Exact solutions and Painlevé analysis of some nonlinear partial differential equations

A Thesis

submitted in partial fulfillment of the
requirement for the award of the degree of

Doctor of Philosophy

in

Mathematics

by

Manjit Singh
(Regd No. 951311004)

to the

School of Mathematics

Thapar Institute of Engineering & Technology,

Patiala-147004, Punjab, India

September, 2017

Declaration

I, Manjit Singh, declare that the thesis entitled, "Exact solutions and Painlevé analysis of some nonlinear partial differential equations" and the work presented in it are my own. I confirm that the ideas and references cited herein have been duly acknowledged.

Manjit Singh
(Regd. No. 951311004)

Attestation by Supervisor

Dr. Rajesh Kumar Gupta
Associate Professor
Central University of Punjab
Bathinda–151001, Punjab, India.

i

Certificate

This is to certify that the thesis entitled "Exact solutions and Painlevé analysis of some nonlinear partial differential equations" submitted by Mr. Manjit Singh in the partial fulfillment of the requirement for the award of degree of Doctor of Philosophy in the School of Mathematics, Thapar Institute of Engineering & Technology, Patiala is a record of candidate's own work carried out under my supervision and guidance. The matter presented in this thesis has not been submitted in part or full for the award of any degree in any other University or Institute.

Attestation by Supervisor

Dr. Rajesh Kumar Gupta
Associate Professor
Central University of Punjab
Bathinda–151001, Punjab, India.

ii

Dedicated to my kids
Gurnoor & Prabhnoor
who waited patiently for me
to come out of my study.
The wait is over.

Acknowledgements

Foremost, I would like to express my sincere gratitude to my supervisor Dr. Rajesh Kumar Gupta for the continuous support of my Ph.D. study and research, for his patience, motivation, enthusiasm, and immense knowledge. His guidance helped me in all the time of research and writing of this thesis. I could not have imagined having a better advisor and mentor for my study.

I must express my gratitude to Prof. Peter J. Olver, School of Mathematics, University of Minnesota, George Bluman, Professor Emeritus, University of British Columbia, Aslı Pekcan, Hacettepe University, Turkey, who have responded to my every single email even on the very elementary question on Lie symmetry analysis, I really appreciate their benevolent service (needless to say, I accept full responsibility for what appears in this thesis).

Completing this work would have been all the more difficult were it not the support provided by Dr. Venu Gopal, Brown University, who provided me necessary research material in the initial stage of my research. I am indebted to him for his help. I am also indebted to my friend Tarun Sachdeva at School of Mathematics, Thapar Institute of Engineering & Technology, Patiala for being all-time dependable and samaritan mate.

This work would never have been completed had I not received two years of study leave from Punjabi University, Patiala. So I am also indebted to Punjabi University, Patiala which generously supports its staff to pursue higher studies.

Manjit Singh

Abstract

The integrability of nonlinear partial differential equations has been addressed in this thesis. The various approaches for complete integrability have been comprehensively exploited for important equations in mathematical physics. Along with usual the Lie symmetry analysis, a detailed discussion on various group classification techniques has been given. More general symmetries for variable coefficient coupled KdV equations have been reported. Based on Lie algebra isomorphism, an obscured technique for Lie algebra classification has been re-introduced to improve existing Lie algebra classifications. Motivated by the powerful Hirota's bilinear method, a new test function has been proposed which generalizes the existing test functions for finding exact solutions of PDEs. The Hirota's bilinear method has also been used to find Bäcklund transformations, Lax system, an infinite number of nontrivial conservation laws. Using direct method and new conservation theorem, nontrivial conservation laws have been constructed for some important PDEs of physical relevance.

The thesis comprises 11 chapters, and the chapter wise detail is given as below:

Chapter 1 The tools such as; Lie symmetries, Painlevé analysis, Hirota's direct method, Bäcklund transformations, Bell polynomials, an infinite set of conservation laws and explicit exact solutions are introduced.

Chapter 2 The degenerate coupled multi-KdV equations for coupled multiplicity $l = 3$ are studied. The equations also known as three fields Kaup-Boussinesq equations are considered for invariant analysis and conservation laws. The classical Lie's symmetry method is used to analyze the symmetries of equations. Based on Killing's form which is invariant of adjoint action, the full classification for Lie algebra is presented. Further, one-dimensional optimal group classification is used to obtain invariant solutions. Besides this, using general theorem proved by Ibragimov, we have found several nonlocal conservation laws for these equations.

Chapter 3 The Klein-Gordon-Zakharov equations describing the propagation of strong turbulence of the Langmuir wave in a high-frequency plasma

have been studied in this chapter. By using symbolic manipulation tool Maple, the classification of symmetry algebra has been carried out, and construction of several nonlocal conservation laws based on a new theorem by Ibragimov is illustrated. Starting with the determination of symmetry algebra, the one and two-dimensional optimal systems are constructed, and optimality is also established using various invariant functions of full adjoint action. Apart from classification and construction of several nonlocal conservation laws, the Painlevé analysis is also performed in a symbolic manner which describes the non-integrability of equations.

Chapter 4 The Painlevé property of Date-Jimbo-Kashiwara-Miwa equation is verified with WTC approach, and the exact solution is obtained using Pickering's algorithm. The Lie point symmetries and three independent similarity reductions have been obtained. Further, two different techniques are employed independently to construct conservation laws. Multipliers in the direct method have given an infinite number of nontrivial local conservation laws, and self-adjointness of equation helped us to construct infinite local conservation laws associated with Lie symmetries. The existence of an infinite number of local conservation laws strongly justifies the complete integrability of equation.

Chapter 5 The variable coefficient KdV system is investigated for general symmetries using compatibility method, and more general symmetries are reported. Several inequivalent reductions are obtained using an optimal system of subalgebras. By using well-known methodologies several traveling wave solutions are also obtained for every reduction.

Chapter 6 The five-dimensional Lie algebra is the main objective of the study, which describes the symmetry properties of an isothermal drift flux model of two-phase flows. The classification provided is based on a method given by Patera and Winternitz. Multi-dimensional optimal systems are obtained for the symmetry algebra of model. Further, using general theorem proved by Ibragimov, we find several nonlocal conservation laws corresponding to every symmetry in the one-dimensional optimal system.

Chapter 7 The multi-dimensional optimal classifications of Lie algebras of some well-known equations under their group of inner automorphism are considered. By writing vector fields of some known Lie algebras in the abstract format, we have proved that there exist explicit isomorphism between Lie algebras and sub-algebras which have already been classified. The isomorphism between Lie algebras is useful in the sense that the classifications of subalgebras of dimension ≤ 4 have previously been carried out in literature. These already available classifications can be used to write classification of any Lie algebra of dimension ≤ 4. As an example, the explicit isomorphism between Lie algebra of variant Boussinesq system and sub-algebra $A_{3,5}^{1/2}$ is proved, and subsequently, optimal sub-algebras up to dimension four are obtained. Besides this, some other examples of Lie algebras are also considered for explicit

isomorphism.

Chapter 8 Based on binary Bell polynomial approach, the bilinear equation and Bäcklund transformations for (3+1)-dimensional Jimbo-Miwa equation are obtained. By virtue of Cole-Hopf transformation, the Lax system is constructed by direct linearization of the coupled system of binary Bell polynomials. Furthermore, infinite conservation laws are obtained from two field condition in a quick and natural way. Finally, a test function of extended three wave method is used to construct multisoliton solutions via the bilinear equation.

Chapter 9 In this chapter too, we use binary Bell polynomials approach, and Maple computer algebra package PDEBellII for constructing bilinear equations for some nonlinear evolution equations. Bilinear equations are then used to construct exact solutions using novel test function. Symbolic manipulation program Maple has been used to carry out tedious calculations involved, a simple Maple code is also given in form of an appendix. The exact solutions obtained using the novel test function enriches the solution structure of well-known evolution equations.

Chapter 10 The truncated Laurent expansion is used to obtain the bilinear equation of a nonlinear evolution equation. As an application of Hirota's method, multisoliton solutions are constructed from the bilinear equation. Extending the application of Hirota's method and employing multidimensional Riemann theta function, one and two-periodic wave solutions are also obtained in a straightforward manner. The asymptotic behavior of one and two-periodic wave solutions under small amplitude limits is presented, and their relations with traveling and non-stationary solutions are also demonstrated.

Chapter 11 This chapter briefly outlines the study carried in this thesis. Findings related to Lie symmetries, group classifications, integrability, and exact solutions are addressed.

List of Publications

The content of this thesis has either been published or it is communicated for possible publication. Here in following, we show a list of our results.

1. M. Singh, R.K. Gupta, Bäcklund transformations, Lax system, conservation laws and multisoliton solutions for Jimbo-Miwa equation with Bell-polynomials, Communications in Nonlinear Science and Numerical Simulation, **37**: 362–373, 2016.
   SCI Impact Factor: 3.239

2. M. Singh, New exact solutions for (3+ 1)-dimensional Jimbo-Miwa equation. Nonlinear Dynamics 84.2 (2016): 875-880.
   SCI Impact Factor: 4.339

3. M. Singh, R.K. Gupta, Exact solutions for nonlinear evolution equations using novel test function, Nonlinear Dynamics, **86(2)**: 1171–1182, 2016.
   SCI Impact Factor: 4.339

4. R.K. Gupta, M. Singh, Nonclassical symmetries and similarity solutions of variable coefficient coupled KdV system using compatibility method, Nonlinear Dynamics, **87(3)**: 1543–1152, 2016.
   SCI Impact Factor: 4.339

5. R.K. Gupta, M. Singh, On group classification and nonlocal conservation laws for a multi phase flow model, International Journal of Applied and Computational Mathematics, International Journal of Applied and Computational Mathematics, **3**(4): 3925-3935.
   A Springer Journal Publishing from 2015

6. M. Singh, R.K. Gupta, Soliton and quasi-periodic wave solutions for b-type Kadomtsev-Petviashvili equation, Indian Journal of Physics, **91**(11): 1345-1354.
   SCI Impact Factor: 0.988.

7. R.K. Gupta, M. Singh, On invariant analysis and conservation laws for degenerate coupled multi-KdV equations for multiplicity $l = 3$, Pramana–Journal of Physics, Accepted

SCI Impact Factor: 0.699.

8. M. Singh, R.K. Gupta, Group classification, conservation laws and Painlevé analysis for Klein–Gordon–Zakharov equations in (3+1)–dimension, Pramana–Journal of Physics, In Press..
   SCI Impact Factor: 0.699.

9. M. Singh, R.K. Gupta, On Painlevé analysis, symmetry group and conservation laws of Date-Jimbo-Kashiwara-Miwa equation, International Journal of Applied and Computational Mathematics, **4**(3): 88.
   A Springer Journal Publishing from 2015

10. R.K. Gupta, M. Singh, Multi-dimensional optimal system of certain low dimensional Lie algebras, Communicated.

Contents

1 Introduction **1**
1.1 Lie theory . . . 1
1.1.1 Group of transformations . . . 2
1.1.2 Infinitesimal transformations . . . 4
1.1.3 Solution of ODE using Lie symmetries . . . 6
1.1.4 Determination of symmetry group . . . 7
1.1.5 Group invariant solutions . . . 10
1.1.6 Lie algebra and its classification techniques . . . 12
1.2 Notion of integrability . . . 16
1.2.1 Painlevé analysis . . . 16
1.2.2 Hirota's bilinear formalism . . . 19
1.2.3 Bäcklund transformations . . . 22
1.2.4 Bell polynomials . . . 25
1.2.5 Conservation laws . . . 28
1.3 Exact solutions . . . 30
1.3.1 Exact solutions from invariant Painlevé analysis . . . 30
1.3.2 Exact soliton solutions . . . 31
1.3.3 Riccati equation mapping method . . . 34
1.3.4 $\frac{G'}{G}$-expansion method . . . 35
1.3.5 Exact solutions using test function . . . 36

2 On invariant analysis and conservation laws for degenerate coupled multi-KdV equations for multiplicity $l = 3$ **38**
2.1 Introduction . . . 38
2.2 Lie symmetry analysis of Kaup-Boussinesq equations (2.3) . . . 39
2.2.1 Construction of an optimal system for Lie algebra (2.8) . . . 40
2.2.2 Symmetry reductions and invariant solutions . . . 43
2.3 Conservation laws . . . 48

3 Group classification, conservation laws and Painlevé analysis for Klein–Gordon–Zakharov equations in (3+1)–dimension **53**
3.1 Introduction . . . 53
3.2 Symmetry group of the KGZ equations . . . 54

3.2.1 Construction of invariant functions . . . 55
3.2.2 Construction of an optimal system . . . 56
3.2.3 Two-dimensional optimal system . . . 58
3.3 Conservation laws . . . 59
3.4 Painlevé analysis of KGZ equation . . . 61

4 On Painlevé analysis, symmetry group and conservation laws of Date-Jimbo-Kashiwara-Miwa equation 65
4.1 Introduction . . . 65
4.2 Truncated Painlevé expansions and exact solutions . . . 67
4.3 Lie symmetries of DJKM equation (4.1) . . . 70
4.4 Conservation laws . . . 72
4.4.1 Conservation laws using direct method . . . 72
4.4.2 Conservation laws using strict self-adjointness . . . 76

5 Nonclassical symmetries and similarity solutions of variable coefficient coupled KdV system using compatibility method 81
5.1 Introduction . . . 81
5.2 Symmetries of KdV system (5.1) . . . 82
5.3 Reduction of system (5.1) and some exact solutions . . . 84

6 Multi-dimensional optimal system of certain low dimensional Lie algebras 93
6.1 Introduction . . . 93
6.2 Lie algebra of variant Boussinesq system in abstract manner . . . 94

7 On group classification and nonlocal conservation laws for a multi phase flow model 104
7.1 Introduction . . . 104
7.2 Explicit isomorphism for abstract Lie algebra of model (7.1) . . . 105
7.2.1 Construction of an optimal system . . . 107
7.3 Nonlocal conservation laws for the model (7.1) . . . 110

8 Bäcklund transformations, Lax system, conservation laws and solitary wave solutions for Jimbo-Miwa equation with Bell-polynomials 115
8.1 Introduction . . . 115
8.2 Bilinear form, bilinear Bäcklund transformations and Lax system . . . 116
8.2.1 Bilinear representation . . . 116
8.2.2 Bilinear Bäcklund transformations . . . 117
8.2.3 Lax system . . . 118
8.3 Infinite conservation laws . . . 118
8.4 Multi-wave solutions using three wave method . . . 120

9 Exact solutions for nonlinear evolution equations using novel test function 128
9.1 Introduction . . . 128
9.2 Bilinear equations using Maple package PDEBellII . . . 129
9.3 Exact solutions using novel test function . . . 131

9.3.1 Exact solutions for (9.1) . . . 131
9.3.2 Exact solutions for (9.6) . . . 132
9.3.3 Exact solutions for (9.11) . . . 134
9.3.4 Exact solutions for (9.16) . . . 135

10 Traveling and quasi-periodic wave solutions for b-type Kadomtsev-Petviashvili equation 140
10.1 Introduction . . . 140
10.2 Hirota bilinear method . . . 141
10.2.1 Bilinear form . . . 142
10.3 Multi-periodic wave solutions using Riemann theta function . . . 143
10.3.1 Riemann theta function . . . 143

11 Conclusion and outlook 152
11.1 Lie symmetries . . . 152
11.2 Group classification . . . 153
11.3 Integrability . . . 153
11.3.1 Painlevé analysis . . . 153
11.3.2 Hirota's direct method . . . 154
11.3.3 Bell polynomials . . . 154
11.3.4 Conservation laws . . . 155
11.3.5 Exact solutions . . . 156
11.4 Future scope and study . . . 156

A Appendices 158
A.1 Coefficients . . . 158
A.2 Maple Code . . . 160

B Appendices 162
B.1 The Maple code for writing KGZ equation (3.1) in real format . . . 162
B.2 Finding Symmetry algebra for equations (3.2) . . . 162
B.3 Maple code for full adjoint action . . . 163
B.4 Full adjoint action . . . 164

List of Figures

1.1 The figure displaying composition of transformations (1.1) and (1.2). 3

4.1 Plots are drawn for similarity variables (4.26) and (4.29) at given instant of time. The parallel straight lines are invariant curves with respect to translation in space. 72

8.1 Setting all arbitrary parameters to unity, figures (a) kink wave profile of solutions (8.31), (b) two solitary wave profile of solutions (8.33) and (c) periodic wave profile of solutions (8.35) on $x-t$ plane 123

8.2 Setting all arbitrary parameters to unity, figure (a) two solitary wave profile of solution (8.37) on $x-t$ plane, figure (b) kinky wave profile of solution (8.39) on $x-z$ plane and figure (c) kink periodic wave profile of solution (8.41) on $x-t$ plane 125

8.3 Setting all arbitrary parameter to unity except $a_3 = 2$, figure (a) soliton-like and figure (b) one solitary wave profile of solution (8.43) on $y-z$ plane and $x-y$ plane respectively, figure (c) two solitary wave profile of solution (8.45). 126

9.1 Figure 9.1a; divergent behaviour of solution (9.22). Figure 9.1b; kink wave solution (9.23). Figure 9.1c; kink wave solution (9.24). All plots are in $x-t$ domain. 136

9.2 Figure 9.2a; kink wave solution (9.25). Figure 9.2b; kink wave solution (9.26). Figure 9.2c; singular solution (9.27). All plots are in $x-t$ domain. 136

9.3 Figure 9.3a; kinky periodic wave profile given by $\Re(u)$ for solution (9.28). Figure 9.3b; smooth profile given by $|u|$ for solution (9.28). Figure 9.3c; kink wave profile given by $\Re(u)$ for solution (9.29). All plots are in $x-t$ domain. 137

9.4 Figure 9.4a; soliton like profile given by $|u|$ for solution (9.29). Figure 9.4b; kinky periodic profile given by $\Re(u)$ for solution (9.29). Figure 9.4c; smooth profile given by $\Re(u)$ for solution (9.30). All plots are in $x-t$ domain except in case of plot (b) where domain is $y-t$. 137

9.5 Figure 9.5a; smooth profile given by $|u|$ for solution (9.30). Figure 9.5b; kink wave profile given by $\Re(u)$ for solution (9.31). Figure 9.5c; soliton like profile given by $|u|$ for solution (9.31). All plots are in $x-t$ domain. 137
9.6 Figure 9.6a; plot given by $\Re(u)$ for solution (9.32). Figure 9.6b; singular profile given by $|u|$ for solution (9.33). Figure 9.6c; plot given by $\Re(u)$ for solution (9.34). All plots are in $x-t$ domain. . . . 138
9.7 Figure 9.7a; periodic wave solution (9.35). Figure 9.7b; periodic wave solution (9.36). Figure 9.7c; kink wave solution (9.37). The plots (a) and (b) are in $y-t$ domain whereas the plot (c) is drawn in $x-t$ domain. 138

10.1 One-Periodic wave solution of equation (10.1) via expression (10.11) for parametric choices $\alpha_1 = 0.23, \omega_1 = -0.3, \delta_1 = 0, \tau_1 = \imath$ and $u_0 = 0$. From Figure 10.1–Figure 10.2, it is evidently clear that every one-periodic wave solution is one-dimensional and can be viewed as a superposition of overlapping solitary waves. Figure 10.1a represents a perspective view of $\Re(u)$ in the $x-t$ plane. Figure 10.1b represents the corresponding contour plot. Figure 10.1c represents the wave propagation along x-direction 148
10.2 Figure 10.2a represents perspective view of $\Re(u)$ in the $y-t$ plane. Figure 10.2b represents the corresponding contour plot. Figure 10.2c represents the wave propagation along the y-direction. 149
10.3 A degenerate two-periodic wave solution of equation (10.1) via expression (10.13) for parametric choices $\alpha_1 = 0.3, \alpha_2 = 0.35, \omega_1 = -0.3, \omega_2 = -0.25, \delta_1 = 0, \delta_2 = 0, \tau_{11} = \imath, \tau_{12} = 2\imath, \tau_{22} = 0.5\imath$. Figure 10.3a represents perspective view of $\Re(u)$ in the $x-t$ plane. Figure 10.3b represents the corresponding contour plot. Figure 10.3c represents the wave propagation along x-direction. 149
10.4 Figure 10.4a represents the perspective view of $\Re(u)$ on $y-t$ plane. Figure 10.4b represents the corresponding contour plot. Figure 10.4c represents the wave propagation along the y-direction. 149
10.5 For parametric choices $\alpha_1 = 0.45, \alpha_2 = 0.4, \omega_1 = -0.1, \omega_2 = -0.15, \tau_{11} = \imath, \tau_{12} = \imath$ and $\tau_{22} = 0.5\imath$, the Figure 10.5a, Figure 10.5b and Figure 10.5c give two-periodic wave solution on $x-t$, $y-t$ and $x-y$ planes respectively. 150
10.6 For parametric choices $\alpha_1 = 0.6, \alpha_2 = 0.5, \omega_1 = -0.2, \omega_2 = -0.1, \tau_{11} = \imath, \tau_{12} = \imath$ and $\tau_{22} = 0.5\imath$, the Figure 10.6a, Figure 10.6b and Figure 10.6c give two-periodic wave solution on $x-t$, $y-t$ and $x-y$ planes respectively 150

List of Tables

1.1 Commutator table. 13

3.1 Adjoint table based on formula (2.14). 57
3.2 Invariants of full adjoint action (3.10). 58
3.3 Invariants of general vector $\lambda X_1 + \mu X_2$ in $M = \langle X_1, X_2 \rangle$. 59

6.1 Classification of algebra $\mathfrak{g} = L_5 \oplus L_4 \oplus L_3$. 99

7.1 List of non-splitting extensions of e_5, where N_a is running over subalgebras (7.9). 109
7.2 Nonlocal conservation laws for model (7.1), where $\epsilon = \pm 1$. 112

1 Introduction

Do not worry about your difficulties in Mathematics. I can assure you mine are still greater.

Albert Einstein

1.1 Lie theory[1]

The application of the Lie theory to the study of partial differential equations (PDEs) has its origins in the original investigations of Sophus Lie. The Lie theory is based on an inspirational idea taken from the theory of algebraic equations given by Galois, which uses the finite groups to solve algebraic equations of degree two, three, and four, and to show that the general polynomial equation of degree greater than four could not be solved by radicals. It was Ludwig Sylow, whose lectures on Galois theory have made a profound impression on Lie, and motivated him to develop a similar approach to solve and simplify ordinary differential equations by studying the structure of invariance group. Some researchers believe that the origins of Lie theory are geometric and stem from the view of Felix Klein (1849–1925) that geometry of space is determined by the group of its symmetries. As Gilmore writes [85]; "Lie groups are beautiful, important, and useful because they have one foot in each of the two great divisions of mathematics-algebra and geometry. Their algebraic properties derive from the group axioms. Their geometric properties derive from the identification of group operations with points in a topological space. The rigidity of their structure comes from the continuity requirements of the group composition and inversion maps". Unfortunately, soon after the inception of Lie theory in 1873–74, it went into obscurity due to several reasons like; the local character of results and incapability in the construction of general solutions for the system under consideration except for the case of ordinary differential equations. Most importantly, the earlier researchers like Wilhelm Killing (1847–1923), Élie Cartan (1869–1951), Hermann Weyl (1885–1955) immensely contributed to structure and

[1]Most of the content in this section is inspired from several textbooks [34, 38, 54, 114, 175].

representation of Lie groups, but they failed to notice the concept of group invariant solutions. Breakthroughs in Lie theory achieved only after the year 1940 when L.I.Sedov and G.Birkhoff did groundbreaking work on a general theory of dimensional analysis [174, 191]. L.I.Sedov is also known for his Sedov similarity solution for a blast wave, which he devised in World War II. Birkhoff and Sedov were first to realize that the local group transformations can be used to construct group invariant solutions of PDEs by the reduction in a number of independent variables. Finally, Ovsiannikov's comprehensive study [177] on group invariant solutions proved the power of Lie theory, and he obtained explicit solutions for a complicated system of PDEs. The Lie's theory gained momentum when researchers like Bluman, Cole [37,38], Ames [9] and others [20,49,52,98,115,172,190,193,251] start obtaining new results for important equations in mathematical physics. In the following sections, we will a give a brief introduction to Lie's theory [34, 38, 54, 114, 175].

1.1.1 Group of transformations

The set of transformations:

$$x_1 = \phi(x, y, a),\ y_1 = \psi(x, y, a),\ a \in G \subset \mathbb{R}, \tag{1.1}$$

determined by some value of parameter a, constitute a group if the successive application of one of the transformation is also a transformation belonging to set (1.1). For example, for some other arbitrary parameter value b, the second transformation may be written as follows:

$$x_2 = \phi(x_1, y_1, b),\ y_2 = \psi(x_1, y_1, b). \tag{1.2}$$

The transformation (1.1) forms a group if elimination of x_1, y_1 between (1.1) and (1.2), i.e.

$$x_2 = \phi(\phi(x, y, a), \psi(x, y, a), b),\ y_2 = \psi(\phi(x, y, a), \psi(x, y, a), b), \tag{1.3}$$

reduce identically to

$$x_2 = \phi(x, y, c),\ y_2 = \psi(x, y, c), \tag{1.4}$$

where c is a function of a and b. The transformations (1.1) are continuous functions of group parameter a, and this makes them a useful analytical tool. For starting value $a = a_0$ of group parameter, x and y can be transformed continuously, that is, a small change in group parameter a produces small changes in x and y. To illustrate further, we give three basic geometrical examples of group transformations.

Example 1.1.1. Translations in x and y coordinates:

$$x_1 = x + 2a,\ y_1 = y + 3a. \tag{1.5}$$

The set of following transformations for parameter b:

$$x_2 = x_1 + 2b,\ y_2 = y_1 + 3b. \tag{1.6}$$

The successive result of transformations (1.5) and (1.6) is given by

$$x_2 = x + 2c,\ y_2 = y + 3c, \tag{1.7}$$

where $c = a + b$. Hence, all translations of type (1.5) form a group.

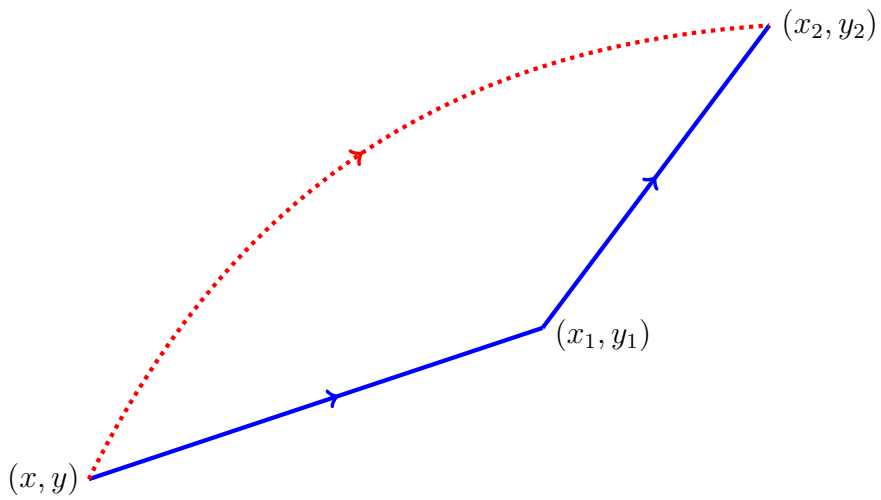


Figure 1.1: The figure displaying composition of transformations (1.1) and (1.2).

Example 1.1.2. Rotation of axes:

$$x_1 = x\cos a - y\sin a,\ \ y_1 = x\sin a + y\cos a. \tag{1.8}$$

The set of following transformations for parameter b:

$$x_2 = x_1\cos b - y_1\sin b,\ \ y_2 = x_1\sin b + y_1\cos b. \tag{1.9}$$

The elimination of x_1, y_1 between (1.8) and (1.9) yields the following result

$$x_2 = x\cos c - y\sin c,\ \ y_2 = x\sin c + y\cos c, \tag{1.10}$$

where $c = a + b$. Hence, all rotations of type (1.8) form a group.

Example 1.1.3. Scaling of axes:

$$x_1 = ax,\ \ y_1 = a^2 y. \tag{1.11}$$

The set of following transformations for parameter b:

$$x_2 = bx_1,\ \ y_2 = b^2 y_1. \tag{1.12}$$

The elimination of x_1, y_1 between (1.11) and (1.12) yields the following result

$$x_2 = cx,\ \ y_2 = c^2 y, \tag{1.13}$$

where $c = ab$. Hence, all scalings of type (1.11) form a group.

Now we define transformations (1.1) in formal group terminology. We denote transformations (1.1) by $\mathbf{T}_a$ and $\mathbf{T}_b$ for group parameters a and b respectively. Therefore, the successive transformations may be written as

$$\mathbf{T}_a\mathbf{T}_b = \mathbf{T}_c = \mathbf{T}_{f(a,b)}, \tag{1.14}$$

where $c = f(a, b)$ is defining law of composition of parameters a and b such that following group properties hold,

- *Closure property.* For any two elements a and b of $G \subset \mathbb{R}$, $f(a, b) \in G$.

- *Associative property.* For any three elements $a, b, c \in G$:

$$f(a, f(b, c)) = f(f(a, b), c).$$

- *Identity element.* There exist a unique element e of G, such that for any a of G:

$$f(a, e) = f(e, a) = a$$

- *Inverse element.* For any element a there exist a unique inverse a^{-1}, such that

$$f(a, a^{-1}) = f(a^{-1}, a) = e.$$

We now define the set of transformations (1.1) in more general setting. Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ be an element in region $D \subset \mathbb{R}^n$. We represent transformations (1.1) in n-dimensional space by bijective mapping $\mathbf{T}_\epsilon(\mathbf{x}) : D \to D$, for parameter ϵ in the set $G \subset \mathbb{R}$ along with $f(\epsilon, \delta)$, which is defining law of composition of $\epsilon, \delta \in G$. The transformations $\mathbf{T}_\epsilon(\mathbf{x})$ form a one parameter group of transformations if following hold:

(i) G forms a group with composition law f.

(ii) For identity element ϵ_0, $\mathbf{T}_{\epsilon_0}(\mathbf{x}) = \mathbf{x}$.

(iii) For successive transformations $\mathbf{T}_\delta(\mathbf{T}_\epsilon(\mathbf{x})) = \mathbf{T}_{f(\epsilon,\delta)}(\mathbf{x})$.

When all the above properties are satisfied, we call $\mathbf{T}_\epsilon(\mathbf{x})$ as one-parameter Lie group of continuous transformations, where ϵ is the group parameter. So, we define Lie group of transformation

Definition 1.1.1. A one-parameter (ϵ) Lie group of transformations are defined by

$$\mathbf{x}^* = \mathbf{T}_\epsilon(\mathbf{x}), \tag{1.15}$$

where $\epsilon = \epsilon_0$ is an identity element of the group generated by composition f. The transformations (1.15) are often called as symmetries, and sometime local transformations as the group conditions need to satisfy only in some neighbourhood of ϵ_0.

1.1.2 Infinitesimal transformations

Since the Lie group of transformations (1.15) are continuous for group parameter ϵ, so by Taylor's theorem

$$\begin{aligned}\mathbf{T}_\epsilon(\mathbf{x}) &= \mathbf{x} + \epsilon \left(\frac{\partial \mathbf{T}_\epsilon}{\partial \epsilon}\right)_{\epsilon=\epsilon_0} + \frac{1}{2}\epsilon^2 \left(\frac{\partial^2 \mathbf{T}_\epsilon}{\partial \epsilon^2}\right)_{\epsilon=\epsilon_0} + \ldots \\ &= \mathbf{x} + \epsilon \left(\frac{\partial \mathbf{T}_\epsilon}{\partial \epsilon}\right)_{\epsilon=\epsilon_0} + O(\epsilon^2),\end{aligned} \tag{1.16}$$

where $\epsilon = \epsilon_0$ is the group identity. We use notation

$$\xi(\mathbf{x}) = \left(\frac{\partial \mathbf{T}_\epsilon}{\partial \epsilon}\right)_{\epsilon=\epsilon_0}, \tag{1.17}$$

which are called *infinitesimals*, and neglecting higher powers of ϵ in (1.16), the transformations $\mathbf{x} + \epsilon\,\xi(\mathbf{x})$ are called *infinitesimal transformations*.

Theorem 1.1.1. (**Lie's first theorem**). The Lie group of transformations (1.15) are equivalent to an initial value problem

$$\frac{\partial \mathbf{T}}{\partial \epsilon} = \xi(\mathbf{x}), \tag{1.18}$$

with

$$\mathbf{x} = \mathbf{T}_\epsilon, \text{ when } \epsilon = \epsilon_0. \tag{1.19}$$

Based on definition (1.17), the *infinitesimals* for transformations (1.5), (1.8) and (1.11) may be written as

- Translations: $\xi = 2, \eta = 3\,; \epsilon_0 = 0$.
- Rotations: $\xi = -y, \eta = x\,; \epsilon_0 = 0$.
- Scaling: $\xi = x, \eta = 2y\,; \epsilon_0 = 1$.

Let $F(\mathbf{x}) = F(x_1, x_2, \ldots, x_n)$ be any analytic function, then effect of transformations (1.15) on it is to replace it with $F(\mathbf{x}+\epsilon\,\xi(\mathbf{x}))$, which can be expanded using Taylor's theorem

$$F(\mathbf{x} + \epsilon\,\xi(\mathbf{x})) = F(\mathbf{x}) + \left(\sum_{i=1}^{n} \xi_i(\mathbf{x}) \frac{\partial F}{\partial x_i}\right) \epsilon + \ldots,$$

that is to say

$$\Delta F = F(\mathbf{x} + \epsilon\,\xi(\mathbf{x})) - F(\mathbf{x}) = X(F)\epsilon + \ldots,$$

where

$$X = \sum_{i=1}^{n} \xi_i(\mathbf{x}) \frac{\partial}{\partial x_i}, \tag{1.20}$$

is called *infinitesimal generator*. For the transformations (1.5), (1.8) and (1.11), the infinitesimals generators are,

$$X_1 = 2\,\frac{\partial}{\partial x} + 3\,\frac{\partial}{\partial y},\; X_2 = -y\,\frac{\partial}{\partial x} + x\,\frac{\partial}{\partial y},\; X_3 = x\,\frac{\partial}{\partial x} + 2y\,\frac{\partial}{\partial y},$$

respectively. As we can see that the Lie group of transformations are characterised by infinitesimals $\xi(\mathbf{x})$, therefore, the notation $X(F) = \left(\sum_{i=1}^{n} \xi_i(\mathbf{x}) \frac{\partial}{\partial x_i}\right) F$ is sufficient to represent infinitesimal transformations (1.15).

Theorem 1.1.2. The one-parameter Lie group of transformations $\mathbf{T}_\epsilon(\mathbf{x})$ are equivalent to

$$\mathbf{T}_\epsilon(\mathbf{x}) \equiv \mathbf{x}^* = \mathbf{x} + \epsilon\,\xi(\mathbf{x}) = e^{\epsilon X}\mathbf{x} = \mathbf{x} + \epsilon X\mathbf{x} + \frac{1}{2}\epsilon^2 X^2\mathbf{x} + \ldots, \tag{1.21}$$

where the infinitesimal generator X is defined by the relation (1.20), and for an analytic function $F(\mathbf{x})$, $F(\mathbf{x}^*) = F(e^{\epsilon X}\mathbf{x}) = e^{\epsilon X}F(\mathbf{x})$.

Definition 1.1.2. An analytic function $F(\mathbf{x})$ is the invariant of Lie group of transformations $T_\epsilon(\mathbf{x})$, if and only if

$$F(\mathbf{x}^*) = F(\mathbf{x}). \tag{1.22}$$

Theorem 1.1.3. An analytic function $F(\mathbf{x})$ is the invariant under Lie group of transformations $T_\epsilon(\mathbf{x})$, if and only if

$$XF(\mathbf{x}) \equiv 0, \tag{1.23}$$

where infinitesimal generator X is defined by relation (1.20). The constraint (1.23) is called invariance criterion and it is quite useful in finding invariant functions (1.22) of Lie group of transformations. For example, for rotational transformations (1.8), when $X = -y\,\frac{\partial}{\partial x} + x\,\frac{\partial}{\partial y}$, the constraint (1.23) reduces to following system of ordinary differential equations

$$\frac{dx}{-y} = \frac{dy}{x} = \frac{dF}{0}. \tag{1.24}$$

The solution of which is an invariant $F = \Theta(x^2 + y^2)$ under rotation of axes.

1.1.3 Solution of ODE using Lie symmetries

In this section, we shall briefly discuss the application of Lie symmetries for solving ordinary differential equations by the change of coordinates.

Definition 1.1.3. For any given one-parameter Lie group of transformations (1.21), it is possible to make change in coordinates; $\mathbf{x} \to \mathbf{y(x)}$, these are called canonical coordinates, and such coordinates become,

$$\begin{aligned} y_i^* &= y_i,\ i = 1, \ldots, n-1, \\ y_n^* &= y_n + \epsilon. \end{aligned} \tag{1.25}$$

Theorem 1.1.4. The Lie group of transformations (1.21) are equivalent to canonical coordinates satisfying (1.25), if and only if

$$\begin{aligned} Xy_i &= 0,\ i = 1, \ldots, n-1, \\ Xy_n &= 1. \end{aligned} \tag{1.26}$$

For 2-dimensional case, the ODEs (1.26) can be written as follow:

$$\frac{dx}{\xi(x,y)} = \frac{dy}{\eta(x,y)} = \frac{ds}{1}, \tag{1.27}$$

whose solution gives canonical coordinates $r(x,y) = c_1$ and $s(x,y) = c_2$. Such coordinates can then transform the following ODE

$$\frac{dy}{dx} = f(x,y), \tag{1.28}$$

into

$$\frac{ds}{dr} = \frac{s_x + s_y f(x,y)}{r_x + r_y f(x,y)},\ \ \eta(x,y) \neq \xi(x,y) f. \tag{1.29}$$

The above procedure can similarly be extended for reduction in order of ODE by one and detailed discussion of it can be seen in the Ref. [34].

Example 1.1.4. Consider Riccati equation:

$$\frac{dy}{dx} = xy^2 - \frac{2y}{x} - \frac{1}{x^3},\ x \neq 0, \tag{1.30}$$

which admits scaling symmetry

$$x^* = e^{\epsilon}x,\ y^* = e^{-2\epsilon}y. \tag{1.31}$$

Solution of ODEs (1.27) gives the following canonical coordinates

$$r(x,y) = x^2y,\ s(x,y) = \ln|x|. \tag{1.32}$$

These coordinates transform the equation (1.30) into following canonical form

$$\frac{ds}{dr} = \frac{1}{r^2-1}, \tag{1.33}$$

which can be solved in a straightforward manner, and final solution for equation (1.30) is thus given by

$$y = \frac{c+x^2}{x^2(c-x^2)}. \tag{1.34}$$

1.1.4 Determination of symmetry group

The symmetry group of a system of differential equations is the largest group of transformations acting on the space of dependent and independent variables with the property that it transforms solutions of system to other solutions. The determination of symmetry group basically relies on the prolongation of infinitesimal generator (1.20) to the spaces of derivatives of dependent variables in the system.

Definition 1.1.4. A one-parameter Lie group of point transformations is a group of transformations of the form,

$$\begin{aligned} \mathbf{x}^* &= \mathbf{T}_\epsilon(\mathbf{x},\mathbf{u}), \\ \mathbf{u}^* &= \mathbf{U}_\epsilon(\mathbf{x},\mathbf{u}), \end{aligned} \tag{1.35}$$

acting on the space of n-independent variables $\mathbf{x} = (x_1,\ldots,x_n)$ and m-dependent variables $\mathbf{u} = (u_1,\ldots,u_m)$.

Definition 1.1.5. (Multi-parameter Lie group) Lie group of transformations,

$$\begin{aligned} \mathbf{x}^* &= \mathbf{T}_{\boldsymbol{\epsilon}}(\mathbf{x},\mathbf{u}), \\ \mathbf{u}^* &= \mathbf{U}_{\boldsymbol{\epsilon}}(\mathbf{x},\mathbf{u}), \end{aligned} \tag{1.36}$$

constitute r-parameter group of transformations for $\boldsymbol{\epsilon} = (\epsilon_1,\ldots,\epsilon_r)$. The detailed discussion on the topic can be seen in section 2.5.1 of the Ref. [34].

In view of transformations (1.35), the infinitesimal generator (1.20) may be written as follows:

$$X = \sum_{i=1}^{n} \xi_i(\mathbf{x},\mathbf{u})\frac{\partial}{\partial x_i} + \sum_{\mu=1}^{m} \eta^\mu(\mathbf{x},\mathbf{u})\frac{\partial}{\partial u^\mu}. \tag{1.37}$$

The kth-order prolongation of (1.37) can be achieved by extending the transformations (1.35) from space $(\mathbf{x}, \mathbf{u})$ to jet space $(\mathbf{x}, \mathbf{u}, \partial\mathbf{u}, \ldots, \partial^k\mathbf{u})$ where $\partial^k\mathbf{u}$ denotes all kth-order partial derivatives of $\mathbf{u}$ with respect to $\mathbf{x}$. The kth-order prolongation of infinitesimal generator (1.37) is given by

$$X^{(k)} = \sum_{i=1}^{n} \xi_i(\mathbf{x}, \mathbf{u}) \frac{\partial}{\partial x_i} + \sum_{\mu=1}^{m} \eta^\mu(\mathbf{x}, \mathbf{u}) \frac{\partial}{\partial u^\mu} + \sum_{\mu=1}^{m} \sum_{i=1}^{n} \eta_i^{(1)\mu}(\mathbf{x}, \mathbf{u}, \partial\mathbf{u}) \frac{\partial}{\partial u_i^\mu} + \ldots$$
$$+ \sum_{\mu=1}^{m} \sum_{i_1=1}^{n} \ldots \sum_{i_k=1}^{n} \eta_{i_1 \ldots i_k}^{(k)\mu}(\mathbf{x}, \mathbf{u}, \partial\mathbf{u}, \ldots, \partial^k\mathbf{u}) \frac{\partial}{\partial u_{i_1 \ldots i_k}^\mu}, \ k \geq 1. \tag{1.38}$$

with the extended infinitesimals $\eta_{i_1 \ldots i_k}^{(k)\mu}$ given by

$$\eta_i^{(1)\mu} = \sum_{i=1}^{n} D_i \eta^\mu - \sum_{i=1}^{n} \sum_{j=1}^{n} (D_i \xi_j) u_j^\mu, \tag{1.39}$$

$$\eta_{i_1 \ldots i_k}^{(k)\mu} = \sum_{i_1=1}^{n} \ldots \sum_{i_k=1}^{n} D_{i_k} \eta_{i_1 \ldots i_{k-1}}^{(k-1)\mu} - \sum_{i_1=1}^{n} \ldots \sum_{i_k=1}^{n} \sum_{j=1}^{n} (D_{i_k} \xi_j) u_{i_1 \ldots i_{k-1} j}^\mu, \ k \geq 2, \tag{1.40}$$

where the total derivative operator is defined by

$$D_i = \frac{\partial}{\partial x_i} + u_i^\mu \frac{\partial}{\partial u^\mu} + u_{ij}^\mu \frac{\partial}{\partial u_j^\mu} + \cdots + u_{i i_1 \ldots i_n}^\mu \frac{\partial}{\partial u_{i_1 \ldots i_n}^\mu} + \ldots, \tag{1.41}$$

where Einstein's notations are used to avoid repeated summations, that is, $\frac{\partial}{\partial x_i} = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}$, etc.

Definition 1.1.6. (Invariance and infinitesimal characterization). The system of PDEs

$$R^\mu(\mathbf{x}, \mathbf{u}, \partial\mathbf{u}, \ldots, \partial^k\mathbf{u}) = 0, \ \mu = 1, \ldots m, \tag{1.42}$$

admits Lie group of transformations (1.35), if and only if

$$X^{(k)} R^\mu(\mathbf{x}, \mathbf{u}, \partial\mathbf{u}, \ldots, \partial^k\mathbf{u})\Big|_{(1.42)} \equiv 0. \tag{1.43}$$

The action of prolongation $X^{(k)}$ in (1.43) is simply to reduce PDEs (1.42) into overdetermined system of linear PDEs in infinitesimals ξ_i and η^μ. Similarly, the invariance criterion (1.43) can also be applied to ODEs by considering one independent and one dependent variable in (1.35).

Example 1.1.5. The ODE

$$y'' = \frac{y'^2}{y} - y^2, \tag{1.44}$$

arises in the study of swimming micro-organisms [114]. The invariance criterion (1.43) is the following linearized symmetry condition

$$\eta_{xx} + (2\,\eta_{xy} - \xi_{xx})y' + (\eta_{yy} - 2\,\xi_{xy})y'^2 - \xi_{yy}y'^3 + (\eta_y - 2\,\xi_x - 3\,\xi_y y') \left(\frac{y'^2}{y} - y^2\right)$$

$$= \eta \left(-\frac{y'^2}{y^2} - 2\,y \right) + \left(\eta_x + (\eta_y - \xi_x)y' - \xi_y y'^2 \right) \left(\frac{2\,y'}{y} \right). \tag{1.45}$$

On equating the coefficients of derivatives of y to zero, we obtain determining equations for infinitesimals as follows:

$$\xi_{yy} + \frac{1}{y}\,\xi_y = 0, \tag{1.46a}$$

$$\eta_{yy} - 2\,\xi_{xy} - \frac{1}{y}\eta_y + \frac{1}{y^2}\eta = 0, \tag{1.46b}$$

$$2\,\eta_{xy} - \xi_{xx} + 3\,y^2\xi_y - \frac{2}{y}\eta_x = 0, \tag{1.46c}$$

$$\eta_{xx} - y^2(\eta_y - 2\,\xi_x) + 2\,y\eta = 0. \tag{1.46d}$$

The general solution of determining equations (1.46) is given by

$$\xi = c_1 + c_2 x,\ \eta = -2c_2\,y.$$

Thus, we have translational and scaling symmetries for ODE (1.44).

Example 1.1.6. For symmetries of PDEs, we consider [34]

$$u_t = u_x^2. \tag{1.47}$$

The first order prolongation (1.38) of infinitesimal generator is as follows:

$$\begin{aligned} X^{(1)} =& \xi(x,t,u)\frac{\partial}{\partial x} + \tau(x,t,u)\frac{\partial}{\partial t} + \eta(x,t,u)\frac{\partial}{\partial u} \\ &+ \eta^{(1)x}(x,t,u)\frac{\partial}{\partial u_x} + \eta^{(1)t}(x,t,u)\frac{\partial}{\partial u_t}. \end{aligned} \tag{1.48}$$

The invariance criterion (1.43) yields

$$\eta^{(1)t} = 2\,u_x\eta^{(1)x}, \tag{1.49}$$

where $\eta^{(1)x}$ and $\eta^{(1)t}$ are first order prolongations of infinitesimal η given by the formula (1.39). Expanding (1.49) and eliminating u_t using (1.47), we have

$$\begin{aligned} &\eta_t - \xi_t u_x + (\eta_u - \tau_t)u_x^2 - \xi_u u_x^3 - \tau_u u_x^4 \\ &\quad = 2\,u_x(\eta_x + (\eta_u - \xi_x)u_x - (\xi_x + \tau_x)u_x^2 - \tau_u u_x^3). \end{aligned} \tag{1.50}$$

On equating the coefficients of derivatives of u to zero, we get

$$\tau_u = 0, \tag{1.51a}$$

$$\xi_u + 2\,\tau_x = 0, \tag{1.51b}$$

$$\eta_u + \tau_t - 2\,\xi_x = 0, \tag{1.51c}$$

$$\xi_t + 2\,\eta_x = 0, \tag{1.51d}$$

$$\eta_t = 0. \tag{1.51e}$$

On solving determining equations (1.51), we get

$$\begin{aligned} \xi =& -4c_1\,xt - 2c_2\,t + c_4\left(\frac{1}{2}x^2 - 2\,ut\right) + c_6\,x + c_7 - 4c_8\,xu - 2c_9\,u, \\ \tau =& -4c_1\,t^2 + c_4\,xt + c_5\,t + c_8\,x^2 + c_9\,x + c_{10}, \\ \eta =& c_1\,x^2 + c_2\,x + c_3 + c_4\,xu - c_5\,u + 2c_6\,u - 4c_8\,u^2 \end{aligned}$$

1.1.5 Group invariant solutions

We learnt that the system of PDEs (1.42) are invariant under Lie group of transformations (1.35), if and only if, the surfaces (1.42) are invariant under kth-order prolongation (1.38). Here, it is quite interesting that the kth-order prolongation (1.38) reads the system of PDEs (1.42) as surfaces in hyper-space with extended coordinates $(\mathbf{x}, \mathbf{u}, \partial\mathbf{u}, \ldots, \partial^k\mathbf{u})$ and invariance criterion (1.43) is analogous to tangential condition in calculus, where the infinitesimals constitute components of tangential vector field which is in present case is the infinitesimal generator (1.37). All this leads to the construction of special type of solutions for system (1.42), and such solutions are called group invariant or similarity solutions.

Definition 1.1.7. The solutions of system of PDEs (1.42) which are invariant under Lie group of transformations (1.35) are called group invariant solutions. Such solutions can precisely be constructed from any group of transformations by reducing the number of variables in the system.

Thus, the solution $\mathbf{u} = \Theta(\mathbf{x})$ is an invariant solution of system (1.42), if and only if

(i) $X(\mathbf{u} - \Theta(\mathbf{x})) \equiv 0$, when $\mathbf{u} = \Theta(\mathbf{x})$, i.e.

$$\sum_{i=1}^{n} \xi_i(\mathbf{x}, \Theta(\mathbf{x})) \frac{\partial \Theta(\mathbf{x})}{\partial x_i} = \eta(\mathbf{x}, \Theta(\mathbf{x})). \tag{1.52}$$

(ii) $\mathbf{u} = \Theta(\mathbf{x})$ solves system (1.42).

The linear differential equation (1.52) is called invariant surface condition for group invariant solutions resulting from invariance condition (1.43). In following, we briefly outline two methods which aid in construction of group invariant solutions arising from invariant surface condition.

- Invariant form method.
- Direct substitution method.

In the first method, the invariant surface condition (1.52) is written in characteristic form

$$\frac{dx_1}{\xi_1(\mathbf{x}, \mathbf{u})} = \frac{dx_2}{\xi_2(\mathbf{x}, \mathbf{u})} = \cdots = \frac{dx_n}{\xi_n(\mathbf{x}, \mathbf{u})} = \frac{d\mathbf{u}}{\eta(\mathbf{x}, \mathbf{u})}. \tag{1.53}$$

Solutions of (1.53) are n-independent functions $y_1(\mathbf{x}, \mathbf{u}), y_2(\mathbf{x}, \mathbf{u}), \ldots, y_{n-1}(\mathbf{x}, \mathbf{u})$ and $\mathbf{v}(\mathbf{x}, \mathbf{u})$. These independent functions aid in reduction of number of independent variables in system (1.42) by one, so that the invariant solution is written by

$$\mathbf{v}(\mathbf{x}, \mathbf{u}) = \Phi(y_1(\mathbf{x}, \mathbf{u}), y_2(\mathbf{x}, \mathbf{u}), \ldots, y_{n-1}(\mathbf{x}, \mathbf{u})), \tag{1.54}$$

where $y_1, y_2, \ldots, y_{n-1}$ are called similarity variables and $\mathbf{v}$ as reduction field. On the other hand, when the characteristics equations (1.53) are difficult to solve then second method is useful. In second method, the invariant surface condition (1.52) is written as

$$\frac{\partial \mathbf{u}}{\partial x_n} = \frac{\eta(\mathbf{x}, \mathbf{u})}{\xi_n(\mathbf{x}, \mathbf{u})} - \sum_{i=1}^{n-1} \frac{\xi_i(\mathbf{x}, \mathbf{u})}{\xi_n(\mathbf{x}, \mathbf{u})} \frac{\partial \mathbf{u}}{\partial x_i}, \ \xi_n(\mathbf{x}, \mathbf{u}) \neq 0. \tag{1.55}$$

It follows from (1.55) that derivatives of $\mathbf{u}$ with respect to x_n can be eliminated from system (1.42) and it is quite evident that the procedure in direct substitution method avoids integration of characteristic equations (1.53). Therefore, the direct method is computationally more efficient than invariant form method. Nevertheless, both of these methods efficiently yield group invariant solutions for system (1.42) subjected to its invariance under Lie group of transformations (1.35).

Example 1.1.7. In this example, we illustrate both of the above methods for construction of group invariant solutions for equation discussed in Example 1.1.6. For infinitesimals

$$\xi = -4\,xt,\ \tau = -4\,t^2,\ \eta = x^2,$$

the invariant surface condition (1.52) may be written as follows:

$$-4\,xt\frac{\partial u}{\partial x} - 4\,t^2\frac{\partial u}{\partial t} = x^2. \tag{1.56}$$

(i) Invariant form method. The characteristics equations (1.53) may be written as follows:

$$\frac{dx}{-4\,xt} = \frac{dt}{-4\,t^2} = \frac{du}{x^2}. \tag{1.57}$$

Solution of characteristics equations (1.57) gives two invariants

$$\zeta = \frac{x}{t},\ v = u + \frac{x^2}{4\,t} \tag{1.58}$$

from which we obtain following invariant form

$$u = -\frac{x^2}{4\,t} + \phi\left(\frac{x}{t}\right). \tag{1.59}$$

Substitution of (1.59) into (1.47) gives

$$\phi'\left(\frac{x}{t}\right) = 0, \tag{1.60}$$

which gives group invariant solution for PDE (1.47)

$$u = -\frac{x^2}{4\,t} + c. \tag{1.61}$$

(ii) Direct substitution method. We re-write the invariant surface condition (1.56) as follows:

$$\frac{\partial u}{\partial t} = -\frac{1}{4\,t^2}\left(x^2 + 4\,xt\frac{\partial u}{\partial x}\right). \tag{1.62}$$

On substitution into the equation (1.47) gives

$$\left(\frac{\partial u}{\partial x}\right)^2 + \frac{1}{4\,t^2}\left(x^2 + 4\,xt\frac{\partial u}{\partial x}\right) = 0. \tag{1.63}$$

The general solution of ODE (1.63) is given by

$$u = -\frac{x^2}{4\,t} + \phi(t). \tag{1.64}$$

Substitution of (1.64) will give $\phi'(t) = 0$, which in turn gives same group invariant solution (1.61) for PDE (1.47).

1.1.6 Lie algebra and its classification techniques

Let G be a Lie group of transformations (1.35), then there exists finite dimensional vector space of fields precisely given by the infinitesimals generators (1.37), this vector space is called Lie algebra. In following, we get familiarize with Lie algebra in formal manner.

Definition 1.1.8. Consider Lie group of transformations (1.35) and infinitesimal generators X and Y defined by (1.37), the commutator of infinitesimal generators X and Y is defined by

$$[X, Y] = XY - YX. \tag{1.65}$$

Then, Lie algebra $\mathscr{L}$ is the vector space of all infinitesimal generators of the type (1.37) together with the bilinear operation $[.,.] : \mathscr{L} \times \mathscr{L} \to \mathscr{L}$ called Lie bracket which satisfies following properties.

(i) Bilinearity.

$$[\alpha X + \beta Y, Z] = \alpha[X, Z] + \beta[Y, Z], \tag{1.66}$$

for reals $\alpha, \beta \in \mathbb{R}$.

(ii) Skew-symmetry.

$$[X, Y] = -[Y, X] \tag{1.67}$$

(iii) Jacobi identity.

$$[X, [Y, Z]] + [Z, [X, Y]] + [Y, [Z, X]] = 0, \tag{1.68}$$

for all $X, Y, W \in \mathscr{L}$.

Definition 1.1.9. A vector sub-space $\mathscr{M}$ is said to be subalgebra of $\mathscr{L}$, if it is closed under Lie bracket and $[X, Y] \in \mathscr{M}$ for all $X \in \mathscr{M}$ and $Y \in \mathscr{L}$.

Theorem 1.1.5. (**Lie's second theorem**.) Let $\mathscr{L}$ be any r-dimensional Lie algebra generated from Lie group of transformations (1.35) with basis $\{X_1, \ldots, X_r\}$, then there exist constants C_{ij}^k called structure constants, such that

$$[X_i, X_j] = \sum_{k=1}^{r} C_{ij}^k X_k, \; i, j = 1, \ldots, r, \tag{1.69}$$

again belong to $\mathscr{L}$.

Proof. The theorem is proved in [86]. □

It is pertinent to mention that the complete information about Lie algebra $\mathscr{L}$ is stored in structure constants C_{ij}^k i.e. if structure constants are known then Lie algebra $\mathscr{L}$ can be completely recovered from (1.69).

Definition 1.1.10. (Commutator table). Let $\mathscr{L}$ be a r-dimensional Lie algebra, then commutator table is the table containing all the structure constants shown

in relation (1.69). For example, consider Lie algebra of famous Korteweg-de Vries equations [175] spanned by vector fields

$$\begin{aligned} X_1 &= \frac{\partial}{\partial x}, && \text{space translation,} \\ X_2 &= \frac{\partial}{\partial t}, && \text{time translation,} \\ X_3 &= t\,\frac{\partial}{\partial x} + \frac{\partial}{\partial u}, && \text{Galilean boost,} \\ X_4 &= x\frac{\partial}{\partial x} + 3\,t\frac{\partial}{\partial t} - 2\,u\frac{\partial}{\partial u}, && \text{scaling.} \end{aligned} \tag{1.70}$$

$[.,.]$	X_1	X_2	X_3	X_4
X_1	0	0	0	X_1
X_2	0	0	X_1	$3\,X_2$
X_3	0	$-X_1$	0	$-2\,X_3$
X_4	$-X_1$	$-3\,X_2$	$2\,X_3$	0

Table 1.1: Commutator table.

Thus, for example, structure constants are given as follows:

$$C^1_{14} = 1 = C^1_{41},\ C^1_{23} = 1 = C^1_{32},$$
$$C^2_{24} = 3 = C^2_{42},\ C^3_{34} = -2 = C^3_{43}$$

Definition 1.1.11. (Solvable Lie algebra). A r-dimensional Lie algebra $\mathscr{L}$ is solvable if there exist a chain subalgebras

$$\mathscr{L}_1 \subset \mathscr{L}_2 \subset \ldots \subset \mathscr{L}_{r-1} \subset \mathscr{L}_r = \mathscr{L}, \tag{1.71}$$

where suffix denotes the dimension of Lie algebra. The existence of solvability of Lie algebra can also be realised from commutator table i.e.

$$[X_i, X_j] = \sum_{k=1}^{j-1} C^k_{ij} X_k,\ i < j.$$

The term solvability of Lie algebra plays important role in reduction of equation under multi-parameter Lie group of transformations (1.36). It has been observed in several textbooks [34, 86, 175, 177] that the invariance of n-th order equation under r-parameter group will not always imply reduction to $(n-r)$-th order equation. However, if the algebra is solvable then reduction to $(n-r)$-th order is achievable.
Theorem 1.1.6. (**Lie's third theorem**.) The structure constants obey

(i) Skew-symmetry.

$$C^k_{ij} = -C^k_{ji}. \tag{1.72}$$

(ii) Jacobi-Identity.

$$\sum_{k=1}^{r} \left[C^k_{ij}C^m_{kj} + C^k_{li}C^m_{kj} + C^k_{jl}C^m_{ki}\right] = 0. \tag{1.73}$$

Proof. The theorem is proved in [86]. □

As mentioned in [86], that there is a one-way connection between Lie groups and Lie algebras i.e. there corresponds a Lie algebra for every Lie group, but the both way connection can only be realised from the following theorems.
Theorem 1.1.7. Let $\mathscr{L}$ be a r-dimensional abstract Lie algebra over field $\mathbb{R}$. Then there is a simply connected r-dimensional Lie group G whose Lie algebra is isomorphic to $\mathscr{L}$.

Proof. The theorem is proved in [86]. □

This mean, the correspondence between Lie groups and Lie algebras is not 1-1. That is, many Lie groups may have same Lie algebra. But among all the Lie groups with same Lie algebra only one would be simply connected, therefore, there is a 1-1 correspondence between simply connected Lie groups and Lie algebras. This suggests that the classification problem for Lie groups is reduced to the classification of underlying Lie algebra.

Why Lie algebra classification is needed? In Section 1.1.5, we have discussed about construction of group invariant solutions by solving invariant surface condition (1.42), and we have seen that any sub-group of full group G is capable of generating group invariant solution. As we can see, there may be infinite number of such sub-groups and hence infinite number of group invariant solutions. Unfortunately, most of such group invariant solutions would be related through some transformation in full group, that is, equivalent group invariant solutions! To resolve this situation, the Lie algebra classification (or simply group classification) is needed.

It is interesting to note that vector field (1.37) is capable of generating Lie group of point transformations through exponentiation

$$\tilde{\Psi} = \exp(\epsilon X)\,\Psi \tag{1.74}$$

which is actually original Lie group of point transformations (1.36). Now we come to inequivalent group invariant solutions: We see that the two group invariant solutions Θ_1 and Θ_2 are essentially different if it is impossible to connect them through group transformation (1.74). The collection of all group-invariant solutions, therefore, separates into equivalence classes, and we seek a minimal list of group generators in the simplest form that will span these classes. Such a list is called an *optimal system* [53].

Let Θ be a group invariant solution which is invariant under sub-group $H \subset G$ and $h\Theta$ be the corresponding action of transformation (1.74) for $h \in H$, with generators from sub-algebra $\mathscr{H} \subset \mathscr{L}$. Since Θ is invariant under sub-group H, therefore, $\Theta = h\Theta$ for all $h \in H$. Consider transformed solution $\tilde{\Theta} = g\Theta$, for $g \in G$. For $g \neq h$, it is quite easy to see that $\tilde{\Theta} \neq \Theta$, that is, $\tilde{\Theta}$ and Θ are inequivalent group invariant solutions. So, one may ask, under which sub-group of G, the solution $\tilde{\Theta}$ is invariant? This would be equivalent to finding a sub-group $K \subset G$ such that $\tilde{\Theta} = k\tilde{\Theta}$ for all $k \in K$. This gives

$$\tilde{\Theta} = g\Theta = gh\Theta = ghg^{-1}(g\Theta) = ghg^{-1}\tilde{\Theta}$$

which implies

$$k = ghg^{-1}.$$

Therefore, Θ is invariant under sub-group H and $\tilde{\Theta}$ is invariant sub-group K with the property that $K = \{ghg^{-1}; g \in G, h \in H\}$. This sub-group is called adjoint group or conjugate group. All this concludes that the two group invariant solutions are essentially different if they are invariant under conjugate groups or adjoint groups. This motivates the introduction of adjoint actions, if one wants to carry out the classification of Lie algebra instead of Lie groups. Needless to mention, classifications of Lie algebra and Lie groups would be equivalent.

Definition 1.1.12. Let $X, Y \in \mathscr{L}$, then the adjoint transformation is defined as

$$\mathrm{Ad}_{\exp(\epsilon X)}(Y) = \exp(-\epsilon X) Y \exp(\epsilon X), \tag{1.75}$$

where the adjoint transformation (1.75) may be reconstructed using Campbell-Hausdorff formula as

$$\mathrm{Ad}_{\exp(\epsilon X)}(Y) = Y - \epsilon[X, Y] + \frac{\epsilon^2}{2}[X, [X, Y]] - \ldots, \tag{1.76}$$

where $[.,.]$ is the Lie bracket defined by (1.65). The relation (1.76) also defines infinitesimal adjoint action [86]

$$ad_X(Y) = \frac{d}{d\epsilon}\mathrm{Ad}_{\exp(\epsilon X)}(Y)\Big|_{\epsilon=0} = [Y, X], \tag{1.77}$$

where $[.,.]$ is usual Lie bracket and $\mathrm{Ad}_{\exp(\epsilon X)}(Y)$ can be recovered from infinitesimal adjoint $ad_X(Y)$ by exponentiation. The matrix version of $ad_X(Y)$ may also be written as

$$(ad_X(Y))_{ij} = \sum_{k=1}^{r} C_{ij}^k X_k, \; i, j = 1, \ldots, r, \tag{1.78}$$

which defines symmetric bilinear form $K : \mathscr{L} \times \mathscr{L} \to \mathbb{R}$ defined by

$$K\langle X, Y\rangle = trace(ad(X) \circ ad(Y)),$$

is called Killing's form. The Killing form has an interesting property that it remains invariant under full adjoint action, and interestingly this property prevents the classification from being wild.

In present study, we have focussed on two independent Lie algebra classification techniques.

(i) Olver's technique.

(ii) Patera-Winternitz's technique.

The Olver's approach [175] consists of applying successive adjoint transformations on a general element in Lie algebra $\mathscr{L}$ and to simplify it as much as possible. On contrary to Olver's technique which relies only on fragments of the theory of Lie algebras, the approach of Patera-Winternitz [183] is very much sophisticated, and they have classified all real Lie algebras of dim ≤ 4 under group of inner automorphism [182]. Since the isomorphic Lie algebras have same classifications, so based on Lie algebra isomorphism, we have demonstrated the use of already existing classifications for Lie algebras with dim ≤ 4 in Chapter 6 and Chapter 7. The rest of the detail about the construction of optimal systems using Killing's form which is invariants of full adjoint action has been discussed in Chapters 2 and 3.

1.2 Notion of integrability

The integrability of PDEs is quite an elusive term. The exact solvability and globally regular solutions are popular beliefs for integrability of nonlinear systems, whereas, in contrast, the non-integrability implies that the system can not be solved exactly and has irregular solutions [74]. The Painlevé property, Bäcklund transformations, Hamiltonian structures, Lax pairs, infinitely many symmetries, the existence of multi-soliton solutions and an infinite set of conservation laws are main indicators of the complete integrability of the nonlinear systems. Although none of the indicators is sufficient to claim for complete integrability, it depends, in which context, the integrability is being sought. Nevertheless, Painlevé property, Bäcklund transformations, Lax pair, soliton solutions and the existence of the infinite conservations laws are strong indicators of complete integrability.

In the series of papers [247, 248, 250], it has been proved beyond doubt that the Painlevé property is powerful tool for investigating integrable properties of the system. One can derive all integrable properties like Bäcklund transformations, Lax pairs, soliton solutions and even conservation laws just by studying Painlevé property of the system. In Chapter 8, we have described the construction of all these integrable properties by using truncated Laurent expansion arising from Painlevé analysis.

1.2.1 Painlevé analysis

In 1887, Picard posed a problem that what is form of second order ODE whose solution has no movable singularity other than poles? This problem was solved by Painlevé in 1888. He had given fifty types of second order ODE admitting said property and out of fifty, forty-four are solvable in term of known functions and remaining six defines new transcendental functions, these six type of nonlinear equations are popularly known as Painlevé type equations [4]. It has been observed by several researchers that, the similarity reductions of several important mathematical equations like; Korteweg-de Vries equation, modified Korteweg-de Vries equation, Sine-Gordon equation, and Boussinesq equation are of Painlevé type. This made the researchers like; Ablowitz, Ramani, Segur, Hasting, and Mcleod [5–8] to conjecture:

> *Every ordinary differential equation which arises as a similarity reduction of a completely integrable partial differential equation is of Painlevé type, perhaps after a transformation of variables.*

This conjecture serves as a useful tool for testing Painlevé property of PDEs, that is, if the similarity reduction of the PDE to an ordinary differential equation, which is not of Painlevé type, then the equation would not be completely integrable. Unfortunately, the process of similarity reduction, though is algorithmic, is a kind of wasteful adventure, since one can not check all the possible similarity reductions for Painlevé testing. Instead, the algorithm of Weiss-Tabor-Carnevale [250] is quite straightforward and avoid undesirable similarity reductions. We shall discuss this algorithm in following sections, but before that, we need to define some basic terms used in the algorithm.

Definition 1.2.1. [56]. Many types of singularities can arise in the study of differential equations. The singularities which occur in the solutions of a linear differential equation can be predicted by inspection of the coefficients of the equation. Their positions are fixed by the singularities of the coefficients. However, for nonlinear differential equations, singularities can "spontaneously evolve", with positions depending upon constants of integration and, therefore, upon initial conditions. Such singularities are called movable singularities.

For example, solutions of the hypergeometric equation

$$x(1-x)\frac{d^2y}{dx^2} + [c-(a+b+1)x]\frac{dy}{dx} - ab\,y = 0,$$

has fixed singularities at the coefficients $x = 0, 1$, and ∞. It is a general property of linear ordinary differential equations that any singularities of their solutions are fixed [4].

But for nonlinear differential equations, the situation is completely different. For example, consider an ordinary differential equation

$$\frac{dy}{dx} + y^2 = 0,$$

for which, the general solution is

$$y(x) = \frac{1}{x - x_0},$$

where the constant of integration $x = x_0$ is a singularity, which is actually a pole. Since the location of singularity depends on the choice of integration constant which can be anywhere in the plane, therefore, the singularity is movable. Needless to mention, the singularity of nonlinear equations can be of both types, movable as well as fixed. We give two more examples from Ref. [141]

$$\frac{dy}{dx} - \exp(y) = 0, \tag{1.79a}$$

$$\frac{dy}{dx} + y(\log y)^2 = 0. \tag{1.79b}$$

In case of ODE (1.79a), the general solution is $y = \log(x - x_0)$, that is, a movable logarithmic branch, and in second case (1.79b), the general solution is $y = \exp\left(\frac{1}{x-x_0}\right)$ and therefore movable essential singularity.

So, we have a definition of Painlevé property.

Definition 1.2.2. [249]. An ordinary differential equation is said to have Painlevé property when every solution is single-valued, except at the fixed singularities of the coefficients. That is Painlevé property requires that the movable singularities are no worse than poles.

Weiss-Tabor-Carnevale algorithm. The Painlevé analysis developed by Weiss-Tabor-Carnevale [250] is most powerful, yet algorithmic procedure, for verifying

Painlevé property of PDEs. The procedure involves expansion of dependent variable in a Laurent expansion about singularity manifold. The expansion about singular manifold is so useful that the procedure gives rise to Lax pairs and Bäcklund transformations in a straightforward manner. Subsequently, the Hirota formulation through truncated Laurent expansion can also be followed for soliton solutions. Let the nonlinear PDE be defined by

$$R(\mathbf{x}, \partial^k u) = 0 \tag{1.80}$$

where $\partial^k u$ defines all partial derivatives of u with respect to $\mathbf{x} = (x_1, \ldots, x_n)$ up to order k. The algorithm starts with assuming Laurent expansion

$$u = \phi^\alpha \sum_{j=0}^{\infty} u_j \, \phi^j \tag{1.81}$$

of dependent variable $u = u(\mathbf{x})$ about singular manifold $\phi(\mathbf{x}) = 0$, where $\mathbf{x} = (x_1, \ldots, x_n)$. Substitution of (1.81) into (1.80) requires

(i) α is an integer.

(ii) ϕ is analytic function of $\mathbf{x} = (x_1, \ldots, x_n)$.

(iii) The equations for u_j have self-consistent solutions.

If the above conditions are proved, then there will be a valid Laurent expansion (1.81) and hence affirmative Painlevé property. In following, we describe the algorithmic procedure of Weiss-Tabor-Carnevale in a stepwise manner for PDEs.

Step 1.2.1. Leading order behavior.

Typically, this algorithm relies on the dominant behavior of solution in the neighborhood of singularity manifold and it should be sufficiently small in the form of

$$u(\mathbf{x}) = u_0 \phi^\alpha. \tag{1.82}$$

On substituting (1.82) into (1.80) and balancing most dominant terms gives u_0 and α. As an illustrative example, take the case of famous KdV equation from Ref. [250]

$$u_t + u u_x + \sigma \, u_{xxx} = 0. \tag{1.83}$$

Substituting the ansätz (1.82) directly into (1.83) and balancing the most dominant terms uu_x and u_{xxx}, gives the leading order $\alpha = -2$ and $u_0 = -12\sigma \, \phi_x^2$. For highly nonlinear equations, one may expect different leading orders called branches and the single valuedness of Laurent expansion (1.81) must be tested for every branch separately.

Step 1.2.2. Determining resonance.

The resonance points are the values of j for which arbitrary functions enter in the Laurent expansion (1.81). Determination of such points starts with substitution of

$$u = u_j \phi^{j-2} \tag{1.84}$$

into (1.83) followed by collecting coefficients of power ϕ^j, all this yields

$$(j+1)(j-4)(j-6)u_j = F(u_{j-1}, \ldots, u_0, \phi_t, \phi_x, \phi_{xx}, \ldots). \tag{1.85}$$

The relation (1.85) suggests that, u_j will be arbitrary for $j = -1, 4, 6$ and these points are called resonance points and the resonance $j = -1$ signifies the arbitrariness of singularity manifold $\phi = 0$.

Step 1.2.3. Checking compatibility conditions.

For the valid Laurent expansion (1.81), the relations of u_j for $j = 4, 6$ must vanish identically. However, if the equations for u_j are not consistent at resonant points, then to make (1.81) a valid expansion, we need to introduce logarithmic term of the type $\phi^j \log \phi$, but all this will lead to a multi-valued phi-series. For the present equation (1.83), which we have taken for illustration purpose, the relation (1.85) yields

$$\begin{aligned}
&j = 0, \quad u_0 = -12\sigma\, \phi_x^2, \\
&j = 1, \quad u_1 = 12\sigma\, \phi_{xx}, \\
&j = 2, \quad \phi_x \phi_t + \phi_x^2 u_2 + 4\sigma\, \phi_x \phi_{xxx} - 3\sigma\, \phi_{xx}^2 = 0, \\
&j = 3, \quad \phi_{xt} + \phi_{xx} u_2 - \phi_x^2 u_3 + \sigma\, \phi_{xxx} = 0, \\
&j = 4, \quad (\phi_{xt} + \phi_{xx} u_2 - \phi_x^2 u_3 + \sigma\, \phi_{xxx})_x = 0.
\end{aligned}$$

The last equation shows that the compatibility condition at $j = 4$ is identically satisfied and compatibility condition at resonance $j = 6$ can be verified by setting $\phi(x, t) = x - \psi(t)$. This assumption is in accordance with the implicit function theorem on $\phi(x, t) = 0$, that is, x can be expressed as a function of t. This assumption drastically simplifies calculations for compatibility testing.

Step 1.2.4. Painlevé property $\Rightarrow$ Integrability. This is the final step for proving Painlevé property and hence integrability of the PDE under study. We need to look back to see if the following conditions are qualified by the equation.

(i) The leading order should be an integer, and in the case of the rational order, the equation should be suitably transformed so as to make leading order integer. If there is irrational leading order, then the equation will be non-Painlevé and hence non-integrable. In such case, there will be complicated singularity structure in the complex domain.

