SOME *n*-COLOR COMPOSITION

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Submitted by
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UNDER
THE GUIDANCE OF
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DEDICATED

TO

GOD, MY PARENTS AND SUPERVISOR

CERTIFICATE

I hereby certify that the work which is being presented in the thesis entitled **"On Some n-Color Compositions"** in partial fulfillment of the requirements for the award of degree of Master of Science, School of Mathematics and Computer Applications (SMCA), Thapar University, Patiala is an authentic record of my own work carried out under the supervision of Dr. Meenakshi Rana.

The matter presented in this thesis has not been submitted for the award of any other degree of this or any other university.

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This is to certify that the above statement made by the candidate is correct and true to the best of my knowledge.

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ABSTRACT

In this thesis, we studied ordered partitions, called compositions which were first defined by MacMahon, "P.A MacMahon, Combinatory Analysis, AMS Chelsea Publishing, 2001." Chapter 1 is devoted to basic definitions and preliminary results on classical partitions, compositions and Fibonacci numbers. Chapter 2 discuss the generating functions of n-Color Compositions and some recurrence formulas related to compositions. In Chapter 3, we have studied an identity analouges to Euler's partition identity and some new combinatorial properties of n-color compositions.

CONTENTS

1. INTRODUCTION

1.1 Introduction

The Theory of Partition is important branch of ***Additive Number Theory***. The Number Theory is a branch of pure mathematics for the study of integers. It is also called as "The Queen of Mathematics". The concept of partition of non negative integers also belong to Combinatorics. Partitions first appeared in a letter written by Lebnitz in 1669 to John Bernoulli, asking him if he had investigated the number of ways in which given number can be expressed as sum of two or more integers.

The real development of partition theory started with Euler[6]. It was he who first discovered the important properties of the partition function. The theory has been further developed by many other famous mathematicians Jacobi, Ramanujan, Sylvester, Gauss.

The purpose of the thesis is to study ordered partitions, called compositions. The first chapter is devoted to basic definitions and preliminary results.

1.1.1 Definition of Partition

Definition 1.1.1: A partition of a positive integer n is the finite non increasing sequence of positive integers $a_1, a_2, a_3, ..., a_r$ such that $\sum_{i=1}^{r} a_i = n$. the a_i are called ***parts*** or ***summands*** of partitions, denoted by p(n), the partition of n.

Tab. 1.1: (Partitions of some integers)

n	$p(n)$	Relevant Partitions
1	1	1
2	2	2, 1+1
3	3	1+1+1, 2+1, 3
4	5	1+1+1+1, 2+2, 3+1, 2+1+1, 4
5	7	1+1+1+1+1, 3+2, 2+2+1, 4+1, 2+1+1+1, 3+1+1, 5

Remark 1. $p(n) = 0$ for $n < 0$ as we cannot write a negative number into sum of positive integers.

Also $p(0) = 1$.

Remark 2. We use an exponent to denote that the part is repeated a certain number of times in partition.

Example 1.

$$\begin{aligned} p(2) &= 2; \quad 2 = (2),\ 1+1 = (1^2); \\ p(3) &= 3; \quad 3 = (3),\ 2+1 = (12),\ 1+1+1 = (1^3); \\ p(4) &= 5; \quad 4 = (4),\ 3+1 = (13),\ 2+2 = (2^2), \\ & \quad 2+1+1 = (1^2 2),\ 1+1+1+1 = (1^4). \end{aligned}$$

1.2 Generating Functions

Definition 1.2.1: Binomial Theorem

$$(x+y)^n = \sum_{r=0}^{\infty} \binom{n}{r} x^{n-r} y^r$$

where n and r are positive integer and $x, y \in \Re$(real numbers).

Definition 1.2.2: Generalized Binomial Theorem

$$(1-x)^{-r} = \sum_{n=0}^{\infty} \frac{(r)_n x^n}{n!}$$

where x is a complex variable, n and r are positive integers.

Definition 1.2.3: The generating function of $p(n)$ is given by

$$\sum_{n=0}^{\infty} p(n) = \frac{1}{(q;q)_\infty}$$

where $|q| < 1$ and $(q;q)_\infty$ is q - rising factorial defined by

$$(a;q)_n = \prod_{i=0}^{\infty} \frac{(1-aq^i)}{(1-aq^{i+n})}.$$

If n is a positive integer, then

$$(a;q)_n = (1-a)(1-aq)(1-aq^2)\cdots(1-aq^{n-1}),$$

$$(a;q)_\infty = (1-a)(1-aq)(1-aq^2)\cdots.$$

and when $a = q$

$$(q;q)_n = \prod_{i=1}^{n} (1-q^i),$$

also

$$(a,q)_0 = 1.$$

Definition 1.2.4: Let P denote the set of all partitions and Let P(S,n) denote the number of partition of n that belong to a subset S of the set P of all partition.

Let D(n) denote the number of partitions of n into distinct parts, then the generating function for this is given by;

$$\sum_{n=0}^{\infty} D(n)q^n = \prod_{n=1}^{\infty}(1+q^n) = (-q;q)_{\infty}.$$

Example 2. Consider D the set of all partitions with distinct parts then

$$P(D,7) = 5,$$

the relevant partitions are 7, 6+1, 5+2, 4+3, 4+2+1.

Definition 1.2.5: Let O(n) denote the number of partitions of n into odd parts, then the generating function for this is

$$\sum_{n=0}^{\infty} O(n)q^n = \prod_{n=1}^{\infty} \frac{1}{(1-q^{2n-1})} = \frac{1}{(q;q^2)_{\infty}}.$$

Example 3. Consider O the set of all partitions with odd parts then

$$P(O,7) = 5,$$

the relevant partitions are 7, 5+1+1, 3+3+1, 3+1+1+1+1, 1+1+1+1+1+1+1.

1.3 Graphical Representation of Partitions

Definition 1.3.1: The Ferrers graph of a partition t_1, t_2,. . . ,t_i of n is a set of rows of equi-spaced dots aligned on the left where the *jth* row has t_j dots.

