STUDY OF VARIABLE ORDERING STRUCTURE

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Submitted by

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Certificate

I hereby certify that the dissertation entitled, **"Study Of Variable Ordering Structure"**, which is being submitted by **Miss. Shivani Saini** (Roll no. 301403018), presented in partial fulfillment of the requirements for the award of the degree of Master of Science (Mathematics and Computing) in the School of Mathematics, Thapar University, Patiala, is a bonafide work carried out under the supervision of **Dr. Vikas Sharma**. The part of the work presented in this dissertation no part of this thesis has been submitted for the award of any other degree.

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Contents

Abstract

The work being presented in the present thesis is devoted to the study of Variable Ordering Structure.

The chapterwise summary of the thesis is as follows:

Chapter 1 is introductory in nature. This chapter includes solution concept of vector optimization problem and set optimization problem, various examples have been discussed to elaborate these concepts. This chapter also introduces variable ordering structure.

In Chapter 2, we have studied the solution concept of vector optimization problem with variable ordering structure. The aim of this chapter is to present various basic and important results of vector optimization problem with variable ordering structure.

In Chapter 3, we have taken the idea of variable ordering structure to set optimization problem. And we collect several optimality notions based on the two binary relations introduced in Chapter 1. We compare these new concepts with known concepts in partially ordered spaces.

Chapter 1

Introduction Of Vector And Set Optimization

1.1 Introduction

An optimization problem consist of maximizing or minimizing a function over some feasible region, mathematically an optimization problem can be defined as

$$\max/\min \ f(x)$$

subject to $$g_i(x) \leq 0, \ \ i = 1, 2, \ldots, n$$

where $x \in R^n$. In case the objective function is single, it is straight forward to define the notion of an optimal solution. A feasible element with smallest (or largest) value of objective function is an optimal solution.

Some times situation arises, where one needs to consider more then one objective functions for example, someone wants to buy a house near the work place. The first and foremost consideration is the cost of the house. Even though the preference would be for a house with least cost, this is not the only criterion for decision as one may wish to have a house with at least five rooms, preferably well furnished, situated very close to the place of work, built in the near past, facilities like school, hospital and shopping malls are in the vicinity of the house and so on. The objectives are many and some are conflicting in nature; for instance, a house at a posh area with all facilities costs much more in comparison with a house in undeveloped area. In such situation one deals with more than one objective at a time and a compromise solution is ultimately found which need not be unique. Problems of this type are termed as multiobjective / multicriteria / vector optimization problems. Mathematically, a vector optimization (VP) Problem can be defined as:

$$\max / \min_{x \in S} \; f(x) = (f_1(x), f_2(x), ..., f_k(x)) \qquad (VP)$$

where $f_i : X \subseteq R^n \to R, \; i = 1, 2, ...k$ and $S \subseteq X$. The set S is called the feasible set for the Problem (VP). In this chapter we have discussed the solution criteria of vector optimization problem and set optimization problem. And also introduced the concept of variable ordering structure.

1.2 Some Preliminaries And Notations

Definition 1.2.1 *Let X be an arbitrary nonempty set with a binary relation $\leq$. Let $A, \; B, \; C \in X$ be arbitrary chosen. The binary relation is said to be*

(i) $reflexive$ if $A \leq A$.

(ii) transitive if $A \leq B$ and $B \leq C$ imply $A \leq C$.

(iii) symmetric if $A \leq B$ implies $B \leq A$.

(iv) antisymmetric if $A \leq B$ and $B \leq A$ imply $A = B$.

Definition 1.2.2 *The binary relation $\leq$ is said to be*

(i) a pre-order if it is reflexive and transitive.

(ii) a partial order if it is reflexive, transitive and antisymmetric or in other words, if it is a pre-order that is antisymmetric.

(iii) an equivalence relation if it is reflexive, transitive and symmetric.

When the relation $\leq$ is a pre order / a partial order, we say that X is a pre-ordered / partially ordered set.

Definition 1.2.3 *Let Y be a real vector space. A nonempty set $A \subseteq Y$ is said to be convex if for $0 \leq \lambda \leq 1$, $\lambda x + (1-\lambda)u \in A \ \forall x,\ u \in A$, see Fig 1.1.*

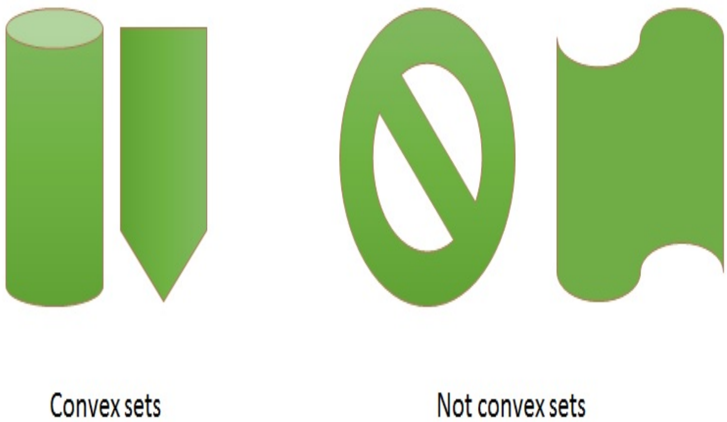

Figure 1.1: Convex Set

The intersection of arbitrarily many convex sets of a real vector space is convex. If S and T are nonempty convex subsets of a real vector space X, then the algebraic sum $\alpha S + \beta T$ is convex for all $\alpha, \beta \ \in \mathbb{R}$. Consequently, for every $\bar{x} \in X$ the translated set $\{\bar{x}\} + S$ is convex as well.

Definition 1.2.4 *Let K be a nonempty subset of a real vector space B. Then K is said to be a cone if $x \in K,\ \lambda \geq 0 \ \Rightarrow \ \lambda x \in K$, see Fig 1.2.*

Definition 1.2.5 *A cone K is said to be pointed if $K \bigcap (-K) = \{0\}$, see Fig 1.3.*

Definition 1.2.6 *Let S be a nonempty subset of a real linear space Y.*

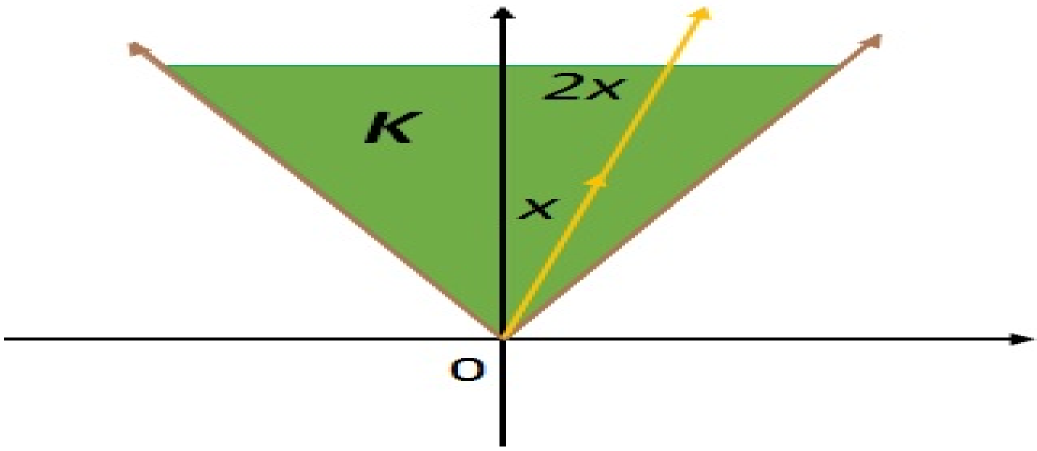

Figure 1.2: Cone

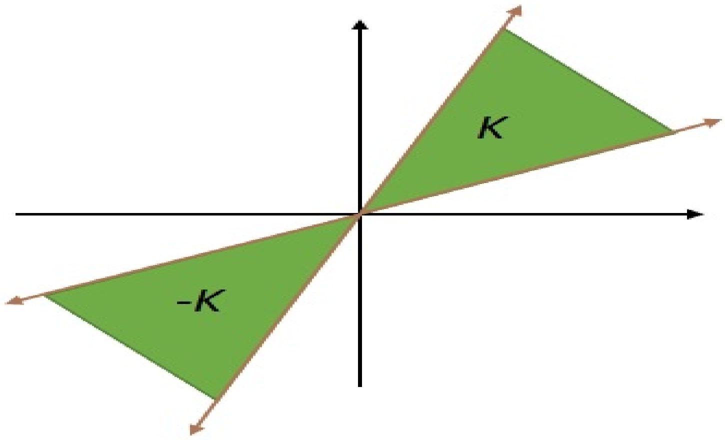

Figure 1.3: Pointed Cone

(a) *The set* $cor(S) := \{\bar{y} \in S|$ *for every* $y \in Y$ *there is a* $\bar{\lambda} > 0$ *with* $\bar{y} + \lambda y \in S$ *for all* $\lambda \in [0, \bar{\lambda}]\}$ *is called the algebraic interior of* S*, see Fig 1.4*

(b) *The set* S *with* $S = cor(S)$ *is called algebraically open.*

(c) *The set of all elements of* X *which do not belong to* $cor(S)$ *and* $cor(X \backslash S)$ *is called the algebraic boundary of* S.

Lemma 1.2.1 *A cone* C *in a real linear space is convex if and only if* $C +$

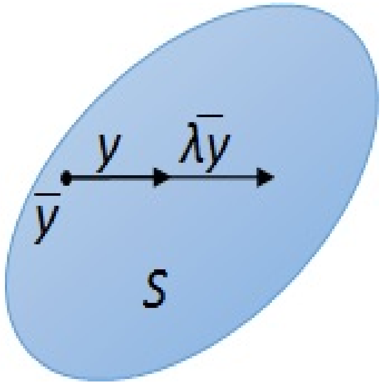

Figure 1.4: $\bar{y} \in cor(S)$

$C \subset C$.

1.3 Vector Optimization

In vector optimization one investigates optimal elements of a set in a space. The problem of determining these optimal elements, if they exist at all, is called a vector optimization problem. Mathematicaly an vector optimization problem can be defined as

$$\max f(x) = (f_1(x), f_2(x), ..., f_k(x))$$
$$\text{subject to } x \in S$$

where $f_i : X \subseteq R^n \to R,\ i = 1, 2, ...k$ and $S \subseteq X$. The set S is called the feasible set. We use the following notations:

For $x, y \in R^n$

,$x \geqq y \quad \Leftrightarrow \quad x - y \in R^n_+ \quad \Leftrightarrow \quad x_j \geq y_j, \forall j = 1, 2, ...n;$

$x \geq y \quad \Leftrightarrow \quad x - y \in R^n_+ \setminus \{0\} \quad \Leftrightarrow \quad x_j \geq y_j, \forall j = 1, 2, ...n,\ x_i > y_i,$ for some i ;

$x > y \quad \Leftrightarrow \quad x - y \in intR^n_+ \quad \Leftrightarrow \quad x_j > y_j, \forall j = 1, 2, ...n;$

$x \ngeqq y \quad \Leftrightarrow \quad x - y \notin R^n_+ \quad \Leftrightarrow$ negation of $x \geqq y$;

$x \ngeq y \quad \Leftrightarrow \quad x - y \notin R^n_+ \setminus \{0\} \quad \Leftrightarrow$ negation of $x \geq y$;

$x \ngtr y \quad \Leftrightarrow \quad x - y \notin intR^n_+ \quad \Leftrightarrow$ negation of $x > y$;

where $R^n_+ = \{(x_1, x_2, \ldots x_n) \mid x_i \geq 0,\ i = 1, 2, \ldots n\}$.

1.3.1 Optimality Concepts

Let Y denote a real linear space that is pre-ordered by some convex cone $K \subset Y$ and let A denote some nonempty subset of Y. In general, one is mainly interested in minimal and maximal elements of the set A.

Definition 1.3.1 *(a) An element $\bar{y} \in A$ is called a minimal element of the set A, if*

$$(\{\bar{y}\} - K) \cap A \subset \{\bar{y}\} + K$$

. (b) An element $\bar{y} \in A$ is called a maximal element of the set A, if

$$(\{\bar{y}\} + K) \cap A \subset \{\bar{y}\} - K.$$

If the ordering cone K is pointed, then the above inclusions can be replaced by the set equations

$$(\{\bar{y}\} - K) \cap S = \{\bar{y}\} \quad (or:\ y \leq_K \bar{y},\ y \in A\ \Rightarrow\ y = \bar{y})$$

and

$$(\{\bar{y}\} + K) \cap S = \{\bar{y}\} \quad (or:\ \bar{y} \leq_K y,\ y \in A\ \Rightarrow\ y = \bar{y}),$$

respectively. Since every maximal element of A is also minimal with respect to the partial ordering induced by the convex cone $-K$, without loss of generality it is sufficient to study the minimality notion.

Lemma 1.3.1 *Let S be a nonempty subset of a partially ordered linear space with an ordering cone K.*

(a) If the ordering cone K is pointed, then every minimal element of the set $S + K$ is also a minimal element of the set S.

(b) Every minimal element of the set S is also a minimal element of the set $S + K$.

Proof

(a) Let $\bar{y} \in S + K$ be an arbitrary minimal element of the set $S + K$. If we assume that $\bar{y} \notin S$, then there is an element $y \neq \bar{y}$ with $y \in S$ and $\bar{y} \in \{y\} + K$. Consequently, we get $y \in (\{\bar{y}\} - K) \cap (S + K)$ which contradicts the assumption that $\bar{y}$ is a minimal element of the set $S + K$. Hence, we obtain $\bar{y} \in S \subset S + K$ and, therefore, $\bar{y}$ is also a minimal element of the set S.

(b) Take an arbitrary minimal element $\bar{y} \in S$ of the set S, and choose any $y \in (\{\bar{y}\} - K) \cap (S + K)$. Then there are elements $s \in S$ and $k \in K$ so that $x = s + k$. Consequently, we obtain $s = y - k \in \{\bar{y}\} - K$, and since $\bar{y}$ is a minimal element of the set S, we conclude $s \in \{\bar{y}\} + K$. But then we get also $y \in \{\bar{y}\} + K$.

