THE TOPOLOGY OF POINTWISE CONVERGENCE

ON C(X)

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Harsimran Kaur

Roll no. 300803008

Under the guidance of

Dr. Pratibha Garg

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School of Mathematics and Computer Applications

Thapar University

Patiala-147004 (PUNJAB)

INDIA

CERTIFICATE

I hereby certify that the work which is being presented in the thesis entitled **"The Topology of Pointwise Convergence on C(X)"** in partial fulfillment of the requirements for the award of degree of Master of Science, School of Mathematics and Computer Applications, Thapar University, Patiala is an authentic record of my own work carried out under the supervision of **Dr. Pratibha Garg**.

The matter presented in this thesis has not been submitted for the award of any other degree of this or any other university.

Harsimran Kaur

(Harsimran Kaur)

Roll no. 300803008

This is to certify that the above statement made by the candidate is correct and true to the best of my knowledge.

Pratibha Garg

(Dr. Pratibha Garg)
Supervisor
Assistant Professor
School of Mathematics & Computer Applications
Thapar University, Patiala.

Countersigned by:

(Dr. S.S. Bhatia) 15/7
Professor & Head
School of Mathematics & Computer Applications
Thapar University
Patiala.

(Dr. R.K. Sharma) 23/7
Dean of Academic Affairs
Thapar University
Patiala.

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Abstract

This dissertation studies the point-open topology on $C(X)$, the set of all continuous real-valued functions defined on a Tychonoff space X. The study includes important properties such as metrizability and submetrizability, various kinds of countability properties and completeness properties of $C(X)$ equipped with the point-open topology.

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Introduction

The set $C(X)$ of all real-valued continuous functions on a Tychonoff space X equipped with the point-open topology is denoted by $C_p(X)$. The idea of topologizing $C(X)$ arose from the notion of convergence of sequences of functions. A sequence of functions (f_n) converges to f in $C_p(X)$ if and only if it converges pointwise on the space X. So the point-open topology on $C(X)$ is also known as the topology of pointwise convergence. If the space X is discrete, $C_p(X)$ coincides with $\mathbb{R}^X$. The topology of pointwise convergence, that is, the point-open topology on $C(X)$ can be considered in three following ways:

We can view the point-open topology on $C(X)$ as a set-open topology. Let $\mathbb{F}(X)$ denote the set of all finite subsets of X. For $A \in \mathbb{F}(X)$ and an open set V of $\mathbb{R}$, define $[A, V] = \{f \in C(X) : f(A) \subseteq V\}$. The collection $\{[A, V] : A \in \mathbb{F}(X),\ V \text{ open in } \mathbb{R}\}$ forms a subbase for the point-open topology on $C(X)$.

The point-open topology on $C(X)$ can also be viewed as a uniform topology. For each $A \in \mathbb{F}(X)$ and $\epsilon > 0$, let $A_\epsilon = \{(f,g) \in C(X) \times C(X) : |f(x) - g(x)| < \epsilon \ \forall \ x \in A\}$. Then it can be verified that the collection $\{A_\epsilon : A \in \mathbb{F}(X), \epsilon > 0\}$ is a base for some uniformity on $C(X)$. This uniformity induces the topology of pointwise convergence on $C(X)$.

The third way enables us to view the point-open topology on $C(X)$ as a locally convex topology. For each $A \in \mathbb{F}(X)$, define seminorm p_A on $C(X)$ as follows: $p_A(f) = \sup\{|f(x)| : x \in A\}$. For each $A \in \mathbb{F}(X)$ and $\epsilon > 0$, let $V_{A,\epsilon} = \{f \in C(X) : p_A(f) < \epsilon\}$. Let $\mathbb{V} = \{V_{A,\epsilon} : A \in \mathbb{F}(X), \epsilon > 0\}$. Then it can be verified that $f + \mathbb{V} = \{f + V : V \in \mathbb{V}\}$ forms a neighborhood base at f. The topology generated by the collection of seminorms $\{p_A : A \in \mathbb{F}(X)\}$ is the point-open topology on $C(X)$.

In this dissertation, we study the following problem:

If P is a topological property, does there exist a topological property Q so that $C_p(X)$ has P if and only if X has Q? For P, we will consider the following topological properties:

(i) submetrizability and metrizability,

(ii) various countability properties such as separability, second countability and Lindelöf property, and

(iii) various completeness properties such as complete metrizability, *Čech*-completeness, almost *Čech*-completeness, sieve-completeness and parti tion-completeness.

This dissertation has been divided into four chapters.

The first chapter defines the point-open topology on $C(X)$ as a set-open topology and then describes two different ways in which this topology can be viewed. In this chapter, we also study some useful maps such as induced map and sum function on the space $C(X)$ equipped with the point-open topology.

The second chapter deals with submetrizability and metrizability of $C_p(X)$. It is well-known that a locally convex Hausdorff space is metrizable if and only if it is first countable. However this chapter shows that even some properties of $C_p(X)$ weaker than first countability are equivalent to the metrizability of $C_p(X)$.

The third chapter focuses on various kinds of countability properties of $C_p(X)$. These properties include separability, second countability and Lindelöf property of $C_p(X)$.

The fourth chapter studies the completeness properties of $C_p(X)$. It has been shown that complete metrizability, *Čech*-completeness, almost *Čech*-completeness, partition-completeness and sieve-completeness are all equivalent on $C_p(X)$.

The following conventions have been used. All spaces are completely regular Hausdorff, that is, Tychonoff. The set $\mathbb{R}$ of real numbers carries the usual topology. The constant zero function on X is denoted by 0_X. We take one numbering for the Definitions, one for the Examples, one for the Remarks and another one for the Propositions, Lemmas, Theorems and Corollaries, each numbering being restricted to its own chapter.

Chapter 1

Preliminaries

In this chapter, we define the point-open topology on $C(X)$, the set of all continuous real-valued functions on a Tychonoff space X. Then we present two different bases for this topology. Further we study two useful maps, induced map and sum function defined on the space $C_p(X)$.

1.1 The Point-Open Topology on $C(X)$

Let $C(X)$ denote the set of all continuous real-valued functions on the space X. Let $\mathbb{F}(X)$ be the set of all finite subsets of X.

For $A \in \mathbb{F}(X)$ and an open set V of $\mathbb{R}$, define $[A, V] = \{f \in C(X) : f(A) \subseteq V\}$. The collection $\{[A, V] : A \in \mathbb{F}(X),\ V \text{ open in } \mathbb{R}\}$ forms a subbase for the point-open topology on $C(X)$. It can be easily proved that the space $C_p(X)$ is a Hausdorff space.

The next result presents another base for the space $C_p(X)$. If $f \in C(X)$, $A \in \mathbb{F}(X)$ and $\epsilon > 0$ is a real number, we define $\langle f, A, \epsilon\rangle = \{g \in C(X) : |f(x) - g(x)| < \epsilon \text{ for all } x \in A\}$.

Theorem 1.1 *For any space X, the collection $\mathbb{B} = \{\langle f, A, \epsilon\rangle : f \in C(X)$, $A \in \mathbb{F}(X)$ and $\epsilon > 0\}$ is a base for the point-open topology on $C(X)$.*

Proof: We shall first prove that every member of $\mathbb{B}$ is open in $C_p(X)$. Let $A = \{x_1, x_2, ..., x_n\}$ and $\epsilon > 0$. Let $h \in \langle f, A, \epsilon\rangle$. Then $|h(x_i) - f(x_i)| < \epsilon$ for all $i = 1, 2, ..., n$. So $h(x_i) \in (f(x_i) - \epsilon, f(x_i) + \epsilon)$ for all $i = 1, 2, ..., n$. Take $V_i = (f(x_i) - \epsilon, f(x_i) + \epsilon)$. Then $h \in [x_i, V_i]$ for all $i = 1, 2, ..., n$. This implies $h \in \cap_{i=1}^{n}[x_i, V_i]$. Suppose $g \in \cap_{i=1}^{n}[x_i, V_i]$. So $g \in [x_i, V_i]$ for all $i = 1, 2, ..., n$. Then $g(x_i) \in V_i$ for all $i = 1, 2, ..., n$. That is, $g(x_i) \in (f(x_i) - \epsilon, f(x_i) + \epsilon)$ for all $i = 1, 2, ..., n$. Therefore, $|g(x_i) - f(x_i)| < \epsilon$ for all $i = 1, 2, ..., n$. So, $g \in \langle f, A, \epsilon\rangle$. Hence, $h \in \cap_{i=1}^{n}[A, V_i] \subseteq \langle f, A, \epsilon\rangle$. Thus, $\langle f, A, \epsilon\rangle$ is open in $C_p(X)$.