Example 4. The Ferrers graph of the partition $\pi = 6+4+3+1$ of 14 as following,

$$\begin{array}{llllll} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & & \\ \cdot & \cdot & \cdot & & & \\ \cdot & & & & & \end{array}$$

by reading this graph horizontally, we see that first row has 6 dots, second row has 4, third row has 3, fourth row has 1.

Definition 1.3.2: The *conjugate of a partition* (π^c) is obtained by interchanging Ferrers diagram's rows with its columns.

Example 5. The Ferrers graph of partition $\pi = 3 + 2 + 2$ of 7 is

$$\begin{array}{ccc} \cdot & \cdot & \cdot \\ \cdot & \cdot & \\ \cdot & \cdot & \end{array}$$

The conjugate of above graph is

$$\begin{array}{ccc} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & & \end{array}$$

So, $\pi^c = 3 + 3 + 1$ of 7.

Definition 1.3.3: A partition is said to be self conjugate if it is identical with its conjugate.

Example 6. Consider the partition $\pi = 3 + 2 + 1$ of 6 then the Ferrers graph is

$$\begin{array}{ccc} \cdot & \cdot & \cdot \\ \cdot & \cdot & \\ \cdot & & \end{array}$$

now if we read this graph vertically we get the same partition $\pi^c = 3 + 2 + 1$, so this partition is self conjugate partition. In other words when $\pi = \pi^c$ then that particular partition is called a self conjugate partition.

1.4 Fibonacci Numbers

Fibonacci numbers and Fibonacci sequence were first introduced in 1202 in Fibonacci's book Liber abaci. Famous Fibonacci problem is – Each month the female of a pair of rabbits give birth to a pair of rabbits(of different sex). Two months later the female of the new pair give birth to a pair of rabbits. Now question is to find the number of pairs of rabbits at the end of the year, if there was one pair of rabbits in the beginning of the year. For this problem the recurrence formula is given below;

let $F(n)$ denotes the number of pairs after n^{th} month, then

$$F_{n+1} = F_n + F_{n-1} \quad ; \quad n \geq 1; F_0 = 1, F_1 = 2.$$

$$\begin{aligned} F_2 &= 3 & F_3 &= 5 \\ F_4 &= 8 & F_5 &= 13 \\ F_6 &= 21 & F_7 &= 34 \\ F_8 &= 55 & F_9 &= 89 \\ F_{10} &= 144 & F_{11} &= 233 \\ F_{12} &= 377. \end{aligned}$$

Definition 1.4.1: Fibonacci numbers are defined by following recurrence relation

$$F_n = F_{n-1} + F_{n-2} \qquad n > 2 \quad \text{such that } F_1 = 1, F_2 = 1 \tag{1.1}$$

$$\text{or, } F_{n+1} = F_n + F_{n-1} \qquad n > 1 \quad \text{such that } F_1 = 1, F_2 = 1 \tag{1.2}$$

1.4.1 Some Theorems on Fibonacci Numbers

Theorem 1.4.1: Neighboring Fibonacci numbers are prime to each other.

$$gcd(F_n, F_{n+1}) = 1, \qquad n \geq 1.$$

Proof. Let $gcd(F_n, F_{n+1}) = d$

then, $d/F_n, \quad d/F_{n+1}$

and $d/F_{n+1} + F_n$

therefore d/F_{n-1}. (using equation (1.2))

Now $d/F_{n-1}, \quad d/F_n$

hence $d/F_n - F_{n-1}$

therefore d/F_{n-2}.

Iterating n times, we get

d/F_1

also $F_1 = 1$

implies $d/1$

hence $d = 1$

which further implies $gcd(F_n, F_{n+1}) = 1$.

Theorem 1.4.2: $F_{m+n} = F_{m-1}F_n + F_mF_{n+1}$

Proof. We will prove the result by induction on n.

For $n = 1$,

$$\begin{aligned} L.H.S &= F_{m+1} \\ R.H.S &= F_{m-1}.F_1 + F_m.F_2 \\ &= F_{m-1}.1 + F_m.1 \\ &= F_{m-1} + F_m \\ &= F_{m+1} \end{aligned}$$

so result is true for $n = 1$.
Assume, it to be true for all 'n'; $n \leq k$.
In particular, for $n = k$;

$$F_{n+k} = F_{m-1}.F_k + F_m.F_{k+1},$$

to prove for $n = k + 1$, i.e.

$$F_{m+k+1} = Fm - 1.F_{k+1} + F_m.F_{k+2} \tag{1.3}$$

we have

$$\begin{aligned}\text{L.H.S} =& F_{m+k+1}\\ =& F_{m+k} + F_{m+k-1}\end{aligned}$$

using recurrence relation of Fibonacci numbers

$$\begin{aligned}=& F_{m-1}F_k + F_mF_{k+1} + F_{m-1}F_{k-1} + F_mF_k\\ =& F_{m-1}(F_k + F_{k-1}) + F_m(F_{k+1} + F_k)\\ =& F_{m-1}F_{k+1} + F_mF_{k+2}\end{aligned}$$

which proves the result.

1.4.2 Properties of Fibonacci Numbers

Some properties of Fibonacci numbers are given below which can be easily proved with the usual notations and mathematical induction.

Property 1.

$$F_1 + F_2 + F_3 + ... + F_n = F_{n+2} - 1.$$

Property 2.

$$F_1 + F_3 + ... + F_{2n-1} = F_{2n}.$$

Property 3.

$$F_2 + F_4 + ... + F_{2n} = F_{2n+1} - 1.$$

Property 4.

$$F_1 - F_2 + F_3 - F_4 + ... + (-1)^{n+1}F_n = (-1)^{n+1}F_{n-1} + 1 \quad ; n > 1.$$

Property 5.

$$F_1^2 + F_2^2 + F_3^2 + ... + F_n^2 = F_n.F_{n+1.}$$

Property 6.

$$F_{n+1}^2 = F_n.F_{n+2} + (-1)^n.$$

1.5 Compositions

In the classical theory of partitions, compositions were first defined by MacMahon [7] as ordered partitions.