Definition 1.3.2 *Let S be a nonempty subset of a partially ordered linear space X with an ordering cone K which has a nonempty algebraic interior.*

(a) An element $\bar{y} \in S$ is called a weakly minimal element of the set S, if

$$(\{\bar{y}\} - cor(K)) \cap S = \emptyset$$

(b) An element $\bar{y} \in S$ is called a weakly maximal element of the set S, if

$$(\{\bar{y}\} + cor(K)) \cap S = \emptyset$$

Definition 1.3.3 *Let S be a nonempty subset of a partially ordered linear space with an ordering cone K.*

(a) An element $\bar{y} \in S$ is called a strongly minimal element of the set S, if

$$S \subset \{\bar{y}\} + K \quad (or : \bar{y} \leq_K y \ \forall y \in S)$$

(b) An element $\bar{x} \in S$ is called a strongly maximal element os the set S, if

$$S \subset \{\bar{y}\} - K \quad (or : y \leq_K \bar{y} \ \forall y \in S)$$

Lemma 1.3.2 *(a) Every strongly minimal element of the set A is also a minimal element of A.*

(b) Let K have a nonempty algebraic interior and $K \neq Y$. Then every minimal element of the set A is also a weakly minimal element of the set A.

Proof

(a) It holds $A \subset \{\bar{y}\} + K$ for any strongly minimal element $\bar{y}$ of A. Thus

$$(\{\bar{y}\} - K) \cap A \subset A \subset \{\bar{y}\} + K$$

(b) The assumption $K \neq Y$ implies $(-cor(K)) \cap K = \emptyset$. Therefore, for an arbitrary minimal element $\bar{y}$ of A it follows

$$\begin{aligned}\emptyset &= (\{\bar{y}\} - cor(K)) \cap (\{\bar{y}\} + K) \\ &= (\{\bar{y}\} - cor(K)) \cap (\{\bar{y}\} - K) \cap A \\ &= (\{\bar{y}\} - cor(K)) \cap A\end{aligned}$$

which means that $\bar{y}$ is also a weakly minimal element of A.

Example 1.3.1 *Let $Y = R^2$ and let a partial order be induced by the cone $K = R^2_+$. Consider the set $A = [0,1] \times [0,1]$. The unique minimal element is 0_{R^2} while all elements of the set $\{(y_1, y_2) \in A |\ y_1 = 0 \vee y_2 = 0\}$ are weakly minimal elements. Note that 0_{R^2} is also a strongly minimal element.*

1.4 Vector Optimization Problem With A Set-Valued Objective Map

Definition 1.4.1 *Let Y be a vector space, partially ordered by a proper pointed convex closed cone K. Let P(Y) $= Y^2$ be the power set of Y. We consider a set valued optimization problem with a general geometric constraint:*

$$\min F(x) \qquad \text{subject to } x \in S, \tag{SP}$$

where S is a subset of X, X is a linear space and the mapping $F : S \rightrightarrows Y$ is a set valued mapping, we use the notations $F(S) = \bigcup_{x \in S} \mathrm{F(x)}$ the image set of F and $domF = \{x \in S |\ F(x) \neq \emptyset\}$.

Now in contrast to single valued functions, for every $\bar{x} \in domF$ there are many distinct values $\bar{y} \in Y$ such that $\bar{y} \in F(\bar{x})$. Hence when studying minimizers of a set valued mapping, we fix one element $\bar{y} \in F(\bar{x})$, and formulate the following solution concept based on the concept of Pareto minimality.

Definition 1.4.2 *A pair* $(\bar{x},\ \bar{y})$ *with* $\bar{x} \in S$ *and* $\bar{y} \in F(\bar{x})$ *is called a minimizer of the Problem (SP) if* $\bar{y}$ *is a minimal element of the set* $F(S)$*, i.e.*

$$(\{\bar{y}\} - K) \bigcap F(S) \subset \{\bar{y}\} + K.$$

Example 1.4.1 *(a) Assume that* $f,\ g : S \to Y$ *are given vector functions. Then* $F : S \rightrightarrows Y$ *with*

$$F(x) := \{y \in Y \mid f(x) \leq_K y \leq_K g(x)\}$$

is a possible set-valued map which may be used as an objective. If $f = g$ *and* K *is pointed, then at every* $x \in S$ *a corresponding image* y *is uniquely determined, otherwise the values of* y *vary in the order interval* $[f(x),\ g(x)]$*.*

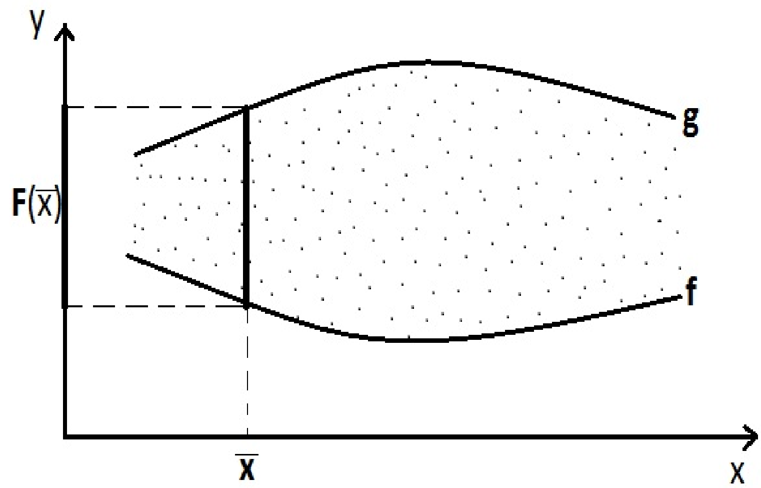

Figure 1.5: Illustration of the set valued map F in Example 3(a)

(b) One special case of the previous example is obtained if a vector function

$\varphi : S \to Y$ is known and the y-values vary around $\varphi(x)$, i.e. we have

$$F(x) := \{y \in Y |\ \varphi(x) - \alpha \leq_K\ y\ \leq_K\ \varphi(x) + \beta\ \}$$

where $\alpha,\ \beta \in K$.

It can be expected that the minimization of the set valued map F in Example 1.4.1 (a) has something to do with the minimization of f. Therefore, under the assumption given in Example 1.4.1 (a) we consider the single-valued vector optimization problem

$$\min_{x \in S} f(x). \tag{1.1}$$

Theorem 1.4.3 *For (SP) Problem, let K be pointed, let $f : S \to Y$ be a given function, and let $F : S \rightrightarrows Y$ be defined as*

$$F(x) := \{y \in Y |\ f(x) \leq_K y\}\ for\ all\ x \in S.$$

(a) If $(\bar{x},\ \bar{y})$ is a minimizer of the Problem (SP), then $\bar{y} = f(\bar{x})$ and $\bar{x}$ is a minimal solution of the Problem (1.1).

(b) If $\bar{x}$ is a minimal solution of the Problem (1.1), Then $(\bar{x},\ f(\bar{x}))$ is a minimizer of the Problem (SP).

The preceding theorem shows that in the special case discussed in Example 1.4.1 (a) the set optimization Problem (SP) is equivalent to the vector optimization Problem (1.1) being simpler than the Problem (SP). Therefore, it is not necessary to work with such a general set-valued theory in this special case. Hence, a general set-valued theory makes only sense for set-valued maps whose lower boundary cannot be described by a function f as it is done in Example 1.4.1 (a). Although the concept of a minimizer of the set-valued Problem (SP) given in Definition 2.1 is of mathematical interest, it cannot be often used in practice.

1.5 Criteria Of Set Optimization

In vector sense a minimizer $(\bar{x},\ \bar{y})$ depends on only certain special element $\bar{y}$ of $F(\bar{x})$ and other elements of $F(\bar{x})$ are ignored. In other words, an element $\bar{x} \in S$ for that there exists at least one element $\bar{y} \in F(\bar{x})$ which is a Pareto minimal point of the image set of F even if there exist many bad elements in $F(\bar{x})$, is a solution of the set-valued optimization Problem (SP). The drawback of this approach is that in most applications only one element $\bar{y}$ does not imply the whole set $F\{\bar{x}\}$ in a certain sense optimal compared to the other sets $F(x),\ s \in S$. For this reason, the solution concept based on vector approach are sometimes improper.

In order to avoid this drawback it is necessary to work with practically relevant order relations for sets. This leads to solution concepts for set-valued optimization problems based on comparisons among values of the set-valued objective map F.

In 1997, Kuroiwa introduced criterion for solution of Problem (SP) called set optimization criterion. This criterion is defined by an appropriate ordering on Y and is based on comparisons among values of F, i.e. the whole image F(x).

In order to understand set relationship, Kuroiwa [35] introduced eight type of relations between two sets in an ordered vector space with respect to a convex cone. This classification is based on two ideas for set-relation. First, with respect to relationship between two vectors a, b $\in Y$, one of the followings holds:

(i) $a \in b + K$ *(equivalently* $b \in\ a - K$*)*;

(ii) $a \notin\ b + K$ *(equivalently* $b \notin\ a - K$*)*;

(iii) $b \in\ a + K$ *(equivalently* $a \in\ b - K$*)*;

(iv) $b \notin\ a + K$ *(equivalently* $a \notin\ b - K$*)*;

These relationships are summarized as $b \leq_K a,\ b \nleq_K a$ *or* $a \leq_K b,\ a \nleq_K$

b, that is, one vector is dominated by the other vector or otherwise. In the case of relationship between a nonempty set $A \subset Y$ and a vector $b \in Y$, a different situation is observed; we have two domination structure

(i) *for all* $a \in A,\ a \leq_K b$;

(ii) *there exists* $a \in A$ *such that* $a \leq_K b$.

Following example shows the above mentioned two domination structures.

Example 1.5.1 *(a) Consider two sets* $A = [-1, -2] \times [1, 2]$, $B = [3, 5] \times [3, 5]$ *and cone* $K = R^2_+$ *for relation (1) see Fig 1.6. Every vector* $b \in B$ *dominates the whole set* A.

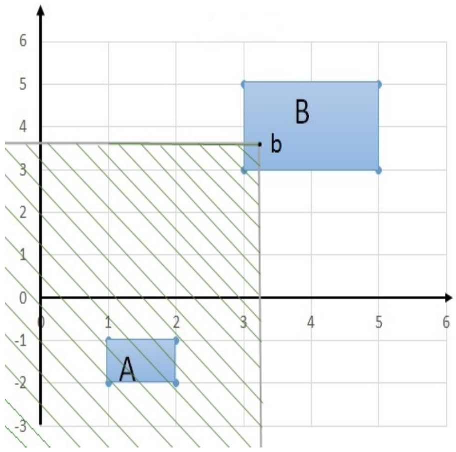

Figure 1.6: Vector b dominates the whole set A

(b) Consider set $A' = [1, 3] \times [2, 4]$, *set* $B' = [3, 5] \times [3, 5]$ *and cone* $K = R^2_+$ *for relation (2) see Fig 1.7. here every vector* $b' \in B'$ *is dominated from below by an vector* $a \in A'$ *(say, a=(2.8,2.8)). But for* $a = (2, 3.5)$ *this condition does not hold.*

Further, these relationships are denoted by b $\in A \bigcap_+ K$ and $b \in A \biguplus K$, respectively, where

$$A \bigcap\nolimits_+ K := \bigcap_{a \in A} a + K \quad and \quad A \biguplus K := \bigcup_{a \in A} a + K.$$

Analogously, we use the following notations for a nonempty set B $\subset Y$:

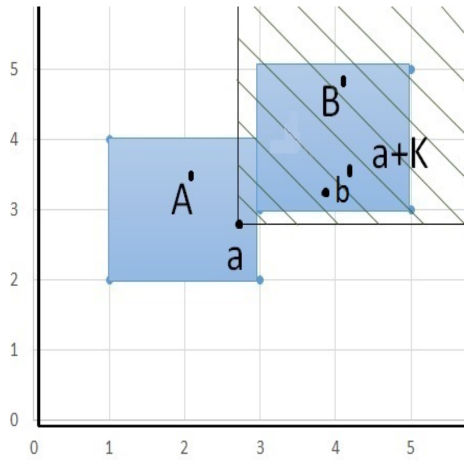

Figure 1.7: Vector b' is dominated from below by an element of the set A'

$$B \underset{-}{\bigcap} K := \bigcap_{b \in B} b - K = B \underset{+}{\bigcap} (-K)$$

and $$B \overset{-}{\bigcup} K := \bigcup_{b \in B} b - K = B \uplus (-K)$$

It is easy to see that $A \underset{+}{\bigcap} K \subset A \uplus K$ and $B \underset{-}{\bigcap} K \subset B \overset{-}{\bigcup} K$, and also that $A \uplus B = A + B$ and $A \overset{-}{\bigcup} B = A - B$. As in Example 1.5.1. (a) $A \underset{+}{\bigcap} K = \{(2, -1) + R^2_+\}$ and $A \uplus K = \{(1, -2) + R^2_+\}$, clearly $(2, -1) + R^2_+ \subset (1, -2) + R^2_+$.

Secondly, we consider the relationship between two nonempty sets in Y, which is strongly concerned with intersection and inclusion in set theory. Given nonempty sets $A,\ B \subset Y$, exactly one of the following conditions holds: **(i)** $A \bigcap B = \emptyset$; **(ii)** $A \bigcap B \neq \emptyset$. By using above two ideas we classify the relationship between two nonempty sets $A,\ B \in Y$ in the sense that A is dominated from above by B or A dominates B from below:

(1) $A \subset (B \underset{-}{\bigcap} K)$ *by* $A \leq_K B$;

(2) $A \bigcap (B \underset{-}{\bigcap} K) \neq \emptyset$ *by* $A \leq_K B$;

(3) $A \uplus K \supset B$ *by* $A \leq_K B$;

(4) $(A \biguplus K) \bigcap B \neq \emptyset$ *by* $A \leq_K B$;

(5) $(A \bigcap_{+} K) \supset B$ *by* $A \leq_K B$;

(6) $(A \bigcap_{+} K) \bigcap B \neq \emptyset$ *by* $A \leq_K B$;

(7) $A \subset B \bigcup^{-} K$ *by* $A \leq_K B$;

(8) $A \bigcap (B \bigcup^{-} K) \neq \emptyset$ *by* $A \leq_K B$;

It may be noted that these eight classifications are not exaustive, in the following we prove that some of them are same.

