Numerical Solutions of Some Partial Differential Equations Using Galerkin-Finite Element Method

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under

the guidance of

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to the

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INDIA

DEDICATED

TO

GOD, MY PARENTS AND MY TEACHERS

CONTENTS

CERTIFICATE

I hereby certify that the work which is being presented in the thesis entitled "**Numerical Solutions of Some Partial Differential Equations Using Galerkin-Finite Element Method**" which is being submitted for the award of degree of master of Science, School of Mathematics and Computer Applications, Thapar University, Patiala is an authentic record of my own work carried out under the supervision of Dr. Ram Jiwari.

The matter presented in the thesis has not been submitted for the award of any other degree of this or any other university.

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This is to certify that the above statement made by the candidate is correct and true to the best of my knowledge.

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ABSTRACT

In this thesis an attempt has been made to solve some parabolic partial differential equations by using finite differences methods. The chapter wise summary of the thesis is as follows

In chapter 2, we consider one-dimensional advection-diffusion parabolic partial differential equation:

$$\frac{\partial u}{\partial t}+c\frac{\partial u}{\partial x}=D\frac{\partial^2 u}{\partial x^2}\ , \qquad 0<x<L,\ 0<t<T$$

The advection–diffusion equation is a parabolic partial differential equation, which describes physical phenomena where energy is transformed inside a physical system due to two processes: advection and diffusion. In this chapter we have developed some finite difference schemes based on weighted average for solving the one dimensional advection–diffusion equation with constant coefficients. In this article, Galerkin-finite element method is proposed to find the numerical solutions of advection-diffusion equation. The equation is generally used to describe mass, heat, energy, velocity, vorticity etc.

In this chapter, Galerkin finite element method is proposed to find the numerical solutions of advection-diffusion equation. In the first step semi discrete finite element model is developed and secondly, time derivative is discritized by weighted average method. Finally, by choosing $\theta = 1/2$ the system is solved by Gauss elimination method. As test problem, three different solutions of advection-diffusion equation are chosen. Maximum absolute errors norm L_∞ are calculated and found that the errors are small and good.

In chapter 3, we consider one-dimensional quasi-linear parabolic partial differential equation:

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}=\frac{1}{\mathrm{Re}}\frac{\partial^2 u}{\partial x^2}, \qquad (x,t)\in\Omega$$

The nonlinear partial differential equation is a homogenous quasi-linear parabolic partial differential equation which encounters in the theory of shock waves, mathematical modelling of turbulent fluid and in continuous stochastic processes. Such type of partial differential equation is introduced by Bateman in 1915 and he proposes the steady-state solution of the problem. In 1948, Burger use the nonlinear partial differential equation to capture some features of turbulent fluid in a channel caused by the interaction of the opposite effects of convection and diffusion, later on it is referred as Burgers' equation. The structure of Burgers' equation is similar to that of Navier-Stoke's equations due to the presence of the non-linear convection term and the occurrence of the diffusion term with viscosity coefficient. The study of the general properties of the Burgers' equation has attracted attention of scientific community due to its applications in the various fields such as gas dynamics, heat conduction, elasticity, etc

In this chapter, a numerical algorithm for the solution of the burger's equation based on Galerkin method employing linear finite elements is developed. The performance of this algorithm is investigated b comparing solutions to two well known problems with data available in literature. The new method produces highly accurate numerical solutions for burger's equation even for small value of viscosity coefficient. The method does, in fact, produce more accurate results then many of the other methods.

Chapter 1

Introduction

1.1 Numerical Solution of Partial Differential Equations

Partial differential equations (PDEs) form the basis of very many mathematical models of physical, chemical and biological phenomena, and more recently their use has spread into economics, financial forecasting, image processing and other fields. The vast majority of PDEs model cannot be solved analytically. So, to investigate the predictions of PDE models of such phenomena it is often necessary to approximate their solution numerically. In most cases, the approximate solution is represented by functional values at certain discrete points (grid points or mesh points). There seems a bridge between the derivatives in the PDE and the functional values at the grid points. The numerical technique is such a bridge, and the corresponding approximate solution is termed the numerical solution. Currently, there are many numerical techniques available in the literature. Among them, the finite difference (FD), finite element (FE), and finite volume (FV) methods fall under the category of low order methods, whereas spectral and pseudo spectral methods are considered global methods. Sometimes the latter two methods are considered as subsets of the method of weighted residuals.

1.1.1 Finite Difference Methods

Finite difference methods are widely dominant in the numerical solution of PDEs and their application. The finite difference (FD) methods are based on the Taylor series expansion or the polynomial approximation. A finite difference method proceeds by replacing the partial derivatives in the PDEs by finite difference approximations. This gives a large algebraic system of equations to be solved in place of the partial differential equation. That is, the partial derivatives in PDEs are written in terms of discrete quantities of dependent and independent variables, resulting in simultaneous algebraic equations with all unknowns prescribed at discrete mesh points or grid points for the entire domain. Appropriate types of differencing schemes and suitable methods of solution are chosen in different applications. For example, in fluid dynamics applications, depending upon the particular physics of the flows, which may include in viscid, viscous, incompressible, compressible, irrigational, rotational, laminar, turbulent, supersonic, or hypersonic flows, finite difference schemes are written to conform to these different physical phenomena. The formulation of FD methods in one dimensional is simple but for multidimensional problems, meshes must be structured in either two or three dimensions. Curved meshes must be transformed into orthogonal Cartesian meshes. The challenge in analyzing finite difference methods for new classes of problems is often to find an appropriate definition of stability that allow one to prove convergence and to estimate the error in approximation.

Finite difference methods discredited the governing PDE directly using their strong form. Although it is most straight forward way to obtain the discrete system equations, but it is difficult to handle the typical boundary conditions. For a problem domain with complex geometry, the discretization of the geometry and the application of the natural and essential boundary conditions can seldom be done automatically by a computer program with no human involvement.

