COMBINATORICS OF MOCK THETA FUNCTIONS AND q-SERIES

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by

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Declaration

I hereby declare that thesis titled *"Combinatorics of Mock Theta Functions and q-Series"* submitted for the award of the degree of Doctor of Philosophy in the School of Mathematics, Thapar Institute of Engineering and Technology, Patiala, is true and original record of my own independent and original research work carried out under the supervision of Dr. Meenakshi Rana, Associate Professor at School of Mathematics, Thapar Institute of Engineering and Technology, Patiala, India. The matter embodied in this thesis has not been submitted in part or full to any other university or institute for the award of any degree in India or abroad and that the ideas and references cited herein have been duly acknowledged.

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Certificate

This is to certify that the thesis titled *"Combinatorics of Mock Theta Functions and q-Series"* which is being submitted by Ms. Shruti, in fulfillment of the requirement for the award of the degree of Doctor of Philosophy in the School of Mathematics, Thapar Institute of Engineering and Technology, Patiala, is a record of the candidate's own independent and original research work carried out under my supervision. The matter embodied in this thesis has not been submitted in part or full to any university or institute for the award of a degree.

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List of Publications

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3. Sharma, S. and Rana, M. On mock theta functions and weight-attached Frobenius partitions. *The Ramanujan Journal*, 50(2):289-303, 2019.
4. Sharma, S. and Rana, M. Interpreting some fifth and sixth order mock theta functions by attaching weights. *Journal of the Ramanujan Mathematical Society*, 34(4):401-410, 2019.
5. Sharma, S. and Rana, M. A simple combinatorial treatment for the q-series interpreted by Agarwal (Accepted). *Utilitas Mathematica.*
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Abstract

This thesis predominantly studies mock theta functions combinatorially. However, there are some chapters dwelling into the combinatorics of q-series also. The combinatorial tools employed in these studies are $(n+t)$–color partitions (A. K. Agarwal and G. E. Andrews. Rogers-Ramanujan identities for partitions with "N copies of N". *Journal of Combinatorial Theory, Series A*, 45:40–49, 1987), generalized Frobenius partitions (G. E. Andrews. Generalized Frobenius partitions. *American Mathematical Society*, 301, 1984) and lattice paths (A. K. Agarwal and D. M. Bressoud. Lattice paths and multiple basic hypergeometric series. *Pacific Journal of Mathematics*, 136:209–228, 1989).

Here, we find interpretations of twenty five mock theta functions by employing above mentioned tools and a novel idea of attaching weights to the partitions generated by the unsigned version of mock theta functions. To obtain interpretations of some mock theta functions in terms of lattice paths, the terminology of paths given by Agarwal and Bressoud has been modified by introducing backward horizontal steps. With these modifications the formed lattice paths naturally correspond to the n–color compositions.

In addition to above, we provide combinatorial interpretations of some generalized q-series. Firstly, combinatorial interpretations of seven generalized q-series are obtained which have been earlier interpreted in terms of split $(n+t)$–color partitions. Secondly, we explore the concept of hook differences (Andrews, G. E., Baxter, R., Bressoud, D. M., Burge, W. H., Forrester, P. and Viennot, G. Partitions with prescribed hook differences. *European Journal of Combinatorics*, 8(4):341350, 1987) which led to the generalization of the successive rank theorem to an identity involving partitions with prescribed hook differences. This identity involves a complex product but in some particular cases, it reduces to a simple triple or quintuple product. Our investigation in this thesis shows that one of these reductions involving triple product provides $(n+t)$–color analogue of Gordon's theorem which is also a generalization of the result of Agarwal and Andrews (A. K. Agarwal and G. E. Andrews. Rogers-Ramanujan identities for partitions with “N copies of N”. *Journal of Combinatorial Theory, Series A*, 45:40–49, 1987). A similar manipulation of another such reduction involving quintuple product provides $(n+t)$–color partitions for a quintuple product, that initially arose in an identity due to Sills.

Contents

Chapter 1

Introduction

The origin of the theory of q-series perhaps dates back to Euler [59] and Gauss [62]. In the absence of a satisfactory definition for q-series, we agree to define a q-series as the one which contains summands of the form

$$(a;q)_n := (1-a)(1-aq)\dots(1-aq^{n-1}) \text{ for } n \geq 0,$$

where $(a;q)_0 := 1$. Also, there are frequent occurrences of the infinite q-products

$$(a;q)_\infty := \lim_{n\to\infty} (a;q)_n,$$

in the theory of theta functions and q-series. We, here, consider the general theta function $f(a,b)$ as defined by Ramanujan.

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{\frac{n(n+1)}{2}} b^{\frac{n(n-1)}{2}}, \ |ab| < 1. \tag{1.1}$$

The theory of theta functions probably began with Gauss. The most beautiful theorem of the theory of theta functions is the Jacobi triple product identity given by

$$f(a;b) = (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty, \ |ab| < 1. \tag{1.2}$$

For $a = -q$ and $b = -q^2$, (1.1) reduces to

$$f(-q;q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{n(3n-1)}{2}}.$$

Using (1.2),

$$f(-q;q^2) = (q;q^3)_\infty (q^2;q^3)_\infty (q^3;q^3)_\infty = (q;q)_\infty.$$

Thus, we obtain the following famous theorem due to Euler which is known as Euler's pentagonal number theorem:

$$\sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{n(3n-1)}{2}} = (q;q)_\infty. \tag{1.3}$$

Euler's pentagonal number theorem is perhaps the first instance where the combinatorics of q-series appears. Although this theorem is as old as the theory of q-series but it gained importance only after Sylvester's new proof [114], Franklin's combinatorial proof [61] and MacMahon's proof [82]. Sylvester is considered to be the founder of modern combinatorial theory of partitions. In this thesis, most of the combinatorial tools employed to study combinatorics of q-series are also partition-theoretic.

1.1 Partitions and q-series identities

The concept of partitions first arose when Leibnitz asked Bernoulli about the number of expressions of a positive integer as the sum of other positive integers. To put it more formally, a *partition* of a positive integer s is a nonincreasing sequence of positive integers whose sum is s. For example, there are 7 partitions of 5 given by: 5, 4+1, 3+2, 3+1+1, 2+2+1, 2+1+1+1, 1+1+1+1+1. Euler was the first to establish the generating function for the partition function $p(s)$ ($p(s)$:=no. of partitions of s) as

$$\sum_{s=0}^{\infty} p(s) q^s = \frac{1}{(q;q)_\infty}.$$

Now, we are ready to dwell into some combinatorics of q-series and we start with the combinatorial version of the Euler's pentagonal number theorem.

Theorem 1.1.1. *Let $D_e(s)$ denote the number of partitions of s into an even number of distinct parts and $D_o(s)$ denote the number of partitions of s into an odd number of distinct parts. Then*

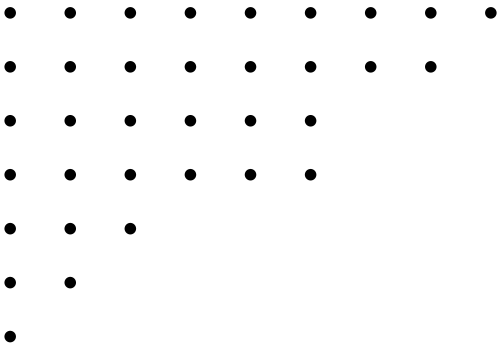
$$D_e(s) - D_o(s) = \begin{cases} (-1)^j & \text{if } n = j(3j \pm 1)/2, \\ 0 & \text{otherwise.} \end{cases}$$

Euler also found the following two identities for $\frac{1}{(q;q)_\infty}$:

$$\sum_{n=0}^{\infty} \frac{q^n}{(q;q)_n} = \frac{1}{(q;q)_\infty}, \tag{1.4}$$

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n^2} = \frac{1}{(q;q)_\infty}. \tag{1.5}$$

To prove identity (1.4), observe that $\frac{1}{(q;q)_n}$ generates partitions into parts $\leq n$ and q^n generates n. In this way, $\frac{q^n}{(q;q)_n}$ provides all the partitions with n as the largest part and the result follows by summing over all possible values of n. To prove identity (1.5), we need to know more about partitions, especially the graphical representation of a partition by Ferrers graph. *Ferrers graph* of a partition $\lambda = \lambda_1 + \lambda_2 + \cdots + \lambda_k$ is a set of rows of left aligned and equally spaced dots, where i^{th} row has λ_i dots. For example, Ferrers graph of the partition 9+8+6+6+3+2+1 is



The conjugate λ^c of a partition λ is obtained by interchanging the rows and columns of Ferrers graph. Thus, the conjugate of 9+8+6+6+3+2+1 is 7+6+5+4+4+4+2+2+1.

The largest square of dots contained in the Ferrers graph of a partition is called *Durfee square.* It is quite straightforward to see that every partition can be uniquely written as $d^2 + \lambda_1 + \lambda_2$, where d^2 is the number of dots in the Durfee square, λ_1 is the partition corresponding to the dots below Durfee square and λ_2 is the conjugate of the partition corresponding to the dots in the right of the Durfee square. Thus, both λ_1 and λ_2 are partitions into parts not exceeding d and the generating function for such partitions is given by $\frac{1}{(q;q)_d}$. Therefore, the following function generates all the partitions with Durfee square of size d:

$$\frac{q^{d^2}}{(q;q)_d^2}.$$

Summing over all the possible values of d, we obtain (1.5). Another very simple and useful q-identity which admits a simple combinatorial interpretation is the following theorem due to Euler:

$$\frac{1}{(q;q^2)_\infty} = (-q;q)_\infty. \tag{1.6}$$

It equates the count of partitions of a positive integer n into odd parts to the count of partitions of n into distinct parts. The two most famous q-series identities which influenced research in many areas of mathematics and physics are the Rogers-Ramanujan identities given by

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} = \prod_{n=0}^{\infty} (1 - q^{5n-1})^{-1}(1 - q^{5n-4})^{-1}, \tag{1.7}$$

$$\sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q;q)_n} = \prod_{n=0}^{\infty} (1 - q^{5n-2})^{-1}(1 - q^{5n-3})^{-1}. \tag{1.8}$$

These identities were first discovered by Rogers [96] but were appreciated only after Ramanujan rediscovered these in 1913. Some of the useful texts on the history of these identities are found in [24, 27, 90]. MacMahon [83] provided the partition-theoretic interpretations of the Rogers-Ramanujan identities given by (1.7) and (1.8) in the following theorems respectively:

Theorem 1.1.2. *The number of partitions of s into parts with minimal difference* 2 *equals the number of partitions of s into parts which are congruent to* $\pm 1 \pmod 5$.

Theorem 1.1.3. *The number of partitions of s with minimal part 2 and minimal difference 2 equals the number of partitions of s into parts which are congruent to* $\pm 2 \pmod 5$.

In 1961, Gordon [64] extended the combinatorial version of the Rogers-Ramanujan identities to arbitrary odd modulus of minimum value 5.

Theorem 1.1.4. *For $1 \leq i \leq k$, $k \geq 2$, let $A_{k,i}(s)$ denote the number of partitions of s of the form $(a_1, a_2, \ldots, a_r)$, where $a_j - a_{j+k-1} \geq 2$ and at most $i-1$ of the a_j equal 1. Let $B_{k,i}(s)$ denote the number of partitions of s into parts $\not\equiv 0, \pm i \pmod{2k+1}$. Then $A_{k,i}(s) = B_{k,i}(s)$ for all s.*

Thirteen years later, Andrews [21] found the following analytic version of the Gordon's theorem or the generalized version of Rogers-Ramanujan identities:

Theorem 1.1.5. *For $1 \leq i \leq k$, $k \geq 2$, $|q| < 1$*

$$\sum_{n_1, n_2, \ldots, n_{k-1} \geq 0} \frac{q^{N_1^2 + N_2^2 + \cdots + N_{k-1}^2 + N_i + N_{i+1} + \cdots + N_{k-1}}}{(q;q)_{n_1} (q;q)_{n_2} \cdots (q;q)_{n_{k-1}}} = \prod_{\substack{n=1 \\ n \not\equiv 0, \pm i \pmod{2k+1}}}^{\infty} (1-q^n)^{-1}, \tag{1.9}$$

where $N_j = n_j + n_{j+1} + \cdots + n_{k-1}$.

Bressoud [40] further generalized this result to include even modulus also. Apart from Rogers-Ramanujan identities, there are many other q-series identities which have been studied combinatorially in terms of partitions [9, 53, 63, 65, 77, 112, 113].

1.2 Frobenius representation and hook differences

As explained in Section 1.1, the conjugate of a partition is obtained by interchanging rows and columns in the corresponding Ferrers graph. Frobenius wanted to express the partitions in such a way that the conjugate can be written without drawing the Ferrers graph. However, he made use of the Ferrers graph to obtain such an expression. He deleted the main diagonal of, say, r dots from Ferrers graph of a partition λ of s. Then, he enumerated the dots to the right of the main diagonal in rows and below the main diagonal in columns. Thus, he obtained two strictly decreasing sequences, say, $(c_1, c_2, \ldots, c_r)$ and $(d_1, d_2, \ldots, d_r)$ of nonnegative integers and represented these by a $2 \times r$ array as follows

$$\begin{pmatrix} c_1 & c_2 & \cdots & c_r \\ d_1 & d_2 & \cdots & d_r \end{pmatrix},$$

where

$$s = r + \sum_{i=1}^{r} c_i + \sum_{i=1}^{r} d_i.$$

This representation of a partition is known as *Frobenius representation.* For example, the Frobenius representation for the partition 9+8+6+6+3+2+1 is

$$\begin{pmatrix} 8 & 6 & 3 & 2 \\ 6 & 4 & 2 & 0 \end{pmatrix}.$$

It can be very easily seen that the conjugate of a partition can be obtained by simply interchanging the two rows in the Frobenius representation.

In order to prove the following partition congruences of Ramanujan, Dyson [56] introduced a new partition statistic named rank:

$$\begin{aligned} p(5s+4) &\equiv 0 \pmod{5}, \\ p(7s+5) &\equiv 0 \pmod{7}, \end{aligned}$$

The largest part minus the number of parts of a partition is defined as the *rank* of partition. Thus, for example, the rank of above partition is 2. Later in [33], Atkin extended this concept by defining successive ranks. *Successive ranks* are defined at the main diagonal of the Ferrers graph of a partition and the successive rank at the i^{th} node on the main diagonal is the number of nodes in the i^{th} row minus the number of nodes in the i^{th} column. These can be obtained by simply subtracting the second element from the first element in each column of the Frobenius representation of a partition. In [58], Andrews et al. further extended the concept of successive ranks to every node in the Ferrers graph and termed these as hook differences.

Definition 1.2.1. *Let π be a partition whose Ferrers graph has a node in the i^{th} row and j^{th} column, call this the $(i,j)^{th}$ node. The hook difference at the $(i,j)^{th}$ node is the number of nodes in the i^{th} row of π minus the number of nodes in the j^{th} column of π.*

Definition 1.2.2. *The $(i,j)^{th}$ node is said to lie on diagonal d if $i-j=d$.*

1.3 $(n+t)$–color partitions

In [58], Andrews et al. proved a partition identity involving hook differences which generalizes the successive ranks theorem (discussed in detail in Chapter 8). Agarwal and Andrews [10], considered a special case of the general identity proved in [58] and rephrased the conditions on hook differences in terms of the conditions on the corresponding Frobenius symbol. They further mapped these partitions to certain restricted partitions from a new class of partitions introduced by them which are known as $(n+t)$–color partitions.

Definition 1.3.1. *An $(n+t)$–color partition, $t \geq 0$ (also called a partition with "$(n+t)$ copies of n") is a partition in which a part of size n, $n \geq 0$, can occur in $(n+t)$ different colors denoted by $n_1, n_2, \ldots, n_{n+t}$. Note that for $t > 0$ zeros can occur but cannot repeat.*

Example 1.3.1. *The partitions of 2 with "$n+1$ copies of n" are given by*

$$2_3,\ 2_3+0_1,\ 1_2+1_2,\ 1_2+1_2+0_1,$$

$$2_2,\ 2_2+0_1,\ 1_2+1_1,\ 1_2+1_1+0_1,$$

$$2_1,\ 2_1+0_1,\ 1_1+1_1,\ 1_1+1_1+0_1.$$

Remark 1.3.1. *The parts of $n-$color partitions follow the lexicographic order as given below:*

$$1_1 < 2_1 < 2_2 < 3_1 < 3_2 < 3_3 < 4_1 < 4_2 < 4_3 < 4_4 < \cdots .$$

Definition 1.3.2. *The weighted difference of two parts $(a_i)_{b_i}$ and $(a_j)_{b_j}(a_i \geq a_j)$ in an $(n+t)-$color partition $(a_1)_{b_1}+(a_2)_{b_2}+\cdots+(a_p)_{b_p}$ such that $(a_1)_{b_1} \geq (a_2)_{b_2} \geq \cdots \geq (a_p)_{b_p}$, is $a_i - b_i - a_j - b_j$ and denoted by $(w.d)_{i,j}$.*

Remark 1.3.2. *The original notation for weighted difference in* [10] *is $(((a_i)_{b_i} - (a_j)_{b_j}))$. For brevity we have replaced it by $(w.d)_{i,j}$.*

The results proved in [10] are considered as Rogers-Ramanujan identities for $(n+t)$–color partitions. In Chapter 8, we further generalize these results to obtain $(n+t)$–color analogue of Gordon's theorem.
In [11], Agarwal et al. succeeded in finding analytic counterpart of the theorem proved in [10], which came out to be a multiple summation Rogers-Ramanujan type identity. Identities similar in form to Rogers-Ramanujan identities are known as Rogers-Ramanujan type identities. Many Rogers-Ramanujan type identities were obtained by Rogers in [96, 97]. Bailey simplified and generalized the idea of Rogers in [34, 35] and obtained a number of identities of Rogers-Ramanujan type. Slater [107, 108] and Bailey only considered single summation and infinite product identities. However, Andrews [21] was the first to discover multiple summation Rogers-Ramanujan type identity while studying the analytic counterpart to Gordon's combinatorial generalization of Rogers-Ramanujan identities. Thereafter a number of authors [26, 40, 104, 105, 106, 116, 117] obtained multiple summation Rogers-Ramanujan type identities.

1.4 Lattice paths

In the generalized Rogers-Ramanujan identity, given by (1.9), the product side clearly generates partitions into parts $\not\equiv 0, \pm i \pmod{2k+1}$ but the interpretation of summation side in terms of Gordon's frequency conditions is not at all obvious. Andrews obtained its interpretation in terms of successive Durfee squares [23] and Bressoud [40] found a complex bijection between the partitions with successive Durfee square condition and partitions with Gordon's frequency conditions. Andrews [20] also proved that the partitions of s into parts $\not\equiv 0, \pm i \pmod{2k+1}$ are equinumerous with those partitions of s whose successive ranks lie in the interval $[-i+2, 2k-i-1]$. The connections between these identities were clarified by Burge in [45, 46] along with the method to interpret other multiple summation identities. Agarwal and Bressoud [12, 42] rephrased the idea of Burge in the form of lattice paths and used this technique to obtain the interpretations of a number of multiple summation series in terms of lattice paths. In order to connect lattice paths to overpartitions [54], Corteel and Mallet [55] modified the terminology of lattice paths given by Agarwal and Bressoud.

Definition 1.4.1. [12, 54] *Lattice paths of finite length lying in the first quadrant, beginning on the Y–axis and terminating on the X–axis with the following steps are considered:*

North–East NE($\nearrow$): *from* (i,j) *to* $(i+1, j+1)$.

South–East SE($\searrow$): *from* (i,j) *to* $(i+1, j-1)$, *only allowed if* $j > 0$.

South S($\downarrow$): *from* (i,j) *to* $(i, j-1)$, *only allowed if* $j \geq 1$ (*S step can only occur after a* NE *step*)

Horizontal H($\rightarrow$): *from* $(i,0)$ *to* $(i+1, 0)$.

All our lattice paths are either empty or terminate with a southeast step from $(i,1)$ *to* $(i+1,0)$ *or south step from* $(i,1)$ *to* $(i,0)$.

In describing the lattice paths, following terminology is used:

Peak: *A vertex preceded by a North–East step and followed by a South–East step* (NESE *peak*) *or by a South step* (NES *peak*).

Valley: *A vertex preceded by a* S *step or* SE *step and followed by a* NE *step. Note that a* S *step or* SE *step followed by* H *step followed by a* NE *step does not constitute a valley.*

Mountain: *A section of the path which starts on either the X–axis or Y–axis, which ends on the X–axis and which does not touch the X–axis anywhere in between the end points. Every mountain has at least one peak and may have more than one.*

Plain: *A section of the path consisting of only* H *steps which starts on the Y–axis or*

at a vertex preceded by a SE *step or at a vertex preceded by a* S *step and ends at a vertex followed by a* NE *step.*
Height *of a vertex is its* Y*–coordinate.*
Weight *of a vertex is its* X*–coordinate.*
Weight of a Lattice Path *is the sum of the weights of its peaks. We will denote weight of a path* P *by* $|P|$*.*

If there is no south step then these paths are same as the paths introduced by Agarwal and Bressoud. Agarwal and Bressoud also linked their paths with restricted $(n+t)$–color partitions by encoding each path with the sequence of weights of peaks subscripted by the height of respective peak. Thus, we see that the path given by Figure 1-1 corresponds to the n–color partition $(1_1, 4_2, 7_3, 13_2, 17_4)$. This connection helped in obtaining the interpretation of many q-series identities in terms of $(n+t)$–color partitions [1, 2, 3, 4, 69] and lattice paths [13, 17, 101].

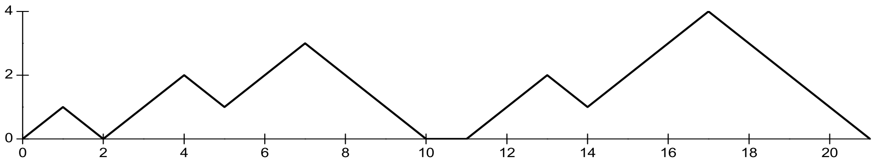


Figure 1-1 – lattice path corresponding to $(1_1, 4_2, 7_3, 13_2, 17_4)$

1.5 Mock theta functions

Mock theta functions introduced by Ramanujan in his Last Letter to Hardy are also considered to be certain special q-series. A formal definition of mock theta functions as given in [31] is:

Definition 1.5.1. *A mock theta function is a function* $F(q)$ *defined by q-series convergent for* $|q| < 1$ *such that*

(i) $F(q)$ *has infinitely many exponential singularities at roots of unity.*

(ii) *For every root of unity* ξ*, there exists a theta function* $\Theta_\xi(q)$ *for which* $F(q) - \Theta_\xi(q)$ *is bounded under the radial approach of* $q \to \xi$*.*

(iii) *A single theta function can not work for all* ξ i.e *there is no single theta function which differs from* $F(q)$ *by a trivial function.*

Ramanujan introduced 17 mock theta functions of order 3, 5 and 7 classified into four groups. Although Ramanujan did not say anything about order, it appears from the known identities between them that they are related to the numbers 3, 5 and 7. Therefore, the order of mock theta function can be considered to be a mere label for convenience. The first group of mock theta functions of order 3 consists of following mock theta functions:

$$f(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q;q)_n^2}, \tag{1.10}$$

$$\phi(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q^2;q^2)_n}, \tag{1.11}$$

$$\psi(q) := \sum_{n=1}^{\infty} \frac{q^{n^2}}{(q;q^2)_n}, \tag{1.12}$$

$$\chi(q) := \sum_{n=0}^{\infty} \frac{(-q;q)_n q^{n^2}}{(-q^3;q^3)_n}. \tag{1.13}$$

Ramanujan did not prove that these are actually mock theta functions. Watson [118] was the first to obtain modular transformation laws for Ramanujan's third order mock theta functions except $\chi(q)$. Watson was able to prove that third order mock theta functions are not theta functions with the help of these transformation laws. He also introduced following three mock theta functions of order 3:

$$\omega(q) := \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{(q;q^2)_{n+1}^2}, \tag{1.14}$$

$$\nu(q) := \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(-q;q^2)_{n+1}}, \tag{1.15}$$

$$\rho(q) := \sum_{n=0}^{\infty} \frac{(-q,q^2)_{n+1} q^{2n(n+1)}}{(q^3;q^6)_{n+1}}. \tag{1.16}$$

The transformation laws for $\chi(q)$ and $\rho(q)$ were proved by Gordon and McIntosh in

[67] and involve two more mock theta functions of order 3:

$$\xi(q) := 1 + 2\sum_{n=1}^{\infty} \frac{q^{6n(n-1)}}{(q;q^6)_n(q^5;q^6)_n}, \tag{1.17}$$

$$\sigma(q) := \sum_{n=1}^{\infty} \frac{q^{3n(n-1)}}{(-q;q^3)_n(-q^2;q^3)_n}. \tag{1.18}$$

In [119], Watson considered the fifth order mock theta functions of Ramanujan. Ramanujan divided his fifth order mock theta functions into the following two groups:

Group A

$$f_0(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q;q)_n}, \tag{1.19}$$

$$\phi_0(q) := \sum_{n=0}^{\infty} q^{n^2}(-q;q^2)_n, \tag{1.20}$$

$$\psi_0(q) := \sum_{n=1}^{\infty} q^{n(n+1)/2}(-q;q)_{n-1}, \tag{1.21}$$

$$F_0(q) := \sum_{n=0}^{\infty} \frac{q^{2n^2}}{(q;q^2)_n}, \tag{1.22}$$

$$\chi_0(q) := \sum_{n=0}^{\infty} \frac{q^n}{(q^{n+1};q)_n}. \tag{1.23}$$

Group B

$$f_1(q) := \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(-q;q)_n}, \tag{1.24}$$

$$\phi_1(q) := \sum_{n=0}^{\infty} q^{(n+1)^2}(-q;q^2)_n, \tag{1.25}$$

$$\psi_1(q) := \sum_{n=0}^{\infty} q^{n(n+1)/2}(-q;q)_n, \tag{1.26}$$

$$F_1(q) := \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{(q;q^2)_{n+1}}, \tag{1.27}$$

$$\chi_1(q) := \sum_{n=0}^{\infty} \frac{q^n}{(q^{n+1};q)_{n+1}}. \tag{1.28}$$

Watson was not able to obtain transformation laws for above functions, so he chose to handle these in a different manner. He proved the linear relations, asserted by

Ramanujan, among the functions of Group A and also established similar relations among functions of Group B. In [102, 103], Selberg dealt with the following functions of order 7 and proved that these are actually mock theta functions:

$$\mathscr{F}_0(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q^{n+1};q)_n}, \tag{1.29}$$

$$\mathscr{F}_1(q) := \sum_{n=1}^{\infty} \frac{q^{n^2}}{(q^{n};q)_n}, \tag{1.30}$$

$$\mathscr{F}_2(q) := \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q^{n+1};q)_{n+1}}. \tag{1.31}$$

Andrews discovery of Lost Notebook [91] triggered an extensive research on mock theta functions. Andrews in [22] pointed out two identities from Lost Notebook and remarked that the awareness of these results would have enabled Watson to prove that fifth order functions are actually mock theta functions. In fact, the Lost Notebook contains ten such identities which are termed as Mock Theta Conjectures. Andrews and Garvan [30] showed that the truth of Mock Theta Conjectures leads to the fact that corresponding functions are mock theta functions and reduced the problem to proving only two conjectures which were proved by Hickerson [76]. Hickerson [75] also found and proved mock theta conjectures for functions of order 7.