Theorem 1.5.1 *Condition (1) is equivalent to Condition (5).*

Proof $A \subset (B \bigcap_{-} K)$ *by* $A \leq_K B$;

$\Rightarrow \quad \forall a \in A \;\; a \in B \bigcap_{-} K$

$\Rightarrow \quad a \in \bigcap_{b \in B} b - K$

$\Rightarrow \quad a \in b - K \qquad \forall b \in B, \; \forall a \in A$

$\Rightarrow \quad b \in a + K \qquad \forall b \in B, \; \forall a \in A$

$\Rightarrow \quad b \in A \bigcap_{+} K \qquad \forall b \in B$

$\Rightarrow \quad B \subset A \bigcap_{+} K$

conversly

let $B \subset A \bigcap_{+} K$

$\Rightarrow \quad b \in A \bigcap_{+} K \qquad \forall b \in B$

$\Rightarrow \quad b \in a + K \qquad \forall b \in B, \; \forall a \in A$

$\Rightarrow \quad a \in b - K \qquad \forall b \in B, \; \forall a \in A$

$\Rightarrow \quad a \in \bigcap_{b \in B} b - K$

$\Rightarrow \quad \forall a \in A \;\; a \in B \bigcap_{-} K$

$A \subset (B \bigcap_{-} K)$ *by* $A \leq_K B$;

In the same way we can prove that Condition (4) is equivalent to Condition (8).

Finally, we get following six kinds of relations between two sets w.r.t ordering cone K.

(1) $(A \bigcap_{+} K) \supset B$ *by* $A \leq_K B$;

(2) $A \bigcap (B \bigcap_{-} K) \neq \emptyset$ *by* $A \leq_K B$;

(3) $A \uplus K \supset B$ *by* $A \leq_K B$;

(4) $(A \bigcap_{+} K) \bigcap B \neq \emptyset$ *by* $A \leq_K B$;

(5) $A \subset B \overset{-}{\bigcup} K$ *by* $A \leq_K B$;

(6) $(A \uplus K) \bigcap B \neq \emptyset$ *by* $A \leq_K B$;

Theorem 1.5.2 *The statement* $A \bigcap (B \bigcap_{-} K) \neq \emptyset$ *by* $A \leq_K B$ *is equivalent to* "$\exists a \in A$ *such that* $\forall b \in B,\ a \leq_K b$".

Proof Given that $A \bigcap (B \bigcap_{-} K) \neq \emptyset$ *by* $A \leq_K B$

$\Rightarrow \quad \exists a \in A$ and $a \in B \bigcap_{-} K$

$\Rightarrow \quad a \in \bigcap_{b \in B} b - K$

$\Rightarrow \quad a \in b - K \qquad \forall b \in B$

$\Rightarrow \quad a \leq_K b \qquad \forall b \in B.$

conversely

$\exists a \in A$ such that $\forall b \in B\ a \leq_K b$

$\Rightarrow \quad a \in b - K \qquad \forall b \in B$ and for some $a \in A$

$\Rightarrow \quad a \in \bigcap_{b \in B} b - K$

$\Rightarrow \quad a \in B \bigcap_{-} K$

also $a \in A$

so $a \in A \bigcap (B \underset{-}{\bigcap} K)$ Thus $A \bigcap (B \underset{-}{\bigcap} K) \neq \emptyset$.

In the following table equivalent statements to the above six relations are mentioned, and can be proved in the same way as first relation is shown in the above theorem.

$(A \underset{+}{\bigcap} K) \supset B$ *by* $A \leq_K B$	$\forall a \in A,\ \forall b \in B,\ a \leq_K b$
$A \bigcap (B \underset{-}{\bigcap} K) \neq \emptyset$ *by* $A \leq_K B$	$\exists a \in A$ *such that* $\forall b \in B,\ a \leq_K b$
$A \uplus K \supset B$ *by* $A \leq_K B$	$\forall b \in B,\ \exists a \in A$ *such that* $a \leq_K b$
$(A \underset{+}{\bigcap} K) \bigcap B \neq \emptyset$ *by* $A \leq_K B$	$\exists b \in B$ *such that* $\forall a \in A,\ a \leq_K b$
$A \subset B \overset{-}{\bigcup} K$ *by* $A \leq_K B$	$\forall a \in A,\ \exists b \in B$ *such that* $a \leq_K b$
$(A \uplus K) \bigcap B \neq \emptyset$ *by* $A \leq_K B$	$\exists a \in A,\ \exists b \in B$ *such that* $a \leq_K b$

It is clear that (1) implies (2) and (4), (2) implies (3), (4) implies (5), and each of (3) and (5) implies (6). As shown in Fig 1.9.

Following example shows that the relation (1) implies relation (2).

Example 1.5.2 *Consider the sets* $A = x \times x^2, \ -3 \leq x \leq 0,$ *and* $B = [0, 3] \times [10, 11]$ *and cone* $K = R^2_+$

As shown in Fig 1.8 relation (1) holds since this condition holds for all $a \in A$ *thus holds for a particular* $\bar{a} \in A$ *also.*

Definition 1.5.3 *An element* $x_0 \in S$ *is said to be*

(i) l-minimal solution of (SP) if $F(x) \leq^l F(x_0)$ *and* $x \in S$ *implies* $F(x_0) \leq^l F(x)$.

(ii) u-minimal solution of (SP) if $F(x) \leq^u F(x_0)$ *and* $x \in S$ *implies* $F(x_0) \leq^u F(x)$.

By the observation of the inequalities above, we can see that the notion of l-minimal solution is based on comparisons among sets of minimal elements

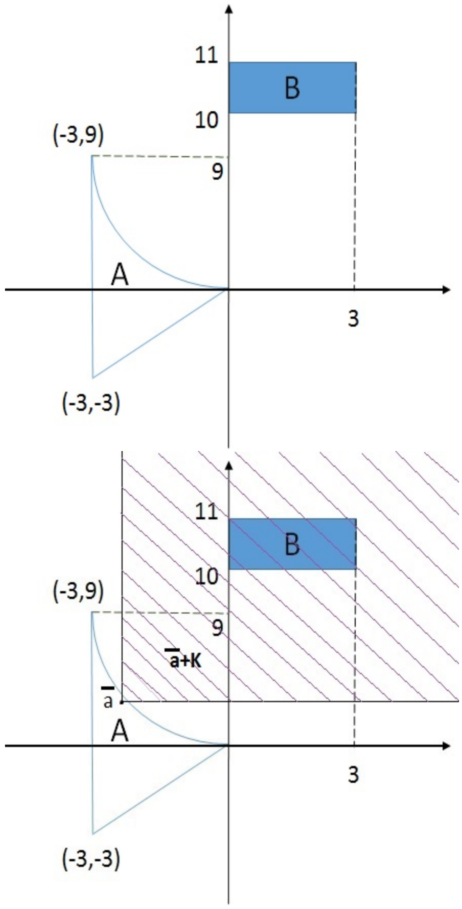

Figure 1.8: Relation (1) implies Relation (2)

of values of F, and u-minimal is among sets of maximal elements. When an element x_0 is one of such solutions of (SP), there does not exist $x \in S$ such that $F(x)$ is strictly smaller that $F(x_0)$ in a certain sense.

1.5.1 Relationship between Solution Concepts

Here we study the relationship between different solution concepts in set valued optimization. Let Y be a linear vector space, partially ordered by a proper pointed convex closed cone K, X a linear space S a subset of X and $F : X \rightrightarrows Y$. We consider the set valued optimization Problem (SP):

$$\min F(x) \text{ subject to } x \in S.$$

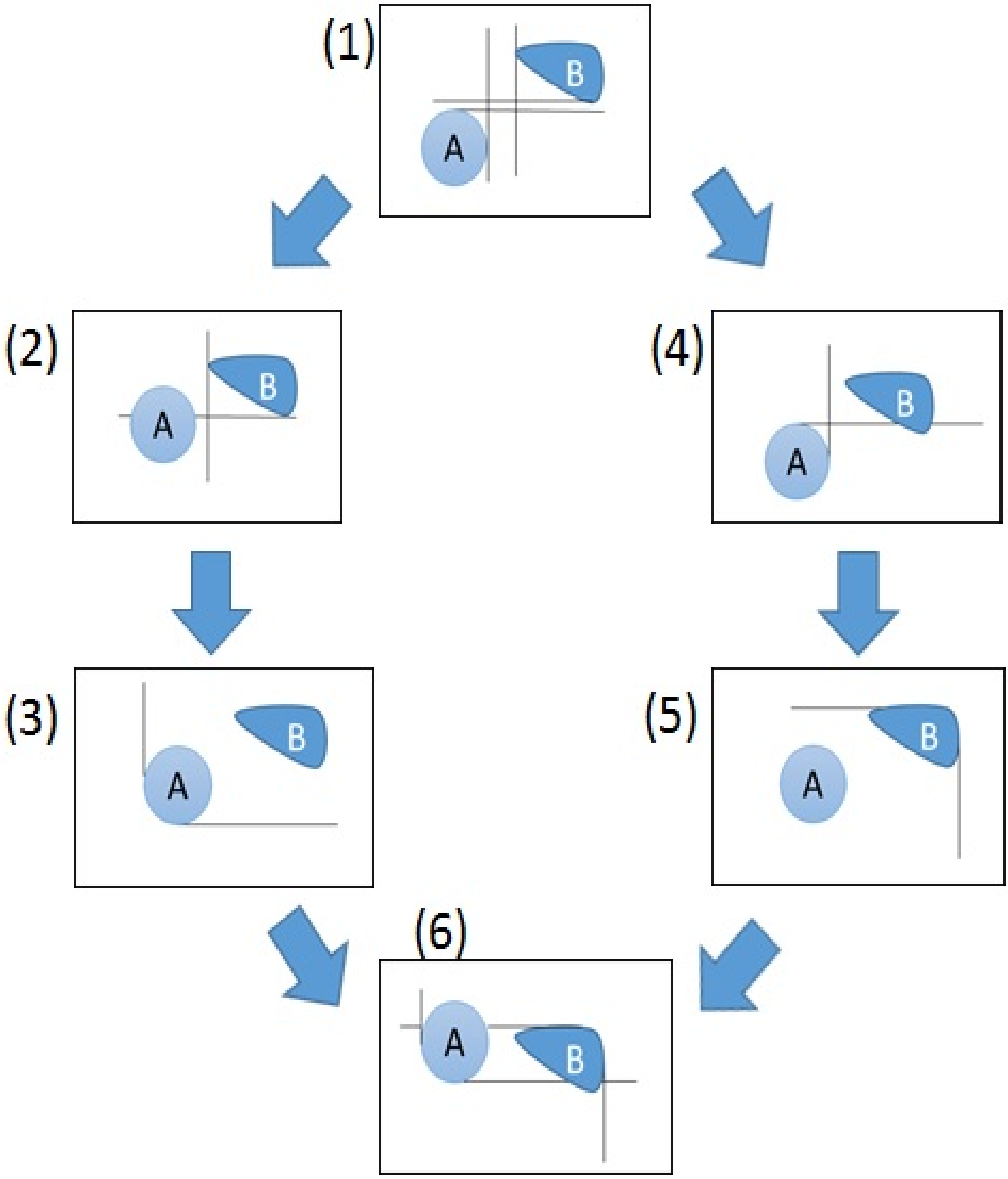

Figure 1.9: Classification of set relations with respect to the order

The differences between the solution concepts based on set approach and the solution based on vector approach are discussed below in examples.

Example 1.5.3 *Let S = [0,1], $Y = R^2$, and K=$R^2_+ = \{(y_1, y_2)|\ y_i \geq 0\}$.*

We consider the following set minimization problems:

(i)$F : S \rightarrow\ 2^y$ *defined by*

$$F(x) = \begin{cases} [(1,0),(0,1)] & if\ x = 0; \\ [(1-x,x),(1,1)] & if\ x \in (0,1]. \end{cases}$$

where [(a,b),(c,d)] is the line segment between (a,b) and (c,d). Then l-minimal solution is only x=0, however, in ordinary vector optimization, all elements in [0,1] are solutions.

(ii)$G : S \rightarrow\ 2^y$ *defined by*

$$G(x) = \begin{cases} [(1,\frac{1}{3}),(\frac{1}{3},1)] & if\ x = 0; \\ [(1-x,x),(1,1)] & if\ x \in (0,1]. \end{cases}.$$

Then in ordinary vector optimization, the set of solutions is (0,1], however, the set of l-minimal solutions is [0,1].

Example 1.5.4 *(Vector-Valued Game) We consider a vector-valued two-person game; Let A and B be nonempty sets, and let f be a map from A×B to Y. Assume that player 1 chooses a ∈A first, player 2 chooses b∈B next, and then f(a,b) is the loss for player 1. When player 2 is cooperative toward player 1, player 1 may choose a l-minimal solution of the following set-valued optimization Problem (VG):*

$$(VG) \qquad minimize f(a,B)$$
$$subject\ to\ a\ \in A.$$

when player 2 is non cooperative, to be exact, player 2 wills player 1's loss, then player 1 should choose a u-minimal solution of (VG).

1.6 Variable Ordering Structure

In scalar-valued optimization the definition of an optimal solution is quite natural: let S be some feasible set in a real linear space X and let $f : S \to R$ be the objective function which should be minimized over S. Then $\bar{x} \in S$ is called an optimal solution of

$$\min_{x \in S} f(x)$$

if $f(x) \leq f(x)$ for all $x \in S$.
The natural ordering $\leq$ describes here a total ordering in R, i.e. all elements in R are comparable using this ordering.

However, when we consider vector optimization problems, i.e. optimization problems with a vector-valued objective map $f : S \to Y$ mapping in some real linear space Y, then the ordering of the elements in Y is not as obvious as in the scalar-valued case. In general, not a total ordering is given but at most a partial ordering, which is a reflexive and transitive binary relation that is compatible with the linear structure of the space. Such a partial ordering can be represented by a convex cone, a so-called ordering cone. Relations which are not defined by one single convex cone but by a family of convex cones, one for each element in Y, are denoted variable ordering structures. These families of cones describe a set-valued map which associates with each element in Y a convex cone in Y. Despite the name variable ordering, transitivity and compatibility with the linear structure of the space are generally not given.