Let $A \in \mathbb{F}(X)$ and V be open in $\mathbb{R}$. Suppose $A = \{x_1, x_2, ..., x_n\}$. Let $f \in [A, V]$. Therefore, for each $i = 1, 2, ..., n$, $f(x_i) \in V$. Then there exists $\epsilon_i > 0$ such that $f(x_i) \in (f(x_i) - \epsilon_i, f(x_i) + \epsilon_i) \subseteq V$. Let $\epsilon = \min\{\epsilon_1, \epsilon_2, ..., \epsilon_n\}$. Then $f(x_i) \subseteq \cup_{i=1}^{n}(f(x_i) - \epsilon, f(x_i) + \epsilon) \subseteq V$. Now, we claim that $\langle f, A, \epsilon\rangle \subseteq [A, V]$. Let $g \in \langle f, A, \epsilon\rangle$. So we have $|g(x_i) - f(x_i)| < \epsilon$ for all $i = 1, 2, ..., n$. Then $g(x_i) \in (f(x_i) - \epsilon, f(x_i) + \epsilon) \subseteq V$. This implies $g(x_i) \in V$ for all $i = 1, 2, ..., n$. Thus, we have $g \in [A, V]$. Hence, $\langle f, A, \epsilon\rangle \subseteq [A, V]$.

Now, let $W = [A_1, V_1] \cap [A_2, V_2] \cap ... \cap [A_n, V_n]$. Suppose $f \in W$. There exists $\epsilon_i > 0$ such that $\langle f, A_i, \epsilon_i \rangle \subseteq [A_i, V_i]$ for $1 \leqslant i \leqslant n$. Take $A = \cup_{i=1}^{n} A_i$ and $\epsilon = \min\{\epsilon_1, \epsilon_2, ..., \epsilon_n\}$. Then we have $\langle f, A, \epsilon \rangle \subseteq \cap_{i=1}^{n} \langle f, A_i, \epsilon_i \rangle \subseteq \cap_{i=1}^{n} [A_i, V_i] = W$. Thus, we have $f \in \langle f, A, \epsilon \rangle \subseteq W$. Hence, the collection $\mathbb{B}$ forms a base for the point-open topology on $C(X)$. □

Now, for each $A \in \mathbb{F}(X)$, define the seminorm p_A on $C(X)$ by

$$p_A(f) = \sup\{|f(x)| : x \in A\}$$

For each $A \in \mathbb{F}(X)$ and $\epsilon > 0$, let $V_{A,\epsilon} = \{f \in C(X) : p_A(f) < \epsilon\}$. Let $\mathbb{V} = \{V_{A,\epsilon} : A \in \mathbb{F}(X), \epsilon > 0\}$.

Theorem 1.2 *For any space X, the collection $f + \mathbb{V} = \{f + V : V \in \mathbb{V}\}$ forms a neighborhood base at f for the point-open topology on $C(X)$. Consequently, $C_p(X)$ is a locally convex space.*

Proof: Let A be a finite subset of X. Suppose $h \in f + V_{A,\epsilon}$ is arbitrary. Then $h - f \in V_{A,\epsilon}$. That is, $p_A(h - f) < \epsilon$. This implies that $\sup\{|h(x) - f(x)| : x \in A\} < \epsilon$. So $|h(x) - f(x)| < \epsilon$ for all $x \in A$. Thus $h \in \langle f, A, \epsilon \rangle$. Now to see that $\langle f, A, \epsilon \rangle \subseteq f + V_{A,\epsilon}$, let $g \in \langle f, A, \epsilon \rangle$. Then $|g(x) - f(x)| < \epsilon$ for all $x \in A$. So $\sup\{|g(x) - f(x)| : x \in A\} < \epsilon$, which implies that $p_A(g - f) < \epsilon$. So $g - f \in V_{A,\epsilon}$. That is, $g \in f + V_{A,\epsilon}$. Hence $h \in \langle f, A, \epsilon \rangle \subseteq f + V_{A,\epsilon}$. Therefore, $f + V_{A,\epsilon}$ is open in $C_p(X)$.

Let $f \in C(X)$, A be a finite subset of X and $\epsilon > 0$. Suppose $g \in f + V_{A,\epsilon}$. Then $g - f \in V_{A,\epsilon}$. So, $p_A(g - f) < \epsilon$ for $x \in A$. This implies that

$\sup\{|g(x) - f(x)| : x \in A\} < \epsilon$, that is, $|g(x) - f(x)| < \epsilon$ for all $x \in A$. Thus $g \in \langle f, A, \epsilon \rangle$ by which we have that $f \in f + V_{A,\epsilon} \subseteq \langle f, A, \epsilon \rangle$. This shows $f + \mathbb{V}$ forms a neighborhood base at f for $C_p(X)$. □

Since every locally convex space is a Tychonoff space, so we have the following corollary.

Corollary 1.3 *For any space X, $C_p(X)$ is a Tychonoff space.*

Theorem 1.4 *For any finite set X, $C_p(X)$ is metrizable.*

Proof: Let X be a finite topological space. Define $\rho : C(X) \times C(X) \longrightarrow \mathbb{R}^+ \cup \{0\}$ by $\rho((f, g)) = \sup\{|f(x) - g(x)| : x \in X\}$. Then ρ is a complete metric on $C(X)$. We claim that the topology τ_ρ generated by ρ and the point-open topology are equivalent. To see this, let $f \in C(X)$ and let $X = \{x_1, x_2, ..., x_n\}$. Let $B(f, \epsilon)$ be a basic open set in $C(X)$ with the topology τ_ρ. Let $U_i = (f(x_i) - \epsilon, f(x_i) + \epsilon)$. Take $W = [x_1, U_1] \cap [x_2, U_2] \cap ... \cap [x_n, U_n]$. Then W is open in the point-open topology. Also note that $f \in W \subseteq B(f, \epsilon)$. So $B(f, \epsilon)$ is open in $C_p(X)$.

Now let $[A, V]$ be a subbasic open set in $C_p(X)$, where $A = \{x_1, x_2, ..., x_n\}$. Let $f \in [A, V]$. So $f(x_i) \in V$ for all $i = 1, 2, ..., n$. Then $\exists\ \epsilon_i$ such that $f(x_i) \in (f(x_i) - \epsilon_i, f(x_i) + \epsilon_i)$ for each $i = 1, 2, ..., n$. Take $\epsilon = \min\{\epsilon_1, \epsilon_2, ..., \epsilon_n\}$. Then $f \in B(f, \epsilon) \subseteq [A, V]$. Thus $[A, V]$ is open in $C(X)$ with the topology τ_ρ. Hence $C_p(X)$ is completely metrizable. □

Let $\mathbb{R}^X$ be the set of all real-valued functions defined on a space X. Here, $\mathbb{R}^X$ is equipped with the point-open topology which is same as the product topology on $\mathbb{R}^X$. Let $A \in \mathbb{F}(X)$ and V be open in $\mathbb{R}$. We define $[A,V]_F = \{f \in \mathbb{R}^X : f(A) \subseteq V\}$. Then the collection $\{[A,V]_F : A \in \mathbb{F}(X), V \text{ open in } \mathbb{R}\}$ forms a subbase for the point-open topology on $\mathbb{R}^X$.

The following theorem can be proved in a similar manner as Theorem 1.1.

Theorem 1.5 *For any space X, the collection $\mathbb{B} = \{\langle f, A, \epsilon\rangle_F : f \in \mathbb{R}^X, A \in \mathbb{F}(X)$ and $\epsilon > 0\}$ is a base for the point-open topology on $\mathbb{R}^X$.*

Theorem 1.6 *For any space X, $C_p(X)$ is dense in $\mathbb{R}^X$.*

Proof: Let $f \in \mathbb{R}^X$, $A \in \mathbb{F}(X)$ and $\epsilon > 0$ be given. Since A is finite, $f|_A : A \longrightarrow \mathbb{R}$ is continuous. Now, as A is finite, A is compact and so we can have a continuous extension of $f|_A$. Let $g : X \longrightarrow \mathbb{R}$ be the continuous extension of $f|_A$ to X. Since $f|_A = g|_A$, $|g(x) - f(x)| = 0$ for all $x \in A$. Then $g \in \langle f, A, \epsilon\rangle_F$. Thus $g \in \langle f, A, \epsilon\rangle_F \cap C_p(X)$, which gives $\langle f, A, \epsilon\rangle_F \cap C_p(X) \neq \emptyset$. Hence, $C_p(X)$ is dense in $\mathbb{R}^X$. □

1.2 Natural Maps

In this section, we study some natural maps on $C_p(X)$. One of the most useful tools in function spaces is the following concept of induced map.