The number of compositions of n is denoted by $C(n)$.

Example 7. For $n = 4$,

$$p(n) = 5$$

and

$$C(n) = 8$$

$p(n) = 4, 31, 22, 211, 1111$ and $C(n) = 4, 31, 13, 211, 121, 112, 1111$.

1.5.1 Generating Function of Compositions

Definition 1.5.1: Let $C^m(n)$ be the compositions of n into m parts. Then

$$\begin{aligned}\sum_{n=0}^{\infty} C^m(n)q^n =&(q^1 + q^2 + q^3 + \cdots)^m \\ =&\frac{q^m}{(1-q)^m}.\end{aligned} \tag{1.4}$$

Definition 1.5.2: Let the number of compositions of n into m parts each part $\leq s$ is denoted by

$$C_s^m(n)$$

then $C_s^m(q)$ is generating function of $C_s^m(n)$

$$= \sum_{n=1}^{\infty} C_s^m(n)q^n$$

$$\begin{aligned}C_s^m(q) =&(q + q^2 + q^3 + \cdots + q^s)^m \\ =&\left(\frac{q - q^{s+1}}{1-q}\right)^m.\end{aligned} \tag{1.5}$$

Definition 1.5.3: Let $C_s(q)$ be the generating function for the composition into parts $\leq s$. Then

$$C_s(q) =\frac{q - q^{s+1}}{1 - 2q + q^{s+1}}. \tag{1.6}$$

1.5.2 Some Theorems on Compositions

Theorem 1.5.1:

$$C^m(n) = \binom{n-1}{m-1} = \frac{(n-1)!}{(m-1)!(n-m)!}$$

where $C^m(n)$ is the number of compositions of n into m parts.

Proof.

$$\begin{aligned}
\sum_{n\geq m} C^m(n)q^n =&(q+q^2+q^3+\cdots)^m q^m \\
=&\left(\frac{q}{1-q}\right)^m \\
=&q^m(1-q)^{-m} \\
=&\sum_{r=0}^{\infty}\frac{(m)_r q^r}{r!} \qquad \text{(using generalized binomial theorem)} \\
=&\sum_{r=0}^{\infty}\frac{(m-1)!(m)_r q^r}{(m-1)!r!} \\
=&q^m\sum_{r=0}^{\infty}\frac{(m+r-1)!}{(r)!(m-1)!}q^r \\
=&q^m\sum_{r=0}^{\infty}\binom{m+r-1}{r}q^m \\
=&\sum_{r=0}^{\infty}\binom{m+r-1}{r}q^{r+m} \\
=&\sum_{n\geq m}\binom{n-1}{n-m}q^n
\end{aligned}$$

now comparing coefficients of q^n

$$\begin{aligned}
C^m(n) =&\binom{n-1}{n-m} \\
=&\binom{n-1}{m-1}.
\end{aligned}$$

Example 8. We know that compositions of n into m parts is given by

$$C^m(n) = \binom{n-1}{n-m}.$$

For $n = 4$, $m = 1$

$$C^1(4) = \binom{3}{0} = 1,$$

the relevant composition of 4 is 4.

For $n = 4$, $m = 2$

$$C^2(4) = \binom{3}{1} = 3,$$

the relevant compositions of 4 are 31, 13, 22.

For $n = 4$, $m = 3$

$$C^3(4) = \binom{3}{2} = 3,$$

the relevant compositions of 4 are 112, 121, 211.

For $n = 4$, $m = 4$

$$C^4(4) = \binom{3}{3} = 3,$$

the relevant composition of 4 is 1111.

Theorem 1.5.2: $C(n) = 2^{n-1}$ where $C(n)$ is number of compositions of n.

Proof. 1^{st} **Method**:

$$\begin{aligned}
C(n) &= \sum_{m=1}^{\infty} C^m(n) \\
&= \sum_{m=1}^{\infty} \binom{n-1}{m-1} \qquad (\binom{n}{m} = 0 \quad for \;\; m > n) \\
&= \sum_{m=1}^{n} \binom{n-1}{m-1} \\
&= \binom{n-1}{0} + \binom{n-1}{1} + \binom{n-1}{2} + \cdots + \binom{n-1}{n-1} \\
&= 2^{n-1}.
\end{aligned}$$

2^{nd} **Method**: Note,

$$\sum_{n=1}^{\infty} C^m(n) q^n = \frac{q^m}{(1-q)^m}$$

and

$$\begin{aligned}
\sum_{n=1}^{\infty} C(n) q^n &= \sum_{n=1}^{\infty} \left[\sum_{m=1}^{\infty} C^m(n) \right] q^n \\
&= \sum_{m=1}^{\infty} \left[\sum_{n=1}^{\infty} C^m(n) q^n \right] \\
&= \sum_{m=1}^{\infty} \left(\frac{q}{1-q} \right)^m \\
&= \sum_{m=0}^{\infty} \left(\frac{q}{1-q} \right)^m - 1 \\
&= \left(\frac{1}{1 - \frac{q}{1-q}} \right) - 1 \\
&= \frac{q}{1-2q} \\
&= q \sum_{n=0}^{\infty} (2q)^n
\end{aligned}$$

now comparing coefficient of q^n on both side, we have

$$C(n) = 2^{n-1}$$

hence the result.

Example 9. For $n = 4$

$$c(4) = 8$$

and the relevant compositions of $n = 4$ are 4, 31 , 22 , 211 , 1^4 , 13 , $1^2 2$, 121.

1.6 Generating Function of Fibonacci numbers

To derive generating function for Fibonacci numbers we use Binet's formula corresponding to n^{th} Fibonacci number given by

$$F_n = \frac{x^n - y^n}{x - y},$$

where $x = \frac{1+\sqrt{5}}{2}$ and $y = \frac{1-\sqrt{5}}{2}$.
Consider right hand side of above as

$$A_n = \frac{x^n - y^n}{x - y},$$

Our purpose is to show that A_n and F_n satisfies the same recurrence formula and initial conditions.