Definition 1.6.1 *A convex cone which characterizes a partial ordering on a real linear space is called an ordering cone.*

Example 1.6.1 *The sets* $K_1 := R^2_+$ *and* $K_1 := \{x \in R^2 |\ 2x_1 + x_2 \leqslant 0,\ -0.5x_1 - x_2 \leqslant 0\}$ *are convex cones and may thus be ordering cones in* R_2*, while the set* $K_1 \cup K_2$ *is a cone which is not convex, thus not an ordering* *cone.*

In classical vector optimization, partial ordering is used to define the concept of optimality. In general this concept allow to compare elements $x \in X$ with an element $y \in Y$ with respect to the ordering cone defined to the element y, alternatively, w.r.t. the ordering cone defined to each element x which should be compared. In order to use these concepts we need to define a convex cone $D(y) \subset Y$ (and a partial ordering) to each element $y \in Y$. Therefore we assume a set-valued map $D : Y \to 2^Y$ to be given with $D(y)$ a nonempty convex cone for all $y \in Y$. The idea of variable ordering structure was firstly introduced in the first publication in mathematics in the 1970's. It was assumed that there is a set-valued map with cone values that associates with each element of the linear space and ordering. A candidate element is called a nondominated element if it is not dominated by other reference elements w.r.t the corresponding ordering of these other ones. In addition to the notion of nondominated elements, also consider another notion of optimal elements in case of a variable ordering structure. Namely, a candidate element is called a minimal element if it is not dominated by any other reference element w.r.t the ordering of the candidate element. These two concepts are equivalent only in the case of a nonvariable ordering.

1.7 Literature Review

In 1881, F.Y. Edgeworth [4] and in 1906, V. Pareto [34] introduced the concept of vector optimization. Therefore, efficient elements are also called EdgeworthPareto optimal points. They have already given the definition of the standard optimality concept in multiobjective optimization. But in mathematics this branch of optimization has started with the famous paper of H.W. Kuhn and A.W. Tucker(1951). Since about the end of the 1960's research in vector optimization is effectively made. In different books authors have given different names to an efficient element in a partially ordered space, for instance the books by Jahn [13] or by Eichfelder [6], where these optimal elements are denoted as minimal elements.This naming was first suggested by Stadler [19].

Condition (1.3.1) is also used to define optimal elements for an arbitrary set A with $0_Y \in A$ (Bergstresser et al. [1] and Weidner [22]), or even with $0_Y \notin A$ (Weidner [21]). A literature survey about the usage of the concept given in (2.2) for K as a set, a cone or the nonnegative orthant we refer to Engau [8]. The definitions of weaker (1.3.2) and stronger (1.3.3) optimality notions in partially ordered spaces can also be found in most introductory books on vector optimization. Ha provides in [10, Definition 21.3] an extensive collection of the different weaker and stronger concepts known in the literature. The definition of proper efficiency according to Henig is (of course) by Henig [11]. The definition according to Benson is from [2] while the definition according to Borwein is from [3]. For instance in [18] several examples are provided for showing the differences between the various proper optimality notions. For studies on the relations between the different notions of proper optimality in a partially ordered space we refer to [9, 15].

Set optimization means optimization of sets or set-valued maps. It is an extension of vector optimization to the set-valued case. In the last two decades there has been an increasing interest in set optimization. We can find an optimal solution of the set optimization problem by vector approach. Also another approach exists which is based on binary set-relations in the power set of the space. For example, there is a binary relation known as set less or KNY order relation which has been independently introduced by Young [26] and Nishnianidze [17] and has been presented by Kuroiwa [16] in a slightly modified form. Kuroiwa considered eight kinds of relationships (in fact, six different relationships) between two sets in an ordered vector space with respect to a convex cone, which are regarded as modifications of the order relation on n ordered space. Recently, several new binary relations for defining solutions to set-valued optimization problems have been proposed by Jahn and Ha [14].

The concept of ordering structure and the definition of a nondominated element was first given by Yu in 1974 ([23], [1]). Further generalized in 1976 by Bergstresser, Yu and Charnes [24]. It is also shortly mentioned in the book by Sawaragi et al. [18], by Chew in [7] and by Weidner in [20] as well as in a more recent survey about advances in preference modeling by Wiecek [25]. This structure is defined by a cone-valued map that associates to each element a cone of preferred or dominated directions. The main motive of examining vector optimization problems with a variable ordering structure is due to its application in image registration in medical engineering[27]. In [28], Engau examined the role of variable domination structures in preference modeling. He gave examples showing the limitations of preference modeling using only one ordering cone and studied variable ordering structures defined by convex cones containing the nonnegative orthant and which are symmetric in some sense. Yu in [24] gave examples of variable domination cones to illustrate the importance of variable ordering structures in modeling the preferences of decision makers adequately [7]. As variable ordering structures are becoming more relevant in applications, it is important to develop a general theory as a basis for further developments. Huang et al. in [12] already carried out theoretical examinations in the context of vector complementarity problems. Further the concept of variable ordering structure is taken to set optimization problems. There are three approaches for defining solution concepts in set optimization: the vector approach[[29, 30], the set approach [31, 14, 16] and lattice approach [32]. Kobis [33] used variable ordering structure for set optimization problems she took the set approach and introduced binary relations to compare sets w.r.t. a variable ordering structure based on one specific set relation. Chen et al. [5] considered a vector approach to set optimization with a variable ordering structure.

Chapter 2

Vector Optimization With Variable Ordering Structure

2.1 Introduction

In vector optimization, one assumes the criterion space to be partially ordered by a nontrivial convex cone. The general concepts of ordering structures were introduced by Yu in 1974 in terms of domination structures and further generalized in 1976 by Bergstresser, Charnes, and Yu [1]. This structure is defined by a cone-valued map that associates to each element a cone of prefered or dominated directions. In vector optimization with a nonvariable ordering structure, it is assumed, that a partial ordering $\leq_K$ in a real linear space Y is given by a nontrivial convex cone $K \subset Y$. Then we have $x \leq_K y$ for $y - x \in K$. An efficient element $\bar{y} \in A$, where A is a nonempty subset of Y w.r.t the cone $K \subset Y$, is defined by the condition $(\{\bar{y}\} - K) \cap A \subset \{\bar{y}\} + K$. For pointed cones, this definition reduces to

$$(\{\bar{y}\} - K) \cap A = \{\bar{y}\} \tag{2.1}$$

above condition is equivalent to the fact that there is no $y \in A$ with

$$\bar{y} \in \{y\} + K \setminus \{0_Y\}. \tag{2.2}$$

We always assume the set K to be a cone. In (2.2), the cone K represents the set of dominated directions of the element y, while in (2.1) the cone $-K$ denotes the set of preferred directions of the element $\bar{y}$. In case of a nonvariable ordering structure, efficient elements are equivalent to the non-dominated elements, or to the most preferred elements thus the two concepts (2.1) and (2.2) coincide, but in case of variable ordering structure these two concepts of preference and domination are different. Let Y be a real linear space a variable ordering structure is defined as a set-valued map $D : Y \rightrightarrows Y$ with $D(y)$ a nontrivial pointed convex cone for all $y \in Y$. The two relations $\leq_1$ and $\leq_2$ and the definition of a variable ordering structure is defined as follows. Let $y,\ \bar{y} \in Y$. We define

$$y \leq_1 \bar{y} \quad iff\ \bar{y} \in \{y\} + D(\bar{y}) \tag{2.3}$$

or

$$y \leq_2 \bar{y} \quad iff\ \bar{y} \in \{y\} + D(y) \tag{2.4}$$

If elements in the space Y are compared using the binary relation (2.3) or (2.4), then the cone-valued map D is called an ordering map and it defines a variable ordering on Y. Relation (2.3) means, that an element $\bar{y}$ is worse than y, if $\bar{y}$ is dominated by y. according to the relation (2.4), an element y is better than another element $\bar{y}$, if $y \in \{\bar{y}\} - D(y)$, thus y is preferred to $\bar{y}$. These two relations leads to the following two optimality notions.

2.2 Optimality Notions

Definition 2.2.1 *An element $\bar{y} \in A$ is a nondominated element of the set A w.r.t the ordering map D, iff no $y \in A$ exists such that*

$$\bar{y} \in \{y\} + D(y)\backslash\{0_Y\}, \tag{2.5}$$

or equivalently,

$$\bar{y} \notin \bigcup_{y \in A} \{y\} + \big(D(y)\backslash\{0_Y\}\big). \tag{2.6}$$

In above definition the cone $D(y) = \{d \in Y |\ y + d$ is dominated by $y\} \cup \{0_y\}$, also called domination cone, is the set of dominated directions or domination factors for each element $y \in Y$. Where as $D(y) = \{d \in Y |\ y - d$ is preferred by $y\} \cup \{0_y\}$ interpreted as the set of preferred directions.

Definition 2.2.2 *An element $\bar{y} \in A$ is called a minimal element of the set A w.r.t the ordering map D, iff*

$$(\{\bar{y}\} - D(\bar{y})) \cap A = \{\bar{y}\}. \tag{2.7}$$

i.e. if there does not exists any element Y in A, which is preferred to $\bar{y}$, then $\bar{y}$ is called a minimal element of A w.r.t D.

For $D(y) \equiv K$ both Definitions 2.2.1 and 2.2.2 are equivalent i.e. the minimal elements are precisely the nondominated elements. In case of nonvariable ordering structure minimal element, nondominated or efficient elements of a set A w.r.t the cone K are all equivalent. However in general the concepts are not equivalent and the one does not imply the other.

Example 2.2.1 *Let $Y = R^2$, $A = \{y \in R^2 |\ \| y \|_2 \leq 1\}$, and let $D : R^2 \rightrightarrows R^2$ be defined by*

$$D(y) := \begin{cases} R^2_+, & for\ y \in R^2\backslash\{(0,-1),(-1,0)\}, \\ \{z \in R^2 |\ z_1 \leq 0,\ z_2 \geq 0\}, & for\ y = (0,-1), \\ \{z \in R^2 |\ z_1 \geq 0,\ z_2 \leq 0\}, & for\ y = (-1,0) \end{cases}$$

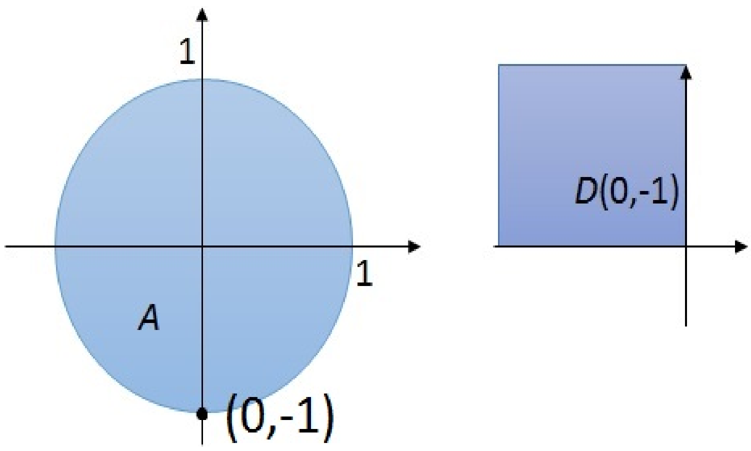

Figure 2.1: Cone D(y)

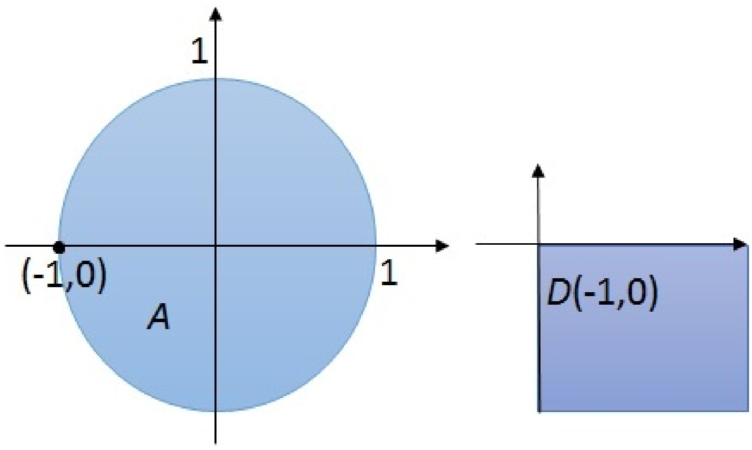

Figure 2.2: Cone D(y)

Then all elements of the set $\{y \in R^2| \ \parallel y \parallel_2 = 1, \ y_1 \leq 0, \ y_2 \geq 0\}$ *are minimal elements w.r.t.* D *since all these elements satisfies the property of minimal elements, consider the element* $(0,-1)$ *now since* $(0,-1) - D(0,-1) \cap A = \{(0,-1)\}$*, so* $(0,-1)$ *is a minimal element thus on the same lines we can see that the set* $\{y \in R^2| \parallel y \parallel_2 = 1, \ y_1 \leq 0, \ y_2 \geq 0\}$ *is the set of all minimal elements. But there is no nondominated element of* A *w.r.t.* D*.*

Example 2.2.2 *Let* $Y = R^2$*,* $A = \{y \in R^2| \ \parallel y \parallel_2 \leq 1\}$*, and let* $D : R^2 \rightrightarrows$

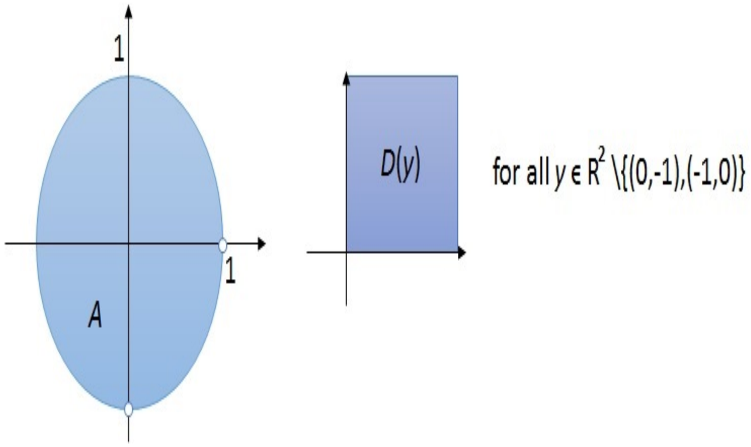

Figure 2.3: Cone D(y)

