NUMERICAL SOLUTION OF MHD EQUATIONS BY SPECTRAL QUASILINEARIZATION METHOD

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Certificate

It is certified that the work contained in this thesis entitled **"NUMERICAL SOLUTION OF MHD EQUATIONS BY SPECTRAL QUASILINEARIZATION METHOD"** in partial fulfillment of the requirements for the award of degree of Master of Science in "Mathematics and Computing" to the School of Mathematics, Thapar University, Patiala is an authentic record of my own work studied under the supervision of Dr. Raj Nandkeolyar. *The matter embodied in this thesis has not been submitted by me for the award of any other degree of this or any other University/Institute.*

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This is to certify that the above statement made by the candidate is correct and true to the best of my knowledge.

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Abstract

Differential equations play an important role in modeling many natural and engineering processes such as ecology, aerospace engineering, fluid engineering etc. There are several methods which help us in finding the analytical solution of the differential equations in simple situations. During the mathematical modeling processe, as we get closer towards the real life situations, the system of differential equations become more and more complex and finding an analytical solution becomes almost impossible, such as in the case of Navier-Stokes' equations governing the fluid flow of the most realistic fluid. In such situations we are left with the option of using a numerical technique to obtain an approximate solution of the system.There are several methods which can help us in solving non-linear ordinary differential equations such as finite difference method, finite element method, shooting technique coupled with some classical method etc. During the recent past spectral methods were widely used for solving non-linear ordinary differential equations.

The aim of the present thesis is to investigate the effectiveness of the spectral method based quasilinearization technique known as spectral quasilinearization method (SQLM) in solving similar ordinary non-linear differential equations arising in the magnetohydrodynamics (MHD) flow along a stretching cylinder.

Chapter 1 is introductory which presents a brief overview of differential equations, their classifications and a short account of the history of spectral methods.

In chapter 2 the numerical solution of the problem considered by Mukhopadhyay [5]using spectral quasilinearization method (SQLM) is presented. The main idea of the study presented in chapter 2 is to compare the results obtained by SQLM with those obtained by Mukhopadhyay [5]. It is observed that the results of Mukhopadhyay [5] are not as accurate as they claim to be.

Chapter 3 is an extension of the research study carried out by Mukhopadhyay [5] considering the effects of induced magnetic field. The main objective of present work is to examine the effectiveness of the SQLM in solving the non linear ordinary differential equations governing the model. It is concluded that the spectral quasilinearization method is an effective technique to deal with such type of fluid flow problems.

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Dedicated To My Parents…

Chapter 1

A Brief Overview of Differential Equations and Spectral Methods

1.1 Introduction

Differential equations are an important part of many disciplines, like engineering, physics, biology, economics and mathematics. Differential equations arise while modeling many natural and man made processes and the complexity of these methods increase as we get closer to the real life situations. In these cases the models are governed by nonlinear ordinary or partial differential equations. Due to the complexity and non linearity it is not easy to find the exact solution . So numerical methods have been developed to find the approximate solution of the models.
A brief overview of the differential equations are presented in the following sections.

1.2 Differential Equation

Definition 1.1. An equation involving derivatives or differentials of one or more dependent variables with respect to one or more independent variables is called a differential equation.

Examples of differential equations are:

- dy=(x+sinx)dx

- $\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = xy$

1.2.1 Order of a Differential Equation

The order of the highest order derivative involved in a differential equation is called the order of the differential equation.

1.2.2 Degree of a Differential Equation

The degree of a differential equation is the power of the highest derivative.
Example:

- $dy = (x + sinx)dx$
  Order:first order, degree:1.
- $y = \sqrt{x}\frac{dy}{dx} + \frac{k}{\frac{dy}{dx}}$
  Order:first order, degree:2.

1.2.3 Types of Differential Equations

1. Ordinary Differential Equations.
2. Partial Differential Equations.

1.2.4 Ordinary Differential Equations

A differential equation involving derivatives with respect to a single independent variable is called an ordinary differential equation.

- $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 0$

1.2.5 Classification of Ordinary Differential Equations:

1. Linear Differential Equations:
   A differential equation is called linear if every dependent variable and every derivative involved occurs in the first degree only and there is no product of dependent

variable and derivative.
The general form of a linear differential equations of n^{th} order is:

$$\frac{d^n(y)}{dx^n} + P_1\frac{d^{n-1}y}{dx^{n-1}} + P_2\frac{d^{n-2}y}{dx^{n-2}} + \ldots + P_n y = X$$

where$P_1, P_2 ... P_n$ and X are either constants or functions of x.
The general form of linear differential equation of first order is

$$\frac{dy}{dx} + Py = Q$$

where P and Q are either constants or functions of x only. Such equations are also known as **Leibnitz form of linear equations.**
Example:
$\frac{dy}{dx} + \frac{y}{x} = x^2$

2. Non Linear Differential Equations:
A differential equation which is not of the form of linear differential equation is called non linear differential equation . In other words the differential equation is not linear in the dependent variables and its derivatives is called nonlinear differential equation. Example:
$y = x\frac{dy}{dx} + a\sqrt{1 + \frac{dy}{dx}^2}$

1.2.6 Partial Differential Equations

A differential equation involving derivatives with respect to more than one independent variable is called a partial differential equation.Partial differential equations are often used to construct models of the most basic theories underlying physics and engineering. For example, the system of partial differential equations known as Maxwell's equations can be written on the back of a post card, yet from these equations one can derive the entire theory of electricity and magnetism, including light. They included applications of fourier analysis to the study of constant coefficient equations especially in Laplace, heat and wave equation. The partial differential equations can be used to describe a wide variety of phenomena such as sound, heat, electrostatics, electrodynamics, fluid flow, elasticity, or quantum mechanics.
Example:

- $\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = xy$
- $\frac{\partial^2 z}{\partial^2 x} + \frac{\partial^2 z}{\partial^2 y} = 0$

1.2.7 Classification of Partial Differential Equations

1. Linear Partial Differential Equations:

   A partial differential equation of first order for an unknown function $z(x, y)$ is said to be linear if it can be expressed in the form
   $a(x, y)\frac{\partial z(x,y)}{\partial x} + b(x, y)\frac{\partial z(x,y)}{\partial y} + c(x, y)z(x, y) = d(x, y)$

2. Quasi-linear Partial Differential Equations:

   A partial differential equation of first order for an unknown function $z(x, y)$ is said to be quasi-linear if it can be expressed in the form
   $a(x, y)z(x, y)\frac{\partial z(x,y)}{\partial x} + b(x, y)z(x, y)\frac{\partial z(x,y)}{\partial y} = c(x, y, z(x, y))$

   Example:
   $\frac{\partial z}{\partial x} + z^3\frac{\partial z}{\partial y} = xyz^2$

3. Semi-linear Partial Differential Equations:

   A partial differential equation of first order for an unknown function $z(x, y)$ is said to be semi-linear if it can be expressed in the form
   $a(x, y)\frac{\partial z(x,y)}{\partial x} + b(x, y)\frac{\partial z(x,y)}{\partial y} = c(x, y, z(x, y))$

   Example:
   $\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = xyz^2$

4. Non-Linear Partial Differential Equations: A first order partial differential equation which does not come under the above three types is known as non-linear partial differential equation.
   Example:
   $(\frac{\partial z}{\partial x})^2 + (\frac{\partial z}{\partial y})^2 = 0$

1.2.8 Homogeneous and Non Homogeneous

A partial differential equation is said to be homogeneous if each term of the equation contains either the dependent variable or one of its derivatives. Otherwise it is said to be non-homogeneous.
Example:

$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = k(\frac{\partial^2 u}{\partial t^2})$ - homogeneous

$(\frac{\partial^2 u}{\partial y^2})^3 + (\frac{\partial^3 u}{\partial x^3})^2 + 8u = 0$ - non homogeneous

1.3 Numerical Solution of Ordinary Differential Equations

There are several natural and man made situations where the equations governing the phenomena are nonlinear in nature and the exact solutions of these nonlinear equations are not always possible. In such situations we can apply the numerical techniques and obtain approximate solutions of such problems. To name a few, Taylor Series method, Euler Modified method and Runge-Kutta methods, can be used to solve initial value problem. The boundary value problem can be solved using Shooting technique, with the methods applicable for initial value problems.
Our main concentration would be on the spectral based quasilinearization method. In order to have a brief idea in this method it is worth while to understand some basics of spectral methods.