If $f : X \longrightarrow Y$ is a continuous map, we define the *induced map* $f^* : C(Y) \longrightarrow C(X)$ by $f^*(g) = g \circ f$ for all $g \in C(Y)$. In this section, we study this induced map f^* when both $C(X)$ and $C(Y)$ are equipped with the point-open topology.

In order to have the first result, we need the definition of an almost onto map. A map $f : X \longrightarrow Y$, where X is any nonempty set and Y is a topological space, is called *almost onto* if $f(X)$ is dense in Y.

Theorem 1.7 *Let $f : X \longrightarrow Y$ be a continuous map between two spaces X and Y. Then*

(i) $f^* : C_p(Y) \longrightarrow C_p(X)$ is continuous.

(ii) $f^* : C(Y) \longrightarrow C(X)$ is one-one if and only if f is almost onto.

(iii) If $f^* : C(Y) \longrightarrow C_p(X)$ is almost onto, then f is one-one.

Proof: (i). Suppose $g \in C_p(Y)$. Let $\langle f^*(g), A, \epsilon\rangle$ be a neighborhood of $f^*(g)$ in $C_p(X)$. Since $f^*\langle g, f(A), \epsilon\rangle \subseteq \langle f^*(g), A, \epsilon\rangle$, f^* is continuous.

(ii). Suppose $f^* : C(Y) \longrightarrow C(X)$ is one-one. We have to show that $f(X)$ is dense in Y. Suppose $f(X)$ is not dense in Y. Let W be open in Y such that $W \neq \emptyset$ and $W \cap f(X) = \emptyset$. Choose $y_\circ \in W$. Since Y is Tychonoff, choose $g \in C(Y)$ such that $g(y_\circ) = 1$ and $g(Y \setminus W) = \{0\}$. It follows that $g(f(X)) = \{0\}$. This means that $f^*(g)(x) = 0$ for all $x \in X$. Therefore, if 0_X and 0_Y denote the zero functions of $C(X)$ and $C(Y)$ respectively, then

$f^*(g) = 0_X = f^*(0_Y)$. The injectivity of f^* implies that $g = 0_Y$ which is impossible as $g(y_\circ) = 1$. Thus, $f(X)$ is dense in Y. Hence f is almost onto.

Conversely, suppose f is almost onto. We have to prove that f^* is one-one. Let g_1 and g_2 be in $C(Y)$ with $f^*(g_1) = f^*(g_2)$. Since g_1 and g_2 are continuous, it suffices to show that $g_1 = g_2$ on the dense subspace $f(X)$ of Y. So let $y \in f(X)$. Then $\exists$ some $x \in X$ such that $y = f(x)$. Now, $g_1(y) = g_1(f(x)) = f^*(g_1)(x) = f^*(g_2)(x) = g_2(f(x)) = g_2(y)$. Since this is true for all y in $f(X)$, we conclude that $g_1 = g_2$ on the space Y. Therefore, f^* is one-one.

(iii). Suppose that f^* is almost onto. Let x and y be two distinct points of X. We have to show that $f(x) \neq f(y)$. Since X is Tychonoff, choose $g \in C(X)$ such that $g(x) \neq g(y)$ in $\mathbb{R}$. Let V and W be disjoint open neighborhoods of $g(x)$ and $g(y)$ respectively. Take $L = [x, V] \cap [y, W]$ which is a nonempty open set in $C_p(X)$. Since f^* is almost onto, $L \cap f^*(C(Y)) \neq \emptyset$. So, let $h \in L \cap f^*(C(Y))$. Then $h = t \circ f$ for some $t \in C(Y)$. Since $h \in L$, $h(x) \neq h(y)$. So, $t(f(x)) \neq t(f(y))$, which implies that $f(x) \neq f(y)$. This shows that f is one-one. $\square$

Another kind of useful map on function spaces is the sum function. Let $\{X_\alpha : \alpha \in \Gamma\}$ be a family of topological spaces. If $\oplus\{X_\alpha : \alpha \in \Gamma\}$ denotes their topological sum then the *sum function* s is defined by $s : C_p(\oplus\{X_\alpha : \alpha \in \Gamma\}) \longrightarrow \prod\{C_p(X_\alpha) : \alpha \in \Gamma\}$ where $s(f) = (f|_{X_\alpha})_\alpha$ for each $f \in C_p(\oplus\{X_\alpha : \alpha \in \Gamma\})$. Here, $\prod\{C_p(X_\alpha) : \alpha \in \Gamma\}$ is the cartesian product of the $C_p(X_\alpha)'$s with the Tychonoff product topology.

Proposition 1.8 *The sum function* $s : C_p(\oplus\{X_\alpha : \alpha \in \Gamma\}) \longrightarrow \prod\{C_p(X_\alpha) : \alpha \in \Gamma\}$ *is a homeomorphism.*

Proof: First we shall prove that s is a bijection. Define $v : \prod\{C_p(X_\alpha) : \alpha \in \Gamma\} \longrightarrow C_p(\oplus\{X_\alpha : \alpha \in \Gamma\})$ by $v((g_\alpha)_{\alpha\in\Gamma}) = g_\alpha$ on X_α for each $\alpha \in \Gamma$ and each $(g_\alpha)_{\alpha\in\Gamma}$ in $\prod\{C_p(X_\alpha) : \alpha \in \Gamma\}$. It is easy to see that $v \circ s$ is the identity map on $C_p(\oplus\{X_\alpha : \alpha \in \Gamma\})$ and $s \circ v$ is the identity map on $\prod\{C_p(X_\alpha) : \alpha \in \Gamma\}$. Thus s is a bijection.

Next we shall prove that s and s^{-1} are continuous. Let $[A, V]_\oplus = \{f \in C(\oplus\{X_\alpha : \alpha \in \Gamma\}) : f(A) \subseteq V\}$ where A is finite subset of $\oplus\{X_\alpha : \alpha \in \Gamma\}$ and V is open in $\mathbb{R}$ and let $[A, V]_\alpha = \{f \in C(X_\alpha) : f(A) \subseteq V\}$ where A is finite subset of X_α and V is open in $\mathbb{R}$. Let $\pi_\alpha : \prod\{C_p(X_\alpha) : \alpha \in \Gamma\} \longrightarrow C_p(X_\alpha)$ be the α-th projection. Now by Theorem 8.8 in [8], $s : C_p(\oplus\{X_\alpha : \alpha \in \Gamma\}) \longrightarrow \prod\{C_p(X_\alpha) : \alpha \in \Gamma\}$ is continuous if and only if $\pi_\alpha \circ s : C_p(\oplus\{X_\alpha : \alpha \in \Gamma\}) \longrightarrow C_p(X_\alpha)$ is continuous for each $\alpha \in \Gamma$. Let $[A, V]_\alpha$ be a subbasic open set in $C_p(X_\alpha)$ where $A \in \mathbb{F}(X_\alpha)$ and V is open in $\mathbb{R}$. Then $(\pi_\alpha \circ s)^{-1}([A, V]_\alpha) = s^{-1}(\pi_\alpha^{-1}([A, V]_\alpha)) = s^{-1}\pi_\alpha^{-1}([A, V]_\alpha) = [A, V]_\oplus$. Thus $\pi_\alpha \circ s$ is continuous and hence s is continuous.

Now $s^{-1} : \prod\{C_p(X_\alpha) : \alpha \in \Gamma\} \longrightarrow C_p(\oplus\{X_\alpha : \alpha \in \Gamma\})$. Let $[A, V]_\oplus$ be a subbasic open set in $C_p(\oplus\{X_\alpha : \alpha \in \Gamma\})$ where A is finite subset of $\oplus\{X_\alpha : \alpha \in \Gamma\}$ and V is open in $\mathbb{R}$. Since A is finite subset of $\oplus\{X_\alpha : \alpha \in \Gamma\}$,

$A \cap X_\alpha = \emptyset$ for all but finitely many α. Suppose $A \cap X_{\alpha_i} \neq \emptyset$, $1 \leqslant i \leqslant n$. Let $A_i = A \cap X_{\alpha_i}$. Then each A_i is finite subset of X_{α_i}, $1 \leqslant i \leqslant n$. Note that $(s^{-1})^{-1}([A, V]_\oplus) = s([A, V]_\oplus) = \pi_{\alpha_1}^{-1}([A_1, V]_{\alpha_1}) \cap ... \cap \pi_{\alpha_n}^{-1}([A_n, V]_{\alpha_n})$. So s^{-1} is also continuous and hence s is a homeomorphism. $\square$

Chapter 2

Submetrizability and Metrizability of $C_p(X)$

This chapter deals with the study of metrizability and submetrizability properties of $C_p(X)$. It is well-known that a locally convex Hausdorff space is metrizable if and only if it is first countable. But this chapter shows that even some properties of $C_p(X)$ weaker than first countability are equivalent to the metrizability of $C_p(X)$.