1.1.2 Finite Volume Method

Finite volume methods (FVMs) form a relatively general class of discretizations for certain types of partial differential equations. These methods start from balance equations over local control volumes, e.g., the conservation of mass in diffusion problems. When these conservation equations are integrated by parts over each control volume, certain terms yield integrals over the boundary of the control volume. For example, mass conservation can be written as a combination of source terms inside the control volume and fluxes across its boundary. Of course the fluxes between neighboring control volumes are coupled. If this natural coupling of boundary fluxes is included in the discretization, then the local conservation laws satisfied by the continuous problem are guaranteed to hold locally also for the discrete problem. This is an important aspect of FVMs that makes them suitable for the numerical treatment of, e.g., problems in fluid dynamics. Another valuable property is that when FVMs are applied to elliptic problems that satisfy a boundary maximum principle, they yield discretizations that satisfy a discrete boundary maximum principle even on fairly general grids.

FVMs were proposed originally as a means of generating finite difference methods on general grids. Today, however, while FVMs can be interpreted as finite difference schemes, their convergence analysis are usually facilitates by the construction of a related finite element method and a study of its convergence properties. The fundamental idea of the finite volume method can be implemented in various ways in the construction of the control volumes, in the localization of the degree of freedom, and in the discretization of the fluxes through the boundaries of the control volumes. There are two basically two classes of FVM. First, in *cell-centered* methods each control volume that surrounds a grid point has no vertices of the original triangulation lying on its boundary. The second approach, *vertex-centered* methods, uses vertices of the underlying triangulation as vertices of control volumes.

1.1.3 Method of Weighted Residuals

The methods of weighted residuals are the approximate methods which determine the solution of the differential equation in the form of functions which are closed in some sense to the exact solution. Consider a differential equation

$$\ell(u) = 0 \tag{1.1}$$

with initial condition, $I(u) = 0$, and boundary condition, $S(u) = 0$. The solution of differential equation, $U(x)$ is approximated by a finite series of functions $\phi_k(x)$ as follows:

$$U(x) = U_0(x) + \sum_{k=1}^{N} a_k \, \phi_k(x) \tag{1.2}$$

Where $\phi_k(x)$ are the basis or trial functions, a_k are the coefficients to be determined that satisfy the differential equation, and N are the number of functions. The form of $U_0(x)$ is chosen to satisfy the boundary and the initial conditions exactly. There is another approach in which exact solutions of the differential equation are known and these are added together to and the boundary conditions are satisfied satisfy the boundary conditions approximately. It is also possible to formulate a method in which the differential equation approximately.

In general, the approximate solution does not satisfy the partial differential equation exactly, and substituting its value results in a residual, *R*,

$$R(x, a_1, a_2, ..., a_N) = \ell(U(x)) \tag{1.3}$$

Which in turn is minimized in some sense? For a given N the $a_k's$ are chosen by requiring that an integration of the weighted residual over the domain is zero. Thus

$$\langle W_k(x), R \rangle = 0. \tag{1.4}$$

By letting $k = 1, 2, ...N$ a system of equations involving only $a_k's$ is obtained. For unsteady partial differential equation this would be a system of ordinary differential equations, for steady problems a system of algebraic equations obtained. Different choices of $W_k(x)$ give rise to the different methods within the class. Some of these methods are:

Galerkin Method

One of the most important weighted residual methods was invented by the Russian mathematician Boris Grigoryevich Gale kin. In the Gale kin method the weighting functions are chosen to be

$$W_k(x) = \phi_k(x) \tag{1.5}$$

i.e., the weighting functions are from the same family as the trial function in equation (1.2). Thus the residual becomes orthogonal to the space spanned by the trial functions.

In traditional Galerkin method each of the trial functions should satisfy the boundary condition but in spectral Tau method the trail functions need not satisfy the boundary condition instead, a supplementary set of equations is used to apply the boundary condition. A generalization of Galerkin method is Petrov-Galerkin method, in which, the weighting functions are different from trial functions.

Collocation Method

One approach to determining the function $U(x)$ is to require that this function satisfy the differential equation at some finite set of collocation points. i.e.

$$W_k(x) = \delta(x - x_k) \tag{1.6}$$

where δ is the Dirac delta function. Thus the collocation method sets $R = 0$ at $(x - x_k)$. Since there are N free parameters and also K boundary conditions (say) that need to be satisfied, we can only hope to satisfy the differential equation at some set of $N - K$ collocation points. This may yield a dense system of equations unless basis function is chosen carefully.

Sub-domain Method

This method can be considered a modification of the collocation method. The idea is to force the weighted residual to zero not just at fixed points in the domain, but over various subsections of the domain. To accomplish this, the weight functions are set to unity, and the integral over the entire domain is broken into a number of sub-domains sufficient to evaluate all unknown parameters.

Least Square Method

The basic idea of Least-Square is that the residual is minimized in a certain norm. The inner product of the governing equations is constructed, which are then differentiated with respect to the nodal values of the variables. A general Least-Square formulation is the following minimization problem:

$$S = \min \int_X \left| R(x, a_1, a_2, ..., a_N) \right|^2 dx \tag{1.7}$$

In order to achieve a minimum of this scalar function, the derivatives of S with respect to all unknown parameters must be zero. That is,

$$\frac{\partial S}{\partial a_k} = 2 \int_X R(x) \frac{\partial R}{\partial a_k} dx = 0 \tag{1.8}$$

Or

$$\int_X R(x) \frac{\partial R}{\partial a_k} dx = 0 .$$

1.1.4 Differential Quadrature Method

The differential quadrature method (DQM) is a higher order numerical technique for solving partial differential equations. In the nineteen century, most of the numerical simulations of engineering problems can be carried out by the low order FD, FE, and FV methods using a large number of grid points. In some practical applications, however, numerical solutions of PDEs are required at only a few specified points in the physical domain. To achieve an acceptable degree of accuracy, low order methods still require the use of a large number of grid points to obtain accurate solutions at these specified points. In seeking an efficient discretization technique to obtain accurate numerical solutions using a considerably small number of grid points, Richard Bellman and his associates [14] introduced the method of differential quadrature in the early 1970s. The DQM, akin to the conventional integral quadrature method, approximates the partial derivative of a function at any location by a linear summation of all the function values along a mesh line. The key procedure in the differential quadrature application lies in the determination of the weighting coefficients. Initially, Bellman and his associates proposed two methods to compute the weighting coefficients for the first order derivative. The first method is based on an ill-conditioned algebraic equation system. The second method uses a simple algebraic formulation, but the coordinates of the grid points are fixed by the roots of the shifted Legendre polynomial. In earlier applications of the DQM, Bellman's first method was usually used because it allows the use of an arbitrary grid point distribution. However, since the algebraic equation system of this method is ill-conditioned, the number of the grid points usually used is less than 13. This drawback limits the application of the DQM.