1.4 Brief History of Spectral Method

During 1960s the methods of finite differences and finite elements were developed, however the spectral methods were mostly developed during 1970s. For spectral methods, some concept are as old as expansion and interpolation. The term spectral refers to the solution of defining equations on a grid of discrete points, y_i, and the solutions, $f(y_i)$, as determined at the grid points. If we want to find the solution of an partial or ordinary differential equation to high accuracy, then the spectral methods are the best tool.They can usually achieve ten digits of accuracy where a finite difference and finite element method usually achieve two or three digits accuracy. In spectral methods, the main idea is the introduction of a differentiation matrix D which is used to approximate the derivatives of the unknown variables at nodes.The fundamentals of spectral methods is well presented by Trefethen [7]

1.4.1 Differentiation Matrices

If we have a set of grid points y_j with function values $v(y_j)$ to approximate the derivative of v. Usually we use the finite difference method. From finite difference method we shall motivate with spectral methods.
Consider a uniform grid $\{y_1, y_2, ..., y_N\}$ with $y_{j+1} - y_j = h$ for each j, and set of corresponding data values $\{v_1, ...v_N\}$
Let ω_j be the approximation to $v'(y_j)$. The standard finite difference scheme is

$$\omega_j = \frac{v_{j+1} - v_{j-1}}{2h} \tag{1.1}$$

which can be derived by the Taylor expansions of $v(y_{j+1})$ and $v(y_{j-1})$. Let us suppose, the problem is periodic and take $v_0 = v_n$ and $v_1 = v_{n+1}$ then we can represent the discrete differentiation process as a matrix vector multiplication.

$$\begin{pmatrix} \omega_1 \\ \\ \vdots \\ \\ \omega_N \end{pmatrix} = h^{-1} \begin{pmatrix} 0 & \frac{1}{2} & & & -\frac{1}{2} \\ -\frac{1}{2} & 0 & \ddots & & \\ & & \ddots & & \\ & & & 0 & \frac{1}{2} \\ \frac{1}{2} & & \ddots & -\frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ \\ \vdots \\ \\ v_N \end{pmatrix} \tag{1.2}$$

here the diagonal entries are constant so this matrix is known as Toeplitz. There is another way to derive (1.1) and (1.2) by local interpolation and differentiation.
For j=1,2,...,N.

- Let a_j be an unique polynomials whose degree is less than equals to 2 with $a_j(y_{j-1}) = v_{j-1}, a_j(y_j) = v_j$ and $a_j(y_{j+1}) = v_{j+1}$.
- Set $\omega_j = a_j'(y_j)$.

when j is fixed the interpolant a_j is given by

$$a_j(y) = v_{j-1}b_{-1}(y) + v_j b_0(y) + v_{j+1}b_j$$

where $b_{-1}(y) = \frac{(y-y_j)(y-y_{j+1})}{2h^2}$, $b_0(y) = -\frac{(y-y_{j-1})(y-y_{j+1})}{h^2}$ and $b_1 = \frac{(y-y_{j-1})(y-y_j)}{2h^2}$ differentiate and at $y = y_j$ we can evaluate (1.1) equation. Similarly by local interpolation we can calculate the higher order differential equations. Here is a example of fourth order differential equation.
For j=1,2,..N.

- Let a_j be an unique polynomials whose degree is less than equals to four with $a_j(y_{j\pm2}) = v_{j\pm2}, a_j(y_{j\pm1}) = v_{j\pm1}$ and $a_j(y_j) = v_j$.
- Set $\omega_j = a_j'(y_j)$.

Assuming that the data is periodic, we have the following matrix-product vector.

$$\begin{pmatrix} \omega_1 \\ \\ \vdots \\ \\ \omega_N \end{pmatrix} = h^{-1} \begin{pmatrix} & & \ddots & & & \frac{1}{12} & -\frac{2}{3} \\ & & \ddots & -\frac{1}{2} & & \frac{1}{12} & \\ & & \ddots & \frac{2}{3} & \ddots & & \\ & & \ddots & 0 & \ddots & & \\ & & \ddots & -\frac{2}{3} & \ddots & & \\ -\frac{1}{12} & & & \frac{1}{12} & \ddots & & \\ \frac{2}{3} & -\frac{1}{12} & & & \ddots & & \end{pmatrix} \begin{pmatrix} v_1 \\ \\ \vdots \\ \\ v_N \end{pmatrix} \tag{1.3}$$

This is a pentadiagonal matrix, in first example we have a tridiagonal matrix but the matrices (1.2) and (1.3) both are differentiation matrices having order of accuracy 2 and

4 respectively. The v_j obtained by sampling the function v. Here $v'(x_j)$ will converges at the rate of $O(h^2)$ and $O(h^4)$. We can verify this result by Taylor series Trefethen [7].

1.4.2 Chebyshev Differentiation Matrices

We have the Chebyshev points

$$y_j = cos\left(\frac{j\pi}{N}\right) \quad \text{where } j = 1, 2, 3..., N \tag{1.4}$$

We shall use these points to construct Chebyshev differentiation matrices and apply these matrices to differentiate some functions. The scheme can be generated by these steps. Given a grid function w defined on the points,we get a discrete derivative v in two steps.

- Let a_j be an unique polynomials whose degree is less than equals to N with $a(x_j) = v_j, 0 \leq j \leq N$.
- Set $v_j = a'_j(y_j)$.

This operation is linear, will be shown by multiplication with $(N+1) \times (N+1)$ matrix which is denoted by D_N.

$$v = D_N w \tag{1.5}$$

where N is positive integer either even or odd .When N is even then we will use Fourier expansion in place of Chebyshev differentiation matrix.
Before moving to the general case we will done interpolation at $N = 1$ and $N = 2$.
Firstly take the case when $N = 1$. The points of interpolation are $y_0 = 1$ and $y_1 = -1$ and the interpolation of the polynomial through w_0 and w_1, written in Lagrange form is

$$a_y = \frac{1}{2}(1+y)w_0 + \frac{1}{2}(1-y)w_1$$

on taking the derivative

$$a_y = \frac{1}{2}w_0 - \frac{1}{2}w_1$$

from the above formula we get D_1 matrix of 2×2 whose first column have constant entries $\frac{1}{2}$ and second column have entries $-\frac{1}{2}$:

$$D_1 = \begin{pmatrix} \frac{1}{2} & \frac{-1}{2} \\ \\ \frac{1}{2} & \frac{-1}{2} \\ \\ \end{pmatrix} \tag{1.6}$$