(ii) In a case of an integral leading order, the sufficient number of arbitrary functions must enter into the Laurent expansion (1.81).

(iii) The above conditions should hold for every branch.

Subject to the fulfillment of above-mentioned conditions, the PDE under study will be completely integrable in the context of Painlevé property. If the Painlevé property is not proved, then one can explore equation for Hirota's bilinear formulation and Bäcklund transformations through truncated Laurent expansion. We shall also demonstrate the construction of exact solutions from the invariant Painlevé analysis. All this, we shall discuss in Section 1.2.2, Section 1.2.3 and Section 1.3.1.

1.2.2 Hirota's bilinear formalism[2]

In soliton theory, there are two popular methods to investigate nonlinear evolution equations with soliton properties: The inverse scattering transform (IST) [82]

[2]In writing this section we benefitted from textbook [164] and references therein.

and the Hirota direct method [102]. The IST is a powerful tool to analyze the integrability of equation, but it uses powerful analytical methods and makes strong assumptions about the equation. On the other hand, the Hirota's direct method does not require heavy machinery of analytical mathematics and yields multi-soliton solutions in a straightforward manner. In following, we define some basic terms used in Hirota's direct method, and for simplicity in presentation, we confined ourself to (x,t)-space only.

Definition 1.2.3. The Hirota's bilinear operator is a bilinear D-operator which acts on the pair of function $f(x,t)$ and $g(x,t)$, and is defined by

$$D_x^m D_t^n\{f\cdot g\} = (\partial_x - \partial_{x'})^m(\partial_t - \partial_{t'})^n\Big|_{x=x',t=t'}, \tag{1.86}$$

where $f(x,t)$ and $g(x,t)$ are differentiable functions and m,n are natural numbers. The definition of D-operator can be extended to include any number of dimensions and its action is similar to Leibnitz rule except for crucial negative sign.

Below are some interesting observations for D-operator [105].

(i) $D_x^m D_t^n\{f\cdot g\} = D_t^n D_x^m\{f\cdot g\} = D_x^{m-1}D_t^n D_x\{f\cdot g\}$.

(ii) $D_x^m D_t^n\{f\cdot 1\} = D_x^m D_t^n\{1\cdot f\} = \partial_x^m\partial_t^n f$.

(iii) $D_x^m D_t^n\{f\cdot g\} = (-1)^{m+n}D_x^m D_t^n\{g\cdot f\}$

(iv) $D_x^m D_t^n\{f\cdot f\} = 0$, if and only if, $m+n=$ odd,

furthermore, for arbitrary independent variable x, a recursive definition of D-operator is

$$D_x^n\{f\cdot g\} = \begin{cases} fg & ;\text{for } n=0,\\ D_x^{n-1}\{f_x\cdot g\} - D_x^{n-1}\{f\cdot g_x\} & ;\text{for integer } n.\end{cases}$$

Based on above recursive formula and general definition (1.86) of D-operator, we give illustration of the operation

$$\begin{aligned} D_x\{f\cdot g\} &= f_x g - f g_x,\\ D_x D_t\{f\cdot g\} &= f_{xt}g - f_x g_t - f_t g_x + f g_{xt}.\end{aligned}$$

Theorem 1.2.1. The following formula is fundamental property of Hirota's D-operator:

$$\exp(\epsilon\, D_x)\{a(x)\cdot b(x)\} = a(x+\epsilon)b(x-\epsilon) \tag{1.87}$$

Proof. We prove this formula as follows:

$$\begin{aligned} &\exp(\epsilon\, D_x)\{a(x)\cdot b(x)\}\\ &\qquad = \sum_{n=0}^{\infty}\frac{\epsilon^n}{n!}\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^n a(x)b(x')\Big|_{x=x'} \quad \text{by definition (1.86)}\\ &\qquad = \sum_{n=0}^{\infty}\frac{\epsilon^n}{n!}\sum_{s=0}^{n}(-1)^{n-s}\binom{n}{s}\frac{\partial^s a}{\partial x^s}\frac{\partial^{n-s} b}{\partial x^{n-s}}\\ &\qquad = \sum_{s=0}^{\infty}\sum_{n=s}^{\infty}\frac{(-1)^{n-s}\epsilon^n}{s!\,(n-s)!}\frac{\partial^s a}{\partial x^s}\frac{\partial^{n-s} b}{\partial x^{n-s}}, \text{ exchanging order of sum}\end{aligned}$$

$$= \sum_{s=0}^{\infty} \frac{\epsilon^s}{s!} \frac{\partial^s a}{\partial x^s} \sum_{n=0}^{\infty} \frac{(-\epsilon)^n}{n!} \frac{\partial^n b}{\partial x^n}, \text{ replacing } n \text{ by } n+s$$
$$= a(x+\epsilon) b(x-\epsilon).$$

Corollary 1.2.2. Setting $a(x) = \exp(p_1\, x)$ and $b(x) = \exp(p_2\, x)$ and comparing power ϵ^n in (1.87), yields an interesting result

$$D_x^n \{\exp(p_1\, x) \cdot \exp(p_2\, x)\} = (p_1 - p_2)^n \exp(p_1 + p_2)x. \tag{1.88}$$

The result (1.88) can easily be extended in multiple dimension. Consider $F(D_x, D_t)$ as a polynomial in D-operator

$$F(D_x, D_t)\{\exp(p_1\, x + q_1\, t) \cdot \exp(p_2\, x + q_2\, t)\}$$
$$= F(p_1 - p_2, q_1 - q_2) \exp[(p_1 + p_2)\, x + (q_1 + q_2)\, t]. \tag{1.89}$$

□

The formula (1.89) is very useful for deriving N-soliton solutions. There are several other formulae relating D-operator in the different forms; those formulae can be seen in Ref. [103].

Bilinearization. The process of transforming nonlinear equation into an equation in Hirota's D-operator is called Bilinearization. We illustrate this process by taking example of famous KdV equation

$$u_t + 6\, u u_x + u_{xxx} = 0. \tag{1.90}$$

The Painlevé analysis of the similar equation in Section 1.2.1 suggests the following dependent variable transformation

$$u(x,t) = 2\, (\log \phi)_{xx}, \tag{1.91}$$

which can be obtained from Laurent expansion (1.81) by truncating at constant level. Such truncation read as

$$u(x,t) = \frac{u_0}{\phi^2} + \frac{u_1}{\phi} + u_2, \tag{1.92}$$

where $u_0 = -2\, \phi_x^2$ and $u_1 = 2\, \phi_{xx}$, setting $u_2 = 0$ as seed solution gives (1.91). Substituting (1.91) into (1.90) and integrating with respect to x, we obtain

$$\underbrace{\phi_{xt}\phi - \phi_x \phi_t}_{D_x D_t\{\phi\cdot\phi\}} + \underbrace{\phi_{xxxx}\phi - 4\, \phi_{xxx}\phi_x + 3\, (\phi_{xx})^2}_{D_x^4\{\phi\cdot\phi\}} = 0, \tag{1.93}$$

where the constant of integration is set to zero. The equation (1.93) transforms to bilinear equation for KdV equation (1.90), which is given by

$$(D_x D_t + D_x^4)\phi \cdot \phi = 0. \tag{1.94}$$

In following, we give bilinear equations of some important equations in mathematical physics.

(i) The Kadomtsev-Petviashvili (KP) equation:

$$(u_t - 6\,uu_x + u_{xxx})_x + 3\,u_{yy} = 0$$

dependent variable transformation $u = -2\,(\log\phi)_{xx}$ gives

$$\underbrace{\phi_{xt}\phi - \phi_x\phi_t}_{D_xD_t\{\phi\cdot\phi\}} + \underbrace{\phi_{xxxx}\phi - 4\,\phi_{xxx}\phi_x + 3\,(\phi_{xx})^2}_{D_x^4\{\phi\cdot\phi\}} + \underbrace{3\,\phi_{yy} - 3\,\phi_y^2}_{3\,D_y^2\{\phi\cdot\phi\}} = 0.$$

The Hirota bilinear form of the KP is as follows:

$$(D_xD_t + D_x^4 + 3\,D_y^2)\{\phi\cdot\phi\} = 0.$$

(ii) The Boussinesq equation:

$$u_{tt} - u_{xx} - 3\,(u^2)_{xx} - u_{xxxx} = 0,$$

dependent variable transformation $u = 2\,(\log\phi)_{xx}$ gives

$$\underbrace{3\,\phi_{tt} - 3\,\phi_t^2}_{D_t^2\{\phi\cdot\phi\}}\underbrace{-3\,\phi_{xx} + 3\,\phi_x^2}_{-D_x^2\{\phi\cdot\phi\}}\underbrace{-2\,\phi_{xxxx}\phi + 8\,\phi_{xxx}\phi_x - 6\,\phi_{xx}^2}_{-D_x^4\{\phi\cdot\phi\}} = 0.$$

The Hirota bilinear form of the Boussinesq equation is as follows:

$$(D_t^2 - D_x^2 - D_x^4)\{\phi\cdot\phi\} = 0.$$

(iii) The Sawada-Kotera equation:

$$u_t + 15\,(u^3 + uu_{xx}) + u_{xxxxx} = 0$$

dependent variable transformation $u = 2\,(\log\phi)_{xx}$ gives

$$\underbrace{\phi_{xt}\phi - \phi_x\phi_t}_{D_xD_t\{\phi\cdot\phi\}} + \underbrace{2\,\phi_{xxxxxx}\phi - 12\,\phi_{xxxxx}\phi_x + 30\,\phi_{xxxx}\phi_{xx} - 20\,\phi_{xxx}^2}_{D_x^6\{\phi\cdot\phi\}} = 0.$$

The Hirota bilinear form of the Sawada-Kotera equation is as follows:

$$(D_xD_t + D_x^6)\{\phi\cdot\phi\} = 0.$$

Once the bilinear equation of nonlinear equation is written, one can go for construction of exact soliton solutions in a straightforward manner. For more details, the Section 1.3.2 can be seen.

1.2.3 Bäcklund transformations

In the past couple of decades, a class of transformations discovered by Bäcklund over a century ago has emerged as an important tool in the study of a wide range of nonlinear PDEs in physics. Bäcklund transformations are significant because invariance under Bäcklund transformations can be used to generate infinite sequences of solutions of certain nonlinear equations by purely algebraic procedures called nonlinear superposition principles. Furthermore, Bäcklund transformations can also be used to relate certain nonlinear equations to their canonical forms of the lower order whose properties are well-known [145, 188].

Definition 1.2.4. ([15, 57, 188]) Bäcklund transformation between two PDEs

$$R_1(x,t,u) = 0, \tag{1.95a}$$
$$R_2(x',t',v) = 0, \tag{1.95b}$$

is a pair of relations

$$F_j(x,t,x',t',u,v) = 0,\ j = 1,2 \tag{1.96}$$

with some transformations between (x,t) and (x',t'), in which F_j depends on the derivatives of $u(x,t)$ and $v(x',t')$, such that the elimination of u between (1.96) yield (1.95b) and the elimination of v between (1.96) yield (1.95a). In case the two PDEs (1.95) are same, then Bäcklund transformations are called auto-Bäcklund transformations. The practical use of an auto-Bäcklund transformations is to build a new solution from an old one, even if this seed solution is trivial.

Theorem 1.2.3. The sine-Gordon equation:

$$u_{xt} = \sin u \tag{1.97}$$

is invariant under Bäcklund transformations

$$v_x = F_1(u,u_x,v) = u_x - 2a\,\sin\left[\frac{u+v}{2}\right], \tag{1.98a}$$
$$v_t = F_2(u,u_t,v) = -u_t + \frac{2}{a}\,\sin\left[\frac{u-v}{2}\right],\ (x,t) = (x',t'). \tag{1.98b}$$

Proof. A detailed proof is given in Ref. [15, Chap. 1, pp. 20]. □

The integrability condition

$$\frac{\partial F_1}{\partial t} = \frac{\partial F_2}{\partial x} \tag{1.99}$$

in (1.98) shows that

$$u_{xt} - a\,(u_t + v_t)\cos\left[\frac{u+v}{2}\right] = -u_{xt} + \frac{1}{a}(u_x - v_x)\cos\left[\frac{u-v}{2}\right] \tag{1.100}$$

which after some calculations gives sine-Gordon equation (1.97). In following, we give some more examples of Bäcklund transformations of few important equations in mathematical physics.

Example 1.2.1. The KdV equation

$$u_t + 6\,uu_x + u_{xxx} = 0, \tag{1.101}$$

admits Bäcklund transformations

$$u_x + v_x = a - \frac{1}{a}(u-v)^2, \tag{1.102a}$$
$$u_t + v_t = (u-v)(u_{xx} - v_{xx}) - 2\,(u_x^2 + 2\,u_xv_x + v_x^2). \tag{1.102b}$$

Example 1.2.2. The Liouville equation:

$$u_{xt} + \alpha \exp(u) = 0 \tag{1.103}$$

admits Bäcklund transformations

$$u_x - v_x = a\alpha \exp\left[\frac{u+v}{2}\right], \tag{1.104a}$$

$$u_t + v_t = \frac{2}{a} \exp\left[\frac{u-v}{2}\right]. \tag{1.104b}$$

Beside the classical method [15], the Bäcklund transformations can also be constructed from usual Painlevé analysis and Hirota's direct method.

Bäcklund transformations using Painlevé analysis. The construction of Bäcklund transformations using classical method involves the cumbersome practice of solving determining equations which are sometimes very difficult to integrate, but truncated expansion from Painlevé analysis yields Bäcklund transformations in a straightforward manner. The procedure involves truncation of Laurent expansion (1.81) by suitably setting resonance functions to zero. The details can be seen in the Chapter 4.

Bäcklund transformations using Hirota's direct method. In addition to truncated Laurent expansion, the Bäcklund transformations can also be derived from the bilinear form of nonlinear PDEs. We have discussed such bilinear equations in Section 1.2.2. Once the bilinear equation is constructed, the exchange formulae [103, See formulae VI to VIII-3] can be used to derive Bäcklund transformations. In following, we give some examples of Bäcklund transformations of some well-known equations in mathematical physics.

Example 1.2.3. The Swada-Kotera equation:

$$D_x(D_t + D_x^5)f \cdot f = 0. \tag{1.105}$$

The Bäcklund transformations

$$D_x^3 f' \cdot f' = \lambda\, f' f, \tag{1.106a}$$

$$\left(D_t - \frac{15}{2}\lambda\, D_x^2 - \frac{3}{2}\, D_x^5\right) f' \cdot f = 0. \tag{1.106b}$$

Example 1.2.4. The Boussinesq equation:

$$(D_t^2 - D_x^2 - D_x^4)f \cdot f = 0. \tag{1.107}$$

The Bäcklund transformations

$$(D_t + a\, D_x^2) f' \cdot f = 0, \tag{1.108a}$$

$$\left(a\, D_x D_t + D_x + D_x^3\right) f' \cdot f = 0, \tag{1.108b}$$

where $a^2 = -3$.

Example 1.2.5. Kadomtsev-Petviashvili equation:

$$(-4\,D_xD_t + 3\,D_y^2 + D_x^4)f\cdot f = 0. \tag{1.109}$$

The Bäcklund transformations

$$(D_y - D_x^2)f'\cdot f = 0, \tag{1.110a}$$
$$\left(3\,D_xD_y - 4\,D_t + D_x^3\right) f'\cdot f = 0. \tag{1.110b}$$

In Chapter 8, the bilinear Bäcklund transformations are derived for Jimbo-Miwa equation by employing Bell polynomials.

Application of Bäcklund transformations. One of the most important application of the Bäcklund transformations is the construction of new solutions from known solution. One can construct a sequence of non-trivial solutions even from trivial solution. We start with re-writing the Bäcklund transformations (1.98)

$$\left(\frac{u_n - u_{n-1}}{2}\right)_x = a_n\,\sin\left[\frac{u_n + u_{n-1}}{2}\right], \tag{1.111a}$$
$$\left(\frac{u_n + u_{n-1}}{2}\right)_t = \frac{1}{a_n}\,\sin\left[\frac{u_n - u_{n-1}}{2}\right]. \tag{1.111b}$$

On taking a seed solution $u_0 = 0$, (1.111) leads to a solution

$$u_1 = 4\,\arctan\left[\exp(a\,x + \frac{1}{a}\,t + b)\right], \tag{1.112}$$

where b is a constant of integration. Once the starting solution u_1 is determined, infinite sequence of additional solutions may be generated, the procedure for such construction is described in Ref. [188, Sec. 1.2].

1.2.4 Bell polynomials

In this section, we present some basic concepts and expressions related to Bell polynomials [22, 87]. Later on, we shall see how these polynomials are helpful in constructing bilinear form for equation (8.1).
Let f be a C^∞ function of x and consider $f_n = \partial_x^n f(x)$, then one dimensional Bell polynomial [22] is defined as follows:

$$Y_{nx}(f) = Y_n(f_1, f_2, ..., f_{nx}) = e^{-f}\partial_x^n e^f, \quad n = 1, 2, \ldots, \tag{1.113}$$

from which it follows that

$$Y_x = f_x, \quad Y_{2x} = f_{2x} + f_x^2, \quad Y_{3x} = f_{3x} + 3f_xf_{2x} + f_x^3, \ldots. \tag{1.114}$$

The expressions (1.114) are obtained by partitional representation also known as Faà di Bruno formula defined as follows:

$$Y_{nx}(f) = \sum \frac{n!}{b_1!\,b_2!\ldots b_n!}\left(\frac{f_x}{1!}\right)^{b_1}\left(\frac{f_{2x}}{2!}\right)^{b_2}\ldots\left(\frac{f_{nx}}{n!}\right)^{b_n} \tag{1.115}$$

where sum run over all partitions of $n = b_1 + 2b_2 + \cdots + nb_n$.
To illustrate how this formula (1.115) works, we take for example, $n = 3$ and look for non-negative solutions of $b_1 + 2b_2 + 3b_3 = 3$. We can have $b_3 = 1, b_1 = b_2 = 0$, such that first term of sum (1.115) is

$$\frac{3!}{0!\,0!\,1!}\left(\frac{f_x}{1!}\right)^0\left(\frac{f_{2x}}{2!}\right)^0\left(\frac{f_{3x}}{3!}\right)^1 = f_{3x}. \tag{1.116}$$

The second non-negative solution to partition is $b_1 = b_2 = 1$, $b_3 = 0$, such that the second term of sum (1.115) is

$$\frac{3!}{1!\,1!\,0!}\left(\frac{f_x}{1!}\right)^1\left(\frac{f_{2x}}{2!}\right)^1\left(\frac{f_{3x}}{3!}\right)^0 = 3f_x f_{2x}. \tag{1.117}$$

Third and only other solution to partition is $b_1 = 3, b_2 = b_3 = 0$, such that the third and last term of sum (1.115) is

$$\frac{3!}{3!\,0!\,0!}\left(\frac{f_x}{1!}\right)^3\left(\frac{f_{2x}}{2!}\right)^0\left(\frac{f_{3x}}{3!}\right)^0 = f_x^3 \tag{1.118}$$

by taking sum of (1.116)-(1.118), the third relation in (1.114) for Bell polynomial can be obtained.
The dimension of Bell polynomials can be extended in similar way by considering multi-variable C^∞ function $f = f(x_1, x_2, ..., x_n)$, the multi-dimensional(or simply generalised) Bell polynomial is defined as follows:

$$Y_{n_1x_1,...,n_lx_l}(f) = Y_{n_1,...,n_l}(f_{r_1x_1,...,r_lx_l}) = e^{-f}\partial_{x_1}^{n_1}...\partial_{x_l}^{n_l}e^f \tag{1.119}$$

where we have considered $f_{r_1x_1,...,r_lx_l} = \partial_{x_1}^{r_1}...\partial_{x_l}^{r_l}f$, $r_i = 0, 1, ..., n_i$ and $i = 1, 2, ..., l$. These polynomials admit generalised Faà di Bruno formula

$$Y_{n_1x_1,...,n_lx_l}(f) = \sum \frac{n_1!\,n_2!\cdots n_l!}{b_1!\,b_2!\cdots b_k!}\prod_{j=1}^{k}\left(\frac{f_{r_{1j},...,r_{lj}}}{r_{1j}!,...,r_{lj}!}\right)^{b_j} \tag{1.120}$$

where sum runs over all partitions $[(r_{11}, ..., r_{l1})^{b_1}, ..., (r_{1k}, ..., r_{lk})^{b_k}]$ of $(n_1, ..., n_l)$. Using partitional formula (1.120), first few Bell polynomials can be written as follows:

$$\begin{aligned} &Y_{x_1}(f) = f_{x_1}, \quad Y_{2x_1}(f) = f_{2x_1} + f_{x_1}^2,\ Y_{x_1,x_2}(f) = f_{x_1,x_2} + f_{x_1}f_{x_2} \\ &Y_{3x_1}(f) = f_{3x_1} + 3f_{x_1}f_{2x_1} + f_{x_1}^3, \\ &Y_{2x_1,x_2} = f_{2x_1,x_2} + f_{2x_1}f_{x_2} + 2f_{x_1,x_2}f_{x_1} + f_{x_1}^2 f_{x_2}, ... \end{aligned} \tag{1.121}$$

Based on the use of multi-dimensional Bell polynomials, the multi-dimensional binary Bell polynomials can be defined as follows:

$$\mathscr{Y}_{n_1x_1,...,n_lx_l}(V, W) = Y_{n_1x_1,...,n_lx_l}(f)|_{f_{r_1x_1,...,r_lx_l}} = \begin{cases} V_{r_1x_1,...,r_lx_l}, & \sum_{i=1}^{l} r_i \text{ is odd;} \\ W_{r_1x_1,...,r_lx_l}, & \sum_{i=1}^{l} r_i \text{ is even.} \end{cases} \tag{1.122}$$

Based on the above definition, the first few lowest order binary Bell polynomials are

$$\begin{aligned}
\mathscr{Y}_{x}(V) &= V_x, \\
\mathscr{Y}_{2x}(V,W) &= W_{2x} + V_x^2, \\
\mathscr{Y}_{x,t}(V,W) &= W_{x,t} + V_x V_t \\
\mathscr{Y}_{3x}(V,W) &= V_{3x} + 3V_x W_{2x} + V_x^3, \\
\mathscr{Y}_{2x,t}(V,W) &= V_{2x,t} + V_t W_{2x} + 2V_x W_{x,t} + V_x^2 V_t \\
\mathscr{Y}_{4x}(V,W) &= W_{4x} + 4V_x V_{3x} + 3W_{2x}^2 + 6V_x^2 W_{2x} + V_x^4, ...
\end{aligned} \tag{1.123}$$

The next identity (see for e.g. [87]) that we are going to present will establish a direct link between the binary Bell polynomials and the standard Hirota D-operator, it is given by

$$\mathscr{Y}_{n_1x_1,...,n_lx_l}(V,W)\Big|_{V=\log\frac{f}{g},W=\log fg} = (fg)^{-1}D_{x_1}^{n_1}\cdots D_{x_l}^{n_l} f\cdot g \tag{1.124}$$

where the D-operator is defined by[105]

$$\begin{aligned}
&D_{x_1}^{n_1}\cdots D_{x_l}^{n_l} f\cdot g = \\
&\quad (\partial_{x_1}-\partial_{x_1'})^{n_1}\cdots(\partial_{x_l}-\partial_{x_l'})^{n_l} f(x_1,...,x_l)\cdot g(x_1',...,x_l')|_{x_1'=x_1,...,x_l'=x_l}
\end{aligned} \tag{1.125}$$

For $f=g$, the identity (1.124) reduces to particular case

$$\begin{aligned}
(f)^{-2}D_{x_1}^{n_1}\cdots D_{x_l}^{n_l} f\cdot f &= \mathscr{Y}_{n_1x_1,...,n_lx_l}(0,q=2\log f) \\
&= \begin{cases} 0, & \sum_{i=1}^{l} n_i \quad \text{is odd} \\ \mathscr{P}_{n_1x_1,...,n_lx_l}(q), & \sum_{i=1}^{l} n_i \quad \text{is even} \end{cases}
\end{aligned} \tag{1.126}$$

where these even ordered $\mathscr{Y}$-polynomials are called $\mathscr{P}$-polynomials and first few of them are given by following formulas

$$\mathscr{P}_{2x}(q) = q_{2x}, \quad \mathscr{P}_{x,t}(q) = q_{x,t}, \quad \mathscr{P}_{3x,t}(q) = q_{3x,t} + 3q_{x,t}q_{2x}, ... \tag{1.127}$$

The binary Bell polynomials $\mathscr{Y}_{n_1x_1,...,n_lx_l}(V,W)$ can be separated into generalised Bell polynomials $Y_{n_1x_1,...,n_lx_l}(V)$ and $\mathscr{P}$-polynomials

$$\begin{aligned}
(fg)^{-1}D_{x_1}^{n_1}\cdots D_{x_l}^{n_l} f\cdot g &= \mathscr{Y}_{n_1x_1,...,n_lx_l}(V,W)\Big|_{V=\log\frac{f}{g},W=\log fg} \\
&= \mathscr{Y}_{n_1x_1,...,n_lx_l}(V,V+q)\Big|_{V=\log\frac{f}{g},q=2\log g} \\
&= \sum_{r_1=0}^{n_1}\cdots\sum_{r_l=0}^{n_l}\prod_{i=1}^{l}\binom{n_i}{r_i}\mathscr{P}_{r_1x_1,...,r_lx_l}(q)Y_{(n_1-r_1)x_1,...,(n_l-r_l)x_l}(V)
\end{aligned} \tag{1.128}$$

It is pertinent to mention that, the generalised Bell polynomial can be linearised by Cole-Hopf transformation $V=\log\psi$ given as follows:

$$Y_{n_1x_1,...,n_lx_l}(V=\log\psi) = \frac{\psi_{n_1x_1,...,n_lx_l}}{\psi}. \tag{1.129}$$

Such that the equation (1.128) can be written as follows:

$$(fg)^{-1}D_{x_1}^{n_1}\cdots D_{x_l}^{n_l}f\cdot g|_{g=\exp(\frac{q}{2}),\frac{f}{g}=\psi}$$
$$=\psi^{-1}\sum_{r_1=0}^{n_1}\cdots\sum_{r_l=0}^{n_l}\prod_{i=1}^{l}\begin{pmatrix} n_i \\ r_i \end{pmatrix}\mathscr{P}_{r_1x_1,\ldots,r_lx_l}(q)\psi_{(n_1-r_1)x_1,\ldots,(n_l-r_l)x_l}. \tag{1.130}$$

Through formula (1.130), one can construct associated Lax system for non-linear equations in shortest possible way. We have used this formula to construct Lax system for Jimbo-Miwa equation in Chapter 8 at (8.15).

1.2.5 Conservation laws

In the study of PDEs, conservation laws play vital role in the identification of integrability and linearization, constants of motion, analysis of solutions, and numerical solution methods. Consequently, an important problem is how to calculate all of the conservation laws for given differential equations [13]. The Noether's approach is the finest and natural technique for finding conservation laws. This approach relies on the setup of suitable Lagrangian for the differential equation, such that, for each Noether symmetry an explicit conservation law can be constructed by a formula. Since this technique completely relies on the existence of Lagrangian, so the scope of application is limited. There exist several other methods for finding conservation laws without the requirement of Lagrangian, and the most reliable approach is the direct method [12–14, 35]. The direct method uses multipliers [175] to give an algorithmic method for finding the local conservation laws for PDEs expressed in a standard Cauchy-Kovalevskaya form.

We have applied the direct method to Date-Jimbo-Kashiwara-Miwa equation in Chapter 4 to construct several local conservation laws. The second method is based on recently given theorem [117] by Ibragimov, and the detailed discussion and applications are presented in Chapter 2, Chapter 4 and Chapter 7. In addition to these two methods, we have also constructed an infinite set of conservation laws from Bäcklund transformations using Bell polynomials in Section 8.3 of Chapter 8.

Besides these three popular methods, there are other methods for constructing conservation laws, like; an approach due to Laplace in which divergence equation is directly solved over solution space of differential equation, approach of Steudel [212] in which the conservation laws are written in characteristics form, variational approach presented in [175], partial Noether approach introduced by Kara and Mahomed [129] etc.

In following, we give some definitions and terms which are essential for the construction of conservation laws using the direct method of multipliers.
Definition 1.2.5. ([36]). A local conservation law for system of PDEs (1.42) is a divergence expression

$$\mathrm{Div}\,\Psi[\mathbf{u}]=D_1\Psi^1[\mathbf{u}]+\cdots+D_n\Psi^n[\mathbf{u}]=0 \tag{1.131}$$

where sub-scripts and super-scripts are corresponding to n-independent variables $\mathbf{x} = (x_1, \ldots, x_n)$ and D_i is total derivative operator defined by (1.41). The components $\Psi^i[\mathbf{u}] = \Psi^i(\mathbf{x}, \mathbf{u}, \partial\mathbf{u}, \ldots, \partial^l\mathbf{u})$ are called fluxes of conservation laws and highest derivative order l is called order of a conservation law. For non-trivial conservation laws, the divergence expression (1.131) must vanish on the solution space of system of PDEs (1.42). The famous KdV equation

$$u_t + uu_x + u_{xxx} = 0, \tag{1.132}$$

admits non-trivial conservation laws

$$D_x\left(\frac{1}{2}u^2 + u_{xx}\right) + D_t(u) = 0, \tag{1.133a}$$

$$D_x\left(\frac{1}{3}u^3 + uu_{xx} - \frac{1}{2}u_x^2\right) + D_t\left(\frac{1}{2}u^2\right) = 0, \tag{1.133b}$$

$$D_x\left(\frac{1}{8}u^4 - uu_x^2 + \frac{1}{2}(u^2u_{xx} + u_{xx}^2) - u_xu_{xxx}\right) + D_t\left(\frac{1}{6}u^3 - \frac{1}{2}u_x^2\right). \tag{1.133c}$$

The conservation laws will be trivial, if

(i) The fluxes $\Psi^i[\mathbf{u}]$ vanishes on the solution space of system of PDEs (1.42),

(ii) Conservation law is just a differential identity.

For example, consider system of PDEs

$$v_x = u,\ v_t = F(u)u_x. \tag{1.134}$$

Conservation law

$$D_x(2(v_t - F(u)u_x)) + D_t(u(u - v_x)) = 0 \tag{1.135}$$

is a trivial conservation law, since each flux vanishes on the solution space. On the other hand, the conservation law

$$D_x(u_{xt}) - D_t(u_{xx}) = 0, \tag{1.136}$$

is just a differential identity and therefore, the conservation law is trivial.

Definition 1.2.6. The two conservation laws $\mathrm{Div}\,\Psi[\mathbf{u}] = 0$ and $\mathrm{Div}\,\Phi[\mathbf{u}] = 0$ are equivalent if $\mathrm{Div}(\Psi[\mathbf{u}] - \Phi[\mathbf{u}]) = 0$ is a trivial conservation law.

The rest of the details about construction of conservation laws using direct method is presented in Section 4.4 of Chapter 4.

Remark 1.2.4. In the present study, we have constructed conservation laws using the direct method, and the method based on new conservation theorem. Working on this thesis, we have learnt from the comments of Anco [11], that, the method based on new conservation theorem by Ibragimov is actually a rephrasing of direct method. Since the Ibragimov's method sometimes gives trivial conservation laws, therefore, among these two methods, we trust on the direct method, and when the Ibragimov's method will give trivial conservation laws is left for future study.

1.3 Exact solutions

1.3.1 Exact solutions from invariant Painlevé analysis

In the Section 1.2.1, we have discussed WTC algorithm for verifying integrability of nonlinear equations by establishing the existence of valid Laurent expansion (1.81). In this section, we discuss application of invariant Painlevé analysis [55] by using Pickering's algorithm [184] to derive exact solutions for the PDE

$$R(x,t,\partial^k u) = 0 \tag{1.137}$$

in independent variables (x,t) and dependent variable u and which is of order k.

In Ref. [55], Conte proved that the whole Painlevé analysis is invariant under the group of homographic transformations of singularity manifold, and

$$\chi = \left(\frac{\phi_x}{\phi} - \frac{\phi_{xx}}{2\phi_x}\right)^{-1} \tag{1.138}$$

proved to the best expansion variable to replace ϕ in the Laurent expansion (1.81). Therefore, the Laurent expansion of Painlevé analysis can be written in the new format

$$u = \chi^{\alpha} \sum_{j=0}^{\infty} u_j \, \chi^j. \tag{1.139}$$

The gradients of χ are

$$\begin{aligned} \chi_x &= 1 + \frac{1}{2} S\chi^2, \\ \chi_t &= -C + C_x\chi - \frac{1}{2}(CS + C_{xx})\chi^2, \end{aligned} \tag{1.140}$$

where

$$S = \frac{\phi_{xxx}}{\phi_x} - \frac{3}{2}\left(\frac{\phi_{xx}}{\phi_x}\right)^2, \; C = -\frac{\phi_y}{\phi_x}. \tag{1.141}$$

The quantities S and C are called *Schwarzian* of ϕ. Best thing about these quantities is that, these are invariant under Möbius group of homographic transformations

$$\phi \mapsto \frac{a\phi + b}{c\phi + d}, \; ad - bc \neq 0, \tag{1.142}$$

where a, b, c, d are arbitrary constants. The cross derivatives condition on Riccati system (1.140) is

$$S_t + C_{xxx} + 2\,SC_x + S_xC = 0. \tag{1.143}$$

Remark 1.3.1. The extension of Riccati system (1.140) to $(2+1)$-dimension can be considered as follows:

$$\begin{aligned} \chi_x &= 1 + \frac{1}{2} S\chi^2, \\ \chi_y &= -C + C_x\chi - \frac{1}{2}(CS + C_{xx})\chi^2, \\ \chi_t &= -K + K_x\chi - \frac{1}{2}(KS + K_{xx})\chi^2, \end{aligned} \tag{1.144}$$

where

$$S = \frac{\phi_{xxx}}{\phi_x} - \frac{3}{2}\left(\frac{\phi_{xx}}{\phi_x}\right)^2,\ C = -\frac{\phi_y}{\phi_x},\ K = -\frac{\phi_t}{\phi_x}, \tag{1.145}$$

and the cross derivative condition yields

$$S_t + K_{xxx} + 2\,SK_x + S_xK = 0 \tag{1.146a}$$
$$S_y + C_{xxx} + 2\,SC_x + S_xC = 0 \tag{1.146b}$$
$$K_y - C_t + CK_x - KC_x = 0. \tag{1.146c}$$

The construction of exact solutions proceed with considering truncated Laurent expansion (1.139) at the leading order α, which is

$$u = \chi^\alpha \sum_{j=0}^{\alpha} u_j\,\chi^j. \tag{1.147}$$

Substituting the truncation (1.147) into (1.137) and vanishing the coefficient of different powers of χ, we can find complicated over-determined system for $u_j,\ j = 0, \ldots \alpha$, S and C. Solving such system along with the following solution

$$\chi(x,t) = \sqrt{\frac{2}{S}}\tan\left[\sqrt{\frac{S}{2}}(x - C\,t + c_1)\right] \tag{1.148}$$

of Riccati system (1.140) will provide exact solution for the PDE (1.137).

1.3.2 Exact soliton solutions

In this section, we shall realize the importance of formula (1.89) for constructing exact soliton solutions from bilinear equations. Consider general bilinear equation

$$F(D)\{\phi\cdot\phi\} = 0, \tag{1.149}$$

where $F(D) = F(D_x, D_t)$ is polynomial in Hirota's D-operator. To construct exact soliton solutions, we take perturbation expansion as suggested in [104].

$$\phi = \sum_{i=1}^{N} \epsilon^i \phi_i,\ \phi_0 = 1. \tag{1.150}$$

Substitution of (1.150) into (1.149) gives

$$\begin{aligned}\epsilon F(D)(\phi_1\cdot 1 + 1\cdot\phi_1) + \epsilon^2 F(D)(\phi_2\cdot 1 + \phi_1\cdot\phi_1 + 1\cdot\phi_2)\\ + \epsilon^3 F(D)(\phi_3\cdot 1 + \phi_2\cdot\phi_1 + \phi_1\cdot\phi_2 + 1\cdot\phi_3)\epsilon^3 + \ldots\\ + \epsilon^n F(D)\sum_{j=0}^{n}\phi_{n-j}\cdot\phi_j + \cdots = 0\end{aligned} \tag{1.151}$$

The aim is to construct exact soliton solutions from relation (1.151), for which, the perturbation expansion (1.150) have to truncate at some fixed level. But this will lead to the approximation of the exact solution. Fortunately, with the suitable choices of $\phi_1, \phi_2, \ldots$, one can force expansion (1.150) to truncate at desired level.

One soliton solution. For one soliton solution, we take

$$\phi_1 = e^{\eta_1},\ \eta_1 = \alpha_1 x + \beta_1 t + \gamma_1, \tag{1.152}$$

where ϕ_1 assumed to be exponential, since the base of the solitary wave is not a superposition of plane waves [105], and parameter α_1 indicates the amplitude of wave and γ_1 is phase constant which determines the position of the wave.

Substituting (1.152) into (1.151) with the help of formula (1.89), the coefficient of ϵ^1 read as

$$2F(D)\{\phi_1 \cdot 1\} = 2F(D)\{e^{\eta_1} \cdot 1\} = 2F(\partial)e^{\eta_1} = 0, \tag{1.153}$$

which gives a relation $F(\alpha_1, \beta_1) = 0$, called as dispersion relation. Similarly, the coefficient of ϵ^2 read as

$$F(D)(\phi_2 \cdot 1 + e^{\eta_1} \cdot e^{\eta_1} + 1 \cdot \phi_2) = 0,$$

which implies

$$2\,F(D)\{\phi_2 \cdot 1\} = -F(D)\{e^{\eta_1} \cdot e^{\eta_1}\} = 0, \tag{1.154}$$

using formula (1.89). Therefore, we may set $\phi_2 = 0$, and detailed calculations shows that the coefficients of $\epsilon^3, \epsilon^4, \ldots$ vanish identically. This completes the construction of one soliton solution, which may be recovered from dependent variable transformation by replacing $\phi = \phi_1$.

Two soliton solution. For two soliton solution, we take

$$\phi_1 = e^{\eta_1} + e^{\eta_2},\ \eta_1 = \alpha_1 x + \beta_1 t + \gamma_1, \eta_2 = \alpha_2 x + \beta_2 t + \gamma_2. \tag{1.155}$$

On substituting (1.155) into (1.151), the coefficient of ϵ^1 read as

$$2F(D)\{\phi_1 \cdot 1\} = 2\,F(\partial)\{e^{\eta_1} + e^{\eta_2}\} = 0, \tag{1.156}$$

which gives dispersion relations $F(\alpha_i, \beta_i) = 0$ for $i = 1, 2$. The coefficient of ϵ^2 reads as

$$\begin{aligned} F(D)\{&\phi_2 \cdot 1 + \phi_1 \cdot \phi_1 + 1 \cdot \phi_2\} \\ &= 2F(\partial)\phi_2 + 2F(D)\{e^{\eta_1} \cdot e^{\eta_2}\} \\ &= 2F(\partial)\phi_2 + 2F(\alpha_1 - \alpha_2, \beta_1 - \beta_2)\, e^{\eta_1 + \eta_2} = 0, \end{aligned} \tag{1.157}$$

where one must have $f_2 = A_{12}\, e^{\eta_1 + \eta_2}$, and on back substituting f_2 into (1.157).

$$A_{12} = -\frac{F(\alpha_1 - \alpha_2, \beta_1 - \beta_2)}{F(\alpha_1 + \alpha_2, \beta_1 + \beta_2)}, \tag{1.158}$$

where A_{12} is phase shift between interacting solitons and coefficients of ϵ^3 is zero and of ϵ^4 vanish identically.

Three soliton solution. The computations related to three soliton solutions are quite tough, so in this case we take

$$\phi_1 = e^{\eta_1} + e^{\eta_2} + e^{\eta_3},\ \eta_i = \alpha_i x + \beta_i t + \gamma_i, \quad i = 1, 2, 3. \tag{1.159}$$

On substituting (1.159) into (1.151), the coefficient of ϵ^1 gives dispersion relations $F(\alpha_i, \beta_i) = 0$ for $i = 1, 2, 3$. The coefficient of ϵ^2 reads as

$$F(D)\{\phi_2 \cdot 1 + 1 \cdot \phi_2 + \phi_1 \cdot \phi_1\} = 2F(\partial)\phi_2 + F(D)\{\phi_1 \cdot \phi_1\} = 0, \tag{1.160}$$

where

$$\begin{aligned} F(D)\left\{\phi_1 \cdot \phi_1\right\} &= F(D)\left\{e^{\eta_1} \cdot e^{\eta_1} + e^{\eta_2} \cdot e^{\eta_2} + e^{\eta_3} \cdot e^{\eta_3}\right\} + 2F(D)\left\{\sum_{i \neq j} e^{\eta_i + \eta_j}\right\} \\ &= \sum_{i=1}^{3} F(p_i - p_i)e^{2\eta_i} + 2F(p_1 - p_2)e^{\eta_1 + \eta_2} \\ &\quad + 2F(p_1 - p_3)e^{\eta_1 + \eta_3} + 2F(p_2 - p_3)e^{\eta_2 + \eta_3}, \end{aligned} \tag{1.161}$$

where $F(p_i - p_j) = F(\alpha_i - \alpha_j, \beta_i - \beta_j)$ for $i, j = 1, 2, 3$. On substituting (1.161) into (1.160), we realize that

$$\phi_2 = A_{12}e^{\eta_1 + \eta_2} + A_{13}e^{\eta_1 + \eta_3} + A_{23}e^{\eta_2 + \eta_3}. \tag{1.162}$$

Therefore, solution of (1.160) yields

$$A_{ij} = -\frac{F(p_i - p_j)}{F(p_i + p_j)}, \tag{1.163}$$

for $i, j = 1, 2, 3$ and $i < j$. Thus, ϕ_2 is completely determined and the coefficient of ϵ^3 yields a relation

$$2F(\partial)\phi_3 + 2F(D)\{\phi_1 \cdot \phi_2\} = 0, \tag{1.164}$$

which after some calculation gives

$$\begin{aligned} -F(\partial)\phi_3 = &[A_{12}F(p_3 - p_1 - p_2) + A_{13}F(p_2 - p_1 - p_3) \\ &+ A_{23}F(p_1 - p_2 - p_3)]e^{\eta_1 + \eta_2 + \eta_3}, \end{aligned} \tag{1.165}$$

from equation (1.165), one can guess that

$$\phi_3 = B\, e^{\eta_1 + \eta_2 + \eta_3}, \tag{1.166}$$

from (1.166), back substituting ϕ_3 into (1.165), we finds that

$$B = -\frac{A_{12}F(p_3 - p_1 - p_2) + A_{13}F(p_2 - p_1 - p_3) + A_{23}F(p_1 - p_2 - p_3)}{F(p_1 + p_2 + p_3)}. \tag{1.167}$$

In order to make the coefficient of ϵ^4 zero, we must have

$$B = A_{12}A_{13}A_{23}. \tag{1.168}$$

Combining (1.167) and (1.168), we obtain three soliton condition

$$A_{12}A_{13}A_{23}F(p_1+p_2+p_3)+A_{12}F(p_3-p_1-p_2) \\ +A_{13}F(p_2-p_1-p_3)+A_{23}F(p_1-p_2-p_3)=0 \tag{1.169}$$

and coefficients of ϵ^5 and ϵ^6 also vanishes identically. The condition (1.169) is sufficient for the existence of three soliton solutions. In principle, the existence of three soliton solution is a week condition for complete integrability [100], however, existence of three soliton solution is strong indicator for existence of N-soliton solutions and existence of N-soliton solutions is sign of complete integrability in context of Hirota's formalism [101](see also [187]).

1.3.3 Riccati equation mapping method

In this section, we give some basic introduction to Riccati equation mapping method. For clear and concise illustration, we take equation in $(1+1)$-dimension

$$R(x,t,u,\partial u,\ldots,\partial^k u)=0, \tag{1.170}$$

which of order k and assume solution of (1.170) in the form

$$u(x,t)=\sum_{i=0}^{n} a_i\phi^i(\xi),\ \xi=\alpha\,x+\beta\,t \tag{1.171}$$

where n is required to be a positive integer, which can be fixed by balancing the highest order linear term with that of nonlinear term in (1.170). However, n may not be positive for all PDEs, nevertheless, by making appropriate transformations in (1.170), desired value of n can be achieved. For example, in case of KdV equation, the order of nonlinear term uu_x is $n+(n+1)$ and of highest order linear term $u_{x,x,x}$ is $n+3$, on equating both these orders $n=2$ can be obtained.

The key point of this method is to take advantage of Riccati equation by taking new variable $\phi(\xi)$ as solution of Riccati equation

$$\phi'(\xi)=r+p\phi(\xi)+q\phi^2(\xi), \tag{1.172}$$

where r,p and q are arbitrary real constants. In contrast to Ref. [268], wherein (1.172) is wrongly interpreted as generalised Riccati equation, but here we prefer calling it as Riccati equation only. Depending on the values of p,q and r, the Riccati equation (1.172) admits several types of different solutions [268].

Differentiating (1.172) with respect to ξ, we get

$$\phi''(\xi)=pr+p^2\phi\left(\xi\right)+3\,pq\phi\left(\xi\right)^2+2\,q\phi\left(\xi\right)r+2\,q^2\phi\left(\xi\right)^3 \tag{1.173a}$$

$$\phi'''(\xi)=p^2r+p^3\phi\left(\xi\right)+7\,p^2q\phi\left(\xi\right)^2+8\,pq\phi\left(\xi\right)r+12\,pq^2\phi\left(\xi\right)^3+2\,qr^2 \\ +8\,q^2r\phi\left(\xi\right)^2+6\,q^3\phi\left(\xi\right)^4, \tag{1.173b}$$

and so on. Substitution of (1.171) in (1.170) and using derivatives (1.172) and (1.173) will give a large expression in powers of ϕ. Vanishing the coefficients of powers of ϕ gives a system of algebraic equations in a_i. Such system of algebraic

equations can be easily solved in any symbolic manipulation program like "Maple" and "Mathematica".

For the general solution of Riccati equation (1.172), we use of transformation $\phi = \frac{1}{2}\nu - \frac{p}{2q}$, which transforms the Riccati equation to standard format as follows:

$$\nu_\xi = \nu^2 + \frac{4rq - p^2}{4}, \tag{1.174}$$

for $\nu = \frac{\psi_\xi}{\psi}$, the Riccati equation further simplifies to

$$\psi_{\xi\xi} - \frac{4rq - p^2}{4}\,\psi = 0, \tag{1.175}$$

which is second order linear differential equation that can be easily solved for general solution of Riccati equation (1.172), which we obtained as follows:

$$\phi(\xi) = \frac{\sqrt{4qr - p^2}}{2q}\left[\frac{k_1 \mathrm{e}^{\frac{\sqrt{4qr-p^2}\xi}{2}} - k_2 \mathrm{e}^{\frac{-\sqrt{4qr-p^2}\xi}{2}}}{k_1 \mathrm{e}^{\frac{\sqrt{4qr-p^2}\xi}{2}} + k_2 \mathrm{e}^{\frac{-\sqrt{4qr-p^2}\xi}{2}}}\right] - \frac{p}{2q}. \tag{1.176}$$

Keeping in mind the suggestions made by Kudryashov [135], careful selection of arbitrary constants k_1 and k_2, and based on values of p, q and r, the Riccati equation (1.172) admits variety of solution like; kink, soliton and periodic solutions. In Chapter 5 (see (5.35)), we have corrected several errors in Ref. [268] related to dependency of solutions and only 4 independent solutions are written instead of 27 solutions.

1.3.4 $\frac{G'}{G}$-expansion method

The $\frac{G'}{G}$-expansion method was firstly proposed by Wang et al. [235]. Since the method is well known in the literature, so we will not give details about the algorithms of this method. Rather we prefer to briefly outline main steps of the algorithm. Just like Riccati equation mapping method, it would be sufficient to illustrate the method by taking equation of the type (1.170).

The method starts with assuming travelling wave solution $u = u(\xi),\ \xi = x - \alpha\, t$ of the equation (1.170), followed by substitution

$$u(\xi) = a_m \left(\frac{G'}{G}\right)^m + a_{m-1}\left(\frac{G'}{G}\right)^{m-1} + \ldots, \tag{1.177}$$

where m can be determined just like the way we did in the case of Riccati method and $G = G(\xi)$ is a solution of second order ordinary differential equation

$$G'' + \lambda\, G' + \mu\, G = 0, \tag{1.178}$$

where λ and μ are arbitrary real constants. The substitution of (1.177) into (1.170) and vanishing the coefficients of powers of $\left(\frac{G'}{G}\right)$ will give system of algebraic equations in $a_1, a_2, \ldots$ and so on. Solving such system of algebraic equations and looking for solutions of (1.178) for cases like;

(i) $\lambda^2 - 4\,\mu > 0$

(ii) $\lambda^2 - 4\,\mu < 0$

(iii) $\lambda^2 - 4\,\mu = 0$

will give three types of traveling wave solutions for equation (1.170).

Remark 1.3.2. Looking at the expansions (1.171) and (1.177), and the second order ordinary differential equations (1.175) and (1.178), the methods; Riccati equation mapping method and $\frac{G'}{G}$-expansion method are equivalent, since the first order derivative in (1.178) can be easily eliminated through appropriate transformation (e.g. use $G(\xi) = H(\xi)\exp(\lambda\,\xi)$). Therefore, both methods will give similar solutions, and moreover, Riccati equation mapping method will give one additional solution. It is pertinent to mention that, the thumb rule for identification of similar solutions is to check their physical structure by actually plotting them within common domain. For more details on such type of similar solutions, the Refs. [135, 136, 186] can be seen.

1.3.5 Exact solutions using test function

The bilinearization discussed in Section 1.2.2 is a powerful tool for generating explicit exact solutions in a straightforward manner. To illustrate the construction of exact solutions in simplest possible manner, we can again consider equation of the type (1.170). Suppose we have the bilinear equation

$$D_x^m D_t^n \{f \cdot f\} = 0, \tag{1.179}$$

which may be constructed using some dependent variable transformation. The next step to is to assume f in the form

$$f = \mathrm{e}^{-\xi_1} + \delta_1 \tan(\xi_2) + \delta_2 \tanh(\xi_3) + \delta_3 \mathrm{e}^{\xi_1}, \tag{1.180}$$

where

$$\xi_i = \alpha_i x + \beta_i t, i = 1, 2, 3.$$

Substitution of (1.180) into (1.179) will give large expression in term of $e^{\pm\xi_1}, e^{\pm\xi_1}\tan(\xi_2)$, $e^{\pm\xi_1}\tanh(\xi_3), \tan(\xi_2)\tanh(\xi_3)$ etc., which can handled with "Maple" program. Since the substitution is a cumbersome practice when carried by hand, so we have provided Maple code in Appendix A.2. Vanishing the coefficients of the independent terms will give system of algebraic equations in α_i, β_i and δ_i. Such system of algebraic equations can be easily solved to give variety of solutions, the details are presented in Chapter 9.

2

On invariant analysis and conservation laws for degenerate coupled multi-KdV equations for multiplicity $l = 3$[1]

Mathematics is a game played according to certain simple rules with meaningless marks on paper.

David Hilbert

2.1 Introduction

Recently, degenerate coupled Korteweg-de Vries equations for coupled multiplicity $l = 2, 3, 4$ have been considered for traveling wave solutions by Gürses and Pekcan [83, 97]. The multi-system of Kaup-Boussinesq equations [16, 17] is given as follow:

$$\begin{aligned} u_t &= \frac{3}{2}\,uu_x + q^2_x, \\ q^2_t &= q^2 u_x + \frac{1}{2}\,uq^2_x + q^3_x, \\ &\vdots \\ q^{l-1}_t &= q^{l-1}u_x + \frac{1}{2}\,uq^{l-1}_x + v_x, \\ v_t &= -\frac{1}{4}u_{xxx} + vu_x + \frac{1}{2}\,uv_x, \end{aligned} \tag{2.1}$$

where $q^1 = u$ and $q^l = v$. For $l = 2$, the system (2.1) reduces to the following system

$$\begin{aligned} u_t &= \frac{3}{2}\,uu_x + v_x, \\ v_t &= -\frac{1}{4}\,u_{xxx} + vu_x + \frac{1}{2}\,uu_x. \end{aligned} \tag{2.2}$$

[1]The content of this chapter is accepted for publication in Pramana–Journal of Physics: The article is in production [94]

The system (2.2) has been studied in detail by Gürses and Pekcan [97] for traveling wave solutions and they have proved that there exists no asymptotically vanishing traveling wave solution for the system. Wazwaz [243] has studied generalized version of system (2.2) for multiple-soliton solutions. The system (2.2) is also known as Kaup-Boussinesq system, it exhibits shallow water wave characteristics in the same approximation as the well-known Boussinesq equation in the lowest order in small parameters controlling weak dispersion and nonlinearity effects [67, 121, 122, 130]. Moreover, the function $v(x,t)$ denotes the height of the water surface above a horizontal bottom, whereas the function $u(x,t)$ denotes its velocity averaged over the depth. The Kaup-Boussinesq system (2.2) corresponds to the case when the gravity force dominates over the capillary one, and it is completely integrable [28, 126, 127].

For $l = 3$, the system (2.1) has the following form:

$$\begin{aligned} &u_t - \frac{3}{2}uu_x - v_x = 0, \\ &v_t - vu_x - \frac{1}{2}uv_x - w_x = 0, \\ &w_t + \frac{1}{4}u_{xxx} - wu_x - \frac{1}{2}uw_x = 0. \end{aligned} \tag{2.3}$$

The system of equations (2.3) which are also known as three field Kaup-Boussinesq equations [96] have been discussed for traveling wave solutions in the Ref. [83], wherein the authors have given a general approach to solve equations (2.1) for $l \geq 3$. Subsequently, by using bifurcation analysis, Li and Chen [148] have given complete parametric representations of traveling wave solutions of system (2.1) for $l = 2, 3, 4$, which were earlier missing in the work of Gürses and Pekcan [83, 97]. In literature, we have noticed that the equations (2.3) have not been completely analyzed, so in this work, we propose to investigate equations (2.3) for Lie's symmetry analysis and the conservation laws using recently proposed new theorem by Ibragimov [117].

2.2 Lie symmetry analysis of Kaup-Boussinesq equations (2.3)

In order to identify the Lie point symmetries for equations (2.3), we follow standard procedure given in Ref. [34, 177]. Consider one parameter local Lie group of point transformations,

$$\begin{aligned} \tilde{x} &= x + \epsilon\,\xi(x,t,u,v,w) + O(\epsilon^2), \\ \tilde{t} &= t + \epsilon\,\tau(x,t,u,v,w) + O(\epsilon^2), \\ \tilde{u} &= u + \epsilon\,\eta_1(x,t,u,v,w) + O(\epsilon^2), \\ \tilde{v} &= v + \epsilon\,\eta_2(x,t,u,v,w) + O(\epsilon^2), \\ \tilde{w} &= w + \epsilon\,\eta_3(x,t,u,v,w) + O(\epsilon^2), \end{aligned} \tag{2.4}$$

where ϵ being group parameter. The invariance of equations (2.3) under symmetry transformations (2.4) give an overdetermined system of linear PDEs in $\xi, \tau, \eta_1, \eta_2$ and η_3. This overdetermined system may be derived by considering the associated

vector field, which may be expressed as follows:

$$V = \xi\frac{\partial}{\partial x} + \tau\frac{\partial}{\partial t} + \eta_1\frac{\partial}{\partial u} + \eta_2\frac{\partial}{\partial v} + \eta_3\frac{\partial}{\partial w}. \tag{2.5}$$

Third order prolongation of vector field (2.5) when applied in following manner

$$V^{(3)}(\Delta)|_{(2.3)} = 0, \text{ here } \Delta \text{ is system (2.3)}, \tag{2.6}$$

will give following infinitesimals of symmetry transformation

$$\begin{aligned} &\xi = -\frac{5c_3}{6}t + \frac{3c_1}{5}x + c_4,\ \tau = c_1 t + c_2, \\ &\eta_1 = -\frac{2c_1}{5}u + c_3,\ \eta_2 = -\frac{4c_1}{5}v - \frac{2c_3}{3}u,\ \eta_3 = -\frac{c_3}{3}v - \frac{6c_1}{5}w. \end{aligned} \tag{2.7}$$

Infinitesimals (2.7) give four dimensional Lie algebra

$$\begin{aligned} V_1 &= \frac{\partial}{\partial t},\ V_2 = \frac{\partial}{\partial x},\ \text{(translation)} \\ V_3 &= -\frac{5t}{6}\frac{\partial}{\partial x} + \frac{\partial}{\partial u} - \frac{2u}{3}\frac{\partial}{\partial v} - \frac{v}{3}\frac{\partial}{\partial w},\ \text{(Galilean boost)} \\ V_4 &= \frac{3x}{5}\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} - \frac{2u}{5}\frac{\partial}{\partial u} - \frac{4v}{5}\frac{\partial}{\partial v} - \frac{6w}{5}\frac{\partial}{\partial w}\,\text{(dilation)}. \end{aligned} \tag{2.8}$$

The non-zero Lie commutations of Lie algebra (2.8) can be obtained as follow:

$$[V_1, V_3] = -\frac{5V_2}{6},\ [V_1, V_4] = V_1,\ [V_2, V_4] = \frac{3V_2}{5},\ [V_3, V_4] = -\frac{2V_3}{5}. \tag{2.9}$$

The Lie brackets (2.9) show that the Lie algebra (2.8) is solvable.

2.2.1 Construction of an optimal system for Lie algebra (2.8)

In symmetry analysis, it is well-known fact that, whenever PDE or system of PDEs admit symmetry group (or group of invariant transformations), then one can find group invariant solutions corresponding to each sub-group by reducing the number of independent variables in the original system. There could exist infinitely many such sub-groups and hence infinitely many group invariant solutions. But most of those group invariant solutions would be equivalent by some transformation in full symmetry group. In order to minimize the search of the inequivalent group invariant solutions under transformations in full symmetry group, the concept of the optimal system has been introduced. Although, the classification of the Lie algebras by the use of adjoint transformations was known to Lie himself, nevertheless, Ovsiannikov [177] was first to use Lie group classification to derive inequivalent group invariant solutions. Ovsiannikov used a global adjoint matrix to construct optimal system and he further extended his technique to derive multi-dimensional optimal system. In the construction of the two-dimensional optimal system, Galas [79] made some modifications in the technique of Ovsiannikov by selecting elements from the normalizer of the one-dimensional optimal system.

Apart from these techniques, the method of classifying the sub-algebras proposed by Patera *et al.* [183] is par excellence (see Ref. [90] for recent applications). In

their subsequent work [182], they have classified all the real Lie algebras of dim $\leq$ 4 under the group of inner automorphisms. It is worth mentioning that the Lie algebras of dimension greater than 4 have also been fully classified. For example, Turkowski [230] has classified all six dimensional solvable Lie algebras containing four-dimensional nilradical. He has also classified all Lie algebras of dimension up to 9. Chou *et al.* [47, 48] suggested a slightly modified technique of classifying Lie algebras. They have constructed different varieties of invariants of the group of inner automorphism including the numerical and conditional invariants. Despite the early work [175] on group classification by using adjoint actions (or identification of equivalence classes based on sign of Killing's form), the work of Chou *et al.* is very useful as their additional invariants help to confirm the optimality. Their technique confirms the completeness and mutual inequivalence of representatives of the sub-algebras. In the present chapter, we shall use only Killing's form as invariant for detecting all the representatives of inequivalent sub-algebras.

Definition 2.2.1. The symmetric bilinear form φ on the space of Lie algebra $\mathscr{L}$, that is, the mapping $\varphi : \mathscr{L} \times \mathscr{L} \to \mathbb{R}$ is called *invariant* (relative to group $Int\ \mathscr{L}$) if for any inner automorphism $A \in Int\ \mathscr{L}$, and for any $V_1, V_2 \in \mathscr{L}$, following is holds true

$$\varphi\langle A\langle V_1\rangle, A\langle V_2\rangle\rangle = \varphi\langle V_1, V_2\rangle.$$

In term of adjoint representation Ad_g, the real function φ on Lie algebra $\mathscr{L}$ is invariant if and only if $\varphi(Ad_g(V)) = \varphi(V)$, for all $V \in \mathscr{L}$ and $g \in G$ (group generated by $\mathscr{L}$).

Definition 2.2.2. Let $\mathscr{L}$ be a Lie algebra, and $V \in \mathscr{L}$. Then the adjoint transformation defined by V is the linear transformation $ad(V) : \mathscr{L} \to \mathscr{L}$ defined as follows:

$$ad(V)(W) = [V, W], \text{ for all } W \in \mathscr{L},$$

here $[\cdot, \cdot]$ is usual Lie bracket. The exponential of $ad(X)$ is denoted by $Ad(X)$ and is a Lie algebra isomorphism. The symmetric bilinear form $K : \mathscr{L} \times \mathscr{L} \to \mathbb{R}$ is defined as follows:

$$K(V, W) = trace(ad(V) \cdot ad(W)), \tag{2.10}$$

where $K(V, W)$ is called Killing's form. This Killing's form is invariant of group of inner automorphism $Int\mathscr{L}$. The importance of this form shall be realized in construction of optimal system of sub-algebras.

Lemma 2.2.1. Let $V = \sum_{i=1}^{4} a_i V_i$ be the general element of Lie algebra $\mathscr{L}^4$ given by (2.8) and $a_1, \ldots, a_4 \in \mathbb{R}$. The invariant function φ is of the form $f(a_4)$, here f is arbitrary function.

Proof. The general invariant function φ can be obtained by solving the following system of linear PDEs

$$\begin{aligned} &a_4\frac{\partial\varphi}{\partial a_2} = 0,\ a_4\frac{\partial\varphi}{\partial a_1} - \frac{5a_3}{6}\frac{\partial\varphi}{\partial a_2} = 0,\ -\frac{2a_4}{5}\frac{\partial\varphi}{\partial a_3} + \frac{5a_1}{6}\frac{\partial\varphi}{\partial a_2} = 0, \\ &-a_1\frac{\partial\varphi}{\partial a_1} + \frac{2a_3}{5}\frac{\partial\varphi}{\partial a_3} - \frac{3a_2}{5}\frac{\partial\varphi}{\partial a_2} = 0. \end{aligned} \tag{2.11}$$

A straightforward solution of above system is $f(a_4)$, for arbitrary function f. The procedure for the construction of system of PDEs (2.11) has been discussed in detail in the Ref. [110]. □

Lemma 2.2.2. The Killing's form is particular invariant which can be derived from general solution of system (2.11) For Lie algebra (2.8), $K(V,W) = \frac{38}{25}a_4^2$.

Proof. The direct computations show that

$$ad(V) = \begin{bmatrix} -a_4 & 0 & 0 & a_1 \\ \frac{5a_3}{6} & -\frac{3a_4}{5} & -\frac{5a_1}{6} & \frac{3a_2}{5} \\ 0 & 0 & \frac{2a_4}{5} & -\frac{2a_3}{5} \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{2.12}$$

By definition of Killing's form, $K(V,W) = trace(ad(V)\cdot ad(V)) = \frac{38}{25}a_4^2$. □

In search of group invariant solutions, one ought to be careful of the instances where two group invariant solutions can be recovered from each other by some transformation in the full symmetry group. For example, the two group invariant solutions Ψ_1 and Ψ_2 are called as essentially inequivalent, if it is impossible to connect these solutions by transformation $\tilde{\psi} = \exp\left[\sum_{i=1}^{4} a_i V_i\right]\psi$. In this manner, the group invariant solutions separate into equivalence classes, and the collection of generators corresponding these classes would constitute an optimal system. In order to find such equivalence classes, define adjoint operator

$$\mathrm{Ad}_{\exp(\epsilon V)}(W) = \exp(-\epsilon\, V) W \exp(\epsilon\, V) = \tilde{W}(\epsilon). \tag{2.13}$$

The adjoint transformation (2.13) can be written through Lie brackets using Campbell-Hausdorff formula, which defined as follows:

$$\mathrm{Ad}_{\exp(\epsilon V)}(W) = W - \epsilon[V,W] + \frac{\epsilon^2}{2}[V,[V,W]] - \ldots, \tag{2.14}$$

where $[.,.]$ is Lie bracket defined by (2.9). Let $V = \sum_{i=1}^{4} a_i V_i$, the based on the Lie brackets defined at (2.9) and formula (2.14), straightforward calculations show that

$$\mathrm{Ad}_{\exp(\epsilon_3 v_3)}\mathrm{Ad}_{\exp(\epsilon_4 v_4)}\mathrm{Ad}_{\exp(\epsilon_1 v_1)}\mathrm{Ad}_{\exp(\epsilon_2 v_2)}(V) = \sum_{i=1}^{4} \tilde{a}_i V_i. \tag{2.15}$$

The full adjoint transformation (2.15) in matrix notation is given as follows:

$$A = \begin{bmatrix} \mathrm{e}^{\epsilon_4} & 0 & 0 & -\epsilon_1 \mathrm{e}^{\epsilon_4} \\ -\frac{5\epsilon_3}{6}\mathrm{e}^{\epsilon_4} & \mathrm{e}^{\frac{3\epsilon_4}{5}} & \frac{5\epsilon_1}{6}\mathrm{e}^{\frac{3\epsilon_4}{5}} & -\frac{3\epsilon_2}{5}\mathrm{e}^{\frac{3\epsilon_4}{5}} + \frac{5\epsilon_1\epsilon_3}{6}\mathrm{e}^{\epsilon_4} \\ 0 & 0 & \mathrm{e}^{-\frac{2\epsilon_4}{5}} & \frac{2\epsilon_3}{5} \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{2.16}$$

The construction of adjoint matrix (2.16) is discussed in the Ref. [177], where the coefficients $\tilde{a}_1, \ldots, \tilde{a}_4$ in (2.15) are given by

$$\begin{aligned}
\tilde{a}_1 &= -a_4\epsilon_1 \mathrm{e}^{\epsilon_4} + a_1 \mathrm{e}^{\epsilon_4}, \\
\tilde{a}_2 &= \frac{5a_3\epsilon_1}{6}\,\mathrm{e}^{\frac{3\epsilon_4}{5}} - \frac{3a_4\epsilon_2}{5}\,\mathrm{e}^{\frac{3\epsilon_4}{5}} + a_2\mathrm{e}^{\frac{3\epsilon_4}{5}} + \frac{5a_4\epsilon_1\epsilon_3}{6}\,\mathrm{e}^{\epsilon_4} - \frac{5a_1\epsilon_3}{6}\,\mathrm{e}^{\epsilon_4}, \\
\tilde{a}_3 &= \frac{2a_4\epsilon_3}{5} + a_3\mathrm{e}^{-\frac{2\epsilon_4}{5}}, \\
\tilde{a}_4 &= a_4.
\end{aligned} \tag{2.17}$$

The last equation in (2.17) agrees with the invariance of Killing's form under full adjoint transformation (2.15).

Theorem 2.2.3. The one -dimensional optimal system corresponding to the Lie algebra (2.7) is $\{V_1, V_2, V_3, V_4, \alpha\, V_1 \pm V_3\}$.

Proof. Let $V = \sum_{i=1}^{4} a_i V_i$ and $K = \frac{38}{25}a_4^2$. We have following cases for K

Case 1. For $K \neq 0$, we take $a_4 = 1$. Choosing $\epsilon_4 = 0$, system of equations (2.17) become

$$\begin{aligned}
\tilde{a}_1 &= -\epsilon_1 + a_1, \\
\tilde{a}_2 &= \frac{5a_3\epsilon_1}{6} - \frac{3\epsilon_2}{5} + a_2 + \frac{5\epsilon_1\epsilon_3}{6} - \frac{5a_1\epsilon_3}{6}, \\
\tilde{a}_3 &= \frac{2\epsilon_3}{5} + a_3, \\
\tilde{a}_4 &= 1.
\end{aligned}$$

The selection, $\epsilon_1 = a_1, \epsilon_2 = \frac{25\,a_3a_1}{18} + \frac{5a_2}{3}, \epsilon_3 = -\frac{5a_3}{2}$, give $\tilde{a}_1 = \tilde{a}_2 = \tilde{a}_3 = 0$. We obtain final simplification $V = V_4$

Case 2. For $K = 0$, we have to take $a_4 = 0$.

(1) $a_3 = 1$. Choosing $\epsilon_4 = \frac{5}{2}\,\ln(a_3)$ gives $\tilde{a}_3 = \pm 1$, and appropriate selection of ϵ_1, ϵ_3 give $\tilde{a}_1 = a_1 a_3{}^{\frac{5}{2}}$ and $\tilde{a}_2 = 0$. We obtain final simplification $V = \alpha\, V_1 \pm V_3$, $\alpha = a_1 a_3{}^{\frac{5}{2}}$

(2) $a_3 = 0$. The system (2.17) reduce to $\tilde{a}_1 = a_1\mathrm{e}^{\epsilon_4}, \tilde{a}_2 = a_2\mathrm{e}^{\frac{3\epsilon_4}{5}} - \frac{5a_1\epsilon_3}{6}\,\mathrm{e}^{\epsilon_4}$. By taking $\epsilon_3 = \frac{6a_2\mathrm{e}^{-\frac{2\epsilon_4}{5}}}{5a_1}$, we obtain final simplification $V = V_1$.

(3) $a_3 = 0, a_1 = 0$. In this case, we obtain a straightforward simplification $V = V_2$.

(4) $a_3 \neq 0, a_1 = 0$. $\tilde{a}_2 = \frac{5a_3\epsilon_1}{6}\,\mathrm{e}^{\frac{3\epsilon_4}{5}} + a_2\mathrm{e}^{\frac{3\epsilon_4}{5}}, \tilde{a}_3 = a_3\mathrm{e}^{-\frac{2\epsilon_4}{5}}$. By taking $\epsilon_1 = -\frac{6a_2}{5a_3}$, we obtain simplification $V = V_3$.

□

2.2.2 Symmetry reductions and invariant solutions

By virtue of vector fields V_1 and V_2, we can see that the equations (2.3) admit symmetry in space and time translation. So instead of separate reductions for both symmetries, it would be more appropriate to consider reduction under thier sum

i.e. $V_1 + c\,V_2$. Therefore, for simialrity variable $\xi = x - c\,t$, we have similarity transformations $u = F(\xi), v = G(\xi)$ and $w = H(\xi)$. On substituting into system (2.3), we obtain

$$-c\,F_\xi - \frac{3}{2}\,FF_\xi - G_\xi = 0, \tag{2.18a}$$

$$-c\,G_\xi - GF_\xi - \frac{1}{2}\,FG_\xi - H_\xi = 0, \tag{2.18b}$$

$$-c\,H_\xi + \frac{1}{4}\,F_{\xi,\xi,\xi} - HF_\xi - \frac{1}{2}\,FH_\xi = 0. \tag{2.18c}$$

Integrating (2.18a) with respect to ξ gives

$$G = -c\,F - \frac{3}{4}\,F^2 + d_1, \text{ here } d_1 \text{ is constant of integration.} \tag{2.19}$$

Again, substituting G into (2.18b) gives

$$H_\xi = (c^2 - d_1)F_\xi + 3c\,FF_\xi + \frac{3}{2}F^2F_\xi.$$

Integrating once with respect to ξ yields

$$H = (c^2 - d_1)F + \frac{3c}{2}F^2 + \frac{1}{2}F^3 + d_2. \tag{2.20}$$

On substituting (2.19) and (2.20) into (2.18c) give

$$\begin{aligned} -c^3\,F_\xi + cd_1\,F_\xi - \frac{9}{2}\,c^2FF_\xi - \frac{9}{2}\,cF_\xi F^2 + \frac{1}{4}\,F_{\xi,\xi,\xi} \\ + \frac{3}{2}d_1\,F_\xi F - \frac{5}{4}\,F_\xi F^3 - d_2\,F_\xi = 0. \end{aligned} \tag{2.21}$$

Integrating (2.21) with respect to ξ and then second integration after using integrating factor F_ξ gives

$$(F_\xi)^2 = \frac{1}{2}\,F^5 + 3c\,F^4 + (6c^2 - 2d_1)\,F^3 + 4(c^3 - cd_1 + d_2)\,F^2 + 8d_3\,F + 8d_4, \tag{2.22}$$

here c, d_1, d_2, d_3, d_4 are constants of integration. Detailed discussion about solution of (2.22) can be seen in the Ref. [83]. The reductions corresponding to rest of vectors fields have been classified in following cases.

Case 3. Reduction under subalgebra V_3.

- Similarity variables.

$$\begin{aligned} \xi &= t, \\ u &= F\,(t) - \frac{6x}{5t}, \\ v &= G\,(t) + \frac{4ux}{5t} + \frac{12\,x^2}{25\,t^2}, \\ w &= H\,(t) + \frac{2vx}{5t} - \frac{4\,ux^2}{25\,t^2} - \frac{8\,x^3}{125\,t^3}. \end{aligned} \tag{2.23}$$

- Reduced system. Substituting (2.23) into (2.3), the reduced system is obtained as follows:

$$\begin{aligned}
&tF_t + F = 0,\\
&2t\,F^2 - 5t^2\,G_t - 4tx\,F_t - 4x\,F - 4t\,G = 0,\\
&4tx\,F^2 + 5t^2\,FG - 25t^3\,H_t - 10t^2x\,G_t - 4tx^2\,F_t\\
&- 4x^2\,F - 8tx\,G - 30t^2\,H = 0.
\end{aligned} \tag{2.24}$$

- Similarity solutions.

$$\begin{aligned}
u =& \frac{c_1}{t} - \frac{6x}{5t},\\
v =& -\frac{{c_1}^2}{3t^2} + \frac{c_2}{t^{4/5}} + \frac{4c_1x}{5t^2} - \frac{12\,x^2}{25\,t^2},\\
w =& -\frac{c_1c_2}{3t^{9/5}} + \frac{{c_1}^3}{27t^3} + \frac{c_3}{t^{6/5}} - \frac{2x{c_1}^2}{15t^3}\\
&+ \frac{2xc_2}{5t^{9/5}} + \frac{4\,c_1x^2}{25\,t^3} - \frac{8\,x^3}{125\,t^3}.
\end{aligned} \tag{2.25}$$

where c_1, c_2 and c_3 are constants of integration.

Case 4. Reduction under subalgebra V_4.