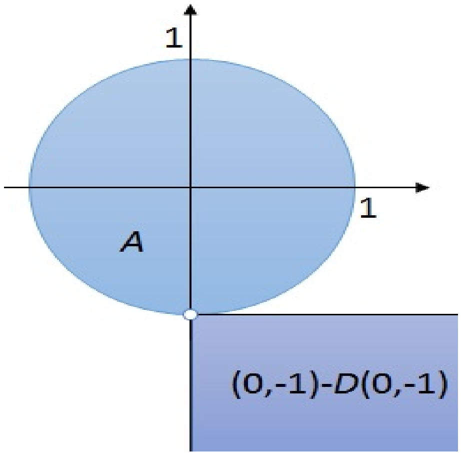

Figure 2.4: (0,-1) is minimal element

R^2 be defined by

$$D(y) := \begin{cases} R^2_+, & for\ y \in R^2 \backslash \{(0,-1)\}, \\ \{z \in R^2 | z_1 + z_2 \geq 0,\ z_1 \geq 0\}, & for\ y = (0,-1); \end{cases}$$

Then (0,-1) is a nondominated element of A w.r.t. D, but not a minimal element of A w.r.t. D, but not a minimal element of A w.r.t. D. The set of all nondominated elements of A w.r.t. D is $\{(y_1, y_2) \in R^2 |\ y_1 \in$

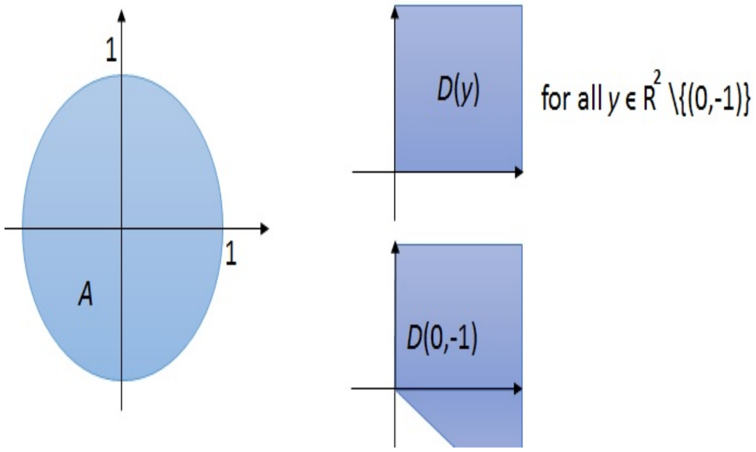

Figure 2.5: Set A and the Cones D(y)

$[-1, 0],\ y_2 = -\sqrt{1-y_1^2}\}$, *and the set of all minimal elements of* A *w.r.t.* D *is* $\{(y_1, y_2) \in R^2 |\ y_1 \in [-1, 0[,\ y_2 = -\sqrt{1-y_1^2}\}$.

Lemma 2.2.1 *(i) If* $\bar{y}$ *is a minimal element of* A *w.r.t.* D *and* $D(y) \subset D(\bar{y})$ *for all* $y \in A$*, then* $\bar{y}$ *is also a nondominated element of* A *w.r.t.* D*.*

(ii) If $\bar{y}$ *is a nondominated element of* A *w.r.t.* D *and* $D(\bar{y}) \subset D(y)$ *for all* $y \in A$*, then* $\bar{y}$ *is also a minimal element of* A *w.r.t.* D*.*

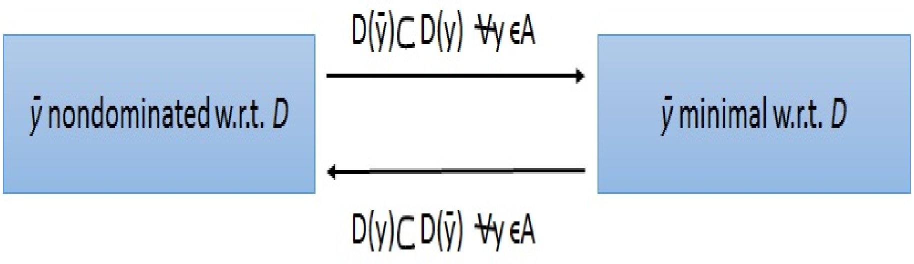

Figure 2.6: Relation between nondominated and minimal element of a set A

Definition 2.2.3 *(a) Assume* $cor(D(y)) \neq 0$ *for all* $y \in A$*. An element* $\bar{y} \in A$ *is a weakly nondominated element of the set* A *w.r.t.* D *if no*

$y \in A$ exists such that

$$\bar{y} \in \{y\} + cor(D(y)).$$

(b) An element $\bar{y} \in A$ is a strongly nondominated element of the set A w.r.t. D if

$$\bar{y} \in \{y\} - D(y) \text{ for all } y \in A.$$

(c) Assume $cor(D(\bar{y})) \neq 0$ for some element $\bar{y} \in A$. Then $\bar{y}$ is a weakly minimal element of the set A w.r.t. D if no $y \in A$ exists such that

$$\bar{y} \in y + cor(D(\bar{y})),$$

i.e. if

$$(\{\bar{y}\} - cor(D(\bar{y}))) \cap A = \emptyset.$$

(d) An element $\bar{y} \in A$ is a strongly minimal element of the set A w.r.t. D if

$$A \subset \{\bar{y}\} + D(\bar{y}).$$

From the definitions of minimal and weakly minimal elements we can directly conclude that

(i) An element $\bar{y}$ is a minimal element of A w.r.t. D if and only if it is a efficient element of A in the linear space Y partially ordered by the cone $K := D(\bar{y})$.

(ii) Let $cor(D(\bar{y})) \neq \emptyset$ for some $\bar{y} \in A$. The element $\bar{y}$ is a weakly minimal element of A w.r.t. D if and only if it is a weakly efficient element of A in the linear space Y partially ordered by the cone $K := D(\bar{y})$.

Lemma 2.2.2 *Let $D(A)$ be convex and pointed.*

(i) Any efficient element of A in the linear space Y partially ordered by the cone $K := D(A)$ is also a nondominated element of A w.r.t. D.

(ii) Let $cor(D(y)) \neq \emptyset$ for all $y \in A$. Any weakly efficient element of A in the linear space Y partially ordered by $K := D(A)$ is also weakly nondominated element of A w.r.t. D.

Proof

(i) If $\bar{y}$ is efficient, then $(\{\bar{y}\} - D(A)) \cap A = \{\bar{y}\}$, i.e. for any $y \in A \backslash \{\bar{y}\}$ it holds $y \notin \{\bar{y}\} - D(A)$ and thus $\bar{y} \notin \{y\} + D(y)$.

(ii) If $\bar{y}$ is weakly efficient, then $(\{\bar{y}\} - cor(D(A))) \cap A = \emptyset$, i.e. for any $y \in A$ it holds $y \notin \{\bar{y}\} - cor(D(A))$ and thus $\bar{y} \notin \{y\} + cor(D(y))$.

Lemma 2.2.3 *(i) Any nondominated element of A w.r.t. D is also an efficient element of A with the linear space Y partially ordered by $K := \cap_{y \in A} D(y)$.*

(ii) Let $cor(D(y)) \neq \emptyset$ for all $y \in A$ and set $K := \cap_{y \in A} D(y)$. If $cor(K) \neq \emptyset$ then any weakly nondominated element of A w.r.t. D is also a weakly efficient element of A with the linear space Y partially ordered by K.

Proof

(i) $\bar{y}$ nondominated of A w.r.t. D is equivalent to

$$\bar{y} \notin \{y\} + D(y) \backslash \{0_Y\} \text{ for all } y \in A,$$

and hence $\bar{y} \notin \{y\} + K \backslash \{0_Y\}$ for all $y \in A$ or

$$(\{\bar{y}\} - K) \cap A = \{\bar{y}\}$$

(ii) $\bar{y}$ weakly nondominated of A w.r.t. D is equivalent to

$$\bar{y} \notin \{y\} + cor(D(y)) \text{ for all } y \in A$$

and hence $\bar{y} \notin \{y\} + cor(K)$ for all $y \in A$ or

$$(\{\bar{y}\} - cor(K)) \cap A = \emptyset$$

2.3 Characterization Of Optimal Elements

Lemma 2.3.1 *Let $cor(D(y))$ be nonempty for all $y \in A$.*

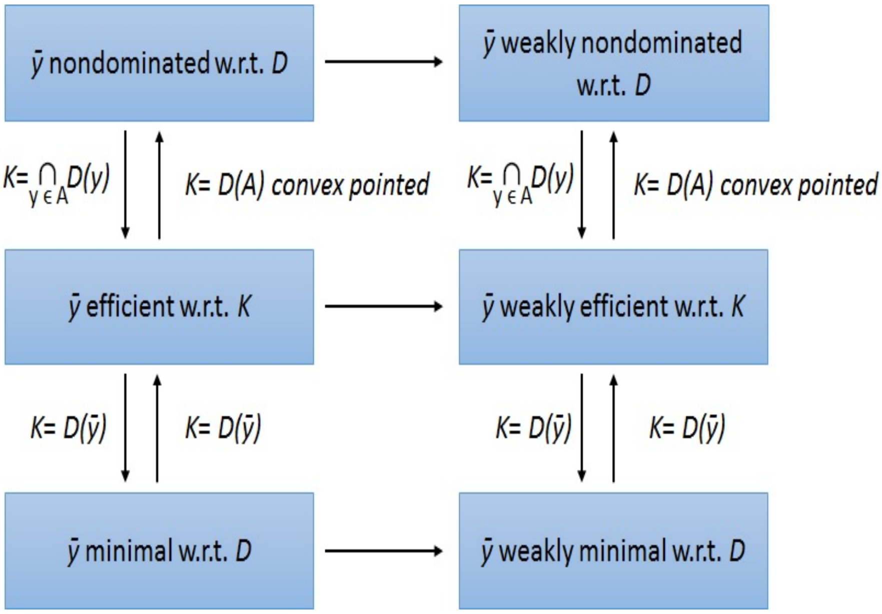

Figure 2.7: Results of Lemmas under the assumption $cor(D(y)) \neq \emptyset$ and $cor(K) \neq \emptyset$

(i) If $\cap_{y \in A} cor(D(y)) \neq \emptyset$ and $\bar{y} \in A$ is a weakly nondominated element of A w.r.t. D, then $\bar{y} \in \delta A$.

(ii) If $\bar{y} \in A$ is a weakly minimal element of A w.r.t. D, then $\bar{y} \in \delta A$.

Proof

(i) We assume that $\bar{y} \in cor(A)$. Let $d \in \cap_{y \in A} cor(D(y))$. Then $d \neq 0_Y$ and there exists $\lambda > 0$ with $\bar{y} - \lambda d \in A \backslash \{\bar{y}\}$. As

$$-\lambda d \in -\cap_{y \in A} cor(D(y)) \subset -cor(D(\bar{y} - \lambda d))$$

, we have $\bar{y} - \lambda d \in A \cap (\{\bar{y}\} - cor(D(\bar{y} - \lambda d)))$ or $\bar{y} \in \bar{y} - \lambda d + cor(D(\bar{y} - \lambda d))$, being a contradiction to $\bar{y}$ weakly nondominated.

(ii) Follows directly from remark (an element $\bar{y}$ is a minimal element of A w.r.t. D if and only if it is an efficient element of A in the space Y partially ordered by the cone $K := D(\bar{y})$), but can also be easily shown by choosing any $d \in cor(D(\bar{y}))$. Then the proof is analogous to (a).

Example 2.3.1 *For the set* $A = [1,3] \times [1,3] \subset Y = R^2$ *and the cone-valued map* $D : R^2 \rightrightarrows R^2$ *with*

$$D(y) := \begin{cases} R^2_+ & for\ all\ y \in R^2\ with\ y_1 \geq 2, \\ \{z \in R^2 |\ z_1 \leq 0,\ z_2 \geq 0\}, & else, \end{cases}$$

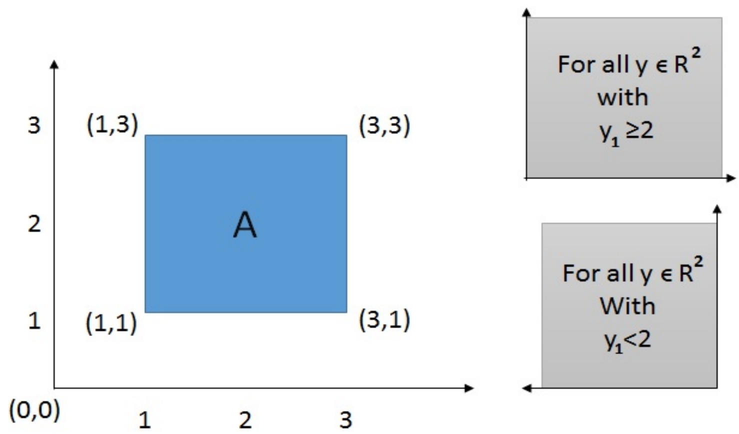

Figure 2.8: Set A and cones D(y)

The line segment $[(2,1),(2,3)]$ *is the set of weakly nondominated elements.* $\bar{y} = (2,2)$ *is a weakly nondominated element of* A *w.r.t.* D *since* $\nexists$ *any* $y \in A$ *such that* $(2,2) \in \{y\} + D(y) \backslash \{0_y\}$*, but* $\bar{y} \notin \delta A$*. That means its not necessary that the nondominated element should lie on the boundary of the set* A*.*

Lemma 2.3.2 *Define the set* $M := \bigcup_{y \in A} (\{y\} + D(y))$

(a) *(i) If* $\bar{y} \in M$ *is a nondominated element of* M *w.r.t.* D*, then* $\bar{y} \in A$ *and* $\bar{y}$ *is also a nondominated element of* A *w.r.t.* D*.*

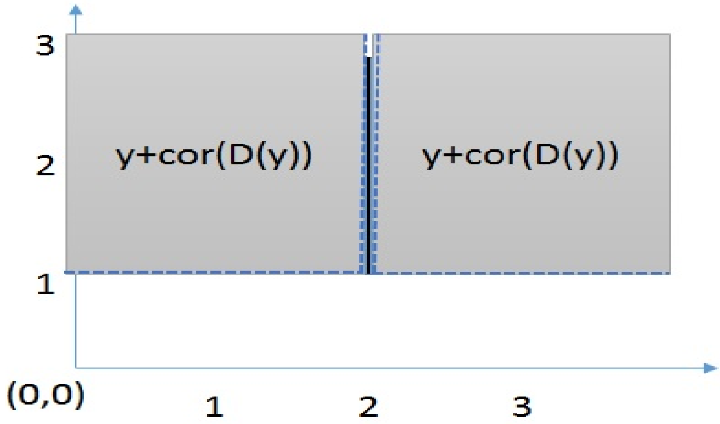

Figure 2.9: Weakly nondominated elements

(ii) *If $\bar{y} \in A$ is a nondominated element of A w.r.t. D, and if $D(y + d) \subset D(y)$ for all $y \in A$ and for all $d \in D(y)$, then $\bar{y}$ is a nondominated element of M w.r.t. D.*

(b) (i) *) If $\bar{y} \in A$ is a minimal element of M w.r.t. D, then it is also a minimal element of A w.r.t. D.*

(ii) *If $\bar{y} \in A$ is a minimal element of A w.r.t. D and if $D(y) \subset D(\bar{y})$ for all $y \in A$, then $\bar{y}$ is also a minimal element of M w.r.t D.*

Proof

(a) (i) If $\bar{y} \in M \backslash A$, then $\bar{y} \in \{y\} + (D(y) \backslash \{0_Y\})$ for some $y \in A \subset M$ in contradiction to $\bar{y}$ a nondominated element of M w.r.t. D. Thus, $\bar{y} \in A$. Due to $A \subset M$, $\bar{y}$ is then also a nondominated element of A w.r.t. D.