Recall from equation (1.1)

$$F_n = F_{n-1} + F_{n-2} \qquad n > 2 \quad \text{such that } F_1 = 1, F_2 = 1.$$

Now we prove similar recurrence formula and initial conditions for A_n.

$$\begin{aligned} \text{for } n = 1, \qquad A_1 =& \frac{x - y}{x - y} \\ =& 1 \\ \text{for } n = 2, \qquad A_2 =& \frac{x^2 - y^2}{x - y} \\ =& x + y \\ =& 1 \end{aligned}$$

$$\text{so,} \qquad A_1 = A_2 = 1$$

thus A_n satisfies same initial conditions as that of F_n.
Now to prove the recurrence formula for A_n, consider

$$t^2 = t + 1$$

the roots of above equation are $x = \frac{1+\sqrt{5}}{2}$, $y = \frac{1-\sqrt{5}}{2}$ and thus,

$$x + y = 1 \tag{1.7}$$

$$xy = -1 \tag{1.8}$$

also

$$x^2 = x + 1 \tag{1.9}$$

$$y^2 = y + 1. \tag{1.10}$$

Thus multiply (1.9) by x^{n-1} and (1.10) by y^{n-1} respectively we get,

$$\begin{aligned} x^{n+1} &= x^n + x^{n-1} \\ y^{n+1} &= y^n + y^{n-1} \end{aligned}$$

on subtracting the above two equations we get

$$x^{n+1} - y^{n+1} = (x^n - y^n) + (x^{n+1} - y^{n-1}),$$

now divide by $(x - y)$ on both sides we get,

$$\left(\frac{x^{n+1} - y^{n+1}}{x - y}\right) = \left(\frac{x^n - y^n}{x - y}\right) + \left(\frac{x^{n-1} - y^{n-1}}{x - y}\right)$$

$$\text{so,} \quad A_{n+1} = A_n + A_{n-1}$$

$$\text{hence} \quad A_n = F_n.$$

Theorem 1.6.1:

$$\sum_{n=0}^{\infty} F_{n+1} q^n = \frac{1}{1 - q - q^2}$$

Proof

$$\text{Consider} \quad \frac{1}{1-qx} - \frac{1}{1-qy};$$

$$\text{then,} \quad \frac{1}{1-qx} - \frac{1}{1-qy} = \left(\sum_{n=0}^{\infty}(qx)^n\right) - \left(\sum_{n=0}^{\infty}(qy)^n\right)$$

$$= \sum_{n=0}^{\infty}(x^n - y^n)q^n \tag{1.11}$$

$$\text{also,} \quad \frac{1}{1-qx} - \frac{1}{1-qy} = \frac{-qy+qx}{(1-qx)(1-qy)}$$

$$= \frac{q(x-y)}{1-q-q^2} \quad \text{(using (1.7) and (1.8))}$$

$$\text{thus,} \quad \sum_{n=0}^{\infty}(x^n - y^n)q^n = \frac{q(x-y)}{1-q-q^2}$$

dividing both sides by $(x-y)$, we have

$$\frac{q}{1-q-q^2} = \sum_{n=0}^{\infty} \frac{(x^n - y^n)}{(x-y)} \ q^n$$

$$= \sum_{n=0}^{\infty} A_n q^n$$

$$= \sum_{n=0}^{\infty} F_n q^n$$

$$= \sum_{n=1}^{\infty} F_n q^n$$

$$= \sum_{n=0}^{\infty} F_{n+1} \ q^{n+1}$$

$$= q\sum_{n=0}^{\infty} F_{n+1} \ q^n$$

$$\text{thus, we get} \quad \sum_{n=0}^{\infty} F_{n+1} q^n = \frac{1}{1-q-q^2}.$$

1.7 Fibonacci Numbers and Compositions

Problem 1.8.1 Number of compositions of n into parts ≤ 2 is equal to F_{n+}.

$$\text{i.e} \quad C_2(n) = F_{n+1}, \quad \text{for } n \geq 1.$$

Solution

$$
\begin{aligned}
C_2(n) =& \sum_{n=0}^{\infty} C_2(n) q^n \\
=& \frac{q - q^{2+1}}{1 - 2q + q^{2+1}} \qquad \text{(using(1.6))} \\
=& \frac{q(1+q^2)}{1 - q - q^2} \\
=& \frac{1}{1 - q - q^2} - 1 \\
=& \sum_{n=0}^{\infty} F_{n+1} q^n - 1 \qquad \text{(Theorem (1.6.1))} \\
=& \sum_{n=1}^{\infty} F_{n+1} q^n + F_1 q^0 - 1
\end{aligned}
$$

comparing coefficients of q^n on both sides, we have

$$C_n(2) = F_{n+1}.$$

Example 10. Let $n = 4$, all relevant compositions of 4 are

$$4, 31, 13, 22, 121, 211, 112, 1111.$$

Relevant composition of 4 with parts ≤ 2 are

$$22, 121, 211, 112, 1111.$$

$$C_4(2) = 5 = F_5 = F_{4+1}.$$

Problem 1.8.2 The number of compositions of n in which no one's appears is F_{n-1}, for $n \geq 2$.