1.1.5 Finite Element Method

Finite element method (FEM) represents a powerful and general class of techniques for the approximate solution of partial differential equations. The basic idea in the FEM is to find the solution of a complicated problem by replacing it by a simpler one. Since the actual problem is replaced by a simpler one in finding the solution, we will be able to find only an approximate solution rather than the exact solution. This method is mostly used for the accurate solution of complex engineering problems with abundant software available commercially. FEM was first developed in 1956 for the analysis of aircraft structural problems. Thereafter, within a decade, the potentialities of the method for the solution of different types of applied science and engineering problems were recognized. Over the years, the FEM technique has been so well established that today it is considered to be one of the best methods for solving a wide variety of practical problems efficiently. In fact, the method has become one of the active research areas for applied mathematicians. Based on the variational principle, basic procedures of the FEM include: obtaining functional (variational expressions) from corresponding differential equations, dividing interested region into small elements, constructing interpolation model for each element, assembling all elements' contributions to the global system, and finally solving the global-matrix problems. The systematic generality of FEM makes it possible to construct a general-purposed computer program for a wide range of problems. In this method, the region is divided into subregions (elements), which could be different shapes i.e. triangular, rectangular, curvilinear, ring, or infinite. In addition, mixed element shapes and / or different base-function orders can be used simultaneously in one problem, depending on required computational accuracy. Moreover, nonuniform unstructured meshes and adaptive meshing procedures can be employed to significantly improve the accuracy and efficiency of FEM programs. Furthermore, FEM scheme can be established not only by the variational method but also by the Galerkin method or the least squares method, so FEM can still be used even though a variational principle does not exit or cannot be identified. Boundary conditions can be easily applied once the mesh generation is done. However, the pre-and post-processes of the computed set up always play an important role for a good FEM program. Many researchers have been using finite element method for the solutions of PDEs since 1956. Gerisch *et al.* have used high-order linearly implicit two-step peer - finite element methods for time dependent PDEs successfully.

1.2 Methodology for Solving a Differential Equation by Finite Element Method

The three fundamental steps of the finite element method are:

- Divide the whole domain into parts. The domain of the problem is represented by collection of simple sub domains, called finite elements. The collection of finite element is called finite element mesh.
- Over each part, seek an approximation to the solution as linear combination of nodal values and approximation functions, and derive the algebraic relations among nodal values of solution over each part called nodes.
- Assemble the parts and obtain the solution to the whole using continuity of the physical quantities.

Example 1: Consider the differential equation

$$-\frac{d^2u}{dx^2} - u + x^2 = 0 \qquad , u(0)=0 \quad , \quad u'(1) = 1$$

For weighted residual method, ϕ_0 and ϕ_1 should satisfy the following conditions:

$$\phi_0(0) = 0, \quad \phi_0'(1) = 1$$
$$\phi_1(0) = 0, \quad \phi_1'(1) = 0$$

For the choice of algebraic polynomials, we assume $\phi_0(x) = a + bx$ and use 2 conditions

On ϕ_0 determine the constants a and b. we obtain

$$\phi_0(x) = x$$

Since there are two homogeneous conditions, we must assume at least three parameter polynomial to obtain a non-zero function, $\phi_1 = a + bx + cx^2$. Using the conditions on ϕ_i ,we obtain

$$\phi_1 = cx(2 - x)$$

The constant c can be set to unity, we assume

$$\phi_2 = a + bx + dx^3 \qquad or \qquad \phi_2 = a + cx^2 + dx^3$$

For second choice of ϕ_2, we obtain

$$\phi_2 = x^2\left(1 - \frac{2}{3}x\right)$$

The residual in the approximation of the equation is

$$R = -\left(0 + \sum_{i=1}^{N} c_i \frac{d^2\phi}{dx^2}\right) - \left(\phi_0 + \sum_{i=1}^{N} c_i\phi_i\right) + x^2$$
$$= c_1\left(2 - 2x + 2x^2\right) + c_2\left(-2 + 4x - x^2 + \frac{2}{3}x^3\right) - x + x^2$$

Taking $\psi_i = \phi_i$, we have

$$\int_0^1 x(2-x)R\,dx = 0, \qquad \int_0^1 x^2\left(1 - \frac{2}{3}x\right)R\,dx = 0$$

$$\frac{4}{5}c_1 + \frac{28}{45}c_2 - \frac{7}{60} = 0, \qquad \frac{17}{90}c_1 + \frac{29}{315}c_2 - \frac{1}{36} = 0$$

Hence, the solution becomes $\left(with\ c_1 = \frac{623}{4306}, c_2 = \frac{21}{4306}\right)$,

$$U_G = 1.2894x - 0.1398x^2 - 0.00325x^3$$

1.3 Organisation of Thesis

In this thesis an attempt has been made to solve some partial differential equations by using some Galerkin finite element methods. The chapter wise summary of the thesis is as follows.

In chapter 2, we consider one-dimensional convection-diffusion parabolic partial differential equation:

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = D\frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < L,\ 0 < t < T$$

The convection–diffusion equation is a parabolic partial differential equation, which describes physical phenomena where energy is transformed inside a physical system due to two processes: convection and diffusion. The term convection means the movement of molecules within fluids, whereas, diffusion describes the spread of particles through random motion from regions of higher concentration to regions of lower concentration.

In this chapter, Galerkin finite element method is proposed to find the numerical solutions of advection-diffusion equation. The equation is generally used to describe mass, heat, energy, velocity, vorticity etc. In the first step semi discrete finite element model is developed and secondly, time derivative is discritized by weighted average method. Finally, by choosing $\theta = 1/2$ the system is solved by Gauss elimination method. As test problem, three different solutions of advection-diffusion equation are chosen. Maximum absolute errors norm L_∞ are calculated and found that the errors are small and good.