- Similarity variables.

$$\xi = \frac{t}{x^{\frac{5}{3}}},\ u = \frac{1}{x^{\frac{2}{3}}}F(\xi),\ v = \frac{1}{x^{\frac{4}{3}}}G(\xi),\ w = \frac{1}{x^2}H(\xi). \tag{2.26}$$

- Reduced system. Substituting (2.26) into (2.3), the reduced system is obtained as follows:

$$\begin{aligned}
&15\,F\xi\,F_\xi + 6\,F^2 + 10\,\xi\,G_\xi + 8\,G + 6\,F_\xi = 0,\\
&5\,F\xi\,G_\xi + 10\,G\xi\,F_\xi + 8\,FG + 10\,\xi\,H_\xi + 12\,H + 6\,G_\xi = 0,\\
&125\,\xi^3F_{\xi,\xi,\xi} - 90\,F\xi\,H_\xi - 180\,H\xi\,F_\xi + 750\,\xi^2F_{\xi,\xi}\\
&- 180\,FH + 830\,\xi\,F_\xi + 80\,F - 108\,H_\xi = 0.
\end{aligned} \tag{2.27}$$

- Similarity solutions.

$$\begin{aligned}
u =& \frac{1}{x^{\frac{2}{3}}}\sum_{n=0}^{\infty}P_n\left(\frac{t}{x^{\frac{5}{3}}}\right)^n,\ v = \frac{1}{x^{\frac{4}{3}}}\sum_{n=0}^{\infty}Q_n\left(\frac{t}{x^{\frac{5}{3}}}\right)^n,\\
w =& \frac{1}{x^2}\sum_{n=0}^{\infty}R_n\left(\frac{t}{x^{\frac{5}{3}}}\right)^n,
\end{aligned} \tag{2.28}$$

where the coefficients P_n, Q_n and R_n are obtained in Theorem 2.2.4.

Case 5. Reduction under subalgebra $\alpha\,V_1 + V_3$.

- Similarity variables.

$$\begin{aligned}
\xi &= \frac{12\,\alpha\,x}{5} + t^2,\\
u &= -F(\xi) - \frac{t}{\alpha},\\
v &= G(\xi) - \frac{2tF(\xi)}{3\alpha} + \frac{4x}{5\alpha},\\
w &= H(\xi) - \frac{t\,G(\xi)}{3\alpha} - \frac{4x\,F(\xi)}{15\,\alpha} - \frac{4\,tx}{45\,\alpha^2} - \frac{2\,t\xi}{27\,\alpha^3}.
\end{aligned} \tag{2.29}$$

- Reduced system. Substituting (2.29) into (2.3), the reduced system is obtained as follows:

$$\begin{aligned}
&18\,F\alpha^2F_\xi + 12\,\alpha^2G_\xi - 1 = 0,\\
&9\,F\alpha^3G_\xi + 18\,G\alpha^3F_\xi + 18\,\alpha^3H_\xi + 4\,\alpha\,\xi\,F_\xi + 6\,F\alpha = 0,\\
&3888\,\alpha^6F_{\xi,\xi,\xi} - 1350\,F\alpha^4H_\xi - 2700\,H\alpha^4F_\xi + 450\,F\alpha^2\xi\,F_\xi\\
&+ 150\,F^2\alpha^2 - 375\,G\alpha^2 - 125\,\xi = 0.
\end{aligned} \tag{2.30}$$

- Similarity solutions.

$$\begin{aligned}
u &= -\sum_{n=0}^{\infty} P_n \left(\frac{12\,\alpha\,x}{5} + t^2\right)^n - \frac{t}{\alpha},\\
v &= \sum_{n=0}^{\infty} Q_n \left(\frac{12\,\alpha\,x}{5} + t^2\right)^n - \frac{2t}{3\alpha}\sum_{n=0}^{\infty} P_n \left(\frac{12\,\alpha\,x}{5} + t^2\right)^n + \frac{4x}{5\alpha},\\
w &= \sum_{n=0}^{\infty} R_n \left(\frac{12\,\alpha\,x}{5} + t^2\right)^n - \frac{t}{3\alpha}\sum_{n=0}^{\infty} Q_n \left(\frac{12\,\alpha\,x}{5} + t^2\right)^n\\
&\quad - \frac{4x}{15\,\alpha}\sum_{n=0}^{\infty} P_n \left(\frac{12\,\alpha\,x}{5} + t^2\right)^n - \frac{4\,tx}{45\,\alpha^2} - \frac{2\,t\left(\frac{12\,\alpha\,x}{5} + t^2\right)}{27\,\alpha^3},
\end{aligned} \tag{2.31}$$

  where the coefficients P_n, Q_n and R_n are obtained in following Theorem 2.2.5.

For similarity solutions of reductions corresponding to vector V_4 and $\alpha\,V_1 + V_3$ we seek power series solution of the following form

$$F = \sum_{n=0}^{\infty} P_n\,\xi^n,\ G = \sum_{n=0}^{\infty} Q_n\,\xi^n,\ H = \sum_{n=0}^{\infty} R_n\,\xi^n, \tag{2.32}$$

here P_n, Q_n and R_n are unknown coefficients of power series that need to be determined later. On substituting (2.32) into reductions corresponding to respective vector fields we have following theorems:

Theorem 2.2.4. Substitution of power series (2.32) into reductions corresponding

to vector field V_4 gives following recurrence relations

$$\begin{aligned}
P_{n+1} =& -\frac{1}{6(n+1)}\Big[10\,nQ_n + 15\sum_{k=0}^{n}(n-k)\,P_kP_{n-k} + 6\sum_{k=0}^{n}P_kP_{n-k} + 8\,Q_n\Big]\\
Q_{n+1} =& -\frac{1}{6(n+1)}\Big[10\,nR_n + 5\sum_{k=0}^{n}(n-k)\,P_kQ_{n-k} + 10\sum_{k=0}^{n}(n-k)\,Q_kP_{n-k}\\
&+ 8\sum_{k=0}^{n}P_kQ_{n-k} + 12\,R_n\Big]\\
R_{n+1} =& \frac{-1}{108\,(n+1)}\Big[-125\,n^3P_n - 375\,n^2P_n - 330\,nP_n + 180\sum_{k=0}^{n}P_kR_{n-k}\\
&+ 90\sum_{k=0}^{n}(n-k)\,P_kR_{n-k} + 180\sum_{k=0}^{n}(n-k)\,R_kP_{n-k} - 80\,P_n\Big],
\end{aligned} \tag{2.33}$$

here P_0, Q_0, R_0 ought to be taken as arbitrary and

$$P_1 = -{P_0}^2 - \frac{4\,Q_0}{3}, Q_1 = -\frac{4\,P_0Q_0}{3} - 2\,R_0, R_1 = -\frac{5\,P_0R_0}{3} + \frac{20\,P_0}{27} \tag{2.34}$$

Proof. For brevity, we have omitted detailed calculations. Results in form of power series solutions for system (2.3) corresponding to reductions under vector field V_4 are presented in Case 4. □

Theorem 2.2.5. Substitution of power series (2.32) into reductions corresponding to vector field $\alpha\,V_1 + V_3$ gives following recurrence relations

$$\begin{aligned}
P_{n+3} =& \frac{-1}{1296\,\alpha^4\,(n^3+6\,n^2+11\,n+6)}\Big[-450\,\alpha^2\sum_{k=0}^{n}(n-k+1)\,P_kR_{n-k+1}\\
&- 900\,\alpha^2\sum_{k=0}^{n}(n-k+1)\,R_kP_{n-k+1} + 50\sum_{k=0}^{n}P_kP_{n-k}\\
&+ 150\sum_{k=0}^{n}(n-k)\,P_kP_{n-k} - 125\,Q_n\Big],\\
Q_{n+1} =& \frac{-3}{2\,(n+1)}\sum_{k=0}^{n}(n-k+1)\,P_kP_{n-k+1},\\
R_{n+1} =& \frac{-1}{18\,\alpha^2\,(n+1)}\Big[9\,\alpha^2\sum_{k=0}^{n}(n-k+1)\,P_kQ_{n-k+1} + 18\,\alpha^2\sum_{k=0}^{n}(n-k+1)\\
&\times Q_kP_{n-k+1} + 4\,nP_n + 6\,P_n\Big]
\end{aligned} \tag{2.35}$$

and

$$
\begin{aligned}
P_3 =& \frac{1350\,\alpha^2 {P_0}^3 P_1 - 1800\,\alpha^2 P_0 P_1 Q_0 + 3600\,\alpha^2 P_1 R_0 - 875\,{P_0}^2 + 500\,Q_0}{31104\,\alpha^4},\\
Q_1 =& \frac{1 - 18\,\alpha^2 P_0 P_1}{12\alpha^2},\\
R_1 =& \frac{6\,\alpha^2 {P_0}^2 P_1 - 8\,\alpha^2 P_1 Q_0 - 3\,P_0}{8\alpha^2},\\
P_4 =& \frac{1}{186624\,\alpha^6}[4050\,\alpha^4 {P_0}^3 P_2 + 2025\,\alpha^4 {P_0}^2 {P_1}^2 - 4050\,\alpha^4 P_0 P_1 Q_1 - 5400\,\alpha^4 P_0 P_2 Q_0\\
&+ 8100\,\alpha^4 P_1 R_1 + 10800\,\alpha^4 P_2 R_0 - 3000\,\alpha^2 P_0 P_1 + 750\,\alpha^2 Q_1 + 250],\\
Q_2 =& -\frac{3\,P_0 P_2}{2} - \frac{3\,{P_1}^2}{4},\\
R_2 =& \frac{54\,\alpha^2 {P_0}^2 P_2 + 27\,\alpha^2 P_0 {P_1}^2 - 54\,\alpha^2 P_1 Q_1 - 72\,\alpha^2 P_2 Q_0 - 20\,P_1}{72\,\alpha^2},
\end{aligned}
\tag{2.36}
$$

here P_0, Q_0, R_0, P_1, P_2 ought to be taken as arbitrary.

Proof. Again for brevity, we have omitted detailed calculations. Results in form of power series solutions for system (2.3) corresponding to reductions under vector field $\alpha\, V_1 + V_3$ are presented in Case 5. □

2.3 Conservation laws

It is well-known fact that the Noether's theorem gives conservation laws for a system only when it has variational principle. To establish conservation laws for a system without variational structure, Ibragimov [117] has given a new theorem based on the concept of adjoint equations for nonlinear equations. In the recent literature, many authors have applied the theorem of Ibragimov to derive conservations laws. For instance, in the Ref. [118], it was proved that the Camassa-Holm is strictly self-adjoint and conservation laws were also obtained without classical Lagrangians. In theRef. [77], authors have constructed some conservation laws for nonlinear self-adjoint class of the generalized fifth-order equation, such as general Kawahara equation, modified Kawahara equation and simplified modified Kawahara equation. Johnpillai and Khalique [125] have applied the same theorem to derive conservation laws for time-dependent variable coefficients generalized KdV equation. For further details about application of Ibragimov's theorem, see the recent work [3, 39, 40, 81, 90, 223, 233, 234].
Based on the theory developed in [117] and notations adopted therein, define formal Lagrangian for the equations (2.3) in the following manner

$$
\begin{aligned}
I =& \phi(x,t)\left(u_t - \frac{3}{2}uu_x - v_x\right) + \psi(x,t)\left(v_t - vu_x - \frac{1}{2}uv_x - w_x\right)\\
&+ \theta(x,t)\left(w_t + \frac{1}{4}u_{xxx} - wu_x - \frac{1}{2}uw_x\right),
\end{aligned}
\tag{2.37}
$$

where $\phi(x,t), \psi(x,t)$, and $\theta(x,t)$ are new dependent variables (nonlocal). The adjoint equations for (2.3) can be written as follow:

$$F^* = \frac{\delta I}{\delta u} = 0,\ G^* = \frac{\delta I}{\delta v} = 0,\ H^* = \frac{\delta I}{\delta w} = 0, \tag{2.38}$$

where we have used variational derivative $\frac{\delta}{\delta u^\alpha}$ defined by the following relation

$$\frac{\delta}{\delta u^\alpha} = \frac{\partial}{\partial u^\alpha} + \sum_{s=1}^{\infty}(-1)^s D_{i_1}\dots D_{i_s}\frac{\partial}{\partial u^\alpha_{i_1\dots i_s}}. \tag{2.39}$$

On substituting Lagrangian (2.37) into (2.38) and using relation (2.39), we obtain following adjoint equations

$$\begin{aligned} F^* =& \frac{1}{2}\,\psi v_x + \frac{1}{2}\,\theta w_x + \frac{3}{2}\,\phi_x u + \psi_x v + \theta_x w - \theta_t - \frac{1}{4}\,\theta_{xxx} = 0,\\ G^* =& -\frac{1}{2}\,\psi u_x + \phi_x + \frac{1}{2}\,\psi_x u - \psi_t = 0,\\ H^* =& -\frac{1}{2}\,\theta u_x + \psi_x + \frac{1}{2}\,\theta_x u - \theta_t = 0. \end{aligned} \tag{2.40}$$

For conservation laws, we shall use the following theorem proved in [117].
Theorem 2.3.1. Any infinitesimal symmetry (*Lie point, Lie Bäcklund, nonlocal*)

$$V = \xi^i(x,u,u_{(1)},\dots)\,\frac{\partial}{\partial x^i} + \eta^\alpha(x,u,u_{(1)},\dots)\frac{\partial}{\partial u^\alpha}$$

of equations (2.3) leads to conservation laws $D_i(C^i) = 0$ constructed by the formula

$$\begin{aligned} C^i =& \xi^i I + W^\alpha\left[\frac{\partial I}{\partial u^\alpha_i} - D_j\left(\frac{\partial I}{\partial u^\alpha_{ij}}\right) + D_jD_k\left(\frac{\partial I}{\partial u^\alpha_{ijk}}\right) - \dots\right]\\ &+ D_j(W^\alpha)\left[\frac{\partial I}{\partial u^\alpha_{ij}} - D_k\left(\frac{\partial I}{\partial u^\alpha_{ijk}}\right) + \dots\right] + D_jD_k(W^\alpha)\left[\frac{\partial I}{\partial u^\alpha_{ijk}} - \dots\right], \end{aligned} \tag{2.41}$$

where $W^\alpha = \eta^\alpha - \xi^j u^\alpha_j$ (Lie's characteristic functions) and I is Lagrangian defined by (2.37).

The relation (2.41) can be simplified by writing Lagrangian I with respect to all mixed derivatives $u^\alpha_{ij}, u^\alpha_{ijk}, \dots$ in symmetric manner. We obtain

$$\begin{aligned} C^x =& \xi\, I + W^{(1)}\left[\frac{\partial I}{\partial u_x} - D_x\left(\frac{\partial I}{\partial u_{xx}}\right) + D_x^2\left(\frac{\partial I}{\partial u_{xxx}}\right)\right] + W^{(2)}\,\frac{\partial I}{\partial v_x} + W^{(3)}\,\frac{\partial I}{\partial w_x}\\ &+ D_x(W^{(1)})\left[\left(\frac{\partial I}{\partial u_{xx}}\right) - D_x\left(\frac{\partial I}{\partial u_{xxx}}\right)\right] + D_x^2(W^{(1)})\,\frac{\partial I}{\partial u_{xxx}},\\ C^t =& \tau\, I + W^{(1)}\frac{\partial I}{\partial u_t} + W^{(2)}\frac{\partial I}{\partial v_t} + W^{(3)}\frac{\partial I}{\partial w_t}, \end{aligned} \tag{2.42}$$

where D_i denotes operator of total differentiation:

$$D_i = \frac{\partial}{\partial x^i} + u_i^\alpha \frac{\partial}{\partial u^\alpha} + u_{ij}^\alpha \frac{\partial}{\partial u_j^\alpha} + \dots .$$

The rest of details about notations can be seen in the Ref. [117].

In following cases, we shall find conserved currents (2.42) corresponding to every symmetry generator of optimal system obtained in Theorem 2.2.3.

Case 2.3.1. For generator $V_1 = \frac{\partial}{\partial t}$, the Lie's characteristic functions are obtained as follow:

$$W^{(1)} = -u_t,\ W^{(2)} = -v_t,\ W^{(3)} = -w_t. \tag{2.43}$$

Substitution of (2.43) into (2.42) yields the following conserved currents

$$\begin{aligned} C^x =& \frac{3}{2}\, u_t\phi\, u + u_t\psi\, v + u_t\theta\, w - \frac{1}{4}\, u_t\theta_{xx} + v_t\phi \\ &+ \frac{1}{2}\, v_t\psi\, u + w_t\psi + \frac{1}{2}\, w_t\theta\, u + \frac{1}{4}\, u_{xt}\theta_x - \frac{1}{4}\, u_{xxt}\theta, \\ C^t =& -\frac{3}{2}\, \phi\, uu_x - \phi\, v_x - \psi\, vu_x - \frac{1}{2}\, \psi\, uv_x - \psi\, w_x \\ &+ \frac{1}{4}\, \theta\, u_{xxx} - \theta\, wu_x - \frac{1}{2}\, \theta\, uw_x. \end{aligned} \tag{2.44}$$

Case 2.3.2. For generator $V_2 = \frac{\partial}{\partial x}$, the Lie's characteristic functions are obtained as follow:

$$W^{(1)} = -u_x,\ W^{(2)} = -v_x,\ W^{(3)} = -w_x. \tag{2.45}$$

Substitution of (2.45) into (2.42) yields the following conserved currents

$$\begin{aligned} C^x =& u_t\phi + v_t\psi + w_t\theta - \frac{1}{4}\, u_x\theta_{xx} + \frac{1}{4}\, u_{x,x}\theta_x, \\ C^t =& -\phi\, u_x - \psi\, v_x - \theta\, w_x. \end{aligned} \tag{2.46}$$

Case 2.3.3. For generator $V_3 = -\frac{5t}{6}\frac{\partial}{\partial x} + \frac{\partial}{\partial u} - \frac{2u}{3}\frac{\partial}{\partial v} - \frac{v}{3}\frac{\partial}{\partial w}$, the Lie's characteristic functions are obtained as follow:

$$W^{(1)} = 1 + \frac{5\,t}{6}u_x,\ W^{(2)} = -\frac{2\,u}{3} + \frac{5\,t}{6}v_x,\ W^{(3)} = -\frac{v}{3} + \frac{5\,t}{6}w_x. \tag{2.47}$$

Substitution of (2.47) into (2.42) yields the following conserved currents

$$\begin{aligned} C^x =& \frac{1}{3}\, \psi\, u^2 - \frac{5}{6}\, tu_t\phi - \frac{5}{6}\, tv_t\psi - \frac{5}{6}\, tw_t\theta + \frac{5\, tu_x\theta_{xx}}{24} + \frac{1}{6}\, v\theta\, u - \frac{5\, tu_{xx}\theta_x}{24} \\ &+ \frac{1}{4}\, \theta_{xx} - \frac{2}{3}\, \psi\, v - \theta\, w - \frac{5}{6}\, \phi\, u, \\ C^t =& \phi + \frac{5}{6}\, \phi\, tu_x - \frac{2}{3}\, \psi\, u + \frac{5}{6}\, \psi\, tv_x - \frac{1}{3}\, v\theta + \frac{5}{6}\, \theta\, tw_x. \end{aligned} \tag{2.48}$$

Case 2.3.4. For generator $V_4 = \frac{3x}{5}\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} - \frac{2u}{5}\frac{\partial}{\partial u} - \frac{4v}{5}\frac{\partial}{\partial v} - \frac{6w}{5}\frac{\partial}{\partial w}$, the Lie's characteristic functions are obtained as follow:

$$\begin{aligned} W^{(1)} =& -\frac{2\,u}{5} - \frac{3\,x}{5}u_x - tu_t,\ W^{(2)} = -\frac{4\,v}{5} - \frac{3\,x}{5}v_x - tv_t, \\ W^{(3)} =& -\frac{6\,w}{5} - \frac{3\,x}{5}w_x - tw_t. \end{aligned} \tag{2.49}$$

Substitution of (2.49) into (2.42) yields the following conserved currents

$$\begin{aligned} C^x =& \frac{3}{5}\,\phi\,u^2 - \frac{1}{10}\,u\theta_{xx} + \frac{4}{5}\,v\phi + \frac{6}{5}\,w\psi + \frac{1}{4}\,\theta_x u_x - \frac{2}{5}\,\theta\,u_{xx} + \frac{3}{2}\,tu_t\phi\,u + tu_t\psi\,v \\ &+ tu_t\theta\,w + \frac{1}{2}\,tv_t\psi\,u + \frac{1}{2}\,tw_t\theta\,u + tv_t\phi + u\theta\,w - \frac{1}{4}\,tu_t\theta_{xx} + \frac{3\,\theta_x x u_{xx}}{20} \\ &+ \frac{1}{4}\,\theta_x tu_{xt} - \frac{1}{4}\,\theta\,tu_{xxt} + \frac{4}{5}\,u\psi\,v + \frac{3}{5}\,x\psi\,v_t + \frac{3}{5}\,x\phi\,u_t - \frac{3\,x u_x\theta_{xx}}{20} \\ &+ \frac{3}{5}\,x\theta\,w_t + tw_t\psi \\ C^t =& -\frac{3}{2}\,t\phi\,uu_x - t\phi\,v_x - t\psi\,vu_x - \frac{1}{2}\,t\psi\,uv_x - t\psi\,w_x + \frac{1}{4}\,tu_{xxx}\theta - t\theta\,wu_x \\ &- \frac{1}{2}\,t\theta\,uw_x - \frac{2}{5}\,\phi\,u - \frac{3}{5}\,\phi\,xu_x - \frac{4}{5}\,\psi\,v - \frac{3}{5}\,\psi\,xv_x - \frac{6}{5}\,\theta\,w - \frac{3}{5}\,\theta\,xw_x. \end{aligned} \tag{2.50}$$

Case 2.3.5. For generator $\alpha\,V_1 + V_3 = -\frac{5t}{6}\frac{\partial}{\partial x} + \alpha\,\frac{\partial}{\partial t} + \frac{\partial}{\partial u} - \frac{2u}{3}\frac{\partial}{\partial v} - \frac{v}{3}\frac{\partial}{\partial w}$, the Lie's characteristic functions are obtained as follow:

$$\begin{aligned} W^{(1)} =& 1 + \frac{5\,t}{6}u_x - \alpha\,u_t,\ W^{(2)} = -\frac{2\,u}{3} + \frac{5\,t}{6}v_x - \alpha\,v_t, \\ W^{(3)} =& -\frac{v}{3} + \frac{5\,t}{6}w_x - \alpha\,w_t. \end{aligned} \tag{2.51}$$

Substitution of (2.51) into (2.42) yields the following conserved currents

$$\begin{aligned} C^x =& \frac{3}{2}\,\alpha\,u_t\phi\,u + \alpha\,u_t\psi\,v + \alpha\,u_t\theta\,w + \frac{1}{2}\,\alpha\,v_t\psi\,u + \frac{1}{2}\,\alpha\,w_t\theta\,u + \frac{1}{6}\,v\theta\,u + \frac{1}{4}\,\theta_x\alpha\,u_{xt} \\ &+ \frac{5\,tu_x\theta_{xx}}{24} - \frac{5}{6}\,tw_t\theta + \alpha\,v_t\phi - \frac{1}{4}\,\alpha\,u_t\theta_{xx} - \frac{1}{4}\,\theta\,\alpha\,u_{xxt} - \frac{5}{6}\,tv_t\psi - \frac{5}{6}\,tu_t\phi \\ &- \frac{5\,tu_{xx}\theta_x}{24} + \alpha\,w_t\psi + \frac{1}{4}\,\theta_{xx} - \frac{5}{6}\,\phi\,u - \frac{2}{3}\,\psi\,v - \theta\,w + \frac{1}{3}\,\psi\,u^2, \\ C^t =& -\frac{3}{2}\,\alpha\,\phi\,uu_x - \alpha\,\phi\,v_x - \alpha\,\psi\,vu_x - \frac{1}{2}\,\alpha\,\psi\,uv_x - \alpha\,\psi\,w_x + \frac{1}{4}\,\alpha\,\theta\,u_{xxx} - \alpha\,\theta\,wu_x \\ &- \frac{1}{2}\,\alpha\,\theta\,uw_x + \phi + \frac{5}{6}\,\phi\,tu_x - \frac{2}{3}\,\psi\,u + \frac{5}{6}\,\psi\,tv_x - \frac{1}{3}\,v\theta + \frac{5}{6}\,\theta\,tw_x, \end{aligned} \tag{2.52}$$

where in all the cases discussed above, the functions $\phi(x,t), \psi(x,t)$ and $\theta(x,t)$ are solutions of adjoint equations (2.40). In similar manner, the conserved currents corresponding to generator $\alpha\,V_1 - V_3 = \frac{5t}{6}\frac{\partial}{\partial x} + \alpha\,\frac{\partial}{\partial t} - \frac{\partial}{\partial u} + \frac{2u}{3}\frac{\partial}{\partial v} + \frac{v}{3}\frac{\partial}{\partial w}$ can also be calculated.

3

Group classification, conservation laws and Painlevé analysis for Klein–Gordon–Zakharov equations in (3+1)–dimension[1]

No human investigation can be called real science if it cannot be demonstrated mathematically.

Leonardo da Vinci

3.1 Introduction

In this chapter, our aim is to study group classification and integrability of Klein-Gordon-Zakharov(KGZ) equations in (3+1)-dimension.

$$\begin{aligned} &\phi_{tt} - \Delta\phi + \phi + \phi\,\psi = 0, \\ &\psi_{tt} - c^2\,\Delta\psi - \Delta|\phi|^2 = 0, \end{aligned} \tag{3.1}$$

where $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is Laplacian operator in $\mathbb{R}^3$ and $\phi = \phi(x, y, z, t), \psi = \psi(x, y, z, t)$, c is the propagation speed of the wave. The system (3.1) describes the propagation of strong turbulence of the Langmuir wave (are rapid oscillations of the electron density in conducting media such as plasmas or metals) in a high frequency plasma [214]. The function ϕ denotes the fast time scale component of electric field raised by electrons and the function ψ denotes the deviation of ion density from its equilibrium. The functions ϕ and ψ are originally real vector valued and real scalar valued, respectively. The KGZ equations have similar shape to Zakharov equations and Klein–Gordon–Schrödinger equations. Ozawa *et al.* [178, 179, 225] have studied well posedness in energy space, global existence and asymptotic behaviour of solutions for the Cauchy problem of the KGZ equations. The equations (3.1) have been studied for exact solutions by various authors [62, 119, 147, 194, 263].

[1]The content of this chapter is accepted for publication in Pramana–Journal of Physics: The article is in production [208]

3.2 Symmetry group of the KGZ equations

The procedure for determining symmetry algebra for PDEs is well-known [34, 93, 95, 175]. The procedure is so algorithmic that it has been successfully implemented in symbolic languages like "Maple" and "Mathematica". The Maple package "PDEtools" written by Cheb-Terrab [46] is so interactive and efficient that it becomes indispensable tool for researchers in the field of PDEs. The PDEtools package can be applied by setting $\phi = u + i\,v$ in (3.1) (for Maple code see Appendix B), we obtain

$$\begin{aligned}
\Delta_1 &= \psi\,u + u + u_{tt} - u_{xx} - u_{yy} - u_{zz} = 0,\\
\Delta_2 &= \psi\,v + v + v_{tt} - v_{xx} - v_{yy} - v_{zz} = 0,\\
\Delta_3 &= \psi_{tt} - c^2\,(\psi_{xx} + \psi_{yy} + \psi_{zz}) - 2\,{u_x}^2 - 2\,u u_{xx} - 2\,{v_x}^2 - 2\,v v_{xx}\\
&\quad - 2\,{u_y}^2 - 2\,u u_{yy} - 2\,{v_y}^2 - 2\,v v_{yy} - 2\,{u_z}^2 - 2\,u u_{zz} - 2\,{v_z}^2 - 2\,v v_{zz} = 0.
\end{aligned} \tag{3.2}$$

The element of symmetry algebra for (3.2) may written as follows:

$$V = \xi^x\,\partial_x + \xi^y\,\partial_y + \xi^z\,\partial_z + \xi^t\,\partial_t + \eta^u\,\partial_u + \eta^v\,\partial_v + \eta^\psi\,\partial_\psi. \tag{3.3}$$

The infinitesimals $\xi's$ and $\eta's$ can be recovered from invariance criterion defined as follows:

$$\mathrm{pr}^{(2)}\,V \cdot \Delta_i\Big|_{\Delta_i=0} = 0,\ i = 1, 2, 3, \tag{3.4}$$

where $\mathrm{pr}^{(2)}$ is the second order prolongation of vector field V and related detailed discussion about prolongation of vector field can be seen in the Ref.[34] (see e.g. Theorem 2.4.2-1). The Maple package PDEtools quickly generates following nine dimensional symmetry algebra

$$\begin{aligned}
V_1 &= x\,\frac{\partial}{\partial x} + y\,\frac{\partial}{\partial y} + z\,\frac{\partial}{\partial z} + t\,\frac{\partial}{\partial t} - u\,\frac{\partial}{\partial u} - v\,\frac{\partial}{\partial v} + (-2\,w - 2)\,\frac{\partial}{\partial w},\\
V_2 &= y\,\frac{\partial}{\partial x} - x\,\frac{\partial}{\partial y}, V_3 = z\,\frac{\partial}{\partial x} - x\,\frac{\partial}{\partial z},\ V_4 = \frac{\partial}{\partial x},\ V_5 = z\,\frac{\partial}{\partial y} - y\,\frac{\partial}{\partial z},\\
V_6 &= \frac{\partial}{\partial y},\ V_7 = \frac{\partial}{\partial z},\ V_8 = \frac{\partial}{\partial t},\ V_9 = v\,\frac{\partial}{\partial u} - u\,\frac{\partial}{\partial v}.
\end{aligned} \tag{3.5}$$

The structure of Lie algebra (3.5) can be properly identified from the non-zero Lie commutation relations

$$\begin{aligned}
&[V_1, V_4] = -V_4, [V_1, V_6] = -V_6, [V_1, V_7] = -V_7, [V_1, V_8] = -V_8, [V_2, V_3] = V_5,\\
&[V_2, V_4] = V_6, [V_2, V_5] = -V_3, [V_2, V_6] = -V_4, [V_3, V_4] = V_7, [V_3, V_5] = V_2,\\
&[V_3, V_7] = -V_4, [V_5, V_6] = V_7, [V_5, V_7] = -V_6.
\end{aligned} \tag{3.6}$$

Notice that the Lie algebra $L = \{V_i,\ i = 1 \ldots 9\}$ defined by (3.5) admits Levi decomposition

$$L = S \rtimes rad(L), \tag{3.7}$$

with semi-simple Levi-subalgebra $S = \{V_2, V_3, V_5\}$ and radical $rad(L) = \{V_1, V_4, V_6, V_7, V_8, V_9\}$. Through the exponentiation, any vector field in Lie algebra (3.5) is capable of generating Lie group of point transformations [53, 175]. We define

$$\tilde{T} = \exp\,(a_i V_i)\,T,\ i = 1 \ldots 9,\ \text{for } T = T(x, y, z, t, u, v, \psi), \tag{3.8}$$

which is known as one parameter group of infinitesimal transformation. The 9-parameter version of Lie group of point transformation may be expressed as follows:

$$\tilde{T} = \exp\left(\sum_{i=1}^{9} a_i V_i\right) T. \tag{3.9}$$

Let $X = \sum_{i=1}^{9} a_i V_i$, based on the Lie brackets defined at (3.6) and formula (2.14), straightforward calculations using Maple programming show that

$$\begin{aligned} &\mathrm{Ad}_{\exp(\epsilon_5 V_5)}\,\mathrm{Ad}_{\exp(\epsilon_3 V_3)}\,\mathrm{Ad}_{\exp(\epsilon_7 V_7)}\,\mathrm{Ad}_{\exp(\epsilon_4 V_4)}\,\mathrm{Ad}_{\exp(\epsilon_6 V_6)}\,\mathrm{Ad}_{\exp(\epsilon_1 V_1)} \\ &\mathrm{Ad}_{\exp(\epsilon_8 V_8)}\,\mathrm{Ad}_{\exp(\epsilon_2 V_2)}\,\mathrm{Ad}_{\exp(\epsilon_9 V_9)}\,(X) = \sum_{i=1}^{9} \tilde{a}_i V_i, \end{aligned} \tag{3.10}$$

where the coefficients $\tilde{a}_i,\ i = 1 \ldots 9$ are given in the Appendix B.4.

3.2.1 Construction of invariant functions

It is evident from the full adjoint action and the Table 3.1 that a_1 and a_9 are invariants. However, as described in [110], the general invariant function can be obtained by solving system of PDEs given by

$$\begin{aligned} &a_1\phi_{a_8} = 0, a_1\phi_{a_4} - a_2\phi_{a_6} - a_3\phi_{a_7} = 0, \\ &a_1\phi_{a_6} + a_2\phi_{a_4} - a_5\phi_{a_7} = 0, a_1\phi_{a_7} + a_3\phi_{a_4} + a_5\phi_{a_6} = 0, \\ &a_2\phi_{a_3} - a_3\phi_{a_2} + a_6\phi_{a_7} - a_7\phi_{a_6} = 0, -a_2\phi_{a_5} + a_4\phi_{a_7} + a_5\phi_{a_2} - a_7\phi_{a_4} = 0, \\ &a_3\phi_{a_5} + a_4\phi_{a_6} - a_5\phi_{a_3} - a_6\phi_{a_4} = 0, -a_4\phi_{a_4} - a_6\phi_{a_6} - a_7\phi_{a_7} - a_8\phi_{a_8} = 0. \end{aligned}$$

The straightforward solution of above system of PDEs shows that the general invariant function must be of the following type

$$\phi = F(a_1, a_2^2 + a_3^2 + a_5^2, a_9), \tag{3.11}$$

where F is arbitrary function.

The particular invariant can also be obtained by the Killing's form. For $X = \sum_{i=1}^{9} a_i V_i$, the straightforward calculations show that

$$K(X, X) = 4a_1^2 - 4a_2^2 - 4a_3^2 - 4a_5^2,\ K(X, X) = \text{Killing form}. \tag{3.12}$$

The Killing's form (see *e.g.* [177]) has been defined at (2.10). he linear transforma-

tion adX is given as follows:

$$adX = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -a_5 & 0 & a_3 & 0 & 0 & 0 & 0 \\ 0 & a_5 & 0 & 0 & -a_2 & 0 & 0 & 0 & 0 \\ a_4 & a_6 & a_7 & -a_1 & 0 & -a_2 & -a_3 & 0 & 0 \\ 0 & -a_3 & a_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_6 & -a_4 & 0 & a_2 & a_7 & -a_1 & -a_5 & 0 & 0 \\ a_7 & 0 & -a_4 & a_3 & -a_6 & a_5 & -a_1 & 0 & 0 \\ a_8 & 0 & 0 & 0 & 0 & 0 & 0 & -a_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} . \tag{3.13}$$

Therefore, the Killing's form gives invariant $K(X) = 4(a_1^2 - a_2^2 - a_3^2 - a_5^2)$ of full adjoint action, such that $K(\mathrm{Ad}\ g(X)) = K(X)$ for $X = \sum_{i=1}^{9} a_i V_i \in L$ and $g \in G$ being the Lie group generated by L.

The various invariants may listed as follow:

$$\begin{aligned} &\eta = a_2^2 + a_3^2 + a_5^2, \quad \text{as evident from the general invariant function (3.11)} \\ &A = a_1, \\ &B = a_9, \\ &C = \begin{cases} 1, & a_2^2 + a_3^2 + a_5^2 \neq 0; \\ 0, & \text{otherwise.} \end{cases} \end{aligned} \tag{3.14}$$

The last invariant C is clear from the full adjoint action given in Appendix B.4. Further it is obvious that

$$\tilde{a}_2^2 + \tilde{a}_3^2 + \tilde{a}_5^2 = a_2^2 + a_3^2 + a_5^2,$$

where $\tilde{a}_2 = \tilde{a}_3 = \tilde{a}_5 = 0$ if and only if $a_2 = a_3 = a_5 = 0$. In similar way

$$D = \begin{cases} 1, & a_4^2 + a_6^2 + a_7^2 + a_8^2 \neq 0,\ a_1 = a_2 = a_3 = a_5 = 0; \\ 0, & \text{otherwise.} \end{cases} \tag{3.15}$$

is also an invariant of the full adjoint action given by Appendix B.4. Further, it is pertinent to mention that the construction of invariants obtained here have been discussed in detail in Ref. [47].

3.2.2 Construction of an optimal system

The detection of invariants of full adjoint action is very helpful in the classification of Lie algebra L into 1-dimensional optimal list of sub-algebras. The invariant η is always positive, but still various restrictions on coefficients a_2, a_3 and a_5 can be discussed. For example, for $a_2 \neq a_3 \neq a_5 \neq 0$, a direct simplification in full adjoint action (3.10) gives

$$\mathrm{Ad}_{\exp(\epsilon_7 V_7)}\, \mathrm{Ad}_{\exp(\epsilon_4 V_4)}\, \mathrm{Ad}_{\exp(\epsilon_6 V_6)}\, \mathrm{Ad}_{\exp(\epsilon_8 V_8)}\, (X) = \sum_{i=1}^{9} \tilde{a}_i V_i, \tag{3.16}$$

where

$$\begin{aligned}
\tilde{a}_4 &= -a_1\epsilon_4 - a_2\epsilon_6 - a_3\epsilon_7 + a_4,\\
\tilde{a}_6 &= -a_1\epsilon_6 + a_2\epsilon_4 - a_5\epsilon_7 + a_6,\\
\tilde{a}_7 &= -a_1\epsilon_7 + a_3\epsilon_4 + a_5\epsilon_6 + a_7,\\
\tilde{a}_8 &= -a_1\epsilon_8 + a_8.
\end{aligned} \tag{3.17}$$

These coefficients vanish on choosing

$$\begin{aligned}
\epsilon_4 &= \frac{{a_1}^2 a_4 - a_1 a_2 a_6 - a_1 a_3 a_7 + a_2 a_5 a_7 - a_3 a_5 a_6 + a_4 {a_5}^2}{a_1\left({a_1}^2 + {a_2}^2 + {a_3}^2 + {a_5}^2\right)},\\
\epsilon_6 &= \frac{{a_1}^2 a_6 + a_1 a_2 a_4 - a_1 a_5 a_7 - a_2 a_3 a_7 + {a_3}^2 a_6 - a_3 a_4 a_5}{a_1\left({a_1}^2 + {a_2}^2 + {a_3}^2 + {a_5}^2\right)},\\
\epsilon_7 &= \frac{{a_1}^2 a_7 + a_1 a_3 a_4 + a_1 a_5 a_6 + {a_2}^2 a_7 - a_2 a_3 a_6 + a_2 a_4 a_5}{a_1\left({a_1}^2 + {a_2}^2 + {a_3}^2 + {a_5}^2\right)}, \epsilon_8 = \frac{a_8}{a_1}.
\end{aligned} \tag{3.18}$$

Further, actions by $\mathrm{Ad}_{\exp(\epsilon_2 V_2)}$ and $\mathrm{Ad}_{\exp(\epsilon_3 V_3)}$ and for suitable selection of ϵ_2 and ϵ_3, X becomes

$$P_1 = a_1 V_1 + a_2 V_2 + a_9 V_9,$$

for some constants a_1, a_2 and a_9. No further simplifications are possible.
Proceeding in similar fashion, the other members of the optimal system can be listed as follow:

Case 3.2.1. $\eta \neq 0$.

$$\begin{aligned}
P_2 &= a_2 V_2 + a_3 V_3 \pm V_6 + a_8 V_8 + a_9 V_9, \quad a_1 = 0,\ a_2 \neq 0,\ a_3 \neq 0\ a_5 \neq 0,\\
P_3 &= a_1 V_1 + a_3 V_3 + a_9 V_9, \quad a_2 = 0,\ a_3 \neq 0\ a_5 \neq 0,\\
P_4 &= a_2 V_2 + a_5 V_5 \pm V_7 + a_8 V_8 + a_9 V_9, \quad a_1 = 0,\ a_2 \neq 0,\ a_3 = 0,\ a_5 \neq 0,\\
P_5 &= \pm V_4 + a_5 V_5 + a_8 V_8 + a_9 V_9, \quad a_1 = 0,\ a_2 = 0,\ a_3 = 0,\ a_5 \neq 0.
\end{aligned}$$

Ad.	V_1	V_2	V_3	V_4	V_5	V_6	V_7	V_8	V_9
V_1	V_1	V_2	V_3	$e^{\epsilon}V_4$	V_5	$e^{\epsilon}V_6$	$e^{\epsilon}V_7$	$e^{\epsilon}V_8$	V_9
V_2	V_1	V_2	$V_3\cos(\epsilon) - V_5\sin(\epsilon)$	$V_4\cos(\epsilon) - V_6\sin(\epsilon)$	$V_5\cos(\epsilon) + V_3\sin(\epsilon)$	$V_6\cos(\epsilon) + V_4\sin(\epsilon)$	V_7	V_8	V_9
V_3	V_1	$V_2\cos(\epsilon) + V_5\sin(\epsilon)$	V_3	$V_4\cos(\epsilon) - V_7\sin(\epsilon)$	$V_5\cos(\epsilon) - V_2\sin(\epsilon)$	V_6	$V_7\cos(\epsilon) + V_4\sin(\epsilon)$	V_8	V_9
V_4	$V_1 - \epsilon V_4$	$V_2 + \epsilon V_6$	$V_3 + \epsilon V_7$	V_4	V_5	V_6	V_7	V_8	V_9
V_5	V_1	$V_2\cos(\epsilon) - V_3\sin(\epsilon)$	$V_3\cos(\epsilon) + V_2\sin(\epsilon)$	V_4	V_5	$V_6\cos(\epsilon) - V_7\sin(\epsilon)$	$V_7\cos(\epsilon) + V_6\sin(\epsilon)$	V_8	V_9
V_6	$V_1 - \epsilon V_6$	$V_2 - \epsilon V_4$	V_3	V_4	$V_5 + \epsilon V_7$	V_6	V_7	V_8	V_9
V_7	$V_1 - \epsilon V_7$	V_2	$V_3 - \epsilon V_4$	V_4	$V_5 - \epsilon V_6$	V_6	V_7	V_8	V_9
V_8	$V_1 - \epsilon V_8$	V_2	V_3	V_4	V_5	V_6	V_7	V_8	V_9
V_9	V_1	V_2	V_3	V_4	V_5	V_6	V_7	V_8	V_9

Table 3.1: Adjoint table based on formula (2.14).

Case 3.2.2. $\eta = 0$.

$$\begin{aligned}
P_6 &= a_1 V_1 + a_9 V_9, \quad a_2 = 0,\ a_3 = 0,\ a_5 = 0,\\
P_7 &= \pm V_7 + a_8 V_8 + a_9 V_9, \quad a_1 = 0,\ a_2 = 0,\ a_3 = 0,\ a_5 = 0.
\end{aligned}$$

Case 3.2.3. Since a_1 and a_9 are invariants of full adjoint action, which is also evident from the Table 3.1. So, the remaining members of optimal system are given by

$$P_8 = V_1, \quad P_9 = V_9.$$

The values of invariants obtained in subsection 3.2.1 are calculated for every member of optimal system $\{P_i,\ i = 1 \ldots 9\}$. These values of invariants are depicted in Table 3.2. The mutual inequivalence among $P_i's$ can be easily seen from this Table 3.2.

	η	A	B	C	D
P_1	a_2^2	a_1	a_9	1	0
P_2	$a_2^2 + a_3^2$	0	a_9	1	0
P_3	a_3^2	a_1	a_9	1	0
P_4	$a_2^2 + a_5^2$	0	a_9	1	0
P_5	a_5^2	0	a_9	1	0
P_6	0	a_1	a_9	0	0
P_7	0	0	a_9	0	1
P_8	0	a_1	0	0	0
P_9	0	0	a_9	0	0

Table 3.2: Invariants of full adjoint action (3.10).

3.2.3 Two-dimensional optimal system

In order to construct two-dimensional optimal list of sub-algebras, we follow procedure developed by Ovsiannikov [177] (see pp. 199), and its implementation by various authors [53, 79, 131, 189]. We list pair of two-dimensional sub-algebras in the form $\langle X, Y\rangle$, such that $[X, Y] = \lambda X + \mu Y$, for $X = P_i,\ i = 1 \ldots 9$. For precise selection, Y can be selected from the normalizer $\mathrm{Nor}_L(X) = \{Y \in L | [X, Y] \in X\}$, that is, $[X, Y] = \lambda X$. As suggested by Galas [79], one can avoid occurrence of X in Y by selecting Y from the factor algebra $\mathrm{Nor}_L(X)/X$, where the normalizer $\mathrm{Nor}_L(X)$ can be obtained by setting

$$\left[X, \sum_{j=1}^{9} a_j V_j\right] = \lambda X, \tag{3.19}$$

where λ is an arbitrary constant. From the relation (3.19), we find all possible non-zero a_i's that constitute $\mathrm{Nor}_L(X)$. The quotient algebra $\mathrm{Nor}_L(X)/X$ can be constructed by removing all linear combinations of X in the normalizer. In following, we give list of two-dimensional sub-algebras with this construction:

$$\begin{aligned}
M_1 &= \langle a_1V_1 + a_2V_2, V_9\rangle,\ a_2 \neq 0, a_9 = 0,\\
M_2 &= \langle a_1V_1 + a_9V_9, b_1V_2 + b_2V_3 + b_3V_5\rangle,\ a_2 = 0, a_9 \neq 0,\\
M_3 &= \langle a_2V_2 + a_3V_3 \pm V_6 + a_8V_8 + a_9V_9, a_2a_3V_2 + a_3^2V_3 + a_2V_7\rangle,\\
M_4 &= \langle a_2V_2 + a_3V_3 \pm V_6 + a_8V_8 + a_9V_9, a_2V_2 + a_3V_3 + V_6\rangle,\\
M_5 &= \langle a_1V_1 + a_3V_3, V_9\rangle,\ a_9 = 0,\\
M_6 &= \langle a_2V_2 + a_5V_5 \pm V_7 + a_8V_8 + a_9V_9, a_5V_5 + a_2V_7\rangle,\\
M_7 &= \langle a_2V_2 + a_5V_5 \pm V_7 + a_8V_8 + a_9V_9, a_2^2V_2 - a_5V_4 + a_2a_5V_5\rangle\\
M_8 &= \langle V_4 + a_5V_5 + a_8V_8, V_9\rangle,\ a_9 = 0,\\
M_9 &= \langle V_4 + a_5V_5 + a_9V_9, V_8\rangle,\ a_8 = 0
\end{aligned}$$

$M_{10} = \langle V_4 + a_5V_5, b_1V_8 + b_2V_9\rangle,\ a_8 = 0, a_9 = 0,$
$M_{11} = \langle V_7 + a_8V_8 + a_9V_9, b_1V_2 + b_2V_4 + b_3V_6\rangle,$
$M_{12} = \langle V_1, b_1V_2 + b_2V_3 + b_3V_5 + b_4V_9\rangle,$
$M_{13} = \langle V_9, P_i^*\rangle,$ here $P_i^* = P_i$(without centre $V_9, i \neq 1, 8, 9$).

It remains to show that M_i's are mutually inequivalent. In view of well known fact [47, 48, 177], any two dimensional subalgebra $\langle X_1, X_2\rangle$ can be written as

$$X = \lambda X_1 + \mu X_2.$$

The subalgebra $\langle X_1, X_2\rangle$ is inequivalent to subalgebra $\langle Y_1, Y_2\rangle$, if it impossible to write $Y_i = \lambda_i X_1 + \mu_i X_2$ for any $\lambda_i, \mu_i,\ i = 1, 2$. As an illustration, we show M_1 and M_8 are inequivalent. On contrary, let $V_4 + a_5V_5 + a_8V_8$ mapped to some $\lambda(a_1V_1 + a_2V_2) + \mu(V_9)$. In term of invariant A, $0 = A(V_4 + a_5V_5 + a_8V_8) = A(\lambda(a_1V_1 + a_2V_2) + \mu(V_9)) = \lambda a_1$; therefore $\lambda = 0$. This further implies that $V_4 + a_5V_5 + a_8V_8$ is equivalent to V_9, which is impossible. Therefore, M_1 and M_8 are inequivalent. Similarly, it is straightforward to see from Table 3.3 that all $\{M_i,\ i = 1 \ldots 12\}$ are inequivalent and inequivalence from M_{13} is justified from Table 3.2.

	η	A	B	C	D
M_1	$(\lambda a_2)^2$	λa_1	μ	1	0
M_2	$\mu^2(b_1^2 + b_2^2 + b_3^2)$	λa_1	λa_9	1	0
M_3	$(\lambda + \mu a_3)^2(a_2^2 + a_3^2)$	0	λa_9	1	0
M_4	$(\lambda + \mu)^2(a_2^2 + a_3^2)$	0	λa_9	1	0
M_5	$(\lambda a_3)^2$	λa_1	μ	1	0
M_6	$(\lambda a_2)^2 + (\lambda + \mu)^2 a_5^2$	0	λa_9	1	0
M_7	$(\lambda + \mu a_2)^2(a_2^2 + a_5^2)$	0	λa_9	1	0
M_8	$(\lambda a_5)^2$	0	μ	1	0
M_9	$(\lambda a_5)^2$	0	λa_9	1	0
M_{10}	$(\lambda a_5)^2$	0	μb_2	1	0
M_{11}	$(\mu b_1)^2$	0	λa_9	1	0
M_{12}	$\mu^2(b_1^2 + b_2^2 + b_3^2)$	λ	μa_9	1	0

Table 3.3: Invariants of general vector $\lambda X_1 + \mu X_2$ in $M = \langle X_1, X_2\rangle$.

3.3 Conservation laws

In this section, for the construction of conservation laws we shall apply recently proposed new theorem by Ibragimov [117]. In accordance with the terminologies

used in Ref. [117], the formal Lagrangian for system (3.2) is given by

$$\begin{aligned} I =& f(x,y,z,t)\Big[\psi\, u + u + u_{tt} - u_{xx} - u_{yy} - u_{zz}\Big] \\ &+ g(x,y,z,t)\Big[\psi\, v + v + v_{tt} - v_{xx} - v_{yy} - v_{zz}\Big] \\ &+ h(x,y,z,t)\Big[\psi_{tt} - c^2\left(\psi_{xx} + \psi_{yy} + \psi_{zz}\right) \\ &- 2\,{u_x}^2 - 2\,u u_{xx} - 2\,{v_x}^2 - 2\,v v_{xx} - 2\,{u_y}^2 - 2\,u u_{yy} \\ &- 2\,{v_y}^2 - 2\,v v_{yy} - 2\,{u_z}^2 - 2\,u u_{zz} - 2\,{v_z}^2 - 2\,v v_{zz}\Big] = 0, \end{aligned} \tag{3.20}$$

where $f(x,y,z,t), g(x,y,z,t)$ and $h(x,y,z,t)$ are new dependent variables nonlocal. The adjoint equations for (3.2) can be written as follow

$$F^* = \frac{\delta I}{\delta u} = 0,\ G^* = \frac{\delta I}{\delta v} = 0,\ H^* = \frac{\delta I}{\delta \psi} = 0, \tag{3.21}$$

here we have used variational derivative $\frac{\delta}{\delta u^\alpha}$ defined at (2.39). On substituting Lagrangian (3.20) into (3.21) and using relation (2.39), we obtain following adjoint equations:

$$\begin{aligned} F^* =& f\left(\psi + 1\right) + h\left(-2\,u_{xx} - 2\,u_{yy} - 2\,u_{zz}\right) + 2\,h u_{xx} + 2\,h u_{yy} \\ &+ 2\,h u_{zz} - f_{xx} - 2\,h_{xx} u - f_{yy} - 2\,h_{yy} u - f_{zz} - 2\,h_{zz} u + f_{tt} = 0, \\ G^* =& g\left(\psi + 1\right) + h\left(-2\,v_{xx} - 2\,v_{yy} - 2\,v_{zz}\right) + 2\,h v_{xx} + 2\,h v_{yy} + 2\,h v_{zz} \\ &- g_{xx} - 2\,h_{xx} v - g_{yy} - 2\,h_{yy} v - g_{zz} - 2\,h_{zz} v + g_{tt} = 0, \\ H^* =& -h_{xx} c^2 - h_{yy} c^2 - h_{zz} c^2 + f u + g v + h_{tt} = 0. \end{aligned} \tag{3.22}$$

Based on the theory developed in Ref. [117], if (3.3) is symmetry generator then it will give following conservation law

$$D_x C^1 + D_y C^2 + D_z C^3 + D_z C^4\Big|_{(3.2)} = 0, \tag{3.23}$$

where the components of conserved densities $C = (C^1, C^2, C^3, C^4)$ are given by the following formula

$$\begin{aligned} C^i =& \xi_i I + W^\alpha \left[\frac{\partial I}{\partial u^\alpha_i} - D_j\left(\frac{\partial I}{\partial u^\alpha_{ij}}\right) + D_j D_k\left(\frac{\partial I}{\partial u^\alpha_{ijk}}\right) - \dots\right] \\ &+ D_j(W^\alpha)\left[\frac{\partial I}{\partial u^\alpha_{ij}} - D_k\left(\frac{\partial I}{\partial u^\alpha_{ijk}}\right) + \dots\right] + D_j D_k(W^\alpha)\left[\frac{\partial I}{\partial u^\alpha_{ijk}} - \dots\right], \end{aligned} \tag{3.24}$$

where $W^\alpha = \eta^\alpha - \xi^j u^\alpha_j$ and I is Lagrangian defined by (3.20), and Einstein summation conventions have been used. The relation (3.24) can be simplified in following manner

$$\begin{aligned} C^1 =& \xi^x I + W^1\left[\frac{\partial I}{\partial u_x} - D_x\left(\frac{\partial I}{\partial u_{xx}}\right)\right] + W^2\left[\frac{\partial I}{\partial v_x} - D_x\left(\frac{\partial I}{\partial v_{xx}}\right)\right] \\ &- W^3 D_x\left(\frac{\partial I}{\partial \psi_{xx}}\right) + D_x(W^1)\frac{\partial I}{\partial u_{xx}} + D_x(W^2)\frac{\partial I}{\partial v_{xx}} + D_x(W^3)\frac{\partial I}{\partial \psi_{xx}}, \end{aligned} \tag{3.25a}$$

$$C^2 = \xi^y I + W^1\left[\frac{\partial I}{\partial u_y} - D_y\left(\frac{\partial I}{\partial u_{yy}}\right)\right] + W^2\left[\frac{\partial I}{\partial v_y} - D_y\left(\frac{\partial I}{\partial v_{yy}}\right)\right]$$
$$- W^3 D_y\left(\frac{\partial I}{\partial \psi_{yy}}\right) + D_y(W^1)\frac{\partial I}{\partial u_{yy}} + D_y(W^2)\frac{\partial I}{\partial v_{yy}} + D_y(W^3)\frac{\partial I}{\partial \psi_{yy}}, \tag{3.25b}$$

$$C^3 = \xi^z I + W^1\left[\frac{\partial I}{\partial u_z} - D_z\left(\frac{\partial I}{\partial u_{zz}}\right)\right] + W^2\left[\frac{\partial I}{\partial v_z} - D_z\left(\frac{\partial I}{\partial v_{zz}}\right)\right]$$
$$- W^3 D_z\left(\frac{\partial I}{\partial \psi_{zz}}\right) + D_z(W^1)\frac{\partial I}{\partial u_{zz}} + D_z(W^2)\frac{\partial I}{\partial v_{zz}} + D_z(W^3)\frac{\partial I}{\partial \psi_{zz}}, \tag{3.25c}$$

$$C^4 = \xi^t I - W^1 D_t\left(\frac{\partial I}{\partial u_{tt}}\right) - W^2 D_t\left(\frac{\partial I}{\partial v_{tt}}\right) - W^3 D_t\left(\frac{\partial I}{\partial \psi_{tt}}\right) + D_t(W^1)\frac{\partial I}{\partial u_{tt}} \tag{3.25d}$$
$$+ D_t(W^2)\frac{\partial I}{\partial v_{tt}} + D_t(W^3)\frac{\partial I}{\partial \psi_{tt}}, \tag{3.25e}$$

where Lie's characteristics functions $W^\alpha, \alpha = 1, 2, 3$ are defined by

$$\begin{aligned} W^1 &= \eta^u - \xi^x u_x - \xi^y u_y - \xi^z u_z - \xi^t u_t, \\ W^2 &= \eta^v - \xi^x v_x - \xi^y v_y - \xi^z v_z - \xi^t v_t, \\ W^3 &= \eta^\psi - \xi^x \psi_x - \xi^y \psi_y - \xi^z \psi_z - \xi^t \psi_t, \end{aligned} \tag{3.26}$$

and D_i denotes operator of total differentiation:

$$D_i = \frac{\partial}{\partial x^i} + u_i^\alpha \frac{\partial}{\partial u^\alpha} + u_{ij}^\alpha \frac{\partial}{\partial u_j^\alpha} + \dots.$$

The rest of details about notations can be seen in Ref. [117].
Further, with the aid of (3.20), (3.25) and (3.26) we may find several nonlocal conservation laws for the equations (3.2) corresponding to the symmetries P_i, $i = 1 \dots 9$. For brevity, we have skipped humongous expressions for conserved currents, nevertheless those expressions can be easily constructed from formulas (3.25) using symbolic manipulation.

3.4 Painlevé analysis of KGZ equation

To investigate the singularity structure of (3.2), the local Laurent expansion in the neighbourhood of noncharacteristic manifold $g(z_1 \dots z_n) = 0$ can be used [250].

The Laurent expansion can be taken in the form

$$u = g^\alpha \sum_{j=0}^{\infty} u_j\, g^j,\ v = g^\alpha \sum_{j=0}^{\infty} v_j\, g^j,\ \psi = g^\alpha \sum_{j=0}^{\infty} \psi_j\, g^j, \quad \text{where } g = g(x, y, z, t). \tag{3.27}$$

The leading order analysis can be carried by letting $u = u_0\, g^{\alpha_1}, v = v_0\, g^{\alpha_2}, \psi = \psi_0\, g^{\alpha_3}$. Inserting into (3.2) and balancing most dominant terms, we obtain four branches as follows:

- First branch

$$\begin{aligned}
\alpha_1 =& -1,\ \alpha_2 = -1,\ \alpha_3 = -2,\\
\psi_0 =& -2(g_t{}^2 - g_x{}^2 - g_y{}^2 - g_z{}^2),\\
u_0 =& \frac{1}{g_x{}^2 + g_y{}^2 + g_z^2}\Big[-\left(g_x{}^2 + g_y{}^2 + g_z{}^2\right)(-2\,g_x{}^2 g_t{}^2 c^2 - 2\,g_y{}^2 g_t{}^2 c^2\\
& - 2\,g_z{}^2 g_t{}^2 c^2 + 2\,g_x{}^4 c^2 + 4\,g_x{}^2 g_y{}^2 c^2 + 4\,g_x{}^2 g_z{}^2 c^2 + 2\,g_y{}^4 c^2\\
& + 4\,g_y{}^2 g_z{}^2 c^2 + 2\,g_z{}^4 c^2 + 2\,g_t{}^4 - 2\,g_x{}^2 g_t{}^2 - 2\,g_y{}^2 g_t{}^2\\
& - 2\,g_z{}^2 g_t{}^2 + g_x{}^2 v_0{}^2 + g_y{}^2 v_0{}^2 + g_z{}^2 v_0{}^2)\Big]^{\frac{1}{2}},
\end{aligned} \tag{3.28}$$

- Second branch

$$\begin{aligned}
\alpha_1 =& -1,\ \alpha_2 = -1,\ \alpha_3 = -2\\
\psi_0 =& -2(g_t{}^2 - g_x{}^2 - g_y{}^2 - g_z{}^2),\\
u_0 =& -\frac{1}{g_x{}^2 + g_y{}^2 + g_z{}^2}\Big[-(g_x{}^2 + g_y{}^2 + g_z{}^2)(-2\,g_x{}^2 g_t{}^2 c^2 - 2\,g_y{}^2 g_t{}^2 c^2\\
& - 2\,g_z{}^2 g_t{}^2 c^2 + 2\,g_x{}^4 c^2 + 4\,g_x{}^2 g_y{}^2 c^2 + 4\,g_x{}^2 g_z{}^2 c^2 + 2\,g_y{}^4 c^2\\
& + 4\,g_y{}^2 g_z{}^2 c^2 + 2\,g_z{}^4 c^2 + 2\,g_t{}^4 - 2\,g_x{}^2 g_t{}^2 - 2\,g_y{}^2 g_t{}^2\\
& - 2\,g_z{}^2 g_t{}^2 + g_x{}^2 v_0{}^2 + g_y{}^2 v_0{}^2 + g_z{}^2 v_0{}^2)\Big]^{\frac{1}{2}},
\end{aligned} \tag{3.29}$$

- Third branch

$$\begin{aligned}
\alpha_1 =& -2,\ \alpha_2 = -2,\ \alpha_3 = -2,\\
\psi_0 =& -6(g_t{}^2 - g_x{}^2 - g_y{}^2 - g_z{}^2),\ u_0 = i\,v_0,
\end{aligned} \tag{3.30}$$

- Fourth branch

$$\begin{aligned}
\alpha_1 =& -2\ \alpha_2 = -2,\ \alpha_3 = -2\\
\psi_0 =& -6(g_t{}^2 - g_x{}^2 - g_y{}^2 - g_z{}^2),\ u_0 = -i\,v_0,
\end{aligned} \tag{3.31}$$

where v_0 is arbitrary. The next step is to determine resonant points corresponding to each branch. For branch given by (3.28), we substitute

$$u = u_0\,g^{-1} + u_j\,g^{j-1},\ v = v_0\,g^{-1} + v_j\,g^{j-1},\ \psi = \psi_0\,g^{-2} + \psi_j\,g^{j-2} \tag{3.32}$$

into (3.2) and on retaining most singular part. The resonance points read as follow:

$$j = -1, 0, 2, 3, 4. \tag{3.33}$$

The first resonance is called Fuch index, it is always there because of the arbitrary nature of singularity manifold $g(x, y, z, t)$. The same resonant points can be calculated for the second branch. Unfortunately, resonant points for the branches (3.30) and (3.31) can not be calculated.
Further, we consider generalized Laurent expansions corresponding to first branch

(3.28) in the following form

$$\begin{aligned} u =& \frac{u_0}{g} + u_1 + u_2\, g + u_3\, g^2 + u_4\, g^3, \\ v =& \frac{v_0}{g} + v_1 + v_2\, g + v_3\, g^2 + v_4\, g^3, \\ \psi =& \frac{\psi_0}{g^2} + \frac{\psi_1}{g} + \psi_2 + \psi_3\, g + \psi_4\, g^2, \end{aligned} \tag{3.34}$$

where $u_j, v_j, \psi_j (j = 0 \ldots 4)$ are analytic functions of (x, y, z, t) in the neighbourhood of singularity manifold g. Substitution of (3.34) into (3.2) leads to complex recursion relations for u_j, v_j, ψ_j. Further simplified analysis can be carried out using Kruskal's ansätz [250]

$$\begin{aligned} & g(x, y, z, t) = x + y + z - h(t) \\ & \quad \text{and} \\ & u_j = u_j(t),\ v_j = v_j(t),\ \psi_j = \psi_j(t), \quad \text{for } j = 0 \ldots 4. \end{aligned} \tag{3.35}$$

This simplification is justifiable by implicit function theorem whenever singular manifold is noncharacteristic. Substituting (3.34) and (3.35) into (3.2) and collecting coefficient of g corresponding to resonance $j = 2$ lead us to the following compatibility condition

$$\begin{aligned} & -216\, {g_t}^5 g_{ttt} c^4 - 108\, {g_t}^4 {g_{tt}}^2 c^4 + 192\, {g_t}^7 g_{ttt} c^2 + 192\, {g_t}^6 {g_{tt}}^2 c^2 - 40\, {g_t}^9 g_{tt,t} \\ & - 20\, {g_t}^8 {g_{tt}}^2 - 144\, {g_t}^3 g_{tt} c^4 + 972\, {g_t}^3 g_{ttt} c^4 + 1566\, {g_t}^2 {g_{tt}}^2 c^4 + 240\, {g_t}^5 g_{tt} c^2 \\ & - 960\, {g_t}^5 g_{ttt} c^2 - 1872\, {g_t}^4 {g_{tt}}^2 c^2 + 212\, {g_t}^7 g_{ttt} + 242\, {g_t}^6 {g_{tt}}^2 - 1512\, g_t g_{tt} c^4 \\ & - 756\, g_t g_{ttt} c^4 - 378\, {g_{tt}}^2 c^4 - 168\, {g_t}^3 g_{tt} c^2 + 984\, {g_t}^3 g_{ttt} c^2 + 1548\, {g_t}^2 {g_{tt}}^2 c^2 \\ & - 244\, {g_t}^5 g_{ttt} - 154\, {g_t}^4 {g_{tt}}^2 + 84\, g_t c^2 g_{ttt} + 42\, {g_{tt}}^2 c^2 - 28\, {g_t}^3 g_{ttt} + 42\, {g_t}^2 {g_{tt}}^2 = 0 \end{aligned}$$

which is clearly not satisfied for an arbitrary g. Therefore, KGZ equation fails to pass the Painlevé test for the first branch (3.28), and it presumably not integrable. Same can also be verified for second branch (3.29).

Since Painlevé property fails at both branches so it is irrelevant to discuss remaining branches (3.30) and (3.31) for Painlevé property.

Remark 3.4.1. In Painlevé analysis, the leading order, and resonant points have been determined using Maple package "wkptest" [89]. Although this package quickly gives out desired results, but due to the complexity of system (3.2), it failed to verify compatibility conditions at resonant points. To encounter this problem we have used Kruskal's ansätz (3.35) which considerably simplified complex algebraic calculations.

4

On Painlevé analysis, symmetry group and conservation laws of Date-Jimbo-Kashiwara-Miwa equation[1]

> Emptiness is everywhere, and it can be calculated, which gives us a great opportunity. I know how to control the universe. So tell me, why should I run for a million.
>
> *Grigori Perelman, The mathematician who declined field medal.*

4.1 Introduction

The study of Painlevé property, Lie symmetries and conserved quantities associated with the symmetries is of theoretical and practical importance in mathematical physics, and investigations may lead to certain integrable properties of PDEs. As far as the integrability is concerned, the Painlevé analysis of PDEs is an efficient tool for revealing the integrable properties of the nonlinear system of differential equations. The algorithmic procedure of Weiss-Tabor-Carnevale [250] which requires expansion of dependent variable in a generalized Laurent expansion about singular manifold is so effective that, the procedure not only establish the integrability of equation but also deduce several properties of system, such as; the Lax pairs, the Bäcklund transformations, the Hirota's equations, the motion of invariants, symmetries and commuting flows, geometrical structure of phase space, and the algebraic properties (symmetries) which establish the integrability of system beyond doubt. Realizing the painful imposition of Painlevé analysis, Conte [55], later in 1989 proved the invariance of Painlevé analysis under the group of homographic transformations of singularity manifold. The invariant Painlevé analysis greatly re-

[1]The content of this chapter is published in International Journal of Applied and Computational Mathematics 4(3) (2018): 88 [207].

duced the task of computations by expressing the coefficients of Laurent series in term of the *Schwarzian* of singularity manifold. In addition to this, Lax pairs which are strong indicators of integrability also shown to be derivable in an algorithmic manner from invariant Painlevé analysis [55].

The second strong indicator of integrability is the existence of infinite conservation laws. A conservation law of a system of PDEs is a divergence expression that vanishes on the solution space of the system. Conservation laws have several applications in the study of PDEs, like for example, conservation laws play a vital role in the analysis of stability, global behavior, existence and uniqueness of solutions. They play an essential role in the development of numerical methods and provide an essential starting point for finding nonlocally related systems and potential variables [36]. There is a well-established technique called Noether's approach for construction of conservation laws from Lagrangian symmetry (point, contact or higher order). Noether showed that if a given system of DEs admits a variational principle, then any one-parameter Lie group of point transformations that leaves invariant the action functional yields a local conservation law [35], but this approach has a list of limitations like; it is applicable to the variational system only, there is difficulty in finding symmetries through action functional and most importantly the Noether's approach is coordinate dependent [33]. Later in 1997, Bluman and Anco have replaced Noether's technique and proposed a direct construction of conservation laws [12], which does not require Lagrangian formulation. They have also implemented their technique in an algorithmic manner in subsequent work for a system of PDEs [13, 14]. The method uses a linear system of determining equations whose solutions yield all multipliers and hence all variational symmetries. This determining system can be solved by the same standard procedure used to solve the determining equations for symmetries. For each multiplier, a corresponding conservation law can then be obtained by various direct integrations [10]. The method is computationally so efficient that it can determine all the admitted conservation laws, analogously to how all admitted symmetries can be found.

On the other hand, the concept of nonlinear self-adjointness for differential equations which has been recently introduced by Ibragimov [117], generalizes the concept of strict self-adjointness and quasi-self-adjointness as well as the usual linear self-adjointness, can also be applied to equations which do not possess variational structure, and conservation laws can be associated with any type of symmetries like, Lie, Lie Bäcklund, nonlocal etc. Moreover, this technique involves differential operations rather integral one as in the case of a direct method and thus easy to implement in an algorithmic manner. Recently, this technique is applied to many important equations in mathematical physics [41, 75–77, 80, 118, 125, 224, 258]. But the recent investigation by Anco [11] has shown that the Ibragimov technique is not a novel way for construction of conservation law rather a special case of the earlier formula using symmetries and adjoint-symmetries. Surprisingly, the general formula proposed by Ibragimov does not always yield all the admitted local conservation laws and sometimes it produces trivial conservation laws.

In the present work, we propose to investigate Date-Jimbo-Kashiwara-Miwa equation or simply DJKM equation

$$u_{xxxxy} + 4\,u_{xxy}u_x + 2\,u_{xxx}u_y + 6\,u_{xy}u_{xx} + u_{yyy} - 2\,u_{xxt} = 0, \tag{4.1}$$

for its complete integrability based on Painlevé property and existence of infinite number of local conservation laws. The equation (4.1) belongs to KP hierarchy [59] which can be obtain from first two bilinear equations using transformation $u = 2\,(\log\tau)_x$. In the Ref. [109], the equation (4.1) is investigated for bilinear Bäcklund transformations and nonlinear superposition formula. After a gap of more than two decades the equation has been recently studied for integrable properties using Bell polynomials [240], and integrable properties such as Bäcklund transformations, Lax system, infinite conservation laws along with multi-shock wave solutions have been reported. We believe this equation has never been investigated for Painlevé analysis, symmetry properties and conservation laws using direct method and nonlinear self-adjointness. In this study, we shall explore all these properties using mentioned techniques.

4.2 Truncated Painlevé expansions and exact solutions

In this section, along with the application of invariant Painlevé analysis, we use Pickering's algorithm [184] to derive the exact solution for equation (4.1). But first, we investigate the singularity structure of equation (4.1) using the scheme of WTC. The local Laurent expansion in a neighbourhood of non-characteristic manifold $\phi(z_1 \dots z_n) = 0$ can be used [250]. The Laurent expansion can be taken in the following form

$$u = \phi^{\alpha} \sum_{j=0}^{\infty} u_j\, \phi^j, \quad \text{where } \phi = \phi(x, y, t). \tag{4.2}$$

The leading order analysis can be carried by setting $u = u_0\, \phi^{\alpha}$. Inserting into equation (4.1) and balancing the most dominant terms, we obtain

$$\alpha = -1,\ u_0 = 2\,\phi_x. \tag{4.3}$$

The next step is to determine resonant points for which we substitute

$$u = u_0\, \phi^{-1} + u_j\, \phi^{j-1}, \tag{4.4}$$

into equation (4.1) and on retaining most singular part, the resonance points obtained as follow:

$$j = -1, 1, 4, 5, 6. \tag{4.5}$$

The first resonance $j = -1$ is called Fuch index. For resonance point $j = 1, 4, 5$ and 6, we need to check if sufficient number of arbitrary functions enter into the Laurent expansion (4.2) when substituted into equation (4.1). To reach this aim, we should insert the Laurent expansion

$$u(x, y, t) = \sum_{j=0}^{\infty} u_j \phi^{j-1}, \tag{4.6}$$

into equation (4.1) and coefficients of powers of ϕ will lead to compatibility conditions at resonance points which need to be satisfied in order to pass Painlevé test. The detailed calculations show that the compatibility conditions are satisfied at all resonance points. For instance, Kruskal's ansätz [250] can be employed i.e. $\phi = x + \psi(y,t)$, which shows that compatibility conditions are satisfied identically at all resonance points. Therefore, we may conclude that the equation (4.1) passes Painlevé's test and is therefore Painlevé integrable. The Bäcklund transformations for equation (4.1) can be obtained by setting resonance functions $u_4 = u_5 = u_6 = 0$ and with further requirement of $u_2 = u_3 = 0$

$$u = \frac{2\,\phi_x}{\phi} + u_1, \tag{4.7}$$

$$\begin{aligned}
&2\,u_{1y}{\phi_x}^3 + 4\,{\phi_x}^2 u_{1x}\phi_y - 2\,{\phi_x}^2\phi_t + 2\,{\phi_x}^2\phi_{xxy} + 2\,\phi_x\phi_y\phi_{xxx} \\
&- 2\,\phi_x\phi_{xy}\phi_{xx} + {\phi_y}^3 - \phi_y{\phi_{xx}}^2 = 0,
\end{aligned} \tag{4.8}$$

$$\begin{aligned}
&24\,{\phi_x}^3 u_{1xy} + 24\,{\phi_x}^2\phi_y u_{1xx} + 48\,u_{1y}{\phi_x}^2\phi_{xx} + 48\,u_{1x}{\phi_x}^2\phi_{xy} + 48\,u_{1x}\phi_x\phi_y\phi_{xx} \\
&- 24\,\phi_x\phi_{xx}\phi_t - 24\,\phi_{xt}{\phi_x}^2 + 24\,\phi_{xxxy}{\phi_x}^2 + 12\,\phi_x\phi_y\phi_{yy} + 12\,\phi_x\phi_{xxxx}\phi_y \\
&+ 12\,\phi_{xy}{\phi_y}^2 - 12\,{\phi_{xx}}^2\phi_{xy} = 0,
\end{aligned} \tag{4.9}$$

$$\begin{aligned}
&- 8\,u_{1xxy}{\phi_x}^2 - 4\,u_{1xxx}\phi_x\phi_y - 36\,\phi_x u_{1xy}\phi_{xx} - 24\,\phi_x\phi_{xy}u_{1xx} - 16\,\phi_{xxx}u_{1y}\phi_x \\
&- 24\,\phi_{xxy}u_{1x}\phi_x - 12\,\phi_y\phi_{xx}u_{1xx} - 8\,u_{1x}\phi_y\phi_{xxx} - 12\,u_{1y}{\phi_{xx}}^2 - 24\,u_{1x}\phi_{xx}\phi_{xy} \\
&+ 4\,\phi_{xxx}\phi_t + 12\,\phi_{xxt}\phi_x - 2\,\phi_x\phi_{yyy} - 10\,\phi_x\phi_{xxxxy} - 6\,\phi_{xyy}\phi_y \\
&- 2\,\phi_{xxxxx}\phi_y + 12\,\phi_{xx}\phi_{xt} - 4\,\phi_{xxxy}\phi_{xx} - 6\,\phi_{xy}\phi_{yy} \\
&- 2\,\phi_{xy}\phi_{xxxx} + 4\,\phi_{xxy}\phi_{xxx} = 0,
\end{aligned} \tag{4.10}$$

$$\begin{aligned}
&8\,u_{1xxy}\phi_{xx} + 4\,u_{1xxx}\phi_{xy} + 12\,u_{1xy}\phi_{xxx} + 12\,\phi_{xxy}u_{1xx} + 4\,\phi_{xxxx}u_{1y} \\
&+ 8\,\phi_{xxxy}u_{1x} - 4\,\phi_{xxxt} + 2\,\phi_{xyyy} + 2\,\phi_{xxxxxy} = 0,
\end{aligned} \tag{4.11}$$

and

$$u_{1xxxxy} + 4\,u_{1xxy}u_{1x} + 2\,u_{1xxx}u_{1y} + 6\,u_{1xy}u_{1xx} + u_{1yyy} - 2\,u_{1xxt} = 0. \tag{4.12}$$

Next we shall find exact solutions for equation (4.1) based on invariant Painlevé analysis [55, 57]. It has been pointed out that the Painlevé expansion (4.2) can also be written in an optimal expansion variable χ as follows:

$$u = \chi^\alpha \sum_{j=0}^{\infty} u_j\,\chi^j, \quad \text{where } \chi = \left(\frac{\phi_x}{\phi} - \frac{\phi_{xx}}{2\phi_x}\right)^{-1}. \tag{4.13}$$

The gradients of χ are given by

$$\begin{aligned}
\chi_x =& 1 + \frac{1}{2}\,S\chi^2, \\
\chi_y =& -C + C_x\chi - \frac{1}{2}(CS + C_{xx})\chi^2, \\
\chi_t =& -K + K_x\chi - \frac{1}{2}(KS + K_{xx})\chi^2,
\end{aligned} \tag{4.14}$$

where

$$S = \frac{\phi_{xxx}}{\phi_x} - \frac{3}{2}\left(\frac{\phi_{xx}}{\phi_x}\right)^2,\ C = -\frac{\phi_y}{\phi_x},\ K = -\frac{\phi_t}{\phi_x}. \tag{4.15}$$

The quantities S, K and C are called *Schwarzian* of ϕ. Best thing about these quantities is that these are invariant under following Möbius group of homographic transformations

$$\phi \mapsto \frac{a\phi + b}{c\phi + d},\ ad - bc \neq 0, \tag{4.16}$$

where a, b, c, d are arbitrary constants. It is easy to prove that the compatibility conditions $\chi_{xt} = \chi_{tx}, \chi_{yt} = \chi_{ty}, \chi_{xy} = \chi_{yx}$ are satisfied identically by virtue of (4.15).