Now we consider the case when $N = 2$. The points of interpolation are $y_0 = 1$ and $y_1 = -1$ and the interpolation of the polynomial through w_0 and w_1, written in Lagrange form is
$a_(y) = \frac{1}{2}y(1+y)w_0 + (1+y)(1-y)w_1 + \frac{1}{2}y(y_1)w_2$
The derivative of this polynomial is linear,
$a'(y) = (y + \frac{1}{2})w_0 - 2yw_1 + (y - \frac{1}{2})w_2$
The differentiation matrix D_2 is the 3×3 matrix whose jth term of this expression at $y = 1, 0$ and -1:

$$D_2 = \begin{pmatrix} \frac{3}{2} & -2 & \frac{1}{2} \\ \\ \frac{1}{2} & 0 & \frac{-1}{2} \\ \\ \frac{-1}{2} & 2 & \frac{-3}{2} \\ \\ \end{pmatrix}. \tag{1.7}$$

For each $N \geq 1$ suppose the rows and columns of the $(N+1) \times (N+1)$ Chebyshev spectral differentiation matrix D_N be indexed from 0 to N. The entries of this matrix are Trefethen [7].

$$(D_N)_{00} = \frac{2N^2+1}{6}, \quad (D_N)_{NN} = -\frac{2N^2+1}{6} \tag{1.8}$$

$$(D_N)_{jj} = \frac{-x_j}{2(1-x_j^2)}, \quad j = 1, 2, 3, ...N-1 \tag{1.9}$$

$$(D_N)_{ij} = \frac{d_i(-1)^{i+j}}{d_j(x_i - x_j)}, \quad i, j = 1, 2, ...N-1. \tag{1.10}$$

here

$$d_i = \begin{pmatrix} 2, & i = 0 \; or \; N \\ 1, & otherwise \end{pmatrix}.$$

With this we can calculate the first five Chebyshev Differentiation matrix.

$$D_1 = \begin{pmatrix} 0.500 & -0.500 \\ 0.500 & -0.500 \end{pmatrix} \tag{1.11}$$

$$D_2 = \begin{pmatrix} 1.500 & -2.000 & 0.500 \\ 0.500 & -0.000 & -0.500 \\ -0.500 & 2.000 & -1.500 \end{pmatrix} \tag{1.12}$$

$$D_3 = \begin{pmatrix} 3.167 & -4.000 & 1.333 & -0.500 \\ 1.000 & -0.333 & -1.000 & 0.333 \\ -0.333 & 1.000 & 0.333 & -1.000 \\ 0.500 & -1.333 & 4.000 & -3.61667 \end{pmatrix} \tag{1.13}$$

$$D_4 = \begin{pmatrix} 5.500 & -6.828 & 2.000 & -1.171 & -0.555 \\ 1.7071 & -0.7071 & -1.4142 & 0.7071 & 0.292 \\ -0.500 & -1.414 & -0.000 & -1.414 & 0.500 \\ 0.292 & -0.7071 & 1.414 & 0.707 & -1.707 \\ -0.500 & 1.171 & -2.000 & 6.828 & -5.500 \end{pmatrix} \tag{1.14}$$

$$D_5 = \begin{pmatrix} 8.500 & 10.472 & 2.894 & -1.527 & 1.105 & -0.500 \\ 10.472 & -1.170 & -2.000 & 0.894 & -0.618 & 0.276 \\ -0.733 & 2.000 & -0.170 & -2.000 & 0.723 & -0.382 \\ -0.276 & 0.618 & -0.894 & 2.000 & 1.170 & -2.6180 \\ 0.500 & -1.105 & 1.527 & -2.894 & 10.472 & -8.500 \\ \end{pmatrix} \tag{1.15}$$

1.4.3 Spectral QuasiLinearization Method (SQLM)

Before brief discussion about the Spectral QuasiLinearization method we have some difinitions:

Definition 1.2. Linearization of a function
To approximate the function $g(y)$ linearly we use Taylor Series expansion by assuming that the higher order derivatives are small. The linear approximation of $g(y)$ using Taylor series expansion to first order about the point $y = b$ is written as

$$\text{g(y)} \cong \text{g(b)+g}'\text{(b)(y-b)}$$

Definition 1.3. Linearization of multi-variable function:
The linear approximation of a function g(x,y,z) using multi-variable Taylor series expansion to first order about the point (k,l,m) can be written as The equation will be like

$$\text{g(x,y,z)} \approx \text{g(k,l,m)} + \frac{\partial g}{\partial x}|_{k,l,m}\text{(x-k)} + \frac{\partial g}{\partial y}|_{k,l,m}\text{(y-l)} + \frac{\partial g}{\partial w}|_{k,l,m}\text{(w-m)}$$

Definition 1.4. Convergence Criteria
An iteration scheme $x_{g+1} = \psi(x_g)$will converges when $\|x_{g+1} - x_g\| < \gamma$ for an arbitrary γ and $\| \ \|$ is some vector norm from this we can check whether the scheme will converges to a fixed point x^*

The spectral quasilinearization method was originally developed by Bellman and Kalaba [2] as a generalization of the Newton-Raphson method and was later used by several authors including RamReddy and Pradeepa [6]. In Spectral based QuasiLinearization Method, first of all we have to linearize the nonlinear terms of given differential equation by using one term multi-variable Taylor series expansion about the previous iteration. This process gives us the iterative scheme. There is need of an initial approximation to begins the iterative scheme. The initial approximation is taken in this way which satisfy the boundary conditions. Then Chebyshev spectral method is used to solve the linearized equations which is based on the Chebyshev polynomials defined on the interval $[-1, 1]$. For this firstly, we transform our given domain $[0, \infty)$ to $[-1, 1]$ by using truncation technique where the problem is solved in the interval $[0, L]$ instead of $[0, \infty)$ by using the mapping

$$\frac{\eta}{L} = \frac{\xi + 1}{2}, \quad -1 \leq \xi \leq 1$$

where L is the scaling parameter used to invoke the boundary conditions at infinity. After that the domain is discretized by using Chebyshev points defined as

$$\xi_j = \cos\left(\frac{j\pi}{N}\right), \quad j = 0, 1, 2, \ldots, N$$

This gives the following matrix equation:

$$A_i f_i = S_i$$

where f_i, S_i are the vectors of size $(N + 1) \times 1$ and A_i is the matrix of size $(N + 1) \times (N + 1)$.

In order to understand the procedure consider the boundary value problem

$$f''' + ff'' - (f')^2 - M^2 f' = 0 \tag{1.16}$$

with boundary conditions

$$f(0) = 0, \quad f'(0) = -1, \quad f'(\infty) = 0$$

where $f = f(\eta)$ and the primes denote differentiation with respect to η and M is an arbitrary parameter.