In order to study the metrizability of $C_p(X)$ in a broader perspective, we first show that a number of properties of $C_p(X)$ are equivalent to its submetrizability. A space X is said to be *submetrizable* if it has a weaker metrizable topology. Equivalently, if there exists a metrizable space Y and a continuous bijection $f : X \longrightarrow Y$ from the space X onto Y.

Definition 2.1 For any space X, if the set $\{(x, x) : x \in X\}$ is a G_δ-set in the product space $X \times X$, then X is said to have a *G_δ-diagonal*.

Lemma 2.1 *Every submetrizable space X has a G_δ-diagonal.*

Proof: Suppose X is submetrizable. Let $f : X \longrightarrow M$ be a continuous bijection from the space X to a metrizable space M. Define the map $F : X \times X \longrightarrow M \times M$ by $F((x,x)) = (f(x), f(x))$ for all $x \in X$. Then F is one-one and continuous. Now $\{(f(x), f(x)) : x \in X\}$ is a closed subset of $M \times M$. Since a closed subset of a metric space is a G_δ-set, $\{(f(x), f(x)) : x \in X\}$ is a G_δ-set in $M \times M$. Then $\{(f(x), f(x)) : x \in X\} = \cap_{i=1}^{n} G_n$ where G_n is open in $M \times M$ for each n. Now by continuity of F, $F^{-1}(G_n)$ is open in $X \times X$ for each n. So $\{(x,x) : x \in X\} = \cap_{i=1}^{n} F^{-1}(G_n)$. This shows that $\{(x,x) : x \in X\}$ is a G_δ-set in $X \times X$. Hence, X has a G_δ-diagonal. $\square$

Definition 2.2 If for any space X, the set $\{(x,x) : x \in X\}$ is a zero-set in the product space $X \times X$, then X is said to have a *zero-set diagonal*.

By making slight modifications in the proof of Lemma 2.1 and using the fact that every closed set is a zero-set in a metric space, we have the following result.

Lemma 2.2 *Every submetrizable space X has a zero-set diagonal.*

Definition 2.3 A space X is called an *E_o-space* if every point in the space is a G_δ-set. Sometimes, E_o-spaces are called pointwise perfect spaces. The submetrizable spaces are E_o-spaces.

Lemma 2.3 *In a submetrizable space X, all countably compact subsets are G_δ-sets. Consequently, all compact subsets and singletons are G_δ-sets.*

Proof: Suppose X is submetrizable. Then there exists a continuous bijection $f : X \longrightarrow M$ from the space X to a metrizable space M. Let A be a countably compact subset of X. Then $f(A)$ is compact in the metric space M. Since a closed set in a metric space is a G_δ-set, $f(A)$ is a G_δ-set in M. That is, $f(A) = \cap_{n=1}^{\infty} G_n$, where G_n is an open subset of M for each n. Or $A = f^{-1}(\cap_{n=1}^{\infty} G_n)$, which gives that $A = \cap_{n=1}^{\infty} f^{-1}(G_n)$. Thus A is a G_δ-set. □

Lemma 2.4 *Every space X which has a G_δ-diagonal is an E_0-space.*

Proof: Suppose X has a G_δ-diagonal. Then the set $\{(x, x) : x \in X\} = \cap_{n=1}^{\infty} G_n$, where each G_n is an open subset of $X \times X$. Let $y \in X$ be fixed. Then $f : X \longrightarrow X \times X$ defined by $f(x) = (y, x)$ is a continuous map. Now $\{y\} = f^{-1}(\{(x, x) : x \in X\}) = f^{-1}\{\cap_{n=1}^{\infty} G_n\} = \cap_{n=1}^{\infty} f^{-1}(G_n)$. Thus, X is an E_0-space. □

Theorem 2.5 *For any space X, the following statements are equivalent.*

(*i*) *$C_p(X)$ is submetrizable.*

(*ii*) *Every countably compact subset of $C_p(X)$ is a G_δ-set in $C_p(X)$.*

(*iii*) *Every compact subset of $C_p(X)$ is a G_δ-set in $C_p(X)$.*

(*iv*) *$C_p(X)$ is an E_0-space.*

(v) $C_p(X)$ *has a* G_δ*-diagonal.*

(vi) X *is separable.*

Proof: $(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv)$ follow from Lemma 2.3. $(v) \Rightarrow (iv)$ follows from Lemma 2.4. Also $(i) \Rightarrow (v)$ follows from Lemma 2.1.

$(iv) \Rightarrow (vi)$: Suppose $C_p(X)$ is an E_0-space. Let $\langle 0_X, A_n, \epsilon_n \rangle$, where each A_n is finite subset of X and $\epsilon_n > 0$, be a sequence of open neighborhoods of zero function 0_X in X such that $\{0_X\} = \bigcap_{n=1}^{\infty} \langle 0_X, A_n, \epsilon_n \rangle$. We shall prove that $X = \overline{\bigcup_{n=1}^{\infty} A_n}$. Suppose it is not so. Let $x_0 \in X \setminus \overline{\bigcup_{n=1}^{\infty} A_n}$. Since X is a Tychonoff space, there exists a continuous function $f : X \longrightarrow \mathbb{R}$ such that $f(x_0) = 1$ and $f(\overline{\bigcup_{n=1}^{\infty} A_n}) = \{0\}$. Since $f(x) = 0$ for all $x \in A_n$, $f \in \langle 0_X, A_n, \epsilon_n \rangle$ for all n. Hence, $f \in \bigcap_{n=1}^{\infty} \langle 0_X, A_n, \epsilon_n \rangle = \{0_X\}$. This implies $f(x) = 0$ for all $x \in X$. But $f(x_0) = 1$, so we arrive at a contradiction. Therefore, $X = \overline{\bigcup_{n=1}^{\infty} A_n}$. Take $A = \overline{\bigcup_{n=1}^{\infty} A_n}$. Then A is countable and $\overline{A} = X$. Thus we have that X is separable.

$(vi) \Rightarrow (i)$: Suppose X is separable. Let A be countable dense subset of X. Suppose $A = \{x_1, x_2, ..., x_n, ...\}$. Take $A_1 = \{x_1\}, A_2 = \{x_1, x_2\}, ..., A_n = \{x_1, x_2, ..., x_n\}, ...$ Let $\oplus\{A_n : n \in \mathbb{N}\}$ be the topological sum of the family $\{A_n : n \in \mathbb{N}\}$. Then $p : \oplus\{A_n : n \in \mathbb{N}\} \longrightarrow X$ is almost onto. Then by Theorem 1.7(ii), the induced map $p^* : C_p(X) \longrightarrow C_p(\oplus\{A_n : n \in \mathbb{N}\})$ is a continuous injection. Now, by Proposition 1.8, $C_p(\oplus\{A_n : n \in \mathbb{N}\})$ is homeomorphic to $\prod\{C_p(A_n) : n \in \mathbb{N}\}$. By Theorem 1.4, $C_p(A_n)$ is metrizable for all n. So, $\prod\{C_p(A_n) : n \in \mathbb{N}\}$ is metrizable. Thus, $C_p(\oplus\{A_n : n \in \mathbb{N}\})$

is metrizable. If we take $M = p^*(C_p(X))$, then p^* is a continuous bijection from $C_p(X)$ to the metrizable space M. Hence we conclude that $C_p(X)$ is submetrizable. □

Example 2.1 *Let $X = \mathbb{R}$. Then $C_p(X)$ is submetrizable.*

Example 2.2 *If X is an uncountable discrete space, then $C_p(X)$ is not submetrizable.*

Now, we proceed further to study the properties that are equivalent to metrizability of $C_p(X)$. But before that we need the following definitions.

Definitions 2.1 By a *π-base* for a space X, we mean any collection $\mathbb{B}$ of non-empty open sets of X with the property that if V is any non-empty open set in X then V contains a member of $\mathbb{B}$. A space X has a *countable local π-base* if for each $x \in X$, there is a countable collection $\mathbb{B}_x$ of non-empty open sets in X such that each neighborhood of x contains some member of $\mathbb{B}_x$. This property is weaker than first countability. Also, if a space has countable π-base then it has a countable local π-base.