Solution Let,
$C'(n)$ denotes the number of compositions of n in which no 1‘s appear.
$C'^{m}(n)$ denotes the number of compositions of n enumerated by $C'(n)$ into exactly m parts.
i.e to show $C'(n) = F_{n-1}$.
From equation (1.4) $C'^{m}(1) = 0$ because compositions of 1 in which no one's appear is zero. So, generating function of $C'^{m}(n)$ is

$$
\begin{aligned}
\sum_{n=2}^{\infty} C'^{m}(n) q^n &= (q^2 + q^3 + \cdots)^m \\
&= q^{2m}(1 + q + q^2 + \cdots)^m
\end{aligned}
$$

$$= q^{2m}\left(\frac{1}{1-q}\right)^m$$
$$= q^{2m}\left(\frac{1}{1-q}\right)^m$$

thus,

$$\sum_{n\geq 2} C'^m(n)q^n = \left(\frac{q^2}{1-q}\right).$$

$$C'(n) = \sum_{n=2}^{\infty} C'(n)q^n$$
$$= \sum_{m=1}^{\infty}\left(\sum_{n=2}^{\infty} C'^m(n)q^n\right)$$

therefore
$$\sum_{n\geq 2} C'(n)q^n = \sum_{m=1}^{\infty}\left(\frac{q^2}{1-q}\right)^m$$
$$= \sum_{m=0}^{\infty}\left(\frac{q^2}{1-q}\right)^m - 1$$
$$= \frac{q^2}{1-q-q^2} \tag{1.12}$$

$$\sum_{n=2}^{\infty} F_{n-1}q^n = \sum_{n=0}^{\infty} F_{n+1}q^{n+2} \qquad \text{(replace } n \text{ by } n+2\text{)}$$
$$= q^2\left(\sum_{n=0}^{\infty} F_{n+1}q^n\right)$$
$$= \frac{q^2}{1-q-q^2}. \qquad \text{(using Theorem (1.6.1))} \tag{1.13}$$

Comparing equation(1.12) and (1.13) we have,

$$\sum_{n\geq 2} C'(n)q^n = \sum_{n\geq 2} F_{n-1}q^n$$

hence, $$C'(n) = F_{n-1}.$$

Example 11. Take $n = 4$, relevant compositions of 4 are

$$4, 31, 13, 22, 211, 121, 112, 1111.$$

So, $$C'(4) = 2.$$

Also, $$F_{n-1} = F_3 = 2.$$

2. N-COLOR COMPOSITIONS

2.1 Introduction

Agarwal and Andrews [2] defined an n-color partitions as "a partition in which a part of size n can appear in n different colors", denoted by subscripts $n_1, n_2, n_3, ..., n_n$ and parts satisfy the order

$$1_1 < 2_1 < 2_2 < 3_1 < 3_2 < 3_3 < 4_1 < 4_2 < 4_3 < 4_4 \cdots .$$

Example 1. For $n = 4$, the relevant n-color partitions of 4 are

$$4_1,\ 4_2,\ 4_3,\ 4_4,$$
$$3_1 1_1,\ 3_2 1_1,\ 3_3 1_1,$$
$$2_1 2_1,\ 2_1 2_2,$$
$$2_1 1_1 1_1,\ 2_2 1_1 1_1,$$
$$1_1 1_1 1_1 1_1.$$

Analogous to MacMahon's ordinary compositions (Chapter 1,Sec(1.5)) Agarwal [1] defined an n-color composition as an n-color ordered partition.

Example 2. For $n = 4$, the relevant n-color composition of 4 are,

$$4_1,\ 4_2,\ 4_3,\ 4_4,$$
$$3_1 1_1,\ 3_2 1_1,\ 3_3 1_1,\ 1_1 3_1,\ 1_1 3_2,\ 1_1 3_3,$$
$$2_1 2_1,\ 2_1 2_2,\ 2_2 2_2,\ 2_2 2_1,$$
$$2_1 1_1 1_1,\ 2_2 1_1 1_1,\ 1_1 2_1 1_1,\ 1_1 1_1 2_1,\ 1_1 2_2 1_1,\ 1_1 1_1 2_2,$$
$$1_1 1_1 1_1 1_1.$$

Definition 2.1.1: An n-color odd composition is an n-color composition with odd parts.

Example 3. For $n = 5$ the relevant n-color odd compositions of 4 are

$$3_1 1_1,\ 3_2 1_1,\ 3_3 1_1,$$
$$1_1 3_1,\ 1_1 3_2,\ 1_1 3_3,$$
$$1_1 1_1 1_1 1_1.$$

Remark 1. n-color composition of ν is denoted by $c(\nu)$.

Remark 2. n-color compositions of ν into m parts is denoted by $c^m(\nu)$.

2.2 Generating Function of n-Color Compositions

Theorem 2.2.1: [1] Let $c^m(q)$ and $c(q)$ denote the enumerative generating functions for $c^m(\nu)$ and $c(\nu)$, respectively. then

$$c^m(q) = \frac{q^m}{(1-q)^{2m}} \tag{2.1}$$

$$c(q) = \frac{q}{1-3q+q^2} \tag{2.2}$$

$$c^m(\nu) = \binom{\nu+m-1}{2m-1} \tag{2.3}$$

$$c(\nu) = F_{2\nu} \tag{2.4}$$

Proof. To prove (2.1)

$$\text{we know that} \quad c^m(q) = \sum_{\nu=1}^{\infty} c^m(\nu)q^\nu$$
$$= (q + 2q^2 + 3q^3 + \cdots)^m$$
$$= \frac{q^m}{(1-q)^{2m}}$$

which proves (2.1).

To prove (2.2)

$$= \sum_{m=1}^{\infty} \frac{q^m}{(1-q)^{2m}}$$
$$= \sum_{m=0}^{\infty} \frac{q^m}{(1-q)^{2m}} - 1$$
$$c(q) = \frac{q}{1-3q+q^2}$$

which proves(2.2).