(ii) Assume that $\bar{y}$ is a nondominated element of A w.r.t. D but not of M, i.e., there exist $y \in A$ and $d_y \in D(y) \backslash \{0_Y\}$ with $\bar{y} \in y + d_y + (D(y + d_y) \backslash \{0_Y\})$. As $D(y)$ is a pointed convex cone, this implies $\bar{y} \in \{y\} + (D(y) \backslash \{0_Y\}) + (D(y + d_y) \backslash \{0_Y\})$

$\subset \{y\} + (D(y)\backslash\{0_Y\}) + (D(y)\backslash\{0_Y\})$

$\subset \{y\} + (D(y)\backslash\{0_Y\})$,

in contradiction to $\bar{y}$ being a nondominated element of A w.r.t. D.

(b) (i) The first implication (i) follows again from $A \subset M$.

(ii) Assume that $\bar{y}$ is a minimal element of A but not of M w.r.t. D, i.e., there exist $y \in A$ and $d_y \in D(y)\backslash\{0_Y\}$ with $y + d_y \in \{\bar{y}\} - (D(\bar{y})\backslash\{0_Y\})$. Again, as $D(\bar{y})$ is a pointed convex cone, this implies

$y \in \{\bar{y}\} - (D(y)\backslash\{0_Y\}) - (D(\bar{y})\backslash\{0_Y\})$

$\subset \{\bar{y}\} - (D(\bar{y})\backslash\{0_Y\}) - (D(\bar{y})\backslash\{0_Y\})$

$\subset \{\bar{y}\} - (D(\bar{y})\backslash\{0_Y\})$,

in contradiction to $\bar{y}$ being a nondominated element of A w.r.t. D.

Example 2.3.2 *Let $Y = R^2$, $A = \{y \in R^2 | \parallel y \parallel_2 \leq 1\}$, and let $D : R^2 \rightrightarrows R^2$ be defined by*

$$D(y) := \begin{cases} R^2_+, & for\ y \in R^2\backslash\{(0,-1)\}, \\ \{z \in R^2 |\ z_1 + z_2 \geq 0,\ z_1 \geq 0\}, & for\ y = (0,-1); \end{cases}$$

Then

$$M = \{(y_1, y_2) \in R^2 |\ y_1 \in [-1, 0],\ y_2 \geq -\sqrt{1 - y_1^2}\} \cup \{(y_1, y_2) \in R^2 |\ y_1 \geq 0,\ y_2 \geq -1 - y_1\};$$

The set of nondominated elements of M w.r.t. the ordering map D is $\{(y_1, y_2) \in R^2 |\ y_1 \in [-1, 0],\ y_2 = -\sqrt{1 - y_1^2}\}$ and thus equals the set of non-dominated elements of A w.r.t. D. It is easy to verify that $D(y + d) \subset D(y)$ for all $y \in A$ and for all $d \in D(y)$. The set of minimal elements of M w.r.t. D is $\{(y1, y2) \in R^2 |\ y_1 \in [-1, 0[,\ y_2 = -\sqrt{1 - y_1^2}\} \cup \{(y_1, y_2) \in$

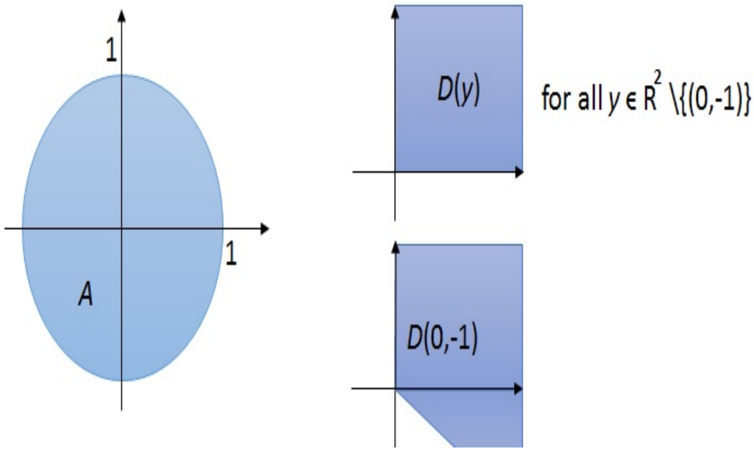

Figure 2.10: Set A and cones D(y)

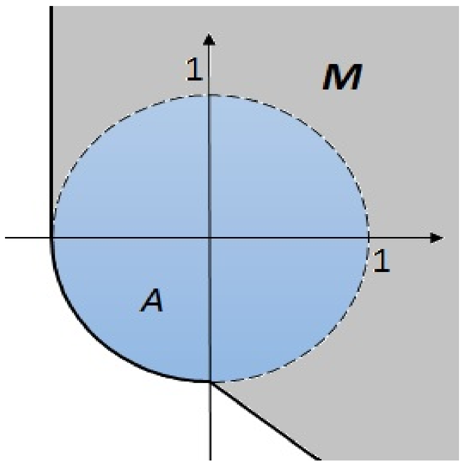

Figure 2.11: Set M

$R^2|\ y_1 > 0,\ y_2 = -1 - y_1\}$. *Hence, the set of minimal elements of* M *w.r.t.* D *intersected with* A *equals the set of minimal elements of* A *w.r.t.* D.

Lemma 2.3.3 *Let* $M := \bigcup_{y \in A}(\{y\} + D(y))$, *If* $\bar{y} \in M$ *is a minimal element of* M *w.r.t.* D *and if* $D(y + d) \supset D(y)$ *for all* $y \in A$ *and for all* $d \in D(y)$, *then* $\bar{y}$ *is also a minimal element of* A *w.r.t.* D.

Proof Using Lemma 2.1.5, it is left to show that $\bar{y} \in A$. Let $\bar{y} \in M \backslash A$, i.e., there exist $\bar{a} \in A$ and $\bar{d}_a \in D(\bar{a}) \backslash \{0_Y\}$ with $\bar{y} = \bar{a} + \bar{d}_a$. Then for

$y := \bar{a} + \frac{1}{2}\bar{d}_a \in M\backslash\{\bar{y}\}$, we obtain

$$y = \bar{y} - \frac{1}{2}\bar{d}_a \in \{\bar{y}\} - \frac{1}{2}(D(\bar{a})\backslash\{0_Y\}) \subset \{\bar{y}\} - (D(\bar{y})\backslash\{0_Y\}),$$

which is the contradiction to the minimality of $\bar{y}$ for the set M w.r.t. D.

In this chapter we have studied about the variable ordering structure in vector optimization. The applicability of variable ordering structures in practice and more intensive study of the known applications is of special importance. In this regard, the study of domination sets instead of domination cones is also important. In next chapter we will take the concept of variable ordering structure to set optimization problem.

Chapter 3

Variable Ordering Structure In Set Optimization

In vector optimization, one minimizes a vector valued function, while the aim is to find elements that are somehow efficient. In vector optimization with a variable ordering structure the partial ordering defined by a convex cone in the image space of the vector-valued objective function is replaced by a variable ordering structure. In this chapter we introduce a variable ordering structure in set optimization.

3.1 Variable Upper Set Less Order Relation

Let Y be a real linear space and let $P(Y) := \{A \subseteq Y|\ A \textit{ is nonempty}\}$ denote the power set of Y.

Definition 3.1.1 *Let $K \subset Y$ be a proper pointed closed convex cone. Then the upper set less order relation for two sets $A,\ B \in P(y)$ is given as*

$$A \preceq^u_K B : \Leftrightarrow \forall a \in A,\ \exists b \in B :\ a \in b - K$$
$$\Leftrightarrow A \subseteq B - K.$$

Definition 3.1.2 *Let $K : Y \rightrightarrows Y$ be a set-valued map such that for every*

$y \in Y$, $K(y)$ is a proper pointed closed convex cone. Then we define for $y_1,\ y_2 \in Y$

$$y_1 \leq_1 y_2 : \quad \Leftrightarrow y_1 \ \in y_2 - K(y_1), \tag{3.1}$$

$$y_1 \leq_2 y_2 : \quad \Leftrightarrow y_1 \ \in y_2 - K(y_2), \tag{3.2}$$

Definition 3.1.3 *Let $A,\ B \in P(y)$, and let $D : Y \times Y \rightrightarrows Y$. Assume that for all $y_1,\ y_2 \in Y$, $D(y_1, y_2)$ is a proper pointed closed convex cone. Then the variable upper set less order relation is defined by*

$$A \preceq^u_D B : \quad \Leftrightarrow \ \forall a \in A,\ \exists b \in B : \ a \in b - D(a, b). \tag{3.3}$$

If D in Definition 3.1.3 is given by a constant cone K, then the above definition is equivalent to Definition 3.1.1. If we replace A by y_1 and B by y_2 in the above definition and if $D_1 : Y \rightrightarrows Y$ is given as a set-valued map that only depends on the first variable such that $D_1(y_1) := D(y_1, y_2)$ for all $y_1,\ y_2 \in Y$, then we have the following equivalence:

$$y_1 \preceq^u_{D_1} y_2 \ \Leftrightarrow \ y_1 \in y_2 - D_1(y_1) \ \Leftrightarrow \ y_1 \leq_1 y_2,$$

where $\leq_1$ is given by equivalence (3.1) for $K = D_1$. If, on the other hand, $D_2 : Y \rightrightarrows Y$ is given as a set-valued map that only depends on the second variable such that $D_2(y_2) := D(y_1,\ y_2)$ for all $y_1,\ y_2 \in Y$, then we have the following equivalence:

$$y_1 \preceq^u_{D_2} y_2 \ \Leftrightarrow \ y_1 \in y_2 - D_2(y_2) \ \Leftrightarrow \ y_1 \leq_2 y_2,$$

where $\leq_2$ is given by (3.2) for $K = D_2$.

3.2 Optimality Notions

Let $D : Y \times Y \rightrightarrows Y$ be a set-valued map such that for all $y_1,\ y_2 \in Y$, $D(y_1, y_2)$ is a proper pointed closed convex cone, and $\mathrm{int}D(y_1, y_2)$ is nonempty. Consider a set valued map $F : X \rightrightarrows Y$ that we want to minimize on $\chi \subseteq X$, where X is a linear space. The optimal solution of minimization problem

$$\min_{x \in \chi} F(x) \tag{P}$$

is given as:

Definition 3.2.1 *$x^0 \in X$ is called an optimal (a strictly optimal / a weakly optimal) solution of the set-valued problem (P) w.r.t. the variable upper set less order relation if*

$$\nexists x \in \chi \backslash \{x^0\} : \ F(x) \preceq^u_E F(x^0) \tag{3.4}$$

which is equivalent to

$$\nexists x \in \chi \backslash \{x^0\} : \forall y \in F(x), \ \exists y^0 \in F(x^0) : \ y \in y^0 - E, \tag{3.5}$$

where $E = D(y, \ y^0) \backslash \{0\}$ ($E = D(y, \ y^0)$, $E = intD(y, \ y^0)$, respectively). If D is given by a constant set-valued map K, then we say that $x^0 \in \chi$ is an optimal (a strictly optimal / a weakly optimal) solution of the set-valued Problem (P) w.r.t. $\preceq^u_K$ if condition (3.4) is fulfilled with $E = C\backslash\{0\}$ (E=C, E=int C, respectively).