A subset A of a space X is said to have *countable character* if there exists a sequence $\{V_n : n \in \mathbb{N}\}$ of open subsets of X, each containing A such that if V is any open set containing A, then $V_n \subseteq V$ for some n. A space X is said to be of *countable type* if each compact set in X is contained in a compact set having countable character. A space X is said to be of *pointwise countable type* if each $x \in X$ is contained in a compact set having countable character.

A space X is called an *r-space* if each point x in X has a sequence $\{W_n : n \in \mathbb{N}\}$ of neighborhoods such that if $x_n \in W_n$ for each n, then the set $\{x_n : n \in \mathbb{N}\}$ is contained in a compact subset of X. A space X is a *q-space* if each point $x \in X$ has a sequence $\{V_n : n \in \mathbb{N}\}$ of neighborhoods such that if $x_n \in V_n$ for each n, then the set $\{x_n : n \in \mathbb{N}\}$ has a cluster point. A pointwise countable space is an r-space, and an r-space is a q-space.

Next, we define M-space which is a property that is stronger than being a q-space. A space X is said to be an *M-space* if there is a metrizable space Y and a surjective quasi-perfect map (a continuous closed map in which inverse images of points are countably compact) from X onto Y.

The following results will be required in order to prove the result on metrizability of $C_p(X)$.

Lemma 2.6 *Let D be a dense subset of a space X and $x \in D$. Then x has a countable local π-base in D if and only if x has a countable local π-base in X.*

Proof: Suppose x has a countable local π-base in D. So there exists a countable collection $\{B_n : n \in \mathbb{N}\}$ of non-empty open sets in D such that any neighborhood of x in D contains B_n for some n. Then for each n, there is an open set V_n in X such that $B_n = V_n \cap D$. Let U be any neighborhood of x in X. Then there exists an open neighborhood U' of x in X such that $U' \subseteq \overline{U'} \subseteq U$. Since $U' \cap D$ is a neighborhood of x in D, there exists some B_n such that $B_n \subseteq U' \cap D$. Since D is dense in X and B_n is open in D,

$\overline{B_n} = \overline{V_n \cap D} = \overline{V_n}$. Then $V_n \subseteq \overline{V_n} = \overline{Bn} \subseteq \overline{U' \cap D} \subseteq \overline{U'} \subseteq U$. Thus, x has a countable local π-base in X. Trivially, if x has a countable local π-base in X then x has a countable local π-base in D. □

The proofs of the following lemmas can be seen in the same way as above.

Lemma 2.7 *A dense subset D of a space X has a countable π-base if and only if X has a countable π-base.*

Lemma 2.8 *Let D be a dense subset of a space X and A be a compact subset of D. Then A has countable character in D if and only if A has countable character in X.*

Since a locally convex space is a homogeneous space, $C_p(X)$ is a homogeneous space. A space X is said to be a *homogeneous space* if for every pair of points $x,\ y \in X$, there exists a homeomorphism of X onto X itself which carries x to y.

Proposition 2.9 *Let X be any space. Then*

(*i*) *$C_p(X)$ has a countable local π-base if and only if $C_p(X)$ contains a dense subspace which has a countable local π-base.*

(*ii*) *$C_p(X)$ is of pointwise countable type if and only if $C_p(X)$ contains a dense subspace of pointwise countable type.*

Proof: (i). Let D be a dense subset of $C_p(X)$ that has a countable local π-base. Let $f \in D$ be arbitrary. So f has a countable π-base in D. Then by Lemma 2.6, f has a countable π-base in $C_p(X)$. Suppose $\{U_n : n \in \mathbb{N}\}$ is the sequence of open sets in $C_p(X)$ such that whenever U is an open set containing f, then $U_n \subseteq U$ for some n. Let $g \in C_p(X)$ be arbitrary. Since $C_p(X)$ is a homogeneous space, there exists a homeomorphism $F : C_p(X) \longrightarrow C_p(X)$ such that $F(f) = g$. Then $\{F(U_n) : n \in \mathbb{N}\}$ is a sequence of open sets in $C_p(X)$. Let W be any open set containing g. This implies $f \in F^{-1}(W)$. Then there exists some $n \in \mathbb{N}$ such that $U_n \subseteq F^{-1}(W)$, that is, $F(U_n) \subseteq W$. Thus, g has a countable π-base in $C_p(X)$.

(ii). The proof is similar to that given in (i). But here we have to use Lemma 2.8 instead of Lemma 2.6. □

Theorem 2.10 *For any space X, the following statements are equivalent.*

(i) *$C_p(X)$ is metrizable.*

(ii) *X is countable.*

(iii) *$C_p(X)$ is first countable.*

(iv) *$C_p(X)$ has a countable local π-base.*

(v) *$C_p(X)$ has a dense subspace which has countable local π-base.*

(vi) *$C_p(X)$ is locally metrizable.*

(vii) *$C_p(X)$ contains a non-empty open metrizable subspace.*

(viii) *$C_p(X)$ is of countable type.*

(ix) *$C_p(X)$ is of point countable type.*

(x) *$C_p(X)$ has a dense subspace of point countable type.*

(xi) *$C_p(X)$ is an M-space.*

(xii) *$C_p(X)$ is an r-space.*

(xiii) *$C_p(X)$ is a q-space.*

Proof: From earlier observations, we have $(i) \Rightarrow (viii) \Rightarrow (ix) \Rightarrow (xii) \Rightarrow (xiii)$ and $(i) \Rightarrow (xi) \Rightarrow (xiii)$. Also $(vi) \Rightarrow (vii)$ and $(i) \Rightarrow (vi) \Rightarrow (iii) \Rightarrow (iv)$ are immediate. By Proposition 2.9, we have $(iv) \Leftrightarrow (v)$ and $(ix) \Leftrightarrow (x)$.

$(xiii) \Rightarrow (ii)$: Suppose that $C_p(X)$ is a q-space. Then there exists a sequence $\{U_n : n \in \mathbb{N}\}$ of neighborhoods of the zero function 0_X in $C_p(X)$ such that if $f_n \in U_n$ for each n, then $\{f_n : n \in \mathbb{N}\}$ has a cluster point in $C_p(X)$.

Since $\{U_n : n \in \mathbb{N}\}$ is a sequence of neighborhoods of 0_X, there exist finite subset A_n of X and $\epsilon_n > 0$ such that $0_X \in \langle 0_X, A_n, \epsilon_n \rangle \subseteq U_n$ for each n. We have to prove that $X = \cup_{n=1}^{\infty} A_n$. Suppose $\exists$ an x such that $x \in X \setminus \cup_{n=1}^{\infty} A_n$. Since A_n is closed and $x \notin A_n$ for each n, there exist $f_n \in C(X)$ such that $f_n(A_n) = \{0\}$ and $f_n(x) = n$ for each n. Then $f_n \in U_n$ for each n. Now, as $\{f_n : n \in \mathbb{N}\}$ has a cluster point in $C_p(X)$, say f, there exists a subsequence $\{f_{n_k}\}$ which converges to f. But $f_{n_k}(x) \geqslant n_k \geqslant n$. So $\{f_{n_k}\}$ cannot converge.

Thus, we arrive at a contradiction. Thus $X = \cup_{n=1}^{\infty} A_n$, which shows that X is countable.

$(iv) \Rightarrow (ii)$: Suppose that $C_p(X)$ has a countable local π-base at the zero function 0_X in $C_p(X)$. Let $\{\langle f_n, A_n, \epsilon_n\rangle : n \in \mathbb{N}\}$ be a countable local π-base in $C_p(X)$. We have to prove that $X = \cup_{n=1}^{\infty} A_n$. Suppose there exists $x \in X \setminus \cup_{n=1}^{\infty} A_n$. Consider the neighborhood $\langle 0_X, \{x\}, 1\rangle$ at the zero function 0_X in $C_p(X)$. Then there exists an $n \in \mathbb{N}$ such that $\langle f_n, A_n, \epsilon_n\rangle \subseteq \langle 0_X, \{x\}, 1\rangle$. Choose $M > 0$ such that $f_n(x) + M > 1$. Now, as X is a Tychonoff space, there exists a continuous function $f : X \longrightarrow \mathbb{R}$ such that $f(A_n) = \{0\}$ and $f(x) = M$. Then $f + f_n \in \langle f_n, A_n, \epsilon_n\rangle$. But $f(x) + f_n(x) = M + f_n(x) > 1$. This shows that $f + f_n \notin \langle 0_X, \{x\}, 1\rangle$. That is, $f + f_n \in \langle f_n, A_n, \epsilon_n\rangle \setminus \langle 0_X, \{x\}, 1\rangle$, which is a contradiction. Hence X is countable.