Ramanujan's Last Letter contained mock theta functions of odd orders only, however functions of even orders exist in the Lost Notebook. A second order mock theta function which appears several times in the Lost Notebook is given by:

$$\mu(q) := \sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2}(q;q^2)_n}{(-q^2;q^2)_n^2}. \tag{1.32}$$

Gordon and McIntosh [85] found two more of order 2 while establishing relationships between two families of mock theta functions:

$$A(q) := \sum_{n=0}^{\infty} \frac{q^{(n+1)^2}(-q;q^2)_n}{(q;q^2)_{n+1}^2} = \sum_{n=0}^{\infty} \frac{q^{(n+1)}(-q^2;q^2)_n}{(q;q^2)_{n+1}}, \tag{1.33}$$

$$B(q) := \sum_{n=0}^{\infty} \frac{q^{n(n+1)}(-q^2;q^2)_n}{(q;q^2)_{n+1}^2} = \sum_{n=0}^{\infty} \frac{q^{n}(-q;q^2)_n}{(q;q^2)_{n+1}}. \tag{1.34}$$

The first family contained the mock theta functions $A(q)$, $B(q)$, $\mu(q)$ and the second family contained the mock theta functions $U_0(q)$, $U_1(q)$, $V_0(q)$, $V_1(q)$ from the following eighth order functions obtained in [66]:

$$S_0(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}(-q;q^2)_n}{(-q^2;q^2)_n}, \tag{1.35}$$

$$S_1(q) := \sum_{n=0}^{\infty} \frac{q^{n(n+2)}(-q;q^2)_n}{(-q^2;q^2)_n}, \tag{1.36}$$

$$T_0(q) := \sum_{n=0}^{\infty} \frac{q^{(n+1)(n+2)}(-q^2;q^2)_n}{(-q;q^2)_{n+1}}, \tag{1.37}$$

$$T_1(q) := \sum_{n=0}^{\infty} \frac{q^{n(n+1)}(-q^2;q^2)_n}{(-q;q^2)_{n+1}}, \tag{1.38}$$

$$U_0(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}(-q;q^2)_n}{(-q^4;q^4)_n}, \tag{1.39}$$

$$U_1(q) := \sum_{n=0}^{\infty} \frac{q^{(n+1)^2}(-q;q^2)_n}{(-q^2;q^4)_{n+1}}, \tag{1.40}$$

$$V_0(q) := -1 + 2\sum_{n=0}^{\infty} \frac{q^{n^2}(-q;q^2)_n}{(q;q^2)_n}, \tag{1.41}$$

$$V_1(q) := \sum_{n=0}^{\infty} \frac{q^{(n+1)^2}(-q;q^2)_n}{(q;q^2)_{n+1}}. \tag{1.42}$$

The following two sixth order mock theta functions are also defined in the Lost Notebook:

$$\phi(q) := \sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2}(q;q^2)_n}{(-q;q)_{2n}}, \tag{1.43}$$

$$\psi(q) := \sum_{n=0}^{\infty} \frac{(-1)^{n-1} q^{n^2}(q;q^2)_{n-1}}{(-q;q)_{2n-1}}. \tag{1.44}$$

Ramanujan provided certain identities relating these functions to the following q-series which are also mock theta functions of order 6:

$$\rho(q) := \sum_{n=0}^{\infty} \frac{(-q;q)_n q^{\binom{n+1}{2}}}{(q;q^2)_{n+1}}, \tag{1.45}$$

$$\sigma(q) := \sum_{n=0}^{\infty} \frac{(-q;q)_n q^{\binom{n+2}{2}}}{(q;q^2)_{n+1}}, \tag{1.46}$$

$$\gamma(q) := \sum_{n=0}^{\infty} \frac{(q;q)_n q^{n^2}}{(q^3;q^3)_n}, \tag{1.47}$$

$$\lambda(q) := \sum_{n=0}^{\infty} \frac{(-1)^n q^n (q;q^2)_n}{(-q;q)_n}, \tag{1.48}$$

$$\mu(q) := \sum_{n=0}^{\infty} \frac{(-1)^n (q;q^2)_n}{(-q;q)_n}. \tag{1.49}$$

The series defining $\mu(q)$ does not converge, however the sequences of its even and odd partial sums converge. Thus, $\mu(q)$ is considered to be the average of these two. If we consider the summands of $\rho(q)$ and $\sigma(q)$ over the negative integers, we obtain $\frac{\lambda(q)}{2}$ and $\frac{\mu(q)}{2}$ respectively. In a similar examination of the summands of $\phi(q)$ and $\psi(q)$, the following two mock theta functions of order 6 are obtained:

$$\phi_-(q) := \sum_{n=1}^{\infty} \frac{q^n (-q;q)_{2n-1}}{(q;q^2)_n}, \tag{1.50}$$

$$\psi_-(q) := \sum_{n=1}^{\infty} \frac{q^n (-q;q)_{2n-2}}{(q;q^2)_n}. \tag{1.51}$$

These functions were also independently discovered by Berndt and Chan [39], and McIntosh [84]. Finally, the mock theta functions of order 10 in the Ramanujan's Lost Notebook are:

$$\phi(q) := \sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}}{(q;q^2)_{n+1}}, \tag{1.52}$$

$$\psi(q) := \sum_{n=0}^{\infty} \frac{q^{(n+1)(n+2)/2}}{(q;q^2)_{n+1}}, \tag{1.53}$$

$$X(q) := \sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2}}{(-q;q)_{2n}}, \tag{1.54}$$

$$\chi(q) := \sum_{n=0}^{\infty} \frac{(-1)^n q^{(n+1)^2}}{(-q;q)_{2n+1}}. \tag{1.55}$$

Ramanujan's eight linear relations connecting these functions were proved by Choi [47, 48, 49, 50]. A comprehensive survey of mock theta functions can be found in [68] where authors have also stated mock theta conjectures for the functions of order other than 5 and 7. The proofs of mock theta conjectures for functions of orders 2 and 3 are available in a recent article of McIntosh [86].

Apart from the study of analytic properties of mock theta functions, a number of mathematicians have invested their efforts in studying the combinatorial aspects of these functions. The most common combinatorial tool employed in the study of mock theta functions is partitions. The first and second mock theta conjectures discussed in [30] are based on the rank of partitions. Mock theta functions themselves may have simple partition-theoretic interpretations. In [19], Andrews, while providing enumerative proofs of certain q-identities, has proved that $\chi_0(q)$ (given by (1.23)) is the generating function for partitions with unique smallest part and largest part at most twice the smallest part. In a similar manner, $q\chi_1(q)$ (where $\chi_1(q)$ is given by(1.28))generates partitions with no part as large as twice the smallest part. Fine [60] interpreted third order mock theta function $\psi(q)$, given by (1.12), as a generating function for partitions into odd parts without gaps. A seventh order mock theta function $\mathscr{F}_1(q)$, given by (1.30), appeared in [28] where the author is studying partitions with initial repetitions. In [29], Andrews et al. while studying certain restricted partitions encountered third order mock theta functions $\omega(q)$, $\nu(q)$ and $\phi(q)$ given by (1.14), (1.15) and (1.11) respectively and the work done in this paper has been carried further by many authors (see [32], [36], [80]). In [51], Choi and Kim interpreted some third and sixth order mock theta functions in terms of n–color partitions and n–color overpartitions It is just a brief account of the extensive study done on the combinatorics of the mock theta functions.

In Chapters 2, 3, 4, 5 and 6, we explore the combinatorics of a number of mock theta functions using $(n+t)$–color partitions, generalized Frobenius partitions and lattice paths. In Chapter 6, we also modify the terminology of lattice paths by introducing backward horizontal step. With these modifications, we have been able to relate these paths to n–color compositions and thus providing new proofs to a number of results on n–color compositions. In Chapter 7, we obtain $(n+t)$–color partition-theoretic interpretations of some generalized q-series which have earlier interpretations in terms of split $(n+t)$–color partitions.

Chapter 2

Interpreting some mock theta functions in terms of $(n+t)$–color partitions and lattice paths

The study of mock theta functions in terms of $(n+t)$–color partitions started with the interpretations of the mock theta functions $\psi(q)$, $\phi_0(q)$, $F_0(q)$ and $\phi_1(q)$ (given by (1.12), (1.20), (1.22) and (1.25) respectively) by Agarwal [7]. He also interpreted the same functions using lattice paths [8]. This work was carried further by the interpretation of mock theta function $F_1(q)$ (given by (1.27)) in [16]. Recently, Sareen and Rana [100] studied tenth order mock theta functions $\phi(q)$ and $\psi(q)$ (given by (1.52) and (1.53)) in terms of $(n+t)$–color partitions.

In [99], Santos et al. introduced new two-line array representation for partitions and Brietzke et al. [43] studied mock theta functions interpreted by Agarwal [7] in terms of these two-line array representation. In addition, they studied eighth order mock theta functions $U_1(q)$, $V_0(q)$ and $V_1(q)$ (given by (1.40)–(1.42)) using this new tool. However, to interpret $V_0(q)$ and $V_1(q)$ they used pairs of partitions. $V_0(q)$ and $V_1(q)$ have also been studied in [18] using split $(n+t)$–color partitions and in [93] using signed partitions. Noticeably, none of the authors have interpreted these functions in terms of $(n+t)$–color partitions, which give rise to the obvious question: Is there any combinatorial interpretation of these functions in terms of $(n+t)$–color partitions? This chapter answers this question in affirmative by providing the interpretations of $V_0(q)$ and $V_1(q)$ in terms of $(n+t)$–color partitions. In addition, we provide the interpretations of third order mock theta function $\rho(q)$ (given by (1.16)), fifth order mock theta functions $\psi_0(q)$, $\psi_1(q)$ (given by (1.21), (1.26)) and sixth order mock theta functions $\rho(q)$, $\sigma(q)$ (given by (1.45), (1.46)). We further translate the $(n+t)$–color partition-theoretic interpretations of all these mock theta functions in terms of lattice paths.

2.1 $(n+t)$–color partition-theoretic interpretations

Remark 2.1.1. *Throughout this section $A_i(m,s)$ denote the partitions enumerated by $A_i(s)$ into exactly m parts.*

First of all, we interpret the third order mock theta function $\rho(q)$ in terms of $(n+t)$–color partitions. We use the symbol $\varrho(q)$ instead of $\rho(q)$ for this function, since we are also dealing with another mock theta function $\rho(q)$ of sixth order. To obtain the combinatorial interpretation of $\varrho(q)$, we first consider the following function:

$$\varrho_1(q) = \sum_{n=0}^{\infty} \frac{(-q;q^2)_{n+1} q^{2(n+1)(n+2)}}{(q^3;q^6)_{n+1}}.$$

Theorem 2.1.1. *For $s \geq 1$, let $A_1(s)$ denote the number of n–color partitions of s such that*

(i) $b_i > 1$ *and* $\equiv 0, 2 \pmod 3$, $1 \leq i \leq p$;

(ii) $a_p = b_p + 2$;

(iii) $(w.d)_{i,i+1} = 0$, $1 \leq i \leq p-1$.

Then

$$\sum_{s=1}^{\infty} A_1(s) q^s = \varrho_1(q).$$

Proof. We split the partitions enumerated by $A_1(m,s)$ into three classes. The partitions of the first class, second class, and the third class satisfy $b_p = 2$, $b_p = 3$, and $b_p > 4$ respectively. To transform the partitions of the first class, delete the least part 4_2 and subtract 4 from each of the remaining parts. The partitions obtained in this manner are all the partitions enumerated by $A_1(m-1, s-4m)$. From the partitions of the second class, delete the least part 5_3 and subtract 6 from each of the remaining parts to get all the partitions enumerated by $A_1(m-1, s-6m+1)$. For the partitions of the third class replace $(a_p)_{b_p}$ with $(a_p-3)_{b_p-3}$ and subtract 6 from each of the remaining parts. The transformed partitions are all the partitions enumerated by $A_1(m, s-6m+3)$. All these transformations are clearly reversible and thus lead to the following recurrence relation:

$$A_1(m,s) = A_1(m-1, s-4m) + A_1(m-1, s-6m+1) + A_1(m, s-6m+3). \quad (2.1)$$

Let

$$f_1(z,q) = \sum_{s=0}^{\infty}\sum_{m=0}^{\infty} A_1(m,s) z^m q^s. \quad (2.2)$$

Substituting for $A_1(m,s)$ from (2.1) into (2.2) and then simplifying, we get

$$f_1(z,q) = zq^4 f_1(zq^4,q) + zq^5 f_1(zq^6,q) + q^{-3} f_1(zq^6,q). \quad (2.3)$$

Substitute $f_1(z,q) = \sum_{n=0}^{\infty} \alpha_n(q) z^n$ into (2.3) and compare the coefficients of z^n on both sides to obtain

$$\alpha_n(q) = \frac{(1+q^{2n-1})q^{4n}}{1-q^{6n-3}} \alpha_{n-1}(q). \quad (2.4)$$

Iterating (2.4) n times and noting that $\alpha_0(q) = 1$, we get

$$\alpha_n(q) = \frac{(-q;q^2)_n q^{2n(n+1)}}{(q^3;q^6)_n}. \quad (2.5)$$

Therefore,

$$f_1(z,q) = \sum_{n=0}^{\infty} \frac{(-q;q^2)_n q^{2n(n+1)}}{(q^3;q^6)_n} z^n. \quad (2.6)$$

Now,

$$\begin{aligned}\sum_{s=0}^{\infty} A_1(s) q^s &= \sum_{s=0}^{\infty}\left(\sum_{m=0}^{\infty} A_1(m,s)\right) q^s \\ &= f_1(1,q) = \sum_{n=0}^{\infty} \frac{(-q;q^2)_n q^{2n(n+1)}}{(q^3;q^6)_n}\end{aligned}$$

$$= \ 1+\sum_{n=0}^{\infty}\frac{(-q;q^2)_{n+1}q^{2(n+1)(n+2)}}{(q^3;q^6)_{n+1}} = 1+\varrho_1(q).$$

□

Theorem 2.1.2. *Let $A_2(s)$ denote the number of $(n+2)$–color partitions of s such that*

(i) $b_i > 1$ *and* $\equiv\ 0,2 \pmod 3$*,* $1 \le i \le p$*;*

(ii) $b_p = a_p + 2$*;*

(iii) $(w.d)_{i,i+1} = 0$*,* $1 \le i \le p-1$*.*

Then

$$\sum_{s=0}^{\infty} A_2(s)q^s = \varrho(q).$$

Proof. We split the partitions enumerated by $A_2(m,s)$ into three classes. The partitions of the first class contain 0_2 as the least part. If we delete this part, we are left with all the partitions enumerated by $A_1(m-1,s)$. The least part of the partitions in the second class is 1_3. Deleting the least part 1_3 and subtracting 2 from each of the remaining parts we get all the partitions enumerated by $A_1(m-1,s-2m+1)$. The partitions in the third class satisfy $a_p > 2$ and are transformed by replacing $(a_p)_{b_p}$ with $(a_p-3)_{b_p-3}$ and subtracting 6 from each of the remaining parts. Thus, we get the partitions enumerated by $A_2(m,s-6m+3)$. Each transformation described above is completely reversible. Therefore, $A_2(m,s)$ satisfies the following recurrence relation:

$$A_2(m,s) = A_1(m-1,s) + A_1(m-1,s-2m+1) + A_2(m,s-6m+3). \tag{2.7}$$

Suppose that

$$f_2(z,q) = \sum_{s=0}^{\infty}\sum_{m=0}^{\infty} A_2(m,s)z^m q^s. \tag{2.8}$$

Substitute $A_2(m,s)$ from (2.7) to (2.8) and simplify to obtain

$$f_2(z,q) = zf_1(z,q) + zqf_1(zq^2,q) + q^{-3}f_2(zq^6,q). \tag{2.9}$$

Substitute $f_1(z,q)=\sum_{n=0}^{\infty}\alpha_n(q)z^n$, $f_2(z,q)=\sum_{n=0}^{\infty}\beta_n(q)z^n$ into (2.9) and compare the coefficients of z^n on both sides to obtain $\beta_0=0$, and for $n\geq 1$

$$\beta_n(q)=\frac{(1+q^{2n-1})}{1-q^{6n-3}}\alpha_{n-1}(q)=\frac{(-q;q^2)_n q^{2n(n-1)}}{(q^3;q^6)_n}. \tag{2.10}$$

Therefore,

$$f_2(z,q)=\sum_{n=1}^{\infty}\frac{(-q;q^2)_n q^{2n(n-1)}}{(q^3;q^6)_n}z^n=\sum_{n=0}^{\infty}\frac{(-q;q^2)_{n+1} q^{2n(n+1)}}{(q^3;q^6)_{n+1}}z^{n+1}. \tag{2.11}$$

Now,

$$\begin{aligned}\sum_{s=0}^{\infty}A_2(s)q^s=&\sum_{s=0}^{\infty}\left(\sum_{m=0}^{\infty}A_2(m,s)\right)q^s=f_2(1,q)\\=&\sum_{n=0}^{\infty}\frac{(-q;q^2)_{n+1}q^{2n(n+1)}}{(q^3;q^6)_{n+1}}.\end{aligned}$$

□

Remark 2.1.2. *For the rest of the theorems in this section, we will just include the proofs up to recurrence relations since the remaining steps are similar as in the proofs of* Theorems 2.1.1 *and* 2.1.2.

Theorem 2.1.3. *Let $A_3(s)$ denote the number of n–color partitions of s such that*

(i) $b_i=1,\ 1\leq i\leq p$;

(ii) $a_p=1$ *or* 2;

(iii) $(w.d)_{i,i+1}=-1$ *or* $0,\ 1\leq i\leq p-1$.

Then

$$\sum_{s=0}^{\infty}A_3(s)q^s=\psi_1(q).$$

Proof. Here, the partitions enumerated by $A_3(m,s)$ split into two classes. The partitions in the first class contain 1_1 as the least part. By deleting 1_1 and subtracting 1 each from rest of the parts, we get a partition of $s-m$ into $m-1$ parts in which the least part is again 1_1 or 2_1 and the weighted difference condition remains undisturbed.

Thus, the transformed partitions are enumerated by $A_3(m-1, s-m)$. The partitions in the second class contain 2_1 as the least part, deleting 2_1 and subtracting 2 from each of the remaining parts, we get the partitions enumerated by $A_3(m-1, s-2m)$. Thus, we get the recurrence relation

$$A_3(m, s) = A_3(m-1, s-m) + A_3(m-1, s-2m).$$

□

To interpret $\psi_0(q)$ consider the following form:

$$\psi_0(q) = 1 + \sum_{n=1}^{\infty} (-q;q)_{n-1} q^{\binom{n+1}{2}}.$$

Theorem 2.1.4. *Let $A_4(s)$ denote the number of n–color partitions of s such that*

(i) $b_i = 1,\ 1 \le i \le p$;

(ii) $a_p = 1$;

(iii) $(w.d)_{i,i+1} = -1$ *or* $0,\ 1 \le i \le p-1$.

Then

$$\sum_{s=0}^{\infty} A_4(s) q^s = \psi_0(q).$$

Proof. Split the partitions enumerated by $A_4(m, s)$ in two classes. For the partitions in the first class $a_{p-1} = 2$. By deleting the least part 1_1 and subtracting 1 from each of the remaining parts, we obtain the partitions enumerated by $A_4(m-1, s-m)$. For the partitions of the second class $a_{p-1} = 3$ and these are transformed by deleting the least part 1_1 and subtracting 2 from each of the remaining parts. Thus, we the get partitions enumerated by $A_4(m-1, s-2m+1)$. Therefore, the recurrence relation for $A_4(m, s)$ is given by

$$A_4(m, s) = A_4(m-1, s-m) + A_4(m-1, s-2m+1).$$

□

Theorem 2.1.5. *Let $A_5(s)$ denote the number of n–color partitions of s such that*

(i) $a_p = b_p$ *or* $b_p + 1$;

(ii) $(w.d)_{i,i+1} = -1$ *or* 0, $1 \leq i \leq p-1$.

Then

$$\sum_{s=0}^{\infty} A_5(s)q^s = \sum_{n=0}^{\infty} \frac{(-q;q)_n q^{\binom{n+1}{2}}}{(q;q^2)_n}.$$

Proof. We split the partitions enumerated by $A_5(m,s)$ into three classes. The first of these contains partitions with 1_1 as the least part and the partitions of this class are enumerated by $A_5(m-1, s-m)$. The partitions of the second class containing 2_1 as the least part can be enumerated by $A_5(m-1, s-2m)$ and the third class containing partitions with $b_p > 1$ is enumerated by $A_5(m, s-2m+1)$. Thus,

$$A_5(m,s) = A_5(m-1, s-m) + A_5(m-1, s-2m) + A_5(m, s-2m+1).$$

□

Theorem 2.1.6. *For $s \geq 1$, let $A_6(s)$ denote the number of n–color partitions of s such that*

(i) $a_p = b_p$;

(ii) $(w.d)_{i,i+1} = -1$ *or* 0, $1 \leq i \leq p-1$.

Then

$$\sum_{s=1}^{\infty} A_6(s)q^s = \sigma(q).$$

Proof. Split the partitions enumerated by $A_6(m,s)$ in two classes. The partitions in the first class are those with 1_1 as their least part and these are transformed by deleting 1_1 and subtracting 1 from each of the remaining parts. Thus, we obtain those partitions for which $a_p = b_p$ or $b_p + 1$ and the weighted difference conditions remain undisturbed, so these partitions are enumerated by $A_5(m-1, s-m)$. The second class contains partitions with $a_p > 1$. Transforming the partitions of this class by replacing the least part with $(a_p-1)_{b_p-1}$ and subtracting 2 from each of the remaining parts, the partitions of this class are enumerated by $A_6(m, s-2m+1)$. Therefore, the following recurrence relation for $A_6(m,s)$ is obtained:

$$A_6(m,s) = A_5(m-1, s-m) + A_6(m, s-2m+1).$$

□

Theorem 2.1.7. *Let $A_7(s)$ denote the number of $(n+1)$–color partitions of s such that*

(i) $b_p = a_p + 1$;

(ii) $(w.d)_{i,i+1} = -1$ *or* 0, $1 \le i \le p-1$.

Then

$$\sum_{s=0}^{\infty} A_7(s) q^s = \rho(q).$$

Proof. The partitions enumerated by $A_7(m,s)$ can be classified into two parts. The first class contains partitions with 0_1 as their least part, deleting the part 0_1 leaves us with the partitions enumerated by $A_5(m-1,s)$. For the partitions in the second class $a_p > 0$, replacing the least part $(a_p)_{b_p}$ with $(a_p-1)_{b_p-1}$ and subtracting 2 from each of the remaining parts, the partitions are transformed to the ones enumerated $A_7(m, s-2m+1)$. Thus, the recurrence relation for $A_7(m,s)$ is

$$A_7(m,s) = A_5(m-1,s) + A_7(m, s-2m+1).$$

□

Theorem 2.1.8. *Let $A_8(s)$ denote the number of n–color partitions of s such that*

(i) $a_p = b_p$ *or* $b_p + 1$;

(ii) $$(w.d)_{i,i+1} = \begin{cases} 0 & \text{if } a_i \equiv b_i,\ a_{i+1} \equiv b_{i+1} \pmod 2, \\ 2 & \text{if } a_i \not\equiv b_i,\ a_{i+1} \not\equiv b_{i+1} \pmod 2, \\ 1 & \text{otherwise.} \end{cases} \quad 1 \le i \le p-1$$

Then

$$\sum_{s=0}^{\infty} A_8(s)q^s = \frac{1}{2}\left(1+V_0(q)\right).$$

Proof. We split the partitions enumerated by $A_8(m,s)$ into three classes, the first containing partitions with 1_1 as the least part, the second containing partitions with 2_1 as the least part, and the third containing partitions with $b_p > 1$. The recurrence relation satisfied by $A_8(m,s)$ is

$$A_8(m,s) = A_8(m-1,s-2m+1) + A_8(m-1,s-4m+2) + A_8(m,s-2m+1).$$

□

Theorem 2.1.9. *For $s \geq 1$, let $A_9(s)$ denote the number of n–color partitions of s such that*

(i) $a_p = b_p$;

(ii) $$(w.d)_{i,i+1} = \begin{cases} 0 & \text{if } a_i \equiv b_i,\ a_{i+1} \equiv b_{i+1} \pmod 2, \\ 2 & \text{if } a_i \not\equiv b_i,\ a_{i+1} \not\equiv b_{i+1} \pmod 2, \quad 1 \leq i \leq p-1 \\ 1 & \qquad\text{otherwise.} \end{cases}$$

Then

$$\sum_{s=0}^{\infty} A_9(s)q^s = V_1(q).$$

Proof. Split the partitions enumerated by $A_9(m,s)$ in two classes depending upon the least part. The first class contains partitions with 1_1 as their least part, which on deleting this part and subtracting 2 from each of the remaining parts yield the partitions enumerated by $A_8(m-1,s-2m+1)$. The second class contains partitions with $a_p > 1$. Replacing $(a_p)_{b_p}$ with $(a_p-1)_{b_p-1}$ and subtracting 2 from each of the remaining parts , these partitions are enumerated by $A_9(m-1,s-2m+1)$. Therefore,

$$A_9(m,s) = A_8(m-1,s-2m+1) + A_9(m,s-2m+1).$$

□

We close this section with an example.

Example 2.1.1. *Coefficient of q^4 in*

$$\rho(q) = 1 + 2q + 3q^3 + 6q^4 + 7q^5 + \cdots ,$$

is 6 *and relevant* $(n+1)$*–color partitions* (*satisfying conditions of* Theorem 2.1.7) *are* 4_5, $4_3 + 0_1$, $4_4 + 0_1$, $3_1 + 1_1 + 0_1$, $3_2 + 1_1 + 0_1$, $3_1 + 1_2$.

2.2 Interpretations in terms of lattice paths

In this section, we provide the combinatorial interpretations in terms of lattice paths for all the mock theta functions discussed in Section 2.1. We provide detailed proof for one mock theta function and outline of proofs for the rest.

Theorem 2.2.1. *Let $L_1(s)$ denote the number of lattice paths of weight s starting at* $(0,2)$ *with all* NESE *peaks. The height of each peak is* > 1 *and* $\equiv 0, 2 \pmod 3$. *There is no valley above height* 0 *and no plain in the path. Then*

$$\sum_{s=0}^{\infty} L_1(s) q^s = \varrho(q).$$

Proof. In $\frac{(-q;q^2)_{n+1} q^{2n(n+1)}}{(q^3;q^6)_{n+1}}$, the factor $q^{2n(n+1)}$ generates the lattice path with $(n+1)$ NESE peaks of height 2 each, starting at $(0,2)$ and ending at $(4n+2, 0)$. The path starts with the two SE steps from $(0,2)$ to $(1,1)$ and $(1,1)$ to $(2,0)$. Also, there is no valley above height 0 and no plain in the path. Thus, the path contains $(n+1)$ peaks with the first peak at $(0,2)$. For $n = 3$, the path begins as shown in Fig 2-1. The factor $(-q;q^2)_{n+1}$ generates odd natural numbers $\leq (2n+1)$, say $u_1 \times 1, u_2 \times 3, \ldots, u_{n+1} \times (2n+1)$, u_i=0 or 1. This factor leads to increase in the height of i^{th} peak by u_{n-i+2}. Each increase by 1 in the height of a given peak increases its weight by 1 and weight of each of the subsequent peak by 2. The path of Fig 2-1 transforms to the path shown in Fig 2-2 for u_1=1, u_2=0, u_3=0, u_4=1. The factor $(q^3;q^6)_{n+1}^{-1}$ generates nonnegative multiples of $(6i-3)$, $1 \leq i \leq n+1$, say

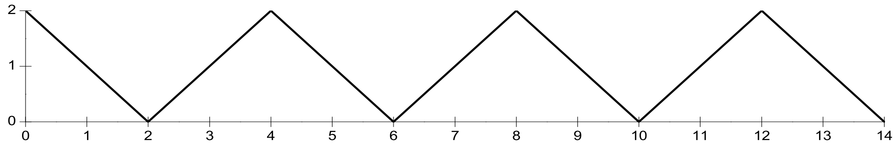


Figure 2-1 – Initial path for $n = 3$

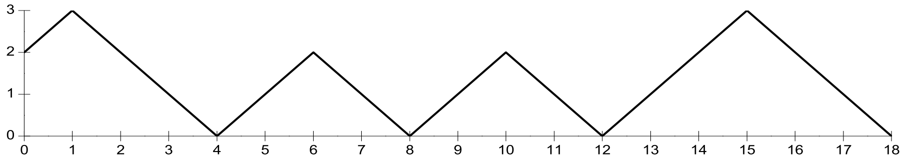


Figure 2-2 – Transformed path for u_1=1, u_2=0, u_3=0, u_4=1

$v_1 \times 3, v_2 \times 9, \ldots, v_{n+1} \times (6n+3)$, $v_i \geq 0$. This factor transforms the path by increasing the height of the i^{th} peak by $3v_{n-i+2}$. The path of Fig 2-2 transforms to the path shown in Fig 2-3 for v_1=2, v_2=0, v_3=1, v_4=0. Clearly, for the paths generated in this manner there is no valley above height 0 and no plain. Also, height of every peak is >1 and since u_i=0 or 1, height of each peak is $\equiv$ 0 or 2 (mod 3). Every path enumerated by $L_1(s)$ is generated in this manner and vice-versa. ☐

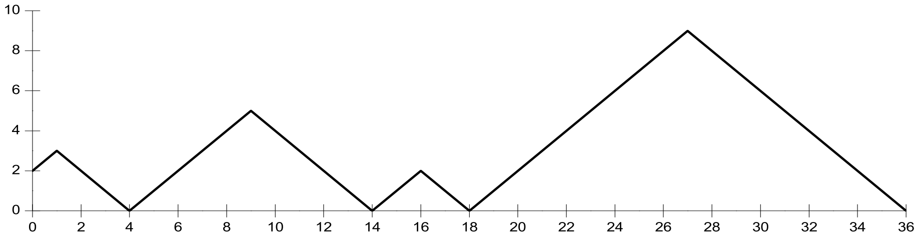


Figure 2-3 – Transformed path for v_1=2, v_2=0, v_3=1, v_4=0

Theorem 2.2.2. *For $s \geq 1$, $\psi_0(q)$ enumerates lattice paths of weight s starting at $(0,0)$ with all* NES *peaks. The height of each peak is* 1*. There is no valley above height* 0 *and no plain in front of the first peak. There is either no plain or a plain of length one in front of rest of the peaks.*

Proof. In the lattice paths generated by $\psi_0(q)$, the factor $q^{\frac{n(n+1)}{2}}$ generates a path starting at $(0,0)$ with n NES peaks of height 1 each and ending at $(n,0)$. The factor

$(-q,q)_{n-1}$ introduces plains of length 1 in front of some or all of the peaks except the first peak. ☐

Theorem 2.2.3. $\psi_1(q)$ *enumerates lattice paths of weight* s *starting at* $(0,0)$ *with all* NES *peaks. The height of each peak is* 1*. There is no valley above height* 0 *and either no plain or a plain of length one in front of the peaks.*

Proof. Here, the paths are generated in the same manner as for $\psi_0(q)$ with the only difference that there can be a plain of length one in front of the first peak also.

☐

Theorem 2.2.4. $\sigma(q)$ *enumerates the lattice paths of weight* s *starting at* $(0,0)$ *with all* NES *peaks. There is no valley above height* 0 *and no plain in front of the first peak. There is either no plain or a plain of length one in front of rest of the peaks.*

Proof. In the lattice paths generated by $\sigma(q)$, the factor $q^{\frac{(n+1)(n+2)}{2}}$ generates the lattice path starting at $(0,0)$ with $(n+1)$ NES peaks of height 1 each and ending at $(n+1,0)$. The factor $(-q;q)_n$ introduces plains in front of all the peaks except the first peak and the factor $(q;q^2)_{n+1}^{-1}$ increases the height of the peaks. ☐

Theorem 2.2.5. $\rho(q)$ *enumerates the lattice paths of weight* s *starting at* $(0,1)$ *with all* NES *peaks. There is no valley above height* 0 *and no plain in front of the first peak. There is either no plain or a plain of length one in front of rest of the peaks.*

Proof. The factor $q^{\frac{n(n+1)}{2}}$ generates a path starting at $(0,0)$ with n NES peaks of height 1 each and ending at $(n,0)$. In addition, we consider a dummy peak at $(0,1)$. The factor $(q;q^2)_{n+1}^{-1}$ increases the height of the peaks and the factor $(-q;q)_n$ introduces plains between the peaks. ☐

Example 2.2.1. *In* Example 2.1.1*, we see that the coefficient of* q^4 *in the expansion of* $\rho(q)$ *is* 6 *and six relevant lattice paths are given in* Fig 2-4.