To prove (2.3)
From (2.1)

$$\sum_{\nu=0}^{\infty} c^m(\nu) q^\nu = q^m(1-q)^{-2m}$$

by using generalized binomial theorem, we have

$$\begin{aligned}
&= q^m \sum_{r=0}^{\infty} \frac{(2m)_r}{r!} q^r \\
&= q^m \sum_{r=0}^{\infty} \binom{2m+r-1}{r} q^r \\
&= \sum_{r=0}^{\infty} \binom{2m+r-1}{r} q^{r+m} \\
&= \sum_{\nu=m}^{\infty} \binom{m+\nu-1}{\nu-m} q^\nu \\
&= \sum_{\nu}^{\infty} \binom{\nu+m-1}{\nu-m} q^\nu \\
&= \sum_{\nu=0}^{\infty} \binom{\nu+m-1}{2m-1} q^\nu
\end{aligned}$$

therefore

$$\sum_{\nu=0}^{\infty} c^m(\nu) q^\nu = \sum_{\nu=0}^{\infty} \binom{\nu+m-1}{2m-1} q^\nu$$

comparing coefficients of q^ν

$$c^m(\nu) = \binom{\nu+m-1}{2m-1}$$

which proves (2.3).

To prove (2.4) Since the right hand side of (2.2) is the generating function for $2n^{th}$ Fibonacci number. Hence

$$c(\nu) = F_{2\nu}.$$

which proves (2.4).

2.3 Main Theorem

Theorem 2.3.1: let $c(O,\nu)$ denotes the number of n-color odd compositions of ν. Then,

$$c(O,1)=1, c(O,2)=1, c(O,3)=4, c(O,4)=7$$

and

$$c(O,\nu)=c(O,\nu-1)+2c(O,\nu-2)-c(o,\nu-3)-c(O,\nu-4) \qquad for \quad \nu>4$$

Proof. (Combinatorial) To prove that

$$c(O,\nu)=c(O,\nu-1)+2c(O,\nu-2)-c(O,\nu-3)-c(O,\nu-4),$$

we split the n-color compositions enumerated by $c(O,\nu)+c(O,\nu+4)$ into four classes;
class(i) contains the compositions enumerated by $c(O,\nu)$ with 1_1 on right,
class(ii) contains the compositions enumerated by $c(O,\nu)$ with 3_3 on the right,
class(iii) contains the compositions enumerated by $c(O,\nu)$ with h_t on the right, $h>1, 1\leq t\leq h-2$(where, h is odd),
class(iv) contains the compositions enumerated by $c(O,\nu)$ with h_t on the right, $h>1, h-1\leq t\leq h$,

Transform the n-color odd compositions in class(i) by deleting 1_1 on the right. This produces n-color compositions enumerated by $c(O,\nu-1)$.
Conversely, for any n-color composition enumerated by $c(O,\nu-1)$ we add 1_1 on the right to produce the elements of class(i). In this way we prove that there are exactly $c(O,\nu-1)$ elements in class(i).

Similarly, we can obtain n-color odd compositions in class(ii) by deleting 3_3 on the right.

Next, transform the n-color odd compositions enumerated by class(iii) by subtracting 2 from h, that is, replacing h_t by $(h-2)_t$. this transformation also establishes the fact that there are exactly $c(O,\nu-2)$ elements in class(iii). This correspondence belongs to one-one.

Finally, transform the elements in class(iv) as follows: subtract 2_2 from h_t on the right when $h>3, h-1\leq t\leq h$, that is, replace h_t by $(h-2)_{(t-2)}$; in this way we will get n-color odd compositions of $\nu-2$ with part $h'_{t'}$ on the right, where, $h'>1, t'\geq h'-1$. After that we replace h_t by $(h-2)_{(t-1)}$ when $h=3, t=2$. This produces n-color odd compositions of $\nu-2$ with part 1_1 on the right. To get the remaining n-color odd compositions from $c(O,\nu-4)$, we add 2 to the right parts, that is, replace h_t by $(h+2)_t$ to get the n-color odd compositions of $(\nu-2)$ with part $h'_{t'}$ on the right, where $h'>1, 1\leq t'\leq h'-2$. We see that the number of n-color odd compositions in class(iv) is also equal to $c(O,nu-2)$. Hence,

$$c(O,\nu)+c(O,\nu+4)=c(O,\nu-1)+2c(O,\nu-2)+c(O,\nu-3)$$

$$c(O,\nu)=c(O,\nu-1)+2c(O,\nu-2)-c(O,\nu-3)-c(O,\nu-4)$$

thus, the proof.

3. AN ANALOUGE OF EULER'S IDENTITY AND NEW COMBINATORIAL PROPERTIES OF N-COLOR COMPOSITIONS

3.1 Introduction

In 1748, Euler gave first partition identity as "The number of partitions of a positive integer n into odd parts equal to the number of its partitions n into distinct parts".
Proof We have,

$$\begin{aligned}\sum_{n=0}^{\infty} p(D,n)q^n =&\prod_{i=1}^{\infty}(1+q^n)\\ =&\prod_{i=1}^{\infty}\frac{1+q^n}{1-q^n}(1-q^n)\\ =&\prod_{i=1}^{\infty}\frac{1-q^{2n}}{1-q^n}\\ =&\prod_{i=1}^{\infty}(1-q^{2n-1})^{-1}\\ =&p(O,n)q^n.\end{aligned}$$

Agarwal [3] proved an analouge of Euler's identity and provided some new combinatorial properties of n-color compositions. Before proceeding to main results, let us recall some definitions.

3.2 Definitions

Definition 3.2.1: (Frobenius[5]) A two rowed array of non negative integers

$$\begin{pmatrix} a_1 & a_2 & \cdots & a_r \\ b_1 & b_2 & \cdots & b_r \end{pmatrix}$$

where $a_1 > a_2 > \cdots > a_r \geq 0,\ b_1 > b_2 > \cdots > b_r \geq 0$ and

$$\nu = r + \sum_{i=1}^{r} a_i + \sum_{i=1}^{r} b_i$$

is called Frobenius partition of ν.