In chapter 3, we consider one-dimensional quasi-linear parabolic partial differential equation:

$$\frac{\partial U}{\partial t} + u\frac{\partial U}{\partial x} - \nu\frac{\partial^2 U}{\partial x^2} = 0 \qquad (x,t) \in \Omega \times [0,T)$$

The nonlinear partial differential equation is a homogenous quasi-linear parabolic partial differential equation which encounters in the theory of shock waves, mathematical modelling of turbulent fluid and in continuous stochastic processes. Such type of partial differential equation is introduced by Bateman in 1915 and he proposes the steady-state solution of the problem. In 1948, Burger use the nonlinear partial differential equation to capture some features of turbulent fluid in a channel caused by the interaction of the opposite effects of convection and diffusion, later on it is referred as Burgers' equation. The structure of Burgers' equation is similar to that of Navier-Stoke's equations due to the presence of the non-linear convection term and the occurrence of the diffusion term with viscosity coefficient. The study of the general properties of the Burgers' equation has attracted attention of scientific community due to its applications in the various fields such as gas dynamics, heat conduction, elasticity, etc

In this chapter, Galerkin-finite element method is proposed for the numerical solution of Burgers' equation. A linear recurrence relationship is found for the numerical solution of resulting system of ordinary differential equations is found vai a Crank-Nocolson approach involving a product approximation. Two test examples are considered in order to check the accuracy of the proposed method. The results show that the proposed method is more accurate.

Chapter 2

Galerkin-Finite Element Method for the Numerical Solution of Advection-Diffusion Equation

2.1. Introduction

Consider the one-dimensional advection diffusion equation

$$\frac{\partial U}{\partial t}-\lambda\frac{\partial^2 U}{\partial x^2}+\alpha\frac{\partial U}{\partial x}\ ,\quad 0\le x\le L\quad ,t\succ 0 \tag{2.1}$$

with initial condition

$$U(x,0)=\phi(x)$$

and boundary condition

$$U(0,t)=f(t)\ ,\ U(L,t)=h(t)$$

where U represents concentration of the pollutant at point x, the advection coefficient α is the velocity of water flow and λ is the diffusion coefficient. The advection-diffusion equation is generally used to describe mass, heat, energy, velocity, and vorticity [1]. The equation has been used as a model equation in many chemistry and engineering problems such as, thermal pollution in river systems [2], flow in porous media [3], dispersion of tracers in porous media [4], the dispersion of dissolved material in estuaries and coastal seas [5], the intrusion of salt water into fresh water aquifers, the spread of pollutants in rivers and streams [6], forced cooling by fluids of solid material such as windings in turbo generators [7], the spread of solute in a liquid flowing through a tube, long-range transport of pollutants in the atmosphere [8], contaminant dispersion in shallow lakes [9], model water transport in soils [10], the absorption of chemicals into beds [11], etc. Isenberg and Gutfinger used the advection-diffusion equation to describe heat transfer in a draining film [12]. Mortan has used the advection-diffusion equation to model some economics and financial forecasting [13]. Besides this, the equation has great importance in civil engineers and hydrologists. Thus, the advection-diffusion equation is very interesting linear partial differential equation from numerical study point of view.Many researchers have developed numerical techniques to study the numerical solutions of advection-diffusion equation. Celia et al [14] have proposed an Eulerian-langrangian localized adjoint method for the advection-diffusion equation. Spalding [15] proposed the hybrid scheme which is a combination of three straight lines to correlate the exact curve. Kakuda and Tosaka [8] used time splitting or fractional steps method for advection-diffusion and viscous fluid flows problems. Nico et al [12] proposed a finite volume upwind scheme for the solution of the linear advection-diffusion equation with sharp gradients in multiple dimensions for the solution of advection-diffusion equation while Neubauer and Bastian [11] used a monotonicity preserving Eulerian-Lagrangian localized adjoint method for advection-diffusion equations.

In this chapter, Galerkin finite element method is proposed to find the numerical solutions of advection-diffusion equation. The equation is generally used to describe mass, heat, energy, velocity, vorticity etc. In the first step semi discrete finite element model is developed and secondly, time derivative is discritized by weighted average method. Finally, by choosing $\theta = 1/2$ the system is solved by Gauss elimination method. As test problem, three different solutions of advection-diffusion equation are chosen. Maximum absolute errors norm L_∞ are calculated and found that the errors are small and good.

2.2. Semi Discrete Finite Element Models

The semi discrete formulation involves approximation of the spatial variation of the dependent variable. The first step involves the construction of the weak form of the given problem over a typical element. In second step, we develop the finite element model by seeking approximation of the solution.

2.2.1. Weak Formulation of the Problem

The weak formulation of the given problem (2.1) over a typical linear $(x_a, x_b)$is given by

$$\int_{x_a}^{x_b} w(x)\left\{\frac{\partial U}{\partial t}-\lambda\frac{\partial^2 U}{\partial x^2}+\alpha\frac{\partial U}{\partial x}\right\}dx = 0; \tag{2.2}$$

where w(x) are arbitrary test functions and may be viewed as the variation in U(x). After reducing the order of integration, we arrive at the following system of equations

$$\int_{x_a}^{x_b} w(x)\left\{\frac{\partial U}{\partial t}+\lambda\frac{\partial^2 U}{\partial x^2}+\alpha\frac{\partial U}{\partial x}\right\}dx = 0; \tag{2.3}$$

2.2.2. Finite Element Formulation of the Problem

The finite-element model may be obtained from equations (2.3) by substituting finite element approximations in the decoupled form

$$U(x,t)=\sum_{j=1}^{N} U_j^{\,e}(t_s)\psi_j^{\,e}(x); \qquad s = 1,2,....... \tag{2.4}$$

Substituting w = $\psi_i(x)$ and (2.4) in equation (2.3) to obtain the *ith* equation of the system, we have

$$\int_{x_a}^{x_b}\left[\psi_i\left(\sum_{j=1}^{N}\frac{U_j}{dt}\psi_j\right)+\alpha\psi_i\left(\sum_{j=1}^{N}U_j\frac{d\psi_j}{dx}\right)+\lambda\frac{d\psi_i}{dx}\left(\sum_{j=1}^{N}U_j\frac{d\psi_j}{dx}\right)\right]dx = 0; \tag{2.5}$$

$$\sum_{j=1}^{N}\left[\left(\int_{x_a}^{x_b}\psi_i\psi_j dx\right)\frac{dU_j}{dt}+\alpha\left(\int_{x_a}^{x_b}\psi_i\frac{d\psi_i}{dx}dx\right)U_j+\lambda\left(\int_{x_a}^{x_b}\frac{d\psi_i}{dx}\frac{d\psi_j}{dx}dx\right)U_j\right]=0 \qquad (2.6)$$