Lemma 3.2.1 *If x^0 is a strictly optimal solution of (P) w.r.t. the variable upper set less order relation, then x^0 is an optimal solution and a weakly optimal solution of (P) w.r.t. the variable upper set less order relation.*

Proof Follows from $intD(y_1, \ y_2) \subseteq D(y_1, y_2)\backslash\{0\} \subseteq D(y_1, y_2)$ for every $y_1, \ y_2 \in Y$

Example 3.2.1 *In this example we will find the optimal solutions of a set-valued optimization problem with respect to the variable upper sets less order relation $\preceq^u_D$. The problem is*

$$\min_{x \in X} F(x) \tag{P_1}$$

with $X = R, \ Y = R^2, \ X = [0, 1]$ and $F : X \rightrightarrows Y$ is given by,

$$F(x) := \begin{cases} [(1, 1), (2, 2)], & if x = 0 \\ [(0, 0), (3, 3)], & if x \in (0, 1), \\ [(1, -1), (1.5, -1.5)], & if x = 1 \end{cases}$$

where $[(a,\ b),\ (c,\ d)] := \{(y^1,\ y^2) \in R^2|\ a \leq y^1 \leq c,\ b \leq y^2 \leq d\}$ *denotes an order interval. Furthermore, the variable ordering is given by*

$$D(y_1, y_2) = \begin{cases} \alpha(1,1) + \beta(0.5,1)|\ \alpha, \beta \geq 0 & if\ y_2 \in [(1,1),(2,2)],\ y_1 \in R, \\ R^2_+ & otherwise. \end{cases}$$

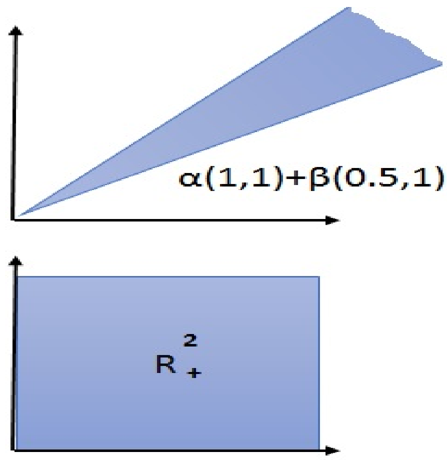

Figure 3.1: Cones $D(y_1, y_2)$

Then the strictly optimal solutions of (P_1) *w.r.t the variable upper set less order relation in the sense of Definition 3.2.1 are* $\chi^0 := \{0, 1\}$*, because*

$$\nexists x \in \chi \backslash \{0\} :\ \forall y \in F(x),\ \exists y^0 \in F(0) :\ y \in y^0 - D(y, y^0) \text{ and}$$

$$\nexists x \in \chi \backslash \{1\} :\ \forall y \in F(x),\ \exists y^0 \in F(1) :\ y \in y^0 - D(y, y^0).$$

If we replace D *by the constant ordering cone* R^2_+*, then* $x^0 = 1$ *is the only strictly optimal solution w.r.t* $\preceq^u_{R^2_+}$ *, since*

$$\nexists x \in \chi \backslash \{1\} :\ \forall y \in F(x),\ \exists y^0 \in F(1) :\ y \in y^0 - R^2_+.$$

$x^0 = 0$ *is not strictly optimal w.r.t* $\preceq^u_{R^2_+}$*, as*

$$\exists x = 1 :\ \forall \in F(1),\ \exists y^0 \in F(0) :\ y \in y^0 - R^2_+.$$

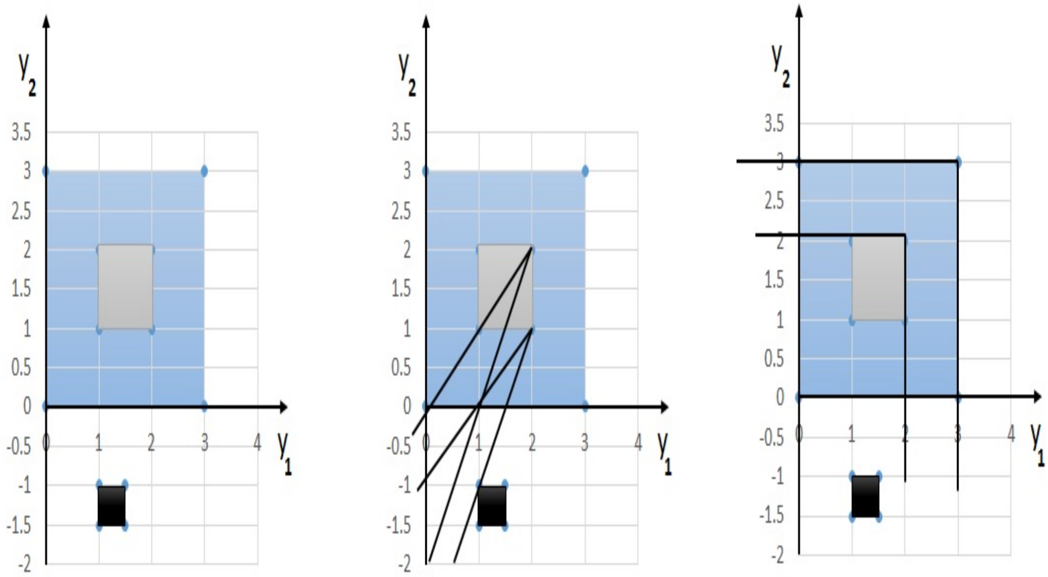

Figure 3.2: Image set of F. The left plot shows the three sets F(0), F(1) and F(x) for $x \in (0, 1)$. The middle image illustrates the variable ordering cone in (2,2) and (2,1). The right plot depicts the problem associated to the (constant) natural ordering cone R^2_+

For the next results we introduce the abbreviations

$$A := F[\chi] := \bigcup_{x \in \chi} F(x) \tag{3.6}$$

$$D[A, A] := \bigcup_{y_1, y_2 \in A} D(y_1, y_2), \tag{3.7}$$

and we assume that $D[A, A]$ is a proper pointed closed convex cone with nonempty interior. Now we have the following theorem:

Theorem 3.2.2 *If $x^0 \in \chi$ is an optimal (a strictly optimal / a weakly optimal) solution of the sat-valued Problem (P) w.r.t $\preceq^u_{D[A,A]}$, then $x^0 \in \chi$ is an optimal (a strictly optimal / a weakly optimal) solution of (P) w.r.t. the variable upper set less order relation.*

Proof let $G = D[A, A] \backslash \{0\}$ ($G = D[A, A]$, $G = intD[A, A]$, respectively). $x^0 \in \chi$ is an optimal (a strictly optimal/ a weakly optimal) solution of (P) w.r.t. $\preceq^u_{D[A,A]}$

$$\Longleftrightarrow \nexists x \in \chi\ x^0 :\ \forall y \in F(x),\ \exists y^0 \in F(x^0) :\ y \in y^0 - G.$$

Now suppose that $x^0 \in \chi$ is not an optimal (a strictly optimal / a weakly optimal) solution of (P) w.r.t the variable upper set less order relation. Then there exists $x \in \chi\ x^0$ such that for $E = D(y, y^0)\backslash\{0\}$ $(E = D(y, y^0),\ E = intD(y, y^0)$, respectively):

$$\forall y \in F(x),\ \exists y^0 \in F(x^0) :\ y \in y^0 - E \subseteq y^0 - G,$$

a contradiction.

The following corollary describes an extension of the preceding theorem.

Corollary 3.2.3 . *Let*

$$\tilde{D} \supset D[A, A] = \bigcup_{y_1, y_2 \in A} (y_1,\ y_2),$$

and suppose that $\tilde{D}$ *is a proper pointed closed convex cone with nonempty interior. If* $x^0 \in \chi$ *is an optimal (a strictly optimal / a weakly optimal) solution of the set-valued Problem (P) w.r.t* $\preceq^u_{\tilde{D}}$*, then* $x^0 \in \chi$ *is an optimal (a strictly optimal / a weakly optimal) solution of (P) w.r.t* $\preceq^u_{D[A,A]}$*. Let*

$$\bar{D} := \bigcap_{y_1, y_2 \in A} D(y_1, y_2), \tag{3.8}$$

and we suppose that int $\bar{D} \neq \emptyset$*.*

Theorem 3.2.4 *If* $x^0 \in \chi$ *is an optimal (a strictly optimal / a weakly optimal) solution of the set-valued Problem (P) w.r.t the variable upper set less order relation, the* $x^0 \in \chi$ *is an optimal (a strictly optimal / a weakly optimal) solution of (P) w.r.t.* $\preceq^u_{\bar{D}}$*.*

Proof Suppose that $x^0 \in \chi$ is not an optimal (a strictly optimal/ a weakly) solution of (P) w.r.t $\preceq^u_{\bar{D}}$, and let $G = \bar{D}\backslash\{0\}$ $(G = \bar{D},\ G = int\bar{D}$, respectively). Then there exists $x \in \chi\backslash\{x^0\}$:

Table 3.1: Interrelation between different optimality notions

x^0 is strictly optimal w.r.t. $\preceq^u_{\widetilde{D}}$	$\Rightarrow$	x^0 is optimal w.r.t. $\preceq^u_{\widetilde{D}}$	$\Rightarrow$	x^0 is weakly optimal w.r.t. $\preceq^u_{\widetilde{D}}$
$\Downarrow$		$\Downarrow$		$\Downarrow$
x^0 is strictly optimal w.r.t. $\preceq^u_{D[A,A]}$	$\Rightarrow$	x^0 is optimal w.r.t. $\preceq^u_{D[A,A]}$	$\Rightarrow$	x^0 is weakly optimal w.r.t. $\preceq^u_{D[A,A]}$
$\Downarrow$		$\Downarrow$		$\Downarrow$
x^0 is strictly optimal w.r.t. the variable upper set less order relation	$\Rightarrow$	x^0 is optimal w.r.t. the variable upper set less order relation	$\Rightarrow$	x^0 is weakly optimal w.r.t. the variable upper set less order relation
$\Downarrow$		$\Downarrow$		$\Downarrow$
x^0 is strictly optimal w.r.t. $\preceq^u_{\bar{D}}$	$\Rightarrow$	x^0 is optimal w.r.t. $\preceq^u_{\bar{D}}$	$\Rightarrow$	x^0 is weakly optimal w.r.t. $\preceq^u_{\bar{D}}$
$\Downarrow$		$\Downarrow$		$\Downarrow$
x^0 is strictly optimal w.r.t. $\preceq^u_{\bar{\bar{D}}}$	$\Rightarrow$	x^0 is optimal w.r.t. $\preceq^u_{\bar{\bar{D}}}$	$\Rightarrow$	x^0 is weakly optimal w.r.t. $\preceq^u_{\bar{\bar{D}}}$

$$\forall y \in F(x),\ \exists y^0 \in F(x^0):\ y \in y^0 - G \subseteq y^0 - E,$$

Where $E = D(y, y^0)\backslash\{0\}$ ($E = D(y, y^0),\ E = intD(y, y^0)$, respectively). But this is a contradiction, as x^0 was assumed to be optimal (strictly optimal / weakly optimal) for(P) w.r.t the variable upper set less order relation. We deduce the following corollary in a similar manner as Corollary 3.2.3.

Corollary 3.2.5 *Let*

$$\bar{\bar{D}} \subset \bar{D} = \bigcap_{y_1, y_2 \in A} D(y_1, y_2),$$

and suppose that $\bar{\bar{D}}$ *is a proper pointed closed convex cone with nonempty interior. If* $x^0 \in \chi$ *is an optimal (a strictly optimal / a weakly optimal) solution of the set-valued Problem (P) w.r.t* $\preceq^u_{\bar{D}}$*, then* $x^0 \in \chi$ *is an optimal (a strictly optimal / a weakly optimal) solution of (P) w.r.t* $\preceq^u_{\bar{\bar{D}}}$*.*

Definition 3.2.6 *A set* $\varepsilon \subset \chi$ *is called externally stable (strictly externally stable / weakly externally stable) if for all* $x \in \chi\backslash\varepsilon_K$ *there exists* $x^0 \in \varepsilon_K$ *such that* $\forall y^0 \in F(x^0),\ \exists y \in F(x):\ y^0 \in y - E$*, where* $E = K\backslash\{0\}$ ($E =$

$K,\ E = intK$, respectively) and $K \subset Y$.

Theorem 3.2.7 *Let $K \subseteq \bar{D} = \bigcap_{y_1,y_2 \in A} D(y_1, y_2)$ and let $\varepsilon_K \subset \chi$ be the set of strictly optimal solutions of (P) w.r.t $\preceq_K^u$. Assume that ε_K is strictly externally stable. Furthermore, assume that*

$$\forall y_1, y_2 \in Y:\ y_1 \in y_2 + K \Longrightarrow\ D(y_1, y^0) \subseteq D(y_2, y^0) \tag{3.9}$$

is fulfilled for every $y^0 \in A$. $x^0 \in \chi$ is an optimal solution of (P) (on χ) w.r.t the variable upper set less order relation if and only if $x^0 \in \chi$ is a strictly optimal solution of (P) on ε_K w.r.t the variable upper set less order relation.

Proof We only prove the direction " $\Longleftarrow$ ", as " $\Longrightarrow$ " is obvious, because $\varepsilon_K \in \chi$. Let $x^0 \in \chi$ be a strictly optimal solution of (P) on ε_K w.r.t the variable upper set less order relation

$$\nexists x \in \varepsilon \backslash \{x^0\} : \forall y \in F(x),\ \exists y^0 \in F(x):\ y \in y^0 - D(y, y^0)$$

Now suppose that x^0 is not a strictly optimal solution of (P) on χ w.r.t. the variable upper set less order relation. Then there exists $x \in \chi \backslash (\varepsilon_K \cup \{x^0\})$ such that

$$\forall y \in F(x),\ \exists y^0 \in F(x^0):\ y \in y^0 - D(y, y^0). \tag{3.10}$$

As ε_K is strictly externally stable, there exists $\hat{x} \in \varepsilon_K$ with

$$\forall \hat{y} \in F(\hat{x}),\ \exists y \in F(x):\ \hat{y} \in y - K. \tag{3.11}$$

this implies $\forall \hat{y} \in F(\hat{x}),\ \exists y \in F(x):\ \hat{y} \in y - K.$

$$\begin{aligned}
&\subseteq y^0 - D(y, y^0) - K \\
&\subseteq y^0 - D(y, y^0) - D(y, y^0),\ \textit{as } K \subseteq D(y, y^0) \\
&\subseteq y^0 - D(y, y^0),\ \textit{as } D(y, y^0) \textit{ is a convex cone} \\
&\subseteq y^0 - D(\hat{y}, y^0).
\end{aligned}$$

which is a contradiction to x^0's strict optimality in ε_K w.r.t. the variable upper set less order relation.

Lemma 3.2.2 *Let some $y^0 \in Y$ be arbitrarily given. The Implication (3.9) holds for $y^0 \in Y$ if and only if $D(y+d, y^0) \subseteq D(y, y^0)$ for all $d \in D(y, y^0)$ and all $y \in Y$.*

Proof As we have defined $K \subseteq \bar{D} = \bigcap_{y_1, y_2 \in A} D(y_1, y_2)$. $y_1 \in y_2 + K$ means that there exists $d \in K \subseteq D(y_1, y_2)$ such that $y_1 = y_2 + d$. Then $D(y_1, y_0) = D(y_2 + d, y^0) \subseteq D(y_2, y^0)$.