Theorem 2.2.6. $\frac{1}{2}(1+V_0(q))$ *enumerates the lattice paths starting at* $(0,0)$ *with all* NESE *peaks. There is no valley above height* 0 *in the path. The length of plain between*

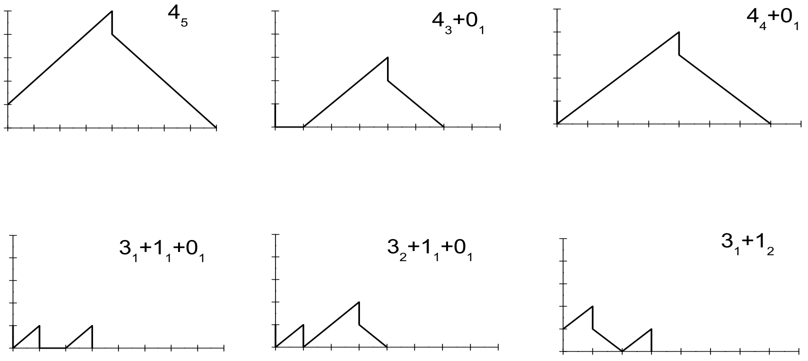


Figure 2-4 – Lattice paths generated by $\rho(q)$ for $s = 4$

two peaks is 0 *if the height and weight are of the same parity for both of the peaks,* 2 *if the height and weight are of opposite parity for both the peaks and* 1 *otherwise. The length of the plain in front of the first peak is* 1 *if the height and weight of the peak are of opposite parity and* 0 *if these are of same parity.*

Proof. In the lattice paths generated by $\frac{1}{2}(1 + V_0(q))$, the factor q^{n^2} generates the lattice path starting at $(0, 0)$ with n NESE peaks of height 1 each and ending at $(2n, 0)$. The factor $(q; q^2)_n^{-1}$ increases the height of the peaks. The factor $(-q; q^2)_n$ generates odd numbers $\leq 2n - 1$, say $u_1 \times 1, u_2 \times 3, \ldots, u_n \times (2n - 1)$, u_i=0 or 1. Introducing a plain of length $u_{n-i+1} + u_{n-i+2}$ in between the i^{th} and $(i + 1)^{th}$ peaks increases the weight of the path by $u_1 \times 1 + u_2 \times 3 + \cdots + u_n \times (2n - 1)$. □

Theorem 2.2.7. $V_1(q)$ *enumerates the lattice paths of weight* ≥ 1 *starting at* $(0, 0)$ *with all* NESE *peaks. There is no valley above height* 0 *in the path. The length of plain between two peaks is* 0 *if the height and weight are of the same parity for both of the peaks,* 2 *if the height and weight are of opposite parity for both the peaks and* 1 *otherwise. There is no plain in front of the first peak.*

Proof. The factor $q^{(n+1)^2}$ generates lattice path starting at $(0, 0)$ with $(n + 1)$ NESE peaks of height 1 each and ending at $(2n + 2, 0)$. The factor $(q; q^2)_{n+1}^{-1}$ increases the height of the peaks and the factor $(-q; q^2)_n$ introduces plains in between the consecutive peaks. □

2.3 Generalized results

In this section, we are presenting the $(n+t)$–color partition-theoretic interpretations of generalized versions of all the mock theta functions discussed in this chapter except $\varrho(q)$. Proofs are omitted here as these can be developed in a similar manner as discussed in Section 2.1.

2.3.1 Generalization of $\psi_0(q)$ and $\psi_1(q)$

Mock theta functions $\psi_0(q)$ and $\psi_1(q)$ are members of the following 2-parameter family of generating functions:

$$\mathscr{F}_1^{(\alpha,k)}(q) = \sum_{n=0}^{\infty} (-q^k;q^k)_n q^{\alpha\binom{n+1}{2}}, \qquad k,\alpha \geq 1. \tag{2.12}$$

Theorem 2.3.1. *For $k,\alpha \geq 1$, $\mathscr{F}_1^{(\alpha,k)}(q)$ generates the n–color partitions satisfying*

(i) $b_i = \alpha$, $1 \leq i \leq p$;

(ii) $a_p = \alpha$ *or* $\alpha + k$;

(iii) $(w.d)_{i,i+1} = -\alpha$ *or* $k-\alpha$, $1 \leq i \leq p-1$.

Interpretation of $\psi_1(q)$ in terms of n–color partitions follows from Theorem 2.3.1 as we see that $\psi_1(q) = \mathscr{F}_1^{(1,1)}(q)$ and interpretation of $\psi_0(q)-1$ can also be obtained from $\mathscr{F}_1^{(1,1)}(q)$ except for the displacement of $n \mapsto n+1$ in $q^{\binom{n+1}{2}}$ which prohibits the occurrence of least part 2_1.

2.3.2 Generalization of $\rho(q)$ and $\sigma(q)$

Mock theta functions $\rho(q)$ and $\sigma(q)$ are members of the following 3-parameter family of generating functions:

$$\mathscr{F}_2^{(\alpha,k,l)}(q) = \sum_{n=0}^{\infty} \frac{(-q^k;q^k)_n q^{\alpha\binom{n+1}{2}}}{(q^l;q^{2l})_n}, \qquad k,\alpha,l \geq 1. \tag{2.13}$$

Theorem 2.3.2. *For* $k, \alpha, l \geq 1$, $\mathscr{F}_2^{(\alpha,k,l)}(q)$ *generates the* n*–color partitions satisfying*

(i) $b_i \geq \alpha$ *and* $\equiv l \,(\mathrm{mod}\, \alpha)$, $1 \leq i \leq p$;

(ii) $a_p = b_p$ *or* $b_p + k$;

(iii) $(w.d)_{i,i+1} = -\alpha$ *or* $k - \alpha$, $1 \leq i \leq p - 1$.

The interpretations of $\rho(q)$ and $\sigma(q)$ can be obtained from $\mathscr{F}_2^{(1,1,1)}(q)$. In case of $\sigma(q)$ the displacement of $n \mapsto n+1$ in $q^{\binom{n+1}{2}}$ and $(q;q^2)_n^{-1}$ prohibits the occurrence of least part with $a_p = b_p + 1$. For $\rho(q)$, the displacement of $n \mapsto n+1$ in $(q;q^2)_n^{-1}$ prohibits the occurrence of least part with $a_p = b_p + 1$ and also shifts the least part with $a_p = b_p$ to $b_p = a_p + 1$.

2.3.3 Generalization of $V_0(q)$ and $V_1(q)$

Mock theta functions $\frac{1}{2}(1+V_0(q))$ and $V_1(q)$ are members of the following 3-parameter family of generating functions:

$$\mathscr{F}_3^{(\alpha,k,l)}(q) = \sum_{n=0}^{\infty} \frac{(-q^k;q^{2k})_n q^{\alpha n^2}}{(q^l;q^{2l})_n}, \qquad k, \alpha, l \geq 1. \tag{2.14}$$

Theorem 2.3.3. *For* $k, \alpha, l \geq 1$ *and* k odd, $\mathscr{F}_3^{(\alpha,k,l)}(q)$ *generates the* n*–color partitions satisfying*

(i) $b_i \geq \alpha$ *and* $\equiv l \,(\mathrm{mod}\, \alpha)$, $1 \leq i \leq p$;

(ii) $a_p = b_p$ *or* $b_p + k$;

(iii) $(w.d)_{i,i+1} = \begin{cases} 0 & \text{if } a_i \equiv b_i,\ a_{i+1} \equiv b_{i+1} \pmod 2, \\ 2k & \text{if } a_i \not\equiv b_i,\ a_{i+1} \not\equiv b_{i+1} \pmod 2, \\ k & \qquad\quad \text{otherwise,} \end{cases} \quad 1 \leq i \leq p-1.$

The interpretation of $\frac{1}{2}(1 + V_0(q))$ follows from Theorem 2.3.3 by substituting $\alpha = k = l = 1$. In case of $V_1(q)$ the displacement of $n \mapsto n+1$ in $q^{\binom{n+1}{2}}$ and $(q;q^2)_n^{-1}$ prohibits the occurrence of least part with $a_p = b_p + 1$.

2.4 Conclusion

With the interpretations of seven mock theta functions provided in this chapter, we have $(n+t)$–color partition-theoretic interpretations of fourteen mock theta functions. However, there are many more mock theta functions which may be interpreted in terms of $(n+t)$–color partitions, lattice paths and F–partitions. We further investigate these problems in the subsequent chapters.

Chapter 3

$(n+t)$–color partition-theoretic interpretations of mock theta functions by attaching weights

With the results of Chapter 2, we now have $(n+t)$–color partition-theoretic interpretations of fourteen mock theta functions. One can easily see that in the expansion of all these mock theta functions, power series in q for all the factors contain only positive coefficients. However, Brietzke et al. [44] have interpreted a number of mock theta functions in terms of two-line arrays [99] which are having some negative coefficients in the expansion. To arrive at the interpretations of such mock theta functions, they attached certain weights to each two-line array generated by the unsigned version of the mock theta function being interpreted. Inspiring from their work, we provide the interpretations of a number of mock theta functions by attaching weights to the $(n+t)$–color partitions generated by the unsigned version of these mock theta functions. We interpret third order mock theta functions $\phi(q)$, $\chi(q)$, $\nu(q)$ (given by (1.11), (1.13), (1.15)), fifth order mock theta functions $f_0(q)$, $f_1(q)$ (given by (1.19) and (1.24)) and sixth order mock theta function $\gamma(q)$ (given by (1.47)). We also interpret all the eighth order mock theta functions of Gordon and McIntosh (given by (1.35)–(1.40)) except $V_0(q)$ and $V_1(q)$ which have already been studied in Chapter 2.

3.1 Main results

Firstly, we give the $(n+t)$–color partition-theoretic interpretations of the unsigned versions of all the mock theta functions being discussed in this chapter.

Theorem 3.1.1. *Let $B_1(s)$ denote the number of n–color partitions of s such that*

(i) $b_i = 1$ *or* $2,\ 1 \le i \le p$;

(ii) $a_p \equiv b_p \pmod 3$;

(iii) $(w.d)_{i,i+1} \ge -1$ *and* $\equiv \begin{cases} 0 \pmod 3 & \text{if } b_{i+1}=1, \\ 2 \pmod 3 & \text{if } b_{i+1}=2, \end{cases} \quad 1 \le i \le p-1.$

Then

$$\sum_{s=0}^{\infty} B_1(s)q^s = \sum_{n=0}^{\infty} \frac{(-q;q)_n q^{n^2}}{(q^3;q^3)_n}.$$

Theorem 3.1.2. *Let $B_2(s)$ denote the number of n–color partitions of s such that*

(i) a_i is odd *and* $b_i = 1,\ 1 \le i \le p$;

(ii) $(w.d)_{i,i+1} \ge 0,\ 1 \le i \le p-1$.

Then

$$\sum_{s=0}^{\infty} B_2(s)q^s = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q^2;q^2)_n}.$$

Theorem 3.1.3. *Let $B_3(s)$ denote the number of $(n+1)$–color partitions of s such that*

(i) $b_p = a_p + 1$;

(ii) $(w.d)_{i,i+1} = 0,\ 1 \le i \le p-1$.

Then

$$\sum_{s=0}^{\infty} B_3(s)q^s = \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q;q^2)_{n+1}}.$$

Theorem 3.1.4. *Let $B_4(s)$ denote the number of n–color partitions of s such that*

(i) $b_i = 1,\ 1 \le i \le p$;

(ii) $(w.d)_{i,i+1} \geq 0,\ 1 \leq i \leq p-1.$

Then

$$\sum_{s=0}^{\infty} B_4(s)q^s = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n}.$$

Theorem 3.1.5. *Let $B_5(s)$ denote the number of n–color partitions of s such that*

(i) $a_i \geq 2$ *and* $b_i = 1,\ 1 \leq i \leq p;$

(ii) $(w.d)_{i,i+1} \geq 0,\ 1 \leq i \leq p-1.$

Then

$$\sum_{s=0}^{\infty} B_5(s)q^s = \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q;q)_n}.$$

Theorem 3.1.6. *Let $B_6(s)$ denote the number of n–color partitions of s such that*

(i) $b_i = \begin{cases} 1 & \text{if } a_i \text{ is odd,} \\ 2 & \text{if } a_i \text{ is even,} \end{cases} \quad 1 \leq i \leq p;$

(ii) $(w.d)_{i,i+1} \geq 0,\ 1 \leq i \leq p-1.$

Then

$$\sum_{s=0}^{\infty} B_6(s)q^s = \sum_{n=0}^{\infty} \frac{(-q;q^2)_n q^{n^2}}{(q^2;q^2)_n}.$$

Theorem 3.1.7. *Let $B_7(s)$ denote the number of n–color partitions of s such that*

(i) $a_i \geq 3,\ 1 \leq i \leq p;$

(ii) $b_i = \begin{cases} 1 & \text{if } a_i \text{ is odd,} \\ 2 & \text{if } a_i \text{ is even,} \end{cases} \quad 1 \leq i \leq p;$

(iii) $(w.d)_{i,i+1} \geq 0,\ 1 \leq i \leq p-1.$

Then

$$\sum_{s=0}^{\infty} B_7(s)q^s = \sum_{n=0}^{\infty} \frac{(-q;q^2)_n q^{n(n+2)}}{(q^2;q^2)_n}.$$

Theorem 3.1.8. *Let $B_8(s)$ denote the number of n–color partitions of s such that*

(i) $a_p = b_p + 1;$

(ii) $(w.d)_{i,i+1} = 0$ *or* 2, $1 \leq i \leq p-1$.

Then

$$\sum_{s=1}^{\infty} B_8(s)q^s = \sum_{n=0}^{\infty} \frac{(-q^2;q^2)_n q^{(n+1)(n+2)}}{(q;q^2)_{n+1}}.$$

Theorem 3.1.9. *Let* $B_9(s)$ *denote the number of* $(n+1)$*–color partitions of* s *such that*

(i) $b_p = a_p + 1$;

(ii) $(w.d)_{i,i+1} = 0$ *or* 2, $1 \leq i \leq p-1$.

Then

$$\sum_{s=0}^{\infty} B_9(s)q^s = \sum_{n=0}^{\infty} \frac{(-q^2;q^2)_n q^{n(n+1)}}{(q;q^2)_{n+1}}.$$

Theorem 3.1.10. *Let* $B_{10}(s)$ *denote the number of* n*–color partitions of* s *such that*

(i) $b_i = 1$ *or* 2, $1 \leq i \leq p$;

(ii) $a_p - b_p \equiv 0 \pmod 4$;

(iii) $(w.d)_{i,i+1} \geq 0$ *and* $\equiv 0 \pmod 4$, $1 \leq i \leq p-1$.

Then

$$\sum_{s=0}^{\infty} B_{10}(s)q^s = \sum_{n=0}^{\infty} \frac{(-q;q^2)_n q^{n^2}}{(q^4;q^4)_n}.$$

Theorem 3.1.11. *Let* $B_{11}(s)$ *denote the number of* n*–color partitions of* s *such that*

(i) b_i is odd, $1 \leq i \leq p$;

(ii) $a_p = b_p$;

(iii) $$(w.d)_{i,i+1} = \begin{cases} 0 & \text{if } a_i \equiv b_i,\ a_{i+1} \equiv b_{i+1} \pmod 2, \\ 2 & \text{if } a_i \not\equiv b_i,\ a_{i+1} \not\equiv b_{i+1} \pmod 2, \quad 1 \leq i \leq p-1. \\ 1 & \qquad \text{otherwise,} \end{cases}$$

Then

$$\sum_{s=1}^{\infty} B_{11}(s)q^s = \sum_{n=0}^{\infty} \frac{(-q;q^2)_n q^{(n+1)^2}}{(q^2;q^4)_{n+1}}.$$

In addition to the above results, we need the following three results to prove Theorems 3.1.3, 3.1.8, 3.1.9 and 3.1.11:

Theorem 3.1.12. *Let $B_{12}(s)$ denote the number of n–color partitions of s such that*

(i) $a_p = b_p + 1$;

(ii) $(w.d)_{i,i+1} = 0$, $1 \leq i \leq p-1$.

Then

$$\sum_{s=0}^{\infty} B_{12}(s) q^s = \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q;q^2)_n}.$$

Theorem 3.1.13. *Let $B_{13}(s)$ denote the number of n–color partitions of s such that*

(i) $a_p = b_p + 1$ *or* $b_p + 3$;

(ii) $(w.d)_{i,i+1} = 0$ *or* 2, $1 \leq i \leq p-1$.

Then

$$\sum_{s=1}^{\infty} B_{13}(s) q^s = \sum_{n=0}^{\infty} \frac{(-q^2;q^2)_{n+1} q^{(n+1)(n+2)}}{(q;q^2)_{n+1}}.$$

Theorem 3.1.14. *Let $B_{14}(s)$ denote the number of n–color partitions of s such that*

(i) b_i is odd, $1 \leq i \leq p$;

(ii) $a_p = b_p$ *or* $b_p + 1$;

(iii) $(w.d)_{i,i+1} = \begin{cases} 0 & \text{if } a_i \equiv b_i,\ a_{i+1} \equiv b_{i+1} \pmod 2, \\ 2 & \text{if } a_i \not\equiv b_i,\ a_{i+1} \not\equiv b_{i+1} \pmod 2, \\ 1 & \qquad \text{otherwise,} \end{cases} \quad 1 \leq i \leq p-1.$

Then

$$\sum_{s=1}^{\infty} B_{14}(s) q^s = \sum_{n=0}^{\infty} \frac{(-q;q^2)_n q^{n^2}}{(q^2;q^4)_n}.$$

Remark 3.1.1. *The generalized versions of* Theorems 3.1.6, 3.1.7 and 3.1.10 *were first proved in* [17, 70].

If the partitions enumerated by $B_i(s)$, $1 \leq i \leq 11$, are counted with certain weight then the generating function for the resulting sequence of numbers is certain mock theta function. For the attached weight $(-1)^t$, the values of t for different $B_i(s)$ along with the corresponding mock theta function are listed in Table 3.1.

Remark 3.1.2. *The unsigned versions of both $\chi(q)$ and $\gamma(q)$are equal and $B_1(s)$ is the enumerator for the partitions generated by common unsigned version.*

Table 3.1 – Enumerator and value of t for different mock theta functions

Enumerator	t	Mock theta function
$B_1(s)$	$\frac{1}{3}(a_1 - \sum_{i=1}^{p} b_i - p + 1)$	$\chi(q)$
$B_2(s)$	$\frac{1}{2}(a_1 + b_1 - 2p)$	$\phi(q)$
$B_3(s)$	$\sum_{i=1}^{p} b_i - p$	$\nu(q)$
$B_4(s)$	$a_1 + b_1 - 2p$	$f_0(q)$
$B_5(s)$	$a_1 - 2p$	$f_1(q)$
$B_1(s)$	$\sum_{i=1}^{p} b_i - p$	$\gamma(q)$
$B_6(s)$	$\frac{1}{2}\left[\sum_{i=1}^{p-1}\{(w.d)_{i,i+1}\} + a_p - b_p\right]$	$S_0(q)$
$B_7(s)$	$\frac{1}{2}\left[\sum_{i=1}^{p-1}\{(w.d)_{i,i+1}\} + a_p - b_p - 2\right]$	$S_1(q)$
$B_8(s)$	$\sum_{i=1}^{p} b_i - p$	$T_0(q)$
$B_9(s)$	$\sum_{i=1}^{p} b_i - p$	$T_1(q)$
$B_{10}(s)$	$\frac{1}{4}\left[\sum_{i=1}^{p-1}\{(w.d)_{i,i+1}\} + a_p - b_p\right]$	$U_0(q)$
$B_{11}(s)$	$\frac{1}{2}(\sum_{i=1}^{p} b_i - p)$	$U_1(q)$

To have a better understanding of the idea, let us take an example:

Example 3.1.1. *Obtaining first few terms in the expansion for*

$$\chi(q) = 1 + q + q^2 + q^6 + q^7 + q^{10} + q^{12} + q^{13} + \cdots,$$

and for the unsigned version

$$\sum_{n=0}^{\infty} \frac{(-q;q)_n q^{n^2}}{(q^3;q^3)_n} = 1 + q + q^2 + 2q^4 + 2q^5 + q^6 + 3q^7 + 2q^8 + 2q^9 + 5q^{10} + 4q^{11} + 5q^{12} + 7q^{13} + \cdots.$$

We see that for $n = 13$ there are seven partitions satisfying the conditions of Theorem 3.1.1 *and the weighted sum of their number is* 1. Table 3.2 *shows that there are four partitions with positive weight and three with negative weight.*

Table 3.2 – Partitions enumerated by $\chi(q)$ for $n = 13$ with weight calculations

Partition	t	Weight
13_1	$\frac{1}{3}(13-1-1+1)$	$+1$
12_1+1_1	$\frac{1}{3}(12-2-2+1)$	-1
11_2+2_2	$\frac{1}{3}(11-4-2+1)$	$+1$
9_1+4_1	$\frac{1}{3}(9-2-2+1)$	$+1$
$9_2+3_1+1_1$	$\frac{1}{3}(9-4-3+1)$	-1
8_2+5_2	$\frac{1}{3}(8-4-2+1)$	-1
$7_2+4_1+2_2$	$\frac{1}{3}(7-5-3+1)$	$+1$

3.2 Proofs

In the proofs of theorems in this section, we follow the method of proof given in [1].

Remark 3.2.1. *Throughout this section, if $B_i(s)$ denote the number of partitions of s with certain specified conditions then $B_i(m, s)$ will denote the partitions of s with the same conditions into m parts and*

$$f_i(z, q) = \sum_{s=0}^{\infty}\sum_{m=0}^{\infty} B_i(m, s) z^m q^s. \tag{3.1}$$

Proof of Theorem 3.1.1. Split the partitions enumerated by $B_1(m, s)$ into three classes. First class contains the partitions which have 1_1 as the least part. Deleting 1_1 and subtracting 2 from all the remaining parts we see that transformed partitions are enumerated by $B_1(m-1, s-2m+1)$ and it is easily seen that this transformation is reversible. Thus, there is a bijection between the partitions of this class and the partitions enumerated by $B_1(m-1, s-2m+1)$. In the second class, partitions contain 2_2 as their least part. By deleting 2_2 and subtracting 3 from the remaining parts we get the partition enumerated by $B_1(m-1, s-3m+1)$. The partitions of second class are in one-one correspondence with the partitions enumerated by

$B_1(m-1, s-3m+1)$. The partitions in third class are the remaining partitions with $a_p > 3$ and these partitions are in one-one correspondence with the partitions enumerated by $B_1(m, s-3m)$. This correspondence is obtained by subtracting 3 from all the parts of the partition. Thus,

$$B_1(m,s) = B_1(m-1, s-2m+1) + B_1(m-1, s-3m+1) + B_1(m, s-3m). \quad (3.2)$$

Now, substituting for $B_1(m,s)$ from (3.2) into (3.1) with $i = 1$ and then simplifying

$$f_1(z,q) = zqf_1(zq^2, q) + zq^2 f_1(zq^3, q) + f_1(zq^3, q). \quad (3.3)$$

Substituting $f_1(z,q) = \sum_{n=0}^{\infty} \delta_n(q) z^n$, and then comparing the coefficient of z^n on both sides of (3.3), we get

$$\delta_n(q) = \frac{(1+q^n)q^{2n-1}}{(1-q^{3n})}\delta_{n-1}(q), \quad n \geq 1. \quad (3.4)$$

Iterating (3.4) n times and considering the fact that $\delta_0(q) = 1$, we find that

$$\delta_n(q) = \frac{(-q;q)_n q^{n^2}}{(q^3;q^3)_n}, \quad n \geq 0. \quad (3.5)$$

Thus,

$$f_1(z,q) = \sum_{n=0}^{\infty} \frac{(-q;q)_n q^{n^2}}{(q^3;q^3)_n} z^n. \quad (3.6)$$

$$\begin{aligned} \sum_{s=0}^{\infty} B_1(s) q^s =& \sum_{s=0}^{\infty} \left(\sum_{m=0}^{\infty} B_1(m,s) \right) q^s \\ =& f_1(1,q) && (3.7) \\ =& \sum_{n=0}^{\infty} \frac{(-q;q)_n q^{n^2}}{(q^3;q^3)_n}. && (3.8) \end{aligned}$$

Hence the proof.

Proof of Theorem 3.1.14. Split the partitions enumerated by $B_{14}(m,s)$ into three classes. Partitions in the first class contain 1_1 as their least part. These partitions are enumerated by $B_{14}(m-1, s-2m+1)$. Second class contains partitions with 2_1 as their least part and are enumerated by $B_{14}(m-1, s-4m+2)$. The partitions in the third class satisfy $b_p > 2$ and these are enumerated by $B_{14}(m, s-4m+2)$. Thus, the recurrence relation for $B_{14}(m, s)$ is given by

$$B_{14}(m, s) = B_{14}(m-1, s-2m+1) + B_{14}(m-1, s-4m+2) + B_{14}(m, s-4m+2), \quad (3.9)$$

and the corresponding q-functional equation is

$$f_{14}(z, q) = zqf_{14}(zq^2, q) + zq^2 f_{14}(zq^4, q) + q^{-2} f_{14}(zq^4, q). \quad (3.10)$$

substituting $f_{14}(z, q) = \sum_{n=0}^{\infty} \lambda_n(q) z^n$ in (3.10) and comparing coefficient of z^n in the expression, we obtain

$$\lambda_n(q) = \frac{(1+q^{2n-1})q^{2n-1}}{(1-q^{4n-2})} \lambda_{n-1}(q), \quad n \geq 1. \quad (3.11)$$

Iterating (3.11) n times and considering the fact that $\lambda_0(q) = 1$, we get

$$\lambda_n(q) = \frac{(-q; q^2)_n q^{n^2}}{(q^2; q^4)_n}, \quad n \geq 0. \quad (3.12)$$

Therefore,

$$f_{14}(z, q) = \sum_{n=0}^{\infty} \frac{(-q; q^2)_n q^{n^2}}{(q^2; q^4)_n} z^n. \quad (3.13)$$

Proceeding further in the same manner as in the proof of Theorem 3.1.1, we get the desired result.

Proof of Theorem 3.1.11. Split the partitions enumerated by $B_{11}(m, s)$ into two classes, first containing the partitions with 1_1 as their least part and second containing the partitions with $a_p > 2$. The partitions in the first class are enumerated by $B_{14}(m-1, s-2m+1)$ and in the second class are enumerated by $B_{11}(m, s-4m+2)$. Thus, the recurrence relation is

$$B_{11}(m, s) = B_{14}(m-1, s-2m+1) + B_{11}(m, s-4m+2), \quad (3.14)$$

and the corresponding q-functional equation is

$$f_{11}(z,q) = zqf_{14}(zq^2,q) + q^{-2}f_{11}(zq^4,q). \tag{3.15}$$

substituting $f_{11}(z,q) = \sum_{n=1}^{\infty}\zeta_n(q)z^n$ in (3.15) and comparing coefficient of z^n in the expression, we obtain

$$\zeta_n(q) = q^{2n-1}\lambda_{n-1}(q) + q^{4n-2}\zeta_n(q), \quad n \geq 1. \tag{3.16}$$

$$\zeta_n(q) = \frac{(-q;q^2)_{n-1}q^{n^2}}{(q^2;q^4)_n}, \quad n \geq 1. \tag{3.17}$$

Therefore,

$$f_{11}(z,q) = \sum_{n=1}^{\infty}\frac{(-q;q^2)_{n-1}q^{n^2}}{(q^2;q^4)_n}z^n. \tag{3.18}$$

$$\begin{aligned}\sum_{s=0}^{\infty}B_{11}(s)q^s =& \sum_{s=0}^{\infty}\left(\sum_{m=1}^{\infty}B_{11}(m,s)\right)q^s \\ =& f_{11}(1,q) \qquad (3.19)\\ =& \sum_{n=1}^{\infty}\frac{(-q;q^2)_{n-1}q^{n^2}}{(q^2;q^4)_n} \qquad (3.20)\\ =& \sum_{n=0}^{\infty}\frac{(-q;q^2)_nq^{(n+1)^2}}{(q^2;q^4)_{n+1}}. \qquad (3.21)\end{aligned}$$

Hence the proof.

The proofs of rest of the theorems are also established by splitting partitions enumerated by $B_i(m,s)$ into classes. Also, it is evident from the proof of Theorems 3.1.1, 3.1.11 and 3.1.14 that classes depend on the least part of the partitions. Therefore, in Table 3.3 we are listing the classes as signified by their least parts. Below we provide the recurrence relations satisfied by $B_i(m,s)$, so that the reader is able to understand the transformations and establish the remaining proofs.

$$\begin{aligned}B_2(m,s) =& B_2(m-1,s-2m+1) + B_2(m,s-2m)\\ B_3(m,s) =& B_{12}(m-1,s) + B_3(m,s-2m+1)\end{aligned}$$

Table 3.3 – Classes for $B_i(m,s)$

Enumerator	Least Part of Partition		
	Class (i)	Class (ii)	Class (iii)
$B_2(m,s)$	1_1	$a_p > 2$	
$B_3(m,s)$	0_1	$a_p > 0$	
$B_4(m,s)$	1_1	$a_p > 1$	
$B_5(m,s)$	2_1	$a_p > 2$	
$B_6(m,s)$	1_1	2_2	$a_p > 2$
$B_7(m,s)$	3_1	4_2	$a_p > 4$
$B_8(m,s)$	2_1	$a_p > 1$	
$B_9(m,s)$	0_1	$b_p \geq 1$	
$B_{10}(m,s)$	1_1	2_2	$a_p > 4$
$B_{12}(m,s)$	2_1	$b_p > 1$	
$B_{13}(m,s)$	2_1	4_1	$b_p > 1$

$$\begin{aligned}
B_4(m,s) =& B_4(m-1,s-2m+1) + B_4(m,s-m)\\
B_5(m,s) =& B_5(m-1,s-2m) + B_5(m,s-m)\\
B_6(m,s) =& B_6(m-1,s-2m+1) + B_6(m-1,s-4m+2) + B_6(m,s-2m)\\
B_7(m,s) =& B_7(m-1,s-2m-1) + B_7(m-1,s-4m) + B_7(m,s-2m)\\
B_8(m,s) =& B_{13}(m-1,s-2m) + B_8(m,s-2m+1)\\
B_9(m,s) =& B_{13}(m-1,s) + B_9(m,s-2m+1)\\
B_{10}(m,s) =& B_{10}(m-1,s-2m+1) + B_{10}(m-1,s-4m+2) + B_{10}(m,s-4m)\\
B_{12}(m,s) =& B_{12}(m-1,s-2m) + B_{12}(m,s-2m+1)\\
B_{13}(m,s) =& B_{13}(m-1,s-2m) + B_{13}(m-1,s-4m) + B_{13}(m,s-2m+1)
\end{aligned}$$

3.3 Generalized results

In continuation of the generalized results provided in Chapter 2, we are interpreting the generalized versions of all the mock theta functions discussed in this chapter.