Firstly we linearize the nonlinear terms of above problem by using one term multi-variable Taylor series expansion about the previous iteration say r, which gives the iterative scheme as follow

$$f'''_{r+1} + f_r f''_{r+1} - (2f'_r + M^2)f'_{r+1} + f''_r f_{r+1} = f_r f''_r - f'^2_r,$$

with boundary conditions

$$f_{r+1}(0) = 0, \;\; f'_{r+1}(0) = -1, \;\; f'_{r+1}(\infty) = 0.$$

After applying the pseudo-spectral method ,we obtain the SQLM matrix form as

$$Af = B$$

where $A = D^3 - M^2 D + diag[f_r]D^2 - diag[2f'_r]D + diag[f''_r]$, and $B = f_r f''_r - f'^2_r$. Moreover, the exact solution of this problem is

$$f(\eta) = \frac{1}{a}(e^{-a\eta} - 1), \;\; a = \sqrt{M^2 - 1}.$$

TABLE 1.1: Comparison of the SQLM results and exact solution of $f''(0)$

M	Iteration	η_∞	N	SQLM	Exact
2	5	7	30	1.73205081	1.73205081
5	5	4	30	4.89897949	4.89897949
10	5	3	40	9.94987437	9.94987437
50	4	2	60	49.98999900	49.98999900

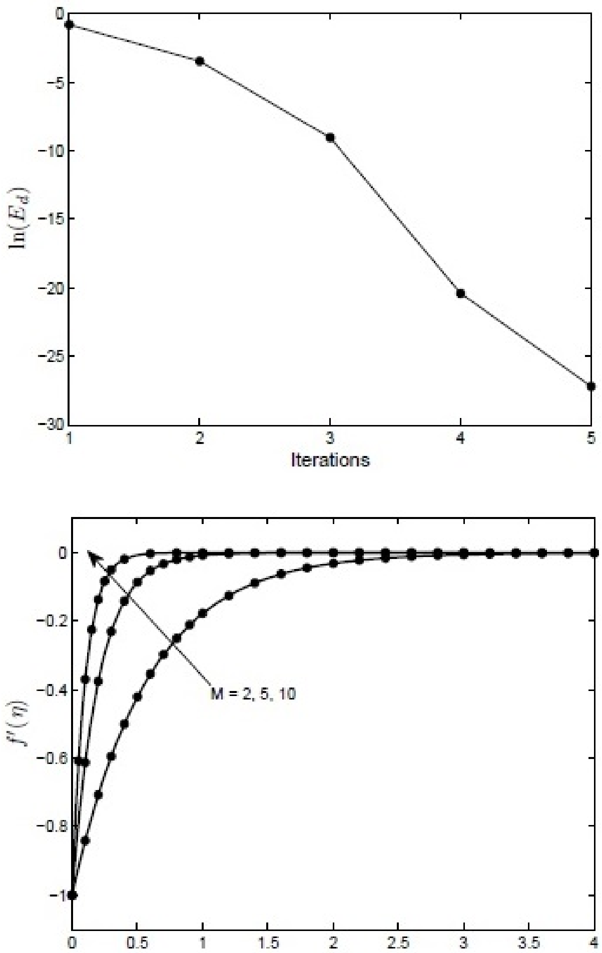

FIGURE 1.1: Velocity profile and Logarithm of the SQLM error (when M = 5) for the 1-Dimensional MHD boundary layer flow using N = 50

Chapter 2

SQLM Solution of MHD Boundary Layer Slip Flow Along a Stretching Cylinder

2.1 Introduction

The present chapter obtains the numerical solution of the problem considered by Mukhopadhyay [5] using SQLM. Mukhopadhyay [5] studied the axi-symmetric flow of a viscous and incompressible fluid along a stretching cylinder in the presence of a uniform magnetic field.The governing non-linear ordinary differential equations in non-dimensional form, were solved subject to the boundary conditions utilizing shooting technique with the help of classical fourth order Runge-Kutta method. Authors have claimed that they have presented the results accurate within 10^{-5}.

The aim of the present work is to reinvestigate the model considered by Mukhopadhyay [5] using Spectral QuasiLinearization Method (SQLM) and establish the validity of the results presented by them.

2.2 The Mathematical Model

The non-dimensional governing ordinary differential equations reported by Mukhopadhyay [5] are:

$$(1+2M\eta)f''' + 2Mf'' + ff'' - f'^2 - d^2f' = 0, \tag{2.1}$$

$$(1+2M\eta)\theta'' + 2M\theta' + Pr(f\theta' - Nf'\theta) = 0, \tag{2.2}$$

Boundary conditions for the given model at $\eta = 0$

$$f' = 1 + Bf'',\ f = 0, \theta = 1 \tag{2.3}$$

when $\eta \to \infty$

$$f' \to 0, \theta \to 0 \tag{2.4}$$

here the primes denote the differentiation with respect to η, B=$B_1\sqrt{\frac{U_0\nu}{L}}$ is the slip parameter, $d^2 = \frac{\sigma B_0^2 L}{\rho U_0}$ is magnetic parameter, M=$\sqrt{\frac{\nu L}{U_0 R^2}}$ is curvature parameter Mukhopadhyay [5].

2.3 The analytical solution for some special cases

The analytic solution of equation (2.1) for M=0, B=0, d=0 is

$$f(\eta) = 1 - e^{-\eta}$$

.

The solution of equation (2.1) for stretching flat plate (M=0) in the presence of magnetic field with no slip boundary condition (for B=0) is given by

$$f(\eta) = \frac{1}{\sqrt{d^2+1}}(1 - e^{-\sqrt{d^2+1}\eta}). \tag{2.5}$$

We can easily find the $f'(\eta) = e^{-\sqrt{d^2+1}\eta}$.
Now we will discuss the case when there is no magnetic field but stretching of flat plat (M=0) and slip at the boundary are present (B=0) then the solution of the equation will be

$$f''' + ff'' + f'^2 - d^2f' = 0 \tag{2.6}$$

The boundary conditions are

$$f(0) = 0, \quad f'(0) = 1 + Bf''(0) \quad \text{and} \quad f'(\infty) \to 0. \tag{2.7}$$

If we suppose the solution in the form $f(\eta) = a + be^{-\alpha\eta}$, then by substituting in equation (2.6) we get a=$\frac{1}{\alpha^2+B\alpha^2}$, b=$-\frac{1}{\alpha+B\alpha^2}$, where α is the root of following equation:

$$B\alpha^3 + \alpha^2 - Bd^2\alpha - 1 - d^2 = 0 \tag{2.8}$$

Then the solution is given by

$$f(\eta) = \frac{1}{\alpha + B\alpha^2} - \frac{1}{\alpha + B\alpha^2}e^{-\alpha\eta} \tag{2.9}$$

Now we convert the equation (2.8) to an equation $\beta^3 + r\beta + s = 0$, where $\beta = \alpha + \frac{1}{3B}$, $r = -\frac{1}{3B^2} - d^2$, $s = \frac{2}{27B^2}\frac{d^2}{3B} - \frac{1+d^2}{B}$. The roots of this equation are β_1 = E+F, $\beta_2 = \frac{-1}{2}$(E+F) + $\frac{\sqrt{3}}{2}$(E-F), $\beta_3 = \frac{-1}{2}$(E+F) $-$ $\frac{\sqrt{3}}{2}$(E-F), where E $= \sqrt[3]{-\frac{s}{2} + \sqrt{G}}$, F $= \sqrt[3]{\frac{-s}{2} - \sqrt{G}}$, $\iota^2 = -1$, G $= (\frac{r}{3})^2 + (\frac{s}{2})^2$

For stretching flat plate with no slip (B=0) and for N=2 the energy equation becomes

$$\theta'' + Prf\theta' - 2Prf'\theta = 0 \tag{2.10}$$

The boundary conditions are

$$\theta(0) = 1, \qquad \theta(\infty) \to 0 \tag{2.11}$$

Now introduce an independent variable,

$$\xi = \frac{-Pr}{\alpha^2}e^{-\alpha\eta}, \tag{2.12}$$

by substituting (2.5) and (2.12) in equation (2.10) we have

$$\xi\frac{d^2\theta}{d\xi^2} + [(1-K) - \xi]\frac{d\theta}{d\xi} + 2\theta = 0, \tag{2.13}$$

and the corresponding boundary conditions become $\theta(\xi = 0) = 1$, $\theta(\xi = \frac{-Pr}{\alpha^2}) = 1$, where K $= \frac{Pr}{\alpha^2}[\alpha^2 - d^2]$.

