k} \geq n_{(i+1),j,k} \ , \ n_{i,j,k} \geq n_{i,(j+1),k} \ , \ n_{i,j,k} \geq n_{i,j,(k+1)} \ , \ \forall i, j, k.$$

As the definition of solid partitions involves three–dimensional arrays of numbers, they are also called three–dimensional partitions in notation where plane partitions are two–dimensional partitions and partitions are one–dimensional partitions. There is a one to one correspondence between plane partitions and solid partitions in which the part magnitude is limited by unity. If we take a plane partition in the xy–plane, we can obtain the corresponding solid partition by replacing each part by a pile of nodes in the direction of z–axis. The plane partition arises by projection of the solid partition upon one of the coordinate planes.

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