3.3.1 Generalization of $\phi(q)$, $f_0(q)$ and $f_1(q)$

Mock theta functions $\phi(q)$, $f_0(q)$ and $f_1(q)$ are members of the following 3-parameter family of generating functions

$$\mathscr{F}_4^{(\alpha,\beta,k)}(q) = \sum_{n=0}^{\infty} \frac{q^{\alpha n^2+\beta n}}{(-q^k;q^k)_n}. \tag{3.22}$$

Theorem 3.3.1. *$\mathscr{F}_4^{(\alpha,\beta,k)}(q)$ generates the n–color partitions satisfying*

(i) $a_i \geq \alpha+\beta$ *and* $b_i = \alpha$, $1 \leq i \leq p$;

(ii) $a_p - b_p \equiv \beta \pmod{k}$;

(iii) $(w.d)_{i,i+1} \geq 0$ *and* $\equiv 0 \pmod{k}$, $1 \leq i \leq p-1$;

(iv) *every partition is counted with weight* $(-1)^t$, *where* $t = \frac{1}{k}(a_1 + b_1 - 2\alpha p)$.

Interpretations of $\phi(q)$, $f_0(q)$ and $f_1(q)$ in terms of n–color partitions follow naturally from Theorem 3.3.1 as we see that $\phi(q) = \mathscr{F}_4^{(1,0,2)}(q)$, $f_0(q) = \mathscr{F}_4^{(1,0,1)}(q)$ and $f_1(q) = \mathscr{F}_4^{(1,1,1)}(q)$.

3.3.2 Generalization of $\chi(q)$ and $\gamma(q)$

Consider the 3-parameter family of generating functions

$$\mathscr{F}_5^{(\alpha,\beta,k)}(q) = \sum_{n=0}^{\infty} \frac{(-q;q)_n q^{\alpha n^2+\beta n}}{(q^k;q^k)_n}. \tag{3.23}$$

Theorem 3.3.2. *$\mathscr{F}_5^{(\alpha,\beta,k)}(q)$ generates the n–color partitions satisfying*

(i) $a_i \geq \alpha+\beta$ *and* $b_i = \alpha$ *or* $\alpha+1$, $1 \leq i \leq p$;

(ii) $a_p - b_p \equiv \beta \pmod{k}$;

(iii) $(w.d)_{i,i+1} \geq -1$ *and* $\equiv \begin{cases} 0 \pmod{k} & \text{if } b_{i+1} = \alpha, \\ k-1 \pmod{k} & \text{if } b_{i+1} = \alpha+1, \end{cases} \quad 1 \leq i \leq p-1.$

The unsigned version of both $\chi(q)$ and $\gamma(q)$ is equal to $\mathscr{F}_5^{(1,0,3)}(q)$. The only difference in the interpretation of $\chi(q)$ from $\gamma(q)$ is in the weight attached to the partitions.

3.3.3 Generalization of $\nu(q)$

Mock theta function $\nu(q)$ is a member of the following 3-parameter family of generating functions with $\alpha = \beta = k = 1$ and a displacement of $n \mapsto n+1$ in the factor $(-q^k; q^{2k})_n$:

$$\mathscr{F}_6^{(\alpha,\beta,k)}(q) = \sum_{n=0}^{\infty} \frac{q^{\alpha n^2+\beta n}}{(-q^k; q^{2k})_n}. \tag{3.24}$$

Theorem 3.3.3. *$\mathscr{F}_6^{(\alpha,\beta,k)}(q)$ generates the n–color partitions satisfying*

(i) $a_i \geq \alpha + \beta$ *and* $b_i \equiv \alpha \pmod{k}$, $1 \leq i \leq p$;

(ii) $a_p - b_p = \beta$;

(iii) $(w.d)_{i,i+1} = 0$;

(iv) *every partition is counted with weight* $(-1)^t$, *where* $t = \frac{1}{k}(\sum_{i=1}^{p} b_i - \alpha p)$.

3.3.4 Generalization of $S_0(q)$, $S_1(q)$ and $U_0(q)$

The n–color partition-theoretic interpretation for the following 4-parameter family of generating functions is:

$$\mathscr{F}_7^{(\alpha,\beta,k,l)}(q) = \sum_{n=0}^{\infty} \frac{(-q^k; q^{2k})_n q^{\alpha n^2+\beta n}}{(-q^l; q^l)_n}. \tag{3.25}$$

Theorem 3.3.4. *$\mathscr{F}_7^{(\alpha,\beta,k,l)}(q)$ is the generating function for the n–color partitions satisfying*

(i) $a_i \geq \alpha + \beta$ *and* $b_i = \alpha$ *or* $\alpha + k$, $1 \leq i \leq p$;

(ii) $a_p - b_p \equiv \beta \pmod{l}$;

(iii) $(((a_i)_{b_i} - (a_{i+1})_{b_{i+1}})) \geq 0$ *and* $\equiv 0 \pmod{l}$, $1 \leq i \leq p-1$;

(iv) *every partition is counted with weight* $(-1)^t$, *where* $t = \frac{1}{l}(\sum_{i=1}^{p-1}(w.d)_{i,i+1} + a_p - b_p - \beta)$.

$S_0(q)$, $S_1(q)$ and $U_0(q)$ can be interpreted by using Theorem 3.3.4 as $S_0(q) = \mathscr{F}_4^{(1,0,1,2)}(q)$, $S_1(q) = \mathscr{F}_4^{(1,2,1,2)}(q)$ and $U_0(q) = \mathscr{F}_7^{(1,0,1,4)}(q)$.

3.3.5 Generalization of $T_0(q)$ and $T_1(q)$

Consider the following 4-parameter family of generating functions:

$$\mathscr{F}_8^{(\alpha,\beta,k,l)}(q)=\sum_{n=0}^{\infty}\frac{(-q^k;q^k)_n q^{\alpha n^2+\beta n}}{(-q^l;q^{2l})_n}. \tag{3.26}$$

Theorem 3.3.5. *$\mathscr{F}_8^{(\alpha,\beta,k,l)}(q)$ is the generating function for n–color partitions satisfying*

(i) $a_i \geq \alpha+\beta$ *and* $b_i \equiv \alpha \pmod{l}$, $1 \leq i \leq p$;

(ii) $a_p - b_p = \beta$ *or* $\beta + k$;

(iii) $(w.d)_{i,i+1} = 0$ *or* k, $1 \leq i \leq p-1$;

(iv) *every partition is counted with weight* $(-1)^t$, *where* $t = \frac{1}{l}(\sum_{i=1}^{p} b_i - \alpha p)$.

Here, $T_0(q)$ is a member of the family $\mathscr{F}_8^{(\alpha,\beta,k,l)}(q)$ with $\alpha=\beta=l=1$ and $k=2$ except for the displacement of $n \mapsto n+1$ in $q^{\alpha n^2+\beta n}$ and $(-q^l;q^{2l})_n$. Similarly, $T_1(q)$ becomes $\mathscr{F}_8^{(1,1,2,1)}(q)$ with the displacement $n \mapsto n+1$ in $(-q^l;q^{2l})_n$.

3.3.6 Generalization of $U_1(q)$

$U_1(q)$ is a member of the family of generating functions

$$\mathscr{F}_9^{(\alpha,k,l)}(q)=\sum_{n=0}^{\infty}\frac{(-q^k;q^{2k})_n q^{\alpha n^2}}{(-q^l;q^{2l})_n},\ k \text{ is odd}, \tag{3.27}$$

with $\alpha = k = 1$, $l = 2$ and a displacement of $n \mapsto n+1$ in the factors $q^{\alpha n^2}$ and $(-q^l;q^{2l})_n$.

Theorem 3.3.6. *$\mathscr{F}_9^{(\alpha,k,l)}(q)$ is the generating function for n–color partitions satisfying*

(i) $a_i \geq \alpha$ *and* $\equiv \alpha \pmod{l}$, $1 \leq i \leq p$;

(ii) $a_p - b_p = 0$ *or* k;

(iii) $$(w.d)_{i,i+1} = \begin{cases} 0 & \text{if } a_i \equiv b_i,\ a_{i+1} \equiv b_{i+1} \pmod 2, \\ 2k & \text{if } a_i \not\equiv b_i,\ a_{i+1} \not\equiv b_{i+1} \pmod 2, \\ k & \text{otherwise}, \end{cases} \quad 1 \leq i \leq p;$$

(iv) *every partition is counted with weight* $(-1)^t$, *where* $t = \frac{1}{l}(\sum_{i=1}^{p} b_i - \alpha p)$.

3.4 Conclusion

The work done in this chapter establishes a bond between the work initiated by Agarwal in [7] and Brietzke et al. in [43, 44]. Further, generalizations of these results help in interpreting a number of q-series from Slater's compendium [108] and Chu-Zhang's compendium [52]. As an example, consider the following two series:

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q^4;q^4)_n}, \tag{3.28}$$

$$\sum_{n=0}^{\infty} \frac{(-q;q^2)_n q^{n(n+2)}}{(q^4;q^4)_n}. \tag{3.29}$$

The interpretation of (3.28) follows from Theorem 3.3.1 with $\alpha = 1$, $\beta = 0$ and $k = 4$ without assigning weight. Interpretation of (3.29) follows from Theorem 3.3.4 with $\alpha = 1$, $\beta = 2$, $k = 1$ and $l = 4$ without assigning weight. The unsigned versions of some members of these families have already been interpreted combinatorially (see [17], for example) in terms of n–color partitions. These interpretations can now be obtained from the interpretations of the generalized versions discussed in this chapter.

Chapter 4

F–partition-theoretic interpretations of mock theta functions by attaching weights

In [25], Andrews introduced the concept of a *generalized Frobenius partition.*

Definition 4.0.1. *A generalized Frobenius partition (or "simply F–partition") of a positive integer s is a two-rowed array of nonnegative integers*

$$\begin{pmatrix} c_1 & c_2 & \dots & c_p \\ d_1 & d_2 & \dots & d_p \end{pmatrix},$$

such that integers are arranged in a nonincreasing order in each row and

$$s = \sum_{i=1}^{p} (c_i + d_i + 1).$$

He also introduced many other general classes of F–partitions among which the partitions with k copies of an integer (k–colored F–partitions) have been most widely studied. Many mathematicians have found congruences for k–colored Frobenius partitions (see [37], [38], [57], [79], [81], [89]). Andrews' F–partitions have also served as a tool to interpret q-series and mock theta functions combinatorially (see [5], [14], [15], [92], [94], [100], [109], [110]). In most of these texts, a bijection is established between certain restricted $(n + t)$–color partitions and F–partitions. In ([15], [94]), the interpretations of q-series in terms of F–partitions are handled by obtaining recurrence relations. In Chapter 2, we have interpreted seven mock theta functions combinatorially in terms of $(n+t)$–color partitions. In Chapter 3, twelve mock theta functions have been interpreted in terms of $(n+t)$–color partitions by attaching weights. In this chapter, we obtain the F–partition-theoretic interpretations of all these nineteen mock theta functions. To obtain the interpretations of twelve mock theta functions studied in Chapter 3, we attach weights to the F–partitions generated by an "unsigned" version of the mock theta function under consideration.

4.1 F–partition-theoretic interpretations of mock theta functions

First of all, we provide an interpretation of an unsigned version of the mock theta function $\chi(q)$ (given by (1.13)) which is obtained by replacing $(-q^3;q^3)_n^{-1}$ with $(q^3;q^3)_n^{-1}$ in the expression for $\chi(q)$.

Theorem 4.1.1. *Let $C_1(s)$ denote the F–partitions of s satisfying*

(i) **(a)** *if $c_p \le d_p$ then $c_p \equiv 0 \pmod 3$,*

(b) *if $c_p > d_p$ then $d_p \equiv 1 \pmod 3$;*

(ii) $c_i = d_i - 1$ *or* d_i *or* $d_i + 1$ *or* $d_i + 2$, $1 \le i \le p$;

(iii) *for* $1 \le i \le p-1$,

(a) *if $c_{i+1} = d_{i+1} - 1$ then*
$$d_{i+1} \equiv \begin{cases} c_i + 1 \pmod 3 & \text{if } c_i \le d_i, \\ d_i \pmod 3 & \text{if } c_i > d_i, \end{cases}$$

(b) *if $c_{i+1} = d_{i+1}$ then*
$$d_{i+1} \equiv \begin{cases} c_i + 2 \pmod 3 & \text{if } c_i \le d_i, \\ d_i + 1 \pmod 3 & \text{if } c_i > d_i, \end{cases}$$

(c) *if $c_{i+1} = d_{i+1} + 1$ then*
$$c_{i+1} \equiv \begin{cases} c_i + 1 \pmod 3 & \text{if } c_i \le d_i, \\ d_i \pmod 3 & \text{if } c_i > d_i, \end{cases}$$

(d) *if $c_{i+1} = d_{i+1} + 2$ then*
$$c_{i+1} \equiv \begin{cases} c_i \pmod 3 & \text{if } c_i \le d_i, \\ d_i + 2 \pmod 3 & \text{if } c_i > d_i. \end{cases}$$

Let $B_1(s)$ denote the n–color partitions of s such that

(iv) $a_p - b_p \equiv 0 \pmod 3$;

(v) $b_i = 1$ *or* 2, $1 \leq i \leq p$;

(vi) $(w.d)_{i,i+1} \geq -1$ *and* $\equiv \begin{cases} 0 \pmod 3 & \text{if } b_i = 1, \\ 2 \pmod 3 & \text{if } b_i = 2, \end{cases} \quad 1 \leq i \leq p-1.$

Then

$$\sum_{s=0}^{\infty} C_1(s) q^s = \sum_{s=0}^{\infty} B_1(s) q^s = \sum_{n=0}^{\infty} \frac{(-q;q)_n q^{n^2}}{(q^3;q^3)_n}.$$

Remark 4.1.1. *We have already proved in* Chapter 3 (Theorem 3.1.1) *that* $\sum_{s=0}^{\infty} B_1(s) q^s = \sum_{n=0}^{\infty} \frac{(-q;q)_n q^{n^2}}{(q^3;q^3)_n}$, *now we will just prove that* $C_1(s) = B_1(s)$.

Proof. We establish a bijection between the partitions enumerated by $C_1(s)$ and $B_1(s)$ such that each column $\begin{pmatrix} c_i \\ d_i \end{pmatrix}$ of F–partition enumerated by $C_1(s)$ is mapped to a single part $(a_i)_{b_i}$ of n–color partition enumerated by $B_1(s)$. The mapping φ and the inverse mapping φ^{-1} are given by

$$\varphi \begin{pmatrix} c_i \\ d_i \end{pmatrix} = \begin{cases} (c_i + d_i + 1)_{d_i - c_i + 1} & \text{if } c_i \leq d_i, \\ (c_i + d_i + 1)_{c_i - d_i} & \text{if } c_i > d_i, \end{cases} \tag{4.1}$$

$$\varphi^{-1}((a_i)_{b_i}) = \begin{cases} \begin{pmatrix} (a_i + b_i - 1)/2 \\ (a_i - b_i - 1)/2 \end{pmatrix} & \text{if } a_i \not\equiv b_i \pmod 2, \\ \begin{pmatrix} (a_i - b_i)/2 \\ (a_i + b_i - 2)/2 \end{pmatrix} & \text{if } a_i \equiv b_i \pmod 2. \end{cases} \tag{4.2}$$

Thus,

$$(w.d)_{i,i+1} = \begin{cases} 2c_i - 2d_{i+1} - 2 & \text{if } c_i \leq d_i,\ c_{i+1} \leq d_{i+1}, \\ 2d_i - 2d_{i+1} - 1 & \text{if } c_i > d_i,\ c_{i+1} \leq d_{i+1}, \\ 2c_i - 2c_{i+1} - 1 & \text{if } c_i \leq d_i,\ c_{i+1} > d_{i+1}, \\ 2d_i - 2c_{i+1} & \text{if } c_i > d_i,\ c_{i+1} > d_{i+1}. \end{cases} \tag{4.3}$$

Using the condition (iii), equations (4.3) imply (vi). By the definition of φ, (i) implies (iv) and (ii) implies (v) clearly.

To see the reverse implications, note that under the inverse mapping φ^{-1}:

$$2c_i - 2c_{i+1} = \begin{cases} (w.d)_{i,i+1} + 2b_{i+1} & \text{if } a_{i+1} \equiv b_{i+1},\ a_i \equiv b_i \pmod 2, \\ (w.d)_{i,i+1} + 2b_{i+1} + 2b_i - 1 & \text{if } a_{i+1} \equiv b_{i+1},\ a_i \not\equiv b_i \pmod 2, \\ (w.d)_{i,i+1} + 1 & \text{if } a_{i+1} \not\equiv b_{i+1},\ a_i \equiv b_i \pmod 2, \\ (w.d)_{i,i+1} + 2b_i & \text{if } a_{i+1} \not\equiv b_{i+1},\ a_i \not\equiv b_i \pmod 2, \end{cases} \tag{4.4}$$

$$2c_i - 2d_{i+1} = \begin{cases} (w.d)_{i,i+1} + 2 & \text{if } a_{i+1} \equiv b_{i+1},\ a_i \equiv b_i \pmod 2, \\ (w.d)_{i,i+1} + 2b_i + 2 & \text{if } a_{i+1} \equiv b_{i+1},\ a_i \not\equiv b_i \pmod 2, \\ (w.d)_{i,i+1} + 2b_{i+1} + 1 & \text{if } a_{i+1} \not\equiv b_{i+1},\ a_i \equiv b_i \pmod 2, \\ (w.d)_{i,i+1} + 2b_{i+1} + 2b_i & \text{if } a_{i+1} \not\equiv b_{i+1},\ a_i \not\equiv b_i \pmod 2, \end{cases} \tag{4.5}$$

$$2d_i - 2c_{i+1} = \begin{cases} (w.d)_{i,i+1} + 2b_i - 2 & \text{if } a_{i+1} \equiv b_{i+1},\ a_i \equiv b_i \pmod 2, \\ (w.d)_{i,i+1} + 2b_{i+1} - 2 & \text{if } a_{i+1} \equiv b_{i+1},\ a_i \not\equiv b_i \pmod 2, \\ (w.d)_{i,i+1} + 2b_i - 1 & \text{if } a_{i+1} \not\equiv b_{i+1},\ a_i \equiv b_i \pmod 2, \\ (w.d)_{i,i+1} & \text{if } a_{i+1} \not\equiv b_{i+1},\ a_i \not\equiv b_i \pmod 2, \end{cases} \tag{4.6}$$

$$2d_i - 2d_{i+1} = \begin{cases} (w.d)_{i,i+1} + 2b_i & \text{if } a_{i+1} \equiv b_{i+1},\ a_i \equiv b_i \pmod 2, \\ (w.d)_{i,i+1} + 1 & \text{if } a_{i+1} \equiv b_{i+1},\ a_i \not\equiv b_i \pmod 2, \\ (w.d)_{i,i+1} + 2b_{i+1} + 2b_i - 1 & \text{if } a_{i+1} \not\equiv b_{i+1},\ a_i \equiv b_i \pmod 2, \\ (w.d)_{i,i+1} + 2b_{i+1} & \text{if } a_{i+1} \not\equiv b_{i+1},\ a_i \not\equiv b_i \pmod 2. \end{cases} \tag{4.7}$$

Condition (vi) together with the equations (4.4)–(4.7) implies (iii). It is also easy to see that (iv) implies (i) and (v) implies (ii). □

In Chapter 3, we obtained the n–color partition-theoretic interpretation of $\chi(q)$ by counting each n–color partition with a weight. Here, we will count each F–partition enumerated by the unsigned version of $\chi(q)$ with the weight $(-1)^h$, where

$$h = \frac{1}{3}\left[c_1 + d_1 - 2(p-1) - \sum_{r}(d_i - c_i) - \sum_{r'}(c_i - d_i - 1)\right],$$

r runs over those values of i for which $d_i = c_i + 1$ and r' runs over those values of i for which $c_i = d_i + 2$. Continuing with the Example 3.1.1, Table 4.1 gives relevant F–partitions for $\chi(q)$ with corresponding weights.

Table 4.1 – Partitions enumerated by $\chi(q)$ for $s = 13$ with weight calculations

Partition	h	Weight
$\begin{pmatrix}6\\6\end{pmatrix}$	$\frac{1}{3}(6+6)$	+1
$\begin{pmatrix}6&0\\5&0\end{pmatrix}$	$\frac{1}{3}(6+5-2)$	−1
$\begin{pmatrix}6&0\\4&1\end{pmatrix}$	$\frac{1}{3}(6+4-2-1-1)$	+1
$\begin{pmatrix}4&2\\4&1\end{pmatrix}$	$\frac{1}{3}(4+4-2)$	+1
$\begin{pmatrix}5&1&0\\3&1&0\end{pmatrix}$	$\frac{1}{3}(5+3-4-1)$	−1
$\begin{pmatrix}3&3\\4&1\end{pmatrix}$	$\frac{1}{3}(3+4-2-1-1)$	−1
$\begin{pmatrix}4&2&0\\2&1&1\end{pmatrix}$	$\frac{1}{3}(4+2-4-1-1)$	+1

The interpretation for the mock theta function $\gamma(q)$ (given by (1.47)) can also be obtained by attaching weight $(-1)^h$ to the count of each F–partition satisfying the conditions of Theorem 4.1.1, where

$$h = \sum_{r}(d_i - c_i) + \sum_{r'}(c_i - d_i - 1),$$

r and r' have same values as given above in case of $\chi(q)$. In Table 4.2, we provide the conditions on F–partitions generated by the unsigned versions of the remaining mock theta functions along with the mapping between relevant F–partitions and corresponding $(n+t)$–color partitions given in Chapters 2 and 3.

Table 4.2 – F–partition-theoretic interpretations with corresponding bijections

Function	F–partitions	Mapping
$\nu(q)$	$c_p = 0$ or 1 $c_i = 2 + d_{i+1}$ $c_i \leq 1 + d_i$	$\varphi\begin{pmatrix}c_i\\d_i\end{pmatrix}=(\mathrm{c}_i+d_i+1)_{d_i-c_i+2}$ $\varphi^{-1}((a_i)_{b_i})=\begin{pmatrix}(\mathrm{a}_i-b_i+1)/2\\(\mathrm{a}_i+b_i-3)/2\end{pmatrix}$
$\varrho(q)$	$c_p = 0$ or 2 $c_i = 3 + d_{i+1}$ $d_i - c_i \geq -1$ and $\equiv 0,\ 2 \pmod 3$	$\varphi\begin{pmatrix}c_i\\d_i\end{pmatrix}=(\mathrm{c}_i+d_i+1)_{d_i-c_i+3}$ $\varphi^{-1}((a_i)_{b_i})=\begin{pmatrix}(\mathrm{a}_i-b_i+2)/2\\(\mathrm{a}_i+b_i-4)/2\end{pmatrix}$

Function	F–partitions	Mapping
$\phi(q)$	c_i even, d_i=0 $c_i \geq 2 + c_{i+1}$	$\varphi\begin{pmatrix} c_i \\ d_i \end{pmatrix}$=$(\mathrm{c}_i + d_i + 1)_{d_i+1}$ $\varphi^{-1}((a_i)_{b_i})$=$\begin{pmatrix} \mathrm{a}_i - b_i \\ \mathrm{b}_i - 1 \end{pmatrix}$
$f_0(q)$	d_i=0 $c_i \geq 2 + c_{i+1}$	Same as for $\phi(q)$
$f_1(q)$	d_i=1 $c_i \geq 2 + c_{i+1}$	$\varphi\begin{pmatrix} c_i \\ d_i \end{pmatrix}$=$(\mathrm{c}_i + d_i + 1)_{d_i}$ $\varphi^{-1}((a_i)_{b_i})$=$\begin{pmatrix} \mathrm{a}_i - b_i - 1 \\ \mathrm{b}_i \end{pmatrix}$
$\psi_0(q) - 1$	$c_p = 0$ d_i=0 $c_i = 1 + c_{i+1}$ or $2 + c_{i+1}$	Same as for $\phi(q)$
$\psi_1(q)$	$c_p = 0$ or 1 d_i=0 $c_i = 1 + c_{i+1}$ or $2 + c_{i+1}$	Same as for $\phi(q)$
$\sigma(q)$	$c_p = 0$ $c_{i+1} \leq d_{i+1} \Rightarrow \mathrm{d}_{i+1} = \begin{cases} c_i - 1 & \text{if } c_i \leq d_i \\ d_i & \text{if } c_i > d_i \end{cases}$ $c_{i+1} > d_{i+1} \Rightarrow \mathrm{c}_{i+1} = \begin{cases} c_i & \text{if } c_i \leq d_i \\ d_i & \text{if } c_i > d_i \end{cases}$	Same as for $\chi(q)$
$\rho(q)$	$c_p > d_p$, $d_p = -1$ $c_{i+1} \leq d_{i+1} \Rightarrow \mathrm{d}_{i+1} = \begin{cases} c_i - 1 & \text{if } c_i \leq d_i \\ d_i & \text{if } c_i > d_i \end{cases}$ $c_{i+1} > d_{i+1} \Rightarrow \mathrm{c}_{i+1} = \begin{cases} c_i & \text{if } c_i \leq d_i \\ d_i & \text{if } c_i > d_i \end{cases}$	Same as for $\chi(q)$
$\frac{1}{2}(1 + V_0(q))$	$c_p = 0$ or $d_p = 0$ or both $c_{i+1} \leq d_{i+1} \Rightarrow \mathrm{d}_{i+1} = \begin{cases} c_i - 1 & \text{if } c_i \leq d_i \\ d_i - 1 & \text{if } c_i > d_i \end{cases}$ $c_{i+1} > d_{i+1} \Rightarrow \mathrm{c}_{i+1} = \begin{cases} c_i - 1 & \text{if } c_i \leq d_i \\ d_i - 1 & \text{if } c_i > d_i \end{cases}$	Same as for $\chi(q)$

Function	F–partitions	Mapping
$V_1(q)$	$c_p = 0$ $c_{i+1} \leq d_{i+1} \Rightarrow c_{i+1} = \begin{cases} c_i - 1 & \text{if } c_i \leq d_i \\ d_i - 1 & \text{if } c_i > d_i \end{cases}$ $c_{i+1} > d_{i+1} \Rightarrow c_{i+1} = \begin{cases} c_i & \text{if } c_i \leq d_i \\ d_i & \text{if } c_i > d_i \end{cases}$	Same as for $\chi(q)$
$U_0(q)$	c_p even $c_i = d_i$ or $d_i - 1$ $c_i - d_{i+1} \geq 1$ and $\equiv 1 \pmod 2$	$\varphi\begin{pmatrix} c_i \\ d_i \end{pmatrix} = (c_i + d_i + 1)_{d_i - c_i + 1}$ $\varphi^{-1}((a_i)_{b_i}) = \begin{pmatrix} (a_i - b_i)/2 \\ (a_i + b_i - 2)/2 \end{pmatrix}$
$T_0(q)$	$c_p = 1$ $c_i \leq 1 + d_i$ $c_i = 2 + d_{i+1}$ or $3 + d_{i+1}$	Same as for $\nu(q)$
$T_1(q)$	$c_p \leq 2$ $c_i \leq 1 + d_i$ $c_i = 2 + d_{i+1}$ or $3 + d_{i+1}$	Same as for $\nu(q)$
$S_0(q)$	$c_i = d_i$ or $d_i - 1$ $c_i \geq 1 + d_{i+1}$	Same as for $U_0(q)$
$S_1(q)$	$c_p, d_p \geq 1$ $c_i = d_i$ or $d_i - 1$ $c_i \geq 1 + d_{i+1}$	Same as for $U_0(q)$

Remark 4.1.2. *In the definition of F–partitions $c_i,\ d_i \geq 0$, but in interpreting the mock theta function $\rho(q)$ in terms of F–partitions we allow $d_p = -1$.*

In a similar way as done for $\chi(q)$, we attach weight (wherever required) $(-1)^h$ to the count of each F–partition generated by the unsigned version of the corresponding mock theta function. Table 4.3 provides the value of h for different mock theta functions.