For exact solutions of equation (4.1), the corresponding truncated expansion for (4.13) reads as

$$u = \frac{u_0}{\chi} + u_1. \tag{4.17}$$

On substituting (4.17) into equation (4.1) along with (4.14) and vanishing different powers of χ, we can obtain complicated over-determined system of equations for u_0, u_1 and S, C, K. It is very difficult to solve such system of equations. However, if we take S, C, K as constants we can derive exact solutions for equation (4.1). Let us denote A_n as coefficient of χ^n, $n = -6, \ldots, 0$.
Relation $A_{-6} = 0$ gives

$$C(-60\,{u_0}^2 + 120\,u_0) = 0,$$

which implies $u_0 = 2$, and A_{-5} vanishes identically. Then, relation $A_{-4} = 0$ gives

$$12\,C^3 - 24\,CS - 24\,K + 48\,Cu_{1x} - 24\,u_{1y} = 0. \tag{4.18}$$

An immediate solution for (4.18) reads as follows:

$$u_1 = \frac{(-C^3 + 2\,CS + 2\,K)\,x}{4C} + F\left(y + \frac{x}{2C}, t\right), \tag{4.19}$$

F is an arbitrary function of its arguments, while A_{-3}, A_{-2}, A_{-1} and A_0 vanish identically if $F = f_1(t)(y + \frac{x}{2C}) + f_2(t)$. The system (4.14) implies

$$\chi = \sqrt{\frac{2}{S}} \tan\left[\sqrt{\frac{S}{2}}\,(x - Cy - Kt + d_0)\right]. \tag{4.20}$$

The relation (4.20) together with truncated Laurent expansion (4.17) and all above results give following solution:

$$\begin{aligned} u =& \frac{\sqrt{2S}}{\tan\left[\sqrt{\frac{S}{2}}\,(x - Cy - Kt + d_0)\right]} + \frac{(-C^3 + 2\,CS + 2\,K)\,x}{4C} \\ &+ f_1(t)\left(y + \frac{x}{2C}\right) + f_2(t), \end{aligned} \tag{4.21}$$

where d_0, S, C, K are arbitrary constants along with arbitrary functions $f_1(t)$ and $f_2(t)$. The detailed structure of solution can be studied by proper selection of arbitrary constants and functions.

4.3 Lie symmetries of DJKM equation (4.1)

In this section, we are interested in finding the Lie point symmetries of the equation (4.1). In order to find the Lie point symmetries, we consider Lie group of point transformations in x, y, t and u defined as follow:

$$\begin{aligned}
\tilde{x} &= x + \epsilon\,\xi_1(x,y,t,u) + O(\epsilon^2),\\
\tilde{y} &= y + \epsilon\,\xi_2(x,y,t,u) + O(\epsilon^2),\\
\tilde{t} &= t + \epsilon\,\xi_3(x,y,t,u) + O(\epsilon^2),\\
\tilde{u} &= u + \epsilon\,\eta(x,y,t,u) + O(\epsilon^2),
\end{aligned} \tag{4.22}$$

where ϵ is group parameter. The associated Lie algebra of infinitesimal symmetries is the set of vector fields of the following form

$$X = \xi_1\frac{\partial}{\partial x} + \xi_2\frac{\partial}{\partial y} + \xi_3\frac{\partial}{\partial t} + \eta\frac{\partial}{\partial u}. \tag{4.23}$$

The main task in the determination of Lie point symmetries is the determination of infinitesimals by linearizing the equation (4.1), which is done by prolongating the vector field (4.23) to the different derivatives that appear in the equation (4.1). The technical details of the procedure are well known [34, 123, 180, 181, 195–197, 211]. Technically, the procedure of the determination of the Lie point symmetries is completely algorithmic, which can be implemented in any symbolic manipulation program like Maxima and Maple. We have used both to cross-validate the results, and the infinitesimals obtained as follow:

$$\begin{aligned}
\xi_1 &= \frac{1}{4}\,x a_t + c,\\
\xi_2 &= \frac{1}{2}\,y a_t + b,\\
\xi_3 &= a,\\
\eta &= -\frac{1}{4}\,xy a_{tt} - \frac{1}{4}u a_t - y c_t - \frac{1}{2}b_t x + d,
\end{aligned} \tag{4.24}$$

where a, b, c and d are arbitrary functions of t. Therefore, the symmetries of equation (4.1) depend on four arbitrary functions. The infinitesimals (4.24) will aid in reductions in number of independent variables in equation (4.1), such reductions can be accomplished by solving characteristics equations associated with vector fields (4.23). In our case the characteristic system is given by

$$\frac{dx}{\xi_1} = \frac{dy}{\xi_2} = \frac{dt}{\xi_3} = \frac{du}{\eta}. \tag{4.25}$$

In following, we shall solve the system (4.25) for the various possibilities.
Case 4.3.1. Invariance under the general similarity reduction for $a \neq 0$.
On solving characteristics equations (4.25), we obtain following results.

- *Similarity variables.*

$$z_1 = \frac{x}{a^{\frac{1}{4}}} - \int \frac{c}{a^{\frac{5}{4}}}\,dt,\ z_2 = \frac{y}{a^{\frac{1}{2}}} - \int \frac{b}{a^{\frac{3}{2}}}\,dt. \tag{4.26}$$

- *Reduction field.*

$$u = \frac{1}{a^{\frac{1}{4}}} \int \left[M_1(t) z_1 + M_2(t) z_1 z_2 + M_3(t) z_2 + M_4(t) \right] dt + \frac{1}{a^{\frac{1}{4}}} F(z_1, z_2) \tag{4.27}$$

where

$$\begin{aligned}
M_1(t) &= \frac{1}{4a^{\frac{5}{4}}} \left(-a^{\frac{5}{4}} a_{tt} \int \frac{b}{a^{\frac{3}{2}}} \, \mathrm{d}t - 2\, a^{\frac{3}{4}} b_t \right) \\
M_2(t) &= -\frac{1}{4}\, a_{tt}, \\
M_3(t) &= \frac{1}{4\, a^{\frac{5}{4}}} \left(-a^{\frac{5}{4}} a_{tt} \int \frac{c}{a^{\frac{5}{4}}} \, \mathrm{d}t - 4\, a c_t \right), \\
M_4(t) &= \frac{1}{4\, a^{\frac{5}{4}}} \left(-a^{\frac{5}{4}} a_{tt} \int \frac{b}{a^{\frac{3}{2}}} \, \mathrm{d}t \int \frac{c}{a^{\frac{5}{4}}} \, \mathrm{d}t - 2\, a^{\frac{3}{4}} b_t \int \frac{c}{a^{\frac{5}{4}}} \, \mathrm{d}t \right. \\
&\qquad \left. - 4\, a c_t \int \frac{b}{a^{\frac{3}{2}}} \, \mathrm{d}t + 4\, d a^{\frac{1}{2}} \right).
\end{aligned}$$

- *Reduction.* Substituting (4.26) and (4.27) into equation (4.1), we obtain reduction

$$\begin{aligned}
8\, F_{z_1 z_1 z_2} F_{z_1} + 4\, F_{z_1 z_1 z_1} F_{z_2} + 12\, F_{z_1 z_1} F_{z_1 z_2} + z_1 F_{z_1 z_1 z_1} + 2\, z_2 F_{z_1 z_1 z_2} \\
+ 3\, F_{z_1 z_1} + 2\, F_{z_2 z_2 z_2} + 2\, F_{z_1 z_1 z_1 z_1 z_2} = 0.
\end{aligned} \tag{4.28}$$

At any given time, the curves along which similarity variables z_1, z_2 are constant, represents straight lines parallel to x-axis and y-axis (see Figure 4.1a).

Case 4.3.2. Invariance under non-uniform translation: $a = 0, b \neq 0, c \neq 0$.

- *Similarity variables.*

$$z_1 = bx - cy, \;\; z_2 = t. \tag{4.29}$$

- *Reduction field.*

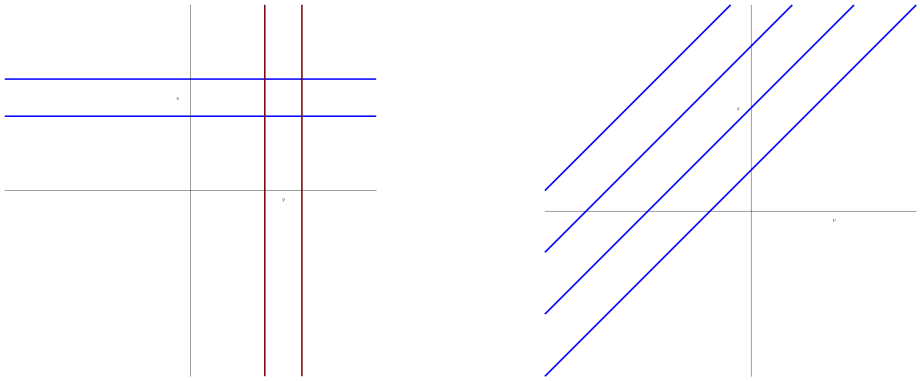
$$u = F(z_1, z_2) - \frac{1}{2\, b^2} \left[\left(b c_t + \frac{1}{2}\, c b_t \right) y^2 + \left(-2\, db + b_t z_1 \right) y \right]. \tag{4.30}$$

- *Reduction.* Substituting (4.29) and (4.30) into equation (4.1), we obtain reduction

$$\begin{aligned}
6\, b^3 c\, F_{z_1 z_1 z_1} F_{z_1} + 6\, b^3 c\, F_{z_1 z_1}^2 + 2\, b^2 F_{z_1 z_1 z_2} + b^4 c\, F_{z_1 z_1 z_1 z_1 z_1} \\
+ 7\, b b_t F_{z_1 z_1} - \left(2\, b^2 d - 3\, b b_t z_1 - c^3 \right) F_{z_1 z_1 z_1} = 0.
\end{aligned} \tag{4.31}$$

At any given time, the curves along which similarity variable z_1 is constant are straight lines in $x - y$ plane, and intercepts are defined by the arbitrary functions b, c (see Figure 4.1b)

Case 4.3.3. Invariance under non-uniform translation along y-axis: $a = 0, b \neq 0, c = 0$.



(a) **(b)**

Figure 4.1: Plots are drawn for similarity variables (4.26) and (4.29) at given instant of time. The parallel straight lines are invariant curves with respect to translation in space.

- *Similarity variables.*

$$z_1 = x,\ z_2 = t. \tag{4.32}$$

- *Reduction field.*

$$u = \frac{y\,(\,b_t x + 2\,d)}{2\,b\,(t)} + F\,(x,t)\,. \tag{4.33}$$

- *Reduction.* Substituting (4.32) and (4.33) into equation (4.1), we obtain reduction

$$\left(\frac{x b_t + 2\,d}{b}\right) F_{xxx} + \frac{3\,b_t}{b} F_{xx} - 2\,F_{xxt} = 0. \tag{4.34}$$

4.4 Conservation laws

In this section, based on two different techniques, we construct local conservation laws for equation (4.1). The first technique that we use is the direct method of Anco and Bluman [12–14] and the second technique is based on new theorem proposed by Ibragimov [117].

4.4.1 Conservation laws using direct method

Before proceeding toward the construction of non-trivial local conservation laws, we need to get familiarize with some of the basic terminologies used in the direct method. In following, we state some of the definitions and theorems.

Definition 4.4.1. The Euler operator with respect to dependent variable u^j is the operator defined by

$$E_{u^j} = \frac{\delta}{\delta u^j} = \frac{\partial}{\partial u^j} + \sum_{s=1}^{\infty} (-1)^s D_{i_1} \dots D_{i_s} \frac{\partial}{\partial u^j_{i_1 \dots i_s}},\ \text{for each } j = 1, \dots, m, \tag{4.35}$$

where D_i is the total differentiation with respect to independent variable $\mathbf{x} = (x^1, \ldots, x^n)$. The Euler operator defined by (4.35) has an interesting property that it can annihilate any divergence expression $D_i\Psi^i(u)$, and this property establishes the following theorem.

Theorem 4.4.1. The equations $E_{u^j}F(x, u, \partial u \ldots, \partial^p u) \equiv 0$ hold true, if and only if

$$F(x, u, \partial u \ldots, \partial^p u) \equiv D_i\Psi^i(x, u, \partial u, \ldots, \partial^{p-1}u),\ i = 1, \ldots, n, \tag{4.36}$$

hold for some functions $\Psi^i(x, u, \partial u, \ldots, \partial^{p-1}u)$.

Proof. The proof of theorem is given in [36]. □

The Theorem 4.4.1 aids in construction of local conservation laws through multipliers. The second theorem is stated as follows.

Theorem 4.4.2. A set of non-singular multipliers $\left\{\Lambda_\sigma(x, u, \partial u, \ldots, \partial^l u)\right\}_{\sigma=1}^N$ yields a local conservation law for PDE system $R^\sigma(x, u, \partial u, \ldots, \partial^k u) = 0$ if and only if

$$E_{u^j}\left(\Lambda_\sigma(x, u, \partial u, \ldots, \partial^l u)R^\sigma(x, u, \partial u, \ldots, \partial^k u)\right) \equiv 0,\ j = 1, \ldots, m. \tag{4.37}$$

Proof. The proof of theorem is given in [36]. □

The set of equations (4.37) are linear determining equations for all local conservation law multipliers up to order l. Solving determining equations (4.37) is same as solving determining equations for infinitesimals in Lie symmetry analysis. When all the multipliers are determined, the divergence expression (4.36) may be written as follows:

$$\Lambda_\sigma(u)R^\sigma[u] = D_i\Psi^i[u]. \tag{4.38}$$

The relation (4.38) gives local conservation laws

$$\mathrm{Div}\,\Psi[u] = 0, \tag{4.39}$$

holding on the solution space of system $R^\sigma[u] = R^\sigma(x, u, \partial u, \ldots, \partial^k u) = 0,\ \sigma = 1, \ldots, N$. For a given set of multipliers, the divergence expression (4.38) may be solved for fluxes $\Psi^i[u]$ in several ways. In one of the way when the multipliers are simple, the fluxes can be obtained by equating derivatives in (4.38), and sometimes inversion of divergence expression (4.38) for fluxes is too complex to handle. In such cases, homotopy operator may be used to invert the divergence expression (4.38). The detailed discussion and implementation of homotopy operator may be seen in Ref. [185] and references therein. Here in following, we will briefly discuss how homotopy operator is used for inversion of divergence expression.

Definition 4.4.2. Let $f(x, y, t, \mathbf{u}^M(x, y, t))$ be an exact differential expression, i.e. its Euler operation is zero with respect to dependent variable $\mathbf{u}(x, y, t)$, then there exist a function F such that $F = \mathrm{Div}^{-1}f = \left(\mathcal{H}^{(x)}_{\mathbf{u}(x,y,t)}f, \mathcal{H}^{(y)}_{\mathbf{u}(x,y,t)}f, \mathcal{H}^{(t)}_{\mathbf{u}(x,y,t)}f\right)$, where the x-component is given by

$$\mathcal{H}^{(x)}_{\mathbf{u}(x,y,t)}f = \int_{\lambda_0}^1 \left(\sum_{j=1}^N \mathcal{I}^{(x)}_{u^j(x,y,t)}f\right)[\lambda\mathbf{u}]\frac{d\lambda}{\lambda}. \tag{4.40}$$

The y- and z-components can be written in a similar way. The x-integrand is given by

$$\mathcal{I}^{(x)}_{u^j(x,y,t)} f = \sum_{k_1=1}^{M_1^j} \sum_{k_2=0}^{M_2^j} \sum_{k_3=0}^{M_3^j} \sum_{i_1=0}^{k_1-1} \sum_{i_2=0}^{k_2} \sum_{i_3=0}^{k_3} \Big(B^{(x)} u^j_{i_1 x i_2 y i_3 t} (-D_x)^{k_1-i_1-1} \times (-D_y)^{k_2-i_2} (-D_t)^{k_3-i_3} \Big) \frac{\partial f}{\partial u^j_{k_1 x k_2 y k_3 t}},$$

where M_1^j, M_2^j and M_3^j are orders of f for the components u^j with respect to x, y and t respectively, and M is maximum order for derivatives on u^j, $j = 1, \ldots, m$. The combinatorial coefficient $B^{(x)} = B(i_1, i_2, i_3, k_1, k_2, k_3)$ is defined as follows:

$$B(i_1, i_2, i_3, k_1, k_2, k_3) = \frac{\binom{i_1+i_2+i_3}{i_1} \binom{i_2+i_3}{i_2} \binom{k_1+k_2+k_3-i_1-i_2-i_3-1}{k_1-i_1-1} \binom{k_2+k_3-i_2-i_3}{k_2-i_2}}{\binom{k_1+k_2+k_3}{k_1} \binom{k_2+k_3}{k_2}}$$

In an analogous manner, the integrands $\mathcal{I}^{(y)}_{u^j(x,y,t)} f$ and $\mathcal{I}^{(t)}_{u^j(x,y,t)} f$ can also be defined, and based on cyclic permutations, the remaining combinatorial coefficients may be defined as $B^{(y)} = B(i_2, i_3, i_1, k_2, k_3, k_1), B^{(t)} = B(i_3, i_1, i_2, k_3, k_1, k_2)$.

Now we shall use the Theorem 4.4.1 to construct conservation law multipliers for equation (4.1). We consider zero order multiplier $\Lambda(x, y, t, u)$, and such multipliers are given by determining equations (4.37), so we have

$$E_u \left(\Lambda(x, y, t, u) \left(u_{xxxxy} + 4\, u_{xxy} u_x + 2\, u_{xxx} u_y + 6\, u_{xy} u_{xx} + u_{yyy} - 2\, u_{xxt} \right) \right) \equiv 0, \tag{4.41}$$

where E_u is Euler operator (4.35) defined for one dependent variable u only. The determining equations (4.41) splits with respect to $u_t, u_x, u_y, u_{xt}, u_{xx}, u_{xy}, u_{yy}, u_{xxt}, u_{xxx}, u_{xxy}, u_{yyy}, u_{xxxx}, u_{xxxy}, u_{xxxxy}$. This yields a linear determining system for $\Lambda(x, y, t, u)$ which can be solved by the same algorithmic method used to solve the determining equations for infinitesimal symmetries. We have obtained the following determining system for multipliers

$$\begin{aligned} &\Lambda_{yyy} = 0,\ \Lambda_{tu} - \Lambda_{xy} = 0,\ \Lambda_{uu} = 0, \\ &\Lambda_{xx} = 0,\ \Lambda_{xu} = 0,\ \Lambda_{yu} = 0, \end{aligned}$$

whose solution yields five multipliers

$$\begin{aligned} \Lambda_1 &= x f_1(t), \\ \Lambda_2 &= xy f_2'(t) + f_2(t) u, \\ \Lambda_3 &= \frac{1}{2} y^2 f_3(t), \\ \Lambda_4 &= y f_4(t), \\ \Lambda_5 &= f_5(t). \end{aligned} \tag{4.42}$$

Each multiplier from set (4.42) determines a local conservation law (4.39) in the format $D_x \Psi_1(x, y, t, u) + D_y \Psi_2(x, y, t, u) + D_t \Psi_3(x, y, t, u) = 0$ with the following characteristic form

$$\begin{aligned} \Lambda(x, y, t, u)(u_{xxxxy} + 4\, u_{xxy} u_x + 2\, u_{xxx} u_y + 6\, u_{xy} u_{xx} + u_{yyy} - 2\, u_{xxt}) \\ = D_x \Psi_1(x, y, t, u) + D_y \Psi_2(x, y, t, u) + D_t \Psi_3(x, y, t, u). \end{aligned} \tag{4.43}$$

The inversion of divergence expression (4.43) can be carried out by 3-dimensional homotopy operator (4.40), and results read as following

- $\Lambda_1 = xf_1(t)$.

$$\begin{aligned}
\Psi_1 =& -\frac{1}{6}\,xf_1\,(t)\,uu_{xxy} + \frac{8}{3}\,xf_1\,(t)\,u_xu_{xy} + \frac{11}{6}xf_1\,(t)\,u_yu_{xx} \\
&+ \frac{1}{3}\,uf\,(t)\,u_{xy} - \frac{5}{3}\,f_1\,(t)\,u_yu_x - \frac{4}{3}\,xf_1\,(t)\,u_{xt} + \frac{4}{5}\,xf_1\,(t)\,u_{xxxy} \\
&+ \frac{2}{3}\,xf_1'(t)u_x + \frac{2}{3}\,f_1\,(t)\,u_t - \frac{3}{5}\,f_1\,(t)\,u_{xxy} - \frac{4}{3}\,f_1'(t)u, \\
\Psi_2 =& \frac{f_1\,(t)}{30}\Big(5\,xuu_{xxx} + 45\,xu_xu_{xx} - 5\,uu_{xx} + 30\,xu_{yy} \\
&+ 6\,xu_{xxxx} - 20\,{u_x}^2 - 6\,u_{xxx}\Big), \\
\Psi_3 =& \frac{2}{3}\,f_1\,(t)\,\Big(u_x - xu_{xx}\Big).
\end{aligned}$$

- $\Lambda_2 = \; xyf_2'(t) + f_2\,(t)\,u.$

$$\begin{aligned}
\Psi_1 =& \frac{2}{5}\,f_2'(t)u_{x,x} + f_2'(t)uu_x + \frac{4}{5}\,f_2\,(t)\,uu_{xxxy} - \frac{4}{3}\,f_2\,(t)\,uu_{xt} \\
&- \frac{1}{9}\,f_2\,(t)\,u^2u_{xxy} - \frac{4}{3}\,yf_2''(t)u + \frac{2}{5}\,f_2\,(t)\,u_{xy}u_{xx} - \frac{3}{5}\,yf_2'(t)u_{xxy} \\
&- \frac{1}{5}\,f_2\,(t)\,u_yu_{xxx} - \frac{3}{5}\,f_2\,(t)\,u_xu_{xxy} + \frac{2}{3}\,f_2\,(t)\,u_xu_t - \frac{14}{9}\,f_2\,(t)\,{u_x}^2u_y \\
&+ \frac{2}{3}\,yf_2'(t)u_t - \frac{1}{5}\,xf_2'(t)u_{xxx} - \frac{2}{3}\,xf_2'(t){u_x}^2 - \frac{1}{6}\,xyf_2'(t)uu_{xxy} \\
&+ \frac{11}{6}\,xyf_2'(t)u_yu_{xx} + \frac{8}{3}\,xyf_2'(t)u_xu_{xy} - \frac{4}{3}\,xyf_2'(t)u_{xt} \\
&+ \frac{4}{5}\,xyf_2'(t)u_{xxxy} - \frac{5}{3}\,yf_2'(t)u_yu_x + \frac{2}{3}\,xyf_2''(t)u_x + \frac{26}{9}\,f_2\,(t)\,uu_xu_{xy} \\
&+ \frac{16}{9}\,f_2\,(t)\,uu_yu_{xx} - \frac{1}{6}\,xf_2'(t)uu_{xx} + \frac{1}{3}\,yf_2'(t)uu_{xy}, \\
\Psi_2 =& \frac{1}{9}\,f_2\,(t)\,u^2u_{xxx} + \frac{4}{3}\,f_2\,(t)\,uu_xu_{xx} - \frac{4}{9}\,f_2\,(t)\,u_x^3 + \frac{1}{6}\,xyf_2'(t)uu_{xxx} \\
&+ \frac{3}{2}\,xyf_2'(t)u_xu_{xx} - \frac{1}{6}\,yf_2'(t)uu_{xx} - \frac{2}{3}\,yf_2'(t)u_x^2 + f_2\,(t)\,uu_{yy} \\
&+ \frac{1}{5}\,f_2\,(t)\,uu_{xxxx} - \frac{1}{5}\,f_2\,(t)\,u_xu_{xxx} - \frac{1}{2}\,f_2\,(t)\,u_y^2 + \frac{1}{10}\,f_2(t)u_{xx}^2 \\
&+ f_2'(t)xyu_{yy} + \frac{1}{5}\,f_2'(t)xyu_{xxxx} - f_2'(t)xu_y - \frac{1}{5}\,f_2'(t)yu_{xxx}, \\
\Psi_3 =& -\frac{2}{3}\,f_2\,(t)\,uu_{xx} + \frac{1}{3}\,f_2\,(t)\,{u_x}^2 - \frac{2}{3}\,f_2'(t)xyu_{xx} + \frac{2}{3}\,f_2'(t)yu_x.
\end{aligned}$$

- $\Lambda_3 = \frac{1}{2}\,y^2f_3(t)$.

$$\begin{aligned}
\Psi_1 =& -\frac{1}{60}\,y\Big(5\,f_3\,(t)\,yuu_{xxy} - 80\,yf_3\,(t)\,u_xu_{xy} - 55\,yf_3\,(t)\,u_{xx}u_y
\end{aligned}$$

$$
\begin{aligned}
&+ 40\, y f_3(t)\, u_{xt} - 24\, y f_3(t)\, u_{xxxy} + 10\, f_3(t)\, u u_{xx} + 40\, f_3(t)\, u_x^2 \\
&- 20\, f_3'(t) y u_x + 12\, f_3(t)\, u_{xxx}\Big), \\
\Psi_2 =& \frac{1}{60} f_3(t) \Big(5\, y^2 u u_{xxx} + 45\, y^2 u_x u_{xx} + 30\, y^2 u_{yy} + 6\, y^2 u_{xxxx} \\
&- 60\, y u_y + 60\, u\Big), \\
\Psi_3 =& -\frac{1}{3} y^2 f_3(t)\, u_{xx}.
\end{aligned}
$$

- $\Lambda_4 = y f_4(t)$.

$$
\begin{aligned}
\Psi_1 =& -\frac{1}{6} y f_4(t)\, u u_{xxy} + \frac{8}{3} y f_4(t)\, u_x u_{xy} + \frac{11}{6} y f_4(t)\, u_{xx} u_y \\
&- \frac{1}{6} f_4(t)\, u u_{xx} - \frac{2}{3} f_4(t)\, u_x^2 - \frac{4}{3} y f_4(t)\, u_{xt} + \frac{4}{5} y f_4(t)\, u_{xxxy} \\
&+ \frac{2}{3} y f_4'(t) u_x - \frac{1}{5} f_4(t)\, u_{xxx}, \\
\Psi_2 =& \frac{1}{30} f_4(t) \Big(5\, u y u_{xxx} + 45\, y u_x u_{xx} + 30\, y u_{yy} + 6\, y u_{xxxx} - 30\, u_y\Big), \\
\Psi_3 =& -\frac{2}{3} y f_4(t)\, u_{xx}.
\end{aligned}
$$

- $\Lambda_5 = f_5(t)$.

$$
\begin{aligned}
\Psi_1 =& -\frac{1}{6} f_5(t)\, u u_{xxy} + \frac{8}{3} f_5(t)\, u_x u_{xy} + \frac{11}{6} f_5(t)\, u_{xx} u_y + \frac{2}{3} f_5'(t) u_x \\
&- \frac{4}{3} f_5(t)\, u_{xt} + \frac{4}{5} f_5(t)\, u_{xxxy}, \\
\Psi_2 =& \frac{1}{30} f_5(t) \Big(5\, u u_{xxx} + 45\, u_x u_{xx} + 30\, u_{yy} + 6\, u_{xxxx}\Big), \\
\Psi_3 =& -\frac{2}{3} f_5(t)\, u_{xx}.
\end{aligned}
$$

Remark 4.4.3. It is pertinent to mention here that, all the conservation laws are non-trivial i.e. the divergence expression $D_x\Psi_1(x,y,t,u) + D_y\Psi_2(x,y,t,u) + D_t\Psi_3(x,y,t,u)$ vanishes on the solution space of equation (4.1), and most importantly, none of the flux vanishes on solution space of equation (4.1).

4.4.2 Conservation laws using strict self-adjointness

Following the details mentioned in chapters 2 and 3, we need to define the formal Lagrangian

$$
I = v(x,y,t)(u_{xxxxy} + 4\, u_{xxy} u_x + 2\, u_{xxx} u_y + 6\, u_{xy} u_{xx} + u_{yyy} - 2\, u_{xxt}), \quad (4.44)
$$

where $v(x,t)$ is new dependent or non-local variable. The adjoint equation for equation (4.1) may be defined as follows:

$$F^* = \frac{\delta I}{\delta u} = 0, \tag{4.45}$$

where the Euler operator (or variational derivative) $\frac{\delta}{\delta u}$ is defined by (4.35).

On substituting Lagrangian (4.44) into (4.45) and using relation (4.35), we obtain adjoint equation:

$$F^* = 4\,v_{xxy}u_x + 2\,v_{xxx}u_y + 2\,v_{xy}u_{xx} + 4\,v_{xx}u_{xy} - 2\,v_{xxt} + v_{yyy} + v_{xxxxy} = 0. \tag{4.46}$$

Remark 4.4.4. The equation (4.1) is strictly self-adjoint, as its adjoint equation (4.46) coincides with it upon setting $v = u$. In the present case, the strict adjointness is identical with usual self-adjointness of linear differential equations. The detailed discussion about the concept of self-adjointness can be seen in the Ref. [116].

Expansion of the formula (2.41) for conserved currents yields

$$\begin{aligned}
\Phi^1 =& \xi_1 I + W\left[\frac{\partial I}{\partial u_x} - D_x\left(\frac{\partial I}{\partial u_{xx}}\right) - D_y\left(\frac{\partial I}{\partial u_{xy}}\right) + D_x^2\left(\frac{\partial I}{\partial u_{xxx}}\right)\right.\\
&\left. + D_xD_y\left(\frac{\partial I}{\partial u_{xxy}}\right) + D_xD_t\left(\frac{\partial I}{\partial u_{xxt}}\right) + D_x^3D_y\left(\frac{\partial I}{\partial u_{xxxxy}}\right)\right] + D_x(W)\\
&\times\left[\frac{\partial I}{\partial u_{xx}} - D_x\left(\frac{\partial I}{\partial u_{xxx}}\right) - D_y\left(\frac{\partial I}{\partial u_{xxy}}\right) - D_t\left(\frac{\partial I}{\partial u_{xxt}}\right) - D_x^2D_y\right.\\
&\left.\times\left(\frac{\partial I}{\partial u_{xxxxy}}\right)\right] + D_y(W)\left[\frac{\partial I}{\partial u_{xy}} - D_x\left(\frac{\partial I}{\partial u_{xxy}}\right) - D_x^3\left(\frac{\partial I}{\partial u_{xxxxy}}\right)\right]\\
&+ D_t(W)\left[\frac{\partial I}{\partial u_{xt}} - D_x\left(\frac{\partial I}{\partial u_{xxt}}\right)\right] + D_x^2(W)\left[\frac{\partial I}{\partial u_{xxx}} + D_xD_y\left(\frac{\partial I}{\partial u_{xxxxy}}\right)\right]\\
&+ D_xD_y(W)\left[\frac{\partial I}{\partial u_{xxy}} + D_x^2\frac{\partial I}{\partial u_{xxxxy}}\right] + D_xD_t(W)\frac{\partial I}{\partial u_{xxt}} - D_x^2D_y(W)\\
&\times D_x\left(\frac{\partial I}{\partial u_{xxxxy}}\right) - D_x^3(W)D_y\left(\frac{\partial I}{\partial u_{xxxxy}}\right) + D_x^3D_y(W)\frac{\partial I}{\partial u_{xxxxy}}\\
\Phi^2 =& \xi_2 I + W\left[\frac{\partial I}{\partial u_y} - D_x\left(\frac{\partial I}{\partial u_{xy}}\right) + D_x^2\left(\frac{\partial I}{\partial u_{xxy}}\right) + D_y^2\left(\frac{\partial I}{\partial u_{yyy}}\right)\right.\\
&\left. + D_x^4\left(\frac{\partial I}{\partial u_{xxxxy}}\right)\right] + D_x(W)\left[\frac{\partial I}{\partial u_{xy}} - D_x\left(\frac{\partial I}{\partial u_{xxy}}\right) - D_x^3\left(\frac{\partial I}{\partial u_{xxxxy}}\right)\right]\\
&- D_y(W)D_y\left(\frac{\partial I}{\partial u_{yyy}}\right) + D_x^2(W)\left[\frac{\partial I}{\partial u_{xxy}} + D_x^2\left(\frac{\partial I}{\partial u_{xxxxy}}\right)\right]\\
&- D_x^3(W)D_x\left(\frac{\partial I}{\partial u_{xxxxy}}\right) + D_x^4(W)\frac{\partial I}{\partial u_{xxxxy}}
\end{aligned}$$

$$\Phi^3 = \xi_3 I + W D_x^2\left(\frac{\partial I}{\partial u_{xxt}}\right) - D_x(W) D_x\left(\frac{\partial I}{\partial u_{xxt}}\right) + D_x^2(W)\frac{\partial I}{\partial u_{xxt}}. \tag{4.47}$$

By virtue of Remark 4.4.4, we can eliminate non-local variable v from the expressions (4.47) for conserved currents to construct local conservation laws for equation (4.1). Substituting $v = u$ into components (4.47), and after reckoning we obtain local conservation laws for equation (4.1) as

$$\begin{aligned}
\Phi^1 = & -\frac{1}{2}\,a_t y u_{xyy} u_{xx} + a_t y u_y u_{xt} - a_t y u_{xy} u_t - a_t y u_x u_{yt} + \frac{1}{2}\,a_t y u_{yy} u_{xxx} \\
& + \frac{1}{2}\,y a_{tt} u_x u_y - 3\,b_t x u_{xy} u_x - \frac{1}{2}\,a_t x u_{xx} u_t + 2\,a_t y u_{yy} {u_x}^2 + \frac{1}{4}\,a_t x u_y u_{xxxx} \\
& + \frac{1}{2}\,a_t y u_x u_{xxyy} + a u_{yt} u_{xxx} + b u_{yy} u_{xxx} - \frac{1}{2}\,a_{tt} u_x u - 2\,b u_x u_{yt} - 2\,a u_x u_{tt} \\
& - 2\,c {u_x}^2 u_{xy} + 4\,a u_{yt} {u_x}^2 + \frac{1}{2}\,a_t u^2 u_{xxy} + 4\,b u_{yy} {u_x}^2 - \frac{7}{4}\,a_t u u_{xxxy} + 2\,b u u_{xyt} \\
& - b u_{xyy} u_{xx} + 2\,y c_t u_{xt} + \frac{7}{4}\,a_t u_x u_{xxy} + \frac{7}{4}\,a_t u_y u_{xxx} - \frac{1}{2}\,b_t x u_{xxxy} + \frac{1}{4}\,a_{tt} x u_{xxx} \\
& + \frac{1}{4}\,y a_{t,t} u_{xxy} - \frac{7}{4}\,a_t u_{xy} u_{xx} - b_{tt} x u_x - \frac{1}{2}\,y a_{tt} u_t - 2\,y c_{tt} u_x + \frac{1}{2}\,a_{tt} x {u_x}^2 \\
& + b_t x u_{xt} - y c_t u_{xxxy} - \frac{7}{2}\,a_t u_x u_t + b_t u_x u_y + 6\,a_t {u_x}^2 u_y + 2\,b_t u u_{x,y} + \frac{7}{2}\,a_t u u_{xt} \\
& + 2\,a u u_{xtt} - a u_{xyt} u_{xx} - a u_{xxt} u_{xy} + a u_x u_{xxyt} + b u_x u_{xxyy} - 2\,c u_{xx} u_t - 2\,b u_{xy} u_t \\
& + a u_{xt} u_{xxy} + 6\,d u_{xy} u_x + 2\,b u_y u_{xt} + a u_y u_{xxxt} - 2\,d u u_{xxy} + \frac{1}{2}\,y a_{ttt} u + c u u_{yyy} \\
& + c u_y u_{xxxx} - a u_t u_{xxxy} - b u u_{xxxyy} - a u u_{xxxyt} + b_{tt} u - 2\,d u_{xt} + d u_{xxxy} + 2\,c_t {u_x}^2 - b_t u_t \\
& + 2\,d_t u_x + \frac{1}{2}\,b_t u_{xxy} + c_t u_{xxx} - \frac{1}{4}\,a_{tt} u_{xx} - 6\,y c_t u_x u_{xy} + \frac{1}{2}\,a_{tt} x y u_{xt} - \frac{1}{4}\,a_{tt} x y u_{xxxy}
\end{aligned}$$

$$\begin{aligned}
& -\frac{1}{2}\,a_{ttt} x y u_x - \frac{1}{2}\,a_t x {u_x}^2 u_{xy} - 4\,b u_{xyy} u u_x + 2\,a u_t u u_{xxy} - 6\,a u_t u_x u_{xy} \\
& - 4\,a u_{xyt} u u_x - 2\,b u_{yy} u u_{xx} + 6\,a u_{xt} u_y u_x - 2\,a u_{xxt} u u_y - \frac{1}{2}\,a_t y u u_{xxxyy} \\
& + b_t x u u_{xxy} + a_t y u u_{xyt} + y a_{tt} u u_{xy} - 2\,a u_{yt} u u_{xx} - 3\,a_t u u_y u_{xx} \\
& - \frac{11}{2}\,a_t u_{xy} u u_x + 2\,y c_t u u_{xxy} + \frac{1}{4}\,a_t x u u_{yyy} + 2\,c u u_x u_{xxy} + 4\,c u u_{xx} u_{xy} \\
& + \frac{3}{2}\,a_t x u_x u_y u_{xx} - \frac{3}{2}\,a_{tt} x y u_{xy} u_x + a_t x u u_{xx} u_{xy} - a_t y u_{yy} u u_{xx} \\
& + \frac{1}{2}\,a_{tt} x y u u_{xxy} - 2\,a_t y u_{xyy} u u_x + \frac{1}{2}\,a_t x u u_x u_{xxy} + 6\,c u_x u_y u_{xx},
\end{aligned}$$

$$
\begin{aligned}
\Phi^2 = & -\frac{3}{4}\,a_t u_{xx}^2 + \frac{1}{2}\,b_t u_{xxx} + 2\,b_t u_x^2 + 2\,a_t u_x^3 + \frac{3}{4}\,a_t u_y^2 + c_t u_y + d u_{yy} + d u_{xxxx} + b u_{xy} u_{xxx} \\
& + a u_{xt} u_{xxx} + c u_y u_{xy} - 2\,b u u_{xxt} + b u u_{yyy} - b u_{xxy} u_{xx} - y c_t u_{xxxx} + \frac{1}{4}\,y a_{tt} u_{xxx} \\
& - \frac{1}{2}\,b_t x u_{xxxx} + \frac{1}{4}\,a_{tt} x u_y + \frac{3}{2}\,a_t u_x u_{xxx} - \frac{1}{2}\,b_t x u_{yy} - y c_t u_{yy} + y a_{tt} u_x^2 + 4\,b u_{xy} u_x^2 \\
& + 4\,a u_{xt} u_x^2 - c u u_{xxxxx} + b u_x u_{xxxy} + a u_x u_{xxxt} - a u u_{xxxxt} + a u_y u_{yt} - c u_x u_{yy} \\
& - b_t u u_{xx} - 2\,c u_{xx}^2 u - b u_y u_{xxxx} - a u_t u_{yy} - a u_t u_{xxxx} - \frac{1}{4}\,a_t u u_{yy} - \frac{3}{2}\,a_t u u_{xxxx} \\
& + 6\,d u_{xx} u_x - a u_{xxt} u_{xx} - 2\,c {u_x}^2 u_{xx} - 3\,a_t y u_y u_{xx} u_x - \frac{3}{2}\,y a_{tt} x u_x u_{xx} - a_t x u_{xxx} u u_x \\
& + a_t y u u_y u_{xxx} + 2\,a_t y u u_{xx} u_{xy} - 2\,a u_{xt} u u_{xx} + 2\,b u u_y u_{xxx} + 4\,b u u_{xx} u_{xy} - 4\,c u_{xxx} u u_x \\
& - 4\,a u_{xxt} u u_x - a_t y u u_{xxt} + \frac{1}{2}\,a_t y u_{xy} u_{xxx} - \frac{1}{2}\,a_t x u_x^2 u_{xx} + \frac{1}{4}\,a_t x u_y u_{xy} + 2\,a_t y u_{xy} u_x^2 \\
& - 6\,y c_t u_x u_{xx} - 3\,b_t x u_x u_{xx} - \frac{1}{2}\,a_t y u_y u_{xxxx} - \frac{1}{4}\,y a_{tt} x u_{yy} - \frac{1}{4}\,a_{tt} x y u_{xxxx} \\
& - \frac{1}{4}\,a_t x u_x u_{yy} + \frac{1}{2}\,a_t y u_x u_{xxxy} - \frac{1}{2}\,a_t y u_{xxy} u_{xx} - 6\,b u_y u_x u_{xx} - 6\,a u_t u_x u_{xx} \\
& - \frac{11}{2}\,a_t u u_x u_{xx} - \frac{1}{2}\,y a_{tt} u u_{xx} - \frac{1}{2}\,a_t x u_{xx}^2 u - \frac{1}{4}\,a_t x u u_{xxxxx} + \frac{1}{2}\,a_t y u u_{yyy},
\end{aligned}
$$

$$
\begin{aligned}
\Phi^3 = & -2\,d u_{xx} - b_t u_x + 6\,a u u_{xx} u_{xy} + a_t y u_{xx} u_y - a_t y u_x u_{xy} + \frac{1}{2}\,a_t x u u_{xxx} \\
& + a_t y u u_{xxy} + \frac{1}{2}\,a_{tt} x y u_{xx} + 4\,a u u_x u_{xxy} + 2\,a u u_y u_{xxx} - a_t {u_x}^2 \\
& + a u u_{yyy} + 2\,b u u_{xxy} + 2\,c u u_{xxx} + 2\,u a_t u_{xx} + a u u_{xxxxy} + 2\,a u_{xx} u_t \\
& - \frac{1}{2}\,y a_{tt} u_x + b_t x u_{xx} + 2\,c_t y u_{xx} - 2\,b u_x u_{xy} - 2\,a u_x u_{xt} + 2\,b u_{xx} u_y.
\end{aligned}
$$

Remark 4.4.5. It is interesting to note that, Stephen C. Anco in his recent paper [11] pointed out that, the technique of Ibragimov is incomplete and sometimes gives trivial conservation laws, but the detailed calculations show that the divergence expression $D_x\Phi^1 + D_y\Phi^2 + D_t\Phi^3$ vanishes on the solution space of equations (4.1) and (4.46), and this confirms the validity of conservation laws. But in the present case, the equation (4.1) is self-adjoint i.e. nonlocal variable v can be replaced by u. So in this way, the conservation laws obtained above are all local conservation laws.

5

Nonclassical symmetries and similarity solutions of variable coefficient coupled KdV system using compatibility method[1]

> The only way to learn mathematics is to do mathematics.
>
> *Paul Halmos*

5.1 Introduction

The inheriting nonlinear character of any physical phenomenon can be better understood by mean of nonlinear PDEs. A rich variety of powerful methods have been proposed for exact solutions, such as tanh method [1, 163], F-expansion method [2,267], Jacobi elliptic method [24,134]. Hirota's bilinear method [104,205,209,210]. Analysis of PDEs using Lie group method is one of the most rigorous way of exploiting symmetry properties of PDEs [61, 91, 92, 137, 138, 151]. There exist several other methodologies for symmetry analysis of PDEs like nonclassical method of Bluman [37,44,140,146], direct method of Clarkson [50,51] etc. Apart from symmetry methods, there are several other tools available to analyze PDEs for analytical solutions [23,30,65,68,167,264–266], and beside analytical techniques, some numerical schemes for solving PDEs are also available [26,27]. Recently Yan and Liu [255] proposed a new method, known as the compatibility method, for exploiting nonclassical symmetries of PDEs by establishing compatibility condition between original PDE and corresponding modified invariant surface condition. In this chapter, we propose to explore symmetries of following variable coefficient KdV system

$$\begin{aligned} &u_t + a_1(t)\, uu_x + a_2(t)\, vv_x + a_3\,(t)u_{xxx} \;=\; 0, \\ &v_t + b_1(t)\, vu_x + b_2(t)\, uv_x + b_3(t)v_{xxx} \;=\; 0, \end{aligned} \tag{5.1}$$

[1]The content of this chapter is published in Nonlinear Dynamics 87(3) (2017): 1543-1552 [95]. The final publication is available at link.springer.com/article/10.1007/s11071-016-3132-3.

using the compatibility method. The KdV system (5.1) can be derived from original KdV equation by use of transformation $u(x,t) \to u(x,t) + i\,v(x,t)$ and separating the real and imaginary parts. The KdV system (5.1) has been investigated by Lei *et al.* [257] for periodic solutions, Wronskian and multi-soliton solutions are constructed using Hirota's bilinear method [262]. Using Fréchet derivative, Singh and Gupta [201] have exploited its variable coefficient version for Lie symmetries. The constant coefficient version of system (5.1) have many forms that have been discussed in literature, for example, Nutku-Oğuz model [173], Fuchssteiner equation [78], Drinfeld-Sokolov model [64], Hirota-Satsuma system [106]. In addition to this, the constants coefficient version of system (5.1) has been derived as a particular case of general coupled KdV system derived from the two layer model of the atmospheric dynamical systems [153] and two-component Bose-Einstein condensates [42]. Even though in past, the KdV system (5.1) and its various constant coefficient versions have been analyzed for Lie symmetries. But we believe that using the compatibility method we can recover some more general symmetries (or nonclassical symmetries in formal manner).

5.2 Symmetries of KdV system (5.1)

In present work, our main aim is to obtain nonclassical symmetries of the KdV system (5.1) using the compatibility method as described in Refs. [255,256]. Therefore, we seek nonclassical symmetry in the following form

$$\begin{aligned} u_t &= r(x,t)\,u_x + s_1(x,t)\,u + p_1(x,t), \\ v_t &= r(x,t)\,v_x + s_2(x,t)\,v + p_2(x,t), \end{aligned} \tag{5.2}$$

where $r(x,t), s_1(x,t), s_2(x,t), p_1(x,t)$ and $p_1(x,t)$ are unknown functions that need to be determined. In order to obtain compatibility condition, we substitute (5.2) into (5.1) and results read as follows:

$$a_1 u u_x + a_2 v v_x + r u_x + u s_1 + a_3 u_{xxx} + p_1 = 0, \tag{5.3a}$$
$$b_2 u v_x + b_1 v u_x + r v_x + v s_2 + b_3 v_{xxx} + p_2 = 0, \tag{5.3b}$$

Thus, the highest order derivatives terms in (5.3) are u_{xxx} and v_{xxx}, below this order no elimination could be possible. Similarly, equality of u_{tt} and v_{tt} between (5.1) and (5.2) gives

$$\begin{aligned} &a_1 u u_{xt} + a_1' u u_x + v a_2 v_{x,t} + a_2' v v_x + a_1 u_t u_x + a_2 v_t v_x + r u_{xt} + u s_1' \\ &+ a_3 u_{xxxt} + r_t u_x + s_1 u_t + a_3' u_{xxx} + p_1' = 0, \end{aligned} \tag{5.4a}$$
$$\begin{aligned} &b_2 u v_{xt} + b_2' u v_x + b_1 v u_{xt} + b_1' v u_x + b_1 u_x v_t + b_2 u_t v_x + r v_{xt} + v s_2' + b_3 v_{xxxt} \\ &+ r' v_x + s_2 v_t + b_3' v_{xxx} + p_2' = 0, \end{aligned} \tag{5.4b}$$

where the primes denote the derivative $\frac{d}{dt}$. Eliminating $u_t, v_t, u_{xt}, v_{xt}, u_{xxx}, v_{xxx}, u_{xxxt}$ and v_{xxxt} from (5.4) with the help of (5.2) and (5.3), we arrive at following determining equations

$$p_2 = 0,\ s_{1x} = 0,\ s_{2x} = 0,\ r_{xx} = 0,\ p_{1t} - \frac{a_3' p_1}{a_3} - 3 r_x p_1 + a_3 p_{1xxx} = 0, \tag{5.5a}$$

$$a_2' - \frac{a_3' a_2}{a_3} + 2a_2 s_2 - 2a_2 r_x - s_1 a_2 = 0,\ s_{1t} - \frac{a_3' s_1}{a_3} - 3r_x s_1 + a_1 p_{1x} = 0, \quad (5.5b)$$

$$r_t - \frac{a_3' r}{a_3} + a_1 p_1 - 3r r_x = 0, b_1' - \frac{b_3' b_1}{b_3} + b_1 s_1 - 2b_1 r_x = 0, \quad (5.5c)$$

$$b_2' - \frac{b_3' b_2}{b_3} + b_2 s_1 - 2b_2 r_x = 0,\ s_{2t} - \frac{b_3' s_2}{b_3} - 3r_x s_2 + p_{1x} b_1 = 0, \quad (5.5d)$$

$$r_t - \frac{b_3' r}{b_3} - 3r r_x + b_2 p_1 = 0,\ a_1' - \frac{a_1 a_3'}{a_3} - 2a_1 r_x + a_1 s_1 = 0. \quad (5.5e)$$

On solving determining equations (5.5), we get

$$\begin{aligned} r &= -\frac{a_3\left(c_1 x + c_3 \int a_1\, dt + c_4\right)}{3c_1 \int a_3\, dt + c_2},\ p_1 = \frac{c_3 a_3}{3c_1 \int a_3\, dt + c_2}, \\ s_1 &= -\frac{c_5 a_3}{3c_1 \int a_3\, dt + c_2},\ s_2 = -\frac{c_6 a_3}{3c_1 \int a_3\, dt + c_2}, \end{aligned} \quad (5.6)$$

where $c_i (i = 1 \ldots 6)$ are arbitrary constants of integration, and a_3 remains arbitrary function of t. The coefficient functions a_1, a_2, b_1, b_2 and b_3 are governed by following constraints

$$\begin{aligned} &\left[\frac{a_1'}{a_1} - \frac{a_3'}{a_3}\right] + a_3\left[\frac{2c_1 - c_5}{3c_1 \int a_3\, dt + c_2}\right] = 0,\ \left[\frac{a_2'}{a_2} - \frac{a_3'}{a_3}\right] + a_3\left[\frac{2c_1 + c_5 - 2c_6}{3c_1 \int a_3\, dt + c_2}\right] = 0, \\ &\left[\frac{b_1'}{b_1} - \frac{a_3'}{a_3}\right] + a_3\left[\frac{2c_1 - c_5}{3c_1 \int a_3\, dt + c_2}\right] = 0,\ \frac{b_3'}{b_3} = \frac{a_3'}{a_3},\ b_2 = a_1. \end{aligned} \quad (5.7)$$

On substituting (5.6) into (5.2), the nonclassical symmetry for variable coefficient KdV system (5.1) is written as follows:

$$u_t + \frac{a_3\left(c_1 x + c_3 \int a_1\, dt + c_4\right)}{3c_1 \int a_3\, dt + c_2} u_x + \frac{c_5 a_3}{3c_1 \int a_3\, dt + c_2} u - \frac{c_3 a_3}{3c_1 \int a_3\, dt + c_2} \equiv 0, \quad (5.8a)$$

$$v_t + \frac{a_3\left(c_1 x + c_3 \int a_1\, dt + c_4\right)}{3c_1 \int a_3\, dt + c_2} v_x + \frac{c_6 a_3}{3c_1 \int a_3\, dt + c_2} v \equiv 0. \quad (5.8b)$$

The corresponding vector field of symmetry (5.8) is given by

$$\begin{aligned} X \equiv &\frac{1}{a_3}\left(3c_1 \int a_3\, dt + c_2\right)\frac{\partial}{\partial t} + \left(c_1 x + c_3 \int a_1\, dt + c_4\right)\frac{\partial}{\partial x} \\ &+ (-c_5 u + c_3)\frac{\partial}{\partial u} - c_6 v \frac{\partial}{\partial v}. \end{aligned} \quad (5.9)$$

The Lie algebra associated with vector field (5.9) is spanned by six vectors

$$\begin{aligned} &X_1 = \left(\frac{3}{a_3}\int a_3\, dt\right)\frac{\partial}{\partial t} + x\frac{\partial}{\partial x}, \quad X_2 = \frac{1}{a_3}\frac{\partial}{\partial t}, \quad X_3 = \left(\int a_1\, dt\right)\frac{\partial}{\partial x} + u\frac{\partial}{\partial u}, \\ &X_4 = \frac{\partial}{\partial x}, \quad X_5 = -u\frac{\partial}{\partial u}, \quad X_6 = -v\frac{\partial}{\partial v}. \end{aligned} \quad (5.10)$$

Remark 5.2.1. Here we want to remark that for $c_3 = 0$, the nonclassical symmetries (5.6) reduce to symmetries obtained in [201]. This is also evident from symmetry operator (5.9), which is more general form of symmetry operator obtained in [201] (see for e.g. symmetry operator V given at (1.7) in [201]). That is, with the aid of the compatibility method we able to generalize symmetries for KdV system (5.1).

5.3 Reduction of system (5.1) and some exact solutions

Based on Olver's procedure [175] for the construction of an optimal system and by solving characteristics equations

$$\frac{dx}{r} = \frac{dt}{-1} = \frac{du}{-(s_1 u + p_1)} = \frac{dv}{-s_2 v}. \tag{5.11}$$

The following reductions for the system (5.1) have been obtained:
Reduction 5.3.1. $X_1 + \epsilon_5 X_5 + \epsilon_6 X_6$.
The similarity variables and coefficient functions corresponding to this subalgebra are obtained as follows:

$$\xi = \frac{x^3}{\int a_3\, dt}, \quad u = F(\xi)\left(\int a_3\, dt\right)^{-\frac{\epsilon_5}{3}}, \quad v = G(\xi)\left(\int a_3\, dt\right)^{-\frac{\epsilon_6}{3}}. \tag{5.12}$$

On solving the constrained conditions (5.7), the results read as follows:

$$a_1 = k_1 a_3 \left(\int a_3\, dt\right)^{-\frac{2}{3}+\frac{\epsilon_5}{3}}, \quad a_2 = k_2 a_3 \left(\int a_3\, dt\right)^{-\frac{2}{3}-\frac{\epsilon_5}{3}+\frac{2\epsilon_6}{3}},$$
$$b_1 = k_3 a_3 \left(\int a_3\, dt\right)^{-\frac{2}{3}+\frac{\epsilon_5}{3}}, \quad b_2 = a_1, \quad b_3 = k_4 a_3,$$

where k_1, k_2, k_3 and k_4 are arbitrary constants of integration. The reduction of system (5.1) obtained as follows:

$$\begin{aligned} &-\xi F' - \frac{1}{3}\epsilon_5 F + 3k_1 \xi^{\frac{2}{3}} F F' + 3k_2 \xi^{\frac{2}{3}} G G' + 27\xi^2 F''' + 54\xi F'' + 6F' = 0, \\ &-\xi G' - \frac{1}{3}\epsilon_6 G + 3k_3 \xi^{\frac{2}{3}} G F' + 3k_1 \xi^{\frac{2}{3}} F G' + 27k_4 \xi^2 G''' + 54k_4 \xi G'' + 6k_4 G' = 0. \end{aligned} \tag{5.13}$$

The traveling wave solution for (5.13) can be found using simple ansätz defined as follows:

$$\begin{aligned} F &= d_0\, \xi^{\frac{1}{3}} + d_1\, \xi^{-\frac{2}{3}}, \\ G &= e_0\, \xi^{\frac{1}{3}} + e_1\, \xi^{-\frac{2}{3}}. \end{aligned} \tag{5.14}$$

On substituting (5.14) into reduced equations (5.13) and solving resulting algebraic equation, we obtain

$$\begin{aligned} d_0 &= \frac{2\,\epsilon_5\epsilon_6 + 5\,\epsilon_5 + 2\,\epsilon_6 - 4}{6k_1\,(3\,\epsilon_5 + \epsilon_6 - 2)}, \\ d_1 &= -\frac{18\,(2\,\epsilon_5\epsilon_6 + 5\,\epsilon_5 + 2\,\epsilon_6 - 4)\,\epsilon_5}{k_1\,(3\,\epsilon_5 + \epsilon_6 - 2)\,(3\,{\epsilon_5}^2 + 4\,\epsilon_5\epsilon_6 - 5\,\epsilon_5 - 2\,\epsilon_6 + 4)}, \\ e_1 &= \frac{36\,(3\,\epsilon_5 - 4 + 2\,\epsilon_6)\,e_0}{3\,{\epsilon_5}^2 + 4\,\epsilon_5\epsilon_6 - 5\,\epsilon_5 - 2\,\epsilon_6 + 4}, \\ k_2 &= \frac{(2\,\epsilon_5\epsilon_6 + 5\,\epsilon_5 + 2\,\epsilon_6 - 4)\,\epsilon_5\,(2\,\epsilon_5 - 1)}{12k_1\,(3\,\epsilon_5 + \epsilon_6 - 2)^2\,{e_0}^2}, \\ k_3 &= \frac{k_1\,(4\,\epsilon_5\epsilon_6 + 2\,{\epsilon_6}^2 + \epsilon_5 - 4\,\epsilon_6)}{2\,\epsilon_5\epsilon_6 + 5\,\epsilon_5 + 2\,\epsilon_6 - 4}, \; k_4 = \frac{3\,(\epsilon_6 + 1)\,\epsilon_5}{3\,{\epsilon_5}^2 + 4\,\epsilon_5\epsilon_6 - 5\,\epsilon_5 - 2\,\epsilon_6 + 4}. \end{aligned} \tag{5.15}$$

By using similarity transformations (5.12) and (5.14), the traveling wave solution for KdV system (5.1) can be written as follows:

$$\begin{aligned} u &= \left(d_0 \left(\frac{x^3}{\int a_3(t)\ \mathrm{d}t} \right)^{\frac{1}{3}} + d_1 \left(\frac{x^3}{\int a_3(t)\ \mathrm{d}t} \right)^{-\frac{2}{3}} \right) \left(\int a_3(t)\ \mathrm{d}t \right)^{-\frac{\epsilon_5}{3}}, \\ v &= \left(e_0 \left(\frac{x^3}{\int a_3(t)\ \mathrm{d}t} \right)^{\frac{1}{3}} + e_1 \left(\frac{x^3}{\int a_3(t)\ \mathrm{d}t} \right)^{-\frac{2}{3}} \right) \left(\int a_3(t)\ \mathrm{d}t \right)^{-\frac{\epsilon_6}{3}}, \end{aligned} \tag{5.16}$$

where e_0 and k_1 are arbitrary; d_0, d_1, e_1, k_2, k_3 and k_4 are defined by (5.15).

Reduction 5.3.2. $X_1 + \epsilon_2 X_2$.

The similarity variables and coefficient functions corresponding to this subalgebra are obtained as follows:

$$\xi = \frac{x}{(3\int a_3\,dt + \epsilon_2)^{\frac{1}{3}}}, \quad u = F(\xi), \quad v = G(\xi). \tag{5.17}$$

On solving the constrained conditions (5.7), the results read as follows:

$$\begin{aligned} a_1 &= \frac{k_1 a_3}{(3\int a_3\,dt + \epsilon_2)^{\frac{2}{3}}}, a_2 = \frac{k_2 a_3}{(3\int a_3\,dt + \epsilon_2)^{\frac{2}{3}}}, \\ b_1 &= \frac{k_3 a_3}{(3\int a_3\,dt + \epsilon_2)^{\frac{2}{3}}}, b_2 = a_1, b_3 = k_4 a_3 \end{aligned}$$

where k_1, k_2, k_3 and k_4 are arbitrary constants of integration. The reduction of system (5.1) obtained as follows:

$$\begin{aligned} &-\xi F' + k_1 F F' + k_2 G G' + F''' = 0, \\ &-\xi G' + k_3 G F' + k_1 F G' + k_4 G''' = 0. \end{aligned} \tag{5.18}$$

For reduction (5.18) we seek solution in the following form

$$\begin{aligned} F &= d_0\,\xi + d_1, \\ G &= e_0\,\xi + e_0. \end{aligned} \tag{5.19}$$

On substituting (5.19) into reduced equation (5.18) and upon solving resulting algebraic equations, we get

$$d_0 = \frac{1}{k_1 + k_3},\ e_1 = -\frac{d_1 e_0 k_1 (k_1 + k_3)}{k_3},\ k_2 = \frac{k_3}{{e_0}^2 \left({k_1}^2 + 2\,k_1 k_3 + {k_3}^2 \right)}. \tag{5.20}$$

On substituting (5.20) into (5.19) and using similarity transformations (5.17), the traveling wave solution for KdV system (5.1) is obtained as follows:

$$u = \frac{x}{\left(3\int a_3(t)\ \mathrm{d}t + \epsilon_2\right)^{\frac{1}{3}} (k_1 + k_3)} + d_1, \tag{5.21a}$$

$$v = \frac{x e_0}{\left(3\int a_3(t)\ \mathrm{d}t + \epsilon_2\right)^{\frac{1}{3}}} - \frac{d_1 e_0 k_1 (k_1 + k_3)}{k_3}, \tag{5.21b}$$

where d_1, e_0, k_1, k_3 and k_4 are arbitrary constants.

Reduction 5.3.3. $X_2 + \epsilon_6 X_6$.
The similarity variables and coefficient functions corresponding to this subalgebra are obtained as follows:

$$\xi = x, \quad u = F(\xi), \quad v = G(\xi) \exp\left(-\int \epsilon_6 a_3 \, dt\right). \tag{5.22}$$

On solving the constrained conditions (5.7), the results read as follows:

$$\begin{aligned} &a_1 = k_1 a_3, \quad a_2 = k_2 a_3 \exp\left(2\int \epsilon_6 a_3 \, dt\right), \\ &b_1 = k_3 a_3, \quad b_2 = k_1 a_3, \quad b_3 = k_4 a_3, \end{aligned}$$

where k_1, k_2, k_3 and k_4 are arbitrary constants of integration. The reduction of system (5.1) obtained as follows:

$$\begin{aligned} &k_1 FF' + k_2 GG' + F''' = 0, \\ &k_3 GF' - \epsilon_6 G + k_1 FG' + k_4 G''' = 0. \end{aligned} \tag{5.23}$$

For traveling wave solutions of reduction (5.23), we use $\left(\frac{G'}{G}\right)$-method as described in [139, 235]. To proceed further, we seek solution of ODEs in the following form

$$\begin{aligned} F = \; & d_m \left(\frac{G'}{G}\right)^m + d_{m-1}\left(\frac{G'}{G}\right)^{m-1} + d_{m-2}\left(\frac{G'}{G}\right)^{m-2} + \ldots, \\ G = \; & e_n \left(\frac{G'}{G}\right)^n + e_{n-1}\left(\frac{G'}{G}\right)^{n-1} + e_{n-2}\left(\frac{G'}{G}\right)^{n-2} + \ldots, \end{aligned} \tag{5.24}$$

where m and n can be found by homogenous balance between highest order derivative term and non-linear term in (5.23). For the present case, we have $m = n = 2$ and $G = G(\xi)$ satisfies following second order linear ODE

$$G'' + \lambda G' + \mu G = 0, \tag{5.25}$$

where λ and μ are constants. Reduction of (5.25) can be written as follows:

$$\frac{G'}{G} = \frac{1}{2}\sqrt{\lambda^2 - 4\mu}\left(\frac{c_1 \cosh\left(\frac{1}{2}\sqrt{\lambda^2 - 4\,\mu}\xi\right) + c_2 \sinh\left(\frac{1}{2}\sqrt{\lambda^2 - 4\,\mu}\xi\right)}{c_1 \sinh\left(\frac{1}{2}\sqrt{\lambda^2 - 4\,\mu}\xi\right) + c_2 \cosh\left(\frac{1}{2}\sqrt{\lambda^2 - 4\,\mu}\xi\right)}\right) - \frac{\lambda}{2}. \tag{5.26}$$

On substituting (5.24) into (5.23), an algebraic system of equations in d_0, d_1, d_2, e_0, e_1 and e_2 can be obtained by equating coefficients of $\left(\frac{G'}{G}\right)$ to zero. To save space, we have omitted this algebraic system and solution to this system can be obtained using Maple as follows:

$$\begin{aligned} &d_1 = \lambda d_2, \; e_0 = \left(\frac{\lambda^2}{12} + \frac{2\mu}{3}\right) e_2, \; e_1 = \lambda e_2, \; \epsilon_6 = 0, \\ &k_1 = 0, \; k_2 = -\frac{12 d_2}{{e_2}^2}, \; k_3 = -\frac{12 k_4}{d_2}. \end{aligned} \tag{5.27}$$

On substituting (5.27) into (5.24) and using similarity transformations (5.22), we deduce that

Case 5.3.1. $\lambda^2 - 4\mu > 0$.

$$u = \left(\frac{d_2\lambda^2}{4} - d_2\mu\right) C - \left(\frac{d_2\lambda^2}{4} - d_0\right),$$
$$v = \left(\frac{e_2\lambda^2}{4} - \mu\, e_2\right) C - \left(\frac{e_2\lambda^2}{6} - \frac{2\mu\, e_2}{3}\right),$$

where $C = \left(\frac{c_1 \cosh\left(\frac{1}{2}\sqrt{\lambda^2-4\,\mu}x\right)+c_2 \sinh\left(\frac{1}{2}\sqrt{\lambda^2-4\,\mu}x\right)}{c_1 \sinh\left(\frac{1}{2}\sqrt{\lambda^2-4\,\mu}x\right)+c_2 \cosh\left(\frac{1}{2}\sqrt{\lambda^2-4\,\mu}x\right)}\right)^2$.

Case 5.3.2. $\lambda^2 - 4\mu < 0$.

$$u = \left(-\frac{d_2\lambda^2}{4} + d_2\mu\right) D - \left(\frac{d_2\lambda^2}{4} - d_0\right),$$
$$v = \left(-\frac{e_2\lambda^2}{4} + \mu\, e_2\right) D - \left(\frac{e_2\lambda^2}{6} - \frac{2\mu\, e_2}{3}\right),$$

where $D = \left(\frac{c_1 \cos\left(\frac{1}{2}\sqrt{-\lambda^2+4\,\mu}x\right)-c_2 \sin\left(\frac{1}{2}\sqrt{-\lambda^2+4\,\mu}x\right)}{c_1 \sin\left(\frac{1}{2}\sqrt{-\lambda^2+4\,\mu}x\right)+c_2 \cos\left(\frac{1}{2}\sqrt{-\lambda^2+4\,\mu}x\right)}\right)^2$.

Case 5.3.3. $\lambda^2 - 4\mu = 0$.

$$u = \frac{\left(-\lambda\, xc_2 - \lambda\, c_1 + 2\, c_2\right)^2 d_2}{\left(2\, xc_2 + 2\, c_1\right)^2} + \frac{\left(-\lambda\, xc_2 - \lambda\, c_1 + 2\, c_2\right) d_2\lambda}{2\, xc_2 + 2\, c_1} + d_0,$$
$$v = \frac{\left(-\lambda\, xc_2 - \lambda\, c_1 + 2\, c_2\right)^2 e_2}{\left(2\, xc_2 + 2\, c_1\right)^2} + \frac{\left(-\lambda\, xc_2 - \lambda\, c_1 + 2\, c_2\right) e_2\lambda}{2\, xc_2 + 2\, c_1} + \frac{\left(\lambda^2 + 8\,\mu\right) e_2}{12},$$

where c_1, c_2, d_0, d_2, e_2 and k_4 are arbitrary constants.

Reduction 5.3.4. $X_2 + \epsilon_5 X_5 + \epsilon_6 X_6$.

The similarity variables and coefficient functions corresponding to this subalgebra are obtained as follows:

$$\xi = x, \quad u = F(\xi) \exp\left(-\int \epsilon_5 a_3\, dt\right), \quad v = G(\xi) \exp\left(-\int \epsilon_6 a_3\, dt\right). \tag{5.28}$$

On solving the constrained conditions (5.7), the results read as follows:

$$a_1 = k_1 a_3 \exp\left(\int \epsilon_5 a_3\, dt\right), \quad a_2 = k_2 a_3 \exp\left(\int \left(2\epsilon_6 - \epsilon_5\right) a_3\, dt\right),$$
$$b_1 = k_3 a_3 \exp\left(\int \epsilon_5 a_3\, dt\right), \quad b_2 = a_1, \quad b_3 = k_4 a_3,$$

where k_1, k_2, k_3 and k_4 are arbitrary constants of integration. The reduction of system (5.1) is obtained as follows:

$$\begin{aligned} &-\epsilon_5 F + k_1 FF' + k_2 GG' + F''' = 0, \\ &-\epsilon_6 G + k_3 GF' + k_1 FG' + k_4 G''' = 0. \end{aligned} \tag{5.29}$$

On repeating procedure of $\left(\frac{G'}{G}\right)$-method as described in previous case of Reduction 5.3.3, we have following traveling wave solutions

Case 5.3.4. $\lambda^2 - 4\mu > 0$.

$$u = \left(\frac{d_2\lambda^2}{4} - d_2\mu\right) C - \frac{d_2\lambda^2}{6} + \frac{2d_2\mu}{3},$$
$$v = \left(\frac{e_2\lambda^2}{4} - e_2\mu\right) C - \frac{2e_2\lambda^2}{3} + \frac{2e_2\mu}{3},$$

where $C = \left(\frac{c_1 \cosh\left(\frac{1}{2}\sqrt{\lambda^2-4\,\mu}x\right)+c_2 \sinh\left(\frac{1}{2}\sqrt{\lambda^2-4\,\mu}x\right)}{c_1 \sinh\left(\frac{1}{2}\sqrt{\lambda^2-4\,\mu}x\right)+c_2 \cosh\left(\frac{1}{2}\sqrt{\lambda^2-4\,\mu}x\right)}\right)^2$.

Case 5.3.5. $\lambda^2 - 4\mu < 0$.

$$u = \left(-\frac{d_2\lambda^2}{4} + d_2\mu\right) D - \frac{d_2\lambda^2}{6} + \frac{2d_2\mu}{3},$$
$$v = \left(-\frac{e_2\lambda^2}{4} + e_2\mu\right) D - \frac{2e_2\lambda^2}{3} + \frac{2e_2\mu}{3},$$

where $D = \left(\frac{c_1 \cos\left(\frac{1}{2}\sqrt{-\lambda^2+4\,\mu}x\right)-c_2 \sin\left(\frac{1}{2}\sqrt{-\lambda^2+4\,\mu}x\right)}{c_1 \sin\left(\frac{1}{2}\sqrt{-\lambda^2+4\,\mu}x\right)+c_2 \cos\left(\frac{1}{2}\sqrt{-\lambda^2+4\,\mu}x\right)}\right)^2$.

Case 5.3.6. $\lambda^2 - 4\mu = 0$.

$$u = \frac{\left(-\lambda\,\xi\,c_2 - \lambda\,c_1 + 2\,c_2\right)^2 d_2}{\left(2\,\xi\,c_2 + 2\,c_1\right)^2} + \frac{\left(-\lambda\,\xi\,c_2 - \lambda\,c_1 + 2\,c_2\right) d_2\lambda}{2\,\xi\,c_2 + 2\,c_1} + \frac{d_2\lambda^2}{12} + \frac{2d_2\mu}{3},$$
$$v = \frac{\left(-\lambda\,\xi\,c_2 - \lambda\,c_1 + 2\,c_2\right)^2 e_2}{\left(2\,\xi\,c_2 + 2\,c_1\right)^2} + \frac{\left(-\lambda\,\xi\,c_2 - \lambda\,c_1 + 2\,c_2\right) e_2\lambda}{2\,\xi\,c_2 + 2\,c_1} + \frac{e_2\lambda^2}{12} + \frac{2e_2\mu}{3},$$

where c_1, c_2, d_2 and e_2 are arbitrary constants.