$(vii) \Rightarrow (vi)$: Let M be a non-empty open subset of $C_p(X)$ such that M is metrizable. Let $h \in M$ and $f \in C_p(X)$. Consider the map $\phi : C_p(X) \longrightarrow C_p(X)$ defined by $\phi(g) = g+f-h \; \forall \, g \in C_p(X)$. Then ϕ is a homeomorphism. Also $f \in \phi(M)$. Since M is metrizable and open in $C_p(X)$, $\phi(M)$ is also metrizable and open in $C_p(X)$. Thus $C_p(X)$ is locally metrizable.

$(ii) \Rightarrow (i)$: Suppose that X is countable. Let $\{x_1, x_2, ...\}$ be an enumeration of X. Take $A_1 = \{x_1\}$, $A_2 = \{x_1, x_2\}$, $A_3 = \{x_1, x_2, x_3\}$, and so on. Now, the locally convex topology on $C(X)$ generated by the countable family of seminorms $\{p_{An} : n \in \mathbb{N}\}$ is weaker than $C_p(X)$. For any finite set A in X, there exists A_n such that $A \subseteq A_n$. This implies that the locally convex topology on $C(X)$ generated by the family of seminorms $\{p_A : A \text{ is finite}\}$ is weaker than the topology generated by the family of seminorms $\{p_{An} : n \in \mathbb{N}\}$. Now, we use a well-known result that if the topology of a locally convex Hausdorff space is generated by a countable family of seminorms, then it is metrizable, (see page 119 in [7]). Thus, we conclude that $C_p(X)$ is metrizable. $\square$

Chapter 3

Countability Properties of $C_p(X)$

The main objective of this chapter is to study the countability properties of $C_p(X)$, which include separability, second countability and Lindelöf property of $C_p(X)$.

3.1 Separability

The first section of this chapter shows that for any space X, $C_p(X)$ is separable if and only if X has separable metrizable compression. If P is a topological property, then a space X has a P-*compression* if and only if there exists a topological space Y having property P and a continuous bijection from X onto Y.

Theorem 3.1 *Let $\mathbb{B}$ and $\mathbb{V}$ be bases for X and $\mathbb{R}$ respectively. If τ is the topology with subbase $\{[B, V] : B \in \mathbb{B}$ and $V \in \mathbb{V}\}$, then $C_p(X) \leqslant C_\tau(X)$.*

Proof: Let $A \in \mathbb{F}(X)$ and $V \in \mathbb{V}$. It suffices to show that $[A, V] \in \tau$. Let $f \in [A, V]$. Then $f(A) \subseteq V$. Let $A = \{x_1, x_2, ..., x_n\}$. Then $f(x_i) \in V$ for all $i = 1, 2, ..., n$. Since f is continuous, there exists $B_{x_i} \in \mathbb{B}$ such that $x_i \in B_{x_i}$ and $f(B_{x_i}) \subseteq V$ for all $i = 1, 2, ..., n$. So $f \in [B_{x_i}, V]$ for all $i = 1, 2, ..., n$. Now, let $W = \cap_{i=1}^{n}[B_{x_i}, V]$, which is open in $C_\tau(X)$. Since $A \subseteq \cup_{i=1}^{n} B_{x_i}$, we have $f \in W \subseteq [A, V]$. Consequently, $C_p(X) \leqslant C_\tau(X)$. □

If $\mathbb{B}$ and $\mathbb{V}$ are countable bases in the above theorem, then $C_\tau(X)$ will have countable base. Thus we have the following corollary.

Corollary 3.2 *If X is second countable, then $C_\tau(X)$ is second countable.*

Since every second countable space is separable, we have the following corollary.

Corollary 3.3 *If X is second countable, then $C_\tau(X)$ is separable. Consequently, $C_p(X)$ is separable.*

A subset $\mathcal{F}$ of $C(X)$ is said to separate points of X if for each $(x, y) \in X \times X$ with $x \neq y$ there exists an $f \in \mathcal{F}$ such that $f(x) \neq f(y)$.

Theorem 3.4 *For a space X, the following statements are equivalent.*

(i) $C_p(X)$ is separable.

(*ii*) *$C_p(X)$ contains a countable subspace that separates points of X.*

(*iii*) *X admits a continuous injection into $\mathbb{R}^{\mathbb{N}}$.*

(*iv*) *X has a separable metrizable compression.*

Proof: $(i) \Rightarrow (ii)$: Suppose $C_p(X)$ is separable. Let D be a countable dense subset of $C_p(X)$. Let $x,\ y \in X$ such that $x \neq y$. So there exists $f \in C(X)$ such that $f(x) \neq f(y)$. Then there exist two disjoint open sets U and V in $\mathbb{R}$, such that $f(x) \in U$ and $f(y) \in V$. This implies $f \in [x, U] \cap [y, V]$. So $[x, U] \cap [y, V] \neq \varnothing$. Since D is dense in $C_p(X)$, $D \cap ([x, U] \cap [y, V]) \neq \varnothing$. Let $g \in D \cap ([x, U] \cap [y, V])$. Thus $g \in D$ and $g(x) \neq g(y)$. This shows that D separates points of X.

$(ii) \Rightarrow (iii)$: Suppose $D = \{f_1, f_2, ...\}$ is a countable subspace of $C_p(X)$ that separates points of X. Define $\phi : X \longrightarrow \mathbb{R}^{\mathbb{N}}$ by $\phi(x) = \{f_1(x), f_2(x), ...\}$. Then $f_n = p_n \circ \phi$ where $p_n : \mathbb{R}^{\mathbb{N}} \longrightarrow \mathbb{R}$ is the n^{th} projection. By Theorem 8.8 in [8], ϕ is also continuous. We have to prove that ϕ is one-one. Let $x,\ y \in X$ such that $x \neq y$. Since D separates points of X, there exists $f_n \in D$ such that $f_n(x) \neq f_n(y)$. Then $\phi(x) = \{f_1(x), f_2(x), ...f_n(x), ...\} \neq \{f_1(y), f_2(y), ...f_n(y), ...\} = \phi(y)$. Therefore, ϕ is one-one. Hence, X admits a continuous injection into $\mathbb{R}^{\mathbb{N}}$.

$(iii) \Rightarrow (iv)$: Let $\phi : X \longrightarrow \mathbb{R}^{\mathbb{N}}$ be a continuous injection. Let $M = \phi(X)$. Then $M \subseteq \mathbb{R}^{\mathbb{N}}$ which implies that M is separable metrizable. Since

$\phi : X \longrightarrow M$ is a continuous bijection, we get M as a separable metrizable compression for X.

$(iv) \Rightarrow (i)$: Suppose X has a separable metrizable compression. Let $f : X \longrightarrow M$ be a continuous bijection where M is a separable metrizable space. Since $f : X \longrightarrow M$ is an injection, so by Theorem 1.7(iii), $f^* : C_p(M) \longrightarrow C_p(X)$ is almost surjective. As M is separable metrizable, M is second countable. Then by Corollary 3.3, $C_p(M)$ is separable. Let D be a countable dense subset of $C_p(M)$. Then $C_p(X) = \overline{f^*(C_p(M))} = \overline{f^*(\overline{D})} \subseteq \overline{\overline{f^*(D)}} = \overline{f^*(D)}$. So $f^*(D)$ is a countable dense subset of $C_p(X)$. Thus, we conclude that $C_p(X)$ is separable. □

Example 3.1 *Suppose X is pseudocompact, but not metrizable. Since a pseudocompact submetrizable space is metrizable, X cannot be submetrizable either. Hence $C_p(X)$ cannot be separable.*

Example 3.2 *Suppose $X = \mathbb{R}_l$, the real line equipped with the lower limit topology. This space is also known as Sorgenfrey line. Since the lower limit topology is stronger than the usual topology on $\mathbb{R}$, $C_p(\mathbb{R}_l)$ is separable. Obviously $C_p(\mathbb{R})$ is also separable.*

3.2 Second Countability

In this section, we present the necessary and sufficient condition for $C_p(X)$ to be second countable.

Theorem 3.5 *For any space X, the following statements are equivalent.*

(i) $C_p(X)$ *is second countable.*

(ii) $C_p(X)$ *has a countable π-base.*

(iii) $C_p(X)$ *contains a dense subspace which has a countable π-base.*

(iv) X *is countable and submetrizable.*

Proof: $(i) \Rightarrow (ii)$ is immediate and by Lemma 2.7, we have $(ii) \Leftrightarrow (iii)$.

$(iv) \Rightarrow (i)$: Suppose that X is countable and submetrizable. Since X is countable, so by Theorem 2.10, $C_p(X)$ is metrizable. Since X is submetrizable, there exists a continuous bijection $f : X \longrightarrow M$, where M is a metrizable space. Since X is countable, M is also countable. Consequently, M is separable. Thus X has a separable metrizable compression. So by Theorem 3.4, $C_p(X)$ is separable. Then $C_p(X)$ is separable metrizable. Hence $C_p(X)$ is second countable.