The equation (2.6) can be written in the matrix form

$$[M]\{\dot{U}\}+[K^1]\{U\}+[K^2]\{U\}=0 \qquad (2.7)$$

where

$$M_{ij}=\int_{x_a}^{x_b}\psi_i\psi_j dx \qquad and \qquad \psi_1^e=1-\frac{\bar{x}}{h_e} \quad and \quad \psi_2^e=\frac{\bar{x}}{h_e}$$

$for \quad (i=1=j)$

$$M_{11}=\int_{x_a}^{x_b}\psi_1\psi_2 dx=\int_{x_a}^{x_b}\left(1-\frac{\bar{x}}{h}\right)\left(1-\frac{\bar{x}}{h}\right)d\bar{x}=\frac{h}{3}$$

$for \quad (i=1,\ j=2 \quad and \quad j=1,\ i=2)$

$$M_{12}=\int_{X_A}^{X_B}\left(1-\frac{\bar{x}}{h}\right)\frac{\bar{x}}{h}d\bar{x}=M_{21}=\frac{h}{6}$$

$for \quad (i=2=j)$

$$M_{22}=\int_{x_a}^{x_b}\frac{\bar{x}}{h}\frac{\bar{x}}{h}d\bar{x}=\frac{h}{6}$$

$$M_{ij}=\frac{h}{6}\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix} \qquad (2.8)$$

$$K_{ij}^{1} = \int_{x_a}^{x_b} \psi_i \frac{d\psi_j}{dx} dx$$

$for(i = 1 = j)$

$$K_{11}^{1} = \int_{x_a}^{x_b} \left(1 - \frac{\bar{x}}{h}\right)\left(-\frac{\bar{x}}{h}\right) d\bar{x} = -\frac{\alpha}{2}$$

$for(i = 1, j = 2)$

$$K_{12}^{1} = \int_{x_a}^{x_b} \left(1 - \frac{\bar{x}}{h}\right)\frac{1}{h} d\bar{x} = \frac{\alpha}{2}$$

$for(i = 2, j = 1)$

$$K_{21}^{1} = \int_{x_a}^{x_b} -\frac{1}{h}\left(\frac{\bar{x}}{h}\right) d\bar{x} = \frac{-\alpha}{2}$$

$for(i = 2 = j)$

$$K_{22}^{1} = \int_{x_a}^{x_b} \frac{\bar{x}}{h}\frac{1}{h} d\bar{x} = \frac{\alpha}{2}$$

so ,

$$K_{ij}^{e} = \frac{\alpha}{2}\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} \tag{2.9}$$

$$K_{ij}^{2} = \int_{x_a}^{x_b} \frac{d\psi_i}{dx}\frac{d\psi_j}{dx} dx$$

$for(i = 1 = j)$

$$K_{11}^{2} = \int_{x_a}^{x_b} \frac{-1}{h}\frac{-1}{h} d\bar{x} = \frac{\lambda}{h}$$

$for(i = 2 = j)$

$$K_{22}^{2} = \int_{x_a}^{x_b} \frac{1}{h}\frac{1}{h} d\bar{x} = \frac{\lambda}{h}$$

$for(i = 1, j = 2, j = 1, i = 2)$

$$K_{12}^{2} = K_{21}^{2} = \int_{x_a}^{x_b} \frac{-1}{h}\frac{1}{h} d\bar{x} = -\frac{\lambda}{h}$$

$$K_{ij}^{2} = \frac{\lambda}{h}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \tag{2.10}$$

$$K = K_{ij}^{1} + K_{ij}^{2}$$

$$= \frac{\alpha}{2}\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} + \frac{\lambda}{h}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Hence the system of equation can be written as

$$[M]\left\{\dot{U}\right\} + [K]\{U\} = 0 \qquad (2.11)$$

Where [K] = [K1] + [K2].

We use the linear piecewise approximation in the space variable and the Galerkin method to obtain the semi discrete approximation to equation (2.1)

where

$$\psi_{i-1}(x) = \frac{x - x_{i-1}}{x_i - x_{i-1}}, \qquad \psi_i(x) = \frac{x_i - x}{x_i - x_{i-1}} \qquad (2.12)$$

We have used the linear piecewise approximation (2.12) and (2.13) to find out the integral in the equation (2.10). Then, the system (2.11) become

$$[M]\left\{\dot{U}\right\} + [K]\{U\} = 0 ,$$

Where

$$[M] = \frac{h}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \quad [K] = \frac{\alpha}{2}\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} + \frac{\lambda}{h}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix},$$

$$\left\{\dot{U}\right\} = \begin{bmatrix} \dot{U}_{i-1} \\ \dot{U}_i \end{bmatrix}, \quad \{U\} = \begin{bmatrix} U_{i-1} \\ U_i \end{bmatrix}. \qquad (2.13)$$

2.3. Fully Discretized Finite Element Equations

We have the system of ordinary differential equations as follows

$$[M]\left\{\dot{U}\right\} + [K]\{U\} = 0 \qquad (2.14)$$

subject to the initial condition

$$\{U\}_0 = \phi(x) = \{U\}_0 , \qquad (2.15)$$

where $\{U\}_0$ denotes the vector of nodal values of U at time t = 0 whereas $\{U\}_0$ denotes the column of nodal values U_{j_0} .

As applied to a vector of time derivatives of the nodal values the weighted average of approximation on the equation (14), we have

$$[M]\left(\frac{\{U\}_{S+1}-\{U\}_S}{\Delta t}\right)+\theta[K]\{U\}_{S+1}+(1-\theta)[K]\{U\}_S=0 \tag{2.16}$$

The equation (2.16) can be written in simple form as

$$([M]+\Delta t\theta[K])\{U\}_{S+1}=[M]\{U\}_S-\Delta t(1-\theta)[K]\{U\}_S\ . \tag{2.17}$$

The algebraic system (2.17) is solved by Gauss elimination method by taking Crank-Nicolson Scheme i.e. $\theta=1/2$ in equation (2.17).