The solution of the equation (2.13) can be written in the form of hypergeometric function as

$$\theta(\xi) = \frac{\xi^{K} F_1(-2+K;1+K;\xi)}{(\frac{-\Pr}{\alpha^2} F_1(-2+K;1+K;\frac{-\Pr}{\alpha^2})}.$$

In terms of η, θ can be expressed as

$$\theta(\eta) = \frac{e_1^{-\alpha K\eta} F_1(-2+K;1+K;\frac{-\Pr}{\alpha^2} e_1^{-\alpha\eta})}{F_1(-2+K;1+K;\frac{-\Pr}{\alpha^2})} \tag{2.14}$$

The heat transfer rate at the surface will be

$$\theta'(0) = -\alpha\, K + \frac{\Pr}{\alpha}\left(\frac{\text{K-2}}{K+1}\right)\frac{F_1(-1+K;2+K;\frac{-\Pr}{\alpha^2})}{F_1(-2+K;1+K;\frac{-\Pr}{\alpha^2})}. \tag{2.15}$$

2.4 SQLM Solution of the Model

The SQLM method begins by linearizing the equations (2.1) and (2.2) using quasilinearization technique. The quasilinearization technique was first proposed by Bellman and Kalaba [2] which is nothing but a generalization of Newton-Raphson method. In the framework of SQLM, linearizing the equations using first order Taylor series approximation about the rth iteration level, we obtain the following iteration scheme

$$\begin{aligned}(1+2M\eta)f'''_{r+1} + (2M+f_r)f''_{r+1} - (2f'_r + d^2)f'_{r+1} \\ +f''_r f_{r+1} = f_r f''_r + f'^2_r,\end{aligned} \tag{2.16}$$

$$\begin{aligned}(1+2M\eta)\theta''_{r+1} + 2M\theta'_{r+1} + \Pr[\theta_r f_{r+1} + f_r\theta'_{r+1} \\ -N(\theta_r f'_{r+1} + f'_r\theta_{r+1})] = \Pr(f_r\theta'_r - Mf'_r\theta_r).\end{aligned} \tag{2.17}$$

The above iteration scheme is to be solved subject to the following boundary conditions:

$$f'_{r+1} = 1 + Bf''_{r+1}, f_{r+1} = 0, \theta_{r+1} = 1 \;\; at\ \eta = 0; \tag{2.18}$$

$$f'_{r+1} \to 0, \quad \theta_{r+1} \to 0 \;\; at\ \eta \to \infty. \tag{2.19}$$

here the primes denote the differentiation with respect to η.The other parameter as reported by Mukhopadhyay [5] are B=$B_1\sqrt{\frac{U_0\nu}{L}}$ is the slip parameter, $d^2 = \frac{\sigma B_0^2 L}{\rho U_0}$ is the magnetic parameter, M=$\sqrt{\frac{\nu L}{U_0 R^2}}$ is curvature parameter.
The equations (2.16) and (2.17) represent a linear system of coupled differential equations and can be solved using some numerical technique.In the present work the iteration scheme subject to the boundary conditions are solved using Chebyshev pseudo-spectral method. The basic idea of the method is to introduce a differentiation matrix D to approximate the derivatives of the unknown functions at the collocation points given by

$$\xi_j = \cos\left(\frac{j\pi}{N}\right), \quad j = 0, 1, 2, \ldots, N$$

where $N + 1$ is the number of collocation points.These points are known as Guass-Lobatto collocation points.The f(η) at the collocation points as the matrix product:

$$\frac{df}{d\eta} = \sum_{k=0}^{N} D_{jk} f(\xi_k) = \mathbf{Df}, \quad j = 0, 1, 2, \ldots, N$$

$$\frac{d\theta}{d\eta} = \sum_{k=0}^{N} D_{jk} \theta(\xi_k) = \mathbf{D}\Theta, \quad j = 0, 1, 2, \ldots, N$$

where $\mathbf{D}=\frac{2D}{L}$ is the differentiation matrix and D is Chebyshev differentiation matrix, L is a constant used to approximate infinity. The higher order derivatives are obtained as powers of **D**, that is

$$f^{(p)} = \mathbf{D}^{(p)}\mathbf{f}$$

$$\theta^{(p)} = \mathbf{D}^{(p)}\Theta$$

Thus, from (2.16) and (2.17), we obtain

$$\begin{aligned} &[(1 + 2M\eta)D^3 + 2MD^2 - \text{diag}(f_r)D^2 - 2\text{diag}(f_r')D - d^2D \\ &\qquad + \text{diag}(f_r'')]f_{r+1} = f_r''f_r - (f_r')^2, \end{aligned} \tag{2.20}$$

$$\begin{aligned} &[(1 + 2M\eta)D^2 + 2MD + \Pr\,\text{diag}(f_r)D - \Pr\,N\,\text{diag}(f_r')]\theta_{r+1} \\ &+[-\Pr\,N\,\text{diag}(\theta_r)D + \Pr\,\text{diag}(\theta_r')]f_{r+1} = \Pr\theta_r'f_r - \Pr\,N\theta_r f_r'. \end{aligned} \tag{2.21}$$

The matrix form of equation (2.20), (2.21)

$$\begin{bmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{bmatrix} \begin{bmatrix} f_{r+1} \\ \theta_{r+1} \end{bmatrix} = \begin{bmatrix} B_1 \\ B_{2}, \end{bmatrix} \tag{2.22}$$

where
$A_{1,1}$=$(1+2M\eta)D^3+2MD^2-\text{diag}(f_r)D^2-2\text{diag}(f'_r)D-d^2D+\text{diag}(f''_r)$,
$A_{1,2}$=0,
$A_{2,1}$=$-\text{Pr N diag}(\theta_r)D+\text{Pr diag}(\theta'_r)$,
$A_{2,2}$=$(1+2M\eta)D^2+2MD+\text{Pr diag}(f'_r)D-\text{PrNdiag}(f'_r)$,
B_1=$f''_r f_r-(f'_r)^2$,
B_2=$\text{Pr}\theta'_r f_r-\text{Pr N }\theta_r f'_r$.

2.5 Results And Discussion

The main objective of the present work is to analyze the results obtained by Mukhopadhyay [5] using Spectral Quasilinearization method. In the original paper by the author [5] utilizes a shooting technique to solve the system of non-linear ordinary differential equations. Author has also presented a comparison of the values of heat transfer rate at the surface $[\theta'(0)]$ for the special case of M=0, d=0, and Pr=1 with the work of Ishak and Nazar [4], Grubka and Bobba [3] and Ali [1] .
In the present chapter, we have presented a comparison of the values of heat transfer rate at the surface $[\theta'(0)]$ obtained by SQLM with these researchers along with the results obtained by Mukhopadhyay [5]. It should be noted that our results obtained using SQLM agree with the results obtained by Ishak and Nazar [4] and Grubka and Bobba [3] up to given number of decimal digits. However, the values obtained by Mukhopadhyay [5] starts differing from the fourth decimal place and hence the claim made by Mukhopadhyay [5] that her results are accurate within 10^{-5} stand false.This fact is further established when we compare our results and the results of Mukhopadhyay [5] for $f''(0)$ when M=0, B=0 with the analytical results. The error in the results reported by Mukhopadhyay [5] suggest that the results are only accurate within second decimal digits. The comparison of exact values of f' and θ obtained by SQLM and analytically are represented in fig (2.1) and (2.2) which also indicate that SQLM is much better option while solving such type of problems.