$(ii) \Rightarrow (iv)$: Suppose $C_p(X)$ has a countable π-base. Then $C_p(X)$ is first countable. So by Theorem 2.10, X is countable. Since $C_p(X)$ has a countable π-base, $C_p(X)$ is separable. By Theorem 3.4, X has a separable metrizable compression. Thus X is submetrizable. Hence we have that X is countable and submetrizable. □

Example 3.3 *If $X = \mathbb{Q}$, then $C_p(X)$ is second countable.*

3.3 Lindelöf Property

In the last section of this chapter, we study the Lindelöf property of $C_p(X)$. In [3], several conditions for $C_p(X)$ to be Lindelöf have been studied.

Theorem 3.6 *Suppose that $C_p(X)$ is Lindelöf and Y is a C-embedded subset of X. Then $C_p(Y)$ is Lindelöf.*

Proof: Consider the inclusion map $i : Y \longrightarrow X$. Then by Theorem 1.7(i), $i^* : C_p(X) \longrightarrow C_p(Y)$ is continuous. Our claim is that i^* is an onto map. To see this, let $g \in C_p(Y)$. Since Y is C-embedded in X, there exists a continuous extension $h : X \longrightarrow \mathbb{R}$ of g. Then $i^*(h) = h \circ i = h|_Y = g$. Thus $i^* : C_p(X) \longrightarrow C_p(Y)$ is a continuous surjection. But a continuous image of a Lindelöf space is also Lindelöf. Hence $C_p(Y)$ is Lindelöf. □

Corollary 3.7 *Suppose that $C_p(X)$ is Lindelöf and let Y be a discrete C-embedded subset of X. Then Y is countable.*

Proof: By Theorem 3.6, $C_p(Y)$ is Lindelöf. Since Y is discrete, $C_p(Y) = \mathbb{R}^Y$. Then $\mathbb{R}^Y$ is Lindelöf, which further implies that $\mathbb{R}^Y$ is normal. Now for each $y \in Y$, let Z_y^+ be the discrete space of positive integers. Then if Y is uncountable, the product space $\prod\{Z_y^+ : y \in Y\}$ is not normal, see [6], Example 103. But $\prod\{Z_y^+ : y \in Y\}$ is closed in $\mathbb{R}^Y$. So if Y is assumed to be uncountable, $\mathbb{R}^Y$ cannot be normal. Thus we have that Y is countable. □

Corollary 3.8 *Suppose that X is normal and $C_p(X)$ is Lindelöf. Then every closed discrete subset of X is countable.*

Proof: Since X is normal, every closed subset of X is C-embedded in X. Then by Corollary 3.7, we have the desired result. □

Corollary 3.9 *Suppose that X is normal and $C_p(X)$ is Lindelöf. Then every discrete family of subsets in X is countable.*

Proof: Let $\{A_\lambda : \lambda \in \Lambda\}$ be a discrete family of nonempty closed sets in X. For each $\lambda \in \Lambda$, let $x_\lambda \in A_\lambda$ and consider the set $Y = \{x_\lambda : \lambda \in \Lambda\}$. Then Y is closed in X. To see this, let $x \in X \setminus Y$. This gives that $x \neq x_\lambda \ \forall \ \lambda \in \Lambda$. Since $\{A_\lambda : \lambda \in \Lambda\}$ is a discrete family, there exists an open set U containing x such that $U \cap A_\lambda \neq \emptyset$ for only one λ, say λ_1. Then $U \cap A_\lambda = \emptyset$ for all $\lambda \neq \lambda_1$. Now for x and x_{λ_1} in X, there exists disjoint open sets V and W in X such that $x \in V$ and $x_{\lambda_1} \in W$. Then $x \in U \cap V \subseteq X \setminus Y$. Our next claim is that Y is discrete in X. For this, let $x_\lambda \in Y$ be arbitrary. Since $x_\lambda \in X$, there exists an open set U containing x_λ such that $U \cap A_\lambda \neq \emptyset$ for only one λ, say λ_1. Then $U \cap A_\lambda = \emptyset$ for all $\lambda \neq \lambda_1$. Then $U \cap Y = \{x_\lambda\}$. Since $U \cap Y$ is open in Y, $\{x_\lambda\}$ is open in Y. So Y is discrete in X. Thus by Corollary 3.8, Y is countable. Since $card Y = card \Lambda$, Λ is also countable. □

Corollary 3.10 *Suppose that $C_p(X)$ is Lindelöf. Then every discrete family of open sets in X is countable.*

Proof: Let $\{A_\lambda : \lambda \in \Lambda\}$ be a discrete family of nonempty open sets in X. For each $\lambda \in \Lambda$, let $x_\lambda \in A_\lambda$ and consider the set $Y = \{x_\lambda : \lambda \in \Lambda\}$. Then as in the Corollary 3.9, we can prove that Y is closed and discrete in

X. Our next claim is that Y is C-embedded in X. To see this, let $f \in C(Y)$. For each $\lambda \in \Lambda$, there exists a $g_\lambda \in C(X)$ such that $g_\lambda(x_\lambda) = f(x_\lambda)$ and $g_\lambda(X \setminus A_\lambda) = \{0\}$. Now the function $g = \sum\{g_\lambda : \lambda \in \Lambda\}$ is continuous on X and $g|_Y = f$. To see this, let $x \in X$. Let $g(x) \in V$ for an open set V in $\mathbb{R}$. Since $x \in X$, there exists an open set U in X such that $x \in U$ and $U \cap A_\lambda \neq \emptyset$ for only one λ, say λ_1. Then $U \cap A_\lambda = \emptyset$ for all $\lambda \neq \lambda_1$. So $g(y) = \sum g_\lambda(y) = g_{\lambda_1}(y)$ for all $y \in U$. Since g_{λ_1} is continuous on X, we have an open set $U' \subseteq U$ in X such that $x \in U' \subseteq U$ and $g_{\lambda_1}(U') \subseteq V$ or $\sum g_\lambda(U') \subseteq V$. This proves the continuity of g. Then by Corollary 3.7, Y is countable. Hence Λ is countable. □

Next we define tightness of a space X, which is used to provide another necessary condition for $C_p(X)$ to be Lindelöf.

Definition 3.1 A space X is said to have *countable tightness* if for each $x \in X$ and $A \subseteq X$ such that $x \in \overline{A}$, there exists a countable subset C of A such that $x \in \overline{C}$.

Theorem 3.11 *If $C_p(X)$ is Lindelöf, then X has countable tightness.*

Proof: Suppose $x_0 \in X$ and $A \subseteq X$ such that $x_0 \in \overline{A}$. Let $U = \{f \in C(X) : f(x_0) = 0\}$. Our claim is that U is closed in $C_p(X)$. Let $h \notin U$. Then $h(x_0) = \epsilon \neq 0$, so $h \in \langle h, \{x_0\}, \epsilon\rangle \subseteq C(X) \setminus U$. This gives that U is closed in $C_p(X)$. Since $C_p(X)$ is Lindelöf, U is also Lindelöf. Next we show that $U \subseteq \bigcup_{a\in A}\langle 0, \{a\}, 1\rangle$. Let $h \notin \bigcup_{a\in A}\langle 0, \{a\}, 1\rangle$, then $|h(a)| \geq 1$ for all $a \in A$. This implies that $|h(x_0)| \geq 1$ as $x_0 \in \overline{A}$, which gives that

$h \notin U$. Since U is Lindelöf, there exists a countable subset C of A such that $U \subseteq \bigcup_{a \in C} \langle 0, \{a\}, 1 \rangle$. Now, what remains to be shown is that $x_0 \in \overline{C}$. For this, suppose $x_0 \notin \overline{C}$. Then there exists a continuous function $f : X \longrightarrow [0,1]$ such that $f(x) = 1 \ \forall \ x \in \overline{C}$ and $f(x_0) = 0$. Then $f \in U \setminus \bigcup_{a \in C} \langle 0, \{a\}, 1 \rangle$, which is a contradiction. Thus, we have that $x_0 \in \overline{C}$. □

Example 3.4 *The ordinal space $[0, \omega_1]$ does not have countable tightness. Hence $C_p([0, \omega_1])$ cannot be Lindelöf. Whereas, if J is an uncountable discrete space, then J has countable tightness but $C_p(J)$ is not Lindelöf. Thus, countable tightness alone is not sufficient for $C_p(X)$ to be Lindelöf.*

Finally we present a sufficient condition for $C_p(X)$ to be Lindelöf.