Remark 4.1.3. *In the case of F–partition-theoretic interpretation of $T_1(q)$, if $c_p \neq 0$ then the virtual column $\begin{pmatrix} 0 \\ -1 \end{pmatrix}$ is considered to be the last column in the partition for weight calculations.*

Table 4.3 – Value of h for different mock theta functions

Function	h
$\nu(q)$	$\sum_{i=1}^{p}(d_i - c_i + 1)$
$\phi(q)$	$\frac{1}{2}(c_1 - 2p + 2)$
$f_0(q)$	c_1
$f_1(q)$	c_1
$U_0(q)$	$\frac{1}{2}\left[\sum_{i=1}^{p-1}(c_i - d_{i+1} - 1) + c_p\right]$
$T_0(q)$	$\sum_{i=1}^{p}(d_i - c_i + 1)$
$T_1(q)$	$\sum_{i=1}^{p}(d_i - c_i + 1)$
$S_0(q)$	$\sum_{i=1}^{p-1}(c_i - d_{i+1} - 1) + c_p$
$S_1(q)$	$\sum_{i=1}^{p-1}(c_i - d_{i+1} - 1) + c_p - 1$

It was not possible to include the mock theta function $U_1(q)$ (given by (1.40)) in the Table 4.2 because of a lengthy set of conditions and mappings. And, the interpretation of its unsigned version is given in the following theorem:

Theorem 4.1.2. *For $s \geq 1$, let $C_2(s)$ denote the* F*–partitions of s satisfying*

(i) *if* $c_p = 0$, d_p is even;

(ii) $c_i \leq d_i + 1$, $1 \leq i \leq p$;

(iii) $$d_{i+1} = \begin{cases} c_i - 1 & \text{if } a_{i+1} \equiv b_{i+1}, a_i \equiv b_i \pmod 2, \\ c_i - 3 & \text{if } a_{i+1} \not\equiv b_{i+1}, a_i \not\equiv b_i \pmod 2, \quad 1 \leq i \leq p-1. \\ c_i - 2 & \qquad \text{otherwise,} \end{cases}$$

Then

$$\sum_{s=1}^{\infty} C_2(s) q^s = \sum_{n=0}^{\infty} \frac{(-q;q^2)_n q^{(n+1)^2}}{(q^2;q^4)_{n+1}}.$$

Proof. A bijection between the partitions enumerated by $C_2(s)$ and $B_{11}(s)$ is given by

$$\varphi\begin{pmatrix} c_i \\ d_i \end{pmatrix} = \begin{cases} (c_i + d_i + 1)_{d_i - c_i + 1} & \text{if } c_i \equiv d_i \pmod 2, \\ (c_i + d_i + 1)_{d_i - c_i + 2} & \text{if } c_i \not\equiv d_i \pmod 2, \end{cases} \tag{4.8}$$

and the inverse mapping is defined to be

$$\varphi^{-1}((a_i)_{b_i}) = \begin{cases} \begin{pmatrix} (a_i-b_i)/2 \\ (a_i+b_i-2)/2 \end{pmatrix} & \text{if } a_i \text{ odd,} \\ \begin{pmatrix} (a_i-b_i+1)/2 \\ (a_i+b_i-3)/2 \end{pmatrix} & \text{if } a_i \text{ even.} \end{cases} \tag{4.9}$$

□

The weight attached to the partitions generated by the unsigned version of $U_1(q)$ is $(-1)^h$ with

$$h = \frac{1}{2}\left[\sum_{r}(d_i-c_i) - \sum_{r'}(d_i-c_i+1)\right],$$

where r runs over those values of i for which $c_i \equiv d_i \pmod 2$ and r' runs over those values of i for which $c_i \not\equiv d_i \pmod 2$.

4.2 Proof by splitting partitions into classes

In this section, we will prove the F–partition-theoretic interpretation for one mock theta function $\varrho(q)$ using the approach of splitting the partitions into classes and thereby arriving at the recurrence relation. We will provide necessary classes and omit the details of proof. However, to accomplish the above we need the following result:

Theorem 4.2.1. *Let $C_3(s)$ denote the number of F–partitions satisfying*

(i) $c_p = 2$;

(ii) $d_i - c_i \geq -1$ *and* $\equiv 0,\ 2 \pmod 3$, $1 \leq i \leq p$;

(iii) $c_i = d_{i+1} + 3,\ 1 \leq i \leq p-1$.

Then

$$\sum_{s=1}^{\infty} C_3(s)q^s = \sum_{n=0}^{\infty} \frac{(-q;q^2)_{n+1}q^{2(n+1)(n+2)}}{(q^3;q^6)_{n+1}}.$$

Remark 4.2.1. *Let $C_3(p, s)$ denote the number of F–partitions enumerated by $C_3(s)$ with the extra condition that there are p columns.*

Proof. We split the partitions enumerated by $C_3(p, s)$ into three classes. The first class contains the partitions with $d_p = 1$. If we delete the last column and subtract 2 from c_i as well as d_i for $1 \leq i \leq p-1$, we get a F–partition of $s - 4p$ into $p - 1$ columns satisfying conditions (i)–(iii). Thus, the partitions of this class are enumerated by $C_3(p-1, s-4p)$.

Similarly, partitions in the second class satisfy $d_p = 2$. Here, by deleting this column and subtracting 3 from c_i as well as d_i for $1 \leq i \leq p-1$, we get a partition enumerated by $C_3(p-1, s-6p+1)$. However, the partitions in the third class satisfy $d_p > 2$. Subtracting 3 from d_p and 3 from each of c_i and d_i for $1 \leq i \leq p-1$, we get a partition enumerated by $C_3(p, s-6p+3)$. Thus, with these reversible transformations we get the following recurrence relation

$$C_3(p, s) = C_3(p-1, s-4p) + C_3(p-1, s-6p+1) + C_3(p, s-6p+3).$$

□

Now, we proceed to our main result.

Theorem 4.2.2. *Let $C_4(s)$ denote the number of F–partitions satisfying*

(i) $c_p = 0$ *or* 2;

(ii) $d_i - c_i \geq -1$ *and* $\equiv 0,\ 2 \pmod 3$, $1 \leq i \leq p$;

(iii) $c_i = d_{i+1} + 3,\ 1 \leq i \leq p-1$.

Then

$$\sum_{s=0}^{\infty} C_4(s) q^s = \varrho(q).$$

Proof. Again, we split the partitions enumerated by $C_4(p, s)$ into three classes, the first of which contains partitions with $c_p = 2$. If we consider these partitions with the virtual column $\begin{pmatrix} 0 \\ -1 \end{pmatrix}$ as their last column, then by simply deleting this virtual column we get a partition enumerated by $C_3(p-1, s)$. Partitions in the second class satisfy $c_p = d_p = 0$ and these are enumerated by $C_3(p-1, s-2p+1)$. Whereas, partitions of the third class are those which satisfy $c_p = 0$ and $d_p > 0$. Partitions of this class are enumerated by $C_4(p, s-6p+3)$. Thus, we arrive at the recurrence relation

$$C_4(p, s) = C_3(p-1, s) + C_3(p-1, s-2p+1) + C_4(p, s-6p+3).$$

□

A similar proof can be developed for a number of mock theta functions discussed in this chapter.

4.3 Generalized results

In this section, we provide interpretations of the generalized versions of some of the mock theta functions discussed in this chapter. We have already given generalized versions in Chapters 2 and 3, here we provide F–partition-theoretic versions.

4.3.1 Generalization of $\psi_0(q) - 1$ and $\psi_1(q)$

Theorem 4.3.1. *For $k, \alpha \geq 1$, $\mathscr{F}_1^{(\alpha,k)}(q)$ generates the F–partitions satisfying:*

(i) $c_p = 0$ *or* k;

(ii) $d_i = \alpha - 1$, $1 \leq i \leq p$;

(iii) $c_i = c_{i+1} + \alpha$ *or* $c_{i+1} + \alpha + k$, $1 \leq i \leq p-1$.

4.3.2 Generalization of $\phi(q)$, $f_0(q)$ and $f_1(q)$

Theorem 4.3.2. $\mathscr{F}_4^{(\alpha,\beta,k)}(q)$ *generates the F–partitions satisfying*

(i) $c_p \geq \beta$ *and* $\equiv \beta \pmod{k}$;

(ii) $d_i = \alpha - 1$, $1 \leq i \leq p$;

(iii) $c_i - c_{i+1} \geq 2\alpha$ *and* $\equiv 2\alpha \pmod{k}$, $1 \leq i \leq p-1$;

(iv) *every partition is counted with weight* $(-1)^t$, *where* $t = \frac{1}{k}(c_1 - \beta - (2p-2)\alpha)$.

4.3.3 Generalization of $\nu(q)$

Theorem 4.3.3. $\mathscr{F}_6^{(\alpha,\beta,k)}(q)$ *generates the* F*–partitions satisfying*

(i) $d_p = 0$;

(ii) $c_i - d_i - \beta + 1 \geq 0$ *and* $\equiv \alpha \pmod{k}$, $1 \leq i \leq p$;

(iii) $d_i = c_{i+1} - \beta + 1$, $1 \leq i \leq p-1$;

(iv) *every partition is counted with weight* $(-1)^t$, *where* $t = \frac{1}{k}\sum_{i=1}^{p}(c_i - d_i - \alpha - \beta + 1)$.

A direct proof as well as the proof by establishing a mapping between $(n+t)$–color partitions and F–partitions for Theorems 4.3.1, 4.3.2 and 4.3.3 can be developed in a similar manner as established for the various theorems of this chapter.

4.4 Conclusion

We now have combinatorial interpretations of nineteen mock theta functions in terms of $(n+t)$–color partitions and F–partitions. This leads to nineteen new combinatorial identities involving $(n+t)$–color partitions and F–partitions. In the subsequent chapters, we explore the combinatorics of some more mock theta functions which will provide another set of such combinatorial identities.

Chapter 5

Interpreting some fifth and sixth order mock theta functions in terms of $(n+t)$–color partitions and F–partitions by attaching weights

A common thread connecting all the mock theta functions interpreted in literature and in Chapters 2, 3 and 4 is that the powers of q occurring in the numerator of q-series, defining the mock theta function, is quadratic in 'n'. In this chapter, in particular we handle those mock theta functions in which these powers of q are linear in 'n'. Therefore, we choose fifth order mock theta functions $\chi_0(q)$ and $\chi_1(q)$ (given by (1.23) and (1.28)) of Ramanujan. In addition, we choose sixth order mock theta functions $\lambda(q)$, $\mu(q)$, $\phi_-(q)$ and $\psi_-(q)$ (given by (1.48)–(1.51)). Since the series defining $\mu(q)$ does not converge, so we take $2\mu(q)$ as the sum of its sequences of even partial sums and odd partial sums as follows:

$$2\mu(q) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{n+1}(1+q^n)(q;q^2)_n}{(-q;q)_{n+1}} \tag{5.1}$$

We interpret these six mock theta functions in terms of $(n+t)$–color partitions and F–partitions. To interpret these functions we make use of two-line arrays to represent $(n+t)$–color partitions and then by mapping these $(n+t)$–color partitions to F–partitions we obtain the corresponding restrictions on the set of F–partitions generated by the mock theta function under consideration.

5.1 Interpretations of mock theta functions $\phi_{-}(q)$ and $\psi_{-}(q)$

First of all, we proceed in a constructive way to obtain the interpretation for

$$\phi_{-}(q) := \sum_{n=1}^{\infty} \frac{q^n(-q;q)_{2n-1}}{(q;q^2)_n} = \sum_{n=1}^{\infty} \frac{q^n(-q;q^2)_n(-q^2;q^2)_{n-1}}{(q;q^2)_n}.$$

In the expression for $\phi_{-}(q)$, q^n generates n parts each equal to 1. $(-q;q^2)_n$ generates odd natural numbers $\leq (2n-1)$, say $u_1 \times 1, u_2 \times 3, \ldots, u_n \times (2n-1)$, u_i=0 or 1. $(-q^2;q^2)_{n-1}$ generates even natural numbers $\leq (2n-2)$, say $v_1 \times 2, v_2 \times 4, \ldots, v_{n-1} \times (2n-2)$, v_i=0 or 1. $(q;q^2)_n^{-1}$ generates multiples of odd natural numbers $\leq (2n-1)$, say $w_1 \times 1, w_2 \times 3, \ldots, w_n \times (2n-1)$, $w_i \geq 0$. Thus, the n–color partition generated by $\phi_{-}(q)$ can be written in two-line array as

$$\begin{pmatrix} \begin{matrix} 1+2(u_n+\cdots+u_2+w_n+\cdots+ \\ +w_2+v_{n-1}+\cdots+v_1)+u_1+w_1 \end{matrix} & \cdots & \begin{matrix} 1+2u_n+2w_n+2v_{n-1} \\ +w_{n-1}+u_{n-1} \end{matrix} & 1+u_n+w_n \\ 1+w_1 & \cdots & 1+w_{n-1} & 1+w_n \end{pmatrix}$$

Here, the first line of the array represents the part and second line represents the corresponding subscript of the n–color partition. Since the last column of two-line array corresponds to the least part $(a_n)_{b_n}$ of the n–color partition, so u_n=0 or 1 implies

$$a_n - b_n = 0 \text{ or } 1. \tag{5.2}$$

The i^{th} and $(i+1)^{th}$ part in the n–color partition are

$$a_i = 1 + 2(u_n + \cdots + u_{i+1} + w_n + \cdots + w_{i+1} + v_{n-1} + \cdots + v_i) + u_i + w_i, \tag{5.3}$$

$$b_i = 1 + w_i, \tag{5.4}$$

$$a_{i+1} = 1 + 2(u_n + \cdots + u_{i+2} + w_n + \cdots + w_{i+2} + v_{n-1} + \cdots + v_{i+1}) + u_{i+1} + w_{i+1}, \tag{5.5}$$

$$b_{i+1} = 1 + w_{i+1}. \tag{5.6}$$

Since $0 \leq u_i, v_i \leq 1$, so by using (5.3)–(5.6) we get

$$(w.d)_{i,i+1} = -2 + u_{i+1} + u_i + 2v_i \leq 2. \tag{5.7}$$

Also,

$$(w.d)_{i,i+1} \geq \begin{cases} -2 & \text{if } a_{i+1} \equiv b_{i+1}, a_i \equiv b_i \pmod 2, \\ 0 & \text{if } a_{i+1} \not\equiv b_{i+1}, a_i \not\equiv b_i \pmod 2, \\ -1 & \text{otherwise.} \end{cases} \tag{5.8}$$

Now, if we map each part $(a_i)_{b_i}$ of n–color partition generated by $\phi_-(q)$ to a single column of the F–partition given by $\begin{pmatrix} c_1 & c_2 & \cdots & c_n \\ d_1 & d_2 & \cdots & d_n \end{pmatrix}$ in the following manner:

$$\varphi((a_i)_{b_i}) = \begin{cases} \begin{pmatrix} (a_i+b_i-1)/2 \\ (a_i-b_i-1)/2 \end{pmatrix} & \text{if } a_i \not\equiv b_i \pmod 2, \\ \begin{pmatrix} (a_i-b_i)/2 \\ (a_i+b_i-2)/2 \end{pmatrix} & \text{if } a_i \equiv b_i \pmod 2. \end{cases} \tag{5.9}$$

Then the inverse map φ^{-1} which maps a column $\begin{pmatrix} c_i \\ d_i \end{pmatrix}$ of F–partition to n–color part $(a_i)_{b_i}$ can be defined as

$$\varphi^{-1}\begin{pmatrix} c_i \\ d_i \end{pmatrix} = \begin{cases} (c_i+d_i+1)_{d_i-c_i+1} & \text{if } c_i \leq d_i, \\ (c_i+d_i+1)_{c_i-d_i} & \text{if } c_i > d_i. \end{cases} \tag{5.10}$$

Thus, under the inverse mapping φ^{-1}

$$(w.d)_{i,i+1} = \begin{cases} 2c_i - 2d_{i+1} - 2 & \text{if } c_i \leq d_i,\ c_{i+1} \leq d_{i+1}, \\ 2d_i - 2d_{i+1} - 1 & \text{if } c_i > d_i,\ c_{i+1} \leq d_{i+1}, \\ 2c_i - 2c_{i+1} - 1 & \text{if } c_i \leq d_i,\ c_{i+1} > d_{i+1}, \\ 2d_i - 2c_{i+1} & \text{if } c_i > d_i,\ c_{i+1} > d_{i+1}. \end{cases} \tag{5.11}$$

And under the mapping φ

$$2c_i - 2c_{i+1} = \begin{cases} (w.d)_{i,i+1} + 2b_{i+1} & \text{if } a_{i+1} \equiv b_{i+1},\ a_i \equiv b_i \pmod 2, \\ (w.d)_{i,i+1} + 2(b_{i+1} + b_i) - 1 & \text{if } a_{i+1} \equiv b_{i+1},\ a_i \not\equiv b_i \pmod 2, \\ (w.d)_{i,i+1} + 1 & \text{if } a_{i+1} \not\equiv b_{i+1},\ a_i \equiv b_i \pmod 2, \\ (w.d)_{i,i+1} + 2b_i & \text{if } a_{i+1} \not\equiv b_{i+1},\ a_i \not\equiv b_i \pmod 2, \end{cases} \tag{5.12}$$

$$2c_i - 2d_{i+1} = \begin{cases} (w.d)_{i,i+1} + 2 & \text{if } a_{i+1} \equiv b_{i+1},\ a_i \equiv b_i \pmod 2, \\ (w.d)_{i,i+1} + 2b_i + 2 & \text{if } a_{i+1} \equiv b_{i+1},\ a_i \not\equiv b_i \pmod 2, \\ (w.d)_{i,i+1} + 2b_{i+1} + 1 & \text{if } a_{i+1} \not\equiv b_{i+1},\ a_i \equiv b_i \pmod 2, \\ (w.d)_{i,i+1} + 2b_{i+1} + 2b_i & \text{if } a_{i+1} \not\equiv b_{i+1},\ a_i \not\equiv b_i \pmod 2, \end{cases} \tag{5.13}$$

$$2d_i - 2c_{i+1} = \begin{cases} (w.d)_{i,i+1} + 2b_i - 2 & \text{if } a_{i+1} \equiv b_{i+1},\ a_i \equiv b_i \pmod 2, \\ (w.d)_{i,i+1} + 2b_{i+1} - 2 & \text{if } a_{i+1} \equiv b_{i+1},\ a_i \not\equiv b_i \pmod 2, \\ (w.d)_{i,i+1} + 2b_i - 1 & \text{if } a_{i+1} \not\equiv b_{i+1},\ a_i \equiv b_i \pmod 2, \\ (w.d)_{i,i+1} & \text{if } a_{i+1} \not\equiv b_{i+1},\ a_i \not\equiv b_i \pmod 2, \end{cases} \tag{5.14}$$

$$2d_i - 2d_{i+1} = \begin{cases} (w.d)_{i,i+1} + 2b_i & \text{if } a_{i+1} \equiv b_{i+1},\ a_i \equiv b_i \pmod 2, \\ (w.d)_{i,i+1} + 1 & \text{if } a_{i+1} \equiv b_{i+1},\ a_i \not\equiv b_i \pmod 2, \\ (w.d)_{i,i+1} + 2b_{i+1} + 2b_i - 1 & \text{if } a_{i+1} \not\equiv b_{i+1},\ a_i \equiv b_i \pmod 2, \\ (w.d)_{i,i+1} + 2b_{i+1} & \text{if } a_{i+1} \not\equiv b_{i+1},\ a_i \not\equiv b_i \pmod 2. \end{cases} \tag{5.15}$$

By analyzing the conditions (5.7) and (5.8) along with (5.12)–(5.15), we find the following restrictions on F–partitions generated by $\phi_{-}(q)$:

$$\left.\begin{array}{llllll} \text{if} & c_i \le d_i & \text{and} & c_{i+1} \le d_{i+1}, & \text{then} & 0 \le c_i - d_{i+1} \le 2, \\ \text{if} & c_i \le d_i & \text{and} & c_{i+1} > d_{i+1}, & \text{then} & 0 \le c_i - c_{i+1} \le 1, \\ \text{if} & c_i > d_i & \text{and} & c_{i+1} \le d_{i+1}, & \text{then} & 0 \le d_i - d_{i+1} \le 1, \\ \text{if} & c_i > d_i & \text{and} & c_{i+1} > d_{i+1}, & \text{then} & 0 \le d_i - c_{i+1} \le 1. \end{array}\right\} \tag{5.16}$$

Also, under the mapping φ, using (5.2) we get

$$c_n \text{ or } d_n = 0. \tag{5.17}$$

Reverse implications follow immediately.

Thus, we see that $\phi_-(q)$ is generating function for certain restricted n–color partitions and F–partitions as stated in the following theorem:

Theorem 5.1.1. *$\phi_-(q)$ is the generating function for n–color partitions satisfying conditions* (5.2), (5.7)–(5.8) *and F–partitions satisfying* (5.17)–(5.16).

Remark 5.1.1. *Another mock theta function $\psi_-(q)$ has similar interpretation as $\phi_-(q)$, the only difference is that in condition* (5.2) *$a_n - b_n$=0 and hence in* (5.17) *c_n=0.*

5.2 Interpretations of mock theta functions by attaching weights

Now, we interpret mock theta functions $\chi_0(q)$, $\chi_1(q)$, $\lambda(q)$ and $2\mu(q)$. In case of these functions, we attach certain weight to the count of each n–color partition as well as F–partition. We are stating the results without proofs here as these can be established in a similar manner as discussed in case of $\phi_-(q)$. Consider the mock theta function

$$\chi_0(q) := \sum_{n=0}^{\infty} \frac{q^n}{(q^{n+1};q)_n} = \sum_{n=0}^{\infty} \frac{q^n}{(q;q^2)_n(-q;q)_n}. \tag{5.18}$$

First of all, we give the interpretation of the following unsigned version of $\chi_0(q)$:

$$\sum_{n=0}^{\infty} \frac{q^n}{(q;q^2)_n(q;q)_n}.$$

Theorem 5.2.1. *Let $D_1(s)$ denote the number of n–color partitions of s satisfying*

$$(w.d)_{i,i+1} \geq -2, \quad 1 \leq i \leq p-1.$$

And $E_1(s)$ *denote the number of* F*–partitions of* s *satisfying*

$$\left.\begin{array}{llllll} \text{if} & c_i \le d_i & \text{and} & c_{i+1} \le d_{i+1}, & \text{then} & c_i \ge d_{i+1}, \\ \text{if} & c_i \le d_i & \text{and} & c_{i+1} > d_{i+1}, & \text{then} & 2c_i \ge 2c_{i+1}-1, \\ \text{if} & c_i > d_i & \text{and} & c_{i+1} \le d_{i+1}, & \text{then} & 2d_i \ge 2d_{i+1}-1, \\ \text{if} & c_i > d_i & \text{and} & c_{i+1} > d_{i+1}, & \text{then} & d_i \ge c_{i+1}-1, \end{array}\right\} 1 \le i \le p-1.$$

Then

$$\sum_{s=0}^{\infty} D_1(s)q^s = \sum_{s=0}^{\infty} E_1(s)q^s = \sum_{n=0}^{\infty} \frac{q^n}{(q;q^2)_n(q;q)_n}.$$

Now, if we attach weight $(-1)^t$, where $t = a_1 - b_1$, to the count of each n–color partition then we see that $\chi_0(q)$ is the generating function for this weighted number of n–color partitions. In a similar manner, we find that $\chi_0(q)$ is the generating function for weighted number of F–partitions, where each F–partition enumerated by $E_1(s)$ is counted with weight $(-1)^h$ and

$$h = \begin{cases} 2c_1 & \text{if } c_1 \le d_1, \\ 2d_1+1 & \text{if } c_1 > d_1. \end{cases}$$

For a better understanding of the idea, we present an example here.

Example 5.2.1. *In the expansion of unsigned version of* $\chi_0(q)$

$$\sum_{n=0}^{\infty} \frac{q^n}{(q;q^2)_n(q;q)_n} = 1 + q + 3q^2 + 6q^3 + 11q^4 + \cdots,$$

the coefficient of q^4 *is* 11 *and in the expansion of*

$$\chi_0(q) = 1 + q + q^2 + 2q^3 + q^4 + \cdots,$$

the coefficient is 1*. Thus, there are* 11 *relevant* n*–color partitions and* F*–partitions for* $\chi_0(q)$*. Out of these* 11 *partitions* 6 *have positive weight and* 5 *have negative weight as shown in* Table 5.1.

Table 5.1 – Partitions enumerated by $\chi_0(q)$ for $s = 4$ with weight calculations

n–color partition	F–partition	Weight
4_1	$\begin{pmatrix} 2 \\ 1 \end{pmatrix}$	-1
4_2	$\begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix}$	$+1$
4_3	$\begin{pmatrix} 3 \\ 0 \end{pmatrix}$	-1
4_4	$\begin{pmatrix} 0 \\ 3 \end{pmatrix}$	$+1$
$3_1 + 1_1$	$\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$	$+1$
$3_2 + 1_1$	$\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$	-1
$3_3 + 1_1$	$\begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}$	$+1$
$2_1 + 2_1$	$\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$	-1
$2_1 + 1_1 + 1_1$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	-1
$2_2 + 1_1 + 1_1$	$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$	$+1$
$1_1 + 1_1 + 1_1 + 1_1$	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$+1$

Mock theta function $\chi_1(q)$ is given by

$$\chi_1(q) := \sum_{n=0}^{\infty} \frac{q^n}{(q^{n+1}; q)_{n+1}} = \sum_{n=0}^{\infty} \frac{q^n}{(q; q^2)_{n+1}(-q; q)_n}.$$

The interpretation of the unsigned version of $\chi_1(q)$ is given in the following theorem:

Theorem 5.2.2. *Let $D_2(s)$ denote the number of $(n+1)$–color partitions of s such that*

(i) $a_p - b_p = -1$;

(ii) $(w.d)_{i,i+1} \geq -2,\ 1 \leq i \leq p-2$;

(iii) $(w.d)_{p-1,p} \geq -1$.

And $E_2(s)$ denote the number of F–partitions of s such that

(iv) $c_p > d_p$ *and* $d_p = -1$;

(v)

$$\left.\begin{array}{lllll} \text{if} & c_i \le d_i & \text{and} & c_{i+1} \le d_{i+1}, & \text{then} \quad c_i \ge d_{i+1}, \\ \text{if} & c_i \le d_i & \text{and} & c_{i+1} > d_{i+1}, & \text{then} \quad 2c_i \ge 2c_{i+1} - 1, \\ \text{if} & c_i > d_i & \text{and} & c_{i+1} \le d_{i+1}, & \text{then} \quad 2d_i \ge 2d_{i+1} - 1, \\ \text{if} & c_i > d_i & \text{and} & c_{i+1} > d_{i+1}, & \text{then} \quad d_i \ge c_{i+1} - 1, \end{array}\right\} 1 \le i \le p-2;$$

(vi)

$$\begin{array}{llll} \text{if} & c_{p-1} \le d_{p-1}, & \text{then} & c_{p-1} \ge c_p, \\ \text{if} & c_{p-1} > d_{p-1}, & \text{then} & 2(d_{p-1} - c_p) \ge -1. \end{array}$$

Then

$$\sum_{s=0}^{\infty} D_2(s)q^s = \sum_{s=0}^{\infty} E_2(s)q^s = \sum_{n=0}^{\infty} \frac{q^n}{(q;q^2)_{n+1}(q;q)_n}.$$

Remark 5.2.1. *In the definition of F–partitions $c_i,\ d_i \ge 0$, but in interpreting the mock theta function $\chi_1(q)$ in terms of F–partitions this condition has to be modified to include $d_p = -1$.*

In a similar manner as for $\chi_0(q)$, the interpretation of $\chi_1(q)$ can also be obtained by attaching weights to the n–color partitions and F–partitions enumerated by $D_2(s)$ and $E_2(s)$ respectively. Here, the value of t is given by

$$t = \begin{cases} 0 & \text{if } p = 1, \\ a_1 - b_1 & \text{if } p > 1, \end{cases}$$

and value of h is as follows:

$$h = \begin{cases} 2c_1 & \text{if } c_1 \le d_1 \text{ and } p > 1, \\ 2d_1 + 1 & \text{if } c_1 > d_1 \text{ and } p > 1, \\ 0 & \text{if } p = 1. \end{cases}$$

Theorem 5.2.3. *Let $D_3(s)$ denote the number of n–color partitions of s such that*

(i) $b_i = 1$ *or* 2, $1 \le i \le p$;

(ii) $(w.d)_{i,i+1} \geq -2$, $1 \leq i \leq p-1$.

And $E_3(s)$ *denote the number of* F*–partitions of* s *such that*

(iii) $-1 \leq c_i - d_i \leq 2$, $1 \leq i \leq p$;

(iv)

$$\left.\begin{array}{llllll} \text{if} & c_i \leq d_i & \text{and} & c_{i+1} \leq d_{i+1}, & \text{then} & c_i \geq d_{i+1}, \\ \text{if} & c_i \leq d_i & \text{and} & c_{i+1} > d_{i+1}, & \text{then} & 2c_i \geq 2c_{i+1} - 1, \\ \text{if} & c_i > d_i & \text{and} & c_{i+1} \leq d_{i+1}, & \text{then} & 2d_i \geq 2d_{i+1} - 1, \\ \text{if} & c_i > d_i & \text{and} & c_{i+1} > d_{i+1}, & \text{then} & d_i \geq c_{i+1} - 1, \end{array}\right\} 1 \leq i \leq p-1.$$

Then

$$\sum_{s=0}^{\infty} D_3(s)q^s = \sum_{s=0}^{\infty} E_3(s)q^s = \sum_{n=0}^{\infty} \frac{q^n(-q;q^2)_n}{(q;q)_n}.$$

Now, by attaching weight $(-1)^t$ to the n–color partitions enumerated by $D_3(s)$ and $(-1)^h$ to the F–partitions enumerated by $E_3(s)$, we get the weighted number of partitions generated by $\lambda(q)$, where

$$t = a_1 - \sum_{i=2}^{p} b_i + 2p - 2,$$

$$h = c_1 + d_1 - \sum_{r} (d_i - c_i) - \sum_{r'} (c_i - d_i - 1) + p,$$

r and r' run over those values of i $(1 \leq i \leq p-1)$ for which $d_i = c_i + 1$ and $c_i = d_i + 2$ respectively.

In the end, we are discussing the function $2\mu(q)$ given by

$$\begin{aligned} 2\mu(q) =& \sum_{n=0}^{\infty} \frac{(-1)^n q^{n+1}(1+q^n)(q;q^2)_n}{(-q;q)_{n+1}} \\ =& \sum_{n=0}^{\infty} \frac{(-1)^n q^{n+1}(q;q^2)_n}{(-q;q)_{n+1}} + \sum_{n=0}^{\infty} \frac{(-1)^n q^{2n+1}(q;q^2)_n}{(-q;q)_{n+1}} \\ =& S_1(q) + S_2(q), \end{aligned} \tag{5.19}$$

where

$$S_1(q)=\sum_{n=0}^{\infty}\frac{(-1)^nq^{n+1}(q;q^2)_n}{(-q;q)_{n+1}},$$

$$S_2(q)=\sum_{n=0}^{\infty}\frac{(-1)^nq^{2n+1}(q;q^2)_n}{(-q;q)_{n+1}}.$$

We interpret the unsigned versions of $S_1(q)$ and $S_2(q)$ in the following theorems and thereby interpret $S_1(q)$ and $S_2(q)$ by attaching weights, which in turn provides the interpretation of $2\mu(q)$.