TABLE 2.1: Value of $[-\theta'(0)]$ for several values of temperature exponent 'N' for flat plate (M=0), D=0 and Pr=1.

N	Ishak and Nazar [4]	Grubka and Bobba [3]	Ali [1]	Mukhopadhyay [5]	Present study
0	0.5820	0.5820	0.5801	0.5821	0.58201050
1	1.0000	1.0000	0.9961	1.0000	1.00000837
2	1.3333	1.3333	1.3269	1.3332	1.33333334

TABLE 2.2: Value of [-$f''(0)$] obtained from numerical method.

d	Analytical solution	Mukhopadhyay [5]	Error in Mukhopadhyay [5]	Present solution	Error
0	1.0000000	0.99005806	0.00994194	1.00000835	0.0000083
0.5	1.1180340	1.1056039	0.012430088	1.11803452	0.00000052
1	1.4142135	1.3943545	0.019859062	1.41421356	0.00000006
1.5	1.802775638	1.7705669	0.032208737	1.80277563	0.0000000007

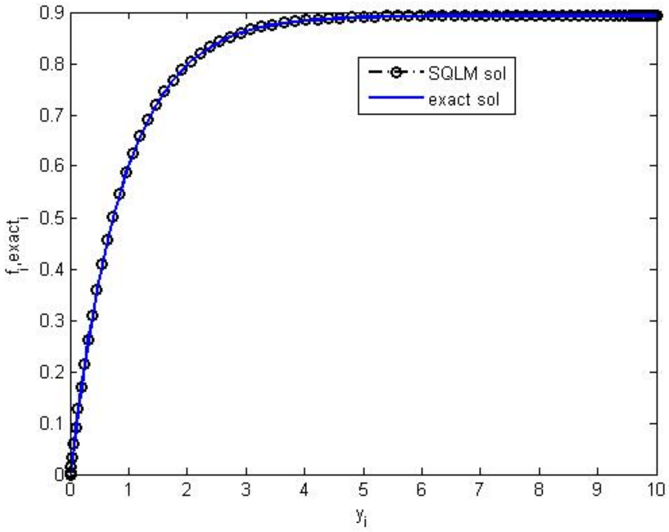

FIGURE 2.1: Comparision of SQLM with exact solution

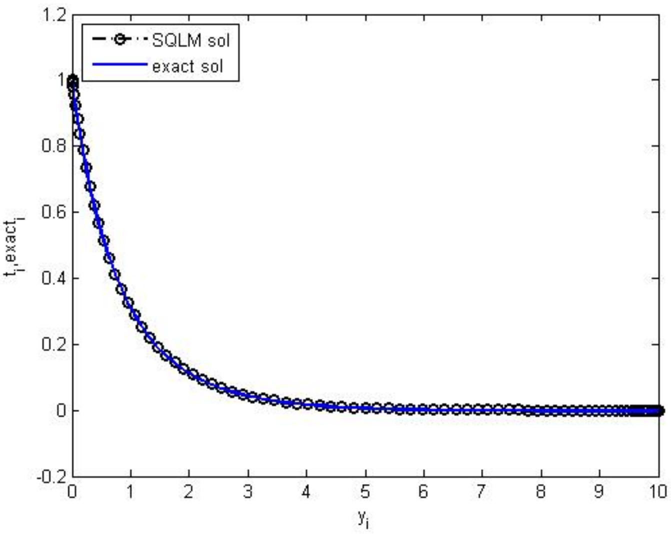

FIGURE 2.2: Comparision of SQLM with exact solution

Chapter 3

On Effectiveness of SQLM for the MHD Flow along a Stretching Cylinder with Induced Magnetic Field

3.1 Introduction

In the present chapter, we extend the research study of Mukhopadhyay [5] taking into account the effects of induced magnetic field. The presence of the induced magnetic field introduces one more equation into the system known as induction equation and the system of equations becomes more complex. Our aim through this chapter would be to study the effectiveness of spectral quasilinearization method in handling this complex system of equations.

3.2 The Mathematical Model

The governing non-linear partial differential equations will be subjected to a similarity transformation and we obtain the following system of non-linear coupled ordinary differential equations:

$$(1+2\gamma\eta)f'''(\eta)+2\gamma f''+ff''-(f')^2+M(g'^2-gg'')=0, \tag{3.1}$$

$$(1+2\gamma\eta)g'''+2\gamma g''+Pr_m fg''-Pr_m gf''=0, \tag{3.2}$$

$$(1+2\gamma\eta)\theta''+2\gamma\theta'+Prf\theta'=0. \tag{3.3}$$

Boudary conditions for the model are,

$$\begin{aligned} f=0, f'=1+Bf''(\eta), g=0, g'=0, \theta=1 \; at \; \eta=0, \\ f'\to 0, \theta\to 0, g'\to 1 \text{ as } \eta\to\infty. \end{aligned} \tag{3.4}$$

3.2.1 Numerical Solution of the Model using SQLM

The SQLM method begins by linearizing the equations (3.5) and (3.6) and (3.7) using quasilinearization technique. The quasilinearization technique was first proposed by Bellman and Kalaba [2] which is nothing but a generalization of Newton-Raphson method. In the framework of SQLM, linearizing the equations using first order Taylor series approximation about the r^{th} iteration level, we obtain the following iteration scheme

$$\begin{aligned} ((1+2\gamma\eta)f'''+2\gamma f''+f_{r+1}f''_r+f''_{r+1}f_r-2f'_{r+1}f'_r)+M(2g'_{r+1}g'_r-g_{r+1}g''_r-g''_{r+1}g_r) \\ =f''_r f_r-(f'_r)^2+M(g'_r)^2-Mg''_r g_r, \end{aligned} \tag{3.5}$$

$$(1+2\gamma\eta)g'''_{r+1}+(2\gamma+Pr_m f_r)g''_{r+1}-(Pr_m f''_r)g_{r+1}-Pr_m g_r f''_{r+1}+Pr_m g''_r f_{r+1}=Pr_m(g''_r f_r+f''_r g_r), \tag{3.6}$$

$$(1+2\gamma\eta)\theta''_{r+1}+2\gamma\theta'_{r+1}+f_r\theta'_{r+1}+Pr\theta'_r f_{r+1}=\theta'_r f_r. \tag{3.7}$$