Reduction 5.3.5. $X_2 + \epsilon_4 X_4$.

The similarity variables and coefficient functions corresponding to this subalgebra are obtained as follows:

$$\xi = x - \int \epsilon_4 a_3\,dt, \quad u = F(\xi), \quad v = G(\xi). \tag{5.30}$$

On solving the constrained conditions (5.7), the results read as follows:

$$a_1 = k_1 a_3, \quad a_2 = k_2 a_3, \quad b_1 = k_3 a_3, \quad b_2 = a_1, \quad b_3 = k_4 a_3,$$

where k_1, k_2, k_3 and k_4 are arbitrary constants of integration. The reduction of system (5.1) is obtained as follows:

$$\begin{aligned} &-\epsilon_4 F' + k_1 F F' + k_2 G G' + F''' = 0 \\ &-\epsilon_4 G' + k_3 G F' + k_1 F G' + k_4 G''' = 0. \end{aligned} \tag{5.31}$$

Motivated by Riccati equation mapping method [268], we seek solution in the form

$$\begin{aligned} F &= \sum_{i=-m}^{i=m} d_i \phi^i(\xi), \\ G &= \sum_{i=-n}^{i=n} e_i \phi^i(\xi), \end{aligned} \tag{5.32}$$

where $\phi'(\xi) = r + p\phi(\xi) + q\phi^2(\xi)$. In (5.31), balancing the highest order derivative term and nonlinear term, we have $m = n = 2$, and on substituting (5.32) into (5.31), set of algebraic equations are obtained by equating powers of $\phi(\xi)$ to zero. Solving such system gives following results:

$$\begin{aligned} &d_{-2} = d_{-1} = 0,\, d_0 = -\frac{p^2 k_1 k_4 + 8\,qrk_1k_4 - \epsilon_4 k_1 - \epsilon_4 k_3}{(k_1 + k_3)\,k_1}, \\ &d_1 = -\frac{12pqk_4}{k_1 + k_3},\, d_2 = -\frac{12q^2k_4}{k_1 + k_3},\, e_{-2} = e_{-1} = 0, e_0 = k\left(\frac{p^2 + 8\,qr}{k_1 + k_3}\right), \\ &e_1 = \left(\frac{12pqk}{k_1 + k_3}\right),\, e_2 = \frac{12kq^2}{k_1 + k_3},\, k = \left(\frac{-k_1{k_4}^2 + k_1k_4 + k_3k_4}{k_2}\right)^{\frac{1}{2}}. \end{aligned} \tag{5.33}$$

On using solution set (5.33) and similarity transformations (5.30), we have following traveling wave solutions

$$\begin{aligned} u &= d_0 + d_1\phi_i(\xi) + d_2\phi_i^2(\xi), \\ v &= e_0 + e_1\phi_i(\xi) + e_2\phi_i^2(\xi), \end{aligned} \tag{5.34}$$

where for $i = 1, 2, 3, 4$.

$$\phi_1(\xi) = -\frac{p}{2q} - \frac{\sqrt{\theta}\tanh\left(\frac{1}{2}\sqrt{\theta}\xi\right)}{2q}, \qquad \theta > 0, \tag{5.35a}$$

and for $\theta < 0$,

$$\phi_2(\xi) = \frac{4r\sin\left(\frac{1}{4}\sqrt{-\theta}\xi\right)\cos\left(\frac{1}{4}\sqrt{-\theta}\xi\right)}{-2\,p\sin\left(\frac{1}{4}\sqrt{-\theta}\xi\right)\cos\left(\frac{1}{4}\sqrt{-\theta}\xi\right) + 2\sqrt{-\theta}\left(\cos\left(\frac{1}{4}\theta\xi\right)\right)^2 - \sqrt{-\theta}}, \tag{5.35b}$$

$$\phi_3(\xi) = -\frac{p\left(\cosh(p\xi) + \sinh(p\xi)\right)}{q\left(d_1 + \cosh(p\xi) + \sinh(p\xi)\right)}, \qquad r = 0, \tag{5.35c}$$

$$\phi_4(\xi) = -\frac{1}{q\xi + d_2}, \qquad r = p = 0, \tag{5.35d}$$

where $\theta = p^2 - 4\,qr$ and $\xi = x - \int \epsilon_4 a_3\,dt$, k_1, k_2, k_3, k_4 and ϵ_4 all are arbitrary constants.

Reduction 5.3.6. $X_2 + \epsilon_4 X_4 + \epsilon_5 X_5 + \epsilon_6 X_6$.

The similarity variables and coefficient functions corresponding to this subalgebra are obtained as follows:

$$\begin{aligned} \xi &= x - \int \epsilon_4 a_3\,dt, \\ u &= F(\xi)\exp\left(-\int \epsilon_5 a_3\,dt\right), \\ v &= G(\xi)\exp\left(-\int \epsilon_6 a_3\,dt\right). \end{aligned} \tag{5.36}$$

On solving the constrained conditions (5.7), the results read as follows:

$$\begin{aligned} &a_1 = k_1 a_3 \exp\left(\int \epsilon_5 a_3\,dt\right), \quad a_2 = k_2 a_3 \exp\left(\int (2\epsilon_6 - \epsilon_5) a_3\,dt\right), \\ &b_1 = k_3 a_3 \exp\left(\int \epsilon_5 a_3\,dt\right), \quad b_2 = k_1 a_3 \exp\left(\int \epsilon_5 a_3\,dt\right), \quad b_3 = k_4 a_3, \end{aligned}$$

where k_1, k_2, k_3 and k_4 are arbitrary constants of integration. The reduction of system (5.1) is obtained as follows:

$$\begin{aligned} &-\epsilon_4 F' - \epsilon_5 F + k_1 F F' + k_2 G G' + F''' = 0 \\ &-\epsilon_4 G' - \epsilon_6 G + k_3 G F' + k_1 F G' + k_4 G''' = 0. \end{aligned} \tag{5.37}$$

For reduction of type (5.37), we seek solution in the following form

$$\begin{aligned} F =&\ d_0 + d_1 \tanh(\xi) + d_2 \tanh^2(\xi), \\ G =&\ e_0 + e_1 \tanh(\xi) + e_2 \tanh^2(\xi). \end{aligned} \tag{5.38}$$

On substituting (5.38) into (5.37) and solving resulting algebraic equation, we get

$$\begin{aligned} &d_1 = 0,\ d_2 = -\frac{12k_4}{k_1 + k_3},\ e_0 = -\frac{8k}{k_1 + k_3},\ e_1 = 0,\ e_2 = \frac{12k}{k_1 + k_3}, \\ &k = \left(\frac{-k_1 {k_4}^2 + k_1 k_4 + k_3 k_4}{k_2}\right)^{\frac{1}{2}}, \\ &\epsilon_4 = \frac{k_1\,(d_0 k_1 + d_0 k_3 - 8\,k_4)}{k_1 + k_3},\ \epsilon_5 = 0,\ \epsilon_6 = 0. \end{aligned} \tag{5.39}$$

Thus with the aid of solutions (5.39) and similarity transformations (5.36), traveling wave solution for KdV system (5.1) can be written as follows:

$$\begin{aligned} u =&\ d_0 - \frac{12k_4}{k_1 + k_3} \tanh^2(x - A(t)), \\ v =&\ -\frac{8k}{k_1 + k_3} + \frac{12k}{k_1 + k_3} \tanh^2(x - A(t)), \end{aligned} \tag{5.40}$$

where $A(t) = \int \frac{k_1(d_0 k_1 + d_0 k_3 - 8\,k_4) a_3(t)}{k_1 + k_3}\,\mathrm{d}t$ and $d_0, k_1, k_3,\ k_4$ are arbitrary constants and k is given in (5.39).

Reduction 5.3.7. X_2.

The similarity variables and coefficient functions corresponding to this subalgebra are obtained as follows:

$$\xi = x, \quad u = F(\xi), \quad v = G(\xi). \tag{5.41}$$

On solving the constrained conditions (5.7), the results read as follows:

$$a_1 = k_1 a_3, \quad a_2 = k_2 a_3, \quad b_1 = k_3 a_3, \quad b_2 = a_1, \quad b_3 = k_4 a_3,$$

where k_1, k_2, k_3 and k_4 are arbitrary constants of integration. The reduction of system (5.1) obtained as follows:

$$\begin{aligned} &k_1 F F' + k_2 G G' + F''' = 0, \\ &k_3 G F' + k_1 F G' + k_4 G''' = 0. \end{aligned} \tag{5.42}$$

For traveling wave solutions of reduction (5.42), we use same procedure of Riccati equation mapping method as described in Reduction 5.3.5. After balancing the highest order derivative term with that of non-linear term, we seek solution of (5.42) in the following form

$$\begin{aligned} F &= d_0 + d_1 \phi(\xi) + d_2 \phi^2(\xi), \\ G &= e_0 + e_1 \phi(\xi) + e_2 \phi^2(\xi), \end{aligned} \tag{5.43}$$

where $\phi'(\xi) = r + p\phi(\xi) + q\phi^2(\xi)$. Substitution of (5.43) into (5.42) yields

$$d_1 = \frac{d_2 p}{q},\ e_0 = -\frac{e_2\,(p^2 d_2 - 12\,q^2 d_0 + 8\,qr d_2 - d_0 d_2 k_3)}{k_3 {d_2}^2},\ e_1 = \frac{e_2 p}{q}, \tag{5.44}$$

$$k_1 = -\frac{12\,q^2 + d_2 k_3}{d_2},\ k_2 = \frac{k_3 {d_2}^2}{{e_2}^2},\ k_4 = 1. \tag{5.45}$$

On using (5.44) and similarity transformations (5.41), traveling wave solutions for KdV system (5.1) can be written as

$$\begin{aligned} u &= d_0 + d_1\phi_i(\xi) + d_2\phi_i^2(\xi), \\ v &= e_0 + e_1\phi_i(\xi) + e_2\phi_i^2(\xi), \end{aligned} \tag{5.46}$$

where for $i = 1, 2, 3, 4$, $\phi_i(\xi)$ are already defined in (5.35) and d_0, d_2, e_2 and k_3 are arbitrary.

Reduction 5.3.8. X_4.

The similarity variables and coefficient functions corresponding to this subalgebra are obtained as follows:

$$\xi = t, \quad u = F(\xi), \quad v = G(\xi). \tag{5.47}$$

On solving the constrained conditions (5.7), the results read as follows:

$$a_1 = \frac{k_1 a_3}{(\int a_3\,dt)^{\frac{2}{3}}}, \quad a_2 = \frac{k_2 a_3}{(\int a_3\,dt)^{\frac{2}{3}}}, \quad b_1 = \frac{k_3 a_3}{(\int a_3\,dt)^{\frac{2}{3}}}, \quad b_2 = a_1, \quad b_3 = k_4 a_3$$

where k_1, k_2, k_3 and k_4 are arbitrary constants of integration. The reduction of system (5.1) obtained as follows:

$$F' = 0, \quad G' = 0. \tag{5.48}$$

The reduction (5.48) is equivalent to constant solution of system (5.1).

Remark 5.3.1. All the calculations and verifications of solutions are handled using Maple program.

6

Multi-dimensional optimal system of certain low dimensional Lie algebras

A mathematician is a blind man in a dark room looking for a black cat which isn't there.

Charles Darwin

6.1 Introduction

This chapter deals with the classification of certain low-dimensional Lie algebras in mathematical physics which are isomorphic to Lie algebras already classified by Patera *et al.* [182]. As an illustrative example, we have taken 5-dimensional Lie algebra of variant Boussinesq system with two central elements. After the removal of central elements, the 3-dimensional Lie algebra is isomorphic to algebra $A_{3,5}^{1/2}$. By using classification of $A_{3,5}^{1/2}$, we have obtained all splitting and non-splitting extensions of central elements to give out the complete classification of 5-dimensional Lie algebra of variant Boussinesq system. Besides this, four examples of Lie algebras which are also shown to be isomorphic to sub-algebras classified by Patera *et al.* and explicit isomorphism that preserves Lie brackets have also been given. In following, some definitions about basic concepts in Lie algebras have been given [90].

Definition 6.1.1. The centralizer of Lie algebra $\mathfrak{p}$ is a sub-algebra $\mathfrak{q}$ such that

$$[p, q] = 0, \quad \text{for all} \quad p \in \mathfrak{p}, \quad q \in \mathfrak{q}, \tag{6.1}$$

that is, elements of centralizer commutes with all the other elements in Lie algebra and this makes central elements invariant under inner automorphism. So that significant amount of work required in the classification can be reduced by removing central elements.

Definition 6.1.2. The normalizer of sub-algebra (or just sub-space) $\mathfrak{q}$ of $\mathfrak{p}$ is defined as follows:

$$\mathrm{Nor}_{\mathfrak{p}}(\mathfrak{q}) = \{x \in \mathfrak{p} \mid [x, \mathfrak{q}] \subset \mathfrak{q}\}, \tag{6.2}$$

or loosely speaking $\mathrm{Nor}_{\mathfrak{p}}(\mathfrak{q})$ is largest sub-algebra of $\mathfrak{p}$ containing $\mathfrak{q}$ as ideal [113].

Definition 6.1.3. A Lie algebra extension $\mathfrak{p}_2$ of $\mathfrak{p}_3$ by $\mathfrak{p}_1$ is a short exact sequence of Lie algebras

$$0 \to \mathfrak{p}_1 \to \mathfrak{p}_2 \xrightarrow{\pi} \mathfrak{p}_3 \to 0. \tag{6.3}$$

The extension is said to be split, if there is a Lie algebra homomorphism $\tau\colon \mathfrak{p}_3 \to \mathfrak{p}_2$ such that $\pi \circ \tau = id_{|\mathfrak{p}_3}$. In particular, a semidirect product $\mathfrak{p}_2 \cong \mathfrak{p}_3 \ltimes \mathfrak{p}_1$ gives a split extension (the direct product $\mathfrak{p}_2 = \mathfrak{p}_3 \oplus \mathfrak{p}_1$ as well, of course). Equivalence classes of such extensions with abelian Lie algebra $\mathfrak{p}_1$ are in bijection with the equivalence classes of 2-cocycles in $\mathfrak{H}^2(\mathfrak{p}_3, \mathfrak{p}_1)$. Here $\mathfrak{p}_1$ is a $\mathfrak{p}_3$-module in a natural way. On contrary if $\mathfrak{p}_2$ can not be written as semi-direct sum of $\mathfrak{p}_1$ and $\mathfrak{p}_3$, the extension is called non-splitting.

6.2 Lie algebra of variant Boussinesq system in abstract manner

The non-zero commutations of five-dimensional Lie algebra for variant Boussinesq system [203] can be written in abstract manner as follows:

$$[e_1, e_2] = e_2, \;\; [e_1, e_3] = 2\, e_3, \tag{6.4}$$

where we have considered abstract Lie algebra $\mathfrak{g} = \{e_1, e_2, e_3, e_4, e_5\}$ for Lie algebra of five vector fields

$$V_1 = \frac{2}{\gamma_1(t)} \left(\int \gamma_1(t)\, dt \right) \partial_t + x\, \partial_x, \;\; V_2 = \partial_x, \;\; V_3 = \frac{1}{\gamma_1(t)}\, \partial_t,$$
$$V_4 = -u\, \partial_u, \;\; V_5 = -v\, \partial_v.$$

It is easy see that the Lie algebra $\mathfrak{g}$ is solvable and can be written in direct sum as; $\mathfrak{g} = L_5 \oplus L_4 \oplus L_3$. Such that $L_3 = \{e_1, e_2, e_3\}$ and $L_4 = \{e_4\}$, $L_5 = \{e_5\}$ are centers of algebra $\mathfrak{g}$. The key point here is that the 3-dimensional sub-algebra L_3 is isomorphic to $A_{3,5}^{1/2}$ and classification of former algebra has been carried out by Patera and Winternitz [182] (see e.g. table I in the reference). We can prove isomorphism between L_3 and $A_{3,5}^{1/2}$ in following manner [60, 84, 90].

Let $(L_3, [.\,,.])$ be Lie algebra with Lie bracket $[.\,,.]$ defined by (6.4) and $\left(A_{3,5}^{1/2}, (.\,,.)\right)$ be Lie algebra with Lie bracket $(.\,,.)$ defined as $(e_1, e_3) = e_1, \;\; (e_2, e_3) = \alpha\, e_2$, here $\alpha = \frac{1}{2}$. Next is to establish isomorphism between L_3 and $A_{3,5}^{1/2}$, so we consider

$$\psi\colon L_3 \longrightarrow A_{3,5}^{1/2}, \quad e_1 \mapsto -2\, e_3, \;\; e_2 \mapsto e_2, \;\; e_3 \mapsto e_1. \tag{6.5}$$

This gives following isomorphism

$$\psi([e_1, e_2]) = \psi(e_2) = e_2 = (e_2, 2e_3) = (-2e_3, e_2) = (\psi(e_1), \psi(e_2)),$$
$$\psi([e_1, e_3]) = \psi(2e_3) = 2e_1 = (e_1, 2e_3) = (-2e_3, e_1) = (\psi(e_1), \psi(e_3)),$$
$$\psi([e_2, e_3]) = 0 = (e_2, e_1) = (\psi(e_2), \psi(e_3)).$$

This shows that, ψ preserves the Lie-brackets. So we can write $L_3 \cong A^{1/2}_{3,5}$. The isomorphism between algebras L_3 and $A^{1/2}_{3,5}$ ensures that the both algebras shall have the same classification. So in following, we shall use classification of algebra $A^{1/2}_{3,5}$ which has already been carried out by Patera and Winternitz [182] and we reproduce the same classification here as follows:

$$\begin{aligned} &\text{1-dim sub-algebras}: \quad \{0\}\,,\ \{e_1\}\,,\ \{e_2\}\,,\ \{e_3\}\,,\ \{e_1+\epsilon\, e_2\}\,, \quad \epsilon=\pm 1,\\ &\text{2-dim sub-algebras}: \quad \{e_1,e_2\}\,,\ \{e_3,e_1\}\,,\ \{e_3,e_2\}\,, \end{aligned} \tag{6.6}$$

where of course sub-algebra $\{0\}$ is trivial but it would be useful in further classification. To obtain full classification of 5-dimensional algebra $\mathfrak{g}$, we first need to find splitting and non-splitting extensions of algebra $L_4 = \{e_4\}$ in algebra $L_4 \oplus L_3$ by systematic procedure described in [182]. Afterwards, we shall go for splitting and non-splitting extensions of $L_5 = \{e_5\}$ in $L_5 \oplus L_4 \oplus L_3$, and that would complete our classification of 5-dimensional Lie algebra $\mathfrak{g}$. For splitting extensions of algebra $L_4 = \{e_4\}$, we follow step 3 of procedure described in [182]. To do this, we need to find all sub-algebras N_a contained in L_3 such that

$$[e_4, N_a] \subseteq N_a, \quad N_a \subseteq L_3. \tag{6.7}$$

Since e_4 is a central element in algebra, so we can run N_a through sub-algebras (6.6). Therefore, the second list of sub-algebras can be obtained as follows:

$$\begin{aligned} &\text{1-dim sub-algebras}: \quad \{e_4\}\,,\\ &\text{2-dim sub-algebras}: \quad \{e_4,e_1\}\,,\ \{e_4,e_2\}\,,\ \{e_4,e_3\}\,,\ \{e_4,e_1+\epsilon\, e_2\}\,,\\ &\text{3-dim sub-algebras}: \quad \{e_4,e_1,e_2\}\,,\ \{e_4,e_3,e_1\}\,,\ \{e_4,e_3,e_2\}\,. \end{aligned} \tag{6.8}$$

For non-splitting extensions of e_4, the step 4 of algorithm given in Ref. [182] may be executed, for which we need to find all sub-algebras of the following type

$$\left\{ e_4 + \sum_{i=1}^{3} x_i e_i, N_a \right\}, \tag{6.9}$$

where N_a is sub-algebra of L_3 with $L_4 \oplus L_3$ normalizer not contained in L_3 and $M = e_4 + \sum_{i=1}^{3} x_i e_i$ can be chosen in such way that it belongs to $\mathrm{Nor}_{L_4 \oplus L_3}(N_a)/N_a$ and the resulting sub-algebra can be simplified under adjoint actions (or precisely group of inner automorphism). Here, again as e_4 is central element and so we can run N_a through sub-algebras (6.6), but, before that we first need to write adjoint actions using non-zero commutations in $L_4 \oplus L_3$

$$\begin{aligned} &\mathrm{Ad}_{\exp(\epsilon\, e_3)}(e_1) = \mathrm{e}^{\epsilon} e_1,\ \ \mathrm{Ad}_{\exp(\epsilon\, e_3)}(e_2) = \mathrm{e}^{\alpha\epsilon} e_2,\\ &\mathrm{Ad}_{\exp(\epsilon\, e_1)}(e_3) = e_3 - \epsilon\, e_1,\ \ \mathrm{Ad}_{\exp(\epsilon\, e_2)}(e_3) = e_3 - \alpha\, \epsilon\, e_2. \end{aligned} \tag{6.10}$$

Case 6.2.1. For $N_a = \{0\}$, the sub-algebra (6.9) can be written as follows:

$$\left\{ e_4 + \sum_{i=1}^{3} x_i e_i \right\}, \tag{6.11}$$

and subsequently under adjoint actions (6.10) this sub-algebra simplified to sub-algebras

$$\{e_4 + x\, e_1\},\ \ \{e_4 + x\, e_2\},\ \ \{e_4 + x\, e_3\},\ \ \{e_4 + x\ (e_1 + \epsilon\, e_2)\},\ \ x \neq 0. \tag{6.12}$$

In sub-algebras (6.12), the second terms are exactly 1-dim sub-algebras given at (6.6) such that first two sub-algebras can be re-scaled to $\{e_4 \pm e_1\}$ and $\{e_4 \pm e_2\}$ under adjoint actions $\mathrm{Ad}_{\exp\left(\log \frac{1}{|x|}\, e_3\right)}(e_1)$ and $\mathrm{Ad}_{\exp\left(\frac{1}{\alpha}\, \log \frac{1}{|x|}\, e_3\right)}(e_2)$ respectively. Remaining two sub-algebras can be re-scaled by factor $1/x$, so we obtain third list of sub-algebras as follows:

$$\begin{aligned} \text{1-dim sub-algebras}: \ & \{e_4 \pm e_1\},\ \{e_4 \pm e_2\},\ \{e_3 + x\, e_4\}, \\ & \{\,(e_1 + \epsilon\, e_2) + x\, e_4\},\ x \neq 0. \end{aligned} \tag{6.13}$$

Case 6.2.2. For $N_a = \{e_1\}$, by Def. 6.1.2 one can check that

$$\mathrm{Nor}_{L_4 \oplus L_3}\left(\{e_1\}\right)/\{e_1\} = \{e_2, e_3, e_4\}. \tag{6.14}$$

So that in sub-algebra (6.9), we can appropriately choose $M = e_4 + x_2\, e_2 + x_3\, e_3$. For $x_3 \neq 0$, M further simplifies to $M = e_4 + x_3\, e_3$ under adjoint operation $\mathrm{Ad}_{\exp(\epsilon\, e_2)}(e_3)$ and for $x_3 = 0$, $M = e_4 + x_2\, e_2$, here x_2 can be scaled to ± 1 by adjoint action $\mathrm{Ad}_{\exp\left(\frac{1}{\alpha}\, \log \frac{1}{|x_2|}\, e_3\right)}(e_2)$. Thus, we obtain fourth list of sub-algebras as follows:

$$\text{2-dim sub-algebras}: \quad \{e_3 + x\, e_4,\, e_1\},\ \{e_2 \pm e_4, e_1\},\ x \neq 0. \tag{6.15}$$

Case 6.2.3. For $N_a = \{e_2\}$,

$$\mathrm{Nor}_{L_4 \oplus L_3}\left(\{e_2\}\right)/\{e_2\} = \{e_1, e_3, e_4\}, \tag{6.16}$$

we can choose $M = e_4 + x_1\, e_1 + x_3\, e_3$, for $x_3 \neq 0$. The adjoint action $\mathrm{Ad}_{\exp(\epsilon\, e_1)}(e_3)$ gives simplification $M = e_4 + x_3\, e_3$. For $x_3 = 0$, $M = e_4 + x_1\, e_1$, where coefficient x_1 scaled to ± 1 by adjoint action $\mathrm{Ad}_{\exp\left(\log \frac{1}{|x_1|}\, e_3\right)}(e_1)$. The fifth list of sub-algebras is given as follows:

$$\text{2-dim sub-algebras}: \quad \{e_3 + x\, e_4,\, e_2\},\ \{e_1 \pm e_4, e_2\},\ x \neq 0. \tag{6.17}$$

Case 6.2.4. For $N_a = \{e_3\}$,

$$\mathrm{Nor}_{L_4 \oplus L_3}\left(\{e_3\}\right)/\{e_3\} = \{e_4\}, \tag{6.18}$$

so that we can choose $M = e_4$. This gives sub-algebra $\{e_4, e_3\}$, which is already listed at (6.8).

Case 6.2.5. For $N_a = \{e_1 + \epsilon\, e_2\}$,

$$\mathrm{Nor}_{L_4 \oplus L_3}\left(\{e_1 + e_2\}\right) = \{e_1, e_2, e_4\}, \tag{6.19}$$

that is, normalizer of $\{e_1 + e_2\}$ is generated by three elements $\{e_1, e_2, e_4\}$. But as we can see for linear span in vector spaces, $\{e_1, e_2, e_4\} = \{e_1 + e_2, e_2, e_4\}$ or $\{e_1, e_1 + e_2, e_4\}$. Therefore we can write

$$\mathrm{Nor}_{L_4 \oplus L_3}\left(\{e_1 + e_2\}\right)/\{e_1 + e_2\} = \{e_2, e_4\} \text{ or } \{e_1, e_4\}, \tag{6.20}$$

so that we can choose $M = e_4 + x\, e_2$ or $e_4 + x\, e_1$ and this gives us sixth list of sub-algebras

$$\text{2-dim sub-algebras}: \quad \{e_1 + x\, e_4, e_1 + \epsilon\, e_2\},\ \{e_2 + x\, e_4, e_1 + \epsilon\, e_2\},\ x \neq 0 \tag{6.21}$$

Case 6.2.6. For $N_a = \{e_1, e_2\}$,

$$\mathrm{Nor}_{L_4\oplus L_3}\left(\{e_1,e_2\}\right)/\{e_1,e_2\} = \{e_3,e_4\}, \tag{6.22}$$

so that we can take $M = e_4 + x\, e_3$ and this gives us sub-algebra after re-scaling by factor $1/x$

$$\text{3-dim sub-algebras}: \quad \{e_3 + x\, e_4, e_1, e_2\}, \;\; x \neq 0. \tag{6.23}$$

For $N_a = \{e_3, e_1\}$ and $\{e_3, e_2\}$, $\mathrm{Nor}_{L_4\oplus L_3}(N_a)/N_a = \{e_4\}$, this would give 3-dimensional sub-algebras already listed at (6.8).

Thus complete list of sub-algebras of $L_4 \oplus L_3$

$$\begin{aligned}
&\{0\},\ \{e_1\},\ \{e_2\},\ \{e_3\},\ \{e_1+\epsilon\, e_2\},\ \{e_1,e_2\},\ \{e_3,e_1\},\ \{e_3,e_2\},\{e_4\},\\
&\{e_4,e_1\},\ \{e_4,e_2\},\ \{e_4,e_3\},\ \{e_4,e_1+\epsilon\, e_2\},\{e_4,e_1,e_2\},\ \{e_4,e_3,e_1\},\ \{e_4,e_3,e_2\},\\
&\{e_4 \pm e_1\},\ \{e_4 \pm e_2\},\ \{e_3 + x\, e_4\},\ \{(e_1+\epsilon\, e_2) + x\, e_4\},\ \{e_3 + x\, e_4,\, e_1\},\\
&\{e_2 \pm e_4, e_1\},\ \{e_3 + x\, e_4,\, e_2\},\ \{e_1 \pm e_4, e_2\},\ \{e_1 + x\, e_4, e_1 + \epsilon\, e_2\},\\
&\{e_2 + x\, e_4, e_1 + \epsilon\, e_2\},\ \{e_3 + x\, e_4, e_1, e_2\},\ \epsilon = \pm 1.
\end{aligned} \tag{6.24}$$

Now we move to classification in $\mathfrak{g} = L_5 \oplus \{L_4 \oplus L_3\}$ keeping in mind that sub-algebra $L_4 \oplus L_3$ has already been classified at (6.24). For splitting extensions of $L_5 = \{e_5\}$ we need to find all sub-algebras N_a contained in $L_4 \oplus L_3$ such that condition (6.7) holds. Here again e_5 being central element of $\mathfrak{g}$ so we obtain sub-algebras

$$\begin{aligned}
&\text{1-dim sub-algebras}: \{e_5\},\\
&\text{2-dim sub-algebras}: \{e_5,e_1\},\ \{e_5,e_2\},\ \{e_5,e_3\},\ \{e_5,e_1+\epsilon e_2\},\\
&\qquad\qquad \{e_5,e_4\},\ \{e_5,e_4 \pm e_1\},\quad \{e_5,e_4 \pm e_2\},\ \{e_5,e_3 + x\, e_4\},\\
&\qquad\qquad \{e_5,(e_1+\epsilon\, e_2) + x\, e_4\},\\
&\text{3-dim sub-algebras}: \{e_5,e_1,e_2\},\ \{e_5,e_3,e_1\},\ \{e_5,e_3,e_2\},\ \{e_5,e_4,e_1\},\\
&\qquad\qquad \{e_5,e_4,e_2\},\{e_5,e_4,e_3\},\ \{e_5,e_4,e_1+\epsilon\, e_2\},\\
&\qquad\qquad \{e_5,e_3 + x\, e_4,e_1\},\{e_5,e_2 \pm e_4,e_1\},\\
&\qquad\qquad \{e_5,e_3 + x\, e_4,e_2\},\ \{e_5,e_1 \pm e_4,e_2\},\\
&\qquad\qquad \{e_5,e_1 + x\, e_4,e_1+\epsilon\, e_2\},\{e_5,e_2 + x\, e_4,e_1+\epsilon\, e_2\},\\
&\text{4-dim sub-algebras}: \{e_5,e_4,e_1,e_2\},\ \{e_5,e_4,e_3,e_1\},\ \{e_5,e_4,e_3,e_2\},\\
&\qquad\qquad \{e_5,e_3 + x\, e_4,e_1,e_2\}.
\end{aligned} \tag{6.25}$$

Finally we need to find following sub-algebras

$$\left\{e_5 + \sum_{i=1}^{4} x_i e_i, N_a\right\} \tag{6.26}$$

for non-splitting extensions of e_5 by running N_a through all sub-algebras of $L_4 \oplus L_3$. For $N_a = \{0\}$, the linear combination $e_5 + \sum_{i=1}^{4} x_i e_i$ can be simplified using adjoint actions (6.10). For $x_4 \neq 0$, this simplifies to

$$\{e_5 + x\, e_4 + e_1\},\ \{e_5 + x\, e_4 + e_2\},\ \{e_5 + x\, e_4 + e_3\},\ \{e_5 + x\, e_4 + e_1 \pm e_2\}, \tag{6.27}$$

and for $x_4 = 0$ we obtain simplification

$$\{e_5 + e_1\},\ \ \{e_5 + e_2\},\ \ \{e_5 + e_3\},\ \ \{e_5 + e_1 \pm e_2\}, \tag{6.28}$$

for remaining N_a in $L_4 \oplus L_3$ we have listed classification in Table 7.2 by running N_a through (6.24)

N_a	$\mathrm{Nor}_{\mathfrak{g}}(N_a)/N_a$	Sub-algebras
$\{e_1\}$	$\{e_2, e_3, e_4, e_5\}$	$\{e_5 + x\,e_4 + y\,e_3, e_1\}$, $\{e_5 + x\,e_4 \pm e_2, e_1\}$
$\{e_2\}$	$\{e_1, e_3, e_4, e_5\}$	$\{e_5 + x\,e_4 + y\,e_3, e_2\}$, $\{e_5 + x\,e_4 \pm e_1, e_2\}$
$\{e_3\}$	$\{e_4, e_5\}$	$\{e_5 + x\,e_4, e_3\}$
$\{e_1 + \epsilon\,e_2\}$	$\{e_1, e_4, e_5\}$, $\{e_2, e_4, e_5\}$	$\{e_5 + x\,e_4 \pm e_1, e_1 + \epsilon\,e_2\}$, $\{e_5 + x\,e_4 \pm e_2, e_1 + \epsilon\,e_2\}$
$\{e_1, e_2\}$	$\{e_3, e_4, e_5\}$	$\{e_5 + x\,e_4 + y\,e_3, e_1, e_2\}$
$\{e_1, e_3\}$	$\{e_4, e_5\}$	$\{e_5 + x\,e_4, e_1, e_3\}$
$\{e_2, e_3\}$	$\{e_4, e_5\}$	$\{e_5 + x\,e_4, e_2, e_3\}$
$\{e_4\}$	$\{e_1, e_2, e_3, e_5\}$	$\{e_5 + y\,e_3, e_4\}$, $\{e_5 + z\,e_2 \pm e_1, e_4\}$
$\{e_1, e_4\}$	$\{e_2, e_3, e_5\}$	$\{e_5 + y\,e_3, e_1, e_4\}$, $\{e_5 \pm e_2, e_1, e_4\}$
$\{e_2, e_4\}$	$\{e_1, e_3, e_5\}$	$\{e_5 + y\,e_3, e_2, e_4\}$, $\{e_5 \pm e_1, e_2, e_4\}$
$\{e_3, e_4\}$	$\{e_5\}$	case already discussed
$\{e_4, e_1 + \epsilon\,e_2\}$	$\{e_2, e_5\}$, $\{e_1, e_5\}$	$\{e_5 \pm e_2, e_4, e_1 + \epsilon\,e_2\}$, $\{e_5 \pm e_1, e_4, e_1 + \epsilon\,e_2\}$
$\{e_1, e_2, e_4\}$	$\{e_3, e_5\}$	sub-algebra of dimension 5
$\{e_1, e_3, e_4\}$	$\{e_5\}$	case already discussed
$\{e_2, e_3, e_4\}$	$\{e_5\}$	case already discussed
$\{e_4 \pm e_1\}$	$\{e_1, e_2, e_5\}$, $\{e_2, e_4, e_5\}$	$\{e_5 + z\,e_2 \pm e_1, e_4 \pm e_1\}$, $\{e_5 + x\,e_4 \pm e_2, e_4 \pm e_1\}$
$\{e_4 \pm e_2\}$	$\{e_1, e_2, e_5\}$, $\{e_1, e_4, e_5\}$	$\{e_5 + z\,e_2 \pm e_1, e_4 \pm e_2\}$, $\{e_5 + x\,e_4 \pm e_1, e_4 \pm e_2\}$
$\{e_3 + x\,e_4\}$	$\{e_3, e_5\}$, $\{e_4, e_5\}$	$\{e_5 + y\,e_3, e_3 + x\,e_4\}$, $\{e_5 + x\,e_4, e_3 + x\,e_4\}$

$\{e_1+\epsilon\, e_2+x\, e_4\}$	$\{e_1,e_4,e_5\}$,	$\{e_5+x\, e_4\pm e_1, e_1+\epsilon\, e_2+x\, e_4\}$,
	$\{e_2,e_4,e_5\}$,	$\{e_5+x\, e_4\pm e_2, e_1+\epsilon\, e_2+x\, e_4\}$,
	$\{e_1,e_2,e_5\}$	$\{e_5+z\, e_2\pm e_1, e_1+\epsilon\, e_2+x\, e_4\}$
$\{e_3+x\, e_4, e_1\}$	$\{e_3,e_5\}$,	$\{e_5+y\, e_3, e_3+x\, e_4, e_1\}$,
	$\{e_4,e_5\}$	$\{e_5+x\, e_4, e_3+x\, e_4, e_1\}$
$\{e_2\pm e_4, e_1\}$	$\{e_2,e_5\}$,	$\{e_5\pm e_2, e_2\pm e_4, e_1\}$,
	$\{e_4,e_5\}$	$\{e_5+x\, e_4, e_2\pm e_4, e_1\}$
$\{e_3+x\, e_4, e_2\}$	$\{e_3,e_5\}$,	$\{e_5+y\, e_3, e_3+x\, e_4, e_2\}$,
	$\{e_4,e_5\}$	$\{e_5+x\, e_4, e_3+x\, e_4, e_2\}$
$\{e_1\pm e_4, e_2\}$	$\{e_1,e_5\}$,	$\{e_5\pm e_1, e_1\pm e_4, e_2\}$,
	$\{e_4,e_5\}$	$\{e_5+x\, e_4, e_1\pm e_4, e_2\}$
$\{e_1+x\, e_4, e_1+\epsilon\, e_2\}$	$\{e_5\}$	case already discussed
$\{e_2+x\, e_4, e_1+\epsilon\, e_2\}$	$\{e_5\}$	case already discussed
$\{e_3+x\, e_4, e_1, e_2\}$	$\{e_3,e_5\}$,	$\{e_5+y\, e_3, e_3+x\, e_4, e_1, e_2\}$,
	$\{e_4,e_5\}$	$\{e_5+x\, e_4, e_3+x\, e_4, e_1, e_2\}$

Table 6.1: Classification of algebra $\mathfrak{g} = L_5 \oplus L_4 \oplus L_3$.

The sub-algebras (6.25), (6.27) and (6.28) along with sub-algebras listed in Table 7.2 is complete list of multi-dimensional optimal list of sub-algebras for variant Boussinesq system. This list includes 1, 2, 3 and 4-dimensional optimal list of sub-algebras. using the inverse of isomorphism (6.5), one can see that 1-dimensional optimal list of sub-algebras reported here, does coincide with sub-algebras reported in [203].

In following, we give examples of Lie algebra of some well-known equations which are isomorphic or coincide with algebras classified in [182].

Example 6.2.1. In similar manner, the 6-dimensional Lie algebra for Hirota-Satsuma coupled KdV system [202]

$$\begin{aligned} &V_1 = \left(\frac{1}{\alpha(t)}\int \alpha(t)\, dt\right)\partial_t + x\,\partial_x,\ \ V_2 = \left(\frac{1}{\alpha(t)}\int \alpha(t)\, dt\right)\partial_t - u\,\partial_u, \\ &V_3 = \frac{1}{\alpha(t)}\,\partial_t, V_4 = \partial_x,\ \ V_5 = -v\,\partial_v,\ \ V_6 = -\phi\,\partial_\phi, \end{aligned} \tag{6.29}$$

can be expressed in abstract manner as $\mathfrak{h} = \{e_1, e_2, e_3, e_4, e_5, e_6\}$ such that the non-zero commutations are given as follows:

$$[e_1, e_3] = e_3,\ \ [e_1, e_4] = e_4,\ \ [e_2, e_3] = e_3,\ V_i \mapsto e_i \text{ (abstractly)},\ i = 1, \ldots, 6. \tag{6.30}$$

where e_5 and e_6 are centers of algebra $\mathfrak{h}$ such that, $\mathfrak{h} = L_6 \oplus L_5 \oplus L_4$, for $L_6 = \{e_6\}, L_5 = \{e_5\}, L_4 = \{e_1, e_2, e_3, e_4\}$. It is easy to see that by replacing e_1 by $e_1 - e_2$ the non-zero commutation relations (6.30) become

$$[e_1, e_4] = e_4,\ \ [e_2, e_3] = e_3. \tag{6.31}$$

Since e_6 and e_5 are centers of algebra $\mathfrak{h}$, so based on (6.31) we can start classification with the 4-dimensional sub-algebra L_4 which is isomorphic to $2\,A_2$ of [182] (see e.g. table II). For proof of isomorphism between L_4 and $2\,A_2$, we consider algebra $(L_4, [.\,,.])$ with Lie bracket $[.\,,.]$ defined by (6.31) and $(2\,A_2, (.\,,.))$ with Lie bracket $(.\,,.)$ defined by non-zero commutations $(e_1, e_2) = e_2, (e_3, e_4) = e_4$. We consider

$$\psi\colon L_4 \longrightarrow 2\,A_2, \quad e_1 \mapsto e_1,\ e_2 \mapsto e_3,\ e_3 \mapsto e_4,\ e_4 \mapsto e_2. \tag{6.32}$$

This gives following isomorphism

$$\begin{aligned} &\psi([e_1, e_4]) = \psi(e_4) = e_2 = (e_1, e_2) = (\psi(e_1), \psi(e_4)),\\ &\psi([e_2, e_3]) = \psi(e_3) = e_4 = (e_3, e_4) = (\psi(e_2), \psi(e_3)). \end{aligned} \tag{6.33}$$

The relations (6.33) show that ψ preserves the Lie brackets. Thus, the isomorphism between L_4 and $2A_2$ ensures that we can write full classification of algebra $\mathfrak{h} = L_6 \oplus L_5 \oplus L_4$ starting with classification of sub-algebra $L_4 (\cong 2\,A_2)$.

Example 6.2.2. Benjamin-Bona-Mahony (BBM) equation [204] admits Lie algebra of vector fields

$$V_1 = x\,\partial_x + \frac{2\,\delta(t)}{\delta'(t)}\,\partial_t,\ V_2 = \partial_x,\ V_3 = u\,\partial_u,\ V_4 = \partial_u, \ \text{ where } .' = \frac{d}{dt}. \tag{6.34}$$

In abstract manner, the vector fields (6.34) have following non-zero commutation relations

$$[e_1, e_2] = e_2,\ [e_3, e_4] = e_4,\ V_i \mapsto e_i \text{ (abstractly)},\ i = 1, \ldots, 4. \tag{6.35}$$

These commutation relations exactly coincides with that of algebra $2A_2$. The algebra $2A_2$ has 1-dimensional optimal list of sub-algebras given as follows:

$$\{e_2\},\ \{e_3\},\ \{e_4\},\ \{e_1 + x\,e_3\},\ \{e_1 + \epsilon\,e_4\},\ \{e_2 + \epsilon\,e_4\},\ \{e_2 + \epsilon\,e_3\}. \tag{6.36}$$

The optimal list of sub-algebras (6.36) is of course much finer than optimal list of sub-algebras reported in [204]. In addition to this, the 2 and 3-dimensional optimal list of sub-algebras have also been obtained as follow:

$$\begin{aligned} \text{2-dim sub-algebras}: &\{e_1 + x\,e_3, e_2\},\ \{e_3 + x\,e_1, e_4\},\ \{e_1 + \epsilon\,e_4, e_2\},\\ &\{e_3 + \epsilon\,e_2, e_4\}, \{e_1 + e_3, e_2 + \epsilon\,e_4\},\ \{e_2, e_3\},\\ &\{e_2, e_4\},\ \{e_1, e_3\},\ \{e_1, e_4\}\\ \text{3-dim sub-algebras}: &\{e_1, e_3, e_2\},\ \{e_1, e_4, e_2\},\ \{e_1, e_3, e_4\},\ \{e_2, e_3, e_4\},\\ &\{e_1 + \epsilon\,e_3, e_2, e_4\},\ \{e_1 + x\,e_3, e_2, e_4\},\ \epsilon = \pm 1. \end{aligned} \tag{6.37}$$

Example 6.2.3. The Navier-Stokes equation [112] admits 4-dimensional Lie algebra of vector fields

$$\begin{aligned} &V_1 = -\,y\,\partial_x + x\,\partial_y,\ V_2 = \frac{x}{2}\,\partial_x + \frac{y}{2}\,\partial_y + t\,\partial_t,\\ &V_3 = -\,yt\,\partial_x + xt\,\partial_y + \frac{x^2 + y^2}{2}\partial_u,\ V_4 = \partial_t. \end{aligned} \tag{6.38}$$

In abstract manner, the non-zero commutation relations for vector fields may be written as follow:

$$[e_2, e_3] = e_3,\ [e_2, e_4] = -e_4,\ [e_3, e_4] = -e_1.\ V_i \mapsto e_i \text{ (abstractly)},\ i = 1, \ldots, 4. \tag{6.39}$$

The commutation relations (6.39) show that the Lie algebra (6.38) is isomorphic to algebra $A_{4.8}$. The following transformations prove this fact

$$e_1 \mapsto e_1,\ e_2 \mapsto e_4,\ e_3 \mapsto e_3,\ e_4 \mapsto e_2. \tag{6.40}$$

As done previously, we can prove that the isomorphism (6.40) actually preserves Lie brackets. Based on new commutation relations, the Navier-Stokes equation admits multi-dimensional optimal list of sub-algebras given as follow:

$$\begin{aligned}
&\text{1-dim sub-algebras :}\{e_1\},\ \{e_2\},\ \{e_3\},\ \{e_2+\epsilon\, e_3\},\ \{e_4+x\, e_1\},\\
&\text{2-dim sub-algebras :}\{e_1,e_2\},\ \{e_1,e_3\},\ \{e_1,e_2+\epsilon\, e_3\},\ \{e_4,e_1\},\\
&\qquad\qquad\qquad\qquad\{e_4+x\, e_1,e_2\},\ \{e_4+x\, e_1,e_3\},\\
&\text{3-dim sub-algebras :}\{e_2,e_3,e_1\},\ \{e_4,e_1,e_2\},\ \{e_4,e_1,e_3\},\ \epsilon=\pm 1.
\end{aligned} \tag{6.41}$$

Looking at 1-dimensional sub-algebra (6.41), one might observe that this algebra is different from sub-algebra reported in [112]. Actually both are identical list, since every element in optimal list of sub-algebras is representative of conjugacy class under inner automorphism. So we may conclude that the 1-dimensional optimal list of sub-algebras written here is same as reported in [112]. Moreover, by (6.41) we able to justify the complete construction of 2 and 3-dimensional optimal list of sub-algebras, description of such construction is missing in [112].

Example 6.2.4. The turbulence model [43] admits 4-dimensional Lie algebra of following vector fields

$$V_1=\partial_x,\ V_2=\partial_t,\ V_3=x\,\partial_x+2u\,\partial_u+2v\,\partial_v,\ V_4=t\,\partial_t-2u\,\partial_u-3v\,\partial_v. \tag{6.42}$$

In abstract manner, the non-zero commutation relations for vector fields (10.40) are given as follow:

$$[e_1,e_3]=e_1,\ [e_2,e_4]=e_2,\ V_i\mapsto e_i\ \text{(abstractly)},\ i=1,\ldots,4. \tag{6.43}$$

which are isomorphic to sub-algebra $2\,A_2$ through the transformations

$$e_1\mapsto -e_2,\ e_2\mapsto -e_4,\ e_3\mapsto -e_1,\ e_4\mapsto -e_3 \tag{6.44}$$

and classification of this subalgebra is already written in Example 6.2.2. Here, we again want to state that the 1-dimensional optimal list of sub-algebras (6.36) for turbulence model is more general than optimal list of sub-algebras obtain in [43]. This means, the present optimal list of sub-algebras would provide more reductions for turbulence model and hence more explicit solutions.

Remark 6.2.1. Throughout the chapter we have taken vector fields of Lie algebra in an abstract manner, that is, the element "e_i" is equivalent to vector field "V_i" and all the multi-dimensional optimal list of sub-algebras that we have written in term of element "e_i" of abstract Lie algebra can be written back in term of vector "V_i" by reverting the explicit isomorphism.

Remark 6.2.2. The reduction in number of independent variables of PDE by 1-dimensional sub-algebra is well known. To perform reduction by 2-dimensional sub-algebra, one need to find invariants of two sub-algebras. For example, we take 2-dimensional sub-algebra $\{e_1,e_2+\epsilon\, e_3\}$ of Navier-Stokes equation from (6.41) and

using isomorphism (6.40), this sub-algebra is equivalent to $\{V_1, V_4 + \epsilon\, V_3\}$, the group invariant for second sub-algebra $V_4 + \epsilon\, V_3$ are

$$\sigma_1 = x^2 + y^2,\ \sigma_2 = 2\ \arctan\left(\frac{x}{y}\right) + t^2,\ \sigma_3 = -\frac{1}{2}(x^2 + y^2)t + u, \quad \epsilon = 1. \tag{6.45}$$

Next, the sub-algebra V_1 can be written in term of invariant (6.45), so we get

$$V_1 \equiv 0\,\partial_{\sigma_1} + (-2)\,\partial_{\sigma_2} + 0\,\partial_{\sigma_3}, \tag{6.46}$$

solving characteristics equations for (6.46), we get final group invariants

$$\xi = x^2 + y^2,\ u = \frac{1}{2}(x^2 + y^2)t + F(\xi), \tag{6.47}$$

using common invariants (6.47) of 2-dimensional sub-algebra, the Navier-Stokes equation actually reduce to ordinary differential equation in ξ as follows:

$$4\gamma\left(\xi^2\, F_{\xi\xi\xi\xi} + 4\xi\, F_{\xi\xi\xi} + 2\, F_{\xi\xi}\right) = 1. \tag{6.48}$$

The standard procedure of reduction by 2-dimensional subalgebra that we have described here can also be extended to 3-dimensional subalgebra. The classification of Lie algebras that we have presented here would give more reductions of PDEs that were missing earlier. The new reduction (6.48) that is missing in [112] justifies this. Some more examples of reductions could also be given here but that is not our major objective in this work.

7

On group classification and nonlocal conservation laws for a multi phase flow model[1]

> The saddest aspect of life right now is that gathers knowledge faster than society gathers wisdom.
>
> *Isaac Asimov*

7.1 Introduction

In fluid mechanics, the multi-phase flow exhibits a very complex dynamical phenomena. The phase can be a solid, a liquid, or a gas. Multiphase flow is the simultaneous flow of several phases. The study of multiphase flow is very important in energy-related industries and applications. The simplest case of multiphase flow is two-phase flow. Two-phase flow can be solid-liquid flow, liquid-liquid flow, gas-solid flow, or gas-liquid flow. The flow of mud, the flow of liquid with suspended solids such as slurries, the motion of the liquid in an aquifers exhibit solid-liquid flow. The flow of two immiscible liquids like oil and water, which is very important in oil recovery processes, is an example of liquid-liquid flow. The power systems especially in nuclear power plants, the steam-water flow is an example of two-phase single-component flow. In the petroleum industry [120, 176], two-phase flow of liquid and gas in pipes is very important, where the study of fluid flows becomes very complex when surface tension affects the dynamics of liquid.

In this chapter, we consider an isothermal no-slip drift-flux model for multiphase flows [19] of the following form:

$$\begin{aligned} &\rho_{1t} + \rho_1 u_x + u\rho_{1x} = 0, \\ &\rho_{2t} + \rho_2 u_x + u\rho_{2x} = 0, \\ &(\rho_1 + \rho_2)(u_t + uu_x) + a^2(\rho_{1x} + \rho_{2x}) = 0. \end{aligned} \tag{7.1}$$

[1]The content of this chapter is published in International Journal of Applied and Computational Mathematics (2017): 3(4), 3925–3935 [90].

Here ρ_1, ρ_2 are densities of respective phases, u is the common velocity, and a being constant which depends on both phases. The model (7.1) first derived from drift flux model [69] by making the assumption that the slip condition has a vanishing slip function. Subsequently, in Ref. [19] authors have investigated pipe network flow governed by isothermal no-slip drift-flux model (7.1). Bira and Sekhar [29] have studied this model for group invariant solutions and later in [192] Sekhar and Satapathy have investigated the same model for classification of admitted Lie algebra into multi-dimensional optimal systems. They have constructed different varieties of invariants of the group of inner automorphism including numerical and conditional invariants. Their work is based on technique developed by Chou *et al.* [47, 48]. Despite the early work [175] on group classification by using adjoint actions(or identification of equivalence classes based on the sign of Killing form), the work of Chou *et al.* is very useful as their additional invariants help to confirm optimality.

As discussed in previous chapters that all the real Lie algebras of dim ≤ 4 are already classified under a group of inner automorphism. In the present case [192], since the admitted Lie algebra is of low dimensionality so carrying group classification seems to be time consuming and wasteful adventure, and that too when classifications are already available. In this chapter, we have demonstrated that the construction of multi-dimensional optimal systems can be carried out in a much finer way than reported earlier work [192].

7.2 Explicit isomorphism for abstract Lie algebra of model (7.1)

The infinitesimal symmetry generators being linear differential operators form a Lie algebra. The procedure of finding an explicit form of generators is very complicated one and may lead to different mathematical expressions for generators of a given differential equation or a system of such equations. Therefore, it is very important to have a constructive approach for verifying whether two sets of generators determine essentially the same, i.e. isomorphic Lie algebras, or different ones. If a symmetry Lie algebra of differential equations can be identified as a member of available tables or databases, one is able to use known properties for further investigations, reduction of the order of ODEs, similarity reduction of PDEs. If the symmetry algebra is isomorphic to a known one it may be also used to find the explicit form of the transformation matrix [84].

The vector fields for model (7.1) obtained in Ref. [192] may be reproduce here as

$$\begin{aligned} &X_1 = \frac{\partial}{\partial x},\ X_2 = x\frac{\partial}{\partial x} + t\frac{\partial}{\partial t},\ X_3 = t\frac{\partial}{\partial x} + \frac{\partial}{\partial u}, \\ &X_4 = \frac{\partial}{\partial t},\ X_5 = \rho_1\frac{\partial}{\partial \rho_1} + \rho_2\frac{\partial}{\partial \rho_2}. \end{aligned} \tag{7.2}$$

The vector fields (7.2) can be written in abstract format with non zero Lie commutations obtained as follows:

$$[e_1, e_2] = e_1, [e_2, e_4] = -e_4, [e_3, e_4] = -e_1, \tag{7.3}$$

where in abstract manner, we have used notations $X_i \mapsto e_i,\ i = 1 \ldots 5$. It is easy to see that $L = \{e_1, e_2, e_3, e_4, e_5\}$ is solvable with e_5 as central element, so we may write direct sum $L = L_4 \oplus \{e_5\}$. Moreover, the central element being structural invariant of Lie algebra, so its removal will not affect the classification, and it can be suitably added to the list when the classification is over. The key point here is that, the 4-dimensional subalgebra L_4 is isomorphic to $A^0_{4,9}$ and the classification of this algebra has been carried out by Patera and Winternitz [182] (see e.g. table II). We can prove an isomorphism between L_4 and $A^0_{4,9}$ based on the technique described in the Refs. [60, 84]. In order to construct isomorphism, we first define a Lemma which shall be beneficial for understanding the concept of isomorphism.

Lemma 7.2.1. Let $\{x_1, \ldots, x_n\}$ be basis for Lie algebra $\mathscr{L}_1$ and $\{y_1, \ldots, y_n\}$ be basis for Lie algebra $\mathscr{L}_2$. Let c^k_{ij} and γ^k_{ij} be structure constants of $\mathscr{L}_1$ and $\mathscr{L}_2$ respectively. If $\psi : \mathscr{L}_1 \to \mathscr{L}_2$ is an isomorphism given by $\psi(x_i) = \sum_j a_{ij} y_j$, then

$$[\psi(x_i), \psi(x_j)] = \sum_{k,l=1}^{n} a_{ik} a_{jl} [y_k, y_l] = \sum_{k,l,m=1}^{n} \gamma^m_{kl} a_{ik} a_{jl} y_m \tag{7.4}$$

and

$$\psi([x_i, x_j]) = \sum_{k=1}^{n} c^k_{ij} \psi(x_k) = \sum_{k,m=1}^{n} c^k_{ij} a_{km} y_m. \tag{7.5}$$

This amounts to solution of algebraic equation in a_{ij}:

$$\sum_{k,l=1}^{n} \gamma^m_{kl} a_{ik} a_{jl} = \sum_{k=1}^{n} c^n_{ij} a_{km},\ \text{for } 1 \leq i < j \leq n,\ 1 \leq m \leq n. \tag{7.6}$$

The algebraic equations (7.6) can be solved by using Gröbner basis calculations to give out explicit isomorphism.

Theorem 7.2.2. Let $(L_4, [.\,,.])$ be the Lie algebra with Lie bracket $[.\,,.]$ defined by (7.3) and $\left(A^0_{4,9}, (.\,,.)\right)$ be the Lie algebra with Lie bracket $(.\,,.)$ defined as $(e_2, e_3) = e_1,\ (e_1, e_4) = e_1,\ (e_2, e_4) = e_2$. Then there exist an isomorphism between L_4 and $A^0_{4,9}$.

Proof. Define an isomorphism $\psi\colon L_4 \longrightarrow A^0_{4,9}$ between L_4 and $A^0_{4,9}$ as follows:

$$\psi(e_1) = -\,e_1,\ \psi(e_2) = e_4,\ \psi(e_3) = e_3,\ \psi(e_4) = -e_2. \tag{7.7}$$

Based on this definition, the relation (7.5) can be written as follows:

$$\begin{aligned}
&\psi([e_1, e_2]) = \psi(e_1) = -e_1 = -(e_1, e_4) = -(-\psi(e_1), \psi(e_2)) = (\psi(e_1), \psi(e_2)),\\
&\psi([e_2, e_4]) = \psi(-e_4) = e_2 = (e_2, e_4) = (-\psi(e_4), \psi(e_2)) = (\psi(e_2), \psi(e_4)),\\
&\psi([e_3, e_4]) = \psi(-e_1) = e_1 = (e_2, e_3) = (-\psi(e_4), \psi(e_3)) = (\psi(e_3), \psi(e_4)).
\end{aligned} \tag{7.8}$$

This shows that ψ preserves the Lie-brackets, so we can write $L_4 \cong A^0_{4,9}$. The isomorphism between algebras L_4 and $A^0_{4,9}$ ensures that the both algebras shall have the same classification. This proves the theorem. □

7.2.1 Construction of an optimal system

In this section, we shall use the explicit isomorphism among the Lie algebras L_4 and $A^0_{4,9}$. In following, it is evident that the classification of algebra $A^0_{4,9}$ has already been carried out by Patera and Winternitz [182] (see *e.g.* table II), and we reproduce the same classification as follows:

$$\begin{aligned}
&\text{1-dim subalgebras}: \{0\},\ \{e_1\},\ \{e_2\},\ \{e_3\},\ \{e_2+\epsilon e_3\},\ \{e_4+xe_3\},\\
&\text{2-dim subalgebras}: \{e_1,e_2\},\ \{e_1,e_3\},\ \{e_1,e_2+\epsilon e_3\},\ \{e_3,e_4\},\\
&\qquad\qquad \{e_4+xe_3,e_1\},\ \{e_4,e_2\},\\
&\text{3-dim subalgebras}: \{e_2,e_3,e_1\},\ \{e_3,e_4,e_1\},\ \{e_4,e_1,e_2\},\\
&\qquad\qquad \{e_4+xe_3,e_1,e_2\},\quad \epsilon=\pm 1,\ x\neq 0.
\end{aligned} \tag{7.9}$$

The subalgebra $\{0\}$ is of course trivial, but it would be useful in further classification. To obtain full classification of 5-dimensional algebra L, we first need to find splitting and non-splitting extensions of subalgebra $\{e_5\}$ in the algebra $L_4 \oplus \{e_5\}$ by systematic procedure described in the Ref. [182] (see also chapter 6). That would complete our classification of 5-dimensional Lie algebra L. For splitting extensions of subalgebra $\{e_5\}$, we follow step 3 of procedure described in the [182]. To do this, we need to find all subalgebras N_a contained in L_4 such that

$$[e_5, N_a] \subseteq N_a, \quad N_a \subseteq L_4. \tag{7.10}$$

Since e_5 is a central element in algebra L so this step is trivial, that is, we can run N_a through subalgebras (7.9) and we obtain the second list of subalgebras as follow:

$$\begin{aligned}
&\text{1-dim subalgebras}: \{e_5\},\\
&\text{2-dim subalgebras}: \{e_5,e_1\},\ \{e_5,e_2\},\ \{e_5,e_3\},\ \{e_5,e_2+\epsilon e_3\},\ \{e_5,e_4+xe_3\},\\
&\text{3-dim subalgebras}: \{e_5,e_1,e_2\},\ \{e_5,e_1,e_3\},\ \{e_5,e_1,e_2+\epsilon e_3\},\ \{e_5,e_3,e_4\},\\
&\qquad\qquad \{e_5,e_4+xe_3,e_1\},\ \{e_5,e_4,e_2\},\\
&\text{4-dim subalgebras}: \{e_5,e_2,e_3,e_1\},\ \{e_5,e_3,e_4,e_1\},\ \{e_5,e_4,e_1,e_2\},\\
&\qquad\qquad \{e_5,e_4+xe_3,e_1,e_2\},\quad \epsilon=\pm 1,\ x\neq 0.
\end{aligned} \tag{7.11}$$

For non-splitting extensions of $\{e_5\}$, we need to find all subalgebras of the following type

$$\left\{ e_5 + \sum_{i=1}^{4} x_i e_i, N_a \right\}, \tag{7.12}$$

where N_a is subalgebra of L_4 with $L_4 \oplus \{e_5\}$ normalizer not contained in L_4 and $M = e_5 + \sum_{i=1}^{4} x_i e_i$ can be chosen in such way that it belongs to $\mathrm{Nor}_{L_4 \oplus \{e_5\}}(N_a)/N_a$ and resulting subalgebra can be simplified under adjoint actions (or precisely group of inner automorphism). Here again as e_5 is central element and so we can run N_a through subalgebras (7.9). But before that, we first need to write non trivial adjoint actions using non-zero commutations in $L_4 \oplus \{e_5\}$

$$\begin{aligned}
&\mathrm{Ad}_{\exp(\alpha\, e_1)}(e_4) = e_4 - \alpha\, e_1,\ \mathrm{Ad}_{\exp(\alpha\, e_2)}(e_3) = e_3 - \alpha\, e_1,\\
&\mathrm{Ad}_{\exp(\alpha\, e_2)}(e_4) = e_4 - \alpha\, e_2,\ \mathrm{Ad}_{\exp(\alpha\, e_3)}(e_2) = e_2 + \alpha\, e_1,\\
&\mathrm{Ad}_{\exp(\alpha\, e_4)}(e_1) = \mathrm{e}^{\alpha} e_1,\ \ \mathrm{Ad}_{\exp(\alpha\, e_4)}(e_2) = \mathrm{e}^{\alpha} e_2.
\end{aligned} \tag{7.13}$$

The adjoint operations listed above can be easily obtained by using command "AdjointExp($-\epsilon_i e_i$)" of LieAlgebra package in Maple 18.
When $N_a = \{0\}$, the subalgebras (7.12) reduce to an extension of one-dimensional subalgebra (7.9) by central element e_5.

$$\begin{aligned}&\{e_5 + x_1e_1\},\ \{e_5 + x_2e_2\},\ \{e_5 + x_3e_3\},\\ &\{e_5 + a(e_2 + \epsilon e_3)\},\ \{e_5 + a(e_4 + xe_3)\},\ a \neq 0.\end{aligned} \tag{7.14}$$

The adjoint action $\mathrm{Ad}_{\exp(\alpha\, e_4)}$ with proper choice of parameter α gives following simplification of first two and fourth subalgebra in (7.14)

$$\{e_5 \pm e_1\},\ \{e_5 \pm e_2\},\ \{e_5 \pm (e_2 + ae_3)\}.$$

Therefore, the first list of one-dimensional non-splitting extensions of $\{e_5\}$ are as follow:

$$\begin{aligned}\text{1-dim subalgebras}:&\{e_5 \pm e_1\},\ \{e_5 \pm e_2\},\ \{e_5 + x_3e_3\},\\ &\{e_5 \pm (e_2 + ae_3)\},\ \{e_5 + a(e_4 + xe_3)\},\ a \neq 0.\end{aligned} \tag{7.15}$$

When $N_a = \{e_1\}$, we may chose $M = e_5 + x_2e_2 + x_3e_3 + x_4e_4$. Through judicious choice of adjoints (7.13), the subalgebra M simplifies into conjugacy classes with representatives $\{e_5 + e_2\},\ \{e_5 + e_3\},\ \{e_5 + e_2 + \epsilon e_3\},\ \{e_5 + e_4 + xe_3\}$. This gives following list of 2-dimensional subalgebra

$$\begin{aligned}\text{2-dim subalgebras}:&\{e_5 + e_2, e_1\},\ \{e_5 + e_3, e_1\},\ \{e_5 + e_2 + \epsilon e_3, e_1\},\\ &\{e_5 + e_4 + xe_3, e_1\}.\end{aligned} \tag{7.16}$$

When $N_a = \{e_2\}$, we may chose $M = e_5 + x_1e_1 + x_4e_4$. For $x_4 \neq 0$, we can use adjoint $\mathrm{Ad}_{\exp(\alpha\, e_1)}(e_4)$ to eliminate e_1 and M reduce to $e_5 + x_4e_4$. We obtain following subalgebra

$$\{e_5 + x_4e_4, e_2\}.$$

For $x_4 = 0$, we may use adjoint $\mathrm{Ad}_{\exp(\alpha\, e_4)}(e_1)$ to scale out x_1 and M reduce to $e_5 \pm e_1$. So we obtain following subalgebra

$$\{e_5 \pm e_1, e_2\}.$$

This gives following list of 2-dimensional subalgebra

$$\text{2-dim subalgebras}: \quad \{e_5 + x_4e_4, e_2\},\ \{e_5 \pm e_1, e_2\},\ x_4 \neq 0. \tag{7.17}$$

The complete list of remaining subalgebras is given in the Table 7.1.

N_a	M	subalgebras
$\{e_3\}$	$e_5 + x_1e_1 + x_4e_4$	$\{e_5 + x_4e_4, e_3\}$, $\{e_5 \pm e_1, e_3\},\ x_4 \neq 0$
$\{e_2 + \epsilon e_3\}$	$e_5 + x_1e_1$	$\{e_5 \pm e_1, e_2 + \epsilon e_3\}$
$\{e_4 + xe_3\}$	e_5	The case is repeating

$\{e_1, e_2\}$	$e_5 + x_3e_3 + x_4e_4$	$\{e_1, e_2, e_5 + x_3e_3\}$, $\{e_1, e_2, e_5 + x_4e_4\}$, $x_3 \neq 0,\ x_4 \neq 0$
$\{e_1, e_3\}$	$e_5 + x_2e_2 + x_4e_4$	$\{e_5 \pm e_2, e_1, e_3\}$, $\{e_5 + x_4e_4, e_1, e_3\},\ x_4 \neq 0$
$\{e_1, e_2 + \epsilon e_3\}$	e_5	The case is repeating
$\{e_3, e_4\}$	e_5	The case is repeating
$\{e_4 + xe_3, e_1\}$	e_5	The case is repeating
$\{e_4, e_2\}$	e_5	This case is repeating
$\{e_2, e_3, e_1\}$	$e_5 + x_4e_4$	$\{e_5 + x_4e_4, e_2, e_3, e_1\}$
$\{e_3, e_4, e_1\}$	e_5	This case is repeating
$\{e_4, e_1, e_2\}$	$e_5 + x_3e_3$	$\{e_5 + x_3e_3, e_4, e_1, e_2\},\ x_3 \neq 0$
$\{e_4 + xe_3, e_1, e_2\}$	e_5	This case is repeating

Table 7.1: List of non-splitting extensions of e_5, where N_a is running over subalgebras (7.9).

To sum up, the complete list of all multi-dimensional optimal subalgebras are given as below:

$$
\begin{aligned}
\text{1-dim subalgebras}: &\{e_5\},\ \{e_5 \pm e_1\},\ \{e_5 \pm e_2\},\ \{e_5 + x_3e_3\},\ \{e_5 \pm (e_2 + ae_3)\},\\
&\{e_5 + a(e_4 + xe_3)\}.\\
\text{2-dim subalgebras}: &\{e_5, e_1\},\ \{e_5, e_2\},\ \{e_5, e_3\},\ \{e_5, e_2 + \epsilon e_3\},\ \{e_5, e_4 + xe_3\},\\
&\{e_5 + e_2, e_1\},\ \{e_5 + e_3, e_1\},\ \{e_5 + e_2 + \epsilon e_3, e_1\},\\
&\{e_5 + e_4 + xe_3, e_1\},\ \{e_5 + x_4e_4, e_2\},\ \{e_5 \pm e_1, e_2\},\\
&\{e_5 + x_4e_4, e_3\},\ \{e_5 \pm e_1, e_3\},\ \{e_5 \pm e_1, e_2 + \epsilon e_3\}.\\
\text{3-dim subalgebras}: &\{e_5, e_1, e_2\},\ \{e_5, e_1, e_3\},\ \{e_5, e_1, e_2 + \epsilon e_3\},\ \{e_5, e_3, e_4\},\\
&\{e_5, e_4 + xe_3, e_1\},\ \{e_5, e_4, e_2\},\ \{e_1, e_2, e_5 + x_3e_3\},\\
&\{e_1, e_2, e_5 + x_4e_4\}, \{e_5 \pm e_2, e_1, e_3\},\ \{e_5 + x_4e_4, e_1, e_3\}.\\
\text{4-dim subalgebras}: &\{e_5, e_2, e_3, e_1\},\ \{e_5, e_3, e_4, e_1\},\ \{e_5, e_4, e_1, e_2\},\\
&\{e_5, e_4 + xe_3, e_1, e_2\},\ \{e_5 + x_4e_4, e_2, e_3, e_1\},\\
&\{e_5 + x_3e_3, e_4, e_1, e_2\}.
\end{aligned}
\tag{7.18}
$$

Based on isomorphism (7.7) and notations $X_i \mapsto e_i,\ i = 1 \ldots 5$, we can transform classification (7.18) into original notations given as follow:

$$
\text{1-dim subalgebras}: \{X_5\},\ \{X_5 \mp X_1\},\ \{X_5 \mp X_4\},\ \{X_5 + x_3X_3\},
$$

$$
\begin{aligned}
&\{X_5 \pm (-X_4 + aX_3)\}\,,\ \{X_5 + a(X_2 + bX_3)\}\,.\\
\text{2-dim subalgebras}:&\{X_5, X_1\}\,,\ \{X_5, X_4\}\,,\ \{X_5, X_3\}\,,\ \{X_5, -X_4 + \epsilon X_3\}\,,\\
&\{X_5, X_2 + bX_3\}\,,\ \{X_5 - X_4, X_1\}\,,\quad \{X_5 + X_3, X_1\}\,,\\
&\{X_5 - X_4 + \epsilon X_3, X_1\}\,,\ \{X_5 + X_2 + bX_3, X_1\}\,,\\
&\{X_5 + x_4X_2, X_4\},\ \{X_5 \mp X_1, X_4\},\ \{X_5 + x_4X_2, X_3\},\\
&\{X_5 \mp X_1, X_3\},\ \{X_5 \mp X_1, -X_4 + \epsilon X_3\}.\\
\text{3-dim subalgebras}:&\{X_5, X_1, X_4\}\,,\ \{X_5, X_1, X_3\}\,,\ \{X_5, X_1, -X_4 + \epsilon X_3\}\,,\\
&\{X_5, X_3, X_2\}\,,\quad \{X_5, X_2 + bX_3, X_1\}\,,\ \{X_5, X_2, X_4\}\,,\\
&\{X_1, X_4, X_5 + x_3X_3\},\ \{X_1, X_4, X_5 + x_4X_2\},\\
&\{X_5 \mp X_4, X_1, X_3\},\ \{X_5 + x_4X_2, X_1, X_3\}.\\
\text{4-dim subalgebras}:&\{X_5, X_4, X_3, X_1\}\,,\ \{X_5, X_3, X_2, X_1\}\,,\ \{X_5, X_2, X_1, X_4\}\,,\\
&\{X_5, X_2 + bX_3, X_1, X_4\}\,,\ \{X_5 + x_4X_2, X_4, X_3, X_1\},\\
&\{X_5 + x_3X_3, X_2, X_1, X_4\}.
\end{aligned}
\tag{7.19}
$$

Here "x" the arbitrary constant in abstract Lie algebra is replaced by arbitrary constant "b".

7.3 Nonlocal conservation laws for the model (7.1)

Define formal Lagrangian for equations (7.1) in the following manner

$$
\begin{aligned}
I =& f(x,t)(\rho_{1t} + \rho_1 u_x + u\rho_{1x} + g(x,t)(\rho_{2t} + \rho_2 u_x + u\rho_{2x})\\
&+ h(x,t)((\rho_1 + \rho_2)(u_t + uu_x) + a^2(\rho_{1x} + \rho_{2x})).
\end{aligned}
\tag{7.20}
$$

where $f(x,t), g(x,t)$ and $h(x,t)$ are new dependent variables (nonlocal). The adjoint equations for (7.1) can be written as

$$
F^* = \frac{\delta I}{\delta u} = 0,\ G^* = \frac{\delta I}{\delta \rho_1} = 0,\ H^* = \frac{\delta I}{\delta \rho_2} = 0,
\tag{7.21}
$$

On substituting Lagrangian (7.20) into (7.21) and using relation (2.39), we obtain following adjoint equations

$$
\begin{aligned}
F^* =& f\rho_{1,x} + g\rho_{2,x} - f_x\rho_1 - g_x\rho_2 - h_x\left(\rho_1 + \rho_2\right)u - \left(hu + f\right)\rho_{1,x}\\
&- \left(hu + g\right)\rho_{1,x} - h_t\left(\rho_1 + \rho_2\right) - h\rho_{1,t} - h\rho_{2,t} = 0,\\
G^* =& h\left(uu_x + u_t\right) - f_x u - h_x a^2 - f_t = 0,\\
H^* =& h\left(uu_x + u_t\right) - g_x u - h_x a^2 - g_t = 0.
\end{aligned}
\tag{7.22}
$$

For nonlocal conservation laws, the relation (2.41) can be simplified by writing Lagrangian I with respect to all mixed derivative $u^\alpha_{ij}, u^\alpha_{ijk}, \ldots$ in symmetric manner. We obtain

$$
\begin{aligned}
C^x =& \xi\, I + W^{(1)}\frac{\partial I}{\partial u_x} + W^{(2)}\frac{\partial I}{\partial \rho_{1,x}} + W^{(3)}\frac{\partial I}{\partial \rho_{2,x}}\\
C^t =& \tau\, I + W^{(1)}\frac{\partial I}{\partial u_t} + W^{(2)}\frac{\partial I}{\partial \rho_{1,t}} + W^{(3)}\frac{\partial I}{\partial \rho_{2,t}},
\end{aligned}
\tag{7.23}
$$

In following Table 7.2, we give conserved currents (7.23) corresponding to every symmetry in one-dimensional optimal system obtained at (7.19).