2.4. Numerical Experiment and Discussion

In this section, we have studied three test examples to check the applicability of the proposed numerical scheme based on finite element method. In order to measure the accuracy of numerical solutions, difference between analytic and numerical solutions at some specified times are computed by using maximum error norm L_∞.

$$L_\infty=\left|U^{exact}-U^{num}\right|_\infty=\max_j\left|U_j^{exact}-U_j^{num}\right|. \tag{2.18}$$

where U^{exact} and U^{num} are exact and numerical solution respectively.

Example 1: In the first test example, the advection-diffusion equation (2.1) is considered with domain $[0,9]$ and the analytical solution

$$U(x,t)=10\exp\left(\frac{-(x-x_0-\alpha t)^2}{2\rho_2}\right) \tag{2.18}$$

The initial and boundary conditions are taken from the analytical solution (18). We have considered $\rho = 264\,m$, $\alpha = 0.5\,m/s$ and $x_0 = 2$. In this example, we have considered two cases. In the first case, we study the purely advection equation by taking $\lambda = 0$. In second case, we take advection-diffusion equation. Figure 1 depicts the absolute errors at different time for first case. Table 1 shows the comparison of numerical and analytic solutions at different time and x with maximum absolute error for the second case. Figure 2 shows the absolute errors for the second case. The Table 1 and Figures 1-2 show that the proposed method has good accuracy.

Example 2: In the second test, the analytical solution of equation (2.1) is given by

$$U(x,t)=100\left(\frac{e^{\frac{Px}{L}-1}}{e^P-1}+\frac{4\pi e^{\frac{Px}{L}}\sinh(P/2)}{e^P-1}\sum_{m=1}^{\infty}A_m+2\pi e^{\frac{Px}{2L}}\sum_{m=1}^{\infty}B_m\right) \tag{2.19}$$

where

$$A_m=(-1)^m\frac{m}{\beta_m}\sin\left(\frac{m\pi x}{L}\right)e^{-\lambda_m t}$$

$$B_m=\left[(-1)^m\frac{m}{\beta_m}\left(1+\frac{P}{\beta_m}\right)e^{-P/2}+\frac{mP}{\beta_m^2}\right]\sin\left(\frac{m\pi x}{L}\right)e^{-\lambda_m t}$$

with

$$\beta_m=\frac{p^2}{4}+(m\pi)^2 \text{ and } \lambda_m=\frac{\alpha^2}{4\lambda}+\frac{m^2\pi^2\lambda}{L^2}$$

Where $P=\frac{\alpha L}{\lambda}$ is the Peclet number.

In the numerical experiment, we have considered the initial and boundary conditions

$$U(x,0)=\frac{100\,x}{L} \quad \text{and} \quad U(0,t)=0,\ U(L,t)=100 \text{ with } L=1.0\,m,\ \alpha=0.1\,m/s,\ \lambda=0.01\,m^2/s.$$

In Table 2, a comparison is made between the analytical solutions and the numerical solution with maximum absolute error. The Figure 3 shows the behaviour of the numerical solutions at different time and from Figure it is clear that as the time increase the profile behaviour of the wave decreases.

Example 3: The analytical solution of the equation (2.1) in the region bounded by $0 \le x \le 1$ is given by

$$U(x,t) = \frac{0.025}{\sqrt{0.000625 + 0.02t}} \exp\left(\frac{-(x - 0.5 - t)^2}{(0.00125 + 0.04t)} \right) \tag{2.20}$$

The initial and boundary conditions are taken from the analytical solution. The values of advection and diffusion coefficients are chosen by $\alpha = 1.0,\ \lambda = 0.01$. The Figures 4-7 show the behaviour of numerical solutions at different times.

2.5 Conclusion

In this chapter, Galerkin finite element method is proposed to find the numerical solutions of advection-diffusion equation. The equation is generally used to describe mass, heat, energy, velocity, vorticity etc. As test problem, three different solutions of advection-diffusion equation are chosen. Maximum errors norm L_∞ are calculated and found that the errors are small and good.

Table 1: Comparison of numerical and analytic solutions of Example 1 for $\alpha = 0.5,\ \lambda = 0.1$ with maximum absolute error.

T	x	Present Method	Exact Solution	Max Error
0.5	1.5	9.99991	9.99993	
	4.5	9.99970	9.99971	1.43479×10^{-5}
	7.5	9.99819	9.99821	
1.0	1.5	9.99968	9.99971	
	4.5	9.99989	9.99993	4.30570×10^{-5}
	7.5	9.99881	9.99885	
3.0	1.5	9.99930	9.99935	
	4.5	9.99993	10.0000	4.45166×10^{-4}
	7.5	9.99928	9.99935	

Table 2: Comparison of numerical and analytic solutions of Example 1 for $\alpha = 0.1,\ \lambda = 0.01$ with maximum absolute error.

T	x	Present Method	Exact Solution	Max Error
3.0	0.25	6.19441	6.19505	
	0.50	22.34930	22.3500	6.4125×10^{-4}
	0.75	46.15160	46.15206	
5.0	0.25	2.72582	2.72601	
	0.50	11.69460	11.69100	3.6142×10^{-4}
	0.75	30.63180	30.63030	
10.0	0.25	0.46021	0.46009	
	0.50	2.60798	2.60830	3.2890×10^{-4}
	0.75	12.78790	12.7904	

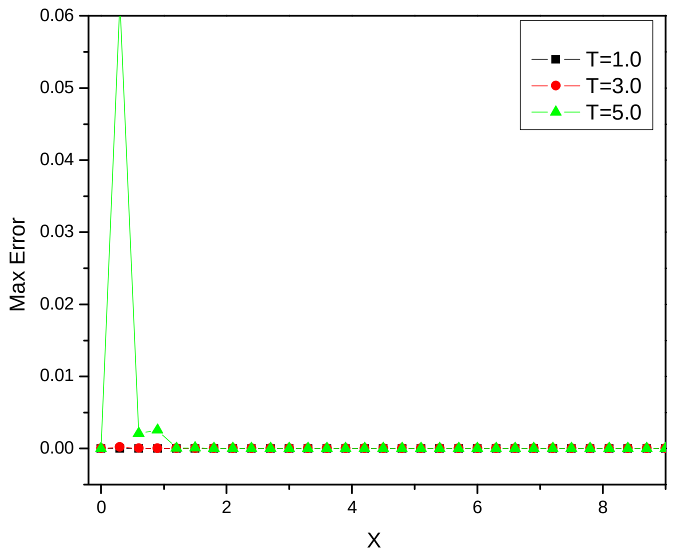

Figure 1: Maximum Errors in Example 1 for $\lambda = 0$ (purely advection equation) at different time.