Theorem 5.2.4. *For $s\geq 1$, let $D_4(s)$ denote the number of n–color partitions of s satisfying*

(i) $b_p=1$;

(ii) $b_i=1$ *or* 2, $1\leq i\leq p$;

(iii) $(w.d)_{i,i+1}\geq -2$, $1\leq i\leq p-1$.

And $E_4(s)$ denote the number of F–partitions satisfying

(iv) $c_p-d_p=0$ *or* 1;

(v) $-1\leq c_i-d_i\leq 2$, $1\leq i\leq p$;

(vi)

$$\left.\begin{array}{llllll}\text{if} & c_i\leq d_i & \text{and} & c_{i+1}\leq d_{i+1}, & \text{then} & c_i\geq d_{i+1},\\ \text{if} & c_i\leq d_i & \text{and} & c_{i+1}> d_{i+1}, & \text{then} & 2c_i\geq 2c_{i+1}-1,\\ \text{if} & c_i> d_i & \text{and} & c_{i+1}\leq d_{i+1}, & \text{then} & 2d_i\geq 2d_{i+1}-1,\\ \text{if} & c_i> d_i & \text{and} & c_{i+1}> d_{i+1}, & \text{then} & d_i\geq c_{i+1}-1,\end{array}\right\}1\leq i\leq p-1.$$

Then

$$\sum_{s=1}^{\infty}D_4(s)q^s=\sum_{s=1}^{\infty}E_4(s)q^s=\sum_{n=0}^{\infty}\frac{q^{n+1}(-q;q^2)_n}{(q;q)_{n+1}}.$$

To get the weighted number of partitions enumerated by $S_1(q)$, $(-1)^t$ is the weight attached to the n–color partitions enumerated by $D_4(s)$, where

$$t=a_1+p-2-\sum_{i=2}^{p-1}(b_i-1),$$

and $(-1)^h$ is the weight attached to the F–partitions enumerated by $E_4(s)$, where

$$h = c_1 + d_1 + p - 1 - \sum_r (d_i - c_i) - \sum_{r'} (c_i - d_i - 1),$$

r and r' run over those values of i $(2 \leq i \leq p-1)$ for which $d_i = c_i + 1$ and $d_i = c_i + 2$ respectively.

Theorem 5.2.5. *For $s \geq 1$, let $D_5(s)$ denote the number of n–color partitions of s satisfying*

(i) $b_p = 1$;

(ii) $b_i = 1$ *or* 2, $1 \leq i \leq p$;

(iii) $(w.d)_{i,i+1} \geq -2$, $1 \leq i \leq p-2$;

(iv) $(w.d)_{p-1,p} \geq -1$.

And $E_5(s)$ denote the number of F–partitions of s satisfying

(v) $c_p - d_p = 0$ *or* 1;

(vi) $-1 \leq c_i - d_i \leq 2$, $1 \leq i \leq p$;

(vii)

$$\left.\begin{array}{llllr} \text{if} & c_i \leq d_i & \text{and} & c_{i+1} \leq d_{i+1}, & \text{then} \quad c_i \geq d_{i+1}, \\ \text{if} & c_i \leq d_i & \text{and} & c_{i+1} > d_{i+1}, & \text{then} \quad 2c_i \geq 2c_{i+1} - 1, \\ \text{if} & c_i > d_i & \text{and} & c_{i+1} \leq d_{i+1}, & \text{then} \quad 2d_i \geq 2d_{i+1} - 1, \\ \text{if} & c_i > d_i & \text{and} & c_{i+1} > d_{i+1}, & \text{then} \quad d_i \geq c_{i+1} - 1, \end{array}\right\} 1 \leq i \leq p-2;$$

(viii)

$$\begin{array}{llllr} \text{if} & c_p = d_p & \text{and} & c_{p-1} \leq d_{p-1}, & \text{then} \quad 2c_{p-1} - 2d_p \geq 1, \\ \text{if} & c_p = d_p & \text{and} & c_{p-1} > d_{p-1}, & \text{then} \quad d_{p-1} \geq d_p, \\ \text{if} & c_p = d_p + 1 & \text{and} & c_{p-1} \leq d_{p-1}, & \text{then} \quad c_{p-1} \geq c_p, \\ \text{if} & c_p = d_p + 1 & \text{and} & c_{p-1} > d_{p-1}, & \text{then} \quad 2d_{p-1} - 2c_p \geq -1. \end{array}$$

Then

$$\sum_{s=1}^{\infty} D_5(s) q^s = \sum_{s=1}^{\infty} E_5(s) q^s = \sum_{n=0}^{\infty} \frac{q^{2n+1}(-q;q^2)_n}{(q;q)_{n+1}}.$$

Now, the weighted number of n–color partitions generated by $S_2(q)$ is obtained by attaching weight $(-1)^t$ to the count of each n–color partition enumerated by $D_5(s)$, where

$$t = \begin{cases} a_1 + p - 1 - \sum_{i=2}^{p-1} (b_i - 1) & \text{if } p > 1, \\ a_1 - b_1 & \text{if } p = 1, \end{cases}$$

and the weighted number of F–partitions generated by $S_2(q)$ is obtained by attaching weight $(-1)^h$ to the count of each F–partition generated by $E_5(s)$, where

$$h = \begin{cases} c_1 + d_1 + p - \sum_{r} (d_i - c_i) - \sum_{r'} (c_i - d_i - 1) & \text{if } p > 1, \\ c_1 + d_1 & \text{if } p = 1, \end{cases}$$

r and r' run over those values of i $(2 \leq i \leq p-1)$ for which $d_i = c_i + 1$ and $d_i = c_i + 2$ respectively.

The number of n–color partitions or F–partitions generated by $2\mu(q)$ is equal to the sum of weighted number of respective partitions generated by $S_1(q)$ and $S_2(q)$.

5.3 Conclusion

In this chapter, we have included interpretations of six mock theta functions in terms of $(n+t)$–color partitions and F–partitions. The interpretations of these mock theta functions in terms of lattice paths cannot be obtained with the existing terminology of paths. Therefore, in the next chapter we introduce some modifications in the existing terminology of these paths so as to obtain these interpretations.

Chapter 6

Lattice paths with backward steps

6.1 Preliminaries

In this chapter, we discuss the mock theta functions of Chapter 5 with lattice paths. As discussed in Chapter 1, Agarwal and Bressoud [12] remarked that the lattice paths defined by them correspond to certain restricted $(n+t)$–color partitions only. It is very obvious that these can not correspond to $(n+t)$–color partitions with repeating parts. As far as mock theta functions of Chapter 5 are concerned, one can easily check that the $(n+t)$–color partitions generated by them can have repeating parts and it is quite clear by having a look at Example 5.2.1. It means that the terminology of lattice paths of Agarwal and Bressoud [12] is insufficient for interpreting these functions. Therefore, in order to handle these functions we modify these paths by introducing backward horizontal step.

Here we will be considering only those paths which have no valley above height zero.

Backward Horizontal Step*: from $(i, 0)$ to $(i-1, 0)$, allowed for $i > 0$ only.*

Now a plain can be of the following two types:

Forward Plain*: same as the plain described earlier.*

Backward Plain*: a section of path that consists of only backward horizontal steps, starts at a vertex preceded by a south-east step and terminates at a vertex followed by a north-east step. It will be considered as a plain of negative length.*

Since we have introduced backward horizontal steps, so a peak can now repeat and a peak having lower weight can occur later in the path. For example, the path shown in

Figure 6-1 corresponds to sequence of peaks $(1_1, 4_1, 3_2, 5_1)$. However, the introduction of backward steps can lead to ambiguity while reading the paths. Thus, it becomes necessary to exhibit the sequence of steps taken to draw the path, as the sequence of steps for a particular path is unique. In this chapter, we promptly use sequence of steps to describe these paths and denote the possible steps with the Greek letters in the following manner:

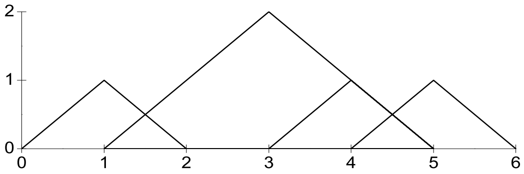


Figure 6-1 – lattice path corresponding to $(1_1, 4_1, 3_2, 5_1)$

North-East-α, South-East-β, Forward Horizontal-γ, Backward Horizontal-δ.

Therefore, the sequence of steps for the path of Figure 6-1 is $\alpha\beta\gamma\alpha\beta\delta\delta\delta\delta\alpha\alpha\beta\beta\delta\alpha\beta$. A α step followed by a β step or an initial β step constitute a peak. Since in our paths there is no valley above height zero, so whenever there exists a peak, α steps are followed by an equal number of β steps and the number of these steps gives the height of that peak. Weight of a peak is equal to the number of steps other than δ steps minus the number of δ steps before that peak. Length of forward plain is the number of γ steps and of the backward plain is the negative of the count of δ steps comprising the plain. Thus, the sequence $\alpha\beta\gamma\alpha\beta\delta\delta\delta\delta\alpha\alpha\beta\beta\delta\alpha\beta$ has first peak with weight 1 and height 1, second peak with weight 4 and height 1, third peak with weight 3 and height 2, and fourth peak with weight 5 and height 1. There is a plain of length 1 between the first and second peak, a plain of length -4 between second and third peak and a plain of length -1 between third and fourth peak.

6.2 Lattice paths with backward steps and mock theta functions

Now, we utilize lattice paths with this modified terminology to obtain the interpretations of mock theta functions discussed in Chapter 5.

Remark 6.2.1. *For interpreting the functions* $\chi_0(q)$, $\chi_1(q)$, $\lambda(q)$ *and* $2\mu(q)$, *we need to attach certain weight to the count of each relevant path which we denote by* $w_{P_i}(P) = (-1)^{t_i}$, $i = 1, 2, 5, 6, 7$, *for a path* $P \in P_i$, *where* P_i *is a set of paths satisfying certain specified conditions and value of* t_i *depends upon set of paths* P_i. *Also define*

$$p_i(s) = \sum_{\substack{|P|=s \\ P \in P_i}} w_{P_i}(P).$$

Remark 6.2.2. *In the sequel,* (x_i, y_i), $1 \leq i \leq p$, *denotes the* i^{th} *peak in the path, where* x_i *and* y_i *are the weight and height of the peak respectively.*

6.2.1 Fifth order mock theta functions

Theorem 6.2.1. *Let* P_1 *denote the set of paths starting at* $(0, 0)$ *with length of plain between any two consecutive peaks* ≥ -2 *and in front of the first peak* ≥ 0. *Then for* $t_1 = x_p - y_p$,

$$\chi_0(q) = \sum_{s=0}^{\infty} p_1(s) q^s.$$

Proof. **Step I.** Begin the construction of path with the factor q^n from $\frac{q^n}{(q;q^2)_n (q;q)_n}$, which generates a path with n peaks at $(1, 1)$ and a plain of length -2 between peaks. An example of such a path with four peaks is shown in Figure 6-2 and the corresponding sequence of steps is $\alpha\beta\delta\delta\alpha\beta\delta\delta\alpha\beta\delta\delta\alpha\beta$.

Step II. Factor $(q; q^2)_n^{-1}$ introduces nonnegative multiples of $1, 3, \ldots, 2n - 1$, say $u_1 \times 1, u_2 \times 3, \ldots, u_n \times (2n - 1)$, $u_i \geq 0$. Transform the path by increasing the number of α as well as β steps of the i^{th} peak by u_{n-i+1}. This increases the weight of each of the $(i + 1)^{th}$ to n^{th} peak by $2u_{n-i+1}$ and of the i^{th} peak by

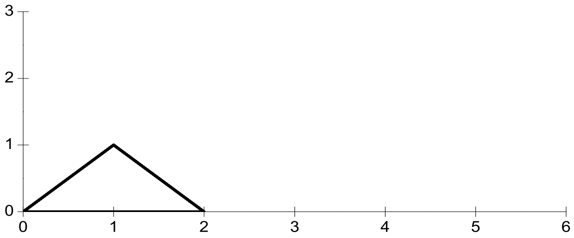


Figure 6-2 – Initial path for $n = 4$

u_{n-i+1}. Thus, for $u_1 = 2$, $u_2 = 1$, $u_3 = 0$ and $u_4 = 1$, the path of Figure 6-2 transforms to the path of Figure 6-3 and corresponding sequence of steps is given by $\alpha\alpha\beta\beta\delta\delta\alpha\beta\delta\delta\alpha\alpha\beta\beta\delta\delta\alpha\alpha\alpha\beta\beta\beta$.

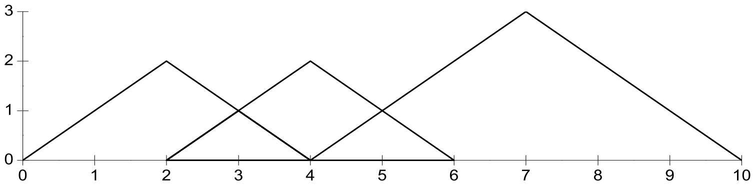


Figure 6-3 – Transformed path for $u_1 = 2$, $u_2 = 1$, $u_3 = 0$, $u_4 = 1$

Step III. The factor $(q;q)_n^{-1}$ generates nonnegative multiples of $1, 2, \ldots, n-1$, say $v_1 \times 1, v_2 \times 2, \ldots, v_n \times n$, $v_i \geq 0$. Introducing v_{n-i+1} γ steps in front of the i^{th} peak increases the weight of each of the i^{th} to n^{th} peaks by v_{n-i+1} (if some δ steps already exist before that peak then each delta step will cancel one γ step and remaining γ or δ steps will appear before that peak). For $v_1 = 1$, $v_2 = 0$, $v_3 = 2$ and $v_4 = 1$, the path of Figure 6-3 transforms to the path of Figure 6-4 with the sequence of steps $\gamma\alpha\alpha\beta\beta\alpha\beta\delta\delta\alpha\alpha\beta\beta\delta\alpha\alpha\alpha\beta\beta\beta$.

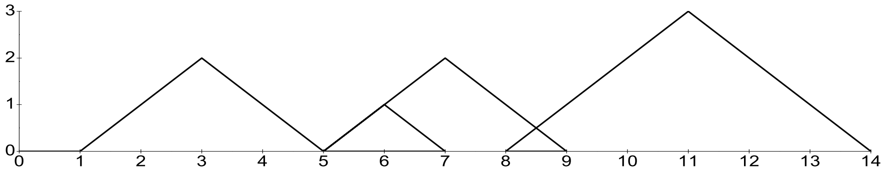


Figure 6-4 – Transformed path for $v_1 = 1$, $v_2 = 0$, $v_3 = 2$, $v_4 = 1$

Initially, there was a plain of length -2 between any two peaks therefore, in the final path the length of plain between any two consecutive peaks will be ≥ -2 and in front of the first peak ≥ 0. Thus, the path belongs to P_1 and every path of P_1 is uniquely generated in this manner.

The last peak (x_n, y_n) is given by

$$\begin{aligned} x_n =& 1 + 2\sum_{i=2}^{n} u_i + u_1 + \sum_{j=1}^{n} v_j, \\ y_n =& 1 + u_1. \end{aligned} \tag{6.1}$$

The series representing $\chi_0(q)$ contains the factor $(-q;q)_n^{-1}$ instead of $(q;q)_n^{-1}$, so following weight is to be attached to the count of each path P generated in the above manner and it is given by

$$w_{P_1}(P) = (-1)^{t_1} = (-1)^{\sum_{j=1}^{n} v_j} = (-1)^{\sum_{j=1}^{n} v_j + 2\sum_{i=2}^{n} u_i}.$$

Hence $t_1 = x_n - y_n$. $\square$

In case of $\chi_1(q)$, an additional peak at $(0,1)$ is considered as the first peak. Therefore, initially there is only one backward step in between the first and second peak which corresponds to weighted difference -1. Thus, the corresponding interpretation of $\chi_1(q)$ is given by the following theorem:

Theorem 6.2.2. *Let P_2 denote the set of paths starting at $(0,1)$ such that length of plain between first and second peak is ≥ -1 and in between the other consecutive peaks is ≥ -2. Then for*

$$t_2 = \begin{cases} 0 & \text{if } p = 1, \\ x_p - y_p & \text{if } p > 1, \end{cases}$$

$$\chi_1(q) = \sum_{s=0}^{\infty} p_2(s) q^s.$$

6.2.2 Sixth order mock theta functions

In this section, we interpret mock theta functions of order six. The first function which we consider here is $\phi_-(q)$.

Theorem 6.2.3. *For $s \geq 1$, let $P_3(s)$ denote the set of paths starting at $(0,0)$ with weight s such that the length of plain in front of the first peak is 0 or 1, between any two consecutive peaks is ≤ 2 and*

$$\geq \begin{cases} -2 & \text{if } x_{i+1} \equiv y_{i+1},\ x_i \equiv y_i \pmod 2, \\ 0 & \text{if } x_{i+1} \not\equiv y_{i+1},\ x_i \not\equiv y_i \pmod 2, \\ -1 & \qquad\qquad \text{otherwise.} \end{cases}$$

Then

$$\phi_-(q) = \sum_{s=1}^{\infty} P_3(s) q^s.$$

Proof. In the construction of path for $\phi_-(q)$, **Step I** and **Step II** proceed exactly in the same manner as in Theorem 6.2.1. In **Step III**, the factor $(-q;q)_n^{-1}$ is replaced by $(-q^2;q^2)_{n-1}$. This factor generates nonnegative multiples of $2, 4, \ldots, 2n-2$, say $v_1 \times 2, v_2 \times 4, \ldots, v_{n-1} \times (2n-2)$, $v_i = 0$ or 1. Introducing $2v_{n-i+1}$ γ steps in front of the i^{th} peak increases weight of each of the i^{th} to n^{th} peak by $2v_{n-i+1}$.
Step IV. The factor $(-q;q^2)_n$ generates either 0 or odd numbers $\leq 2n-1$, say $w_1 \times 1, w_2 \times 3, \ldots, w_n \times (2n-1)$, $w_i = 0$ or 1. Introducing $w_{n-i+1} + w_{n-i+2}$ γ steps in front of the i^{th} peak increases the weight of each of the i^{th} to n^{th} peak by $w_{n-i+1} + w_{n-i+2}$. Thus, the total increase in the weight of the path is $w_1 \times 1 + w_2 \times 3 + \cdots + w_n \times (2n-1)$. It can be easily seen that the way these paths are generated each path of weight s satisfies the conditions of $P_3(s)$. Every path enumerated by $P_3(s)$ is uniquely generated in this manner. ☐

The results for the remaining three mock theta functions of order 6 are presented in the following theorems without proofs as these follow essentially in the similar manner as the previous ones.

Theorem 6.2.4. *For $s \geq 1$, let $P_4(s)$ denote the set of paths from $P_3(s)$ which have no plain in front of the first peak and length of plain in front of the second peak is ≤ 1. Then*

$$\psi_-(q) = \sum_{s=1}^{\infty} P_4(s) q^s.$$

Theorem 6.2.5. *Let P_5 denote the set of paths starting at $(0, 0)$ such that height of each peak is ≤ 2, length of plain between consecutive peaks is ≥ -2 and in front of the first peak is ≥ 0. Then for $t_5 = x_p - \sum_{i=1}^{p-1} y_i + 2p - 2$,*

$$\lambda(q) = \sum_{s=0}^{\infty} p_5(s) q^s.$$

Theorem 6.2.6. *Let P_6 denote the set of paths starting at $(0, 0)$ such that the height of first peak is 1 and rest of the peaks is ≤ 2. Also, length of plain in front of the first peak is ≥ 0 and in front of rest of the peaks is ≥ -2. Let P_7 be the set of those paths of P_6 for which the length of plain between the first and second peak is ≥ -1. Then for*

$$t_6 = x_p + p - 2 - \sum_{i=2}^{p-1} y_i - 1 \quad \textit{and} \quad t_7 = \begin{cases} x_p - y_p & \text{if } p = 1, \\ x_p + p - 1 - \sum_{i=2}^{p-1} y_i - 1 & \text{if } p > 1, \end{cases}$$

$$2\mu(q) = \sum_{s=0}^{\infty} [p_6(s) + p_7(s)] q^s.$$

6.3 Lattice paths with backward steps and n–color compositions

In this section, we will discuss in detail how lattice paths with this modified terminology can be used as an interesting combinatorial tool to prove various results on n–color compositions. But before that we need an overview of the previous work done on n–color compositions.

In [6], Agarwal defined that an n–color composition is an ordered n–color partition. For example, there are six n–color partitions of 3, viz.,

3_1, 3_2, 3_3, $2_1 1_1$, $2_2 1_1$, $1_1 1_1 1_1$,

and eight n–color compositions of 3

3_1, 3_2, 3_3, $2_1 1_1$, $2_2 1_1$, $1_1 2_1$, $1_1 2_2$, $1_1 1_1 1_1$.

Agarwal also presented the combinatorial problem to which n–color compositions are solution. He provided the following generating function and explicit formulae for n–color compositions:

$$C(q) =\frac{q}{1-3q+q^2}, \tag{6.2}$$

$$C(m,q) =\frac{q^m}{(1-q)^{2m}}, \tag{6.3}$$

$$C(m,\nu) =\left(\begin{array}{c} \nu+m-1 \\ 2m-1 \end{array} \right), \tag{6.4}$$

$$C(\nu) =F_{2\nu}, \tag{6.5}$$

where $C(\nu)$ represents the number of n–color compositions of ν, $C(m,\nu)$ represents the number of n–color compositions of ν into m parts and $F_{2\nu}$ is the $(2\nu)^{th}$ Fibonacci number. Using (6.2), it can be proved that $C(\nu)$ satisfies the following initial conditions and recurrence relation:

$$C(1)=1,\ C(2)=3,\ C(\nu)=3C(\nu-1)-C(\nu-2),\ \nu>2. \tag{6.6}$$

In [87], authors defined an n–color self-inverse composition as follows:

Definition 6.3.1. *An n–color composition whose parts read from left to right are identical with when read from right to left is known as n–color self-inverse composition.*

They also provided recurrence relations and generating functions for certain restricted n–color self-inverse composition functions. Later, Guo ([**?** 73]) defined the following restricted n–color composition functions:

$C_e(\nu)$:=number of n–color compositions of ν with all the parts even,

$C_o(\nu)$:=number of n–color compositions of ν with all the parts odd,

$C_{\neq 1}(\nu)$:=number of n–color compositions of ν with parts > 1.

She also obtained recurrence relations, generating functions and explicit formulae for all the above functions. Continuing with the same, she defined n–color even self-inverse compositions and n–color odd self-inverse compositions ([72, 74]).

Hopkins [78] represented n–color compositions in the form of spotted tilings and proved the recurrence relations studied in ([6, 71, 73]) by using spotted tilings as a combinatorial tool. He also used this concept to define conjugable n–color compositions with the help of MacMahon's zig-zag graph and obtained the form of a conjugable n–color composition. Recently in [98], Sachdeva and Agarwal have also studied a number of restricted n–color composition functions, which we study in detail in Section 6.3.3.

In [88], Narang and Agarwal have given lattice paths for n–color compositions. But, those paths are constructed by extending the walks of Van Lint and Wilson [115] and the correspondence is not quite straightforward. However, lattice paths starting at $(0, 0)$ with the modified terminology presented in this chapter correspond naturally to n–color compositions. We try to emphasize this observation by providing the proofs of the known results on n–color compositions in the subsequent sections.

6.3.1 n–color composition function $C(\nu)$

In this section, first of all we provide a new proof, using the paths described in this chapter as a combinatorial tool, for the recurrence relation satisfied by n–color composition function $C(\nu)$.

Remark 6.3.1. *Throughout the remaining chapter, we will label each path with its sequence of steps.*

Theorem 6.3.1. *The number of n–color compositions of ν satisfies the following initial conditions and recurrence relation:*
$C(1) = 1$, $C(2) = 3$ *and* $C(\nu) = 3C(\nu - 1) - C(\nu - 2)$ *for* $\nu > 2$.

Proof. The only possible lattice path of weight 1 is shown in Figure 6-5 and the three possible lattice paths of weight 2 are shown in Figure 6-6. To prove the recurrence relation split the set of n–color compositions of ν, $\nu > 2$, in three classes and perform the following operations:

(i) If the last part is 1_1, then delete this part.

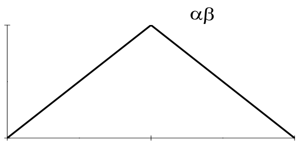


Figure 6-5 – Lattice path for $\nu = 1$

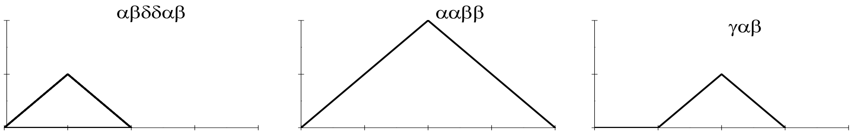


Figure 6-6 – Lattice paths for $\nu = 2$

(ii) If the last part is of the form m_i, $m \neq i$, then replace it with $(m-1)_i$.

(iii) If the last part is of the form k_k, $k > 1$, then replace it with $(k-1)_{k-1}$.

In terms of lattice paths, to perform operation (i) we simply delete the last peak and backward horizontal steps in front of it. Thus, we get all the paths corresponding to n–color compositions enumerated by $C(\nu - 1)$. To perform the reverse operation first introduce backward horizontal steps so as to reach $(0,0)$ and then construct a peak at $(1,1)$ and we get path corresponding to an n–color composition of ν.

To perform operation (ii), we need to decrease the weight of last peak by 1. Following possibilities arise for a path.

(a) If there are some forward horizontal steps in front of the last peak then remove one of these steps as shown in Figure 6-7. It will decrease the weight of last peak by 1.

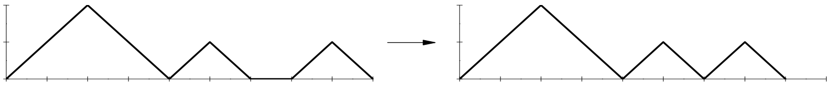


Figure 6-7 – Lattice path transformation of $2_2 5_1 8_1$ to $2_2 5_1 7_1$

(b) If there are some backward horizontal steps in front of the last peak then increase the number of backward horizontal steps by 1 as shown in Figure 6-8.

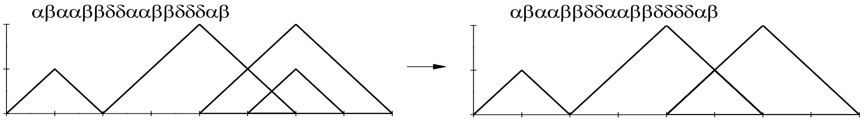


Figure 6-8 – Lattice path transformation of $1_1 4_2 6_2 6_1$ to $1_1 4_2 6_2 5_1$

(c) If there are no horizontal steps in front of the last peak then introduce a backward horizontal step in front of this peak. An example is shown in Figure 6-9.

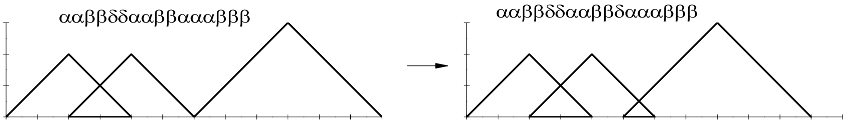


Figure 6-9 – Lattice path transformation of $2_2 4_2 9_3$ to $2_2 4_2 8_3$

In all these cases, the length of plain in front of the last peak decreases by 1. Thus, we obtain all the paths corresponding to n–color compositions of $\nu - 1$. Reverse operations can also be easily performed. Therefore, these paths correspond to the compositions enumerated by $C(\nu - 1)$.

To perform operation (iii), simply decrease the height of last peak by 1 and we obtain all paths corresponding to those n–color compositions of $\nu - 1$ which have last part of the form k_k. These are enumerated by $C(\nu - 1) - C(\nu - 2)$. Reverse operation can be performed by increasing the height of the peak. □

Theorem 6.3.2. [72] *The number of n–color even compositions of ν satisfies the following initial conditions and recurrence relation:*
$C_e(2) = 2$, $C_e(4) = 8$ *and* $C_e(\nu) = 4C_e(\nu - 2) - C_e(\nu - 4)$ *for* $\nu > 4$.

Theorem 6.3.3. [73] *The number of n–color odd compositions of ν satisfies the following initial conditions and recurrence relation:*
$C_o(1) = 1$, $C_o(2) = 1$, $C_o(3) = 4$, $C_o(4) = 7$
and $C_o(\nu) = C_o(\nu - 1) + 2C_o(\nu - 2) + C_o(\nu - 3) - C_o(\nu - 4)$ *for* $\nu > 4$.

Theorem 6.3.4. [73] *The number of n–color compositions of ν with parts* > 1 *satisfies the following initial conditions and recurrence relation:*

$C_{\neq 1}(2) = 2$, $C_{\neq 1}(3) = 3$, $C_{\neq 1}(4) = 8$
and $C_{\neq 1}(\nu) = 2C_{\neq 1}(\nu - 1) + C_{\neq 1}(\nu - 2) - C_{\neq 1}(\nu - 3)$ for $\nu > 4$.

The combinatorial proofs by splitting the relevant set of composition into certain classes for Theorems 6.3.2–6.3.4 are available in [72] and [73]. The proofs by using lattice paths can be developed in a similar manner as in Theorem 6.3.1. So, we are omitting the proofs here.