Symmetry	Conserved currents
X_5	$C^x = a^2h\rho_2 + a^2h\rho_1 + fu\rho_1 + gu\rho_2$ $C^t = f\rho_1 + g\rho_2$
$X_5 + \epsilon\, X_1$	$C^x = a^2h\rho_1 + a^2h\rho_2 + \epsilon\, h\rho_1u_t + \epsilon\, h\rho_2u_t + \epsilon\, f\rho_{1,t}$ $+ \epsilon\, g\rho_{2,t} + fu\rho_1 + gu\rho_2,$ $C^t = -\epsilon\, h\rho_1u_x - \epsilon\, h\rho_2u_x - \epsilon\, f\rho_{1,x} - \epsilon\, g\rho_{2,x} + f\rho_1 + g\rho_2$
$X_5 + \epsilon\, X_4$	$C^x = -a^2\epsilon\, h\rho_{1,t} - a^2\epsilon\, h\rho_{2,t} - \epsilon\, hu\rho_1u_t - \epsilon\, hu\rho_2u_t + a^2h\rho_1$ $+ a^2h\rho_2 - \epsilon\, fu\rho_{1,t} - \epsilon\, f\rho_1u_t - \epsilon\, gu\rho_{2,t} - \epsilon\, g\rho_2u_t$ $+ fu\rho_1 + gu\rho_2,$ $C^t = a^2\epsilon\, h\rho_{1,x} + a^2\epsilon\, h\rho_{1,x} + \epsilon\, hu\rho_1u_x + \epsilon\, hu\rho_2u_x + \epsilon\, fu\rho_{1,x}$ $+ \epsilon\, f\rho_1u_x + \epsilon\, gu\rho_{2,x} + \epsilon\, g\rho_2u_x + f\rho_1 + g\rho_2$
$X_5 + x_3\, X_3$	$C^x = ht\rho_1u_tx_3 + ht\rho_2u_tx_3 + a^2h\rho_1 + a^2h\rho_2 + ft\rho_{1,t}x_3$ $+ gt\rho_{2,t}x_3 + hu\rho_1x_3 + hu\rho_2x_3 + fu\rho_1 + f\rho_1x_3$ $+ gu\rho_2 + g\rho_2x_3,$ $C^t = -ht\rho_1u_xx_3 - ht\rho_2u_xx_3 - ft\rho_{1,x}x_3 - gt\rho_{2,x}x_3$ $+ h\rho_1x_3 + h\rho_2x_3 + f\rho_1 + g\rho_2$
$X_5 - \epsilon\, X_4 + a\epsilon\, X_3$	$C^x = a\epsilon\, ht\rho_1u_t + a\epsilon\, ht\rho_2u_t + a^2\epsilon\, h\rho_{1,t} + a^2\epsilon\, h\rho_{2,t}$ $+ a\epsilon\, ft\rho_{1,t} + a\epsilon\, gt\rho_{2,t} + a\epsilon\, hu\rho_1 + a\epsilon\, hu\rho_2$ $+ \epsilon\, hu\rho_1u_t + \epsilon\, hu\rho_2u_t + a^2h\rho_1 + a^2h\rho_2$ $+ a\epsilon\, f\rho_1 + a\epsilon\, g\rho_2 + \epsilon\, fu\rho_{1,t} + \epsilon\, f\rho_1u_t$ $+ \epsilon\, gu\rho_{2,t} + \epsilon\, g\rho_2u_t + fu\rho_1 + gu\rho_2,$ $C^t = -a\epsilon\, ht\rho_1u_x - a\epsilon\, ht\rho_2u_x - a^2\epsilon\, h\rho_{1,x} - a^2\epsilon\, h\rho_{2,x}$ $- a\epsilon\, ft\rho_{1,x} - a\epsilon\, gt\rho_{2,x} - \epsilon\, hu\rho_1u_x - \epsilon\, hu\rho_2u_x$ $+ a\epsilon\, h\rho_1 + a\epsilon\, h\rho_2 - \epsilon\, fu\rho_{1,x} - \epsilon\, f\rho_1u_x$ $- \epsilon\, gu\rho_{2,x} - \epsilon\, g\rho_2u_x + f\rho_1 + g\rho_2$
$X_5 + a\, X_2 + ab\, X_3$	$C^x = -a^3ht\rho_{1,t} - a^3ht\rho_{2,t} + abht\rho_1u_t + abht\rho_2u_t$ $- ahtu\rho_1u_t - ahtu\rho_2u_t + abft\rho_{1,t} + abgt\rho_{2,t}$ $+ abhu\rho_1 + abhu\rho_2 - aftu\rho_{1,t} - aft\rho_1u_t - agtu\rho_{2,t}$ $- agt\rho_2u_t + ah\rho_1xu_t + ah\rho_2xu_t + a^2h\rho_1$ $+ a^2h\rho_2 + abf\rho_1 + abg\rho_2 + afx\rho_{1,t}$ $+ agx\rho_{2,t} + fu\rho_1 + gu\rho_2,$ $C^t = a^3ht\rho_{1,x} + a^3ht\rho_{2,x} - abht\rho_1u_x - abht\rho_2u_x$ $+ ahtu\rho_1u_x + ahtu\rho_2u_x - abft\rho_{1,x} - abgt\rho_{2,x}$ $+ aftu\rho_{1,x} + aft\rho_1u_x + agtu\rho_{1,x} + agt\rho_2u_x$ $- ah\rho_1xu_x - ah\rho_2xu_x + abh\rho_1 + abh\rho_2$ $- afx\rho_{1,x} - agx\rho_{2,x} + f\rho_1 + g\rho_2$

Table 7.2: Nonlocal conservation laws for model (7.1), where $\epsilon = \pm 1$.

8

Bäcklund transformations, Lax system, conservation laws and multi-wave solutions for Jimbo-Miwa equation with Bell-polynomials[1]

Mathematics is the language with which God wrote the universe.

Galileo Galilei

8.1 Introduction

In the study of non-linear equations, the integrability and integrable properties are intensively investigated in recent the years, but still there exist no specific definition of integrability of nonlinear equations. A model which is integrable under one meaning may not be integrable under other. However, there are indicators like Bäcklund transformations, linearization in term of Lax pairs, infinite set of conservation laws, infinite symmetries, which can serve as indicators of complete integrability of nonlinear equations. If one talk about tools for investigating integrability, the Painlevé analysis [250] is a most sophisticated technique. The Hirota bilinear method [104,105] is a simple and effective technique to examine integrability of nonlinear equations. This method relies on the conversion of the nonlinear equation into bilinear form through some dependent variable transformation. Once the bilinear form is written, then multi-soliton solutions may be constructed in lucid and straightforward manner. Apart from Hirota bilinear method, the procedure based on Bell-polynomials developed by Lambert, Gilson *et al.* [87] is quite revolutionary. The procedure enables the construction of bilinear Bäcklund transformations, Lax pair, and even infinite conservation laws [143, 144].
Recently, the procedure based on Bell-polynomials has been applied to various soliton equations to investigate their integrability [71,107,144,154,198,199,238,239,261], where Bell polynomials are found to be quite useful in the study of integrable prop-

[1]The content of this chapter is published in Communications in Nonlinear Science and Numerical Simulation 37 (2016): 362-373 [210].
The final publication is available at doi.org/10.1016/j.cnsns.2016.01.023

erties of nonlinear equations. In this chapter, we shall investigate (3+1)-dimensional Jimbo-Miwa equation for bilinear Bäcklund transformations, Lax pair and infinite conservation laws. The equation is expressed as follows:

$$u_{xxxy} + 3u_y u_{2x} + 3u_x u_{xy} + 2u_{ty} - 3u_{xz} = 0. \tag{8.1}$$

The equation (8.1) is a second equation that belongs to KP-hierarchy [124] and is shown to be conditionally integrable having two types of solitary wave solutions such that the solitary waves in the second type do not interact elastically [63]. For equation (8.1), the multi-soliton solutions are derived using Hirota bilinear method [111]. Nevertheless, different varieties of exact solutions have also been obtained [70, 160, 213, 241].

8.2 Bilinear form, bilinear Bäcklund transformations and Lax system

8.2.1 Bilinear representation

The Painlevé analysis of equation (8.1) has already been performed in Ref. [252] and it is shown to be non-integrable. Despite the fact that equation (8.1) is non-integrable, it can still be explored for Bäcklund transformations, Lax pairs, multi-soliton solutions and even for infinite set of conservation laws by exploiting the truncated Laurent's expansion, which in present case is given as follows:

$$u = 2\,(\log f)_x + u_1, \tag{8.2}$$

where u_1 is a seed solution of equation (8.1). In virtue of expressions presented in Section 1.2.4, and with the help truncated Painlevé expansion (8.2), the equation (8.1) can be bilinearized with the introduction of potential field $q = 2\,(\log f)$ by setting

$$u = cq_x, \text{ where } u_1 = 0, \tag{8.3}$$

where c being free parameter to be determined later. On substituting (8.3) into (8.1) and integrating once with respect to x, we get

$$E(q) = q_{xxxy} + 3cq_{xy}q_{xx} + 2q_{ty} - 3q_{xz} = 0, \tag{8.4}$$

where constant of integration is taken as zero. In order connect (8.4) with $\mathscr{P}$-polynomials, we must choose $c = 1$. So that using formulae (1.127), we obtain

$$E(q) = \mathscr{P}_{xxxy}(q) + 2\mathscr{P}_{ty}(q) - 3\mathscr{P}_{xz}(q) = 0. \tag{8.5}$$

Through the transformation $q = 2\log f$ and on account of the property (1.126), the equation (8.5) gives bilinear form of equation (8.1) as follows:

$$(D_x^3 D_y + 2D_t D_y - 3D_x D_z) f \cdot f = 0. \tag{8.6}$$

In the following sections, we shall use bilinear form (8.6) to construct Bäckland transformations, Lax system and conservation laws for the equation (8.1).

8.2.2 Bilinear Bäcklund transformations

In order to find bilinear Bäcklund transformations for equation (8.1), suppose $\tilde{q} = 2\log\tilde{f}$ be another solution of (8.5), then corresponding two-field condition can be written as follows:

$$\begin{aligned} E(\tilde{q}) - E(q) = &[\mathscr{P}_{3x,y}(\tilde{q}) + 2\mathscr{P}_{t,y}(\tilde{q}) - 3\mathscr{P}_{x,z}(\tilde{q})] \\ &- [\mathscr{P}_{3x,y}(q) + 2\mathscr{P}_{t,y}(q) - 3\mathscr{P}_{x,z}(q)] = 0. \end{aligned} \tag{8.7}$$

Basically, this two-field condition serves as ansätz for determining Bäcklund transformations under certain constraints.

To proceed further, we introduce new variables

$$W = \log f\tilde{f}, \quad V = \log\frac{\tilde{f}}{f}, \quad \tilde{q} = 2\log\tilde{f} = W + V, \quad q = \log f = W - V. \tag{8.8}$$

So that the equation (8.7) can be written as follows:

$$\begin{aligned} E(\tilde{q}) - E(q) &= [\mathscr{P}_{3x,y}(W+V) - \mathscr{P}_{3x,y}(W-V)] \\ &\quad + 2[\mathscr{P}_{t,y}(W+V) - \mathscr{P}_{t,y}(W-V)] \\ &\quad - 3[\mathscr{P}_{x,z}(W+V) - \mathscr{P}_{x,z}(W-V)] \\ &= 2V_{xxxy} + 6V_{xx}W_{xy} + V_{xy}W_{xx} + 4V_{ty} - 6V_{xz} = 0. \end{aligned} \tag{8.9}$$

Next thing is to express (8.9) in the linear combination of $\mathscr{Y}$-polynomials. For this purpose, we choose following additional constraint of lowest possible weight

$$\mathscr{Y}_{2x}(V,W) - \lambda = 0, \tag{8.10}$$

where λ being arbitrary parameter. On account of (8.10) and after detailed calculations (8.9) reduces to

$$\begin{aligned} 3\partial_x[\mathscr{Y}_{2x,y}(V,W) - 2\mathscr{Y}_z(V,W) + 2\lambda\mathscr{Y}_y(V,W)] \\ - \partial_y[\mathscr{Y}_{3x}(V,W) - 4\mathscr{Y}_t(V,W)] = 0. \end{aligned} \tag{8.11}$$

In virtue of equation (8.11), the final decoupling of two-field condition (8.9) can be written as

$$\mathscr{Y}_{2x}(V,W) - \lambda = 0, \tag{8.12a}$$
$$\mathscr{Y}_{2x,y}(V,W) - 2\mathscr{Y}_z(V,W) + 2\lambda\mathscr{Y}_y(V,W) - \mu = 0, \tag{8.12b}$$
$$\mathscr{Y}_{3x}(V,W) - 4\mathscr{Y}_t(V,W) - \nu = 0, \tag{8.12c}$$

where μ and ν are arbitrary parameters. The equations (8.12) constitute Bäcklund transformations for equation (8.1) expressed in term of Bell-polynomials. With the aid of formula (1.128), these equations can be transformed into bilinear form as follow:

$$(D_x^2 - \lambda)f\cdot g = 0, \tag{8.13a}$$
$$(D_x^2D_y - 2D_z + 2\lambda D_y - \mu)f\cdot g = 0, \tag{8.13b}$$
$$(D_x^3 - 4D_t - \nu)f\cdot g = 0. \tag{8.13c}$$

The above form of Bäcklund transformations for equation (8.1) can also be derived using exchange formulae for Hirota's D-operator [103], but that procedure is not recommended due to highly complicated computations.

8.2.3 Lax system

In this section, we shall linearize Bell-system (8.12) with the aid of formulae (1.128) and (1.129). Using Cole-Hopf transformation $V = \log \psi$, the formula (1.130) read as follows:

$$\mathscr{Y}_t(V) = \frac{\psi_t}{\psi}, \quad \mathscr{Y}_y(V) = \frac{\psi_y}{\psi}, \quad \mathscr{Y}_z(V) = \frac{\psi_z}{\psi}, \quad \mathscr{Y}_{2x}(V,W) = q_{2x} + \frac{\psi_{xx}}{\psi},$$
$$\mathscr{Y}_{2x,y}(V,W) = \frac{q_{xx}\psi_y}{\psi} + \frac{2q_{xy}\psi_x}{\psi} + \frac{\psi_{xxy}}{\psi}, \quad \mathscr{Y}_{3x}(V,W) = \frac{\psi_{xxx}}{\psi} + \frac{3q_{xx}\psi_x}{\psi}. \tag{8.14}$$

With the aid of formulas (8.14), the Bell-system (8.12) linearized into following Lax system

$$\psi_{xx} + (u_x - \lambda)\psi = 0, \tag{8.15a}$$
$$u_x\psi_y + 2u_y\psi_x + \psi_{xxy} - 2\psi_z + 2\lambda\psi_y - \mu\psi = 0, \tag{8.15b}$$
$$\psi_{xxx} + 3u_x\psi_x - 4\psi_t - \nu\psi = 0. \tag{8.15c}$$

8.3 Infinite conservation laws

In physics, the conservation laws are fundamental laws those ensures that certain physical quantity will not change with time during the course of the physical process. Some of the well-known conservation laws in physics are conservation of mass, momentum, energy and electric charge etc. In practice, one can find various expressions for conservation laws. For example, in fluid dynamics, the conservation law is expressed by the equation of continuity, in electromagnetism the equation of continuity expressing the relation between electric charge and current is also a conservation law. Moreover, in the study of nonlinear PDEs, conservation laws provides a method to verify complete integrability and linearization, constants of motion and most importantly analysis of solution. The conservation laws can be derived by mean of Bäcklund transformations [232], symmetry analysis [175], inverse scattering [132].

In this section, we shall construct an infinite set of conservation laws for (3+1)-dimensional Jimbo-Miwa equation (8.1) using Bell polynomials. In order to derive conservation laws for equation (8.1), we re-write the two-field equations (8.10) and (8.11) as follow:

$$\mathscr{Y}_{2x}(V,W) - \lambda = 0, \tag{8.16}$$
$$4\partial_t[\mathscr{Y}_y(V)] + \partial_x[3\mathscr{Y}_{2x,y}(V,W) + 6\lambda\mathscr{Y}_y(V)]$$
$$- \partial_y[\mathscr{Y}_{3x}(V,W)] - 6\partial_z[\mathscr{Y}_x(V,W)] = 0. \tag{8.17}$$

Next, we introduce new potential function,

$$\eta = \frac{\tilde{q}_x - q_x}{2}. \tag{8.18}$$

It follows from equations (8.8) that,

$$V_x = \eta, \quad W_x = q_x + \eta, \quad V_y = \partial_x^{-1}(\eta_y). \tag{8.19}$$

On substituting (8.19) in (8.16) and (8.17), we get following Riccati type equation

$$q_{xx} + \eta_x + \eta^2 - \lambda = 0, \tag{8.20}$$

and divergence type equation,

$$\begin{aligned} &4\partial_t[\partial_x^{-1}(\eta_y)] + 3\partial_x[\eta_{xy} + 2\eta(q_{xy} + \eta_y) + 3\lambda\partial_x^{-1}(\eta_y)] \\ &- \partial_y[\eta_{xx} - 2\eta^3 + 3\eta\lambda] - 6\partial_z[\eta] = 0. \end{aligned} \tag{8.21}$$

Letting $\lambda = \epsilon^2$ and making use of transformation $\eta = \tilde{\eta} + \epsilon$, we obtain

$$\tilde{\eta_x} + \tilde{\eta}^2 + 2\epsilon\tilde{\eta} + q_{xx} = 0, \tag{8.22}$$

$$\begin{aligned} &4\partial_t[\partial_x^{-1}(\tilde{\eta}_y)] + 3\partial_x[\tilde{\eta}_{xy} + 2\tilde{\eta}q_{xy} + 2\tilde{\eta}\tilde{\eta}_y + 2\epsilon q_{xy} + 2\epsilon\tilde{\eta}_y + 3\epsilon^2\partial_x^{-1}(\tilde{\eta}_y)] \\ &- \partial_y[\tilde{\eta}_{xx} - 2\tilde{\eta}^3 - 6\epsilon\tilde{\eta}^2 - 6\epsilon^2\tilde{\eta} + 3\epsilon^2\tilde{\eta}] - 6\partial_z[\tilde{\eta}] = 0. \end{aligned} \tag{8.23}$$

To proceed further, we insert expansion

$$\tilde{\eta} = \sum_{n=1}^{\infty} I_n(q, q_x, ...)\epsilon^{-n} \tag{8.24}$$

into (8.22) and equating coefficients of powers of ϵ, the recursions relations for conserved densities can be obtained as follow:

$$I_1 = -\frac{1}{2}q_{xx} = -\frac{1}{2}u_x, \tag{8.25a}$$

$$I_2 = -\frac{1}{2}I_{1,x} = \frac{1}{4}q_{xxx} = \frac{1}{4}u_{xx}, \tag{8.25b}$$

$$I_3 = -\frac{1}{2}[I_{2,x} + I_1^2] = -\frac{1}{8}[q_{xxxx} + q_{xx}^2] = -\frac{1}{8}[u_{xxx} + u_x^2], \tag{8.25c}$$

$$I_4 = -\frac{1}{2}[I_{3,x} + 2I_1I_2] = \frac{1}{16}[q_{xxxxx} + 4q_{xx}q_{xxx}] = \frac{1}{16}[u_{xxxx} + 4u_xu_{xx}], \tag{8.25d}$$

$$\dots,$$

$$I_{n+1} = -\frac{1}{2}\left[I_{n,x} + \sum_{k=1}^{n} I_kI_{n-k}\right], \quad n = 2, 3, \dots. \tag{8.25e}$$

On substituting again (8.24) into (8.23), we get

$$\begin{aligned} &4\partial_t\left[\partial_x^{-1}\sum_{n=1}^{\infty} I_{n,y}\epsilon^{-n}\right] + 3\partial_x\left[\sum_{n=1}^{\infty} I_{n,x,y}\epsilon^{-n} + 2q_{x,y}\sum_{n=1}^{\infty} I_n\epsilon^{-n}\right. \\ &\left. + 2\sum_{n=1}^{\infty} I_n\epsilon^{-n}\sum_{n=1}^{\infty} I_{n,y}\epsilon^{-n} + 2\epsilon q_{xy} + 2\epsilon\sum_{n=1}^{\infty} I_{n,y}\epsilon^{-n} + 3\epsilon^2\sum_{n=1}^{\infty}\partial_x^{-1}I_{n,y}\epsilon^{-n}\right] \\ &- \partial_y\left[\sum_{n=1}^{\infty} I_{n,xx}\epsilon^{-n} - 2\left(\sum_{n=1}^{\infty} I_n\epsilon^{-n}\right)^3 - 6\epsilon\left(\sum_{n=1}^{\infty} I_n\epsilon^{-n}\right)^2\right. \\ &\left. - 3\epsilon^2\sum_{n=1}^{\infty} I_n\epsilon^{-n}\right] - 6\partial_z\left[\sum_{n=1}^{\infty} I_n\epsilon^{-n}\right] = 0, \end{aligned}$$

from which, we can obtain following conservation laws:

$$4\mathscr{T}_{n,t} + 3\mathscr{X}_{n,x} + \mathscr{Y}_{n,y} + 6\mathscr{Z}_{n,z} = 0, \quad n = 1, 2, \dots, \tag{8.26}$$

from (8.26), with the help of the recursion formulae (8.25) the explicit formulae for conserved densities are given by

$$\mathscr{T}_1 = \partial_x^{-1} I_{1,y} = -\frac{1}{2} u_y, \tag{8.27a}$$

$$\mathscr{T}_2 = \partial_x^{-1} I_{2,y} = \frac{1}{4} u_{xy}, \tag{8.27b}$$

..

$$\mathscr{T}_n = \partial_x^{-1} I_{n,y}, \tag{8.27c}$$

while the first fluxes $\mathscr{X}_n's$ are explicitly given by the following recursion formulae

$$\mathscr{X}_1 = -u_x u_y - \frac{3}{8} u_{xxy} - \frac{3}{4} \partial_x^{-1} [u_x u_{xy}], \tag{8.28a}$$

$$\mathscr{X}_2 = \frac{1}{2} u_{xx} u_y + \frac{3}{16} u_{xxxy} + \frac{3}{4} \partial_x^{-1} [u_{xx} u_{xy} + u_x u_{xxy}], \tag{8.28b}$$

...

$$\mathscr{X}_n = I_{n,x,y} + 2u_y I_n + 2 \sum_{k=1}^{n} I_k I_{(n-k),y} + 2 I_{(n+1),y} + 3 \partial_x^{-1} I_{(n+2),y}, \tag{8.28c}$$

and the second fluxes $\mathscr{Y}_n's$ are

$$\mathscr{Y}_1 = \frac{1}{8} (u_{xxx} + 9u_x^2), \tag{8.29a}$$

$$\mathscr{Y}_2 = -\frac{1}{16} (u_{xxxx} + 12 u_x u_{xx}), \tag{8.29b}$$

...

$$\mathscr{Y}_n = -I_{n,xx} + 2 \sum_{i+j+k=n} I_i I_j I_k + 6 \sum_{k=1}^{n} I_k I_{n-k+1} - 3 I_{n+2}. \tag{8.29c}$$

The remaining fluxes $\mathscr{Z}_n'$s are given by the recursion formulae (8.25). The recursion formulae of $\mathscr{T}_n, \mathscr{X}_n, \mathscr{Y}_n$ and $\mathscr{Z}_n$ as presented above help in the construction of an infinite set of conservation laws for Jimbo-Miwa equation. The first equation of (8.26) is exactly as the equation (8.1) which prove the non-triviality of conservation laws.

8.4 Multi-wave solutions using three wave method

In this section, we shall introduce construction of multi-wave solutions using the well known Hirota's bilinear method. We use test function [200] defined in following manner:

$$f = e^{-\eta_1} + \delta_1 \cos \eta_2 + \delta_2 \cosh \eta_3 + \delta_3 e^{\eta_1}, \tag{8.30}$$

where $u = 2 (\log f)_x$ and $\eta_i = a_i x + b_i y + c_i t + d_i z$, for $i = 1, 2, 3$. On substituting (8.30) into bilinear equations (8.6) and equating coefficients of

$e^{\eta_1}\sin(\eta_2), e^{\eta_1}\sinh(\eta_3),\ e^{\eta_1}\cos(\eta_2), e^{\eta_1}\cosh(\eta_3),\ \sin(\eta_2)\sinh(\eta_3), \cos(\eta_2)\cosh(\eta_3)$ to zero, we obtain system of seven algebraic equation in a_i, b_i, c_i, d_i and $\delta_i (i = 1, 2, 3)$ given as follow:

$$
\begin{aligned}
&\delta_2\delta_1(-2\,b_2c_3 - 2\,c_2b_3 + 3\,a_2d_3 - 3\,a_2b_3{a_3}^2 + 3\,a_3b_2{a_2}^2\\
&+ 3\,a_3d_2 + {a_2}^3b_3 - {a_3}^3b_2) = 0,\\
&\delta_1\delta_2(-3\,d_3a_3 + 3\,d_2a_2 - 3\,{a_2}^2b_3a_3 - 3\,{a_3}^2b_2a_2 + b_3{a_3}^3\\
&+ b_2{a_2}^3 + 2\,b_3c_3 - 2\,b_2c_2 = 0,\\
&\delta_3\delta_1(-3\,d_1a_1 - 3\,{a_1}^2b_2a_2 - 3\,{a_2}^2b_1a_1 + b_2{a_2}^3 - 2\,b_2c_2\\
&+ 3\,d_2a_2 + b_1{a_1}^3 + 2\,c_1b_1 = 0,\\
&\delta_3\delta_2(-3\,d_3a_3 + b_1{a_1}^3 + 2\,c_1b_1 - 3\,d_1a_1 + b_3{a_3}^3 + 2\,b_3c_3\\
&+ 3\,{a_3}^2b_1a_1 + 3\,{a_1}^2b_3a_3) = 0,\\
&\delta_1\delta_3(-3\,a_1d_2 - 3\,a_2d_1 - 3\,a_1b_2{a_2}^2 + 3\,a_2b_1{a_1}^2 + {a_1}^3b_2\\
&- {a_2}^3b_1 + 2\,b_1c_2 + 2\,b_2c_1) = 0,\\
&\delta_2\delta_3({a_3}^3b_1 + {a_1}^3b_3 + 2\,b_1c_3 - 3\,a_3d_1 - 3\,a_1d_3\\
&+ 3\,a_3b_1{a_1}^2 + 3\,a_1b_3{a_3}^2 + 2\,c_1b_3 = 0,\\
&8\,{\delta_1}^2{a_2}^3b_2 + 8\,{\delta_2}^2{a_3}^3b_3 - 4\,{\delta_1}^2c_2b_2 + 4\,{\delta_2}^2c_3b_3 + 6\,{\delta_1}^2d_2a_2\\
&- 6\,{\delta_2}^2d_3a_3 - 24\,\delta_3d_1a_1 + 32\,\delta_3b_1{a_1}^3 + 16\,\delta_3c_1b_1 = 0.
\end{aligned}
$$

This system of algebraic equations can be easily solved with the aid of Maple, so that we have following cases:

Case 8.4.1.

$$
\begin{aligned}
&b_1 = \frac{b_3a_1}{a_3}, \quad d_1 = \frac{b_3\,({a_1}^3 + 2\,c_1 + 3\,a_1{a_3}^2)}{3a_3}, \quad d_3 = \frac{({a_3}^3 + 2\,c_3 + 3\,a_3{a_1}^2)\,b_3}{3a_3},\\
&\delta_1 = 0, \quad \delta_3 = \frac{{a_3}^2{\delta_2}^2}{4{a_1}^2}.
\end{aligned}
$$

Under above constraints, the exact solution to equation (8.1) can be obtained as follows:

$$
u = \frac{-2\,a_1\mathrm{e}^{-\eta_1} + 2\,\delta_2\sinh(\eta_3)\,a_3 + \frac{{a_3}^2{\delta_2}^2}{2a_1}\mathrm{e}^{\eta_1}}{\mathrm{e}^{-\eta_1} + \delta_2\cosh(\eta_3) + \frac{{a_3}^2{\delta_2}^2}{4{a_1}^2}\mathrm{e}^{\eta_1}}, \tag{8.31}
$$

where

$$
\eta_1 = a_1x + \frac{b_3a_1y}{a_3} + c_1t + \frac{b_3\,({a_1}^3 + 2\,c_1 + 3\,a_1{a_3}^2)\,z}{3a_3}, \tag{8.32a}
$$

$$
\eta_3 = a_3x + b_3y + c_3t + \frac{({a_3}^3 + 2\,c_3 + 3\,a_3{a_1}^2)\,b_3z}{3a_3}. \tag{8.32b}
$$

The parameters mentioned in (8.31) and (8.32) are all arbitrary.

Case 8.4.2.

$$
\begin{aligned}
&b_1 = -\frac{a_2b_2}{a_1}, \quad d_1 = -\frac{b_2\,(3\,a_2{a_1}^2 - {a_2}^3 + 2\,c_2)}{3a_1},\\
&d_2 = \frac{b_2\,(-3\,a_1{a_2}^2 + {a_1}^3 + 2\,c_1)}{3a_1}\delta_2 = 0, \quad \delta_3 = \frac{{\delta_1}^2}{4}.
\end{aligned}
$$

The second exact solution to equation (8.1) can be obtained as follows:

$$u = \frac{-2\,a_1 \mathrm{e}^{-\eta_1} - 2\,\delta_1 \sin(\eta_2)\,a_2 + \frac{{\delta_1}^2 a_1}{2}\,\mathrm{e}^{\eta_1}}{\mathrm{e}^{-\eta_1} + \delta_1 \cos(\eta_2) + \frac{{\delta_1}^2}{4}\mathrm{e}^{\eta_1}}, \tag{8.33}$$

where

$$\eta_1 = a_1 x - \frac{a_2 b_2 y}{a_1} + c_1 t - \frac{b_2\left(3\,a_2 {a_1}^2 - {a_2}^3 + 2\,c_2\right) z}{3a_1}, \tag{8.34a}$$

$$\eta_2 = a_2 x + b_2 y + c_2 t + \frac{b_2\left(-3\,a_1 {a_2}^2 + {a_1}^3 + 2\,c_1\right) z}{3a_1}. \tag{8.34b}$$

The parameters mentioned in (8.33) and (8.34) are all arbitrary.

Case 8.4.3.

$$a_1 = 0, \quad b_2 = \frac{2c_1 b_1\left({\delta_1}^2 - 4\,\delta_3\right)}{3{\delta_1}^2 {a_2}^3},$$
$$d_1 = -\frac{b_1\left(3\,{a_2}^6 {\delta_1}^2 - 6\,c_2 {\delta_1}^2 {a_2}^3 - 4\,{c_1}^2 {\delta_1}^2 + 16\,\delta_3 {c_1}^2\right)}{9{a_2}^4 {\delta_1}^2},$$
$$d_2 = -\frac{4c_1 b_1\left(2\,{\delta_1}^2 {a_2}^3 - 2\,{a_2}^3 \delta_3 - c_2 {\delta_1}^2 + 4\,\delta_3 c_2\right)}{9{a_2}^4 {\delta_1}^2}, \quad \delta_2 = 0.$$

The third exact solution to equation (8.1) can be obtained as follows

$$\begin{aligned} u &= -\frac{2\,\delta_1 \sin(\eta_2)\,a_2}{\mathrm{e}^{-\eta_1} + \delta_1 \cos(\eta_2) + \delta_3 \mathrm{e}^{\eta_1}} \\ &= \begin{cases} -\frac{2\delta_1 \sin(\eta_2) a_2}{2\sqrt{\delta_3}\cosh(\eta_1 + \ln\sqrt{\delta_3}) + \delta_1 \cos(\eta_2)}, & \delta_3 > 0; \\ -\frac{2\delta_1 \sin(\eta_2) a_2}{2\sqrt{-\delta_3}\sinh(\eta_1 + \ln\sqrt{-\delta_3}) + \delta_1 \cos(\eta_2)}, & \delta_3 < 0. \end{cases} \end{aligned} \tag{8.35}$$

where

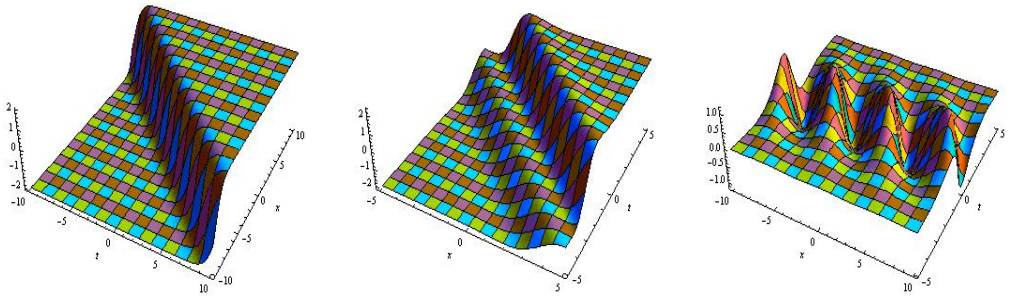
$$\eta_1 = b_1 y + c_1 t - \frac{b_1\left(3\,{a_2}^6 {\delta_1}^2 - 6\,c_2 {\delta_1}^2 {a_2}^3 - 4\,{c_1}^2 {\delta_1}^2 + 16\,\delta_3 {c_1}^2\right) z}{9{a_2}^4 {\delta_1}^2}, \tag{8.36a}$$

$$\begin{aligned} \eta_2 = {} & a_2 x + \frac{2c_1 b_1\left({\delta_1}^2 - 4\,\delta_3\right) y}{3{\delta_1}^2 {a_2}^3} + c_2 t \\ & - \frac{4c_1 b_1\left(2\,{\delta_1}^2 {a_2}^3 - 2\,{a_2}^3 \delta_3 - c_2 {\delta_1}^2 + 4\,\delta_3 c_2\right) z}{9{a_2}^4 {\delta_1}^2}. \end{aligned} \tag{8.36b}$$

The parameters mentioned in (8.35) and (8.36) are all arbitrary.

Case 8.4.4.

$$a_1 = 0, \quad b_2 = \frac{a_2 b_3}{a_3}, \quad c_1 = -\frac{3{a_2}^2 b_3 a_3}{2b_1}, \quad c_2 = \frac{a_2\left({a_3}^3 + 2\,c_3 + a_3 {a_2}^2\right)}{2a_3},$$
$$d_1 = -\frac{-{a_3}^3 {b_1}^2 - 2\,{b_1}^2 c_3 + 3\,a_3 {b_3}^2 {a_2}^2}{3b_1 a_3}, \quad d_2 = \frac{2a_2 b_3\left(2\,{a_3}^3 + c_3\right)}{3{a_3}^2},$$
$$d_3 = -\frac{b_3\left(-2\,c_3 + 3\,a_3 {a_2}^2 - {a_3}^3\right)}{3a_3},$$



(a) **(b)** **(c)**

Figure 8.1: Setting all arbitrary parameters to unity, figures (a) kink wave profile of solutions (8.31), (b) two solitary wave profile of solutions (8.33) and (c) periodic wave profile of solutions (8.35) on $x-t$ plane

$$\delta_3 = \frac{{a_2}^4{\delta_1}^2 + {a_3}^2{\delta_1}^2{a_2}^2 + {a_3}^4{\delta_2}^2 + {a_3}^2{a_2}^2{\delta_2}^2}{4{a_3}^2{a_2}^2}.$$

The fourth exact solution to equation (8.1) can be obtained as follows:

$$\begin{aligned} u &= \frac{-2\,\delta_1 \sin(\eta_2)\,a_2 + 2\,\delta_2 \sinh(\eta_3)\,a_3}{\mathrm{e}^{-\eta_1} + \delta_1 \cos(\eta_2) + \delta_2 \cosh(\eta_3) + \delta_3 \mathrm{e}^{\eta_1}} \\ &= \begin{cases} \frac{-2\delta_1 \sin(\eta_2)a_2 + 2\delta_2 \sinh(\eta_3)a_3}{2\sqrt{\delta_3}\cosh(\eta_1 + \ln\sqrt{\delta_3}) + \delta_1\cos(\eta_2) + \delta_2\cosh(\eta_3)}, & \delta_3 > 0; \\ \frac{-2\delta_1 \sin(\eta_2)a_2 + 2\delta_2 \sinh(\eta_3)a_3}{2\sqrt{-\delta_3}\sinh(\eta_1 + \ln\sqrt{-\delta_3}) + \delta_1\cos(\eta_2) + \delta_2\cosh(\eta_3)}, & \delta_3 < 0. \end{cases} \end{aligned} \tag{8.37}$$

where

$$\eta_1 = b_1 y - \frac{3{a_2}^2 b_3 a_3 t}{2b_1} - \frac{\left(-{a_3}^3{b_1}^2 - 2\,{b_1}^2 c_3 + 3\,a_3 {b_3}^2 {a_2}^2\right) z}{3b_1 a_3}, \tag{8.38a}$$

$$\eta_2 = a_2 x + \frac{a_2 b_3 y}{a_3} + \frac{a_2\left({a_3}^3 + 2\,c_3 + a_3 {a_2}^2\right) t}{2a_3} + \frac{2a_2 b_3\left(2\,{a_3}^3 + c_3\right) z}{3{a_3}^2}, \tag{8.38b}$$

$$\eta_3 = a_3 x + b_3 y + c_3 t - \frac{b_3\left(-2\,c_3 + 3\,a_3 {a_2}^2 - {a_3}^3\right) z}{3a_3}. \tag{8.38c}$$

The parameters mentioned in (8.37) and (8.38) except δ_3 are all arbitrary.

Case 8.4.5.

$$a_1 = 0, \quad a_2 = 0, \quad b_3 = 0, \quad c_2 = 0, \quad d_1 = \frac{b_1\left({a_3}^3 + 2\,c_3\right)}{3a_3},$$

$$d_2 = \frac{b_2\left({a_3}^3 + 2\,c_3\right)}{3a_3}, \quad d_3 = 0.$$

The fifth exact solution can be obtained as follows:

$$\begin{aligned} u &= \frac{2\,\delta_2 \sinh(\eta_3)\,a_3}{\mathrm{e}^{-\eta_1} + \delta_1 \cos(\eta_2) + \delta_2 \cosh(\eta_3) + \delta_3 \mathrm{e}^{\eta_1}} \\ &= \begin{cases} \frac{2\delta_2 \sinh(\eta_3)a_3}{2\sqrt{\delta_3}\cosh(\eta_1 + \ln\sqrt{\delta_3}) + \delta_1\cos(\eta_2) + \delta_2\cosh(\eta_3)}, & \delta_3 > 0; \\ \frac{2\delta_2 \sinh(\eta_3)a_3}{2\sqrt{-\delta_3}\sinh(\eta_1 + \ln\sqrt{-\delta_3}) + \delta_1\cos(\eta_2) + \delta_2\cosh(\eta_3)}, & \delta_3 < 0. \end{cases} \end{aligned} \tag{8.39}$$

where

$$\eta_1 = b_1 y + \frac{b_1 \left({a_3}^3 + 2\, c_3\right) z}{3 a_3}, \tag{8.40a}$$

$$\eta_2 = b_2 y + \frac{b_2 \left({a_3}^3 + 2\, c_3\right) z}{3 a_3}, \tag{8.40b}$$

$$\eta_3 = a_3 x + c_3 t. \tag{8.40c}$$

The parameters mentioned in (8.39) and (8.40) are all arbitrary.

Case 8.4.6.

$$a_2 = 0, \quad b_1 = \frac{b_2 c_2 \left({\delta_1}^2 - 4\,\delta_3\right)}{6 \delta_3 {a_1}^3},$$
$$d_1 = \frac{b_2 c_2 \left(-16\,\delta_3 {a_1}^3 + {a_1}^3 {\delta_1}^2 + 2\, c_1 {\delta_1}^2 - 8\,\delta_3 c_1\right)}{18 {a_1}^4 \delta_3},$$
$$d_2 = \frac{b_2 \left(3\, {a_1}^6 \delta_3 + {\delta_1}^2 {c_2}^2 - 4\, {c_2}^2 \delta_3 + 6\,\delta_3 c_1 {a_1}^3\right)}{9 {a_1}^4 \delta_3}, \quad \delta_2 = 0.$$

The sixth exact solution to equation (8.1) can be obtained as follows:

$$u = \frac{-2\, a_1 \mathrm{e}^{-\eta_1} + 2\,\delta_3 a_1 \mathrm{e}^{\eta_1}}{\mathrm{e}^{-\eta_1} + \delta_1 \cos\left(\eta_2\right) + \delta_3 \mathrm{e}^{\eta_1}} = \begin{cases} -\frac{4 a_1 \sqrt{\delta_3} \sinh(\eta_1 + \ln\sqrt{\delta_3})}{2\sqrt{\delta_3}\cosh(\eta_1 + \ln\sqrt{\delta_3}) + \delta_1 \cos(\eta_2)}, & \delta_3 > 0 \\ -\frac{4 a_1 \sqrt{\delta_3} \cosh(\eta_1 + \ln\sqrt{-\delta_3})}{2\sqrt{-\delta_3}\sinh(\eta_1 + \ln\sqrt{-\delta_3}) + \delta_1 \cos(\eta_2)}, & \delta_3 < 0 \end{cases} \tag{8.41}$$

where

$$\eta_1 = a_1 x + \frac{b_2 c_2 \left({\delta_1}^2 - 4\,\delta_3\right) y}{6 \delta_3 {a_1}^3} + c_1 t + \frac{b_2 c_2 \left(-16\,\delta_3 {a_1}^3 + {a_1}^3 {\delta_1}^2 + 2\, c_1 {\delta_1}^2 - 8\,\delta_3 c_1\right) z}{18 {a_1}^4 \delta_3}, \tag{8.42a}$$

$$\eta_2 = b_2 y + c_2 t + \frac{b_2 \left(3\, {a_1}^6 \delta_3 + {\delta_1}^2 {c_2}^2 - 4\, {c_2}^2 \delta_3 + 6\,\delta_3 c_1 {a_1}^3\right) z}{9 {a_1}^4 \delta_3}, \tag{8.42b}$$

$$\eta_3 = a_3 x + b_3 y + c_3 t + d_3 z. \tag{8.42c}$$

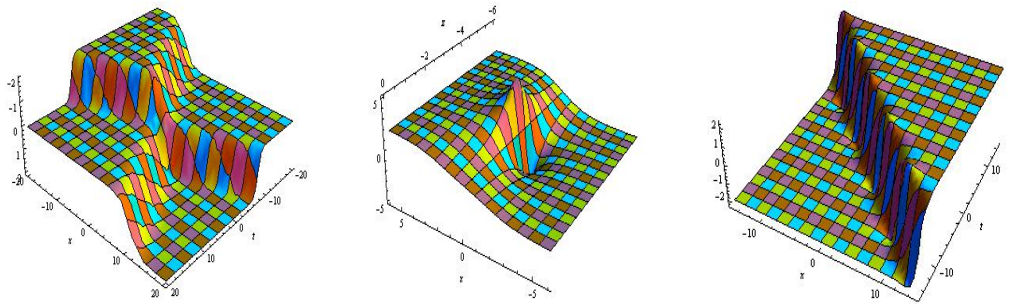
The parameters mentioned in (8.41) and (8.42) are all arbitrary.

Case 8.4.7.

$$a_2 = 0, \quad b_1 = -b_3 \quad c_2 = 0, \quad c_3 = c_1,$$
$$d_1 = -\frac{b_3 \left({a_3}^3 + 2\, c_1\right)}{3 a_3}, \quad d_2 = \frac{b_2 \left({a_3}^3 + 2\, c_1\right)}{3 a_3},$$
$$d_3 = \frac{b_3 \left({a_3}^3 + 2\, c_1\right)}{3 a_3}, \quad \delta_3 = \frac{{\delta_2}^2}{4}.$$

The seventh exact solution to equation (8.1) can be obtained as follows:

$$u = \frac{-2\, a_3 \mathrm{e}^{-\eta_1} + 2\,\delta_2 \sinh\left(\eta_3\right) a_3 + \frac{1}{2}\,{\delta_2}^2 a_3 \mathrm{e}^{\eta_1}}{\mathrm{e}^{-\eta_1} + \delta_1 \cos\left(\eta_2\right) + \delta_2 \cosh\left(\eta_3\right) + \frac{1}{4}\,{\delta_2}^2 \mathrm{e}^{\eta_1}} \tag{8.43}$$



(a) **(b)** **(c)**

Figure 8.2: Setting all arbitrary parameters to unity, figure (a) two solitary wave profile of solution (8.37) on $x-t$ plane, figure (b) kinky wave profile of solution (8.39) on $x-z$ plane and figure (c) kink periodic wave profile of solution (8.41) on $x-t$ plane

where

$$\eta_1 = a_3 x - b_3 y + c_1 t - \frac{b_3\left({a_3}^3 + 2\,c_1\right) z}{3a_3}, \tag{8.44a}$$

$$\eta_2 = b_2 y + \frac{b_2\left({a_3}^3 + 2\,c_1\right) z}{3a_3}, \tag{8.44b}$$

$$\eta_3 = a_3 x + b_3 y + c_1 t + \frac{b_3\left({a_3}^3 + 2\,c_1\right) z}{3a_3}. \tag{8.44c}$$

The parameters mentioned in (8.43) and (8.44) are all arbitrary.

Case 8.4.8.

$$a_2 = 0, \quad b_1 = \frac{a_1 b_3}{a_3}, \quad c_2 = -\frac{3{a_1}^2 a_3 b_3}{2b_2}, \quad c_3 = \frac{a_3\left({a_1}^3 + 2\,c_1 - a_1{a_3}^2\right)}{2a_1},$$
$$d_1 = \frac{b_3\left({a_1}^3 + 2\,c_1 + 3\,a_1{a_3}^2\right)}{3a_3}, \quad d_2 = \frac{{b_2}^2{a_1}^3 - 3\,{a_1}^3{b_3}^2 + 2\,{b_2}^2 c_1}{3a_1 b_2},$$
$$d_3 = \frac{2b_3\left(2\,{a_1}^3 + c_1\right)}{3a_1}, \quad \delta_3 = -\frac{{a_3}^2\left({\delta_1}^2{a_1}^2 - {\delta_2}^2{a_1}^2 + {\delta_2}^2{a_3}^2\right)}{4{a_1}^2\left({a_1}^2 - {a_3}^2\right)}.$$

The eighth exact solution to equation (8.1) can be obtained as follows:

$$\begin{aligned} u &= \frac{-2\,a_1 \mathrm{e}^{-\eta_1} + 2\,\delta_2 \sinh\left(\eta_3\right) a_3 + 2\,a_1 \delta_3 \mathrm{e}^{\eta_1}}{\mathrm{e}^{-\eta_1} + \delta_1 \cos\left(\eta_2\right) + \delta_2 \cosh\left(\eta_3\right) + \delta_3 \mathrm{e}^{\eta_1}} \\ &= \begin{cases} \frac{-4a_1\sqrt{\delta_3}\sinh(\eta_1+\ln\sqrt{\delta_3})+2\delta_2\sinh(\eta_3)a_3}{2\sqrt{\delta_3}\cosh(\eta_1+\ln\sqrt{\delta_3})+\delta_1\cos(\eta_2)+\delta_2\cosh(\eta_3)}, & \delta_3 > 0; \\ \frac{-4a_1\sqrt{-\delta_3}\cosh(\eta_1+\ln\sqrt{-\delta_3})+2\delta_2\sinh(\eta_3)a_3}{2\sqrt{-\delta_3}\sinh(\eta_1+\ln\sqrt{-\delta_3})+\delta_1\cos(\eta_2)+\delta_2\cosh(\eta_3)}, & \delta_3 < 0. \end{cases} \end{aligned} \tag{8.45}$$

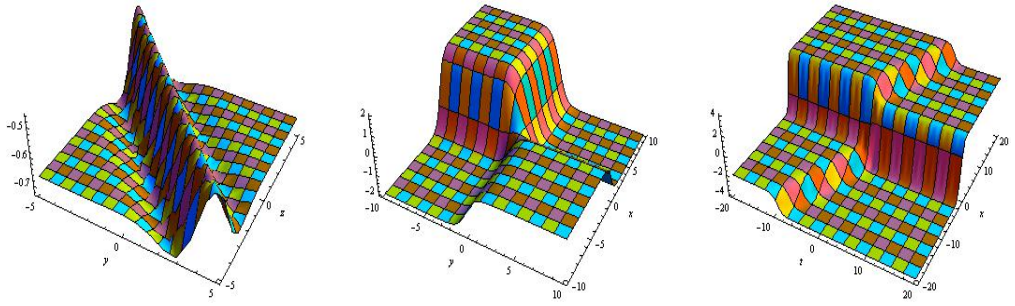
where

$$\eta_1 = a_1 x + \frac{a_1 b_3 y}{a_3} + c_1 t + \frac{b_3\left({a_1}^3 + 2\,c_1 + 3\,a_1{a_3}^2\right) z}{3a_3}, \tag{8.46a}$$

$$\eta_2 = b_2 y - \frac{3{a_1}^2 a_3 b_3 t}{2b_2} + \frac{\left({b_2}^2{a_1}^3 - 3\,{a_1}^3{b_3}^2 + 2\,{b_2}^2 c_1\right) z}{3a_1 b_2}, \tag{8.46b}$$

$$a_3x + b_3y + \frac{a_3\,(a_1{}^3 + 2\,c_1 - a_1a_3{}^2)\,t}{2a_1} + \frac{2b_3\,(2\,a_1{}^3 + c_1)\,z}{3a_1}. \tag{8.46c}$$

The parameters mentioned in (8.45) and (8.46) except δ_3 are all arbitrary.



(a) **(b)** **(c)**

Figure 8.3: Setting all arbitrary parameter to unity except $a_3 = 2$, figure (a) soliton-like and figure (b) one solitary wave profile of solution (8.43) on $y - z$ plane and $x - y$ plane respectively, figure (c) two solitary wave profile of solution (8.45).

9

Exact solutions for nonlinear evolution equations using novel test function[1]

> The mathematical sciences particularly exhibit order, symmetry, and limitation; and these are the greatest forms of the beautiful. .
>
> *Aristotle*

9.1 Introduction

Last two decades have witnessed a considerable increase in number of new methods for solving nonlinear PDEs including those with constant and variable coefficients as well. The introduction of some symbolic manipulation programs like "Maple" and "Mathematica" have greatly helped the researchers to perform complex algebraic calculations. In the early nineties, many known physically relevant PDEs have been investigated for exact solutions using tanh method [162, 163]. The Jacobi elliptic function expansion method [152] followed by its generalizations; the F-expansion method [267] and Exp-function method [21, 99] have played a crucial role in further developments toward the analysis of nonlinear PDEs for exact solutions. The methodologies like generalized Kudryashov method [128] and the functional variable method [45] are also very effective in the construction of exact solutions. Besides these techniques for solving nonlinear PDEs, there exist much more sophisticated methods like Lie group method [34,93], inverse scattering transform [4] and Hirota's bilinear method [104]. Recently, motivated by extended three-wave method, a novel test function [205] for constructing exact solutions has been proposed and consequently, several more general solutions have been reported for (3+1)-dimensional Jimbo-Miwa equation.

In this chapter, based on Bell polynomials approach and Maple computer algebra

[1]The content of this chapter is published in Nonlinear Dynamics 86(2) (2016): 1171-1182 [209]. The final publication is available at link.springer.com/article/10.1007/s11071-016-2955-2.

package PDEBellII [166], the bilinear equations for well-known physically relevant nonlinear evolution equations have been constructed. These bilinear equations are then used to construct exact solutions using novel test function.

9.2 Bilinear equations using Maple package PDEBellII

Recently, Miao *et al.* [166] have proposed Maple package PDEBellII for automatically constructing bilinear equations for nonlinear evolutions using Bell polynomials. We have used this package to automatically construct bilinear equations for following well-known evolution equations:

Example 9.2.1. The (3+1)-dimensional B-type Kadomstev-Petviashvili equation [58, 59, 159, 161, 169],

$$u_{tz} - 3u_{xx}u_y - 3u_x u_{xy} + 3u_{xx} - u_{xxxy} = 0. \tag{9.1}$$

In order to bilinearize equation (9.1), the Maple package PDEBellII automatically detect transformation

$$u = c\, q_x \text{ for } q = q(x, y, z, t). \tag{9.2}$$

So that equation (9.1) transformed to new equation

$$q_{tz} - 3\, c\, q_{xy}q_{xx} + 3\, q_{xx} - q_{xxxy} = 0, \tag{9.3}$$

under constraint $c = 1$ and by the formulae (1.127) of $\mathscr{P}$-polynomials, the new equation (9.3) can be written as follows:

$$\mathscr{P}_{t,z}(q) + 3\mathscr{P}_{2x}(q) - \mathscr{P}_{3x,y}(q) = 0. \tag{9.4}$$

Through the transformation $q = 2\log f$ and on account of property of Bell polynomials (1.126), the equation (9.4) gives following bilinear equation

$$\left(D_t D_z + 3D_x^2 - D_x^3 D_y\right) f \cdot f = 0. \tag{9.5}$$

Example 9.2.2. The (3+1)-dimensional generalized Kadomstev-Petviashvili equation, [158, 161, 242]

$$u_{xxxy} + 3\,(u_x u_y)_x + u_{tx} + u_{ty} - u_{zz} = 0. \tag{9.6}$$

In order to bilinearize equation (9.6), the Maple package PDEBellII automatically detect transformation

$$u = c\, q_x \text{ for } q = q(x, y, z, t). \tag{9.7}$$

So that equation (9.6) transformed to new equation

$$3\, c\, q_{xy}q_{xx} + q_{ty} - q_{zz} + q_{tx} + q_{xxxy} = 0, \tag{9.8}$$

under constraint $c = 1$ and by the formulae (1.127) of $\mathscr{P}$-polynomials, the new equation (9.8) can be written as follows.

$$\mathscr{P}_{3x,y}(q) - \mathscr{P}_{z,z}(q) + \mathscr{P}_{t,y}(q) + \mathscr{P}_{t,x}(q) = 0. \tag{9.9}$$

Through the transformation $q = 2\log f$ and on account of property of Bell polynomials (1.126), the equation (9.9) gives following bilinear equation

$$(D_x^3 D_y + D_t D_x + D_t D_y - D_z^2) f \cdot f = 0. \tag{9.10}$$

Example 9.2.3. The (3+1)-dimensional Kadomstev-Petviashvili equation [260],

$$u_{tx} + 6\,u_x^2 + 6\,uu_{xx} - u_{xxxx} - u_{yy} - u_{zz} = 0. \tag{9.11}$$

In order to bilinearize equation (9.11), the Maple package PDEBellII automatically detect transformation

$$u = c\,q_{x,x} \text{ for } q = q(x, y, z, t). \tag{9.12}$$

So that equation (9.11) transformed to new equation

$$3\,c\,q_{xx}^2 - q_{xxxx} + q_{tx} - q_{yy} - q_{zz} = 0, \tag{9.13}$$

under constraint $c = -1$ and by the formulae (1.127) of $\mathscr{P}$-polynomials, the new equation (9.13) can be written as follows:

$$-\mathscr{P}_{2y}(q) - \mathscr{P}_{2z}(q) - \mathscr{P}_{4x}(q) + \mathscr{P}_{t,x}(q) = 0. \tag{9.14}$$

Through the transformation $q = 2\log f$ and on account of property of Bell polynomials (1.126), the equation (9.14) gives following bilinear equation

$$(-D_x^4 + D_t D_x - D_y^2 - D_z^2) f \cdot f = 0. \tag{9.15}$$

Example 9.2.4. The (3+1)-dimensional generalised shallow water equation [156, 215, 254],

$$u_{xxxy} + 3\,u_{xx}u_y + 3\,u_x u_{xy} - u_{ty} - u_{xz} = 0. \tag{9.16}$$

In order to bilinearize equation (9.16), the Maple package PDEBellII automatically detect transformation

$$u = c\,q_x \text{ for } q = q(x, y, z, t). \tag{9.17}$$

So that equation (9.11) transformed to new equation

$$3\,c\,q_{xy}q_{xx} + q_{xxxy} - q_{ty} - q_{xz} = 0, \tag{9.18}$$

under constraint $c = -1$ and by the formulae (1.127) of $\mathscr{P}$-polynomials, the new equation (9.18) can be written as follows:

$$-\mathscr{P}_{x,z}(q) - \mathscr{P}_{t,y}(q) + \mathscr{P}_{3x,y}(q) = 0. \tag{9.19}$$

Through the transformation $q = 2\log f$ and on account of property of Bell polynomials (1.126), the equation (9.19) gives following bilinear equation

$$(D_x^3 D_y - D_t D_y - D_x D_z) f \cdot f = 0. \tag{9.20}$$

9.3 Exact solutions using novel test function

Recently, one of us has proposed a novel test function [205] for constructing exact solutions of nonlinear evolution equations using Hirota's bilinear equation. In this section, we shall apply this test function for constructing exact solutions for some well-known physically relevant (3+1)-dimensional evolution equations. In order to construct exact solutions for equations (9.1), (9.6), (9.11) and (9.16), we seek novel test function of the following form

$$f = \mathrm{e}^{-\xi_1} + \delta_1 \tan(\xi_2) + \delta_2 \tanh(\xi_3) + \delta_3 \mathrm{e}^{\xi_1}, \tag{9.21}$$

where

$$\xi_i = a_i x + b_i y + c_i t + d_i z, \ i = 1, 2, 3.$$

On substituting (9.21) into bilinear forms (9.5), (9.10), (9.15) and (9.20), a very large expression in coefficients of $e^{\pm\xi_1}, e^{\pm\xi_1}\tan(\xi_2)$, $e^{\pm\xi_1}\tanh(\xi_3), \tan(\xi_2)\tanh(\xi_3)$ etc will be obtained. Such coefficients can be collected into set of algebraic equations with the aid of Maple(refer to Appendix A.2 for code).

9.3.1 Exact solutions for (9.1)

On substituting test function (9.21) into bilinear equation (9.5) and solving resulting algebraic equations give set of exact solutions for (9.1). These solutions have been listed in following cases:

Case 9.3.1. $a_2 = a_3 = 0, b_1 = \frac{a_1 b_2 c_1 + 3c_2}{a_1 c_2}, b_3 = \frac{b_2 c_3}{c_2}, d_1 = \frac{a_1^3 b_2}{c_2}, d_2 = d_3 = 0, \delta_3 = 0,$

$$u = -\frac{2a_1 \mathrm{e}^{-\xi_1}}{\mathrm{e}^{-\xi_1} + \delta_1 \tan(\xi_2) + \delta_2 \tanh(\xi_3)}, \tag{9.22}$$

where

$$\xi_1 = a_1 x + \frac{y(a_1 b_2 c_1 + 3\,c_2)}{a_1 c_2} + \frac{z {a_1}^3 b_2}{c_2} + c_1 t, \ \xi_2 = y b_2 + t c_2, \ \xi_3 = \frac{y b_2 c_3}{c_2} + c_3 t,$$

$a_1, b_2, c_1, c_2, c_3, \delta_1$ and δ_2 are arbitrary constants.

Case 9.3.2. $a_2 = a_3 = 0, b_1 = 0, b_2 = -\frac{3d_2}{a_1 d_1}, b_3 = -\frac{3d_3}{a_1 d_1}, c_1 = -\frac{3a_1^2}{d_1}, c_2 = c_3 = 0,$

$$u = \frac{2a_1\left(-\mathrm{e}^{-\xi_1} + \delta_3 \mathrm{e}^{\xi_1}\right)}{\mathrm{e}^{-\xi_1} + \delta_1 \tan(\xi_2) + \delta_2 \tanh(\xi_3) + \delta_3 \mathrm{e}^{\xi_1}}, \tag{9.23}$$

where

$$\xi_1 = a_1 x + d_1 z - \frac{3t {a_1}^2}{d_1}, \ \xi_2 = -\frac{3y d_2}{a_1 d_1} + d_2 z, \ \xi_3 = -\frac{3y d_3}{a_1 d_1} + d_3 z,$$

$a_1, d_1, d_2, d_3, \delta_1, \delta_2$ and δ_3 are arbitrary constants.

Case 9.3.3. $a_1 = \frac{a_3 d_1}{d_3}, a_2 = 0, b_1 = b_3 = 0, c_1 = -\frac{3a_3^3 d_1}{d_3^2}, c_3 = -\frac{3a_3^2}{d_3}, \delta_1 = 0,$

$$u = -\frac{2a_3\left(\left(\tanh\left(\xi_3\right)\right)^2 d_3\delta_2 - \delta_3 d_1 \mathrm{e}^{\xi_1} + d_1 \mathrm{e}^{-\xi_1} - \delta_2 d_3\right)}{d_3\left(\mathrm{e}^{-\xi_1} + \delta_2 \tanh\left(\xi_3\right) + \delta_3 \mathrm{e}^{\xi_1}\right)}, \tag{9.24}$$

where

$$\xi_1 = \frac{x a_3 d_1}{d_3} + d_1 z - \frac{3t{a_3}^2 d_1}{{d_3}^2},\ \xi_3 = a_3 x + d_3 z - \frac{3t{a_3}^2}{d_3},$$

$a_3, b_3, c_2, d_1, d_2, d_3, \delta_2, \delta_3$ are arbitrary constants.

Case 9.3.4. $c_1 = \frac{a_1^2(4a_1b_1-3)}{d_1},$

$$u = \frac{2\left(-a_1 \mathrm{e}^{-\xi_1} + \delta_3 a_1 \mathrm{e}^{\xi_1}\right)}{\mathrm{e}^{-\xi_1} + \delta_3 \mathrm{e}^{\xi_1}},\quad \xi_1 = a_1 x + b_1 y + d_1 z + \frac{t{a_1}^2\left(4\,a_1 b_1 - 3\right)}{d_1}, \tag{9.25}$$

a_1, b_1, d_1, δ_3 are arbitrary constants.

Case 9.3.5. $a_2 = a_3 = 0, b_1 = b_2 = 0, b_3 = \frac{c_3 d_1}{a_1^3}, c_1 = -\frac{3a_1^2}{d_1}, c_2 = 0, d_2 = d_3 = 0,$

$$u = \frac{2\left(-a_1 \mathrm{e}^{-\xi_1} + \delta_3 a_1 \mathrm{e}^{\xi_1}\right)}{\mathrm{e}^{-\xi_1} + \delta_2 \tanh\left(\xi_3\right) + \delta_3 \mathrm{e}^{\xi_1}}, \tag{9.26}$$

where

$$\xi_1 = a_1 x + d_1 z - \frac{3t{a_1}^2}{d_1},\ \xi_3 = \frac{y c_3 d_1}{{a_1}^3} + c_3 t,$$

$a_1, c_3, d_1, \delta_1, \delta_2, \delta_3$ are arbitrary constants.

9.3.2 Exact solutions for (9.6)

On substituting test function (9.21) into bilinear equation (9.10) and solving resulting algebraic equations give set of exact solutions for (9.6). These solutions have been listed in following cases:

Case 9.3.6. $b_1 = b_2 = b_3 = 0, c_1 = \frac{d_2^2 a_1}{a_2^2}, c_2 = \frac{d_2^2}{a_2}, c_3 = \frac{d_2^2 a_3}{a_2^2}, d_1 = \frac{a_1 d_2}{a_2}, d_3 = \frac{a_3 d_2}{a_2},$

$$\begin{aligned} u = &\frac{1}{\mathrm{e}^{-\xi_1} + \delta_1 \tan\left(\xi_2\right) + \delta_2 \tanh\left(\xi_3\right) + \delta_3 \mathrm{e}^{\xi_1}}\Big[2\,\left(\tan\left(\xi_2\right)\right)^2 a_2 \delta_1 \\ &- 2\,\left(\tanh\left(\xi_3\right)\right)^2 a_3 \delta_2 + 2\,\delta_3 a_1 \mathrm{e}^{\xi_1} - 2\,a_1 \mathrm{e}^{-\xi_1} + 2\,\delta_1 a_2 + 2\,\delta_2 a_3\Big], \end{aligned} \tag{9.27}$$

where

$$\begin{aligned} \xi_1 =& a_1 x + \frac{z a_1 d_2}{a_2} + \frac{t{d_2}^2 a_1}{{a_2}^2}, \\ \xi_2 =& a_2 x + d_2 z + \frac{t{d_2}^2}{a_2}, \\ \xi_3 =& a_3 x + \frac{z a_3 d_2}{a_2} + \frac{t{d_2}^2 a_3}{{a_2}^2}, \end{aligned}$$

$a_1, a_2, a_3, d_2, \delta_1, \delta_2, \delta_3$ are arbitrary constants.

Case 9.3.7. $a_3 = 0, b_1 = -\frac{{a_1}^3 {d_3}^2 + a_1 {c_3}^2 + d_3\left(-{a_1}^3 d_3 + c_3 d_1 + i {a_1}^2 c_3\right) - 2\, d_1 d_3 c_3}{{c_3}^2}, b_3 = \frac{d_3^2}{c_3}, c_1 = \frac{-{a_1}^3 d_3 + c_3 d_1 + i {a_1}^2 c_3}{d_3}, \delta_1 = \delta_3 = 0,$

$$u = -\frac{2 a_1 \mathrm{e}^{-\xi_1}}{\mathrm{e}^{-\xi_1} + \delta_2 \tanh(\xi_3)}, \tag{9.28}$$

where

$$\begin{aligned}\xi_1 =& a_1 x - \frac{y\left({a_1}^3 {d_3}^2 + a_1 {c_3}^2 + d_3\left(-{a_1}^3 d_3 + c_3 d_1 + i {a_1}^2 c_3\right) - 2\, d_1 d_3 c_3\right)}{{c_3}^2} + d_1 z \\ &+ \frac{t\left(-{a_1}^3 d_3 + c_3 d_1 + i {a_1}^2 c_3\right)}{d_3}, \\ \xi_2 =& x a_2 + y b_2 + z d_2 + t c_2,\ \xi_3 = \frac{y {d_3}^2}{c_3} + d_3 z + c_3 t,\end{aligned}$$

$a_1, a_2, b_2, c_2, c_3, d_1, d_2, d_3, \delta_2$ are arbitrary constants.

Case 9.3.8. $a_2 = 0, b_1 = 0, b_2 = \frac{d_2 a_1}{d_1 + i {a_1}^2}, c_1 = \frac{d_1^2}{a_1}, c_2 = \frac{\left(d_1 + i {a_1}^2\right) d_2}{a_1}, \delta_2 = 0,$

$$u = \frac{2\left(-a_1 \mathrm{e}^{-\xi_1} + \delta_3 a_1 \mathrm{e}^{\xi_1}\right)}{\mathrm{e}^{-\xi_1} + \delta_1 \tan(\xi_2) + \delta_3 \mathrm{e}^{\xi_1}}, \tag{9.29}$$

where

$$\begin{aligned}\xi_1 =& a_1 x + d_1 z + \frac{t {d_1}^2}{a_1},\ \xi_2 = \frac{y d_2 a_1}{d_1 + i {a_1}^2} + d_2 z + \frac{t\left(d_1 + i {a_1}^2\right) d_2}{a_1}, \\ \xi_3 =& x a_3 + y b_3 + z d_3 + t c_3,\end{aligned}$$

$a_1, a_3, b_3, c_3, d_1, d_2, d_3, \delta_1, \delta_3$ are arbitrary constants.

Case 9.3.9. $a_2 = 0, b_1 = -\frac{{a_1}^3 {d_2}^2 + a_1 {c_2}^2 + d_2\left(-{a_1}^3 d_2 + c_2 d_1 + i {a_1}^2 c_2\right) - 2\, d_1 d_2 c_2}{{c_2}^2}, b_2 = \frac{d_2^2}{c_2}, c_1 = \frac{-{a_1}^3 d_2 + c_2 d_1 + i {a_1}^2 c_2}{d_2}, \delta_1 = \delta_2 = 0,$

$$u = -\frac{2 a_1 \mathrm{e}^{-\xi_1}}{\mathrm{e}^{-\xi_1} + \delta_1 \tan(\xi_2)}, \tag{9.30}$$

where

$$\begin{aligned}\xi_1 =& a_1 x - \frac{y\left({a_1}^3 {d_2}^2 + a_1 {c_2}^2 + d_2\left(-{a_1}^3 d_2 + c_2 d_1 + i {a_1}^2 c_2\right) - 2\, d_1 d_2 c_2\right)}{{c_2}^2} + d_1 z \\ &+ \frac{t\left(-{a_1}^3 d_2 + c_2 d_1 + i {a_1}^2 c_2\right)}{d_2}, \\ \xi_2 =& \frac{y {d_2}^2}{c_2} + d_2 z + c_2 t,\ \xi_3 = x a_3 + y b_3 + z d_3 + t c_3,\end{aligned}$$

$a_1, a_3, b_3, c_2, c_3, d_1, d_2, d_3, \delta_1$ are arbitrary constants.

Case 9.3.10. $a_3 = 0, b_1 = 0, b_3 = \frac{d_3 a_1}{d_1 + i {a_1}^2}, c_1 = \frac{d_1^2}{a_1}, c_3 = \frac{\left(d_1 + i {a_1}^2\right) d_3}{a_1}, \delta_1 = 0,$

$$u = \frac{2\left(-a_1 \mathrm{e}^{-\xi_1} + \delta_3 a_1 \mathrm{e}^{\xi_1}\right)}{\mathrm{e}^{-\xi_1} + \delta_2 \tanh(\xi_3) + \delta_3 \mathrm{e}^{\xi_1}}, \tag{9.31}$$

where

$$\xi_1 = a_1 x + d_1 z + \frac{t {d_1}^2}{a_1},\ \xi_2 = x a_2 + y b_2 + z d_2 + t c_2,$$
$$\xi_3 = \frac{y d_3 a_1}{d_1 + i {a_1}^2} + d_3 z + \frac{t\left(d_1 + i {a_1}^2\right) d_3}{a_1},$$

$a_1, a_2, b_2, c_2, d_1, d_2, d_3, \delta_2, \delta_3$ are arbitrary constants.

9.3.3 Exact solutions for (9.11)

On substituting test function (9.21) into bilinear equation (9.15) and solving resulting algebraic equations give set of exact solutions for (9.11). These solutions have been listed in following cases:

Case 9.3.11. $a_2 = a_3 = 0, b_2 = i\, d_2, b_3 = i\, d_3, c_1 = \frac{{a_1}^4 + {b_1}^2 + {d_1}^2}{a_1}, c_2 = \frac{2 d_2 (i b_1 + d_1)}{a_1}, c_3 = \frac{2 d_3 (i b_1 + d_1)}{a_1}, \delta_3 = 0,$

$$u = -\frac{2 {a_1}^2 \mathrm{e}^{-\xi_1} \left(\delta_1 \tan(\xi_2) + \delta_2 \tanh(\xi_3)\right)}{\left(\mathrm{e}^{-\xi_1} + \delta_1 \tan(\xi_2) + \delta_2 \tanh(\xi_3)\right)^2}, \tag{9.32}$$

where

$$\xi_1 = a_1 x + b_1 y + d_1 z + \frac{t\left({a_1}^4 + {b_1}^2 + {d_1}^2\right)}{a_1},\ \xi_2 = i y d_2 + d_2 z + \frac{2 t d_2 (i b_1 + d_1)}{a_1},$$
$$\xi_3 = i y d_3 + d_3 z + \frac{2 t d_3 (i b_1 + d_1)}{a_1},$$

$a_1, b_1, d_1, d_2, d_3, \delta_1, \delta_2$ are arbitrary constants.

Case 9.3.12. $a_3 = 0, b_3 = i\, d_3, c_1 = \frac{{a_1}^4 + {b_1}^2 + {d_1}^2}{a_1}, c_3 = 2\,\frac{d_3 (i b_1 + d_1)}{a_1}, \delta_1 = 0, \delta_3 = 0,$

$$u = -\frac{2 {a_1}^2 \mathrm{e}^{-\xi_1} \delta_2 \tanh(\xi_3)}{\left(\mathrm{e}^{-\xi_1} + \delta_2 \tanh(\xi_3)\right)^2}, \tag{9.33}$$

where

$$\xi_1 = a_1 x + b_1 y + d_1 z + \frac{t\left({a_1}^4 + {b_1}^2 + {d_1}^2\right)}{a_1},\ \xi_2 = x a_2 + y b_2 + z d_2 + t c_2,$$
$$\xi_3 = i y d_3 + d_3 z + \frac{2 t d_3 (i b_1 + d_1)}{a_1},$$

$a_1, a_2, b_1, b_2, c_2, d_1, d_2, d_3, \delta_2$ are arbitrary constants.

Case 9.3.13. $a_2 = 0, b_2 = i\, d_2, c_1 = \frac{{a_1}^4 + {b_1}^2 + {d_1}^2}{a_1}, c_2 = \frac{2 d_2 (i b_1 + d_1)}{a_1}, \delta_2 = \delta_3 = 0,$

$$u = -\frac{2 {a_1}^2 \mathrm{e}^{-\xi_1} \delta_1 \tan(\xi_2)}{\left(\mathrm{e}^{-\xi_1} + \delta_1 \tan(\xi_2)\right)^2}, \tag{9.34}$$

where

$$\xi_1 = a_1 x + b_1 y + d_1 z + \frac{t\left({a_1}^4 + {b_1}^2 + {d_1}^2\right)}{a_1},\ \xi_2 = i y d_2 + d_2 z + \frac{2 t d_2 (i b_1 + d_1)}{a_1},$$
$$\xi_3 = x a_3 + y b_3 + z d_3 + t c_3,$$

$a_1, a_3, b_1, b_3, c_3, d_1, d_2, d_3, \delta_1$ are arbitrary constants.

9.3.4 Exact solutions for (9.16)

On substituting test function (9.21) into bilinear equation (9.20) and solving resulting algebraic equations give set of exact solutions for (9.16). These solutions have been listed in following cases:

Case 9.3.14. $a_2 = 0, b_1 = 0, c_1 = \frac{a_1\left({a_1}^2 b_2 - d_2\right)}{b_2}, c_2 = 0, d_1 = 0, \delta_2 = 0,$

$$u = \frac{2\left(-a_1 \mathrm{e}^{-\xi_1} + \delta_3 a_1 \mathrm{e}^{\xi_1}\right)}{\mathrm{e}^{-\xi_1} + \delta_1 \tan\left(\xi_2\right) + \delta_3 \mathrm{e}^{\xi_1}}, \tag{9.35}$$

where

$$\xi_1 = a_1 x + \frac{t a_1 \left({a_1}^2 b_2 - d_2\right)}{b_2},\ \xi_2 = y b_2 + z d_2,\ \xi_3 = x a_3 + y b_3 + z d_3 + t c_3,$$

$a_1, a_3, b_2, b_3, c_3, d_2, d_3, \delta_1, \delta_3$ are arbitrary constants.