The above iteration scheme is to be solved subject to the following boundary conditions:

$$\begin{aligned} \mathrm{f}_{r+1} = 0, \mathrm{f}'_{r+1} = 1 + \mathrm{B}\mathrm{f}''_{r+1}(\eta), \mathrm{g}_{r+1} = 0, \mathrm{g}'_{r+1} = 0, \theta_{r+1} = 1 \; at \; \eta = 0 \\ \mathrm{f}'_{r+1} \to 0, \theta_{r+1} \to 0, \mathrm{g}'_{r+1} \to 1 \text{ as } \eta \to \infty \end{aligned} \tag{3.8}$$

here the prime denotes the differentiation with respect to η. Pr_m is prandtl number,γ is curvature parameter and M is magnetic parameter.
The equations (3.5), (3.6) and (3.7) represent a linear system of coupled differential equations and can be solved using some numerical technique.In the present work the iteration scheme subject to the boundary conditions are solved using Chebyshev pseudo-spectral method. The basic idea of the method is to introduce a differentiation matrix **D** to approximate the derivatives of the unknown functions at the collocation points given by

$$\xi_j = \cos\left(\frac{j\pi}{N}\right), \quad j = 0, 1, 2, \ldots, N,$$

where $N + 1$ is the number of collocation points.These points are known as Guass-Lobatto collocation points. The $\mathrm{f}(\eta)$ at the collocation points as the matrix product:

$$\frac{df}{d\eta} = \sum_{k=0}^{N} D_{jk} f(\xi_k) = \mathbf{Df}, \quad j = 0, 1, 2, \ldots, N,$$

$$\frac{dg}{d\eta} = \sum_{k=0}^{N} D_{jk} g(\xi_k) = \mathbf{Dg}, \quad j = 0, 1, 2, \ldots, N,$$

$$\frac{d\theta}{d\eta} = \sum_{k=0}^{N} D_{jk} \theta(\xi_k) = \mathrm{D}\Theta, \quad j = 0, 1, 2, \ldots, N,$$

where $\mathbf{D} = \frac{2D}{L}$ is the differentiation matrix and D is Chebyshev differentiation matrix, L is a constant used to approximate infinity. The higher order derivatives are obtained as powers of **D**, that is

$$f^{(p)} = \mathbf{D}^{(p)}\mathbf{f}$$

$$g^{(p)} = \mathbf{D}^{(p)}\mathbf{g}$$

$$\theta^{(p)} = \mathbf{D}^{(p)}\Theta$$

Thus, we obtain from equation (3.5), (3.6) and (3.7) as

$$\begin{aligned}[(1+2\gamma\eta)\mathbf{D}^3+2\gamma\mathbf{D}^2+\text{diag}(f_r'')\text{I}+\mathbf{D}^2\text{diag}(f_r)-2\mathbf{D}\text{diag}(f_r')]f_{r+1}\\ +[\text{M}(2\mathbf{D}\text{diag}(g_r')-\text{diag}(g_r'')\text{I}-\mathbf{D}^2\text{diag}(g_r)]g_{r+1}\\ =(f_r''f_r-(f_r')^2)+\text{M}((g_r')^2-g_r''g_r),\end{aligned} \tag{3.9}$$

$$\begin{aligned}[(1+2\gamma\eta)\mathbf{D}^3+2\gamma\mathbf{D}^2+\text{Pr}_m(\text{diag}(f_r)\mathbf{D}^2)\\ -\text{diag}(\text{Pr}_m f_r'')\text{I}]g_{r+1}-\text{Pr}_m[(\text{diag}(g_r)\mathbf{D}^2)+\text{diag}(\text{Pr}_m g_r'')\text{I}]f_{r+1}\\ =\text{Pr}_m(g_r''f_r+f_r''g_r),\end{aligned} \tag{3.10}$$

$$[(1+2\gamma\eta)\mathbf{D}^2+2\gamma\mathbf{D}+\text{diag}(f_r)\mathbf{D}]\theta_{r+1}+[\text{Pr diag}(\theta_r')\text{I}]f_{r+1}=\theta_r'f_r. \tag{3.11}$$

Equations (3.9), (3.10), (3.11) can be represented in the matrix form as

$$\begin{bmatrix} A_{1,1} & A_{1,2} & A_{1,3} \\ A_{2,1} & A_{2,2} & A_{2,3} \\ A_{3,1} & A_{3,2} & A_{3,3} \end{bmatrix} \begin{bmatrix} \mathbf{f}_{r+1} \\ \mathbf{g}_{r+1} \\ \Theta_{r+1} \end{bmatrix} = \begin{bmatrix} \mathbf{B}_1 \\ \mathbf{B}_2 \\ \mathbf{B}_3 \end{bmatrix}. \tag{3.12}$$

where

$A_{1,1}=(1+2\gamma\eta)\text{D}^3+2\gamma\text{D}^2+\text{diag}(f_r'')\text{I}+\text{D}^2\text{diag}(f)_r-2\text{Ddiag}(f_r'),$

$A_{1,2}=\text{M}(2\text{Ddiag}(g_r')-\text{diag}(g_r'')\text{I}-\text{D}^2\text{diag}(g_r),$

$A_{1,3}=0,$

$A_{2,1}=(\text{diag}(g_r)\text{D}^2)+\text{diag}(\text{Pr}_m g_r'')\text{I},$

$A_{2,2}=(1+2\gamma\eta)\text{D}^3+2\gamma\text{D}^2+\text{Pr}_m(\text{diag}(f_r)\text{D}^2)-\text{diag}(\text{Pr}_m f_r'')\text{I},$

$A_{2,3}=0,$

$A_{3,1}=\text{Prdiag}(\theta_r')\text{I},$

$A_{3,2}=0,$

$A_{3,3}=(1+2\gamma\eta)\text{D}^2+2\gamma\text{D}+\text{diag}(f_r)\text{D},$

$B_1=(f_r''f_r-(f_r')^2)+\text{M}((g_r')^2-g_r''g_r),$

$B_2=\text{Pr}_m(g_r''f_r+f_r''g_r),$

$B_3=\theta_r'f_r.$

3.2.2 Results And Discussion

In order to analyze the effectiveness of the SQLM method in solving the system, the values of $f''(0)$, $\theta^{'}(0)$ and $g^{''}(0)$ for B=1, 2 and 3 are presented in table (3.1), (3.2) and (3.3), respectively. It is observed from these three tables that around 10 iterations are sufficient to maintain an accuracy within 10^{-4}. However with the increasing values of B, we may require more number of iterations to achieve the same accuracy.
Table (3.4), (3.5) and (3.6) present the values of the same quantities for different values of B. It is interesting to note that 30 iterations are more than enough to achieve an accuracy within 10^{-10} in the results. Hence we may conclude that the spectral quasilinearization method is an efficient tool to solve such type of problems.

TABLE 3.1: Value of $[f''(0), \theta'(0), g''(0)]$ for B=1, $Pr_m = 0.1$, Pr=0.71 iterations=10.

Iterations	$f''(0)$	$g''(0)$	$\theta'(0)$
1	-1.3690521134	2.6851242897	-3.1883844579
2	-1.0951099415	2.6566473884	-3.3212418904
3	-1.2255363157	2.6664766226	-3.2845008635
4	-1.1877008836	2.6611194834	-3.2937491815
5	-1.1992725200	2.6626869405	-3.2909013444
6	-1.1958995840	2.6622001820	-3.2917275579
7	-1.1968848763	2.6623422083	-3.2914861258
8	-1.1965986023	2.6623006440	-3.2915562382
9	-1.19668174927	2.6623127264	-3.2915358747
10	-1.1966576168	2.6623092160	-3.2915417846

TABLE 3.2: Value of [$f''(0), \theta'(0), g''(0)$] for B=2, $Pr_m = 0.1$, Pr=0.71 iterations=10.

Iterations	$f''(0)$	$g''(0)$	$\theta'(0)$
1	-0.7876959989	2.6790818271	-3.1600213393
2	-0.6073283939	2.6495779472	-3.3133408789
3	-0.7062472874	2.6598636154	-3.2628908528
4	-0.6709236288	2.6531014040	-3.2786920933
5	-0.6848890403	2.6554261720	-3.2723767308
6	-0.6796212269	2.6544885566	-3.2747475909
7	-0.6816419202	2.6548393285	-3.2738367141
8	-0.6808717653	2.6547044218	-3.2741836658
9	-0.6811660414	2.6547557807	-3.2740510637
10	-0.6810537032	2.6547361487	-3.2741016793

TABLE 3.3: Value of [$f''(0), \theta'(0), g''(0)$] for B=3, $Pr_m = 0.1$, Pr=0.71 iterations=10.