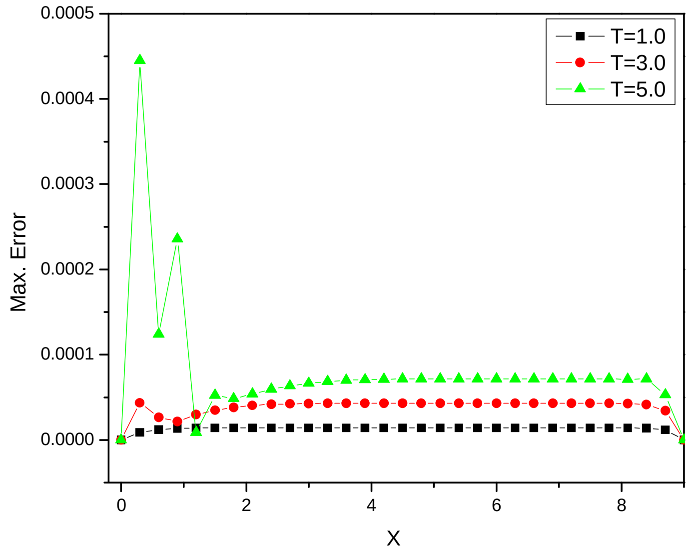

Figure 2: Maximum Errors in Example 1 for $\lambda = 0.1$ (for second case) at different time.

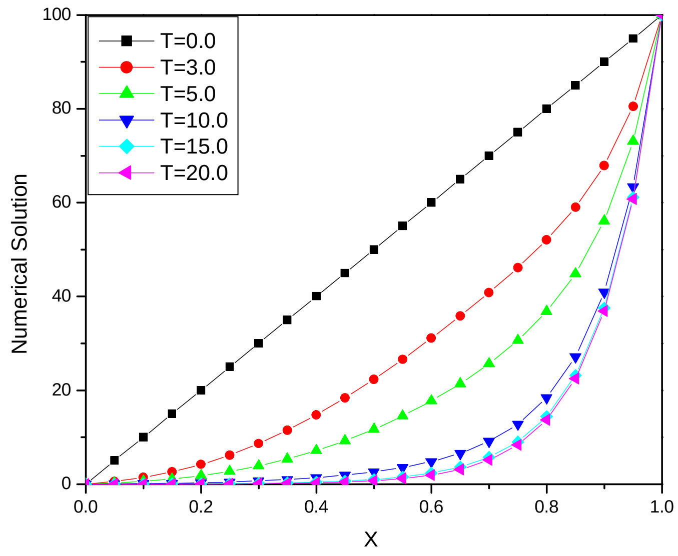

Figure 3: Numerical solutions of Example 2 for $\alpha = 0.1$, $\lambda = 0.01$ at different time.

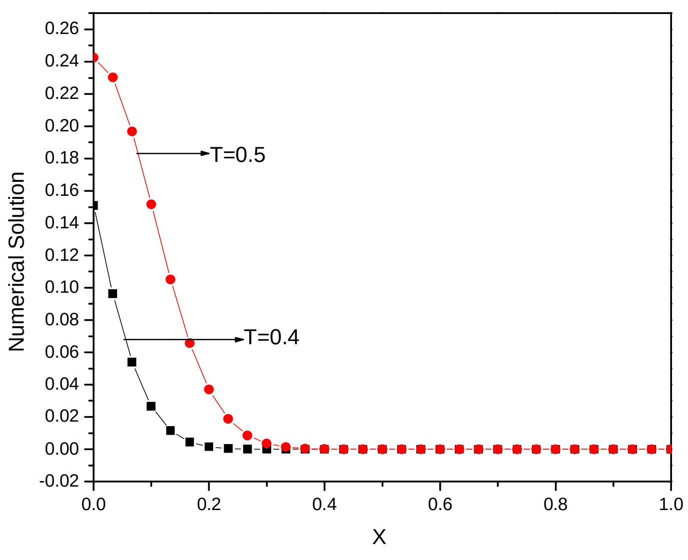

Figure 4: Numerical solutions of Example 3 at different time for $\alpha = 1.0$, $\lambda = 0.01$.

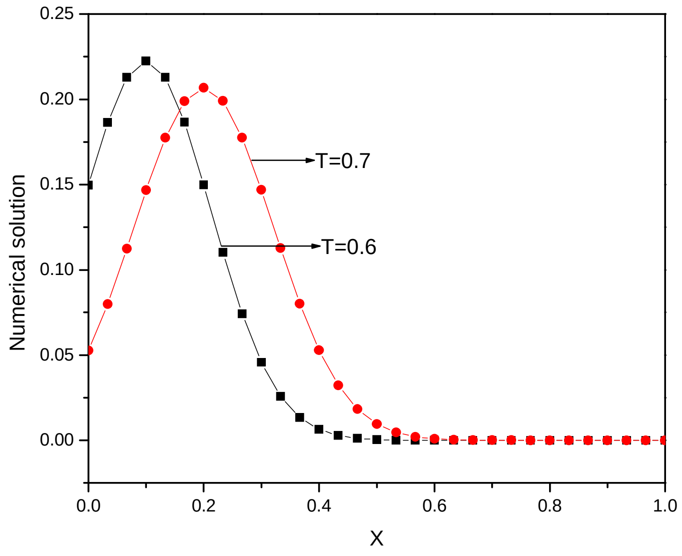

Figure 5: Numerical solutions of Example 3 at different time for $\alpha = 1.0,\ \lambda = 0.01$.