Example 1. The Frobenius notation for $n = 25 = 7 + 7 + 6 + 5$ is given below;

$$\begin{pmatrix} 6 & 5 & 3 & 1 \\ 3 & 2 & 1 & 0 \end{pmatrix}$$

3.3 An Analogue of the Euler's Partition Identity

Theorem 3.3.1: Let $C(O, \nu)$ denote the number of compositions of ν into odd parts and $C^m(O, \nu)$ denote the number of compositions of ν into exactly m odd parts. Let OE_ν denote the number of "odd-even"partitions with largest part ν and OE_ν^m denote the number of "odd-even" partitions with largest part ν into exactly m parts. Then,

$$C^m(O, \nu) = OE_\nu^m \tag{3.1}$$

and

$$C(O, \nu) = OE_\nu \tag{3.2}$$

Proof (combinatorial). Consider a graph of an "odd-even" partition π into m parts with largest part n such that each part 'a' is represented by a row of 'a' dots.
Draw vertical lines from the corner point of each row and measure the distance of each line from its preceding one taking y-axis also into consideration. Since π is an "odd-even" partition (that is, its part alternate in parity starting with the smallest part odd), these distances are all of odd lengths and sum up to the largest part n. Consequently, these distances give rise to a composition of n into exactly m odd parts. Since the correspondence is one-to-one therefore we get

$$C^m(O, \nu) = OE_\nu^m$$

and

$$C(O, \nu) = OE_\nu.$$

Remark 1. We see that (3.2) is very similar in structure to Euler's identity. we call (3.2) an analouge of Euler's identity for compositions.

Remark 2. Obviously (3.2) is an immediate consequence of (3.1).

Remark 3. From equation(1.4) it follows easily that

$$\begin{aligned} \sum_{\nu=0}^{\infty} C^m(O, \nu) =& (q^1 + q^3 + \cdots)^m \\ =& \frac{q}{1 - q - q^2} \end{aligned} \tag{3.3}$$

from the last theorem

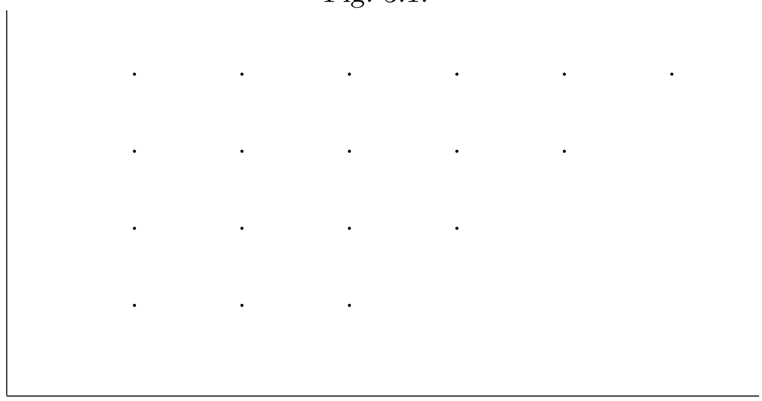
$$\begin{aligned} \sum_{\nu=0}^{\infty} OE_\nu q^\nu =& \sum_{\nu=0}^{\infty} C(O, \nu) q^\nu \\ =& \sum_{m=1}^{\infty} \frac{q^m}{(1 - q^2)^m} \end{aligned} \tag{3.4}$$

hence

$$OE_\nu = C(O,\nu) = F_\nu. \tag{3.5}$$

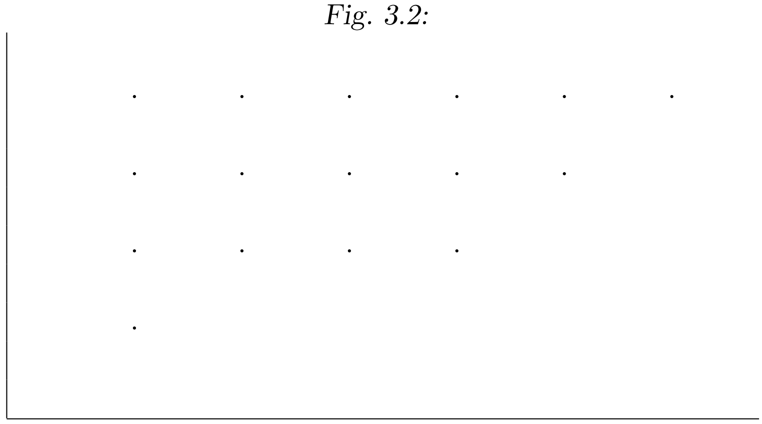
Example 2. Consider $\nu = 6$, $m = 4$, the relevant 'odd-even' partitions OE_6^4 and there respective compositions given by $C^4(O,\nu)$ are as follows:
The one to one correspondence of $6+5+4+3$ into $3+1+1+1$ is shown in figure below

Fig. 3.1:



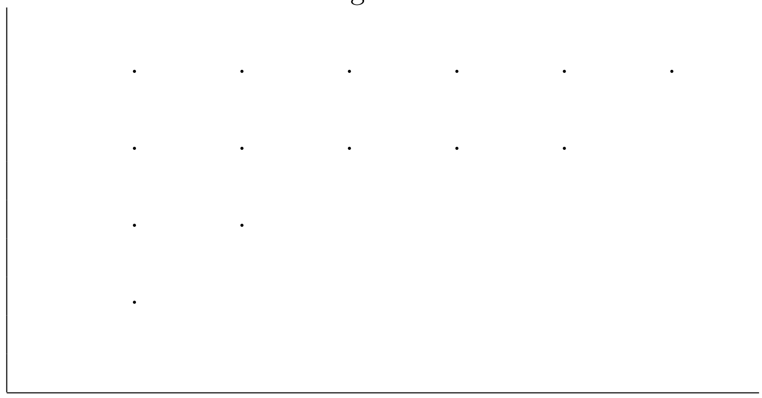
similarly, $6+5+4+1 \to 1+3+1+1$

Fig. 3.2:



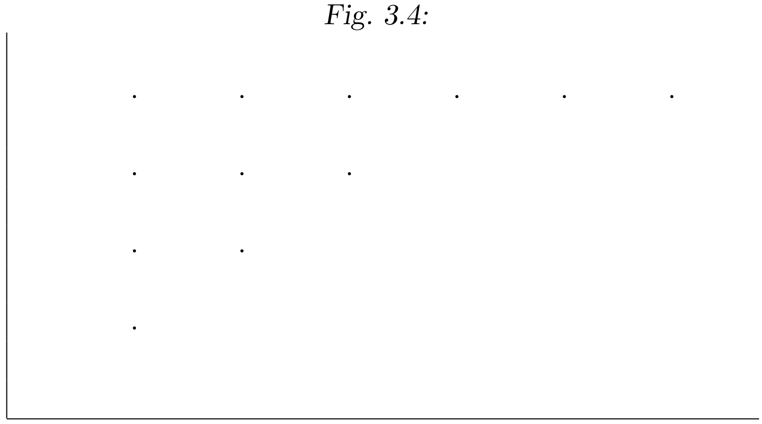
and, $6+5+2+1 \to 1+1+3+1$

Fig. 3.3:



lastly, $6+3+2+1 \to 1+1+1+3$

Fig. 3.4:



3.4 New Combinatorial Properties of n-Color Compositions

We shall now prove the following results:

Theorem 3.4.1: The number of n-color compositions of ν equals the number of "odd-even" partitions with largest part 2ν.

Theorem 3.4.2: The number of n-color compositions of ν equals the number of compositions of ν into odd parts.

Theorem 3.4.3: Let Δ_ν denote the number of "odd-even" partitions with largest part odd and $\leq 2\nu - 1$. Then Δ_ν equals the number of n-color compositions of ν.

Theorem 3.4.4: The number of n-color compositions of ν equals the number of self-conjugate partitions with largest part ν such that in the Frobenius notation

$$\begin{pmatrix} a_1 & a_2 & \cdots & a_r \\ a_1 & a_2 & \cdots & a_r \end{pmatrix}, a_i$$

alternate in parity.

Theorem 3.4.5: The number of n-color compositions ν equals of partitions into an even number of odd parts with largest part $4\nu-1$ such that the parts are alternately $\equiv 3$ and $1(\mathrm{mod}4)$.

proof of Theorem 3.4.1

From Chapter 2, from(2.4) we have,

$$c(\nu) = F_{2\nu}.$$

Also from Remark 3 of Theorem 3.3.1 we have,

$$OE_{\nu} = C(O,\nu) = F_{\nu}.$$

Replacing n by 2ν in above, we get,

$$OE_{2\nu} = C(O,2\nu) = F_{2\nu} = c(\nu). \tag{3.6}$$

Therefore,

$$OE_{2\nu} = c_{\nu}.$$

Hence the result.

Proof of Theorem 3.4.2

Proceeding same as in Theorem 3.4.1, from(3.6), we have,

$$C(O,2\nu) = c(\nu).$$

Proof of Theorem 3.4.3

We have to prove

$$\Delta_{\nu} = c(\nu) \tag{3.7}$$

also from Chapter 2, from(2.4) we have,

$$c(\nu) = F_{2\nu} \tag{3.8}$$

also from (3.5) we have

$$OE_{\nu} = F_{\nu} \tag{3.9}$$

Also using the Property2 Chapter1 Section1.4.2 of Fibonacci numbers

$$F_1 + F_3 + ... + F_{2\nu-1} = F_{2\nu} \tag{3.10}$$

from (3.8), (3.9), (3.10) we get

$$OE_1 + OE_3 + \cdots + OE_{2\nu-1} = c\nu$$
$$\Delta_\nu = c(\nu).$$

Example 3. For $\nu = 3$, the relevant eight partitions are

$$11+9,\ 11+5,\ 11+1,\ 11+9+7+5,\ 11+9+7+1,\ 11+9+3+1,$$
$$11+5+3+1,\ 11+9+7+5+3+1.$$

Proof of Theorem 3.4.4

We establish a bijection between the "odd-even" partitions with the largest part 2ν and the self-conjugate partitions with the largest part 2ν such that in the Frobenius notation

$$\begin{pmatrix} a_1 & a_2 & \cdots & a_r \\ a_1 & a_2 & \cdots & a_r \end{pmatrix},$$

a_i alternate in parity.

Let

$$\pi = a_1 + a_2 + \cdots a_r (2\nu = a_1 > a_2 > \cdots a_r)$$

be an "odd-even" partition with the largest part 2ν. We consider a graph which consists of r successive bends viz., a_1-bend, a_2-bend,$\cdots$ a_r-bend.Here by a k-bend we mean a right-bend containing k dots in the first row as well as in the first column. We see that the graph represents a self-conjugate partition with largest part equal to 2ν such that in the Frobrnius notation

$$\begin{pmatrix} a_1 & a_2 & \cdots & a_r \\ a_1 & a_2 & \cdots & a_r \end{pmatrix},$$

a_i alternate in parity. The correspondence is one-to-one.
Hence the result.

Example 4 For $\nu = 4$, let us consider $\pi = 8 + 7 + 4 + 1$. Then

0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0
0 0 0 0 0 0
0 0 0 0
0 0 0
0 0 0
0 0
0 0

which is a graph of the self-conjugate partition of $8+8+6+4+3+3+2+2$ with the largest part as 8 in such a way that in Frobenius notation

$$\begin{pmatrix} 7 & 6 & 3 & 0 \\ 7 & 6 & 3 & 0 \end{pmatrix},$$

a_i alternate in parity.

Proof of Theorem 3.4.5 Straightening the right-bends in the graph of a self-conjugate partitions of theorem (3.4) we get the number of partitions into an even number of parts with largest part $4\nu - 1$ such that the parts are alternately $\equiv$ 3 and 1(mod4).
Hence the result.

Example 5 The self-conjugate partition $8 + 8 + 6 + 4 + 3 + 3 + 2 + 2$ of Example5 corresponds to the partition $15 + 13 + 7 + 1$ which is a partition of type described in Theorem 3.4.5

Conclusion: In this chapter we have studied the new combinatorial properties of n-color compositions. The idea developed have been further used in obtaining several new combinatorial properties of the n-Color Compositions.

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