6.3.2 n–color self-inverse compositions

Theorem 6.3.5. [87] *Let a_ν, b_ν, c_ν, d_ν denote the number of n–color self-inverse compositions of $2\nu + 1$, 2ν into an even number of parts, 2ν into an odd number of parts and 2ν respectively. The following initial conditions and recurrence relations are satisfied by a_ν, b_ν, c_ν and d_ν:*

(1) *$a_0 = 1$, $a_1 = 4$ and $a_\nu = 3a_{\nu-1} - a_{\nu-2}$ for $\nu \geq 2$,*

(2) *$b_1 = 1$, $b_2 = 3$ and $b_\nu = 3b_{\nu-1} - b_{\nu-2}$ for $\nu > 2$,*

(3) *$c_1 = 2$, $c_2 = 6$ and $c_\nu = 3c_{\nu-1} - c_{\nu-2}$ for $\nu > 2$,*

(4) *$d_1 = 3$, $d_2 = 9$ and $d_\nu = 3d_{\nu-1} - d_{\nu-2}$ for $\nu > 2$.*

Proof. (1) The lattice paths corresponding to n–color self-inverse compositions of 1 and 3 can be easily constructed. Divide the n–color compositions enumerated by a_ν ($\nu \geq 2$) in three classes and perform the following operations.

(i) Compositions with 1_1 as its first and last part. Here, deleting both these parts we get compositions enumerated by $a_{\nu-1}$.

(ii) Compositions with a single part m_i, $m - i \geq 2$ or with m_i, $m \neq i$ as first and last part in case of more than one parts. In this case, by replacing the single part by $(m-2)_i$ and for more than one parts replacing both the first and last part by $(m-1)_i$ we get the compositions enumerated by $a_{\nu-1}$.

(iii) Compositions with a single part m_i, $m - i \leq 1$ or with first and last part m_i, $m = i$, $i > 1$ for more than one parts. In the former case, replace the single part with $(m-2)_{i-2}$ and in the later case replace both the first and last part by $(m-1)_{i-1}$. Thus, the transformed compositions are enumerated by $a_{\nu-1}$ with the proviso that in case of single part m_i, $m-i \leq 1$ and in case of more than one parts the first and last part m_i is such that $m = i$. Therefore, all the transformed compositions are enumerated by $a_{\nu-1} - a_{\nu-2}$.

Now, we explain how the above said operations are performed in terms of lattice paths. To perform operation (i), delete the last peak at (1,1) and remove the backward horizontal steps in front of this peak. Also, delete the first peak at (1,1) and increase the length of plain in front of that second peak by 2. An example is shown in Figure 6-10. Thus, the transformed path corresponds to the n–color self-inverse compositions of $2\nu - 1$ i.e enumerated by $a_{\nu-1}$. To perform operation (ii), decrease the length of the plain in front of the first and last peak by 1 and increase the length of plain in front of the second peak by 1. Thus, it will not change the weight of remaining peaks. In case of a single peak, decrease the length of plain in front of the peak by 2. Figure 6-11 presents an example for transformation of paths in this case. Again, we get all the paths corresponding to n–color compositions enumerated by $a_{\nu-1}$. For performing operation (iii) in case of paths with more than one peak, decrease the height of both the first and last peaks by 1 and increase the length of plain in front of second peak by 2. And, in case of single peak decrease the height of the peak by 2. For example, refer

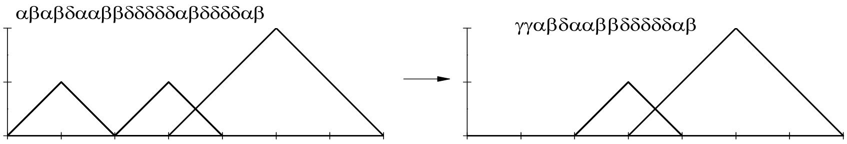


Figure 6-10 – Lattice path transformation of $1_1 3_1 5_2 3_1 1_1$ to $3_1 5_2 3_1$

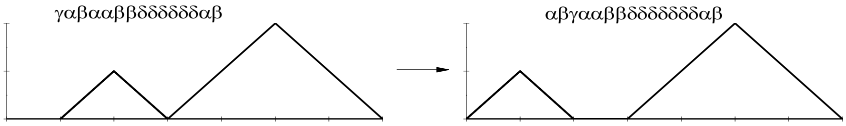


Figure 6-11 – Lattice path transformation of $2_1 5_2 2_1$ to $1_1 5_2 1_1$

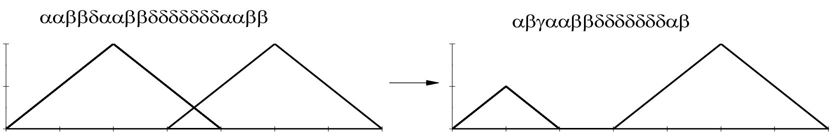


Figure 6-12 – Lattice path transformation of $2_2 5_2 2_2$ to $1_1 5_2 1_1$

to Figure 6-12. The transformed paths are the ones corresponding to compositions enumerated by $a_{\nu-1} - a_{\nu-2}$. All the above said operations are reversible. Hence the proof.

Remaining parts of the theorem have similar constructions and so are omitted. □

Guo [72, 74] has given combinatorial proofs for the following recurrence relations satisfied by n–color even self-inverse compositions and n–color odd self-inverse compositions respectively.

Theorem 6.3.6. *If r_ν and s_ν denote the number of n–color even self-inverse compositions of 4ν and $4\nu + 2$ respectively. Then*

(i) $r_1 = 6$, $r_2 = 24$ *and* $r_\nu = 4r_{\nu-1} - r_{\nu-2}$ *for* $\nu > 2$,

(ii) $s_0 = 2$, $s_1 = 10$ *and* $s_\nu = 4s_{\nu-1} - s_{\nu-2}$ *for* $\nu > 1$.

Theorem 6.3.7. *Let t_ν and u_ν denote the number of n–color odd self-inverse compositions of 2ν and $2\nu + 1$ respectively. Then*

(i) $t_1 = 1$, $t_2 = 1$, $t_3 = 4$, $t_4 = 7$ *and* $t_\nu = t_{\nu-1} + 2t_{\nu-2} + t_{\nu-3} - t_{\nu-4}$ *for* $\nu > 4$,

(ii) $u_0 = 1$, $u_1 = 4$, $u_2 = 9$, $u_3 = 19$ *and* $u_\nu = u_{\nu-1} + 2u_{\nu-2} + u_{\nu-3} - u_{\nu-4}$ *for* $\nu > 3$.

Proofs for Theorems 6.3.6 and 6.3.7 are similar to the proof of Theorem 6.3.5 and are therefore omitted.

6.3.3 Restricted n–color composition functions of Sachdeva and Agarwal

In [98], Sachdeva and Agarwal have introduced the following new restricted n–color composition functions:

$C_1(\nu)$:=Number of n–color compositions of ν into parts with odd subscripts.

$C_2(\nu)$:=Number of n–color compositions of ν into parts with odd subscripts > 1.

$C_3(\nu)$:=Number of n–color compositions of ν into parts with even subscripts.

$C_4(\nu)$:=Number of n–color compositions of ν such that all even parts but odd parts with even subscripts only are allowed.

$C_5(\nu)$:=Number of n–color compositions of ν such that all even parts but odd parts with odd subscripts only are allowed.

$C_6(\nu)$:=Number of n–color compositions of ν such that all odd parts but even parts with even subscripts only are allowed.

$C_7(\nu)$:=Number of n–color compositions of ν such that even parts with even subscripts and odd parts with odd subscripts only are allowed.

$C_8(\nu)$:=Number of n–color compositions of ν such that even parts with even subscripts and odd parts with odd subscripts > 1 are allowed.

$C_9(\nu)$:=Number of n–color compositions of ν such that all odd part but even parts with odd subscripts only are allowed.

They presented explicit formulae, generating functions and recurrence relations for each of these functions. They remarked that all these results can be proved using standard techniques from partition theory. Here, we are going to study the following recurrence relations satisfied by these n–color composition functions:

$$\begin{aligned}
C_1(\nu) =& 2C_1(\nu-1) + C_1(\nu-2) - C_1(\nu-3),\ \nu > 3,\\
C_2(\nu) =& C_2(\nu-1) + C_2(\nu-2),\ \nu > 3,\\
C_3(\nu) =& C_3(\nu-1) + 2C_3(\nu-2) - C_3(\nu-3),\ \nu > 3,\\
C_4(\nu) =& 4C_4(\nu-2) + C_4(\nu-3) - C_4(\nu-4),\ \nu > 4,\\
C_5(\nu) =& C_5(\nu-1) + 4C_5(\nu-2) - C_5(\nu-4),\ \nu > 4,\\
C_6(\nu) =& C_6(\nu-1) + 3C_6(\nu-2) + C_6(\nu-3) - C_6(\nu-4),\ \nu > 4.
\end{aligned}$$

Remark 6.3.2. *$C_1(\nu) = C_7(\nu)$, $C_3(\nu) = C_8(\nu)$ and $C_6(\nu) = C_9(\nu)$.*

For the results proved in previous sections, we were already equipped with the combinatorial proofs of recurrence relations from the earlier studies. However, for the n–color composition functions $C_1(\nu)$ to $C_9(\nu)$ these proofs are not available. The proofs for the recurrence relations satisfied by these functions follow essentially in a similar manner as for the functions discussed in Section 6.3.1, so we just provide the classes in which compositions enumerated by $C_i(\nu)$'s split and the corresponding enumerator for those classes. We present these in Table 6.1, as a systematic proof can be written from there. Here, we present the proof for the recurrence relation satisfied by $C_2(\nu)$ using lattice paths.

Proof. For the compositions of class (i), the last part is m_3 which means that the last peak is at $(m, 3)$. If $m = 3$ then delete this peak and backward horizontal steps in front of it, if any. Also, increase the height of second-to-last peak by 2. If $m > 3$ then, shift this peak to $(m - 1, 3)$ by decreasing the length of plain in front of this peak by 1. Thus, we get all the paths enumerated by $C_2(\nu - 1)$ and the process is easily reversible.
In case of compositions of class (ii), last part is $m_i,\ i > 3$ which means that the last peak of the path is at $(m, i),\ i > 3$. Decreasing the height of this peak by 2, the peak shifts to $(m - 2, i - 2)$ and we get all the paths corresponding to the compositions enumerated $C_2(\nu - 2)$. Hence the proof. □

Rest of the proofs can be developed easily as by now the reader must be familiar with the techniques employed by the authors.

Table 6.1 – Classes for $C_i(\nu),\ 1 \leq i \leq 9$

Enumerator	Class	Last Part	Enumerator
$C_1(\nu)$, $\nu > 3$	(i)	1_1	$C_1(\nu-1)$
	(ii)	m_i, $i > 1$	$C_1(\nu-2)$
	(iii)	m_1, $m > 1$	$C_1(\nu-1) - C_1(\nu-3)$
$C_2(\nu)$, $\nu > 3$	(i)	m_3	$C_2(\nu-1)$
	(ii)	m_i, $i > 3$	$C_2(\nu-2)$
$C_3(\nu)$, $\nu > 3$	(i)	2_2	$C_3(\nu-2)$
	(ii)	m_i, $i > 2$	$C_3(\nu-2)$
	(iii)	m_2, $m > 2$	$C_3(\nu-1) - C_3(\nu-3)$
$C_4(\nu)$, $\nu > 4$	(i)	2_1	$C_4(\nu-2)$
	(ii)	2_2	$C_4(\nu-2)$
	(iii)	3_2	$C_4(\nu-3)$
	(iv)	m_i, $m-i > 1$	$C_4(\nu-2)$
	(v)	m_i, $m > 3$, $m-i \leq 1$	$C_4(\nu-2) - C_4(\nu-4)$
$C_5(\nu)$, $\nu > 4$	(i)	1_1	$C_5(\nu-1)$
	(ii)	2_1	$C_5(\nu-2)$
	(iii)	2_2	$C_5(\nu-2)$
	(iv)	m_i, $m-i > 1$	$C_5(\nu-2)$
	(v)	m_i, $m > 2$, $m-i \leq 1$	$C_5(\nu-2) - C_5(\nu-4)$
$C_6(\nu)$, $\nu > 4$	(i)	1_1	$C_6(\nu-1)$
	(ii)	2_2	$C_6(\nu-2)$
	(iii)	3_2	$C_6(\nu-3)$
	(iv)	m_i, $m-i > 1$	$C_6(\nu-2)$
	(v)	m_i, $i > 2$, $m-i \leq 1$	$C_6(\nu-2) - C_6(\nu-4)$
$C_7(\nu)$, $\nu > 4$	(i)	1_1	$C_7(\nu-1)$
	(ii)	m_i, $m \neq i$	$C_7(\nu-1)$
	(iii)	m_i, $m = i$, $m > 1$	$C_7(\nu-2) - C_7(\nu-3)$
$C_8(\nu)$, $\nu > 3$	(i)	2_2	$C_8(\nu-2)$
	(ii)	m_i, $m \neq i$	$C_8(\nu-2)$
	(iii)	m_i, $m = i$, $m > 2$	$C_8(\nu-1) - C_8(\nu-3)$
$C_9(\nu)$, $\nu > 4$	(i)	1_1	$C_9(\nu-1)$
	(ii)	2_1	$C_9(\nu-2)$
	(iii)	3_2	$C_9(\nu-3)$
	(iv)	m_i, $m-i > 1$	$C_9(\nu-2)$
	(v)	m_i, $i > 2$, $m-i \leq 1$	$C_9(\nu-2) - C_9(\nu-4)$

6.4 Conclusion

As remarked by Agarwal and Bressoud [12], the link between restricted n–color partitions and their lattice paths is somewhat misleading as no weight can repeat in the path while in case of n–color partitions a part can repeat. With the modifications given in this chapter, it is now possible to construct lattice path corresponding to any n–color partition. Agarwal and Bressoud [12] have shifted the valleys while interpreting multiple summation q-series. A valley above height zero between two peaks correspond to even negative weighted difference between the corresponding n–color parts of the partition. However, if we allow valleys above height zero with the paths as modified in this paper then the situation again becomes misleading as backward steps also correspond to negative weighted difference between parts. Thus, the obvious future problem is how to synchronize the concepts of valley above height zero and backward plain so as to obtain interpretations of multiple summation q-series with these paths.

Chapter 7

$(n+t)$–color partition-theoretic interpretations for some generalized q-series

In an endeavor to interpret mock theta functions $V_0(q)$ and $V_1(q)$ (given by (1.41) and (1.42)) with the two factors $(q;q^2)_n$ and $(-q;q^2)_n$ present together, Sood and Agarwal [18] introduced the concept of split $(n+t)$–color partitions.

Definition 7.0.1. [18] *Let $(a_i)_{b_i}$ be a part in an $(n+t)$–color partition of a nonnegative integer s. The color 'b_i' is split into two parts-'the green part' and 'the red part' and denote these by 'g_i' and 'r_i' respectively, such that $1 \leq g_i \leq b_i$, $0 \leq r_i \leq b_i - 1$ and $b_i = g_i + r_i$. An $(n+t)$–color partition in which color of each part is split in this manner is called a split $(n+t)$–color partition.*

They further used split $(n+t)$–color partitions to interpret some generalized q-series [111]. Continuing with the same, Rana et al. [95] interpreted some generalized q-series with split $(n+t)$–color partitions. In this chapter, we prove that the same generalized q-series can be interpreted in terms of $(n+t)$–color partitions.

7.1 Some generalized q-series and split $(n+t)$–color partitions

In this section, we recapitulate the results of two papers [95] and [111]. The series interpreted in these papers are defined below:

Definition 7.1.1. *Let* $S = \{-1, 1, 3, 5, 7\ldots\}$. *For* $|q| < 1$, $k \in S$ *and* $1 \leq i \leq 7$. *Define* $f_i^k(q)$ *by*

$$f_1^k(q) := \sum_{n=0}^{\infty} \frac{(-q;q^2)_n q^{n[1+(n-1)(k+3)/2]}}{(q;q)_{2n}}, \tag{7.1}$$

$$f_2^k(q) := \sum_{n=0}^{\infty} \frac{(-q;q^2)_n q^{n(n+1)(k+3)/2}}{(q;q)_{2n+1}}, \tag{7.2}$$

$$f_3^k(q) := \sum_{n=0}^{\infty} \frac{(-q;q^2)_n q^{n[1+(n+1)(k+3)/2]}}{(q;q)_{2n+1}}, \tag{7.3}$$

$$f_4^k(q) := \sum_{n=0}^{\infty} \frac{(-q;q^2)_n q^{n[1+(n-1)(k+3)/2]}}{(q;q^2)_n (q^4;q^4)_n}, \tag{7.4}$$

$$f_5^k(q) := \sum_{n=0}^{\infty} \frac{(-q;q^2)_n q^{n(n+1)(k+3)/2}}{(q;q^2)_{n+1} (q^4;q^4)_n}, \tag{7.5}$$

$$f_6^k(q) := \sum_{n=0}^{\infty} \frac{(-q;q^2)_n q^{n[1+(n+1)(k+3)/2]}}{(q;q^2)_{n+1} (q^4;q^4)_n}, \tag{7.6}$$

$$f_7^k(q) := \sum_{n=0}^{\infty} \frac{(-q;q^2)_n q^{n[1+1(n+1)(k+3)/2]}}{(q;q^2)_n (q^4;q^4)_n}. \tag{7.7}$$

The interpretations of the generalized q-series (7.1)–(7.7) as given in [111] and [95] are stated in the following theorems respectively:

Theorem 7.1.1. *For* $k \in S$, *let* $M_1^k(s)$ *denote the number of split* n*–color partitions of* s *such that*

(i) $a_i \equiv b_i \pmod 2$, $1 \leq i \leq p$;

(ii) $r_i = 0$ *or* 1, $1 \leq i \leq p$;

(iii) $(w.d)_{i,i+1} > k$, $1 \leq i \leq p-1$.

Then

$$\sum_{s=0}^{\infty} M_1^k(s) q^s = f_1^k(q).$$

Theorem 7.1.2. *For* $k \in S$, *let* $M_2^k(s)$ *denote the number of split* $(n+1)$*–color partitions of* s *such that*

(i) $a_i \not\equiv b_i \pmod 2$, $1 \leq i \leq p$;

(ii) $r_i = 0$ *or* 1, $1 \leq i \leq p$;

(iii) $b_p = a_p + 1$;

(iv) $r_p = 0$;

(v) $(w.d)_{i,i+1} > k$, $1 \le i \le p-1$.

Then

$$\sum_{s=0}^{\infty} M_2^k(s)q^s = f_2^k(q).$$

Theorem 7.1.3. *For $k \in S$, let $M_3^k(s)$ denote the number of split $(n+2)$–color partitions of s such that*

(i) $a_i \equiv b_i \pmod 2$, $1 \le i \le p$;

(ii) $r_i = 0$ *or* 1, $1 \le i \le p$;

(iii) $b_p = a_p + 2$;

(iv) $r_p = 0$;

(v) $(w.d)_{i,i+1} > k$, $1 \le i \le p-1$.

Then

$$\sum_{s=0}^{\infty} M_3^k(s)q^s = f_3^k(q).$$

Theorem 7.1.4. *For $k \in S$, let $M_4^k(s)$ denote the number of split n–color partitions of s such that*

(i) $a_i \equiv b_i \pmod 2$, $1 \le i \le p$;

(ii) $r_i = 0$ *or* 1, $1 \le i \le p$;

(iii) $a_p - b_p \equiv 0 \pmod 4$;

(iv) $(w.d)_{i,i+1} > k$ *and* $\equiv k+1 \pmod 4$, $1 \le i \le p-1$.

Then

$$\sum_{s=0}^{\infty} M_4^k(s)q^s = f_4^k(q).$$

Theorem 7.1.5. *For $k \in S$, let $M_5^k(s)$ denote the number of split $(n+1)$–color partitions of s such that*

(i) $a_i \not\equiv b_i \pmod 2$, $1 \le i \le p$;

(ii) $r_i = 0$ *or* 1, $1 \leq i \leq p$;

(iii) $b_p = a_p + 1$;

(iv) $(w.d)_{i,i+1} > k$ *and* $\equiv k+1 \pmod 4$, $1 \leq i \leq p-1$.

Then

$$\sum_{s=0}^{\infty} M_5^k(s) q^s = f_5^k(q).$$

Theorem 7.1.6. *For* $k \in S$, *let* $M_6^k(s)$ *denote the number of split* $(n+2)$*–color partitions of* s *such that*

(i) $a_i \equiv b_i \pmod 2$, $1 \leq i \leq p$;

(ii) $r_i = 0$ *or* 1, $1 \leq i \leq p$;

(iii) $b_p = a_p + 2$;

(iv) $(w.d)_{i,i+1} > k$ *and* $\equiv k+1 \pmod 4$, $1 \leq i \leq p-1$.

Then

$$\sum_{s=0}^{\infty} M_6^k(s) q^s = f_6^k(q).$$

Theorem 7.1.7. *For* $k \in S$, *let* $M_7^k(s)$ *denote the number of split* n*–color partitions of* s *such that*

(i) $a_i \equiv b_i \pmod 2$, $1 \leq i \leq p$;

(ii) $r_i = 0$ *or* 1, $1 \leq i \leq p$;

(iii) $a_p \geq (k+4)$ *and* $a_p - b_p \equiv k+3 \pmod 4$;

(iv) $(w.d)_{i,i+1} > k$ *and* $\equiv k+1 \pmod 4$, $1 \leq i \leq p-1$.

Then

$$\sum_{s=0}^{\infty} M_7^k(s) q^s = f_7^k(q).$$

7.2 Interpreting (7.1)–(7.7) with $(n+t)$–color partitions

The present study will prove that the generalized q-series (7.1)-(7.7) can be interpreted in terms of $(n+t)$–color partitions. We claim that our results are not only valid for

$k \in S$ but for every integral value of $k \geq -3$. Before stating our main results we need the following terminology:
For any k, we split the set of parts of a general $(n+t)$–color partition $(a_1)_{b_1} + (a_2)_{b_2} + \cdots + (a_p)_{b_p}$ $((a_1)_{b_1} \geq (a_2)_{b_2} \geq \cdots \geq (a_p)_{b_p})$ into two mutually exclusive and exhaustive sets given by

$$S_1^k = \{(a_i)_{b_i} | a_i - b_i - (p-i)(k+3) \equiv 0 \pmod 2\},$$

$$S_2^k = \{(a_i)_{b_i} | a_i - b_i - (p-i)(k+3) \not\equiv 0 \pmod 2\}.$$

Theorem 7.2.1. *For $s \geq 0$, $k \geq -3$ and $1 \leq i \leq p-1$, $f_1^k(q)$ is the generating function for n–color partitions of s satisfying*

$$(w.d)_{i,i+1} \geq \begin{cases} k+1 \text{ and } \equiv k+1 \pmod 2 & \text{if } (a_i)_{b_i}, (a_{i+1})_{b_{i+1}} \in S_1^k, \\ k+3 \text{ and } \equiv k+1 \pmod 2 & \text{if } (a_i)_{b_i}, (a_{i+1})_{b_{i+1}} \in S_2^k, \\ k+2 \text{ and } \not\equiv k+1 \pmod 2 & \text{otherwise.} \end{cases}$$

Theorem 7.2.2. *For $s \geq 0$, $k \geq -3$ and $1 \leq i \leq p-1$, $f_2^k(q)$ is the generating function for n–color partitions of s satisfying*

$$(w.d)_{i,i+1} \geq \begin{cases} k+1 \text{ and } \equiv k+1 \pmod 2 & \text{if } (a_i)_{b_i}, (a_{i+1})_{b_{i+1}} \in S_2^k, \\ k+3 \text{ and } \equiv k+1 \pmod 2 & \text{if } (a_i)_{b_i}, (a_{i+1})_{b_{i+1}} \in S_1^k, \\ k+2 \text{ and } \not\equiv k+1 \pmod 2 & \text{otherwise.} \end{cases}$$

Theorem 7.2.3. *For $s \geq 0$ and $k \geq -3$, $f_3^k(q)$ is the generating function for $(n+2)$–color partitions of s such that $b_p - a_p = 2$ and for $1 \leq i \leq p-1$,*

$$(w.d)_{i,i+1} \geq \begin{cases} k+1 \text{ and } \equiv k+1 \pmod 2 & \text{if } (a_i)_{b_i}, (a_{i+1})_{b_{i+1}} \in S_1^k, \\ k+3 \text{ and } \equiv k+1 \pmod 2 & \text{if } (a_i)_{b_i}, (a_{i+1})_{b_{i+1}} \in S_2^k, \\ k+2 \text{ and } \not\equiv k+1 \pmod 2 & \text{otherwise.} \end{cases}$$

Theorem 7.2.4. *For $s \geq 0$ and $k \geq -3$, $f_4^k(q)$ is the generating function for n–color* *partitions of s such that $a_p - b_p = 0$ or $1 \pmod 4$ and for $1 \leq i \leq p-1$,*

$$(w.d)_{i,i+1} \geq \begin{cases} k+1 \text{ and } \equiv k+1 \pmod 4 & \text{if } (a_i)_{b_i}, (a_{i+1})_{b_{i+1}} \in S_1^k, \\ k+3 \text{ and } \equiv k+3 \pmod 4 & \text{if } (a_i)_{b_i}, (a_{i+1})_{b_{i+1}} \in S_2^k, \\ k+2 \text{ and } \equiv k+2 \pmod 4 & \text{otherwise.} \end{cases}$$

Theorem 7.2.5. *For $s \geq 0$ and $k \geq -3$, $f_5^k(q)$ is the generating function for $(n+1)$–color partitions of s such that $b_p - a_p = 1$ and for $1 \leq i \leq p-1$,*

$$(w.d)_{i,i+1} \geq \begin{cases} k+1 \text{ and } \equiv k+1 \pmod 4 & \text{if } (a_i)_{b_i}, (a_{i+1})_{b_{i+1}} \in S_2^k, \\ k+3 \text{ and } \equiv k+3 \pmod 4 & \text{if } (a_i)_{b_i}, (a_{i+1})_{b_{i+1}} \in S_1^k, \\ k+2 \text{ and } \equiv k+2 \pmod 4 & \text{otherwise.} \end{cases}$$

Theorem 7.2.6. *For $s \geq 0$ and $k \geq -3$, $f_6^k(q)$ is the generating function for $(n+2)$–color partitions of s such that $b_p - a_p = 2$ and for $1 \leq i \leq p-1$,*

$$(w.d)_{i,i+1} \geq \begin{cases} k+1 \text{ and } \equiv k+1 \pmod 4 & \text{if } (a_i)_{b_i}, (a_{i+1})_{b_{i+1}} \in S_1^k, \\ k+3 \text{ and } \equiv k+3 \pmod 4 & \text{if } (a_i)_{b_i}, (a_{i+1})_{b_{i+1}} \in S_2^k, \\ k+2 \text{ and } \equiv k+2 \pmod 4 & \text{otherwise.} \end{cases}$$

To state the n–color partition theoretic interpretation of $f_7^k(q)$ we split the set of parts of a general $(n+t)$–color partition $(a_1)_{b_1} + (a_2)_{b_2} + \cdots + (a_p)_{b_p}$ $((a_1)_{b_1} \geq (a_2)_{b_2} \geq \cdots \geq (a_p)_{b_p})$ into two mutually exclusive and exhaustive sets as follows:

$$S_3^k = \{(a_i)_{b_i} | a_i - b_i - (p-i+1)(k+3) \equiv 0 \pmod 2\},$$

$$S_4^k = \{(a_i)_{b_i} | a_i - b_i - (p-i+1)(k+3) \not\equiv 0 \pmod 2\}.$$

Theorem 7.2.7. *For $s \geq 0$ and $k \geq -3$, $f_7^k(q)$ is the generating function for n–color* *partitions of s such that $a_p - b_p - (k+3) \equiv 0$ or $1 \pmod 4$ and for $1 \le i \le p-1$,*

$$(w.d)_{i,i+1} \ge \begin{cases} k+1 \text{ and } \equiv k+1 \pmod 4 & \text{if } (a_i)_{b_i}, (a_{i+1})_{b_{i+1}} \in S_3^k, \\ k+3 \text{ and } \equiv k+3 \pmod 4 & \text{if } (a_i)_{b_i}, (a_{i+1})_{b_{i+1}} \in S_4^k, \\ k+2 \text{ and } \equiv k+2 \pmod 4 & \text{otherwise.} \end{cases}$$

7.3 Proof of Theorem 7.2.1

In this section, an elaborate proof of Theorem 7.2.1, by following a constructive method, is presented. The representation of $(n+t)$–color partitions in terms of two-line arrays is the key to obtain desired results. As discussed in Chapter 5, in the two-line arrays the first line represents the parts of the partition and second line represents the corresponding subscripts. In the expression

$$f_1^k(q) = \sum_{n=0}^{\infty} \frac{(-q;q^2)_n q^{n[1+\frac{(n-1)(k+3)}{2}]}}{(q;q^2)_n (q^2;q^2)_n},$$

the factor $q^{n[1+\frac{(n-1)(k+3)}{2}]}$ is generating n parts 1, $k+4$, $2k+7$, …, $(n-1)k+(3n-2)$. $(-q;q^2)_n$ generates multiples of $(2i-1)$, $1 \le i \le n$, say $u_1 \times 1$, $u_2 \times 3$, …, $u_n \times (2n-1)$, $u_i = 0$ or 1. $(q;q^2)_n^{-1}$ generates multiples of $(2i-1)$, $1 \le i \le n$, say $v_1 \times 1$, $v_2 \times 3$, …, $v_n \times (2n-1)$, $v_i \ge 0$. $(q^2;q^2)_n^{-1}$ generates multiples of $2i$, $1 \le i \le n$, say $w_1 \times 2$, $w_2 \times 4$, …, $w_n \times 2n$, $w_i \ge 0$. Thus, an n–color partition generated by $f_1^k(q)$ can be written in a two-line array as

$$\begin{pmatrix} \begin{matrix}(n-1)k+3n-2+2(w_n+\cdots+w_1+v_n+\cdots+v_2+ \\ +u_n+\cdots+u_2)+v_1+u_1\end{matrix} & \cdots & \begin{matrix}k+4+2(w_n+w_{n-1}+v_n+ \\ +u_n)+u_{n-1}+v_{n-1}\end{matrix} & \begin{matrix}1+2w_n+ \\ +u_n+v_n\end{matrix} \\ 1+v_1 & \cdots & 1+v_{n-1} & 1+v_n \end{pmatrix}$$

Therefore, the i^{th} and $(i+1)^{th}$ parts in the partition are

$$\begin{aligned} a_i =& (n-i)(k+3)+1+2(w_n+\cdots+w_i+v_n+\cdots+v_{i+1}+u_n+ \\ & +\cdots+u_{i+1})+v_i+u_i, \qquad (7.8) \\ b_i =& 1+v_i, \qquad (7.9) \end{aligned}$$

$$\begin{aligned} a_{i+1} =&(n-i-1)(k+3)+1+2(w_n+\cdots+w_{i+1}+v_p+\cdots+v_{i+2}+ \\ &+u_n+\cdots+u_{i+2})+v_{i+1}+u_{i+1}, \qquad (7.10)\\ b_{i+1} =&1+v_{i+1}. \qquad (7.11)\end{aligned}$$

Thus,

$$(w.d)_{i,i+1} = k+1+2w_{i+1}+u_i+u_{i+1} \qquad (7.12)$$

Since $u_i, u_{i+1} = 0$ or 1, (7.12) implies

$$(w.d)_{i,i+1} \geq \begin{cases} k+1 \text{ and } \equiv k+1 \pmod 2 & \text{if } u_i, u_{i+1} = 0, \\ k+3 \text{ and } \equiv k+1 \pmod 2 & \text{if } u_i, u_{i+1} = 1, \\ k+2 \text{ and } \not\equiv k+1 \pmod 2 & \text{otherwise.} \end{cases} \qquad (7.13)$$

By analyzing (7.8) and (7.9), we see that

$$u_i = 0 \;\Leftrightarrow\; a_i - b_i - (n-i)(k+3) \equiv 0 \pmod 2 \;\Leftrightarrow\; (a_i)_{b_i} \in S_1^k,$$

$$u_i = 1 \;\Leftrightarrow\; a_i - b_i - (n-i)(k+3) \not\equiv 0 \pmod 2 \;\Leftrightarrow\; (a_i)_{b_i} \in S_2^k.$$

Hence the proof.