Case 9.3.15. $a_2 = a_3 = 0, b_1 = 0, c_1 = \frac{a_1\left({a_1}^2 b_2 - d_2\right)}{b_2}, c_2 = c_3 = 0, d_1 = 0, d_3 = \frac{b_3 d_2}{b_2},$

$$u = \frac{2\left(-a_1 \mathrm{e}^{-\xi_1} + \delta_3 a_1 \mathrm{e}^{\xi_1}\right)}{\mathrm{e}^{-\xi_1} + \delta_1 \tan\left(\xi_2\right) + \delta_2 \tanh\left(\xi_3\right) + \delta_3 \mathrm{e}^{\xi_1}}, \tag{9.36}$$

where

$$\xi_1 = a_1 x + \frac{t a_1 \left({a_1}^2 b_2 - d_2\right)}{b_2},\ \xi_2 = y b_2 + z d_2,\ \xi_3 = b_3 y + \frac{z b_3 d_2}{b_2},$$

$a_1, b_2, b_3, d_2, \delta_1, \delta_2, \delta_3$ are arbitrary constants.

Case 9.3.16. $a_2 = 0, c_1 = \frac{a_1\left(4\,{a_1}^2 b_1 - d_1\right)}{b_1}, \delta_1 = \delta_2 = 0,$

$$u = \frac{2\left(-a_1 \mathrm{e}^{-\xi_1} + \delta_3 a_1 \mathrm{e}^{\xi_1}\right)}{\mathrm{e}^{-\xi_1} + \delta_3 \mathrm{e}^{\xi_1}}, \tag{9.37}$$

where

$$\xi_1 = a_1 x + b_1 y + d_1 z + \frac{t a_1 \left(4\,{a_1}^2 b_1 - d_1\right)}{b_1},\ \xi_2 = y b_2 + z d_2 + t c_2,$$
$$\xi_3 = x a_3 + y b_3 + z d_3 + t c_3,$$

$a_1, a_3, b_1, b_2, b_3, c_2, c_3, d_1, d_2, d_3, \delta_3$ are arbitrary constants.

Case 9.3.17. $a_2 = a_3 = 0, b_2 = 0, c_2 = c_3 = 0, d_1 = \frac{b_1\left({a_1}^3 - c_1\right)}{a_1}, d_2 = 0, d_3 = \frac{b_3\left({a_1}^3 - c_1\right)}{a_1}, \delta_3 = 0,$

$$u = -\frac{2 a_1 \mathrm{e}^{-\xi_1}}{\mathrm{e}^{-\xi_1} + \delta_2 \tanh\left(\xi_3\right)}, \tag{9.38}$$

where

$$\xi_1 = a_1 x + b_1 y + \frac{z b_1 \left({a_1}^3 - c_1\right)}{a_1} + c_1 t,\ \xi_3 = b_3 y + \frac{z b_3 \left({a_1}^3 - c_1\right)}{a_1},$$

$a_1, b_1, b_3, c_1, \delta_1, \delta_2$ are arbitrary constants.

Case 9.3.18. $a_2 = 0, a_3 = 0, b_1 = 0, c_3 = 0, d_1 = 0, d_3 = \frac{b_3\left({a_1}^3 - c_1\right)}{a_1}, \delta_1 = 0,$

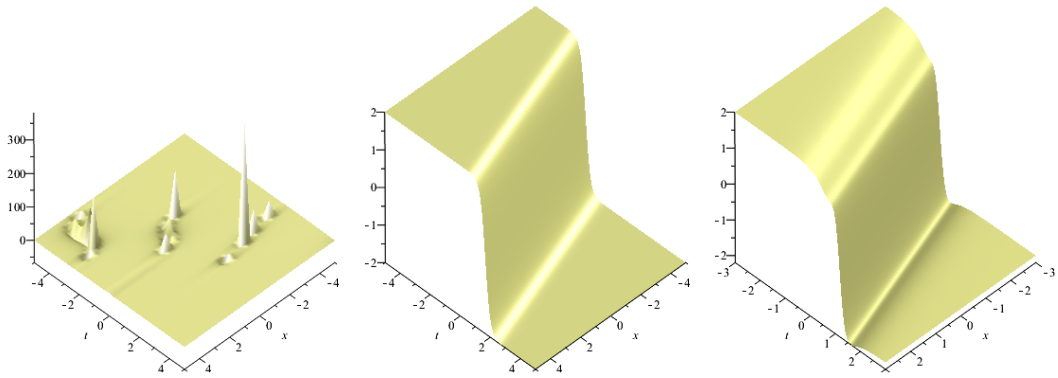
$$u = \frac{2\left(-a_1 \mathrm{e}^{-\xi_1} + \delta_3 a_1 \mathrm{e}^{\xi_1}\right)}{\mathrm{e}^{-\xi_1} + \delta_2 \tanh\left(\xi_3\right) + \delta_3 \mathrm{e}^{\xi_1}}, \tag{9.39}$$

where

$$\xi_1 = x a_1 + t c_1, \ \xi_2 = y b_2 + z d_2 + t c_2, \ \xi_3 = b_3 y + \frac{z b_3\left({a_1}^3 - c_1\right)}{a_1},$$

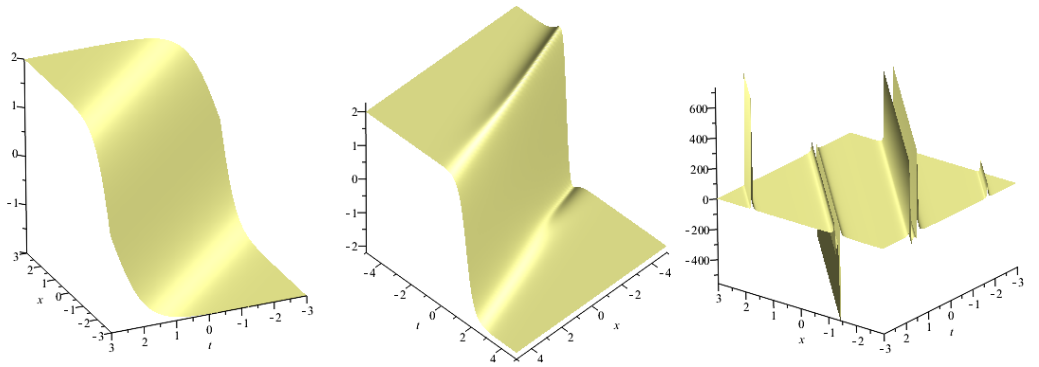
$a_1, b_2, b_3, c_1, c_2, d_2, \delta_2, \delta_3$ are arbitrary constants.

Remark 9.3.1. The solutions that are listed above have actually been verified by the use of Maple. It is interesting to note that the solutions (9.22)-(9.39) involve exponential and trigonometric functions, which means that the obtained solutions are actually complexiton [157]. The plots of these solutions have also been drawn by taking arbitrary constants as unity, see *e.g.* Figure 9.1-Figure 9.7.



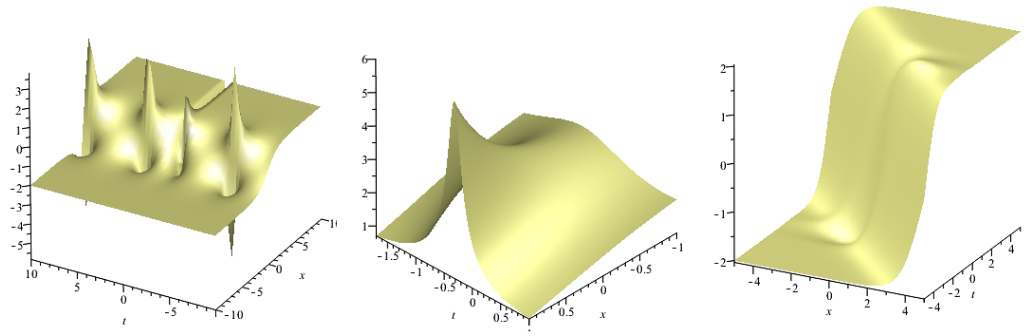
(a) **(b)** **(c)**

Figure 9.1: Figure 9.1a; divergent behaviour of solution (9.22). Figure 9.1b; kink wave solution (9.23). Figure 9.1c; kink wave solution (9.24). All plots are in $x - t$ domain.



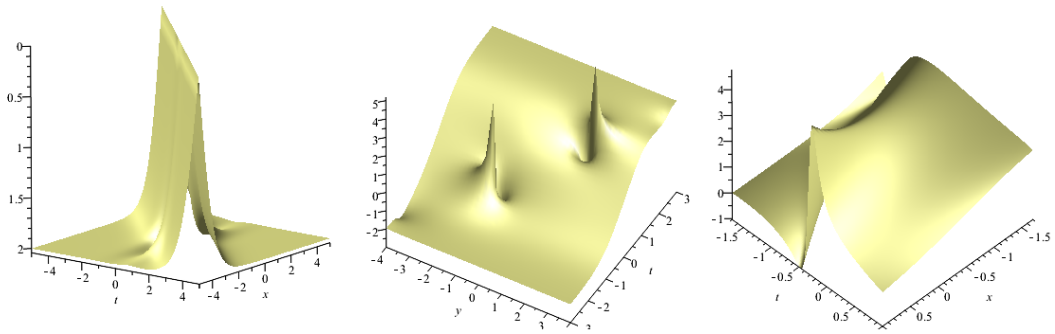
(a) **(b)** **(c)**

Figure 9.2: Figure 9.2a; kink wave solution (9.25). Figure 9.2b; kink wave solution (9.26). Figure 9.2c; singular solution (9.27). All plots are in $x - t$ domain.



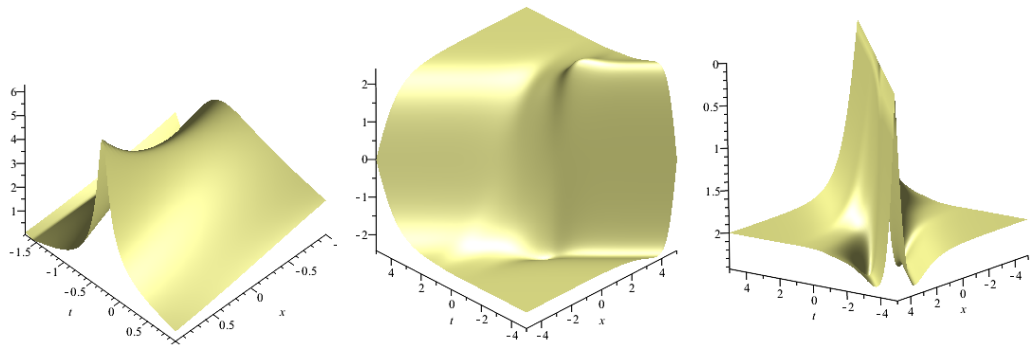
(a) **(b)** **(c)**

Figure 9.3: Figure 9.3a; kinky periodic wave profile given by $\Re(u)$ for solution (9.28). Figure 9.3b; smooth profile given by $|u|$ for solution (9.28). Figure 9.3c; kink wave profile given by $\Re(u)$ for solution (9.29). All plots are in $x-t$ domain.



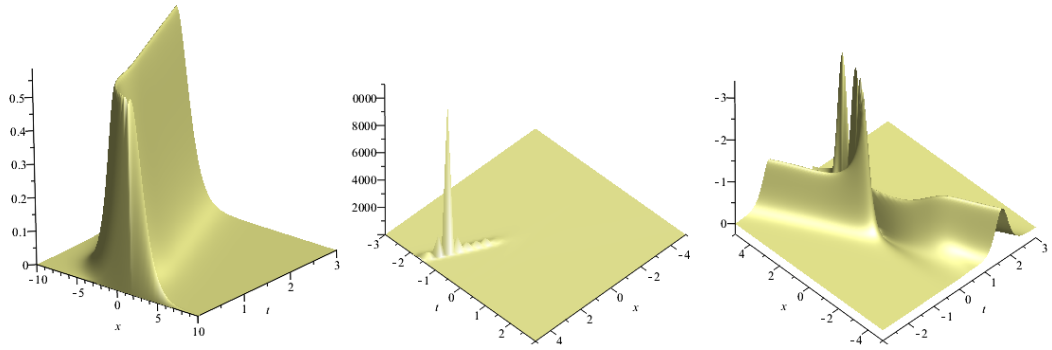
(a) **(b)** **(c)**

Figure 9.4: Figure 9.4a; soliton like profile given by $|u|$ for solution (9.29). Figure 9.4b; kinky periodic profile given by $\Re(u)$ for solution (9.29). Figure 9.4c; smooth profile given by $\Re(u)$ for solution (9.30). All plots are in $x-t$ domain except in case of plot (b) where domain is $y-t$.



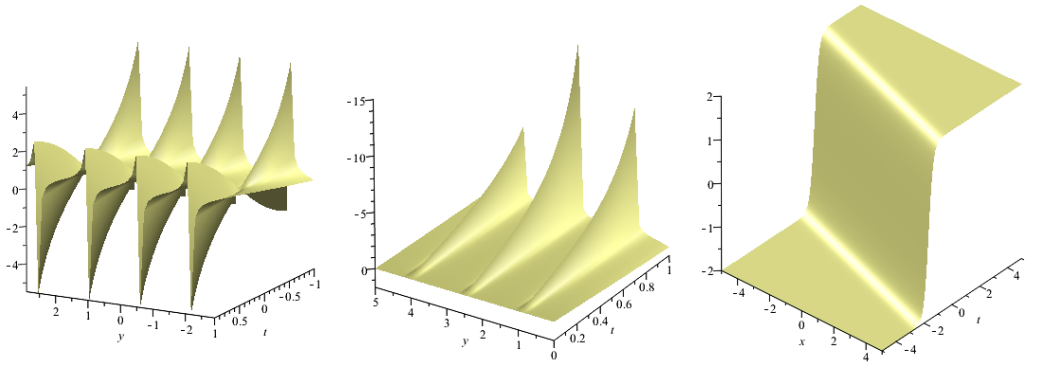
(a) **(b)** **(c)**

Figure 9.5: Figure 9.5a; smooth profile given by $|u|$ for solution (9.30). Figure 9.5b; kink wave profile given by $\Re(u)$ for solution (9.31). Figure 9.5c; soliton like profile given by $|u|$ for solution (9.31). All plots are in $x-t$ domain.



(a) **(b)** **(c)**

Figure 9.6: Figure 9.6a; plot given by $\mathfrak{R}(u)$ for solution (9.32). Figure 9.6b; singular profile given by $|u|$ for solution (9.33). Figure 9.6c; plot given by $\mathfrak{R}(u)$ for solution (9.34). All plots are in $x-t$ domain.



(a) **(b)** **(c)**

Figure 9.7: Figure 9.7a; periodic wave solution (9.35). Figure 9.7b; periodic wave solution (9.36). Figure 9.7c; kink wave solution (9.37). The plots (a) and (b) are in $y-t$ domain whereas the plot (c) is drawn in $x-t$ domain.

10

Traveling and quasi-periodic wave solutions for b-type Kadomtsev-Petviashvili equation[1]

"Obvious" is the most dangerous word in mathematics.

E.T. Bell

10.1 Introduction

The recent developments in nonlinear sciences have shown that the exact solutions are at the center of study for researchers. The wave characteristics in fluid dynamics, plasma physics, elastic media, optical fibre communications are generally modeled into exact solutions of PDEs describing the model under study. These exact solutions are generally found in form of bell-shaped sech solutions, kinked shaped tanh solution, solitons, soliton-offs, dromions etc. Once the exact solution is available, it becomes possible to compare actual experimental results with results obtained using exact solution. Moreover, exact solutions are used to improve stability and reliability of numerical solvers. Therefore, it is important to study exact solutions in order to have a better analytical understanding of physical phenomena in nonlinear sciences

There is a considerable number of methods for finding exact solutions. For example, inverse scattering transform(IST) [82] is one of the most powerful technique developed so far. This technique has been applied to the variety of integrable nonlinear evolution equations describing phenomena in nonlinear optics, the theory of relativity, plasma physics, hydrodynamics etc. But this technique is highly sophisticated and relies on powerful analytical methods. Bäcklund transformations which were first discovered for sine-Gordon equation, may be used to construct series of new solutions from known solutions by purely algebraic procedure known as nonlinear superposition principle [142, 188, 231]. Darboux transformation method

[1]The content of this chapter is published in Indian Journal of Physics (2017): 91(11), 1345-1354 [206].
The final publication is available at link.springer.com/article/10.1007/s12648-017-1035-x.

[149,150,165] is used to construct series of new exactly solvable equations from solvable equations (*e.g.* famous Kadomtsev-Petviashvili was derived in this way), and explicit solutions using Darboux covariance. The Hirota's bilinear method [105,164], which relies on the bilinear equation for the construction of multisoliton solutions in a straightforward manner is also very effective method. The Lie group method [34, 175, 222, 228], in which the group invariant solutions are obtained by reduction in number of independent variables via symmetry transformations. Beside these sophisticated methods for finding exact solutions, there is variety of other important methods like $\frac{G'}{G}$-method, Exp-function method [66], F-expansion method [25], Jacobi elliptic function method [31], semi-inverse variational principle [32], mapping method [133] and several methodologies also exist for constructing solitary wave solutions [216–218, 229, 236, 237].
Among these methods, Hirota's bilinear method is a powerful tool for solving wide class of nonlinear evolution equations. In Hirota's bilinear formalism, an equation is transformed into the bilinear form using some dependent variable transformation. Once the bilinear form is written, N-soliton solutions, Bäcklund transformations, infinite conservation laws may be derived in lucid and straightforward manner [164]. In literature, numerous applications of Hirota's bilinear method for solving nonlinear evolution equations with constant and variable coefficients [18, 73, 88, 166, 205, 209, 210, 239, 244–246, 253, 259, 261] can be seen.
In 1979, Akira Nakamura used multi-dimensional Riemann theta function to extend the application of Hirota's bilinear method to a new level by obtaining exact periodic wave solutions from bilinear equations of important PDEs in physical phenomena [170, 171]. Recently, this method has been successfully applied to variety of nonlinear evolution equations, including variable coefficient Kadomtsev-Petviashvili equation [220], modified Nizhnik-Novikov-Vesselov equation in (2+1)-dimensions [155], (2+1)-dimensions Bogoyavlenskii breaking soliton equation [72], and several other equations [219, 221, 226, 227] have also been investigated for periodic wave solutions. Akira Nakamura demonstrated that, using only the bilinear equation, periodic solutions can be derived in a straightforward manner, and that too without the use of heavy machinery of algebraic mathematics. The periodic solutions obtained in this manner asymptotically tends to soliton solutions [164] under small amplitude limits. Moreover, all the parameter involved in periodic solutions are completely arbitrary. In this study, inspired by work of Akira Nakamura, we shall investigate b-type Kadomtsev-Petviashvili equation

$$(u_{xxxxx} + 60u_x^3 + 5u_{xxt} + u_y + 30u_{xxx}u_x + 30u_xu_t)_x = 5u_{tt}, \tag{10.1}$$

for traveling and multi-periodic wave solutions. This equation has two Painlevé branches and it passes Painlevé test for both branches, and the bilinear form of this equation also satisfies three soliton condition [100]. Therefore, the equation (10.1) is integrable in sense of Painlevé property and Hirota's formalism.

10.2 Hirota bilinear method

The traveling wave ansätz may be used to construct a one-soliton solution. If the bilinear form of an equation is available, then one and two-soliton solutions may be constructed depending on the type of Cole-Hopf transformation. If an equation

is integrable in sense of Hirota's formalism, then it is always possible to construct multi-soliton solutions [101].

In order to apply Hirota's method, the given equation is first transformed into a quadratic equation of derivatives using some dependent variable transformations (also called as Cole-Hopf transformation), and this equation should be expressible in term of Hirota's D-operator defined in subsection 1.2.2 of chapter 1.

10.2.1 Bilinear form

By employing WTC algorithm [250] and truncating the Laurent expansion about singularity manifold at constant level, we find that

$$u = (\log f)_x + u_1, \tag{10.2}$$

where u_1 is a constant or seed solution of equation (10.1). Setting $u_1 = 0$ and substituting (10.2) into (10.1), we obtain

$$(D_x^6 + 5D_x^3 D_t - 5D_t^2 + D_x D_y) f \cdot f = 0, \tag{10.3}$$

where constant of integration has been taken as zero.

Traveling wave solution

For traveling wave solution, we use

$$f = 1 + e^{\eta_1},\ \eta_1 = \alpha_1 x + \beta_1 y + \omega_1 t + \delta_1. \tag{10.4}$$

On substituting (10.4) into the bilinear equation (10.3), we obtain following dispersion relation

$$\alpha_1^6 + 5\alpha_1^3\omega_1 - 5\omega_1^2 + \alpha_1\beta_1 = 0, \tag{10.5}$$

from which traveling wave solution may be written as follows:

$$u = (1 + e^{\eta_1})_x,\ \eta_1 = \alpha_1 x + \beta_1 y + \omega_1 t + \delta_1, \tag{10.6}$$

where parameters $\alpha_1, \beta_1, \omega_1$ and δ_1 are constrained by the dispersion relation (10.5).

Non-stationary solution

For non-stationary solutions, we may use

$$f = 1 + e^{\eta_1} + e^{\eta_2} + e^{\eta_1+\eta_2+A_{12}},\ \eta_i = \alpha_i x + \beta_i y + \omega_i t + \delta_i, i = 1, 2. \tag{10.7}$$

On substituting ansätz (10.7) into the bilinear equation (10.3). The dispersion relation is obtained as follows:

$$\alpha_i^6 + 5\alpha_i^3\omega_i - 5\omega_i^2 + \alpha_i\beta_i = 0,\ i = 1, 2. \tag{10.8}$$

Therefore, non-stationary solution may be written as follows:

$$u = (1 + e^{\eta_1} + e^{\eta_2} + A_{12}e^{\eta_1+\eta_2})_x,\ \eta_i = \alpha_i x + \beta_i y + \omega_i t + \delta_i,\ i = 1, 2. \tag{10.9}$$

Similarly, for three wave superposition, we may use

$$f = 1 + e^{\eta_1} + e^{\eta_2} + A_{12}e^{\eta_1+\eta_2} + A_{13}e^{\eta_1+\eta_3} + A_{23}e^{\eta_2+\eta_3} + A_{123}e^{\eta_1+\eta_2+\eta_3}, \quad (10.10)$$

for $A_{123} = A_{12}A_{13}A_{23}$ and $A_{ij} = \frac{(\alpha_i-\alpha_j)^6+5(\alpha_i-\alpha_j)^3(\omega_i-\omega_j)-5(\omega_i-\omega_j)^2+(\alpha_i-\alpha_j)(\beta_i-\beta_j)}{(\alpha_i+\alpha_j)^6+5(\alpha_i+\alpha_j)^3(\omega_i+\omega_j)-5(\omega_i+\omega_j)^2+(\alpha_i+\alpha_j)(\beta_i+\beta_j)}$, $i, j = 1, 2, 3$, $i < j$.

10.3 Multi-periodic wave solutions using Riemann theta function

In this section, we shall discuss technique developed by Nakamura [170,171] for constructing multi-periodic wave solutions using Hirota's bilinear equation. In order to do so, we have to write the bilinear equation in slightly generalized form. Therefore, we seek the solution of equation (10.1) in the following form

$$u = u_0 + (\log f)_x, \quad (10.11)$$

where u_0 is a constant or seed solution of equation (10.1). On substituting (10.11) into (10.1), we obtain

$$F(D_x, D_y, D_t)f \cdot f = (D_x^6 + 5D_x^3D_t - 5D_t^2 + D_xD_y + c)f \cdot f = 0. \quad (10.12)$$

However, for two-periodic wave solution for equation (10.1), we consider a more general form of the bilinear equation (10.12). So we seek solution of equation (10.1) in the form

$$u = u_0t + (\log f)_x. \quad (10.13)$$

On substituting (10.13) into (10.1), and integrating once, we obtain

$$\begin{aligned} F(D_x, D_y, D_t)f \cdot f = \ & (D_x^6 + 5D_x^3D_t - 5D_t^2 \\ & + 30u_0D_x^2 + D_xD_y + c)f \cdot f = 0, \end{aligned} \quad (10.14)$$

where in both cases (10.12) and (10.14), the constants of integration are taken as non-zero, and we shall realize its importance in subsequent sections.

10.3.1 Riemann theta function

For multi-periodic wave solutions, we consider N-dimensional Riemann theta function defined as follows:

$$f = \vartheta(\eta; \tau) = \sum_{n=-\infty}^{+\infty} e^{2\pi i\langle\eta,n\rangle+\pi i\langle\tau n,n\rangle}, \quad (10.15)$$

where $n = (n_1, n_2, \ldots, n_N)^T$, $\eta = (\eta_1, \eta_2, \ldots, \eta_N)$ for $\eta_i = \alpha_i x + \beta_i y + \omega_i t + \delta_i, i = 1, 2, \ldots, N$, and $\tau = [\tau_{ij}]_{N\times N}$ is symmetric matrix with $\mathrm{Im}(\tau) > 0$ and $\langle\cdot,\cdot\rangle$ is usual inner product. The series (10.15) converges absolutely and uniformly.
The Riemann theta function is a kind of periodic function with two periodicities namely 1, τ, and these two periodicity together can be seen as

$$\vartheta(\eta + a\tau + b; \tau) = e^{-\pi ia^2\tau-2\pi ia\eta}\vartheta(\eta; \tau). \quad (10.16)$$

This function is also known to be entire function with two quasi-periods [168].

Construction of one-periodic solution

For one-periodic solution, we take $N = 1$ in (10.15), so we have,

$$f = \vartheta(\eta_1; \tau) = \sum_{n_1=-\infty}^{\infty} e^{2\pi i n_1 \eta_1 + \pi i n_1^2 \tau}, \ \eta_1 = \alpha_1 x + \beta_1 y + \omega_1 t + \delta_1. \tag{10.17}$$

On substituting (10.17) into the bilinear equation (10.12), and using essential property (1.88) of D-operator along with the procedure described in [170], we obtain

$$\begin{aligned} F(D_x, D_y, D_t) f \cdot f = & \sum_{n_1=-\infty}^{\infty} \sum_{n_1'=-\infty}^{\infty} F(D_x, D_y, D_t) e^{2\pi i n_1 \eta_1 + \pi i n_1^2 \tau} \cdot e^{2\pi i n_1' \eta_1 + \pi i n_1'^2 \tau} \\ = & \sum_{n_1=-\infty}^{\infty} \sum_{n_1'=-\infty}^{\infty} F(2\pi i (n_1 - n_1')\alpha_1, 2\pi i (n_1 - n_1')\beta_1, \\ & 2\pi i (n_1 - n_1')\omega_1) \times e^{2\pi i (n_1 + n_1')\eta_1 + \pi i (n_1^2 + n_1'^2)\tau} \\ = & \sum_{n_1=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} F(2\pi i (2n_1 - m)\alpha_1, 2\pi i (2n_1 - m)\beta_1, \\ & 2\pi i (2n_1 - m)\omega_1) \times e^{2\pi i m \eta_1 + \pi i (n_1^2 + (n_1 - m)^2)\tau} \\ = & \sum_{m=-\infty}^{\infty} \tilde{F}(m) e^{2\pi i m \eta_1}, \end{aligned} \tag{10.18}$$

where $m = n_1 + n_1'$ and $\tilde{F}$ is defined as

$$\begin{aligned} \tilde{F}(m) = & \sum_{n_1=-\infty}^{\infty} F(2\pi i (2n_1 - m)\alpha_1, 2\pi i (2n_1 - m)\beta_1, 2\pi i (2n_1 - m)\omega_1) \\ & \times e^{\pi i \{n_1^2 + (n_1 - m)^2\}\tau} \\ = & \tilde{F}(m-2) e^{2\pi i (m-1)\tau}, \end{aligned} \tag{10.19}$$

where summation index has been shifted by unity. From equation (10.19), we observe that, if $\tilde{F}(0)$ and $\tilde{F}(1)$ are zero, then (10.17) constitute exact one-periodic wave solution for equation (10.1). Thus, we have following system of determining equations

$$\sum_{n_1=-\infty}^{\infty} F[4\pi i n_1 \alpha_1, 4\pi i n_1 \beta_1, 4\pi i n_1 \omega_1] e^{2\pi i n_1^2 \tau} = 0, \tag{10.20}$$

$$\sum_{n_1=-\infty}^{\infty} F[2\pi i (2n_1 - 1)\alpha_1, 2\pi i (2n_1 - 1)\beta_1, 2\pi i (2n_1 - 1)\omega_1] e^{\pi i \{n_1^2 + (n_1 - 1)^2\}\tau} = 0. \tag{10.21}$$

On using expression for $F(D_x, D_y, D_t)$ given at (10.12), we obtain

$$\begin{aligned} & b_1 + \beta_1 a_{11} + c a_{12} = 0, \\ & b_2 + \beta_1 a_{21} + c a_{22} = 0, \end{aligned} \tag{10.22}$$

where new quantities have been introduced, such as following

$$
\begin{aligned}
b_1 &= \sum_{n_1=-\infty}^{\infty} [-\pi^6\alpha_1^6(4n_1)^6 + 5\pi^4\alpha_1^3\omega_1(4n_1)^4 + 5\pi^2\omega_1^2(4n_1)^2]e^{2\pi i n_1^2\tau},\\
a_{11} &= -\sum_{n_1=-\infty}^{\infty} \pi^2\alpha_1(4n_1)^2 e^{2\pi i n_1^2\tau},\ a_{12} = \sum_{n_1=-\infty}^{\infty} e^{2\pi i n_1^2\tau},\\
b_2 &= \sum_{n_1=-\infty}^{\infty} [-\pi^6\alpha_1^6(4n_1-2)^6 + 5\pi^4\alpha_1^3\omega_1(4n_1-2)^4 + 5\pi^2\omega_1^2(4n_1-2)^2]\\
&\quad \times e^{\pi i\{n_1^2+(n_1-1)^2\}\tau},\\
a_{21} &= -\sum_{n_1=-\infty}^{\infty} \pi^2\alpha_1(4n_1-2)^2 e^{\pi i\{n_1^2+(n_1-1)^2\}\tau},\ a_{22} = \sum_{n_1=\infty}^{\infty} e^{\pi i\{n_1^2+(n_1-1)^2\}\tau}.
\end{aligned}
$$

On solving system (10.22) for β_1 and constant of integration c, we find that

$$
\beta_1 = \frac{b_2a_{12} - b_1a_{22}}{a_{11}a_{22} - a_{12}a_{21}},\ c = \frac{b_1a_{21} - b_2a_{11}}{a_{11}a_{22} - a_{12}a_{21}}. \tag{10.23}
$$

Denoting $e^{\pi i\tau} = \wp$, and using expression (10.23) for β_1 in (10.17), we obtain

$$
f = \vartheta(\eta_1;\tau) = 1 + (e^{2\pi i\eta_1} + e^{-2\pi i\eta_1})\wp + (e^{4\pi i\eta_1} + e^{-4\pi i\eta_1})\wp^4 + \ldots, \tag{10.24}
$$

for $\eta_1 = \alpha_1 x + (\frac{b_2a_{12}-b_1a_{22}}{a_{11}a_{22}-a_{12}a_{21}}) \cdot y + \omega_1 t + \delta_1$, where $\alpha_1, \omega_1, \delta_1$ and τ are arbitrary parameters. On substituting expression (10.24) into (10.11), we can obtain exact one-periodic wave solution for equation (10.1) as follows:

$$
u = u_0 + (\log f)_x. \tag{10.25}
$$

Remark 10.3.1. Here the role of constant of integration is evidently clear, which was dropped for traveling wave solutions in the previous section. However, if this constant of integration is set to zero, then equations (10.22) will not be compatible anymore. Moreover, elliptic functions do not satisfy equation with zero constant of integration.

Limiting behaviour of one-periodic solution

In this section, we shall prove that by appropriate limiting procedure, that traveling wave solution (10.6) can be obtained from one-periodic wave solution. For this purpose, we take small amplitude limit $\wp = e^{\pi i\tau} \to 0$ or Im $\tau \to \infty$. We have following expansions in powers of $\wp$

$$
\begin{aligned}
b_1 =& 2\left(-4096\,\pi^6{\alpha_1}^6 + 1280\,\pi^4{\alpha_1}^3\omega_1 + 80\,\pi^2{\omega_1}^2\right)\wp^2\\
&+ 2\left(-262144\,\pi^6{\alpha_1}^6 + 20480\,\pi^4{\alpha_1}^3\omega_1 + 320\,\pi^2{\omega_1}^2\right)\wp^8 + \ldots,\\
b_2 =& 2\left(-64\,\pi^6{\alpha_1}^6 + 80\,\pi^4{\alpha_1}^3\omega_1 + 20\,\pi^2{\omega_1}^2\right)\wp\\
&+ 2\left(-46656\,\pi^6{\alpha_1}^6 + 6480\,\pi^4{\alpha_1}^3\omega_1 + 180\,\pi^2{\omega_1}^2\right)\wp^5 + \ldots,\\
a_{11} =& -32\,\pi^2\alpha_1\wp^2 - 128\,\pi^2\alpha_1\wp^8 - 288\,\pi^2\alpha_1\wp^{18} + \ldots,\\
a_{12} =& 1 + 2\,\wp^2 + 2\,\wp^8 + 2\,\wp^{16} + \ldots,\\
a_{21} =& -8\,\pi^2\alpha_1\wp - 72\,\pi^2\alpha_1\wp^5 - 200\,\pi^2\alpha_1\wp^{13} + \ldots,\\
a_{22} =& 2\,\wp + 2\,\wp^5 + 2\,\wp^{13} + \ldots.
\end{aligned} \tag{10.26}
$$

On substituting expansions (10.26) into (10.23), we obtain

$$\begin{aligned}\beta_1 =& \frac{1}{8\,\pi^2\alpha_1\wp\,(1-6\,\wp^2+9\,\wp^4+10\,\wp^6-30\,\wp^8+\dots)}\Big[8\,\pi^2\wp\{20\,\pi^2{\alpha_1}^3\omega_1\\ &-16\,\pi^4{\alpha_1}^6+5\,{\omega_1}^2+\left(-30\,{\omega_1}^2-600\,\pi^2{\alpha_1}^3\omega_1+2016\,\pi^4{\alpha_1}^6\right)\wp^2+\dots\}\Big]\\ =&20\pi^2\alpha_1^2\omega_1-16\pi^4\alpha_1^5+\frac{5\omega_1^2}{\alpha_1},\ \wp\to 0, \end{aligned} \tag{10.27}$$

$$\begin{aligned}c =& \frac{1}{1-16\,\wp^2+9\,\wp^4+10\,\wp^6-30\,\wp^8+\dots}\Big[-32\,\pi^2\wp^2\{60\,{\alpha_1}^3\pi^2\omega_1\\ &-240\,{\alpha_1}^6\pi^4+\left(-900\,{\alpha_1}^3\pi^2\omega_1+9360\,{\alpha_1}^6\pi^4\right)\wp^4+\dots\}\Big]\\ =&0,\ \wp\to 0. \end{aligned} \tag{10.28}$$

In order to derive traveling wave solution from one-periodic wave solution, we make following conversions:

$$\begin{aligned}&\tilde{\alpha_1} = 2\pi i\alpha_1,\ \tilde{\omega_1} = 2\pi i\omega_1,\\ &\tilde{\beta_1} = 2\pi i\beta_1 = 5\frac{\tilde{\omega_1^2}}{\tilde{\alpha_1}} - \tilde{\alpha_1^5} - 5\tilde{\alpha_1^2}\tilde{\omega_1},\\ &\tilde{\delta_1} = 2\pi i\left(\delta_1+\frac{\tau}{2}\right).\end{aligned} \tag{10.29}$$

On using transformations (10.29), and after detailed calculations, one-periodic wave solution (10.24) is obtained as follows:

$$\begin{aligned}f =& \vartheta(\tilde{\eta_1}) = 1+e^{\tilde{\eta_1}}+(e^{-\tilde{\eta}_1}+e^{2\tilde{\eta_1}})\wp^2+(e^{-\tilde{2}\eta_1}+e^{3\tilde{\eta_1}})\wp^6+\dots\\ =&1+e^{\tilde{\eta_1}},\ \wp\to 0.\end{aligned} \tag{10.30}$$

This limiting solution is nothing but traveling wave solution (10.4) for $\tilde{\eta_1} = 2\pi i\eta_1 + \pi i\tau$.

Construction of two-periodic solution

The two-periodic wave solutions can be obtained by setting $N = 2$ in (10.15), this yields,

$$f = \vartheta(\eta;\tau) = \sum_{n=-\infty}^{+\infty} e^{2\pi i\langle\eta,n\rangle+\pi i\langle\tau n,n\rangle},\ \eta_i = \alpha_i x+\beta_i y+\omega_i t+\delta_i, i = 1,2, \tag{10.31}$$

where $n = (n_1,n_2)^T$, $\eta = (\eta_1,\eta_2)$ and $\tau = \begin{pmatrix}\tau_{11} & \tau_{12}\\ \tau_{21} & \tau_{22}\end{pmatrix}$ for $\mathrm{Im}(\tau_{11},\tau_{22}) > 0$, and $\tau_{ij}(i\neq j)$ are effects representing interaction among periodic waves such that $\tau_{12} = \tau_{21}$.
On substituting (10.31) into (10.14), we obtain

$$\begin{aligned}F(D_x,D_y,D_t)f\cdot f =& \sum_{n,n'=-\infty}^{\infty} F(D_x,D_y,D_t)e^{2\pi i\langle\eta,n\rangle+\pi i\langle\tau n,n\rangle}\cdot e^{2\pi i\langle\eta,n'\rangle+\pi i\langle\tau n',n'\rangle}\\ =& \sum_{n,n'=-\infty}^{\infty} F(2\pi i\langle\alpha,n-n'\rangle,2\pi i\langle\beta,n-n'\rangle,2\pi i\langle\omega,n-n'\rangle)\end{aligned}$$

$$
\begin{aligned}
&\times e^{2\pi i\langle\eta,n+n'\rangle+\pi i(\langle\tau n,n\rangle+\langle\tau n',n'\rangle)}\\
&=\sum_{n,m=-\infty}^{\infty} F(2\pi i\langle\alpha,2n-m\rangle,2\pi i\langle\beta,2n-m\rangle,2\pi i\langle\omega,2n-m\rangle)\\
&\quad\times e^{2\pi i\langle\eta,m\rangle+\pi i(\langle\tau n,n\rangle+\langle\tau(m-n),(m-n)\rangle)}\\
&=\sum_{m_1,m_2=-\infty}^{\infty}\tilde{F}(m_1,m_2)e^{2\pi i\langle\eta,m\rangle}. \qquad (10.32)
\end{aligned}
$$

The relation (10.32) is corresponding to relation (10.18) for $N=1$, where we have the following

$$
\begin{aligned}
\tilde{F}(m_1,m_2)&=\sum_{n=-\infty}^{\infty} F(2\pi i\langle\alpha,2n-m\rangle,2\pi i\langle\beta,2n-m\rangle,2\pi i\langle\omega,2n-m\rangle)\\
&\quad\times e^{\pi i(\langle\tau n,n\rangle+\langle\tau(m-n),(m-n)\rangle)}\\
&=\sum_{n_1,n_2=-\infty}^{\infty} F\Bigg(2\pi i\sum_{j=1}^{2}(2n_j-m_j)\alpha_j,2\pi i\sum_{j=1}^{2}(2n_j-m_j)\beta_j,\\
&\quad 2\pi i\sum_{j=1}^{2}(2n_j-m_j)\omega_j\Bigg)\times e^{\pi i\left(\sum_{j,k=1}^{2}\eta_j\tau_{jk}n_k+(m_j-n_j)\tau_{jk}(m_k-n_k)\right)}.
\end{aligned}
\qquad (10.33)
$$

Shifting the summation index in (10.33) by unity, we obtain

$$
\tilde{F}(m_1,m_2)=\begin{cases}\tilde{F}(m_1-2,m_2)e^{2\pi i(m_1-1)\tau_{11}+2\pi i m_2\tau_{12}}\\ \tilde{F}(m_1,m_2-2)e^{2\pi i m_1\tau_{21}+2\pi i(m_2-1)\tau_{22}}.\end{cases} \qquad (10.34)
$$

The relation (10.34) is corresponding to relation (10.19) for one-periodic wave solution, and hence it is evidently clear that (10.31) will satisfy the bilinear equation for (10.12) if

$$
\tilde{F}(0,0)=0,\quad \tilde{F}(1,0)=0,\quad \tilde{F}(0,1)=0,\quad \tilde{F}(1,1)=0. \qquad (10.35)
$$

The constraints (10.35) constitute system of four linear equations in four parameters β_1,β_2,c and u_0, and other parameters like $\alpha_1,\alpha_2,\omega_1,\omega_2,\tau_{11},\tau_{12}$ and τ_{22} can be freely chosen. Using formula (10.33), the system of equations (10.35) can be written as follow:

$$
\begin{aligned}
&a_{11}\beta_1+a_{12}\beta_2+a_{13}c+a_{14}u_0+b_1=0,\\
&a_{21}\beta_1+a_{22}\beta_2+a_{23}c+a_{24}u_0+b_2=0,\\
&a_{31}\beta_1+a_{32}\beta_2+a_{33}c+a_{34}u_0+b_3=0,\\
&a_{41}\beta_1+a_{42}\beta_2+a_{43}c+a_{44}u_0+b_4=0,
\end{aligned}
\qquad (10.36)
$$

where coefficients and constants in system (10.36) are expansions in $\wp_{11}=e^{\pi i\tau_{11}},\wp_{12}=e^{\pi i\tau_{12}},\wp_{22}=e^{\pi i\tau_{22}}$ (see Appendix A.1). Solving system of equations (10.36) for $\beta_1,\beta_2,c,\ u_0$, and from (10.31), we obtain

$$
\begin{aligned}
f=\vartheta(\eta_1,\eta_2,\tau)=&1+(e^{2\pi i\eta_1}+e^{-2\pi i\eta_1})\wp_{11}+(e^{2\pi i\eta_2}+e^{-2\pi i\eta_2})\wp_{22}\\
&+(e^{2\pi i(\eta_1+\eta_2)}+e^{-2\pi i(\eta_1+\eta_2)})\wp_{11}\wp_{12}^2\wp_{22}+\ldots
\end{aligned}
\qquad (10.37)
$$

Finally, using expression (10.37) in (10.13), we obtain exact two-periodic wave solution for equation (10.1)

$$u = u_0 t + (\log f)_x. \tag{10.38}$$

Limiting behaviour of two-periodic solution

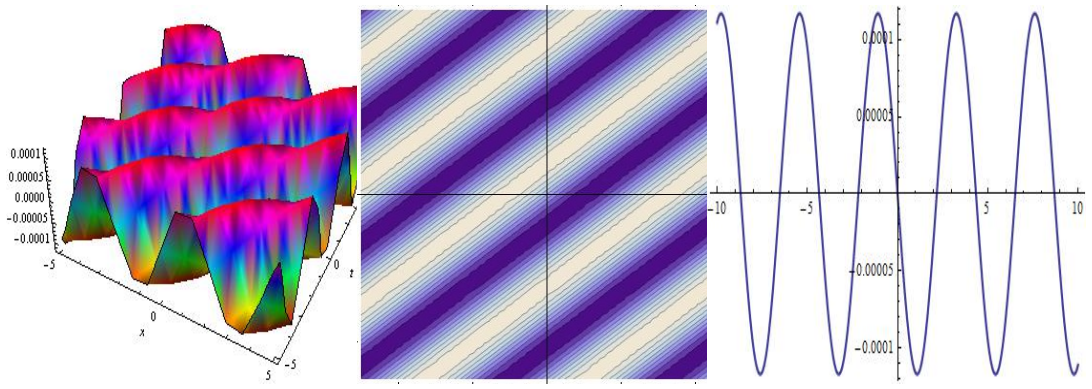
In order to prove limiting behaviour of two-periodic wave solution, we make following transformations:

$$\begin{aligned}
\tilde{\alpha_i} =& 2\pi i\alpha_i,\ \tilde{\omega_i} = 2\pi i\omega_i,\\
\tilde{\beta_i} =& 2\pi i\beta_i = 5\frac{\tilde{\omega_i^2}}{\tilde{\alpha_i}} - \tilde{\alpha_i^5} - 5\tilde{\alpha_i^2}\tilde{\omega_i},\\
\tilde{\delta_i} =& 2\pi i(\delta_1 + \frac{\tau_{ii}}{2}),\ \tau_{12} = \frac{A_{12}}{2\pi i},\ i = 1, 2.
\end{aligned} \tag{10.39}$$

Using above transformations and after detailed calculations, the two-periodic wave solution (10.37) transformed to

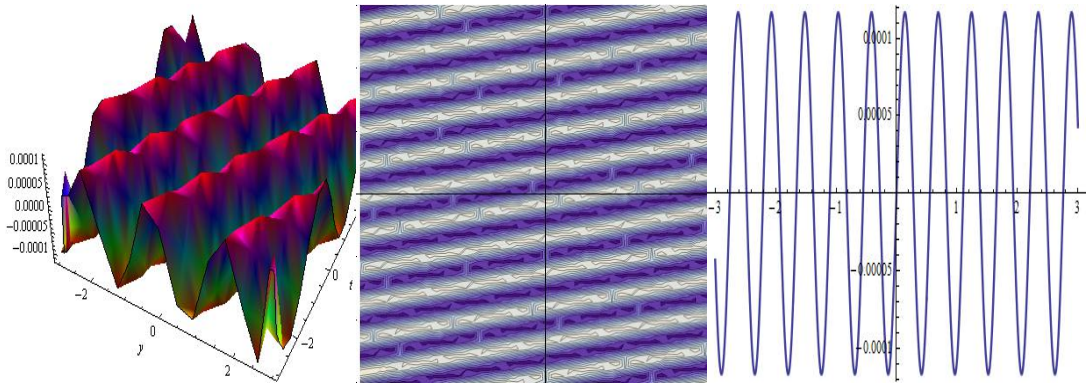
$$\begin{aligned}
f = \vartheta(\tilde{\eta_1}, \tilde{\eta_2}) =& 1 + e^{\tilde{\eta_1}} + e^{\tilde{\eta_2}} + e^{\tilde{\eta_1}+\tilde{\eta_2}+A_{12}} + \wp_{11}^2 e^{-\tilde{\eta_1}}\\
&+ \wp_{22}^2 e^{-\tilde{\eta_2}} + \wp_{11}^2\wp_{22}^2\wp_{12}^2 e^{-\tilde{\eta_1}-\tilde{\eta_2}} + \dots\\
=& 1 + e^{\tilde{\eta_1}} + e^{\tilde{\eta_2}} + e^{\tilde{\eta_1}+\tilde{\eta_2}+A_{12}},\ \wp_{11}, \wp_{12}, \wp_{22} \to 0.
\end{aligned} \tag{10.40}$$

which is same as non-stationary solution (10.7).



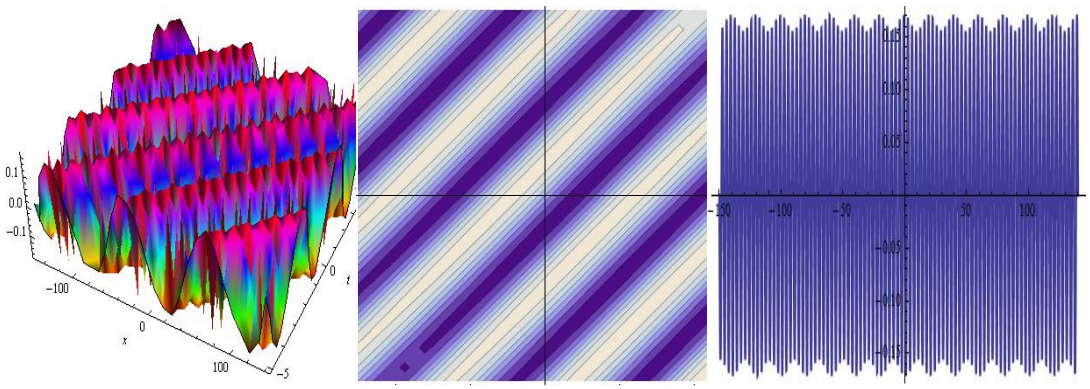
(a) $y = 1$ **(b)** **(c)**

Figure 10.1: One-Periodic wave solution of equation (10.1) via expression (10.11) for parametric choices $\alpha_1 = 0.23, \omega_1 = -0.3, \delta_1 = 0, \tau_1 = \imath$ and $u_0 = 0$. From Figure 10.1–Figure 10.2, it is evidently clear that every one-periodic wave solution is one-dimensional and can be viewed as a superposition of overlapping solitary waves. Figure 10.1a represents a perspective view of $\Re(u)$ in the $x - t$ plane. Figure 10.1b represents the corresponding contour plot. Figure 10.1c represents the wave propagation along x-direction



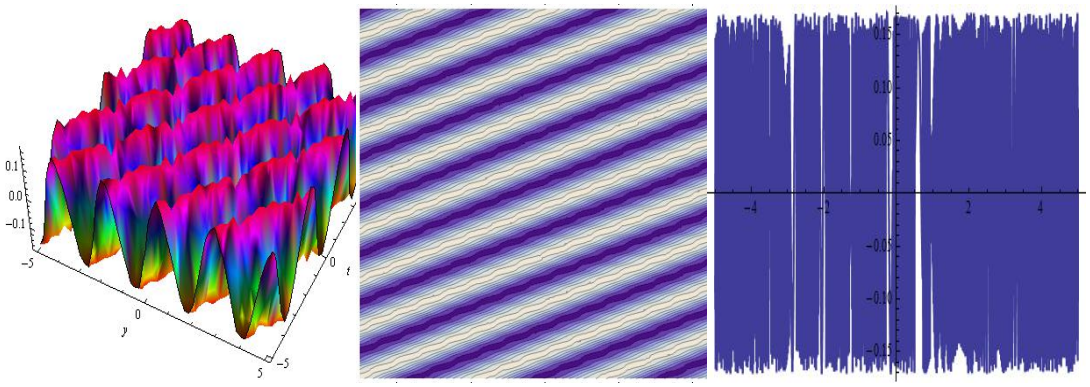
(a) $x = 1$ **(b)** **(c)**

Figure 10.2: Figure 10.2a represents perspective view of $\Re(u)$ in the $y - t$ plane. Figure 10.2b represents the corresponding contour plot. Figure 10.2c represents the wave propagation along the y-direction.



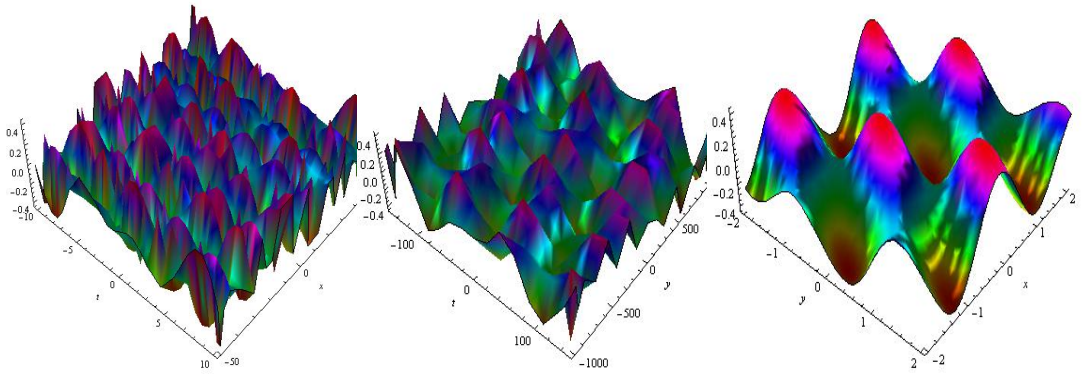
(a) $y = 1$ **(b)** **(c)**

Figure 10.3: A degenerate two-periodic wave solution of equation (10.1) via expression (10.13) for parametric choices $\alpha_1 = 0.3, \alpha_2 = 0.35, \omega_1 = -0.3, \omega_2 = -0.25, \delta_1 = 0, \delta_2 = 0, \tau_{11} = \imath, \tau_{12} = 2\imath, \tau_{22} = 0.5\imath$. Figure 10.3a represents perspective view of $\Re(u)$ in the $x - t$ plane. Figure 10.3b represents the corresponding contour plot. Figure 10.3c represents the wave propagation along x-direction.



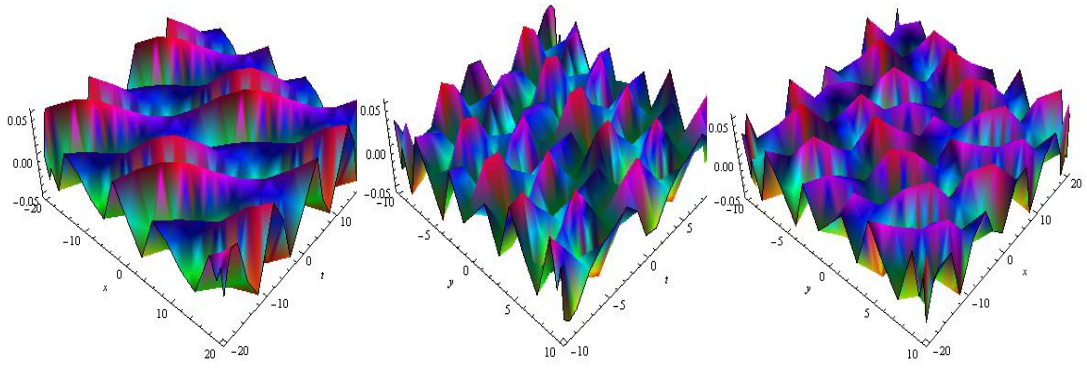
(a) $x = 1$ **(b)** **(c)**

Figure 10.4: Figure 10.4a represents the perspective view of $\Re(u)$ on $y - t$ plane. Figure 10.4b represents the corresponding contour plot. Figure 10.4c represents the wave propagation along the y-direction.



(a) $y = 1$ **(b)** $x = 1$ **(c)** $t = 1$

Figure 10.5: For parametric choices $\alpha_1 = 0.45, \alpha_2 = 0.4, \omega_1 = -0.1, \omega_2 = -0.15, \tau_{11} = \imath, \tau_{12} = \imath$ and $\tau_{22} = 0.5\imath$, the Figure 10.5a, Figure 10.5b and Figure 10.5c give two-periodic wave solution on $x - t$, $y - t$ and $x - y$ planes respectively.



(a) $y = 1$ **(b)** $x = 1$ **(c)** $t = 1$

Figure 10.6: For parametric choices $\alpha_1 = 0.6, \alpha_2 = 0.5, \omega_1 = -0.2, \omega_2 = -0.1, \tau_{11} = \imath, \tau_{12} = \imath$ and $\tau_{22} = 0.5\imath$, the Figure 10.6a, Figure 10.6b and Figure 10.6c give two-periodic wave solution on $x - t$, $y - t$ and $x - y$ planes respectively

11

Conclusion and outlook

11.1 Lie symmetries

- The classical Lie symmetry analysis has been used to analyze three fields Kaup-Boussinesq system (2.3). Based on Killing's form derived in Lemma 2.2.2, the complete classification of Lie algebra (2.8) is obtained in Theorem 2.2.3. Similarity reductions and invariant solutions using power series method are also presented.

- The Lie symmetry analysis is performed in a symbolic manner to investigate the symmetries of Klein-Gordon-Zakharov equations. The corresponding Lie algebra is classified into one and two-dimensional optimal subalgebras. Besides Killing's form, some more invariants are introduced at (3.14) and (3.15) to verify optimality of derived systems.

- The classical Lie symmetry analysis of DJKM equation (4.1) is performed for the first time, and resulting symmetries depends on four arbitrary functions of t. The symmetries are then classified to give out three independent similarity reductions. We able to construct an infinite number of non-trivial local conservation laws using two different techniques and this conclude the integrability of equation (4.1).

- The variable coefficient coupled KdV system has been investigated using a recently introduced compatibility method. We have successfully applied the procedure of compatibility method with the aid of the Maple program without which bulky algebraic equations could have been unmanageable. Bypassing the invariance criterion of Lie group method, six-dimensional symmetry algebra is constructed. From symmetry operator (5.9) for KdV system (5.1), it is evident that we able to recover more general symmetries than reported in earlier work [201]. Pairwise disjointness of sub-algebras through adjoint action has given an optimal system for symmetry algebra and consequently, several inequivalent reductions and their traveling wave solutions are reported.

All the remaining equations beyond chapter 6 are well-known equations in mathematical physics which are already investigated for Lie symmetry analysis. For example, the equations in chapter 6, 7, 8 have been considered for Lie symmetry

analysis in the Refs. [43, 108, 112, 192, 202–204]. In chapter 9, the equations considered in examples 9.2.1-3 are different variants of Kadomstev-Petviashvili equation, and equation in example 9.2.4 is shallow water wave equations. These equations are comprehensively studied for Lie symmetries in the literature. Same is the case with the equation in chapter 10.

11.2 Group classification

- The Lie algebras of vector fields of some of the PDEs in mathematical physics are treated in an abstract manner. We have learned that the Lie algebras of dimension $d \leq 4$ have been fully classified under the group of inner automorphism, so it is of no need to re-classify the Lie algebras which have already been classified. In the case of variant Boussinesq system, we have seen that after removal of central elements, the resulting 3-dimensional subalgebra is isomorphic sub-algebra which has already been classified and subsequently full classification is achieved after inclusion of central elements by using Goursat method. We have noticed that the classification of Lie algebra for variant Boussinesq system is much finer than classification reported earlier. Some other examples of Lie algebras are also considered for explicit isomorphism. Besides this, we have noticed that the 1-dimensional subalgebras for Hirota-Satusma coupled KdV system, Benjamin-Bona-Mahony equation, and turbulence model are not complete. Because in earlier work, the authors have missed some of the sub-algebras during classification. We have completed these classifications by adding new elements to the optimal list which have been missed earlier.

- Essentially any model in physical phenomena admitting Lie algebra of dimension ≤ 4 is isomorphic to Lie algebra given in the list available in Ref. [182]. So, on removing the center of Lie algebra (7.2) we have observed that the resulting subalgebra L_4 is isomorphic to subalgebra $A^0_{4,9}$ and explicit isomorphism is also given. Since subalgebra $A^0_{4,9}$ has already been classified, so it helped us to give a complete classification of Lie algebra (7.2) in a much easier way than reported earlier.

  In short, the importance of Lie algebra in abstract format is demonstrated, that is, the two Lie algebras of vector fields might look different but in term of their commutation relations or structure constants, they may be isomorphic and hence would possess the same classification.

11.3 Integrability

11.3.1 Painlevé analysis

- The package "wkptest" built for "Maple" is used to investigate Painlevé property of equations (3.1) and detailed manual calculations shows that the equations under study are not integrable. Further, the DJKM equation (4.1) is studied for integrability for the first time. The usual analysis based on WTC

framework is performed to check the Painlevé integrability of (4.1) and we have proved that the equation is integrable. Taking the advantage of invariant Painlevé analysis, the Pickering's algorithm is used to derive exact solution (4.21) for the equation (4.1). The reported solution contains two arbitrary functions which enrich the structure of the existing solutions of the equation under study.

Beside this, we have also investigated remaining equations for Painlevé property. In the case of variable coefficient KdV equations (5.1), the resonances at first and second branch are non-integers which imply the failure of Painlevé property. Similarly in chapter 6, due to the occurrence of variable coefficients in variant Boussinesq system, Hirota-Satsuma coupled KdV equations and Benjamin-Bona-Mahony equation the resonance could be non-integer and hence all these three equations fail to pass Painlevé property. In example 6.2.3, the compatibility condition does not hold at resonance $j = 2$, therefore the considered Navier-Stokes equation is not Painlevé integrable. The turbulence model in example 6.2.4 also fails to yield a leading order which again implies uncertain Painlevé property.
In chapter 9, the analysis shows that, for the equations (9.1), (9.6), (9.11) and (9.16), the compatibility condition does hold at resonance $j = 6$. Therefore, all these equations fail to pass Painlevé test and hence non-integrable in sense of Painlevé property. In chapter 10, the equation (10.1) has two branches. At first branch, the resonances are $j = -2, -1, 5, 6, 12$ and the compatibility conditions hold at every resonance point. At the second branch, the resonances at $j = -1, 1, 2, 3, 6, 10$, and in this case too, the compatibility conditions hold at every resonance. Therefore, the equation (10.1) is integrable in sense of Painlevé property.

11.3.2 Hirota's direct method

The Hirota's direct method is used to construct traveling wave and non-stationary solutions of the b-type Kadomtsev-Petviashvili equation, and it is shown that by writing equation (10.1) in bilinear form, multi-periodic wave solutions can be constructed in straightforward manner using multi-periodic(or quasi-periodic) Riemann theta function. Although, calculations are cumbersome but are manageable using symbolic computation tools such as "Maple". Moreover, through detailed analysis, it has also been established that these one and two-periodic wave solution asymptotically tends to traveling wave and non-stationary solutions respectively under small amplitude limit. In addition to this, by taking special parametric values, the Figure 10.1-Figure 10.6 are plotted. It has been shown that every one-periodic wave solution is one-dimensional, and every two-periodic wave solution is spatially periodic in two directions.

11.3.3 Bell polynomials

The Jimbo-Miwa equation (8.1) which is second member of KP hierarchy has been investigated. The Bell-polynomial approach has been employed to investigate inte-

grable properties of this equation. For equation (8.1), we have obtained its bilinear form (8.6), bilinear Bäcklund transformations (8.13), Lax system (8.15) and finally infinite conservation laws (8.26) with (8.25), (8.27), (8.28), (8.29) for equation (8.1) have been derived.

We have shown that the Bell-polynomial approach can be applied in a procedural manner to derive Bäcklund transformations from two field condition (8.7) without using complicated exchange formulae for Hirota D-operator. Subsequently, for Lax system (8.15) Cole-Hopf transformation (1.129) is used to linearise two-field condition (8.12). In addition to this, from Riccati and divergence-type equations (8.22)-(8.23) infinite conservation laws (8.26) have been derived in a natural way. Finally, as illustrated in Figure 8.1, Figure 8.2 and Figure 8.3, a rich class of exact solutions like kink, solitary, periodic wave solutions are also obtained for Jimbo-Miwa equation and we have verified every solution with aid of Maple.

11.3.4 Conservation laws

- **New conservation theorem**. The construction of an infinite set of conservation laws based on the theory of a new conservation theorem [117] is demonstrated. The presented work emphasized the relevance of new conservation theorem by Ibragimov for the construction of conservation laws from Lie symmetries without the formulation of classical Lagrangian. We have successfully applied Ibragimov's approach to find conservation laws for Kaup-Boussinesq equations (2.3), Klein-Gordon-Zakharov(KGZ) equations in (3+1)-dimension (3.1), Date-Jimbo-Kashiwara-Miwa equation (4.1) and drift-flux model (7.1).

  Despite the huge success of new conservation theorem of Ibragimov, the recent comments from Stephen C. Anco [11] confirm the incompleteness of the theorem. In particular, the formulation proposed by Ibragimov can generate trivial conservation laws and does not always yield all non-trivial conservation laws. Under what conditions the Ibragimov's method can give trivial conservation deserve attention and we left this topic for future study.

- **The direct method**. Along with the new conservation theorem, we have also studied the direct method of Bluman and Anco for the construction of nontrivial local conservation laws. We have noticed that the procedure of direct method always incorporates original PDEs while determining multipliers of any order. That is, unlike new conservation theorem, the conservation laws obtained using direct method are always non-trivial. Five non-trivial local conservation laws are reported for DJKM equation (4.1) in Chapter 4.

- **Bäcklund transformations**. The third approach is the construction of the infinite set of conservation laws using Bäcklund transformations. The technique is illustrated by constructing conservation laws for Jimbo-Miwa equation (8.1) in Chapter 8. The procedure illustrates the importance of Bäcklund transformations in the construction of an infinite set of conservation laws.

  The conserved currents obtained in this work can be useful for the better understanding of some physical phenomena modeled by the underlying equations.

11.3.5 Exact solutions

- Based on the Bell polynomials approach and Hirota's bilinear method, we have used novel test function to construct exact solutions for well-known evolutionary equations in (3+1)-dimension. We have used Maple algebra package PDEBellII for constructing bilinear equations, this package greatly helped us to avoid cumbersome calculations for constructing bilinear equations. Moreover, for the construction of exact solutions, we have extensively used Maple programming and basic code for algebraic calculations is given in form of the appendix which we believe may be useful for researchers in the mathematical physics. Although the solutions obtained using novel test function are quite straightforward but novelty in the construction of such exact solutions might pave the way for discussion among the community of researchers for further developments.
- The Hirota's bilinear form of Jimbo-Miwa equation (8.1) is used to construct a variety of new solutions, and physical structures of solutions are also investigated. Several other methods for finding exact solutions are also illustrated in Chapter 5.

11.4 Future scope and study

In this thesis, at the center stage, was the study of integrability of some PDEs. To accomplish this task, several topics like; Lie symmetries, Painlevé analysis, Hirota's bilinear method, conservation laws, Bäcklund transformations, and Lax pairs have been explored. It has been observed that the following areas can also be explored.

In chapter 5, more general symmetries have been obtained using the compatibility method. This method can also be tested on other equations for possible general symmetries. For some PDEs, the connection between symmetries and conservation laws can be explored. Like for example, using contact symmetries, new conservation laws can be constructed from known conservation laws. The recursion operator which provides a rich variety of symmetries can also be studied. Besides this, some PDEs can also be investigated for the nonlocal and adjoint symmetries. The Painlevé analysis has rarely been performed on variable coefficient PDEs, this area can be explored, and some new results may be obtained. In chapter 9, we have proposed a new test function, it can be used to obtain exact solutions for PDEs which have not been explored yet. In chapter 6, we have suggested that it is of no need to re-classify the Lie algebras which have already been classified. As an illustration, we have proved that the 4 and 5-dimensional Lie algebras of five important PDEs are isomorphic to Lie algebras which are already classified. We have also learned that Lie algebras up to dimension 9 have also been fully classified. Using isomorphism, the classifications of higher dimensional Lie algebra can be used to compare with existing classifications.

A

Appendices

A.1 Coefficients

$$a_{11} = \sum_{n=-\infty}^{\infty} (-16\,\pi^2 n_1 n_2 \alpha_2 - 16\,\pi^2 {n_1}^2 \alpha_1) \Im_1(n) \tag{A.1}$$

$$a_{12} = \sum_{n=-\infty}^{\infty} (-16\,\pi^2 n_2 n_1 \alpha_1 - 16\,\pi^2 {n_2}^2 \alpha_2) \Im_1(n), a_{13} = \sum_{n=-\infty}^{\infty} \Im_1(n) \tag{A.2}$$

$$a_{14} = \sum_{n=-\infty}^{\infty} (-480\,\pi^2 {n_1}^2 {\alpha_1}^2 - 960\,\pi^2 n_1 \alpha_1 n_2 \alpha_2 - 480\,\pi^2 {n_2}^2 {\alpha_2}^2) \Im_1(n) \tag{A.3}$$

$$\begin{aligned} b_1 = \sum_{n=-\infty}^{\infty} (&3840\,\pi^4 {n_1}^2 \alpha_1 {n_2}^2 {\alpha_2}^2 \omega_1 + 3840\,\pi^4 n_1 \alpha_1 {n_2}^3 {\alpha_2}^2 \omega_2 - 4096\,\pi^6 {n_1}^6 {\alpha_1}^6 \\ &- 24576\,\pi^6 {n_1}^5 {\alpha_1}^5 n_2 \alpha_2 + \cdots + 1280\,\pi^4 {n_2}^4 {\alpha_2}^3 \omega_2 + 80\,\pi^2 {n_2}^2 {\omega_2}^2) \Im_1(n) \end{aligned} \tag{A.4}$$

$$a_{21} = \sum_{n=-\infty}^{\infty} (16\,\pi^2 \alpha_1 n_1 - 16\,\pi^2 \alpha_1 {n_1}^2 - 16\,\pi^2 n_2 \alpha_2 n_1 + 8\,\pi^2 n_2 \alpha_2 - 4\,\pi^2 \alpha_1) \Im_2(n) \tag{A.5}$$

$$a_{22} = \sum_{n=-\infty}^{\infty} (8\,\pi^2 \alpha_1 n_2 - 16\,\pi^2 \alpha_1 n_1 n_2 - 16\,\pi^2 {n_2}^2 \alpha_2) \Im_2(n), a_{23} = \sum_{n=-\infty}^{\infty} \Im_2(n) \tag{A.6}$$

$$\begin{aligned} a_{24} = \sum_{n=-\infty}^{\infty} (&-480\,\pi^2 {\alpha_1}^2 {n_1}^2 + 480\,\pi^2 {\alpha_1}^2 n_1 - 960\,\pi^2 \alpha_1 n_1 n_2 \alpha_2 \\ &- 120\,\pi^2 {\alpha_1}^2 + 480\,\pi^2 \alpha_1 n_2 \alpha_2 - 480\,\pi^2 {n_2}^2 {\alpha_2}^2) \Im_2(n) \end{aligned} \tag{A.7}$$

$$\begin{aligned} b_2 = \sum_{n=-\infty}^{\infty} (&3840\,\pi^4 {\alpha_1}^2 {n_1}^3 n_2 \alpha_2 \omega_1 - 5760\,\pi^4 {\alpha_1}^2 {n_1}^2 n_2 \alpha_2 \omega_1 \\ &+ 3840\,\pi^4 {\alpha_1}^2 {n_1}^2 {n_2}^2 \alpha_2 \omega_2 + \ldots \\ &+ 160\,\pi^2 \omega_1 n_1 n_2 \omega_2 + 1280\,\pi^4 {\alpha_1}^3 {n_1}^3 n_2 \omega_2) \Im_2(n) \end{aligned} \tag{A.8}$$

$$a_{31} = \sum_{n=-\infty}^{\infty} (-16\,\pi^2 n_2 \alpha_2 n_1 - 16\,\pi^2 \alpha_1 {n_1}^2 + 8\,\pi^2 \alpha_2 n_1) \Im_3(n) \tag{A.9}$$

$$a_{32} = \sum_{n=-\infty}^{\infty} (8\,\pi^2 \alpha_1 n_1 - 16\,\pi^2 {n_2}^2 \alpha_2 + 16\,\pi^2 n_2 \alpha_2 - 16\,\pi^2 \alpha_1 n_1 n_2 - 4\,\pi^2 \alpha_2) \Im_3(n) \tag{A.10}$$

$$a_{33} = \sum_{n=-\infty}^{\infty} \Im_3(n) \tag{A.11}$$

$$a_{34} = \sum_{n=-\infty}^{\infty} (-480\,\pi^2 {n_1}^2 {\alpha_1}^2 - 960\,\pi^2 n_1 \alpha_1 \alpha_2 n_2 + 480\,\pi^2 n_1 \alpha_1 \alpha_2 - 480\,\pi^2 {\alpha_2}^2 {n_2}^2 + 480\,\pi^2 {\alpha_2}^2 n_2 - 120\,\pi^2 {\alpha_2}^2) \Im_3(n) \tag{A.12}$$

$$b_3 = \sum_{n=-\infty}^{\infty} (3840\,\pi^4 {\alpha_1}^2 {n_1}^3 n_2 \alpha_2 \omega_1 + 3840\,\pi^4 {\alpha_1}^2 {n_1}^2 {n_2}^2 \alpha_2 \omega_2 - 3840\,\pi^4 {n_1}^2 {\alpha_1}^2 \alpha_2 n_2 \omega_2 + 3840\,\pi^4 \alpha_1 {n_1}^2 {n_2}^2 {\alpha_2}^2 \omega_1 + \ldots + 1280\,\pi^4 {\alpha_1}^3 {n_1}^3 n_2 \omega_2) \Im_3(n) \tag{A.13}$$

$$a_{41} = \sum_{n=-\infty}^{\infty} (8\,\pi^2 \alpha_2 n_1 + 16\,\pi^2 \alpha_1 n_1 - 16\,\pi^2 \alpha_1 {n_1}^2 - 16\,\pi^2 \alpha_2 n_2 n_1 - 4\,\pi^2 \alpha_2 + 8\,\pi^2 \alpha_2 n_2 - 4\,\pi^2 \alpha_1) \Im_4(n) \tag{A.14}$$

$$a_{42} = \sum_{n=-\infty}^{\infty} (16\,\pi^2 \alpha_2 n_2 - 16\,\pi^2 \alpha_1 n_1 n_2 - 16\,\pi^2 \alpha_2 {n_2}^2 - 4\,\pi^2 \alpha_1 - 4\,\pi^2 \alpha_2 + 8\,\pi^2 \alpha_1 n_2 + 8\,\pi^2 \alpha_1 n_1) \Im_4(n) \tag{A.15}$$

$$a_{43} = \sum_{n=-\infty}^{\infty} \Im_4(n) \tag{A.16}$$

$$a_{44} = \sum_{n=-\infty}^{\infty} (480\,\pi^2 {\alpha_2}^2 n_2 - 240\,\pi^2 \alpha_1 \alpha_2 - 480\,\pi^2 {\alpha_2}^2 {n_2}^2 - 960\,\pi^2 \alpha_1 n_1 \alpha_2 n_2 + 480\,\pi^2 {\alpha_1}^2 n_1 - 120\,\pi^2 {\alpha_1}^2 - 120\,\pi^2 {\alpha_2}^2 + 480\,\pi^2 \alpha_1 \alpha_2 n_2 + 480\,\pi^2 \alpha_1 n_1 \alpha_2 - 480\,\pi^2 {\alpha_1}^2 {n_1}^2) \Im_4(n) \tag{A.17}$$

$$b_4 = \sum_{n=-\infty}^{\infty} (-5760\,\pi^4 {\alpha_1}^2 {n_1}^2 \alpha_2 n_2 \omega_1 + 3840\,\pi^4 {\alpha_1}^2 {n_1}^2 \alpha_2 {n_2}^2 \omega_2 - 3840\,\pi^4 {\alpha_1}^2 {n_1}^2 \alpha_2 n_2 \omega_2 + \ldots - 24576\,\pi^6 {\alpha_1}^5 {n_1}^5 \alpha_2 n_2 + 160\,\pi^2 \omega_1 n_1 \omega_2 n_2) \Im_4(n) \tag{A.18}$$

where

$$\Im_1(n) = \wp_{11}^{n_1^2} \wp_{12}^{2n_1 n_2} \wp_{22}^{n_2^2},\ \Im_2(n) = \wp_{11}^{n_1^2 + (n_1-1)^2} \wp_{12}^{2[n_1 n_2 + (n_1-1) n_2]} \wp_{22}^{2n_2^2}$$

$$\Im_3(n) = \wp_{11}^{2n_1^2} \wp_{12}^{2[n_1 n_2 + n_1 (n_2-1)]} \wp_{22}^{n_2^2 + (n_2-1)^2},$$

$$\Im_4(n) = \wp_{11}^{n_1^2 + (n_1-1)^2} \cdot \wp_{12}^{2[n_1 n_2 + (n_1-1)(n_2-1)]} \cdot \wp_{22}^{n_2^2 + (n_2-1)^2}.$$

A.2 Maple Code

Maple program for computing coefficients in algebraic expression
Step A.2.1. #Define Hirota derivative;

$$\begin{aligned}
\texttt{Hirota_Dyt} :=& \texttt{(f,g)} \to \texttt{f}\cdot\texttt{diff(g,y,t)} - \texttt{diff(f,y)}\cdot\texttt{diff(g,t)} \\
&- \texttt{diff(f,t)}\cdot\texttt{diff(g,y)} + \texttt{diff(f,y,t)}\cdot\texttt{g};\\
\texttt{Hirota_Dtz} :=& \texttt{(f,g)} \to \texttt{f}\cdot\texttt{diff(g,t,z)} - \texttt{diff(f,t)}\cdot\texttt{diff(g,z)} \\
&- \texttt{diff(f,z)}\cdot\texttt{diff(g,t)} + \texttt{diff(f,t,z)}\cdot\texttt{g};\\
\texttt{Hirota_Dty} :=& \texttt{(f,g)} \to \texttt{f}\cdot\texttt{diff(g,t,y)} - \texttt{diff(f,t)}\cdot\texttt{diff(g,y)} \\
&- \texttt{diff(f,y)}\cdot\texttt{diff(g,t)} + \texttt{diff(f,t,y)}\cdot\texttt{g};\\
\texttt{Hirota_Dxx} :=& \texttt{(f,g)} \to \texttt{f}\cdot\texttt{diff(g,x,x)} - 2\,\texttt{diff(f,x)}\cdot\texttt{diff(g,x)} \\
&+ \texttt{diff(f,x,x)}\cdot\texttt{g};\\
\texttt{Hirota_Dyy} :=& \texttt{(f,g)} \to \texttt{f}\cdot\texttt{diff(g,y,y)} - 2\,\texttt{diff(f,y)}\cdot\texttt{diff(g,y)} \\
&+ \texttt{diff(f,y,y)}\cdot\texttt{g};\\
\texttt{Hirota_Dzz} :=& \texttt{(f,g)} \to \texttt{f}\cdot\texttt{diff(g,z,z)} - 2\,\texttt{diff(f,z)}\cdot\texttt{diff(g,z)} \\
&+ \texttt{diff(f,z,z)}\cdot\texttt{g};\\
\texttt{Hirota_Dxxxy} :=& \texttt{(f,g)} \to \texttt{f}\cdot\texttt{diff(g,x\$3,y)} - 3\cdot\texttt{diff(f,x)} \\
&\cdot\texttt{diff(g,x\$2,y)} + 3\cdot\texttt{diff(f,x\$2)}\cdot\texttt{diff(g,x,y)} \\
&- \texttt{diff(f,x\$3)}\cdot\texttt{diff(g,y)} - \texttt{diff(f,y)}\cdot\texttt{diff(g,x\$3)} \\
&+ 3\cdot\texttt{diff(f,x,y)}\cdot\texttt{diff(g,x\$2)} - 3\cdot\texttt{diff(f,x\$2,y)} \\
&\cdot\texttt{diff(g,x)} + \texttt{diff(f,x\$3,y)}\cdot\texttt{g};\\
\texttt{Hirota_Dxxxx} :=& \texttt{(f,g)} \to \texttt{f}\cdot\texttt{diff(g,x\$4)} - 4\cdot\texttt{diff(f,x)}\cdot\texttt{diff(g,x\$3)} \\
&+ 6\cdot\texttt{diff(f,x\$2)}\cdot\texttt{diff(g,x\$2)} \\
&- 4\cdot\texttt{diff(f,x\$3)}\cdot\texttt{diff(g,x)} + \texttt{diff(f,x\$4)}\cdot\texttt{g}
\end{aligned}$$

Step A.2.2. #Define new test function;

$$\begin{aligned}
\texttt{f} :=& \,\texttt{e}^{-\texttt{a}_1\texttt{x}-\texttt{b}_1\texttt{y}-\texttt{c}_1\texttt{t}-\texttt{d}_1\texttt{z}} + \delta_1\cdot\texttt{tan}(\texttt{a}_2\texttt{x}+\texttt{b}_2\texttt{y}+\texttt{c}_2\texttt{t}+\texttt{d}_2\texttt{z}) \\
&+ \delta_2\cdot\texttt{tanh}(\texttt{a}_3\texttt{x}+\texttt{b}_3\texttt{y}+\texttt{c}_3\texttt{t}+\texttt{d}_3\texttt{z})\delta_3\cdot\texttt{e}^{\texttt{a}_1\texttt{x}+\texttt{b}_1\texttt{y}+\texttt{c}_1\texttt{t}+\texttt{d}_1\texttt{z}};
\end{aligned}$$

Step A.2.3. #Substitution of new test function into bilinear equation (9.5) (for example sake);

$$(\texttt{Hirota_Dzt} + 3\cdot\texttt{Hirota_Dxx} - \texttt{Hirota_Dxxxy})\texttt{f}\cdot\texttt{f}$$

#This command will generate large expression in coefficients of $e^{\pm\xi_1}$, $e^{\pm\xi_1}\tan(\xi_2), e^{\pm\xi_1}\tanh(\xi_3), \tan(\xi_2)\tanh(\xi_3)$
Step A.2.4. #Sorting terms in large expression in previous step;

$$\texttt{sort}(\texttt{expr}, \texttt{order} = \texttt{plex}[\texttt{f}_1, \texttt{f}_2, ...], \texttt{ascending});$$

#This command will sort all terms of expression in appropriate order
Step A.2.5. #Collecting coefficients from expression sorted in previous step

$$\texttt{coeffs}(\texttt{collect}(\texttt{expr}, [\texttt{f}_1, \texttt{f}_2, ...], \texttt{distributed}), [\texttt{f}_1, \texttt{f}_2, ...]);$$

#This command will produce list of all coefficients in expression.