Theorem 3.12 *If X is second countable, then $C_p(X)$ is Lindelöf.*

Proof: By Corollary 3.2, we have that $C_\tau(X)$ is second countable. Then $C_\tau(X)$ is Lindelöf, and hence $C_p(X)$ is Lindelöf. □

Chapter 4

Completeness Properties of $C_p(X)$

In this chapter, we study the complete metrizability of $C_p(X)$. We show that complete metrizability, *Čech*-completeness, almost *Čech*-completeness, partition completeness and sieve-completeness are all equivalent on $C_p(X)$.

A space X is called *Čech-complete* if X is a G_δ-set in βX.

Next we study the concept of completeness of a sequence of subsets of X.

Let $\mathcal{U}$ and $\mathcal{V}$ be two collections of subsets of X. Then $\mathcal{U}$ is said to be *controlled* by $\mathcal{V}$ if for each $V \in \mathcal{V}$, there exists some $U \in \mathcal{U}$ such that $U \subseteq V$. A sequence (U_n) of subsets of X is said to be *complete* if every filter base $\mathcal{F}$ on X which is controlled by (U_n) clusters at some $x \in X$. A sequence $(\mathcal{U}_n)$ of collections of subsets of X is called *complete* if (U_n) is a complete sequence of subsets of X whenever $U_n \in \mathcal{U}_n$ for all n.

It has been shown in Theorem 2.8 of [2] that the following statements are equivalent for a Tychonoff space X:

(*i*) X is a G_δ-subset of any Hausdorff space in which it is densely embedded;

(*ii*) X has a complete sequence of open covers; and

(*iii*) X is *Čech*-complete.

A *sieve* on a space X is a sequence of indexed covers $\{U_\alpha : \alpha \in A_n\}_{n\geqslant 0}$ of X (with disjoint A_n), together with the functions $\pi_n : A_{n+1} \longrightarrow A_n$, such that $U_\alpha = X$ for $\alpha \in A_0$ and $U_\alpha = \cup\{U_\beta : \beta \in \pi_n^{-1}(\alpha)\}$ for all α and all n. Such a sieve is called complete if whenever $\alpha_n \in A_n$ with $\pi_n(\alpha_{n+1}) = \alpha_n$ for all n, then the sequence (U_{α_n}) is complete. A sieve $(\{U_\alpha : \alpha \in A_n\}, \pi_n)$ on X is called open if every U_α is open in X. A space is called *sieve-complete* if it has a complete, open sieve. A cover $\mathcal{U}$ of a space X is called *exhaustive* if every nonempty $S \subseteq X$ has a nonempty, relatively open subset of the form $U \cap S$ with $U \in \mathcal{U}$. A sieve $(\{U_\alpha : \alpha \in A_n\}, \pi_n)$ is called exhaustive if $\{U_\beta : \beta \in \pi^{-1}(\alpha)\}$ is an exhaustive cover of U_α for all $\alpha \in A_n$ and for all n. A space X is called *partition-complete* if it has a complete, exhaustive sieve.

A collection $\mathcal{U}$ of subsets of X is called an *almost-cover* of X if $\cup\mathcal{U}$ is dense in X. A space X is said to be *almost Čech-complete* if X has a complete sequence of open almost-covers. Such a space has been simply called almost complete in [4]. Every almost *Čech*-complete space is a Baire space, see Proposition 4.5 in [4].

We know that the point-open topology on $C(X)$ is generated by the uniformity generated by $\{A_\epsilon : A \in \mathbb{F}(X), \epsilon > 0\}$. When this uniformity is complete, then we say that $C_p(X)$ is uniformly complete. This uniform completeness can also be seen as the completeness of a topological group. A topological group E is called *complete* provided that every Cauchy net in E converges to some element in E, where a net (x_α) in E is Cauchy if for every neighborhood U of zero in E, there is an α_0 such that $x_{\alpha_1} - x_{\alpha_2} \in U$ for all $\alpha_1, \alpha_2 \geqslant \alpha_0$ (for E additive). One can check that $C_p(X)$ is uniformly complete if and only it is complete as an additive topological group. Also $C_p(X)$ is completely metrizable if and only if it is complete and metrizable, (see [1], pp. 34, 36).

The following result gives characterization of uniform completeness of $C_p(X)$.

Theorem 4.1 *The space $C_p(X)$ is uniformly complete if and only if X is a discrete space.*

Proof: Suppose that $C_p(X)$ is uniformly complete. In order to prove that X is discrete, it suffices to prove that $C(X) = \mathbb{R}^X$. Now, by Theorem 1.6, $C(X)$ is dense in $\mathbb{R}^X$. Let $f \in \mathbb{R}^X$. Then $f \in \overline{C(X)}$. So by Theorem 11.7 in

[8], $\exists$ a net (f_α) in $C(X)$ such that $f_\alpha \longrightarrow f$ in $\mathbb{R}^X$. Then by Theorem 39.2 in [8], (f_α) is cauchy. Since $C_p(X)$ is uniformly complete, $f_\alpha \longrightarrow g$ in $C_p(X)$. This implies that $f = g$. Thus $C(X) = \mathbb{R}^X$. Hence X is a discrete space.

Conversely, if X is a discrete space, $C_p(X) = \mathbb{R}^X$. Since $\mathbb{R}$ is complete, by Theorem 39.4 in [8] $\mathbb{R}$ is uniformly complete. Consequently, $\mathbb{R}^X$ is uniformly complete, and hence $C_p(X)$ is also uniformly complete.

Theorem 4.2 *For any space X, the following statements are equivalent.*

(i) $C_p(X)$ *is almost Čech-complete.*

(ii) $C_p(X)$ *is partition complete.*

(iii) $C_p(X)$ *is sieve-complete.*

(iv) $C_p(X)$ *is Čech-complete.*

(v) $C_p(X)$ *is completely metrizable.*

(vi) X *is a countable discrete space.*

Proof: $(v) \Rightarrow (iv) \Rightarrow (iii) \Rightarrow (ii) \Rightarrow (i)$ follow from the Proposition 4.4 in [4].

$(vi) \Rightarrow (v)$. Suppose that X is countable discrete space. Since X is countable, by Theorem 2.10, $C_p(X)$ is metrizable. Also, as X is a discrete space, by Theorem 4.1, $C_p(X)$ is uniformly complete. Hence $C_p(X)$ is completely metrizable.

$(i) \Rightarrow (vi)$. Suppose $C_p(X)$ is almost *Čech*-complete. Let D be a dense *Čech*-complete subspace of $C_p(X)$. Since $C_p(X)$ is dense in $\mathbb{R}^X$, D is dense in $\mathbb{R}^X$. Since $\mathbb{R}^X$ contains a dense Baire subspace D, $\mathbb{R}^X$ itself is a Baire space. Also since D is *Čech*-complete, D is a G_δ-subset of $\mathbb{R}^X$. Suppose that X is not a discrete space. Let f be a discontinuous function on X. Define $\Phi : \mathbb{R}^X \longrightarrow \mathbb{R}^X$ by $\Phi(g) = f + g$ for all $g \in \mathbb{R}^X$. Then Φ is a homeomorphism. Since $f \notin C_p(X)$, $\Phi(C_p(X)) \cap C_p(X) = \varnothing$. For, if not, let $h \in \Phi(C_p(X)) \cap C_p(X)$. So $h \in \Phi(C_p(X))$ and $h \in C_p(X)$. This implies that $h = \Phi(g)$ for some $g \in C_p(X)$ and $h = f + g$. Since h and g are continuous functions, so f is also continuous, which is a contradiction to the fact that f is not continuous. Now, $\Phi(D)$ is a dense G_δ-subset of $\mathbb{R}^X$. Then $\Phi(D)$ and D are two disjoint dense G_δ-subsets of the Baire space $\mathbb{R}^X$, which is a contradiction. Thus, we conclude that X is a discrete space.

Now, D being a *Čech*-complete space is of pointwise countable type. So, by Theorem 2.10, X is countable. □

List of Notations

Symbol	**Meaning**
$\forall$, $\exists$	for all, there exists
$\in$	belongs to
$\subseteq$	subset
$\nsubseteq$	not subset
$\mathbb{N}$	the set of natural numbers
$\mathbb{R}$	the set of real numbers
$\mathbb{F}(X)$	the set of all finite subsets of X
$\cup$, $\cap$	union, intersection
$\emptyset$	empty set
$\square$	end of a proof
$X \setminus S$	complement of S in X
$\overline{A}$	the closure of A
G_δ-set	a countable intersection of open sets in a topological space X

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