Corollary 3.2.8 *Let $K \subseteq \bigcap_{y_1, y_2 \in A} D(y_1, y_2) \backslash \{0\}$ ($K \subseteq \bigcap_{y_1, y_2 \in A} intD(y_1, y_2)$, respectively and let $\varepsilon_K \subset \chi$ be the set of optimal (weakly optimal, respectively) solutions of (P) on χ w.r.t. $\preceq_K^u$. Assume that ε_K is externally stable (weakly externally stable, respectively). Furthermore, assume that*

$$y_1 \in y_2 + K \Rightarrow D(y_1, y^0) \backslash \{0\} \subseteq D(y_2, y^0)\ \{0\}$$

($y_1 \in y_2 + K \Rightarrow intD(y_1, y^0) \subseteq intD(y_2, y^0), respectively$) is fulfilled for all $y^0 \in A$. $x^0 \in \chi$ is an optimal (a weakly optimal, respectively) solution of (P) on χ w.r.t. the variable upper set less order relation if and only if $x^0 \in \chi$ is an optimal (a weakly optimal) solution of (P) on ε_K w.r.t. the variable upper set less order relation.

Theorem 3.2.9 *Let $x^0 \in \chi$ be an optimal (a strictly optimal / a weakly optimal) solution of (P) w.r.t. $\preceq^u_{D(y^0, y^0)}$ for every $y^0 \in F(x^0)$. Furthermore, suppose that for all $y \in A$ and for every $y^0 \in F(x^0)$, the inclusion*

$$D(y, y^0) \subseteq D(y^0, y^0) \tag{3.12}$$

holds true. Then $x^0 \in \chi$ is an optimal (a strictly optimal / a weakly optimal) solution of the set-valued Problem (P) w.r.t. the variable upper set less order relation.

Proof Suppose that $x^0 \in \chi$ is not an optimal (a strictly optimal / a weakly optimal) solution of (P) w.r.t. the variable upper set less order relation. Then there exists $x \in \chi \backslash \{x^0\}$ such that

$$\forall y \in F(x), \exists y^0 \in F(x^0) : y \in y^0 - E,$$

where $E = D(y, y^0) \backslash \{0\} (E = D(y, y^0),\ E = intD(y, y^0))$. Furthermore,

$$\forall y \in F(x),\ \exists y^0 \in F(x^0) :\ y \in y^0 - E \subseteq y^0 - G,$$

where $G = D(y^0, y^0) \backslash \{0\} (G = D(y^0, y^0),\ G = intD(y^0, y^0)$, respectively). But this is a contradiction to the assumption.

Theorem 3.2.10 *Define* $\bar{D}(x) := \bigcap_{y \in F(x)} D(y, y)$ *for some* $x \in \chi$. *Let* $x^0 \in \chi$ *be an optimal (a strictly optimal / a weakly optimal) solution of (P) w.r.t. the variable upper set less order relation and suppose that*

$$\forall y \in A, \forall y^0 \in F(x^0) : D(y^0,\ y^0) \subseteq D(y,\ y^0). \tag{3.13}$$

Then $x^0 \in \chi$ *is an optimal (a strictly optimal / a weakly optimal) solution of the set-valued problem (P) w.r.t.* $\preceq^u_{\bar{D}(x^0)}$.

Proof Suppose that x^0 is not an optimal solution of the set-valued problem (P) w.r.t. $\preceq^u_{\bar{D}(x^0)}$. Then there exists some $x \in \chi \backslash \{x^0\}$ such that for $\bar{E} = \bar{D}(x^0) \backslash \{0\}$ $(\bar{E} = \bar{D}(x^0),\ \bar{E} = int\bar{D}(x^0)$, respectively),

$$\forall y \in F(x),\ \exists y^0 \in F(x^0) :\ y \in y^0 - \bar{E},$$

implying that

$$\forall y \in F(x),\ \exists y^0 \in F(x^0) :\ y \in y^0 - G,$$

where $G = D(y^0,\ y^0)\backslash\{0\}$ ($G = D(y^0,\ y^0),\ G = intD(y^0,\ y^0)$, respectively). With Condition (3.7), it follows that

$$\forall y \in F(x),\ \exists y^0 \in F(x^0):\ y \in y^0 - E,$$

where $E = D(y,\ y^0)\backslash\{0\}$ ($E = D(y,\ y^0),\ E = intD(y,\ y^0)$, respectively), in contradiction to the assumption that x^0 is an optimal (strictly optimal / weakly optimal) solution of (P) w.r.t. the variable upper set less order relation.

3.3 Optimal Elements Of Sections

Definition 3.3.1 *A solution set $F(x^0)$ is efficient (strictly efficient, weakly efficient, respectively) w.r.t. the variable upper set less order relation if x^0 is an optimal (a strictly optimal / a weakly optimal) solution of the set-valued Problem (P) w.r.t. the variable upper set less order relation. If D is given by a constant set-valued map K, then we say that the solution set $F(x^0)$ is efficient (strictly efficient, weakly efficient, respectively) w.r.t. $\preceq_K^u$ if $x^0 \in \chi$ is an optimal (a strictly optimal / a weakly optimal) solution of the set-valued Problem (P) w.r.t. $\preceq_K^u$.*

A section in vector optimization is defined as a set $(\bar{y} - K) \cap A$, where $\bar{y} \in Y$, $K \subset Y$ is a proper pointed closed convex cone and $A \subset Y$ is the image set of feasible elements. In vector optimization an efficient element in this section is also an efficient element of the whole set A. Eichfelder has developed corresponding result for vector optimization problems with a variable ordering structure. To analyze this concept in set optimization with the variable upper set less order relation we define a set

$$A_{\bar{y}} := (\bar{y} - D[a,\ a]) \cap A,$$

which is denoted as a section w.r.t. an element $\bar{y} \in Y$.

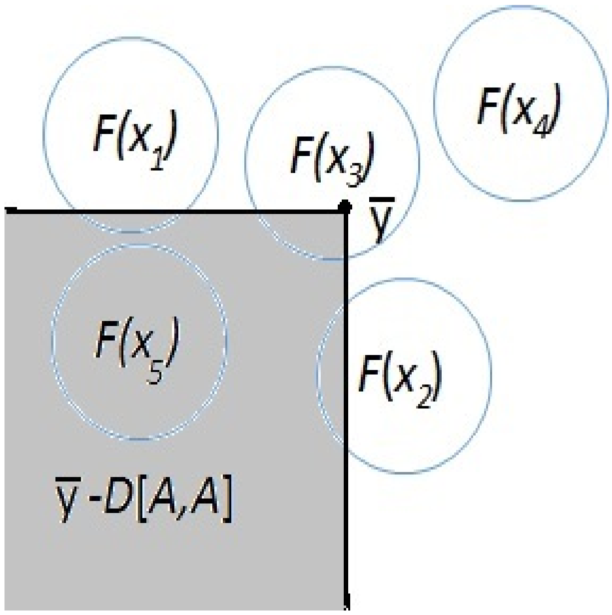

Figure 3.3: Section w.r.t. an element, $(\bar{y} - D[a,\ a]) \cap A$

Theorem 3.3.2 *If $F(x^0) \subset A_{\bar{y}}$ is efficient (strictly efficient / weakly efficient) in $A_{\bar{y}}$ for an element $\bar{y} \in Y$ w.r.t. $\preceq^u_{D[A,\ A]}$, then $F(x^0)$ is efficient (strictly efficient / weakly efficient) in A w.r.t. the variable upper set less order relation.*

Proof Let $F(x^0)$ be efficient (strictly efficient / weakly efficient) in $A_{\bar{y}}$ for an element $\bar{y} \in Y$ w.r.t. $\preceq^u_{D[A,\ A]}$, i.e. there does not exist $x \in \chi \backslash \{x^0\}$ with $F(x) \subset A_{\bar{y}}$ such that for $G = D[A, A]\backslash\{0\}$ $(G = D[A,\ A],\ G = intD[A,\ A]$, respectively)

$$F(x) \preceq^u_G F(x^0),$$

being equivalent to

$$F(x) \subseteq F(x^0) - G.$$

Suppose that $F(x^0)$ is not efficient (strictly efficient / weakly efficient) in A w.r.t. the variable upper set less order relation. Then there exists $x \in$

$\chi\backslash\{x^0\}$ such that for $E = D(y,\ y^0)\backslash\{0\}$ $(E = D(y,\ y^0),\ E = intD(y,\ y^0)$, respectively)

$$F(x) \preceq^u_E F(x^0).$$

implying

$$\forall \in F(x),\ \exists y^0 \in F(x^0):\ y \in y^0 - E \subseteq F(x^0) - G \subseteq A_{\bar{y}} - G,$$

because $f(x^0) \subseteq A_{\bar{y}}$. Since $D[A,\ A]$ is a convex cone, so we conclude with $F(x) \subseteq A_{\bar{y}}$ and $F(x) \subseteq F(x^0) - G$, a contradiction to the inclusion $F(x) \subseteq F(x^0) - G$.

Theorem 3.3.3 *If $F(x^0) \subset A_{\bar{y}}$ for some $\bar{y} \in Y$ and $F = (x^0)$ is efficient (strictly efficient / weakly efficient) in A w.r.t. the variable upper set less order relation, then $F(x^0)$ is efficient (strictly efficient / weakly efficient) in $A_{\bar{y}}$ w.r.t. $\preceq^u_{\bar{D}}$, where $\bar{D}$ is given as $\bar{D} := \bigcap_{y_1,\ y_2 \in A} D(y_1, y_2)$.*

Proof Suppose that $F(x^0)$ is not efficient (strictly efficient / weakly efficient) in $A_{\bar{y}}$ w.r.t. $\preceq^u_{\bar{D}}$. Then there exists $x \in \chi\backslash\{x^0\}$ with $F(x) \subseteq A_{\bar{y}}$ such that for $G = \bar{D}\backslash\{0\}$ $(G = \bar{D},\ G = int\bar{D}$, respectively)

$$F(x) \preceq^u_G F(x^0),$$

implying

$$\forall y \in F(x),\ \exists y^0 \in F(x^0):\ y \in y^0 - G \subseteq y^0 - E,$$

where $E = D(y,\ y^0)\backslash\{0\}$ $(E = D(y,\ y^0),\ E = intD(y,\ y^0)$, respectively), in contradiction to $F(x^0)$'s efficiency (strict efficiency / weak efficiency) in A w.r.t. the variable upper set less order relation.

We can also evaluate sections w.r.t. a set $F(\bar{x}) \subseteq A$. instead of elements $\bar{y} \in Y$, so se introduce a section w.r.t. a set $F(\bar(x)$ as

$$A_{F(\bar{x})} := (F(\bar{x}) - D[A, A]) \cap A$$

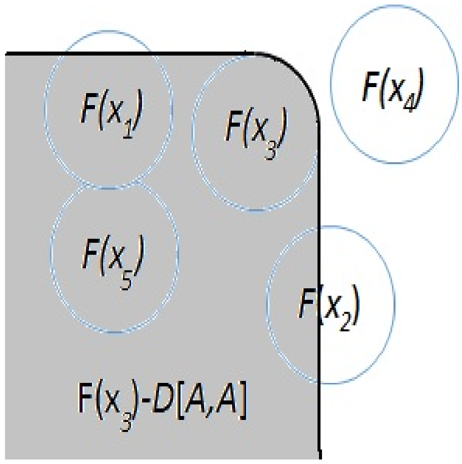

Figure 3.4: Section w.r.t. an element, $(F(\bar{x}) - D[A, A]) \cap A$

Theorem 3.3.4 *If $F(x^0)$ is efficient (strictly efficient / weakly efficient) in $A_{F(\bar{x})}$ for a set $F(\bar{x}) \subset Y$ w.r.t. $\preceq^u_{D[A,A]}$, then $F(x^0)$ is efficient (strictly efficient / weakly efficient) in A w.r.t. the variable upper set less order relation.*

Proof Suppose that $F(x^0)$ is not efficient (strictly efficient / weakly efficient) in A w.r.t. the variable upper set less order relation. Thus, there exist $x \in \chi\{x^0\}$ such that

$$\forall y \in F(x),\ \exists y^0 \in F(x^0):\ y \in y^0 - E,$$

where $E = D(y, y^0)\backslash\{0\}$ $(E = D(y, y^0),\ E = intD(y, y^0)$, respectively). This implies
$\forall y \in F(x),\ \exists y^0 \in F(x^0):\ y \in y^0 - E \subseteq F(x^0) - G \subseteq A_{F(\bar{x})} - G,$
where $G = D[A, A]\backslash\{0\}$ $(G = D[A, A],\ G = intD[A, A]$, respectively), because $F(x^0) \subseteq A_{F(\bar{x})}$. Since $D[A, A]$ is a convex cone, we conclude with $F(x) \subseteq A_{F(\bar{x})}$ and $F(x) \subseteq F(x^0) - G$, in contradiction to the assumption.

Theorem 3.3.5 *If $F(x^0) \subset A_{F(\bar{x})}$ for a set $F(\bar{x}) \subset Y$ and $F(x^0)$ is efficient (strictly efficient / weakly efficient) in A w.r.t. the variable upper set less* *order relation, then $F(x^0)$ is efficient (strictly efficient / weakly efficient) in $A_{F(\bar{x})}$ w.r.t. $\preceq^u_{\bar{D}}$, where $\bar{D}$ is defined as* $\bar{D} := \bigcap_{y_1, y_2 \in A} D(y_1, y_2)$

Proof Suppose that $F(x^0)$ is not efficient (strictly efficient / weakly efficient) in $A_{F(\bar{x})}$ w.r.t. $\preceq^u_{\bar{D}}$. Thus, there exists an element $x \in \chi \backslash \{x^0\}$ with $F(x) \subseteq A_{F(\bar{x})}$ such that

$$\forall y \in F(x),\ \exists y^0 \in F(x^0):\ \ y \in y^0 - G,$$

where $G = \bar{D} \backslash \{0\}$ $(G = \bar{D},\ G = int\bar{D}$, respectively). But then we obtain

$$\forall y \in F(x), \exists y^0 \in F(x^0) : y \in y^0 - G \subseteq y^0 - E,$$

with $E = D(y,\ y^0) \backslash \{0\}$ $(E = D(y,\ y^0),\ E = intD(y, y^0)$, respectively). But this is a contradiction to $F(x^0)'s$ efficiency (strictly efficiency / weakly efficiency) in A w.r.t. the variable upper set less order relation.

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