Iterations	$f''(0)$	$g''(0)$	$\theta'(0)$
1	-0.5529082869	2.6766415053	-3.1485665502
2	-0.4188679035	2.6469489253	-3.3111905670
3	-0.4964169625	2.6573561324	-3.2539035889
4	-0.4665440504	2.6499731414	-3.2733480829
5	-0.4794891657	2.6526515837	-3.2648181443
6	-0.4741533594	2.6514606721	-3.2683163642
7	-0.4764018086	2.6519450825	-3.2668392529
8	-0.4754623828	2.6517400462	-3.2674559068
9	-0.4758563970	2.6518255260	-3.2671971766
10	-0.4756913910	2.6517896444	-3.2673055126

TABLE 3.4: Value of [$f''(0), \theta'(0), g''(0)$] for B=1, $Pr_m = 0.1$, Pr=0.71 iterations=30.

Iterations	$f''(0)$	$g''(0)$	$\theta'(0)$
1	-1.3690521134	2.6851242897	-3.1883844579
2	-1.0951099415	2.6566473884	-3.3212418904
3	-1.2255363157	2.6664766226	-3.2845008635
4	-1.1877008836	2.6611194834	-3.2937491815
5	-1.1992725200	2.6626869405	-3.2909013444
6	-1.1958995840	2.6622001820	-3.2917275579
7	-1.1968848763	2.6623422083	-3.2914861258
8	-1.1965986023	2.6623006440	-3.2915562382
9	-1.1966817492	2.6623127264	-3.2915358747
10	-1.1966576168	2.6623092160	-3.2915417846
11	-1.1966646199	2.6623102349	-3.2915400696
12	-1.1966625879	2.6623099392	-3.2915405672
13	-1.1966631775	2.6623100251	-3.2915404228
14	-1.1966630064	2.6623100002	-3.2915404647
15	-1.1966630561	2.6623100074	-3.2915404526
16	-1.1966630417	2.6623100053	-3.2915404561
17	-1.1966630458	2.6623100059	-3.2915404551
18	-1.1966630446	2.6623100057	-3.2915404554
19	-1.1966630450	2.6623100058	-3.2915404553
20	-1.1966630449	2.6623100058	-3.2915404553
21	-1.1966630449	2.6623100058	-3.2915404553
22	-1.1966630449	2.6623100057	-3.2915404553
23	-1.1966630449	2.6623100057	-3.2915404553
24	-1.1966630449	2.6623100058	-3.2915404553
25	-1.1966630449	2.6623100058	-3.2915404553
26	-1.1966630449	2.6623100058	-3.2915404553
27	-1.1966630449	2.6623100057	-3.2915404553
28	-1.1966630449	2.6623100058	-3.2915404553
29	-1.1966630449	2.6623100058	-3.2915404553
30	-1.1966630449	2.6623100058	-3.2915404553

TABLE 3.5: Value of [f$''(0), \theta'(0)$, g$''(0)$] for B=2, Pr$_m = 0.1$, Pr=0.71 iterations=30.

Iterations	f$''(0)$	g$''(0)$	$\theta'(0)$
1	-0.7876959989	2.6790818271	-3.1600213393
2	-0.6073283939	2.6495779472	-3.3133408789
3	-0.7062472874	2.6598636154	-3.2628908528
4	-0.6709236288	2.6531014040	-3.2786920933
5	-0.6848890403	2.6554261720	-3.2723767308
6	-0.6796212269	2.6544885566	-3.2747475909
7	-0.6816419202	2.6548393285	-3.2738367141
8	-0.6808717653	2.6547044218	-3.2741836658
9	-0.6811660414	2.6547557807	-3.2740510637
10	-0.6810537032	2.6547361487	-3.2741016793
11	-0.6810966034	2.6547436419	-3.2740823493
12	-0.6810802227	2.6547407802	-3.2740897301
13	-0.6810864777	2.6547418729	-3.2740869117
14	-0.6810840893	2.6547414556	-3.2740879879
15	-0.6810850013	2.6547416149	-3.2740875769
16	-0.6810846531	2.6547415541	-3.2740877339
17	-0.6810847860	2.6547415773	-3.2740876739
18	-0.6810847353	2.6547415685	-3.2740876968
19	-0.6810847547	2.6547415718	-3.2740876881
20	-0.6810847472	2.6547415706	-3.2740876914
21	-0.6810847501	2.6547415711	-3.2740876901
22	-0.6810847490	2.6547415709	-3.2740876906
23	-0.6810847494	2.6547415709	-3.2740876904
24	-0.6810847492	2.6547415709	-3.2740876905
25	-0.6810847493	2.6547415709	-3.2740876905
26	-0.6810847493	2.6547415709	-3.2740876905
27	-0.6810847493	2.6547415709	-3.2740876905
28	-0.6810847493	2.6547415709	-3.2740876905
29	-0.6810847493	2.6547415709	-3.2740876905
30	-0.6810847493	2.6547415709	-3.2740876905

TABLE 3.6: Value of [f$''(0), \theta'(0)$, g$''(0)$] for B=3, Pr$_m$ $= 0.1$, Pr=0.71, iterations=30.

Iterations	f$''$(0)	g$''(0)$	$\theta'(0)$
1	-0.5529082869	2.6766415053	-3.1485665502
2	-0.4188679035	2.6469489253	-3.3111905670
3	-0.4964169625	2.6573561324	-3.2539035889
4	-0.4665440504	2.6499731414	-3.2733480829
5	-0.4794891657	2.6526515837	-3.2648181443
6	-0.4741533594	2.6514606721	-3.2683163642
7	-0.4764018086	2.6519450825	-3.2668392529
8	-0.4754623828	2.6517400462	-3.2674559068
9	-0.4758563970	2.6518255260	-3.2671971766
10	-0.4756913910	2.6517896444	-3.2673055126
11	-0.4757605384	2.6518046654	-3.2672601105
12	-0.4757315693	2.6517983698	-3.2672791310
13	-0.4757437072	2.6518010071	-3.2672711614
14	-0.4757386217	2.6517999021	-3.2672745005
15	-0.4757407525	2.6518003650	-3.2672731015
16	-0.4757398597	2.6518001711	-3.2672736876
17	-0.4757402338	2.6518002523	-3.2672734420
18	-0.4757400771	2.6518002183	-3.2672735449
19	-0.4757401427	2.6518002326	-3.2672735018
20	-0.4757401152	2.6518002266	-3.2672735199
21	-0.4757401267	2.6518002291	-3.2672735123
22	-0.4757401219	2.6518002280	-3.2672735155
23	-0.4757401239	2.6518002285	-3.2672735142
24	-0.4757401231	2.6518002283	-3.2672735147
25	-0.4757401234	2.6518002284	-3.2672735145
26	-0.4757401233	2.6518002283	-3.2672735146
27	-0.4757401234	2.6518002283	-3.2672735145
28	-0.4757401233	2.6518002283	-3.2672735146
29	-0.4757401233	2.6518002283	-3.2672735146
30	-0.4757401233	2.6518002283	-3.2672735146

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