Step A.2.6. #Solving algebraic equations

solve($\{$list=$\sim 0\}, \{a_i, b_i, c_i, d_i, \delta_i\})$, i=1..3

#This command will generate set of solutions for $a_i, b_i, c_i, d_i, \delta_i$

B

Appendices

B.1 The Maple code for writing KGZ equation (3.1) in real format

> with(PDEtools):
> DepVars := $[\phi(x,y,z,t), \psi(x,y,z,t), u(x,y,z,t), v(x,y,z,t), f(x,y,z,t)]$:
> alias($\phi = \phi(x,y,z,t), \psi = \psi(x,y,z,t), u = u(x,y,z,t), v = v(x,y,z,t)$,
$f = f(x,y,z,t)$):
> declare($\phi(x,y,z,t)$, $\psi(x,y,z,t), u(x,y,z,t), v(x,y,z,t), f(x,y,z,t)$)
> Δ := $f \to diff(f,x,x) + diff(f,y,y) + diff(f,z,z)$

$$\Delta := f \to \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

> $\phi := u + I \cdot v$:

$$\phi := u + I\,v$$

> PDE1 := $diff(\phi,t,t) - \Delta(\phi) + \phi + \phi \cdot \psi = 0$

$$\text{PDE1} := u_{t,t} + iv_{t,t} - u_{x,x} - iv_{x,x} - u_{y,y} - iv_{y,y} - u_{z,z} - iv_{z,z} + u + iv + (u + iv)\,\psi = 0$$

> PDE2 := $diff(\psi,t,t) - c^2 \cdot \Delta(\psi) - \Delta(evalc(abs(\phi)^2)) = 0$

$$\text{PDE2} := \psi_{t,t} - c^2\,(\psi_{x,x} + \psi_{y,y} + \psi_{z,z}) - 2\,{u_x}^2 - 2\,uu_{x,x} - 2\,{v_x}^2 - 2\,vv_{x,x} - 2\,{u_y}^2 - 2\,uu_{y,y} - 2\,{v_y}^2 - 2\,vv_{y,y} - 2\,{u_z}^2 - 2\,uu_{z,z} - 2\,{v_z}^2 - 2\,vv_{z,z} = 0$$

> PDESys := $evalc\sim([(\text{Re}, \text{Im})\sim(\text{PDE1})])$

$$[u\psi + u + u_{t,t} - u_{x,x} - u_{y,y} - u_{z,z} = 0,\ v\psi + v + v_{t,t} - v_{x,x} - v_{y,y} - v_{z,z} = 0]$$

B.2 Finding Symmetry algebra for equations (3.2)

The programming in same worksheet can be continued as follows:
> Sys := $[\text{PDESys}_1, \text{PDESys}_2, \text{PDE2}]$
> G := $[seq(\xi_j(x,y,z,t,u,v,\psi), j = [x,y,z,t]), seq(\eta_j(x,y,z,t,u,v,\psi), j = [u,v,\psi])]$:

$$[\xi_x(x,y,z,t,u,v,\psi),\xi_y(x,y,z,t,u,v,\psi),\xi_z(x,y,z,t,u,v,\psi),\xi_t(x,y,z,t,u,v,\psi),$$
$$\eta_u(x,y,z,t,u,v,\psi),\eta_v(x,y,z,t,u,v,\psi),\eta_\psi(x,y,z,t,u,v,\psi)]$$

> `declare(G)`

$$\eta(x,y,z,t,u,v,\psi) \text{ will now be displayed as } \eta$$

$$\xi(x,y,z,t,u,v,\psi) \text{ will now be displayed as } \xi$$

> S:=*Infinitesimals* (*Sys*, *displayfunctionality* = *false*)

$$\text{S:=}[[\xi_x=1,\xi_y=0,\xi_z=0,\xi_t=0,\eta_u=0,\eta_v=0,\eta_\psi=0],[\xi_x=0,\xi_y=1,\xi_z=0,\xi_t=0,\eta_u=0,\eta_v=0,\eta_\psi=0],[\xi_x=0,\xi_y=0,\xi_z=1,\xi_t=0,\eta_u=0,\eta_v=0,\eta_\psi=0],[\xi_x=0,\xi_y=0,\xi_z=0,\xi_t=1,\eta_u=0,\eta_v=0,\eta_\psi=0],[\xi_x=y,\xi_y=-x,\xi_z=0,\xi_t=0,\eta_u=0,\eta_v=0,\eta_\psi=0],[\xi_x=z,\xi_y=0,\xi_z=-x,\xi_t=0,\eta_u=0,\eta_v=0,\eta_\psi=0],[\xi_x=0,\xi_y=z,\xi_z=-y,\xi_t=0,\eta_u=0,\eta_v=0,\eta_\psi=0],[\xi_x=0,\xi_y=0,\xi_z=0,\xi_t=0,\eta_u=v,\eta_v=-u,\eta_\psi=0],[\xi_x=x,\xi_y=y,\xi_z=z,\xi_t=t,\eta_u=-u,\eta_v=-v,\eta_\psi=-2\,\psi-2]]$$

> `for` i `from 1 to 9 do` $InfinitesimalGenerator(\text{S}_i,[u,v,\psi])$ `od;`

$$f \to \frac{\partial}{\partial x} f$$

$$f \to \frac{\partial}{\partial y} f$$

$$f \to \frac{\partial}{\partial z} f$$

$$f \to \frac{\partial}{\partial t} f$$

$$f \to y\,\frac{\partial}{\partial x} - x\,\frac{\partial}{\partial y}$$

$$f \to z\,\frac{\partial}{\partial x} - x\,\frac{\partial}{\partial z}$$

$$f \to z\,\frac{\partial}{\partial y} - y\,\frac{\partial}{\partial z}$$

$$f \to v\,\frac{\partial}{\partial u} - u\,\frac{\partial}{\partial v}$$

$$f \to x\,\frac{\partial}{\partial x} + y\,\frac{\partial}{\partial y} + z\,\frac{\partial}{\partial z} + t\,\frac{\partial}{\partial t} - u\,\frac{\partial}{\partial u} - v\,\frac{\partial}{\partial v} + (-2\,w-2)\,\frac{\partial}{\partial w}$$

To test whether a given list of infinitesimals represents a symmetry of a given PDE system, we use following command:
> $map(SymmetryTest,[S],\text{Sys})$

$$[\{0\},\{0\},\{0\},\{0\},\{0\},\{0\},\{0\},\{0\},\{0\}]$$

B.3 Maple code for full adjoint action

Let $M_{i,j}$ be $(i,j)th$ entry in Table 3.1.
> $F := (i,j) \to M_{i,j}$:
> $M := Matrix(9,F)$:
This command will assign M the adjoint matrix equivalent to adjoint Table 3.1.
> $\epsilon := E$:
> $J := [0,9,2,8,1,6,4,7,3,5]$:
The order in J should be reversed when writing full adjoint action.
> $G_0 := add(a_k \cdot V_k, k = 1\ldots 9)$

$$G_0 := a_1V_1 + a_2V_2 + a_3V_3 + a_4V_4 + a_5V_5 + a_6V_6 + a_7V_7 + a_8V_8 + a_9V_9$$

> `for` j `from` 2 `to` $nops(J)$ `do`
$k := J_j$
$F_k := expand(T_k \cdot G_{J_{j-1}})$
`for` ii `to` 9 `do for` jj `to` 9 `do`
$F_k := expand(algsubs(T_{ii} \cdot V_{jj} = M_{ii,jj}, F_k))$
`od od`
$G_k := expand(algsubs(E = \epsilon_k, F_k))$
`od`

This final procedure will produce full adjoint action given by (3.10). For writing full adjoint in matrix format following procedure can be used

> `SM:= proc`(ex, V, A)
`Matrix`$(map(t \to [seq(coeff(t, A_i), i = 1 \ldots nops(A))], map2(coeff, ex, V)))$
`end proc;`
> `SM`$(G_5, [seq(V_i, i = 1 \ldots 9)], [seq(a_i, i = 1 \ldots 9)])$

B.4 Full adjoint action

$$\begin{aligned}
\tilde{a}_1 &= a_1 \\
\tilde{a}_2 &= -\cos(\epsilon_2)\cos(\epsilon_5)\sin(\epsilon_3)\,a_5 + \sin(\epsilon_2)\cos(\epsilon_5)\sin(\epsilon_3)\,a_3 \\
&\quad + \cos(\epsilon_2)\sin(\epsilon_5)\,a_3 + \sin(\epsilon_2)\sin(\epsilon_5)\,a_5 + \cos(\epsilon_5)\cos(\epsilon_3)\,a_2 \\
\tilde{a}_3 &= \cos(\epsilon_2)\sin(\epsilon_5)\sin(\epsilon_3)\,a_5 - \sin(\epsilon_2)\sin(\epsilon_5)\sin(\epsilon_3)\,a_3 \\
&\quad + \cos(\epsilon_2)\cos(\epsilon_5)\,a_3 + \sin(\epsilon_2)\cos(\epsilon_5)\,a_5 - \sin(\epsilon_5)\cos(\epsilon_3)\,a_2 \\
\tilde{a}_4 &= a_4\cos(\epsilon_2)\,\mathrm{e}^{\epsilon_1}\cos(\epsilon_3) + a_3\cos(\epsilon_2)\,\epsilon_4\sin(\epsilon_3) + a_5\cos(\epsilon_2)\,\epsilon_6\sin(\epsilon_3) \\
&\quad - a_3\cos(\epsilon_2)\,\epsilon_7\cos(\epsilon_3) + a_6\sin(\epsilon_2)\,\mathrm{e}^{\epsilon_1}\cos(\epsilon_3) \\
&\quad - a_3\sin(\epsilon_2)\,\epsilon_6\sin(\epsilon_3) + a_5\sin(\epsilon_2)\,\epsilon_4\sin(\epsilon_3) - a_5\sin(\epsilon_2)\,\epsilon_7\cos(\epsilon_3) \\
&\quad + a_7\mathrm{e}^{\epsilon_1}\sin(\epsilon_3) - a_1\epsilon_7\sin(\epsilon_3) - a_1\epsilon_4\cos(\epsilon_3) - a_2\epsilon_6\cos(\epsilon_3) \\
\tilde{a}_5 &= a_5\cos(\epsilon_2)\cos(\epsilon_3) - a_3\sin(\epsilon_2)\cos(\epsilon_3) + a_2\sin(\epsilon_3) \\
\tilde{a}_6 &= -\cos(\epsilon_2)\,\mathrm{e}^{\epsilon_1}\sin(\epsilon_5)\sin(\epsilon_3)\,a_4 + \cos(\epsilon_2)\sin(\epsilon_5)\sin(\epsilon_3)\,a_3\epsilon_7 \\
&\quad + \cos(\epsilon_2)\sin(\epsilon_5)\cos(\epsilon_3)\,a_3\epsilon_4 + \cos(\epsilon_2)\sin(\epsilon_5)\cos(\epsilon_3)\,a_5\epsilon_6 \\
&\quad - \sin(\epsilon_2)\,\mathrm{e}^{\epsilon_1}\sin(\epsilon_5)\sin(\epsilon_3)\,a_6 + \sin(\epsilon_2)\sin(\epsilon_5)\sin(\epsilon_3)\,a_5\epsilon_7 \\
&\quad - \sin(\epsilon_2)\sin(\epsilon_5)\cos(\epsilon_3)\,a_3\epsilon_6 + \sin(\epsilon_2)\sin(\epsilon_5)\cos(\epsilon_3)\,a_5\epsilon_4 \\
&\quad + \cos(\epsilon_2)\,\mathrm{e}^{\epsilon_1}\cos(\epsilon_5)\,a_6 - \cos(\epsilon_2)\cos(\epsilon_5)\,a_5\epsilon_7 - \sin(\epsilon_2)\,\mathrm{e}^{\epsilon_1}\cos(\epsilon_5)\,a_4 \\
&\quad + \sin(\epsilon_2)\cos(\epsilon_5)\,a_3\epsilon_7 + \mathrm{e}^{\epsilon_1}\sin(\epsilon_5)\cos(\epsilon_3)\,a_7 + \sin(\epsilon_5)\sin(\epsilon_3)\,a_1\epsilon_4 \\
&\quad + \sin(\epsilon_5)\sin(\epsilon_3)\,a_2\epsilon_6 - \sin(\epsilon_5)\cos(\epsilon_3)\,a_1\epsilon_7 - \cos(\epsilon_5)\,a_1\epsilon_6 + \cos(\epsilon_5)\,a_2\epsilon_4 \\
\tilde{a}_7 &= -\cos(\epsilon_2)\,\mathrm{e}^{\epsilon_1}\cos(\epsilon_5)\sin(\epsilon_3)\,a_4 + \cos(\epsilon_2)\cos(\epsilon_5)\sin(\epsilon_3)\,a_3\epsilon_7 \\
&\quad + \cos(\epsilon_2)\cos(\epsilon_5)\cos(\epsilon_3)\,a_3\epsilon_4 + \cos(\epsilon_2)\cos(\epsilon_5)\cos(\epsilon_3)\,a_5\epsilon_6 \\
&\quad - \sin(\epsilon_2)\,\mathrm{e}^{\epsilon_1}\cos(\epsilon_5)\sin(\epsilon_3)\,a_6 + \sin(\epsilon_2)\cos(\epsilon_5)\sin(\epsilon_3)\,a_5\epsilon_7 \\
&\quad - \sin(\epsilon_2)\cos(\epsilon_5)\cos(\epsilon_3)\,a_3\epsilon_6 + \sin(\epsilon_2)\cos(\epsilon_5)\cos(\epsilon_3)\,a_5\epsilon_4 \\
&\quad - \cos(\epsilon_2)\,\mathrm{e}^{\epsilon_1}\sin(\epsilon_5)\,a_6 + \cos(\epsilon_2)\sin(\epsilon_5)\,a_5\epsilon_7 + \sin(\epsilon_2)\,\mathrm{e}^{\epsilon_1}\sin(\epsilon_5)\,a_4 \\
&\quad - \sin(\epsilon_2)\sin(\epsilon_5)\,a_3\epsilon_7 + \mathrm{e}^{\epsilon_1}\cos(\epsilon_5)\cos(\epsilon_3)\,a_7 + \cos(\epsilon_5)\sin(\epsilon_3)\,a_1\epsilon_4 \\
&\quad + \cos(\epsilon_5)\sin(\epsilon_3)\,a_2\epsilon_6 - \cos(\epsilon_5)\cos(\epsilon_3)\,a_1\epsilon_7 + \sin(\epsilon_5)\,a_1\epsilon_6 - \sin(\epsilon_5)\,a_2\epsilon_4
\end{aligned}$$

$$\tilde{a}_8 = -\mathrm{e}^{\epsilon_1} a_1 \epsilon_8 + \mathrm{e}^{\epsilon_1} a_8$$
$$\tilde{a}_9 = a_9$$

Bibliography

[1] Abdou, M. New solitons and periodic wave solutions for nonlinear physical models. *Nonlinear Dynamics 52*, 1-2 (2008), 129–136.

[2] Abdou, M. Further improved F-expansion and new exact solutions for nonlinear evolution equations. *Nonlinear Dynamics 52*, 3 (2008), 277–288.

[3] Abdulwahhab, M. A. Exact solutions and conservation laws of the system of two-dimensional viscous Burgers equations. *Communications in Nonlinear Science and Numerical Simulation 39* (2016), 283–299.

[4] Ablowitz, M. J., and Clarkson, P. *Solitons, Nonlinear Evolution Equations and Inverse Scattering*, vol. 149. Cambridge University Press, Cambridge, 1991.

[5] Ablowitz, M. J., Ramani, A., and Segur, H. A connection between nonlinear evolution equations and ordinary differential equations of P-type. I. *Journal of Mathematical Physics 21*, 4 (1980), 715–721.

[6] Ablowitz, M. J., Ramani, A., and Segur, H. A connection between nonlinear evolution equations and ordinary differential equations of P-type. II. *Journal of Mathematical Physics 21*, 5 (1980), 1006–1015.

[7] Ablowitz, M. J., and Segur, H. Asymptotic solutions of the Korteweg-de Vries equation. *Studies in Applied Mathematics 57*, 1 (1977), 13–44.

[8] Ablowitz, M. J., and Segur, H. Exact linearization of a Painlevé transcendent. *Physical Review Letters 38*, 20 (1977), 1103–1106.

[9] Ames, W. F. *Nonlinear Partial Differential Equations in Engineering*, vol. 18. Academic press, New York, 1965.

[10] Anco, S. C. Symmetry properties of conservation laws. *International Journal of Modern Physics B 30*, 28n29 (2016), 1640003.

[11] Anco, S. C. On the incompleteness of Ibragimov's conservation law theorem and its equivalence to a standard formula using symmetries and adjoint-symmetries. *Symmetry 9*, 3 (2017), 33(1–28).

[12] Anco, S. C., and Bluman, G. Direct construction of conservation laws from field equations. *Physical Review Letters 78*, 15 (1997), 2869–2873.

[13] Anco, S. C., and Bluman, G. Direct construction method for conservation laws of partial differential equations part I: Examples of conservation law classifications. *European Journal of Applied Mathematics 13*, 05 (2002), 545–566.

[14] Anco, S. C., and Bluman, G. Direct construction method for conservation laws of partial differential equations part II: General treatment. *European Journal of Applied Mathematics 13*, 05 (2002), 567–585.

[15] Anderson, R. L., and Ibragimov, N. H. *Lie-Bäcklund Transformations in Applications*. SIAM, Philadelphia, 1979.

[16] Antonowicz, M., and Fordy, A. P. Coupled KdV equations with multi-Hamiltonian structures. *Physica D: Nonlinear Phenomena 28*, 3 (1987), 345–357.

[17] Antonowicz, M., and Fordy, A. P. A family of completely integrable multi-Hamiltonian systems. *Physics Letters A 122*, 2 (1987), 95–99.

[18] Awawdeh, F., Jaradat, H. M., and Al Shara, S. Applications of a simplified bilinear method to ion-acoustic solitary waves in plasma. *The European Physical Journal D-Atomic, Molecular, Optical and Plasma Physics 66*, 2 (2012), 40(1–8).

[19] Banda, M. K., Herty, M., and Ngnotchouye, J. M. T. Toward a mathematical analysis for drift-flux multiphase flow models in networks. *SIAM Journal on Scientific Computing 31*, 6 (2010), 4633–4653.

[20] Barenblatt, G., and Cole, J. Similarity, self-similarity and intermediate asymptotics. *Journal of Applied Mechanics 48* (1981), 213–214.

[21] Bekir, A. Multisoliton solutions to Cahn-Allen equation using double Exp-function method. *Physics of Wave Phenomena 20*, 2 (2012), 118–121.

[22] Bell, E. T. Exponential polynomials. *Annals of Mathematics 35*, 2 (1934), 258–277.

[23] Bhrawy, A. H., Abdelkawy, M., Kumar, S., Johnson, S., and Biswas, A. Solitons and other solutions to quantum Zakharov–Kuznetsov equation in quantum magneto-plasmas. *Indian Journal of Physics 87*, 5 (2013), 455–463.

[24] Bhrawy, A. H., Abdelkawy, M. A., and Biswas, A. Cnoidal and snoidal wave solutions to coupled nonlinear wave equations by the extended Jacobi elliptic function method. *Communications in Nonlinear Science and Numerical Simulation 18*, 4 (2013), 915–925.

[25] Bhrawy, A. H., Abdelkawy, M. A., Kumar, S., and Biswas, A. Solitons and other solutions to Kadomtsev–Petviashvili equation of b–type. *Romanian Journal of Physics 58*, 7-8 (2013), 729–748.

[26] Bhrawy, A. H., Doha, E. H., Ezz Eldien, S. S., and Abdelkawy, M. A. A Jacobi spectral collocation scheme based on operational matrix for time-fractional

modified Korteweg-de Vries equations. *CMES: Computer Modeling in Engineering & Sciences 104*, 3 (2015), 185–209.

[27] Bhrawy, A. H., Doha, E. H., Ezz Eldien, S. S., and Abdelkawy, M. A. A numerical technique based on the shifted Legendre polynomials for solving the time-fractional coupled KdV equations. *Calcolo 53*, 1 (2016), 1–17.

[28] Bhrawy, A. H., Tharwat, M., and Abdelkawy, M. A. Integrable system modelling shallow water waves: Kaup–Boussinesq shallow water system. *Indian Journal of Physics 87*, 7 (2013), 665–671.

[29] Bira, B., and Sekhar, T. R. Exact solutions to drift-flux multiphase flow models through Lie group symmetry analysis. *Applied Mathematics and Mechanics 36*, 8 (2015), 1105–1112.

[30] Biswas, A., Bhrawy, A. H., Abdelkawy, M. A., Alshaery, A., and Hilal, E. Symbolic computation of some nonlinear fractional differential equations. *Romanian Journal Physics 59*, 5-6 (2014), 433–442.

[31] Biswas, A., Krishnan, E., Suarez, P., Kara, A., and Kumar, S. Solitary waves and conservation laws of Bona-Chen equations. *Indian Journal of Physics 87*, 2 (2013), 169–175.

[32] Biswas, A., Milovic, D., Kumar, S., and Yildirim, A. Perturbation of shallow water waves by semi-inverse variational principle. *Indian Journal of Physics 87*, 6 (2013), 567–569.

[33] Bluman, G. Connections between symmetries and conservation laws. *Symmetry, Integrability and Geometry: Methods and Applications 1*, 011 (2005), 1–16.

[34] Bluman, G., and Anco, S. C. *Symmetry and Integration Methods for Differential Equations*, vol. 154. Springer-Verlag Inc., New York, 2002.

[35] Bluman, G., Cheviakov, A. F., and Anco, S. C. Construction of conservation laws: How the direct method generalizes Noether's theorem. In *Proceedings of 4th Workshop on Group Analysis of Differential Equations & Integrability* (Larnaca, Cyprus, 2009), vol. 1, pp. 1–23.

[36] Bluman, G., Cheviakov, A. F., and Anco, S. C. *Applications of Symmetry Methods to Partial Differential Equations*, vol. 168. Springer, New York, 2010.

[37] Bluman, G., and Cole, J. The general similarity solution of the heat equation. *Journal of Mathematics and Mechanics 18*, 11 (1969), 1025–1042.

[38] Bluman, G., and Cole, J. *Similarity Methods for Differential Equations*, vol. 13. Springer-Verlag Inc., New York, 1974.

[39] Bozhkov, Y., and Dimas, S. Group classification and conservation laws for a two-dimensional generalized Kuramoto–Sivashinsky equation. *Nonlinear Analysis: Theory, Methods & Applications 84* (2013), 117–135.

[40] Bozhkov, Y., Dimas, S., and Ibragimov, N. H. Conservation laws for a coupled variable-coefficient modified Korteweg–de Vries system in a two-layer fluid

model. *Communications in Nonlinear Science and Numerical Simulation 18*, 5 (2013), 1127–1135.

[41] Bozhkov, Y., Freire, I. L., and Ibragimov, N. H. Group analysis of the Novikov equation. *Computational and Applied Mathematics 33*, 1 (2014), 193–202.

[42] Brazhnyi, V., and Konotop, V. Stable and unstable vector dark solitons of coupled nonlinear Schrödinger equations: Application to two-component Bose-Einstein condensates. *Physical Review E 72*, 2 (2005), 026616.

[43] Bruzón, M., Clarkson, P., Gandarias, M., and Medina, E. The symmetry reductions of a turbulence model. *Journal of Physics A: Mathematical and General 34*, 18 (2001), 3751–3760.

[44] Bruzón, M., and Gandarias, M. Applying a new algorithm to derive non-classical symmetries. *Communications in Nonlinear Science and Numerical Simulation 13*, 3 (2008), 517–523.

[45] Çevikel, A. C., Bekir, A., Akar, M., and San, S. A procedure to construct exact solutions of nonlinear evolution equations. *Pramana Journal of Physics 79*, 3 (2012), 337–344.

[46] Cheb Terrab, E. S., and Von Bülow, K. A computational approach for the analytical solving of partial differential equations. *Computer Physics Communications 90*, 1 (1995), 102–116.

[47] Chou, K. S., Li, G. X., and Qu, C. A note on optimal systems for the heat equation. *Journal of Mathematical Analysis and Applications 261*, 2 (2001), 741–751.

[48] Chou, K. S., and Qu, C. Optimal systems and group classification of (2+1)-dimensional heat equation. *Acta Applicandae Mathematica 83*, 3 (2004), 257–287.

[49] Clarkson, P. New similarity solutions for the modified Boussinesq equation. *Journal of Physics A: Mathematical and General 22*, 13 (1989), 2355–2367.

[50] Clarkson, P. New similarity solutions and Painlevé analysis for the symmetric regularized long wave and the modified Benjamin-Bona-Mahoney equations. *Journal of Physics A: Mathematical and General 22*, 18 (1989), 3821–3848.

[51] Clarkson, P. Nonclassical symmetry reduction of Boussinesq equation. *Chaos, Solitons & Fractals 5*, 2 (1995), 2261–2300.

[52] Clarkson, P., and Kruskal, M. D. New similarity reduction of the Boussinesq equation. *Journal of Mathematical Physics 30*, 10 (1989), 2201–2213.

[53] Coggeshall, S., and Meyer-ter Vehn, J. Group-invariant solutions and optimal systems for multidimensional hydrodynamics. *Journal of Mathematical Physics 33*, 10 (1992), 3585–3601.

[54] Cohen, A. *An Introduction to the Lie Theory of One-Parameter Groups*. DC Health & Company, New York, 1911.

[55] Conte, R. Invariant Painlevé analysis of partial differential equations. *Physics Letters A 140*, 7-8 (1989), 383–390.

[56] Conte, R., Fordy, A. P., and Pickering, A. A perturbative Painlevé approach to nonlinear differential equations. *Physica D: Nonlinear Phenomena 69*, 1-2 (1993), 33–58.

[57] Conte, R. M., and Musette, M. *The Painlevé Handbook*. Springer Science & Business Media, Dordrecht, Netherlands, 2008.

[58] Date, E., Jimbo, M., Kashiwara, M., and Miwa, T. KP hierarchies of orthogonal and symplectic type transformation groups for soliton equations VI. *Journal of the Physical Society of Japan 50*, 11 (1981), 3813–3818.

[59] Date, E., Jimbo, M., Kashiwara, M., and Miwa, T. Transformation groups for soliton equations IV. A new hierarchy of soliton equations of KP-type. *Physica D Nonlinear Phenomena 4* (1982), 343–365.

[60] de Graaf, W. A. Classification of solvable Lie algebras. *Experimental Mathematics 14*, 1 (2005), 15–25.

[61] de la Rosa, R., Gandarias, M., and Bruzón, M. Symmetries and conservation laws of a fifth-order KdV equation with time-dependent coefficients and linear damping. *Nonlinear Dynamics 84*, 1 (2016), 135–141.

[62] Dehghan, M., and Nikpour, A. The solitary wave solution of coupled Klein–Gordon–Zakharov equations via two different numerical methods. *Computer Physics Communications 184*, 9 (2013), 2145–2158.

[63] Dorizzi, B., Grammaticos, B., Ramani, A., and Winternitz, P. Are all the equations of the Kadomtsev–Petviashvili hierarchy integrable? *Journal of Mathematical Physics 27*, 12 (1986), 2848–2852.

[64] Drinfel'd, V. G., and Sokolov, V. V. Lie algebras and equations of Korteweg-de Vries type. *Journal of Soviet Mathematics 30*, 2 (1985), 1975–2036.

[65] Ebadi, G., Fard, N. Y., Bhrawy, A. H., Kumar, S., Triki, H., Yildirim, A., and Biswas, A. Solitons and other solutions to the (3+1)–dimensional extended Kadomtsev–Petviashvili equation with power law nonlinearity. *Romanian Reports in Physics 65*, 1 (2013), 27–62.

[66] Ebadi, G., Mojaver, A., Johnson, S., Kumar, S., and Biswas, A. Dynamics of dispersive topological solitons and its perturbations. *Indian Journal of Physics 86*, 12 (2012), 1115–1129.

[67] El, G., Grimshaw, R., and Pavlov, M. Integrable shallow water equations and undular bores. *Studies in Applied Mathematics 106*, 2 (2001), 157–186.

[68] Eslami, M., Vajargah, B. F., Mirzazadeh, M., and Biswas, A. Application of first integral method to fractional partial differential equations. *Indian Journal of Physics 88*, 2 (2014), 177–184.

[69] Evje, S., and Flåtten, T. On the wave structure of two-phase flow models. *SIAM Journal on Applied Mathematics 67*, 2 (2007), 487–511.

[70] Fan, E. An algebraic method for finding a series of exact solutions to integrable and nonintegrable nonlinear evolution equations. *Journal of Physics A: Mathematical and General 36*, 25 (2003), 7009–7026.

[71] Fan, E. The integrability of non-isospectral and variable-coefficient KdV equation with binary Bell polynomials. *Physics Letters A 375*, 3 (2011), 493–497.

[72] Fan, E., and Hon, Y. Quasiperiodic waves and asymptotic behavior for Bogoyavlenskii breaking soliton equation in (2+1)-dimensions. *Physical Review E 78*, 3 (2008), 036607–13.

[73] Feng, L. L., Tian, S. F., Wang, X. B., and Zhang, T. T. Rogue waves, homoclinic breather waves and soliton waves for the (2+1)-dimensional b-type Kadomtsev–Petviashvili equation. *Applied Mathematics Letters 65* (2017), 90–97.

[74] Flaschka, H., Newell, A., and Tabor, M. Integrability. In *What is Integrability?* Springer, New York, 1991, pp. 73–114.

[75] Freire, I. L. Conservation laws for self-adjoint first-order evolution equation. *Journal of Nonlinear Mathematical Physics 18*, 2 (2011), 279–290.

[76] Freire, I. L. New classes of nonlinearly self-adjoint evolution equations of third-and fifth-order. *Communications in Nonlinear Science and Numerical Simulation 18*, 3 (2013), 493–499.

[77] Freire, I. L., and Sampaio, J. C. S. Nonlinear self-adjointness of a generalized fifth-order KdV equation. *Journal of Physics A: Mathematical and Theoretical 45*, 3 (2011), 032001.

[78] Fuchssteiner, B. The Lie algebra structure of degenerate Hamiltonian and bi-Hamiltonian systems. *Progress of Theoretical Physics 68*, 4 (1982), 1082–1104.

[79] Galas, F., and Richter, E. Exact similarity solutions of ideal MHD equations for plane motions. *Physica D: Nonlinear Phenomena 50*, 2 (1991), 297–307.

[80] Gandarias, M., and Bruzón, M. Some conservation laws for a forced KdV equation. *Nonlinear Analysis: Real World Applications 13*, 6 (2012), 2692–2700.

[81] Gandarias, M. L., and Khalique, C. M. Symmetries, solutions and conservation laws of a class of nonlinear dispersive wave equations. *Communications in Nonlinear Science and Numerical Simulation 32* (2016), 114–121.

[82] Gardner, C., Greene, J., Kruskal, M. D., and Miura, R. M. Method for solving the Korteweg-de Vries equation. *Physical Review Letters 19*, 19 (1967), 1095–1097.

[83] Gürses, M., and Pekcan, A. Traveling wave solutions of degenerate coupled multi-KdV equations. *Journal of Mathematical Physics 57*, 10 (2016), 103507.

[84] Gerdt, V., and Lassner, W. Isomorphism verification for complex and real Lie algebras by Gröbner basis technique. In *Modern Group Analysis: Advanced Analytical and Computational Methods in Mathematical Physics*. Springer, Dordrecht, Netherlands, 1993, pp. 245–254.

[85] Gilmore, R. *Lie groups, Physics, and Geometry: An Introduction for Physicists, Engineers and Chemists*. Cambridge University Press, Cambridge, 2008.

[86] Gilmore, R. *Lie Groups, Lie Algebras, and Some of Their Applications*. John Wiley and Sons, New York, 2012.

[87] Gilson, C., Lambert, F., Nimmo, J., and Willox, R. On the combinatorics of the Hirota D-operators. In *Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences* (1996), vol. 452, The Royal Society, pp. 223–234.

[88] Gui Qiong, X. Painlevé integrability of a generalized fifth-order KdV equation with variable coefficients: Exact solutions and their interactions. *Chinese Physics B 22*, 5 (2013), 050203–08.

[89] Gui Qiong, X., and Zhi Bin, L. Symbolic Computation of the Painlevé test for nonlinear partial differential equations using Maple. *Computer Physics Communications 161*, 1-2 (2004), 65–75.

[90] Gupta, R., and Singh, M. On group classification and nonlocal conservation laws for a multiphase flow model. *International Journal of Applied and Computational Mathematics 3*, 4 (2017), 3925–3935.

[91] Gupta, R. K., and Bansal, A. Similarity reductions and exact solutions of generalized Bretherton equation with time-dependent coefficients. *Nonlinear Dynamics 71*, 1-2 (2013), 1–12.

[92] Gupta, R. K., Kumar, V., and Jiwari, R. Exact and numerical solutions of coupled short pulse equation with time-dependent coefficients. *Nonlinear Dynamics 79*, 1 (2015), 455–464.

[93] Gupta, R. K., and Singh, K. Symmetry analysis and some exact solutions of cylindrically symmetric null fields in general relativity. *Communications in Nonlinear Science and Numerical Simulation 16*, 11 (2011), 4189–4196.

[94] Gupta, R. K., and Singh, M. On invariant analysis and conservation laws for degenerate coupled multi-KdV equations for multiplicity $l = 3$. *Pramana-Journal Physics*, In press.

[95] Gupta, R. K., and Singh, M. Nonclassical symmetries and similarity solutions of variable coefficient coupled KdV system using compatibility method. *Nonlinear Dynamics 87*, 3 (2016), 1543–1552.

[96] Gürses, M. Integrable hierarchy of multi-component Kaup-Boussinesq equations. *arXiv preprint arXiv:1301.4075* (2013).

[97] Gürses, M., and Pekcan, A. Traveling wave solutions of degenerate coupled Korteweg-de Vries equation. *Journal of Mathematical Physics 55*, 9 (2014), 091501.

[98] Hansen, A. G. *Similarity Analysis of Boundary Value Problems in Engineering*. Prentice-Hall, Michigan, 1964.

[99] He, J. H., and Wu, X. H. Exp-function method for nonlinear wave equations. *Chaos, Solitons & Fractals 30*, 3 (2006), 700–708.

[100] Hietarinta, J. Searching for integrable PDEs by testing Hirota three-soliton condition. In *Proceedings of the 1991 International Symposium on Symbolic and Algebraic Computation* (1991), ACM, pp. 295–300.

[101] Hietarinta, J. Hirota bilinear method and soliton solutions. *Phys. AUC 15*, 1 (2005), 31–37.

[102] Hirota, R. Exact solution of the Korteweg-de Vries equation for multiple collisions of solitons. *Physical Review Letters 27*, 18 (1971), 1192–1194.

[103] Hirota, R. A new form of Bäcklund transformations and its relation to the inverse scattering problem. *Progress of Theoretical Physics 52*, 5 (1974), 1498–1512.

[104] Hirota, R. Direct method of finding exact solutions of nonlinear evolution equations. *R. Bullough, P. Caudrey (Eds.), Bäcklund Transformations, Springer, Berlin 515* (1976), 40–68.

[105] Hirota, R. *The Direct Method in Soliton Theory*. Cambridge University Press, Cambridge, 2004.

[106] Hirota, R., and Satsuma, J. Soliton solutions of a coupled Korteweg-de Vries equation. *Physics Letters A 85*, 8-9 (1981), 407–408.

[107] Hon, Y., and Fan, E. Binary Bell polynomial approach to the non-isospectral and variable-coefficient KP equations. *IMA Journal of Applied Mathematics 77*, 2 (2012), 236–251.

[108] Hong-Cai, M. A simple method to generate Lie point symmetry groups of the (3+1)-dimensional Jimbo–Miwa equation. *Chinese Physics Letters 22*, 3 (2005), 554.

[109] Hu, X., and Li, Y. Bäcklund transformation and nonlinear superposition formula of DJKM equation. *Acta Math. Sci 11* (1991), 164–172.

[110] Hu, X., Li, Y., and Chen, Y. A direct algorithm of one-dimensional optimal system for the group invariant solutions. *Journal of Mathematical Physics 56*, 5 (2015), 053504.

[111] Hu, X. B., Wang, D. L., Tam, H. W., and Xue, W. M. Soliton solutions to the Jimbo–Miwa equations and the Fordy–Gibbons–Jimbo–Miwa equation. *Physics Letters A 262*, 4 (1999), 310–320.

[112] Hua, X., Dongb, Z., Huangc, F., and Chena, Y. Symmetry reductions and exact solutions of the (2+1)-dimensional Navier-Stokes equations. *Zeitschrift für Naturforschung A 65*, 6-7 (2010), 504–510.

[113] Humphreys, J. E. *Introduction to Lie Algebras and Representation Theory*, vol. 9. Springer Science & Business Media, New York, 1972.

[114] Hydon, P. E. *Symmetry Methods for Differential Equations*. Cambridge University Press, Cambridge, 2000.

[115] Ibragimov, N. H. *CRC Handbook of Lie Group Analysis of Differential Equations*, vol. 3. CRC press, Boca Raton, 1995.

[116] Ibragimov, N. H. Integrating factors, adjoint equations and Lagrangians. *Journal of Mathematical Analysis and Applications 318*, 2 (2006), 742–757.

[117] Ibragimov, N. H. A new conservation theorem. *Journal of Mathematical Analysis and Applications 333*, 1 (2007), 311–328.

[118] Ibragimov, N. H., Khamitova, R., and Valenti, A. Self-adjointness of a generalized Camassa–Holm equation. *Applied Mathematics and Computation 218*, 6 (2011), 2579–2583.

[119] Ismail, M., and Biswas, A. 1-Soliton solution of the Klein–Gordon–Zakharov equation with power law nonlinearity. *Applied Mathematics and Computation 217*, 8 (2010), 4186–4196.

[120] Issa, R., and Kempf, M. Simulation of slug flow in horizontal and nearly horizontal pipes with the two-fluid model. *International Journal of Multiphase Flow 29*, 1 (2003), 69–95.

[121] Ivanov, R. Two-component integrable systems modelling shallow water waves: The constant vorticity case. *Wave Motion 46*, 6 (2009), 389–396.

[122] Ivanov, R., and Lyons, T. Integrable models for shallow water with energy dependent spectral problems. *Journal of Nonlinear Mathematical Physics 19*, sup1 (2012), 72–88.

[123] Jena, J., and Sharma, V. D. Lie transformation group solutions of non-linear equations describing viscoelastic materials. *International Journal of Engineering Science 35*, 10-11 (1997), 1033–1044.

[124] Jimbo, M., and Miwa, T. Solitons and infinite dimensional Lie algebras. *Publications of the Research Institute for Mathematical Sciences 19*, 3 (1983), 943–1001.

[125] Johnpillai, A., and Khalique, C. M. Conservation laws of KdV equation with time dependent coefficients. *Communications in Nonlinear Science and Numerical Simulation 16*, 8 (2011), 3081–3089.

[126] Kamchatnov, A., Kraenkel, R. A., and Umarov, B. On asymptotic solutions of integrable wave equations. *Physics Letters A 287*, 3 (2001), 223–232.

[127] Kamchatnov, A., Kraenkel, R. A., and Umarov, B. Asymptotic soliton train solutions of Kaup–Boussinesq equations. *Wave Motion 38*, 4 (2003), 355–365.

[128] Kaplan, M., Bekir, A., and Akbulut, A. A generalized Kudryashov method to some nonlinear evolution equations in mathematical physics. *Nonlinear Dynamics 85*, 4 (2016), 2843–2850.

[129] Kara, A., and Mahomed, F. Noether-type symmetries and conservation laws via partial Lagrangians. *Nonlinear Dynamics 45*, 3 (2006), 367–383.

[130] Kaup, D. A higher-order water-wave equation and the method for solving it. *Progress of Theoretical Physics 54*, 2 (1975), 396–408.

[131] Koetz, H. A technique to classify the similarity solutions of nonlinear partial (integro-) differential equations. II. Full optimal subalgebraic systems. *Zeitschrift für Naturforschung A 48*, 4 (1993), 535–550.

[132] Konno, K., Sanuki, H., and Ichikawa, Y. H. Conservation laws of nonlinear-evolution equations. *Progress of Theoretical Physics 52*, 3 (1974), 886–889.

[133] Krishnan, E., Kara, A., Kumar, S., and Biswas, A. Topological solitons, cnoidal waves and conservation laws of coupled wave equations. *Indian Journal of Physics 87*, 12 (2013), 1233–1241.

[134] Krishnan, E., Kumar, S., and Biswas, A. Solitons and other nonlinear waves of the Boussinesq equation. *Nonlinear Dynamics 70*, 2 (2012), 1213–1221.

[135] Kudryashov, N. A. Seven common errors in finding exact solutions of nonlinear differential equations. *Communications in Nonlinear Science and Numerical Simulation 14*, 9 (2009), 3507–3529.

[136] Kudryashov, N. A. A note on the $\left(\frac{G'}{G}\right)$-expansion method. *Applied Mathematics and Computation 217*, 4 (2010), 1755–1758.

[137] Kumar, R., Gupta, R. K., and Bhatia, S. S. Invariant solutions of variable coefficients generalized Gardner equation. *Nonlinear Dynamics 83*, 4 (2016), 2103–2111.

[138] Kumar, S. Painlevé analysis and invariant solutions of Vakhnenko–Parkes (VP) equation with power law nonlinearity. *Nonlinear Dynamics 85*, 2 (Jul 2016), 1275–1279.

[139] Kumar, S., Singh, K., and Gupta, R. K. Coupled Higgs field equation and Hamiltonian amplitude equation: Lie classical approach and $\left(\frac{G'}{G}\right)$–expansion method. *Pramana Journal of Physics 79*, 1 (2012), 41–60.

[140] Kunzinger, M., and Popovych, R. Is a nonclassical symmetry a symmetry? *Proceedings of 4th Workshop on Group Analysis of Differential Equations and Integrable Systems* (2009), 107–120.

[141] Lakshmanan, M., and Sahadevan, R. Painlevé analysis, Lie symmetries, and integrability of coupled nonlinear oscillators of polynomial type. *Physics Reports 224*, 1-2 (1993), 1–93.

[142] Lamb Jr, G. Analytical descriptions of ultrashort optical pulse propagation in a resonant medium. *Reviews of Modern Physics 43*, 2 (1971), 99.

[143] Lambert, F., and Springael, J. On a direct procedure for the disclosure of Lax pairs and Bäcklund transformations. *Chaos, Solitons & Fractals 12*, 14 (2001), 2821–2832.

[144] Lambert, F., and Springael, J. Soliton equations and simple combinatorics. *Acta Applicandae Mathematicae 102*, 2-3 (2008), 147–178.

[145] Lee, K. Bäcklund Transformations and Their Applications (C. Rogers and WF Shadwick). *SIAM Review 26*, 1 (1984), 138–141.

[146] Levi, D., and Winternitz, P. Nonclassical symmetry reduction: Example of the Boussinesq equation. *Journal of Physics A: Mathematical and General 22*, 15 (1989), 2915–2924.

[147] Li, J. Exact explicit travelling wave solutions for (n+1)-dimensional Klein–Gordon–Zakharov equations. *Chaos, Solitons & Fractals 34*, 3 (2007), 867–871.

[148] Li, J., and Chen, F. Bifurcations and exact traveling wave solutions of degenerate coupled multi-KdV equations. *International Journal of Bifurcation and Chaos 26*, 03 (2016), 1650045.

[149] Li, Y., Ma, W., and Zhang, J. E. Darboux transformations of classical Boussinesq system and its new solutions. *Physics Letters A 275*, 1 (2000), 60–66.

[150] Li, Y., and Zhang, J. E. Darboux transformations of classical Boussinesq system and its multi-soliton solutions. *Physics Letters A 284*, 6 (2001), 253–258.

[151] Liu, H., Li, J., and Liu, L. Painlevé analysis, Lie symmetries, and exact solutions for the time-dependent coefficients Gardner equations. *Nonlinear Dynamics 59*, 3 (2010), 497–502.

[152] Liu, S., Fu, Z., Liu, S., and Zhao, Q. Jacobi elliptic function expansion method and periodic wave solutions of nonlinear wave equations . *Physics Letters A 289*, 1-2 (2001), 69–74.

[153] Lou, S. Y., Tong, B., Hu, H., and Tang, X. Coupled KdV equations derived from two-layer fluids. *Journal of Physics A: Mathematical and General 39*, 3 (2005), 513.

[154] Lü, X., Tian, B., Sun, K., and Wang, P. Bell-polynomial manipulations on the Bäcklund transformations and Lax pairs for some soliton equations with one Tau-function. *Journal of Mathematical Physics 51*, 11 (2010), 113506–08.

[155] Luo, L., and Fan, E. Bilinear approach to the quasi-periodic wave solutions of modified Nizhnik–Novikov–Vesselov equation in (2+1)-dimensions. *Physics Letters A 374*, 30 (2010), 3001–3006.

[156] Ma, S. H., Fang, J. P., and Wu, H. Y. Elastic and annihilation solitons of the (3+1)-dimensional generalized shallow water wave system. *Zeitschrift für Naturforschung A 68*, 5 (2013), 350–354.

[157] Ma, W. X. Complexiton solutions to integrable equations. *Nonlinear Analysis: Theory, Methods & Applications 63*, 5 (2005), e2461–e2471.

[158] Ma, W. X., and Abdeljabbar, A. A bilinear Bäcklund transformation of a (3+1)-dimensional generalized KP equation. *Applied Mathematics Letters 25*, 10 (2012), 1500–1504.

[159] Ma, W. X., and Fan, E. Linear superposition principle applying to Hirota bilinear equations. *Computers & Mathematics with Applications 61*, 4 (2011), 950–959.

[160] Ma, W. X., and Lee, J. H. A transformed rational function method and exact solutions to the (3+1)-dimensional Jimbo–Miwa equation. *Chaos, Solitons & Fractals 42*, 3 (2009), 1356–1363.

[161] Ma, W. X., and Zhu, Z. Solving the (3+1)-dimensional generalized KP and BKP equations by the multiple Exp-function algorithm. *Applied Mathematics and Computation 218*, 24 (2012), 11871–11879.

[162] Malfliet, W. Solitary wave solutions of nonlinear wave equations. *American Journal of Physics 60*, 7 (1992), 650–654.

[163] Malfliet, W., and Hereman, W. The tanh method: I. Exact solutions of nonlinear evolution and wave equations. *Physica Scripta 54*, 6 (1996), 563–568.

[164] Matsuno, Y. *Bilinear Transformation Method*, vol. 174. Academic Press INC., Orlando, Florida, 1984.

[165] Matveev, V. B., and Salle, M. *Darboux Transformations and Solitons*. Springer-Verlag, Germany, 1991.

[166] Miao, Q., Wang, Y., Chen, Y., and Yang, Y. PDEBellII: A Maple package for finding bilinear forms, bilinear Bäcklund transformations, Lax pairs and conservation laws of the KdV-type equations. *Computer Physics Communications 185*, 1 (2014), 357–367.

[167] Mirzazadeh, M., Eslami, M., and Biswas, A. Soliton solutions of the generalized Klein–Gordon equation by using $\left(\frac{G'}{G}\right)$–expansion method. *Computational and Applied Mathematics 33*, 3 (2014), 831–839.

[168] Mumford, D. *Tata Lectures on Theta I*, vol. 28. Modern Birkhäuser Classics, Boston, 1983.

[169] Na, L. Bäcklund transformation and multi-soliton solutions for the (3+1)-dimensional BKP equation with Bell polynomials and symbolic computation. *Nonlinear Dynamics 82*, 1-2 (2015), 311–318.

[170] Nakamura, A. A direct method of calculating periodic wave solutions to nonlinear evolution equations. I. Exact two-periodic wave solution. *Journal of the Physical Society of Japan 47*, 5 (1979), 1701–1705.

[171] Nakamura, A. A direct method of calculating periodic wave solutions to nonlinear evolution equations. II. Exact one and two-periodic wave solution of the coupled bilinear equations. *Journal of the Physical Society of Japan 48*, 4 (1980), 1365–1370.

[172] Nucci, M. Group analysis for MHD equations. *Atti del Seminario Matematico e Fisico dell' Universita di Modena e Reggio Emilia 33*, 1 (1984), 21–34.

[173] Nutku, Y. Bi-Hamiltonian structure of a pair of coupled KdV equations. *Il Nuovo Cimento B (1971-1996) 105*, 12 (1990), 1381–1383.

[174] Olver, P. Symmetry groups and group invariant solutions of partial differential equations. *Journal of Differential Geometry 14* (1979), 497–542.

[175] Olver, P. *Applications of Lie Groups to Differential Equations*, vol. 107. Springer-Verlag Inc., New York, 1986.

[176] Osiptsov, A., Sinkov, K., and Spesivtsev, P. Justification of the drift-flux model for two-phase flow in a circular pipe. *Fluid Dynamics 49*, 5 (2014), 614–626.

[177] Ovsiannikov, L. *Group Analysis of Differential Equations*. Academic Press, New York, 1982.

[178] Ozawa, T., Tsutaya, K., and Tsutsumi, Y. Normal form and global solutions for the Klein-Gordon-Zakharov equations. In *Annales de l'IHP Analyse Non Linéaire* (1995), vol. 12, pp. 459–503.

[179] Ozawa, T., Tsutaya, K., and Tsutsumi, Y. Well-posedness in energy space for the Cauchy problem of the Klein-Gordon-Zakharov equations with different propagation speeds in three space dimensions. *Mathematische Annalen 313*, 1 (1999), 127–140.

[180] Pandey, M., Pandey, B., and Sharma, V. D. Symmetry groups and similarity solutions for the system of equations for a viscous compressible fluid. *Applied Mathematics and Computation 215*, 2 (2009), 681–685.

[181] Pandey, M., Radha, R., and Sharma, V. D. Symmetry analysis and exact solutions of magnetogasdynamic equations. *The Quarterly Journal of Mechanics & Applied Mathematics 61*, 3 (2008), 291–310.

[182] Patera, J., and Winternitz, P. Subalgebras of real three and four dimensional Lie algebras. *Journal of Mathematical Physics 18*, 7 (1977), 1449–1455.

[183] Patera, J., Winternitz, P., and Zassenhaus, H. Continuous subgroups of the fundamental groups of physics. I. General method and the Poincaré group. *Journal of Mathematical Physics 16*, 8 (1975), 1597–1614.

[184] Pickering, A. The singular manifold method revisited. *Journal of Mathematical Physics 37*, 4 (1996), 1894–1927.

[185] Poole, D., and Hereman, W. The homotopy operator method for symbolic integration by parts and inversion of divergences with applications. *Applicable Analysis 89*, 4 (2010), 433–455.

[186] Popovych, R. O., and Vaneeva, O. O. More common errors in finding exact solutions of nonlinear differential equations: Part I. *Communications in Nonlinear Science and Numerical Simulation 15*, 12 (2010), 3887–3899.

[187] Ramani, A. Inverse scattering, ordinary differential equations of Painlevé-type, and Hirota bilinear formalism. *Annals of the New York Academy of Sciences 373*, 1 (1981), 54–67.

[188] Rogers, C., and Shadwick, W. F. *Bäcklund transformations and their applications*, vol. 161. Academic Press, New York, 1982.

[189] Ryzhkov, I. I. On the normalizers of subalgebras in an infinite Lie algebra. *Communications in Nonlinear Science and Numerical Simulation 11*, 2 (2006), 172–185.

[190] Sattinger, D. H., and Weaver, O. L. *Lie Groups and Algebras with Applications to Physics, Geometry, and Mechanics*, vol. 61. Springer Science & Business Media, New York, 2013.

[191] Sedov, L. I. *Similarity and Dimensional Methods in Mechanics*, 10 ed. CRC press, Boca Raton, Florida, 1993.

[192] Sekhar, T. R., and Satapathy, P. Group classification for isothermal drift flux model of two phase flows. *Computers & Mathematics with Applications 72*, 5 (2016), 1436–1443.

[193] Seshadri, R., and Na, T. Y. *Group Invariance in Engineering Boundary Value Problems*. Springer Science & Business Media, New York, 2012.

[194] Shang, Y., Huang, Y., and Yuan, W. New exact traveling wave solutions for the Klein–Gordon–Zakharov equations. *Computers & Mathematics with Applications 56*, 5 (2008), 1441–1450.

[195] Sharma, V. D., and Arora, R. Similarity solutions for strong shocks in an ideal gas. *Studies in Applied Mathematics 114*, 4 (2005), 375–394.

[196] Sharma, V. D., and Radha, C. Similarity solutions for converging shocks in a relaxing gas. *International Journal of Engineering Science 33*, 4 (1995), 535–553.

[197] Sharma, V. D., and Radha, R. Exact solutions of Euler equations of ideal gasdynamics via Lie group analysis. *Zeitschrift für Angewandte Mathematik und Physik (ZAMP) 59*, 6 (2008), 1029–1038.

[198] Shen, Y. J., Gao, Y. T., and Yu, X. Bell-polynomial application and multi-soliton solutions for the generalized variable-coefficient Drinfeld–Sokolov–Satsuma–Hirota system in fluids and plasmas. *Physica Scripta 88*, 3 (2013), 035003–07.

[199] Shen, Y. J., Gao, Y. T., Yu, X., Meng, G. Q., and Qin, Y. Bell-polynomial approach applied to the seventh-order Sawada–Kotera–Ito equation. *Applied Mathematics and Computation 227* (2014), 502–508.

[200] Shi, Y., and Li, D. New exact solutions for the (2+1)-dimensional Sawada–Kotera equation. *Computers & Fluids 68* (2012), 88–93.

[201] Singh, K., and Gupta, R. K. On symmetries and invariant solutions of a coupled KdV system with variable coefficients. *International Journal of Mathematics and Mathematical Sciences 2005*, 23 (2005), 3711–3725.

[202] Singh, K., and Gupta, R. K. Lie symmetries and exact solutions of a new generalized Hirota–Satsuma coupled KdV system with variable coefficients. *International Journal of Engineering Science 44*, 3 (2006), 241–255.

[203] Singh, K., and Gupta, R. K. Exact solutions of a variant Boussinesq system. *International Journal of Engineering Science 44*, 18 (2006), 1256–1268.

[204] Singh, K., Gupta, R. K., and Kumar, S. Benjamin–Bona–Mahony (BBM) equation with variable coefficients: Similarity reductions and Painlevé analysis. *Applied Mathematics and Computation 217*, 16 (2011), 7021–7027.

[205] Singh, M. New exact solutions for (3+1)-dimensional Jimbo–Miwa equation. *Nonlinear Dynamics 84*, 2 (2016), 875–880.

[206] Singh, M., and Gupta, R. Soliton and quasi-periodic wave solutions for b-type Kadomtsev–Petviashvili equation. *Indian Journal of Physics 91*, 11 (2017), 1345–1354.

[207] Singh, M., and Gupta, R. On Painlevé analysis, symmetry group and conservation laws of Date–Jimbo–Kashiwara–Miwa equation. *International Journal of Applied and Computational Mathematics 4*, 3 (2018), 88.

[208] Singh, M., and Gupta, R. K. Group classification, conservation laws and Painlevé analysis for Klein–Gordon–Zakharov equations in (3+1)–dimension. *Pramana-Journal Physics*, In press.

[209] Singh, M., and Gupta, R. K. Exact solutions for nonlinear evolution equations using novel test function. *Nonlinear Dynamics 86*, 2 (2016), 1171–1182.

[210] Singh, M., and Gupta, R. K. Bäcklund transformations, Lax system, conservation laws and multisoliton solutions for Jimbo–Miwa equation with Bell-polynomials. *Communications in Nonlinear Science and Numerical Simulation 37* (2016), 362–373.

[211] Stephani, H. *Differential Equations: Their Solution Using Symmetries*. Cambridge University Press, Cambridge, 1989.

[212] Steudel, H. Über die Zuordnung zwischen lnvarianzeigenschaften und Erhaltungssätzen. *Zeitschrift für Naturforschung A 17*, 2 (1962), 129–132.

[213] Tang, X., and Liang, Z. Variable separation solutions for the (3+1)-dimensional Jimbo–Miwa equation. *Physics Letters A 351*, 6 (2006), 398–402.

[214] Thornhill, S. G., and Ter Haar, D. Langmuir turbulence and modulational instability. *Physics Reports 43*, 2 (1978), 43–99.

[215] Tian, B., and Gao, Y. T. Beyond travelling waves: A new algorithm for solving nonlinear evolution equations. *Computer Physics Communications 95*, 2 (1996), 139–142.

[216] Tian, S. F. The mixed coupled nonlinear Schrödinger equation on the half-line via the Fokas method. In *Proc. R. Soc. A* (2016), vol. 472, The Royal Society, pp. 0588–0610.

[217] Tian, S. F. Initial–boundary value problems for the general coupled nonlinear Schrödinger equation on the interval via the Fokas method. *Journal of Differential Equations 262*, 1 (2017), 506–558.

[218] Tian, S. F., Wang, Z., and Zhang, H. Q. Some types of solutions and generalized binary Darboux transformation for the mKP equation with self-consistent sources. *Journal of Mathematical Analysis and Applications 366*, 2 (2010), 646–662.

[219] Tian, S. F., and Zhang, H. Q. Riemann theta functions periodic wave solutions and rational characteristics for the nonlinear equations. *Journal of Mathematical Analysis and Applications 371*, 2 (2010), 585–608.

[220] Tian, S. F., and Zhang, H. Q. On the integrability of a generalized variable-coefficient Kadomtsev–Petviashvili equation. *Journal of Physics A: Mathematical and Theoretical 45*, 5 (2012), 055203–29.

[221] Tian, S. F., and Zhang, H. Q. On the integrability of a generalized variable-coefficient forced Korteweg-de Vries equation in fluids. *Studies in Applied Mathematics 132*, 3 (2014), 212–246.

[222] Tian, S. F., Zhang, Y., Feng, B., and Zhang, H. On the Lie algebras, generalized symmetries and Darboux transformations of the fifth-order evolution equations in shallow water. *Chinese Annals of Mathematics, Series B 36*, 4 (2015), 543–560.

[223] Torrisi, M., and Tracinà, R. Quasi self-adjointness of a class of third order nonlinear dispersive equations. *Nonlinear Analysis: Real World Applications 14*, 3 (2013), 1496–1502.

[224] Tracinà, R., Bruzón, M., Gandarias, M., and Torrisi, M. Nonlinear self-adjointness, conservation laws, exact solutions of a system of dispersive evolution equations. *Communications in Nonlinear Science and Numerical Simulation 19*, 9 (2014), 3036–3043.

[225] Tsutaya, K. Global existence of small amplitude solutions for the Klein-Gordon-Zakharov equations. *Nonlinear Analysis: Theory, Methods & Applications 27*, 12 (1996), 1373–1380.

[226] Tu, J. M., Tian, S. F., Xu, M. J., Ma, P. L., and Zhang, T. T. On periodic wave solutions with asymptotic behaviors to a (3+1)-dimensional generalized b-type Kadomtsev–Petviashvili equation in fluid dynamics. *Computers & Mathematics with Applications 72*, 9 (2016), 2486–2504.

[227] Tu, J. M., Tian, S. F., Xu, M. J., Song, X. Q., and Zhang, T. T. Bäcklund transformation, infinite conservation laws and periodic wave solutions of a generalized (3+1)-dimensional nonlinear wave in liquid with gas bubbles. *Nonlinear Dynamics 83*, 3 (2016), 1199–1215.

[228] Tu, J. M., Tian, S. F., Xu, M. J., and Zhang, T. T. On Lie symmetries, optimal systems and explicit solutions to the Kudryashov–Sinelshchikov equation. *Applied Mathematics and Computation 275* (2016), 345–352.

[229] Tu, J. M., Tian, S. F., Xu, M. J., and Zhang, T. T. Quasi-periodic waves and solitary waves to a generalized KdV-Caudrey-Dodd-Gibbon equation from fluid dynamics. *Taiwanese Journal of Mathematics 20*, 4 (2016), 823–848.

[230] Turkowski, P. Solvable Lie algebras of dimension six. *Journal of Mathematical Physics 31*, 6 (1990), 1344–1350.

[231] Wadati, M. Wave propagation in nonlinear lattice. I. *Journal of the Physical Society of Japan 38*, 3 (1975), 673–680.

[232] Wadati, M., Sanuki, H., and Konno, K. Relationships among inverse method, Bäcklund transformation and an infinite number of conservation laws. *Progress of Theoretical Physics 53*, 2 (1975), 419–436.

[233] Wang, G., and Kara, A. Nonlocal symmetry analysis, explicit solutions and conservation laws for the fourth-order Burgers equation. *Chaos, Solitons & Fractals 81* (2015), 290–298.

[234] Wang, G. W., Liu, X. Q., and Zhang, Y. Y. Symmetry reduction, exact solutions and conservation laws of a new fifth-order nonlinear integrable equation. *Communications in Nonlinear Science and Numerical Simulation 18*, 9 (2013), 2313–2320.

[235] Wang, M., Li, X., and Zhang, J. The $\left(\frac{G'}{G}\right)$–expansion method and travelling wave solutions of nonlinear evolution equations in mathematical physics. *Physics Letters A 372*, 4 (2008), 417–423.

[236] Wang, X. B., Tian, S. F., Qin, C. Y., and Zhang, T. T. Dynamics of the breathers, rogue waves and solitary waves in the (2+1)-dimensional Ito equation. *Applied Mathematics Letters 68* (2017), 40–47.

[237] Wang, X. B., Tian, S. F., Xua, M. J., and Zhang, T. T. On integrability and quasi-periodic wave solutions to a (3+1)-dimensional generalized KdV-like model equation. *Applied Mathematics and Computation 283* (2016), 216–233.

[238] Wang, Y., and Chen, Y. Binary Bell polynomial manipulations on the integrability of a generalized (2+1)-dimensional Korteweg–de Vries equation. *Journal of Mathematical Analysis and Applications 400*, 2 (2013), 624–634.

[239] Wang, Y., Temuer, C., and Yang, Y. Integrability for the generalised variable-coefficient fifth-order Korteweg–de Vries equation with Bell polynomials. *Applied Mathematics Letters 29* (2014), 13–19.

[240] Wang, Y. H., Wang, H., and Temuer, C. Lax pair, conservation laws, and multi-shock wave solutions of the DJKM equation with Bell polynomials and symbolic computation. *Nonlinear Dynamics 78*, 2 (2014), 1101–1107.

[241] Wazwaz, A. M. Multiple-soliton solutions for the Calogero–Bogoyavlenskii–Schiff, Jimbo–Miwa and YTSF equations. *Applied Mathematics and Computation 203*, 2 (2008), 592–597.

[242] Wazwaz, A. M. Multiple-soliton solutions for a (3+1)-dimensional generalized KP equation. *Communications in Nonlinear Science and Numerical Simulation 17*, 2 (2012), 491–495.

[243] Wazwaz, A. M. The generalized Kaup–Boussinesq equation: Multiple soliton solutions. *Waves in Random and Complex Media 25*, 4 (2015), 473–481.

[244] Wazwaz, A. M. A seventh-order member of KdV6 hierarchy and its (2+1)-dimensional extensions. *Modern Physics Letters B 30*, 17 (2016), 1650198.

[245] Wazwaz, A. M. The simplified Hirota method for studying three extended higher-order KdV-type equations. *Journal of Ocean Engineering and Science 1*, 3 (2016), 181–185.

[246] Wazwaz, A. M. A variety of (3+1)-dimensional KdV equations derived by using the KdV recursion operator. *Indian Journal of Physics 90*, 5 (2016), 577–582.

[247] Weiss, J. The Painlevé property of partial differential equations. II : Bäcklund transformation, Lax pair, and Schwarzian derivative. *Journal of Mathematical Physics 24*, 6 (1983), 1405–1413.

[248] Weiss, J. Bäcklund transformation and the Painlevé property. *Journal of Mathematical Physics 27*, 5 (1986), 1293–1305.

[249] Weiss, J. Bäcklund transformations and the Painlevé property. In *Partially Intergrable Evolution Equations in Physics*. Springer, 1990, pp. 375–411.

[250] Weiss, J., Tabor, M., and Carnevale, G. The Painlevé property for partial differential equations. *Journal of Mathematical Physics 24*, 3 (1983), 522–526.

[251] Winternitz, P. Lie groups and solutions of nonlinear differential equations. In *Nonlinear Phenomena*. Springer, Berlin, Heidelberg, 1983, pp. 263–305.

[252] Xu, G. The soliton solutions, dromions of the Kadomtsev–Petviashvili and Jimbo–Miwa equations in (3+1)-dimensions. *Chaos, Solitons & Fractals 30*, 1 (2006), 71–76.

[253] Xu, G. Q., and Wazwaz, A. M. New bilinearization, Bäcklund transformation and infinite conservation laws for the KdV6 equation with Bell polynomials. *Mathematical Methods in the Applied Sciences 39*, 10 (2015), 2716–2721.

[254] Ya Ning, T., Wen-Xiu, M., and Wei, X. Grammian and Pfaffian solutions as well as Pfaffianization for a (3+1)-dimensional generalized shallow water equation. *Chinese Physics B 21*, 7 (2012), 70212–70218.

[255] Yan, Z., and Liu, X. Symmetry and similarity solutions of variable coefficients generalized Zakharov–Kuznetsov equation. *Applied Mathematics and Computation 180*, 1 (2006), 288–294.

[256] Yan, Z., Zhou, J., and Liu, X. Symmetry reductions and similarity solutions of the (3+1)-dimensional breaking soliton equation. *Applied Mathematics and Computation 201*, 1 (2008), 333–339.

[257] Yang, L., Yang, K., and Luo, H. Complex version KdV equation and the periods solution. *Physics Letters A 267*, 5 (2000), 331–334.

[258] Yaşar, E., and Özer, T. Conservation laws for one-layer shallow water wave systems. *Nonlinear Analysis: Real World Applications 11*, 2 (2010), 838–848.

[259] Yu, X., Gao, Y., Sun, Z., and Liu, Y. N-soliton solutions, Bäcklund transformation and Lax pair for a generalized variable-coefficient fifth-order Korteweg–de Vries equation. *Physica Scripta 81*, 4 (2010), 045402–06.

[260] Zhang, S., and Xia, T. A further improved extended Fan sub-equation method and its application to the (3+1)-dimensional Kadomstev–Petviashvili equation. *Physics Letters A 356*, 2 (2006), 119–123.

[261] Zhang, T. T., Ma, P. L., Xu, M. J., Zhang, X. Y., and Tian, S. F. On Bell polynomials approach to the integrability of a (3+1)-dimensional generalized Kadomtsev–Petviashvili equation. *Modern Physics Letters B 29*, 12 (2015), 1550051.

[262] Zhang, Y., Lv, Y., Ye, L., and Zhao, H. The exact solutions to the complex KdV equation. *Physics Letters A 367*, 6 (2007), 465–472.

[263] Zhen, H. L., Tian, B., Sun, Y., Chai, J., and Wen, X. Y. Solitons and chaos of the Klein-Gordon-Zakharov system in a high-frequency plasma. *Physics of Plasmas (1994-present) 22*, 10 (2015), 102304.

[264] Zhou, Q., Liu, L., Zhang, H., Mirzazadeh, M., Bhrawy, A. H., Zerrad, E., Moshokoa, S., and Biswas, A. Dark and singular optical solitons with competing nonlocal nonlinearities. *Optica Applicata 46*, 1 (2016), 79–86.

[265] Zhou, Q., Mirzazadeh, M., Zerrad, E., Biswas, A., and Belic, M. Bright, dark, and singular solitons in optical fibers with spatio-temporal dispersion and spatially dependent coefficients. *Journal of Modern Optics 63*, 10 (2016), 950–954.

[266] Zhou, Q., Zhong, Y., Mirzazadeh, M., Bhrawy, A. H., Zerrad, E., and Biswas, A. Thirring combo-solitons with cubic nonlinearity and spatio-temporal dispersion. *Waves in Random and Complex Media 26*, 2 (2016), 204–210.

[267] Zhou, Y., Wang, M., and Wang, Y. Periodic wave solutions to a coupled KdV equations with variable coefficients. *Physics Letters A 308*, 1 (2003), 31–36.

[268] Zhu, S. D. The generalizing Riccati equation mapping method in non-linear evolution equation: Application to (2+1)-dimensional Boiti–Leon–Pempinelle equation. *Chaos, Solitons & Fractals 37*, 5 (2008), 1335–1342.