Proofs for rest of the theorems can be established in a similar manner.

7.4 Some deductions from Theorems 7.2.1–7.2.7

The following identities from Slater's compendium and Chu-Zhang's compendium can be interpreted in terms of $(n+t)$–color partitions by assigning some particular values to k in Theorems 7.2.1–7.2.3, 7.2.5 and 7.2.6, respectively:

$$\sum_{n=0}^{\infty} \frac{(-q;q^2)_n q^{n^2}}{(q,q)_{2n}} = \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}[-q^2,-q^4,q^6;q^6]_\infty, \qquad ([108, I(29)], [52, I(44)])$$

$$\sum_{n=0}^{\infty} \frac{(-q,q^2)_n q^{n(n+1)}}{(q,q)_{2n+1}} = \frac{1}{(q;q)_\infty}[q^4,q^8,q^{12};q^{12}]_\infty, \qquad ([108, I(51)], [52, I(104)])$$

$$\sum_{n=0}^{\infty}\frac{(-q,q^2)_n q^{n(n+2)}}{(q,q)_{2n+1}}=\frac{1}{(q;q)_\infty}[q^2,q^{10},q^{12};q^{12}]_\infty, \qquad ([108, \text{I}(50)], [52, \text{I}(41)])$$

$$\sum_{n=0}^{\infty}\frac{(-q,q^2)_n q^{2n(n+1)}}{(q^4,q^4)_n(q,q^2)_{n+1}}=\frac{1}{(q^2;q^2)_\infty}[-q,-q^5,q^6;q^6]_\infty, \qquad ([108, \text{I}(27)], [52, \text{I}(113)])$$

$$\sum_{n=0}^{\infty}\frac{(-q,q^2)_n q^{n(n+2)}}{(q^4,q^4)_n(q,q^2)_{n+1}}=\frac{1}{(q^2;q^2)_\infty}[-q,-q^5,q^6;q^6]_\infty. \qquad ([52, \text{I}(144)])$$

There are some other identities which can also be interpreted in terms of $(n+t)$–color partitions with attached weights using Theorems 7.2.4-7.2.7. To obtain the interpretations of these identities certain weight is attached to each $(n+t)$–color partition generated by their unsigned version. As an example, let us consider the following identity:

$$\sum_{n=0}^{\infty}\frac{(-1)^n(q,q^2)_n q^{n^2}}{(q^4,q^4)_n(-q,q^2)_n}=\frac{(q;q^2)_\infty}{(q^2;q^2)_\infty}[q^5,-q^2,-q^3;q^5]_\infty. \qquad ([108, \text{I}(21)], [52, \text{I}(29)])$$

Theorem 7.2.4 gives the interpretation of the following unsigned version of L.H.S of this identity with $k=-1$:

$$\sum_{n=0}^{\infty}\frac{(-q,q^2)_n q^{n^2}}{(q^4,q^4)_n(q,q^2)_n}.$$

Now, by attaching weight $(-1)^h$ to each n–color partition generated by the unsigned version, where

$$h=\begin{cases}\frac{1}{2}(a_1+b_1) & \text{if } (a_1)_{b_1}\in S_1^{-1},\\ \frac{1}{2}(a_1+b_1+1) & \text{if } (a_1)_{b_1}\in S_2^{-1},\end{cases}$$

we get the interpretation of the original identity. Similarly, the identities

$$\sum_{n=0}^{\infty}\frac{(q,q^2)_n q^{2n(n+1)}}{(q^4,q^4)_n(-q,q^2)_{n+1}}=\frac{(q;q^2)_\infty}{(q^2;q^2)_\infty}[q^3,-q^3,-q^3;q^3]_\infty, \qquad ([52, \text{I}(5)])$$

$$\sum_{n=0}^{\infty}\frac{(-1)^n(q,q^2)_n q^{n(n+2)}}{(q^4,q^4)_n(-q,q^2)_{n+1}}=\frac{(q;q^2)_\infty}{(q^2;q^2)_\infty}[q^5,-q^5,-q^5;q^5]_\infty, \qquad ([52, \text{I}(25)])$$

$$\sum_{n=0}^{\infty}\frac{(-1)^n(q,q^2)_n q^{n(n+2)}}{(q^4,q^4)_n(-q,q^2)_n}=\frac{(q;q^2)_\infty}{(q^2;q^2)_\infty}[q^5,-q,-q^4;q^5]_\infty, \qquad ([52, \text{I}(27)])$$

can be interpreted by attaching weight $(-1)^h$ to each $(n+t)$–color partition generated by their unsigned version, where respective values of h for these identities are

$$h = \begin{cases} \frac{1}{2}(a_1+b_1+4)-2p & \text{if } (a_1)_{b_1} \in S_1^1, \\ \frac{1}{2}(a_1+b_1+3)-2p & \text{if } (a_1)_{b_1} \in S_2^1, \end{cases}$$

$$h = \begin{cases} \frac{1}{2}(a_1+b_1)-1 & \text{if } (a_1)_{b_1} \in S_1^{-1}, \\ \frac{1}{2}(a_1+b_1+1)-1 & \text{if } (a_1)_{b_1} \in S_2^{-1}, \end{cases}$$

$$h = \begin{cases} \frac{1}{2}(a_1+b_1-2) & \text{if } (a_1)_{b_1} \in S_3^{-1}, \\ \frac{1}{2}(a_1+b_1-1) & \text{if } (a_1)_{b_1} \in S_4^{-1}. \end{cases}$$

7.5 Conclusion

Our results in conjunction with the results from ([94, 95, 111]) provide infinitely many three-way combinatorial identities. Also, it is evident that in some particular cases, our results lead to four way combinatorial interpretations of some Rogers-Ramanujan type identities. In ([94, 95]), only unsigned versions of the identities from Chu-Zhang's compendium have been interpreted, but we are providing the interpretations of original identities by attaching weights to partitions generated by unsigned versions. In the light of this work, one can try to find the interpretations of some other identities from Slater's compendium [108] and Chu-Zhang's compendium [52] by attaching weights.

Chapter 8

$(n+t)$–color analogue of Gordon's theorem

As discussed in Chapter 1, Andrews et al. generalized the concept of successive ranks to hook differences. By using the notion of hook differences they generalized the following successive ranks theorem ([20], [41]):

Theorem 8.0.1. *Let* $P_{K,i}(s)$ *be the number of partitions of* s *with successive ranks in the interval* $[-i+2, K-i-2]$ *and* $Q_{K,i}(s)$ *be the number of partitions of* s *into parts* $\not\equiv 0, \pm i \pmod{K}$. *Then for* $1 \leq i \leq \frac{K}{2}$

$$P_{K,i}(s) = Q_{K,i}(s) \quad \forall\ s.$$

The above theorem imposes bound on successive ranks only i.e hook differences on diagonal 0. The extension of this result by Andrews et al. [58] includes two diagonals (not necessarily different) and imposes an upper bound on hook differences of the first and a lower bound on the second. The partitions appearing in their general identity are defined below:

Definition 8.0.1. $p_{K,i}(\alpha, \beta; s)$ *defines the number of partitions of* s *with hook differences on the diagonals* $1-\beta$ *and* $\alpha-1$ *as* $\geq -i+\beta+1$ *and* $\leq K-i-\alpha-1$ *respectively.*

Although the general identity of [58] includes a complex product, in some particular cases it reduces to a simple triple or quintuple product. We are particularly interested in two of these reductions, first of which is stated here.

Theorem 8.0.2. *For* $1 \leq i, \alpha \leq \frac{K}{2}$ *and* $2i \neq K$*,*

$$\sum_{s=0}^{\infty} p_{K,i}(\alpha, \alpha; s) q^s = \prod_{\substack{s=1 \\ s \not\equiv 0, \pm \alpha i \pmod{\alpha k}}}^{\infty} (1-q^s)^{-1}. \tag{8.1}$$

In [10], Agarwal and Andrews rephrased a special case $\alpha = 2$ of this identity in terms of the corresponding Frobenius symbol and mapped the partitions enumerated by the left hand side to certain restricted $(n+t)$–color partitions to obtain an identity of the Rogers-Ramanujan type. In fact, the concept of $(n+t)$–color partitions was first explicitly used in the same manuscript. The general partition identity obtained in [10] is stated below:

Theorem 8.0.3. *For* $1 \leq l \leq k-1$ *and* $k \geq 2$*, let* $\mathscr{A}_l^{(2)}(k,s)$ *be the number of* $(n+l)$*–color partitions of* s *such that* $b_i = a_i + l$ *for some* i*, all pairs of parts satisfy*

$$(w.d)_{i,j} \geq -2 \min\{b_i - 1, b_j - 1, k-3\}, \tag{8.2}$$

and if $(w.d)_{i,j}$ *is* nonpositive *then it is* even*. Further, suppose that* $\mathscr{B}_l^{(2)}(k,s)$ *be the number of partitions of* s *into parts* $\not\equiv 0, \pm 2(k-l) \pmod{4k+2}$*. Then*

$$\mathscr{A}_l^{(2)}(k,s) = \mathscr{B}_l^{(2)}(k,s) \quad \forall\, s \geq 0.$$

The above theorem modifies to the following in case of $l = 0$:

Theorem 8.0.4. *Let* $\mathscr{A}_0^{(2)}(k,s)$ *be the number of* n*–color partitions of* s *such that if weighted differences of any pair of summands is* nonpositive *then it is* even *and satisfies* (8.2)*. Then*

$$\mathscr{A}_0^{(2)}(k,s) = \mathscr{B}_0^{(2)}(k,s) \quad \forall\, s \geq 0.$$

Our foremost idea was to examine $p_{K,i}(\alpha, \alpha; s)$ of Theorem 8.0.2 for $\alpha = 3$ and k odd. We not only succeeded in obtaining results for $\alpha = 3$ but were also able to generalize these for any value of α. For $K = 2k+1$, the right hand side of equation (8.1) is a triple product identity involving modulus $2\alpha k+\alpha$. We know that the combinatorial generalization of Rogers-Ramanujan identities due to Gordon (Theorem 1.1.4) involves modulus $2k+1$. Recently, Sills [106] gave the following mild extension of Gordon's theorem which involves modulus $2\alpha k+\alpha$:

Theorem 8.0.5. *For $1 \leq i \leq k$, $\alpha \geq 1$, the number of partitions of s such that α appears at most $i-1$ times as a part and total number of occurrences of any two consecutive multiples of α together is at most $k-1$ equals the number of partitions of s into parts $\not\equiv 0, \pm\alpha i \pmod{2\alpha k+\alpha}$.*

However, both Gordon's theorem and its extension use only ordinary partitions. Our main result here involves $(n+t)$–color partitions and hence is named as $(n+t)$–color analogue of Gordon's theorem.

Another reduction from the general identity of [58] which we utilize to obtain a n–color partition identity for a quintuple product is the following theorem:

Theorem 8.0.6. *For $\alpha+\beta<3K$ and $R=2K(\alpha+\beta)$*

$$\sum_{s=0}^{\infty} p_{3K,K}(\alpha,\beta;s)q^s = \prod_{\substack{s=1 \\ s\not\equiv 0,\pm K\beta \ (\mathrm{mod}\ R) \\ s\not\equiv \pm 2K\alpha \ (\mathrm{mod}\ 2R)}}^{\infty} (1-q^s)^{-1}. \tag{8.3}$$

8.1 Main results

In this section, first we deduce a result for $\alpha=3$ from Theorem 8.0.2 and thereby generalizing for any value of α to obtain $(n+t)$–color analogue of Gordon's theorem.

Theorem 8.1.1. *For $1 \leq l \leq k-2$, $k \geq 3$, let $\mathscr{A}_l^{(3)}(k,s)$ be the number of $(n+l+1)$–color partitions of s satisfying*

(i) $b_p = a_p + l + 1$;

(ii) $b_i \geq 2$, $1 \leq i \leq p$;

(iii) $(w.d)_{i,i+1}$ *takes* odd *values* ≥ -1 *and* even *values* $\geq -2\min\{b_i-1, b_{i+1}-1\}$, $1 \leq i \leq p-1$;

(iv) $(w.d)_{i,i+2}$ *is* nonpositive *if it is* even *and* $\geq -2\ \min\{b_i-2, b_{i+2}-2, k-4\}$, $1 \leq i \leq p-2$.

And, $\mathscr{B}_l^{(3)}(k,s)$ *denote the number of partitions of* s *into parts* $\not\equiv 0, \pm 3(k-l) \pmod{6k+3}$. *Then*

$$\mathscr{A}_l^{(3)}(k,s) = \mathscr{B}_l^{(3)}(k,s), \qquad \forall\ s \geq 0.$$

Proof. Substituting $\alpha = 3$, $K = 2k+1$ and $i = k-l$ in equation (8.1)

$$\sum_{s=0}^{\infty} p_{2k+1,k-l}(3,3;s)q^s = \prod_{\substack{s=1 \\ s\not\equiv 0,\pm 3(k-l) \pmod{6k+3}}}^{\infty} (1-q^s)^{-1}. \tag{8.4}$$

The Frobenius representation of the partitions enumerated by $p_{2k+1,k-l}(3,3;s)$ satisfy the following conditions:

$$c_i > c_{i+1} \geq 0,\ d_i > d_{i+1} \geq 0 \quad \forall\ 1 \leq i \leq p-1, \tag{8.5}$$

$$\min\{c_i - d_{i+2} - l - 1, d_i - c_{i+2} + l\} \geq 5 - k \quad \forall\ 1 \leq i \leq p-2, \tag{8.6}$$

$$\left.\begin{array}{lll} \text{if } c_p = 0 & \text{then} & c_{p-1} = 1 \text{ or } \geq 4-(k-l), \\ \text{if } c_p \neq 0 & \text{then} & c_p = 1 \text{ or} \geq 4-(k-l) \quad \text{and} \quad c_{p-1} \geq 5-(k-l). \end{array}\right\} \tag{8.7}$$

Now, we define a mapping on the set of partitions satisfying conditions (8.5)–(8.7) which assigns a colored part to each column of the Frobenius representation as follows:

$$\phi\begin{pmatrix} c_i \\ d_i \end{pmatrix} = \begin{cases} (c_i+d_i+1)_{c_i-d_i-l+1} & \text{if } c_i > d_i + l, \\ (c_i+d_i+1)_{d_i-c_i+l+2} & \text{if } c_i \leq d_i + l. \end{cases} \tag{8.8}$$

The inverse mapping is defined as

$$\phi^{-1}((a_i)_{b_i}) = \begin{cases} \begin{pmatrix} (a_i+b_i+l-2)/2 \\ (a_i-b_i-l)/2 \end{pmatrix} & \text{if } a_i \equiv b_i + l \pmod 2, \\ \begin{pmatrix} (a_i-b_i+l+1)/2 \\ (a_i+b_i-l-3)/2 \end{pmatrix} & \text{if } a_i \not\equiv b_i + l \pmod 2. \end{cases} \tag{8.9}$$

For any two colored parts $(a_i)_{b_i}$, $(a_j)_{b_j}$ $(i<j)$ corresponding to the columns $\begin{pmatrix} c_i \\ d_i \end{pmatrix}$ and $\begin{pmatrix} c_j \\ d_j \end{pmatrix}$

$$(w.d)_{i,j} = \begin{cases} 2(d_i - c_j + l - 1) & \text{if } c_i > d_i + l,\ c_j > d_j + l, \\ 2(d_i - d_j) - 3 & \text{if } c_i > d_i + l,\ c_j \le d_j + l, \\ 2(c_i - c_j) - 3 & \text{if } c_i \le d_i + l,\ c_j > d_j + l, \\ 2(c_j - d_j - l - 2) & \text{if } c_i \le d_i + l,\ c_j \le d_j + l, \end{cases} \tag{8.10}$$

$$= \begin{cases} -2(b_j - 1) + 2(d_i - d_j - 1) & \text{if } c_i > d_i + l,\ c_j > d_j + l, \\ 2(d_i - d_j) - 3 & \text{if } c_i > d_i + l,\ c_j \le d_j + l, \\ 2(c_i - c_j) - 3 & \text{if } c_i \le d_i + l,\ c_j > d_j + l, \\ -2(b_j - 1) + 2(c_i - c_j - 1) & \text{if } c_i \le d_i + l,\ c_j \le d_j + l, \end{cases} \tag{8.11}$$

$$= \begin{cases} -2(b_i - 1) + 2(c_i - c_j - 1) & \text{if } c_i > d_i + l,\ c_j > d_j + l, \\ 2(d_i - d_j) - 3 & \text{if } c_i > d_i + l,\ c_j \le d_j + l, \\ 2(c_i - c_j) - 3 & \text{if } c_i \le d_i + l,\ c_j > d_j + l, \\ -2(b_i - 1) + 2(d_i - d_j - 1) & \text{if } c_i \le d_i + l,\ c_j \le d_j + l. \end{cases} \tag{8.12}$$

The condition (ii) of the theorem is clearly implied by the way mapping ϕ is defined. An examination of equations (8.10)–(8.12) for $j = i+1$ under (8.5) provides the condition (iii) of the theorem. Again examining (8.10)–(8.12) for $j = i+2$ under the conditions (8.5) and (8.6), we get the condition (iv) of the theorem.

Now to fulfill (8.7), we assume that $c_p = 0$, if it is not so we include a dummy column $\begin{pmatrix} 0 \\ -1 \end{pmatrix}$ since in this case (8.7) becomes a tautology. Fixing $c_p = 0$ ensures that least part satisfies condition (i) of the theorem. It also implies that u is at least $l+1$. Also, $(w.d)_{i,i+1} \ge -2(b_{i+1} - 1)$ implies that

$$a_i - b_i \ge a_{i+1} - b_{i+1} + 2.$$

Therefore, the value of $a_i - b_i$ is least for $i = p$ which in turn implies that $u = l + 1$. The reverse implications can be easily proved once we have the following equations in hand:

$$2(c_i-c_j)=\begin{cases}(w.d)_{i,j}+2b_i & \text{if } a_i\equiv b_i+l,\ a_j\equiv b_j+l \pmod 2,\\ (w.d)_{i,j}+2(b_i+b_j)-3 & \text{if } a_i\equiv b_i+l,\ a_j\not\equiv b_j+l \pmod 2,\\ (w.d)_{i,j}+3 & \text{if } a_i\not\equiv b_i+l,\ a_j\equiv b_j+l \pmod 2,\\ (w.d)_{i,j}+2b_j & \text{if } a_i\not\equiv b_i+l,\ a_j\not\equiv b_j+l \pmod 2,\end{cases} \tag{8.13}$$

$$2(d_i-d_j)=\begin{cases}(w.d)_{i,j}+2b_j & \text{if } a_i\equiv b_i+l,\ a_j\equiv b_j+l \pmod 2,\\ (w.d)_{i,j}+3 & \text{if } a_i\equiv b_i+l,\ a_j\not\equiv b_j+l \pmod 2,\\ (w.d)_{i,j}+2(b_i+b_j)-3 & \text{if } a_i\not\equiv b_i+l,\ a_j\equiv b_j+l \pmod 2,\\ (w.d)_{i,j}+2b_i & \text{if } a_i\not\equiv b_i+l,\ a_j\not\equiv b_j+l \pmod 2,\end{cases} \tag{8.14}$$

$$2(d_i-c_j+l)=\begin{cases}(w.d)_{i,j}+2 & \text{if } a_i\equiv b_i+l,\ a_j\equiv b_j+l \pmod 2,\\ (w.d)_{i,j}+2b_j-1 & \text{if } a_i\equiv b_i+l,\ a_j\not\equiv b_j+l \pmod 2,\\ (w.d)_{i,j}+2b_i-1 & \text{if } a_i\not\equiv b_i+l,\ a_j\equiv b_j+l \pmod 2,\\ (w.d)_{i,j}+2(b_i+b_j)-4 & \text{if } a_i\not\equiv b_i+l,\ a_j\not\equiv b_j+l \pmod 2,\end{cases} \tag{8.15}$$

$$2(c_i-d_j-l)=\begin{cases}(w.d)_{i,j}+2(b_i+b_j) & \text{if } a_i\equiv b_i+l,\ a_j\equiv b_j+l \pmod 2,\\ (w.d)_{i,j}+2b_i+1 & \text{if } a_i\equiv b_i+l,\ a_j\not\equiv b_j+l \pmod 2,\\ (w.d)_{i,j}+2b_j+1 & \text{if } a_i\not\equiv b_i+l,\ a_j\equiv b_j+l \pmod 2,\\ (w.d)_{i,j}+4 & \text{if } a_i\not\equiv b_i+l,\ a_j\not\equiv b_j+l \pmod 2,\end{cases} \tag{8.16}$$

□

Corollary 8.1.2. *For $l=0$,* Theorem 8.1.1 *is true with the modification in condition* (i) *that $b_p=a_p+1$ may or may not be true.*

Corollary 8.1.3. *For $l=k-1$,* Theorem 8.1.1 *is true with the additional condition $a_{p-1}+l\neq b_{p-1}+3$.*

To extend Theorem 8.1.1 for any value of $\alpha\geq 2$, the bijective map defined by ϕ

is modified as below:

$$\varphi\begin{pmatrix} c_i \\ d_i \end{pmatrix} = \begin{cases} (c_i+d_i+1)_{c_i-d_i-l+\alpha-2} & \text{if } c_i > d_i+l, \\ (c_i+d_i+1)_{d_i-c_i+\alpha+l-1} & \text{if } c_i \leq d_i+l. \end{cases} \tag{8.17}$$

$$\varphi^{-1}((a_i)_{b_i}) = \begin{cases} \begin{pmatrix} (a_i+b_i+l-\alpha+1)/2 \\ (a_i-b_i-l+\alpha-3)/2 \end{pmatrix} & \text{if } a_i-b_i \not\equiv l-\alpha \pmod 2, \\ \begin{pmatrix} (a_i-b_i+l+\alpha-2)/2 \\ (a_i+b_i-l-\alpha)/2 \end{pmatrix} & \text{if } a_i-b_i \equiv l-\alpha \pmod 2. \end{cases} \tag{8.18}$$

By using the above map and following same steps as in Theorem 8.1.1, we obtain the following $(n+t)$–color analogue of Gordon's theorem:

Theorem 8.1.4. *For* $1 \leq l \leq k-\alpha+1$, $k \geq \alpha$, *let* $\mathscr{A}_l^{(\alpha)}(k,s)$ *be the number of* $(n+\alpha+l-2)$*–color partitions of* s *satisfying*

(i) $b_p = a_p + \alpha + l - 2$;

(ii) $b_i \geq \alpha - 1$, $1 \leq i \leq p$;

(iii) $(w.d)_{i,i+1}$ *takes* odd *values* $\geq 5-2\alpha$ *and* even *values* $\geq -2\min\{b_i-1, b_{i+1}-1\}$, $1 \leq i \leq p-1$.

(iv) $(w.d)_{i,i+2}$ *is* nonpositive *if it is* even *and* $\geq -2\min\{b_i-\alpha+1, b_{i+2}-\alpha+1, k-\alpha-1\}$, $1 \leq i \leq p-2$.

And, $\mathscr{B}_l^{(\alpha)}(k,s)$ *denote the number of partitions of* s *into parts* $\not\equiv 0, \pm\alpha(k-l) \pmod{2\alpha k+\alpha}$. *Then*

$$\mathscr{A}_l^{(\alpha)}(k,s) = \mathscr{B}_l^{(\alpha)}(k,s) \qquad \forall\, s \geq 0.$$

8.2 n–color partitions for a quintuple product

The partitions arising from the product side of the following quintuple product identity due to Sills ([104], p. 400, Eq. (1.7)) are in one-to-one correspondence with certain restricted n–color partitions:

$$\sum_{n_1,n_2,\dots,n_k,j\geq 0}\frac{q^{N_1^2+N_2^2+\cdots+N_k^2}\left(\frac{n_k-j+1}{3}\right)}{(q;q)_{n_1}(q;q)_{n_2}\dots(q;q)_{n_k}(q;q)_j(q;q)_{2n_k-j}}=\frac{(q^k,q^{5k},q^{6k};q^{6k})(q^{4k},q^{8k};q^{12k})}{(q;q)_\infty}, \tag{8.19}$$

where $N_i=n_i+n_{i+1}+\cdots+n_k$ and $\left(\frac{n}{p}\right)$ is the Legendre symbol. This result is also accomplished in a similar manner as in Theorem 8.1.1 by substituting $K=k$, $\alpha=2$ and $\beta=1$ in equation (8.3). The mapping used to obtain the desired result is

$$\phi\begin{pmatrix}c_i\\ d_i\end{pmatrix}=\begin{cases}(c_i+d_i+1)_{c_i-d_i} & \text{if } c_i>d_i,\\ (c_i+d_i+1)_{d_i-c_i+1} & \text{if } c_i\leq d_i.\end{cases}$$

And, the inverse mapping is defined as

$$\phi^{-1}((a_i)_{b_i})=\begin{cases}\begin{pmatrix}(a_i+b_i-1)/2\\ (a_i-b_i-1)/2\end{pmatrix} & \text{if } a_i\not\equiv b_i \pmod 2,\\ \begin{pmatrix}(a_i-b_i)/2\\ (a_i+b_i-2)/2\end{pmatrix} & \text{if } a_i\equiv b_i \pmod 2.\end{cases}$$

Thus, we obtain the following result:

Theorem 8.2.1. *For* $k\geq 2$, *let* $\mathscr{C}(k,s)$ *be the number of* n*–color partitions of* s *satisfying*

(i) *if* $a_i\equiv b_i \pmod 2$ *then* $b_i\leq k-1$, $1\leq i\leq p$;

(ii) $(w.d)_{i,i+1}$ *is* nonpositive *if it is* even *and* $\geq -2\min\{b_i-1,b_{i+1}-1\}$, $1\leq i\leq p-1$;

(iii) *if* $a_i\not\equiv b_i$ *and* $a_{i+1}\not\equiv b_{i+1}$ *then* $(w.d)_{i,i+1}\geq 8-4k$, $1\leq i\leq p-1$.

And, $\mathscr{D}(k,s)$ *denote the number of partitions of* s *into parts* $\not\equiv 0,\pm k \pmod{6k}$ *and* $\not\equiv \pm 4k \pmod{12k}$. *Then*

$$\mathscr{C}(k,s)=\mathscr{D}(k,s)\qquad\forall\ s\geq 0.$$

8.3 Conclusion

The analytic counterpart to the identities proved in [10] was found by Agarwal et al. [11] using the concept of Bailey lattice. However for two particular cases, Identities (4.12) and (4.13) of Verma and Jain [116] can also be considered as their analytic counterparts. Also, there are two Identities (4.18) and (4.19) in [116] involving triple product modulo $6k+3$. Again, in two particular cases (i.e. for $l=0$ and $l=k-1$) these identities can be considered as analytic counterparts to Corollaries 8.1.2 and 8.1.3. Using Identity (4.17) of [116], we found the following analytic counterpart of Theorem 8.1.1 for the case $l=1$:

$$
\begin{aligned}
1+\sum_{n=0}^{\infty}\sum_{\substack{r_1,r_2,\dots,r_{k-4}=0\\(n,r_1,\dots,r_{k-4})\neq(0,0,\dots,0)}}^{\infty} &\frac{(q^3;q^3)_{n+2M_{k-4}-1}\,q^{3(M_1^2+\cdots+M_{k-5}^2)+n^2+12M_{k-4}^2+6nM_{k-4}}}{(q;q)_n(q^3;q^3)_{r_1}\dots(q^3;q^3)_{r_{k-4}}}\\
&\cdot\frac{\left(1-(1-q^{3r_{k-4}})q^{6(M_1+\cdots+M_{k-5})+3(k-1)-12}\right)}{(q;q)_{2n+6M_{k-4}-1}}\\
&=\prod_{\substack{n=1\\n\not\equiv0,\pm3(k-1)\ (\mathrm{mod}\ 6k+3)}}^{\infty}(1-q^n)^{-1},
\end{aligned}
\tag{8.20}
$$

where $M_i=r_1+r_2+\cdots+r_i$. Thus following the results of Verma and Jain [116], we get a separate identity for each value of l. It will be interesting to find a similar analytic counterpart for Theorem 8.1.1. However, the approach of Agarwal et al. [11] seems to be more appropriate for these cases as it gives two generalized identities as analytic counterpart to the results of [10]. Thus, an obvious future problem arising from this work is to find the analytic counterpart to Theorem 8.1.4. Also, it will further generalize the result of Theorem 5.2 of [11].

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