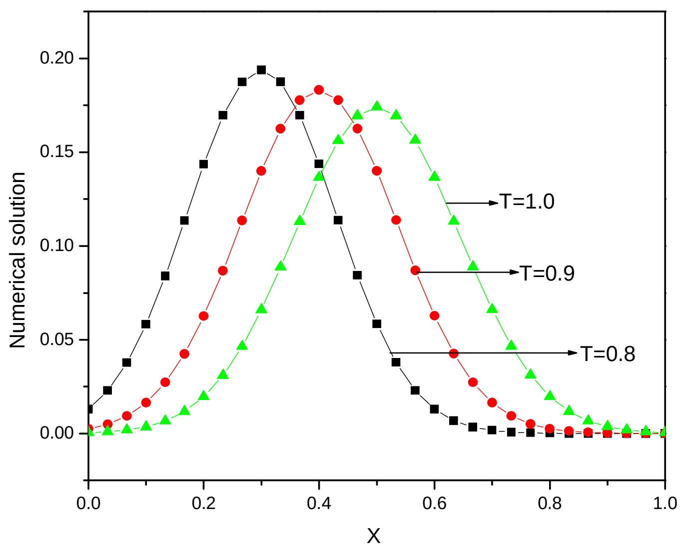

Figure 6: Numerical solutions of Example 3 at different time for $\alpha = 1.0,\ \lambda = 0.01$.

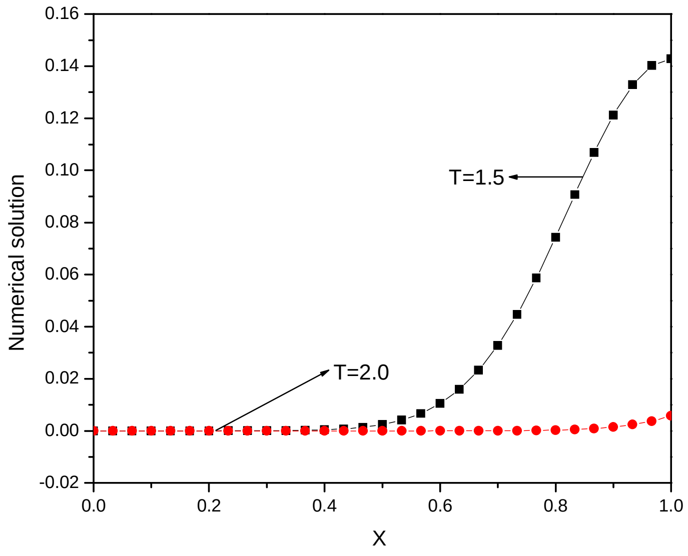

Figure 7: Numerical solutions of Example 3 at different time for $\alpha = 1.0,\ \lambda = 0.01$.

Chapter 3

Numerical Solution of Burger's Equation by Using Galerkin Finite Element Method

3.1 Introduction

Consider one-dimensional quasi-linear parabolic partial differential equation:

$$\frac{\partial U}{\partial t}+u\frac{\partial U}{\partial x}-\nu\frac{\partial^2 U}{\partial x^2}=0 \qquad (x,t)\in\Omega\times[0,T) \tag{3.1}$$

where

$$\Omega=(0,1)\times(0,t]$$

with initial condition

$$U(x,0)=f(x) \qquad 0<x<1 \tag{3.2}$$

and boundary conditions

$$U(0,t)=g_1(t) \qquad 0\le t\le T \tag{3.3}$$

$$U(1,t)=g_2(t) \qquad 0\le t\le T \tag{3.4}$$

where $\nu=\frac{1}{R}$ and R is the Reynolds number and $f, g_1 \, and \, g_2$ are the sufficiently smooth given functions.

The nonlinear partial differential equation (1) is a homogenous quasi-linear parabolic partial differential equation which encounters in the theory of shock waves, mathematical modeling of turbulent fluid and in continuous stochastic processes. Such type of partial differential equation is introduced by Bateman [16] in 1915 and he proposes the steady-state solution of the problem. In 1948, Burger use the nonlinear partial differential equation to capture some features of turbulent fluid in a channel caused by the interaction of the opposite effects of convection and diffusion, later on it is referred as Burgers' equation. The structure of Burgers' equation is similar to that of Navier-Stoke's equations due to the presence of the non-linear convection term and the occurrence of the diffusion term with viscosity coefficient. The study of the general properties of the Burgers' equation has attracted attention of scientific community due to its applications in the various fields such as gas dynamics, heat conduction, elasticity, etc.

The study of the solution of Burgers' equation has been carried out for last half Century and still it is an active area of research to develop better numerical schemes to approximate its solution. In 1965, Holf and Cole [18] propose a transformation known as Holf-Cole transformation to solve the Burgers' equation. In 1972, Benton and Platzman [19] published a number of distinct solutions to the initial value problems for the Burgers' equation in the infinite domain as well as in the finite domain. Caldwell and Smith [20] use finite difference and cubic spline finite element methods to solve Burgers' equation. Evans et al. [21] introduce the group-explicit method and Kakuda et al. [22] propose a generalized boundary element approach to solve Burgers' equation. Ali et al. [23] use a cubic B-spline finite element method based on a collocation formulation to solve Burgers' equation. Mittal et al. [24] present a numerical approximation based on one dimensional Fourier expansion with time dependent coefficients. Gardner et al. [25] apply Petrov-Galerkin method with quadratic B-spline spatial finite elements and use a least squares technique using linear space-time finite elements [26]. In [27], Ozis and Ozdes generate a sequence of approximate solutions based on variational approach which converges to the exact solution. In [28], Kutluay et al. transform the Burgers' equation to linear heat equation using Hopf-Cole transformation and then use explicit finite difference and exact explicit finite difference methods to solve the transformed linear heat equation with Neumann boundary conditions. In [29], Kutluay et al.

reduce Burgers’ equation to a pentadiagonal matrix system by applying the classical weighted residual method over the finite elements which is solved by a variant of Thomas algorithm together with an iteration process at each time step. Ozis et al. [30] use a finite element approach for numerical solution of Burgers' equation. Kadalbajoo et al. [31] propose a parameter uniform numerical method to solve Burgers’ equation with small coefficient of viscosity and establish robust error estimate. Kadalbajoo et al. [32] use Crank-Nicolson finite difference method on the transformed linear heat equation with Neumann boundary conditions and the method is proved to be unconditionally stable. Recently, Kannan and Wang [33] have developed a high order spectral volume method using the Hopf–Cole transformation for the numerical solution of Burgers' equation while Altiparmak and Özis [34] used factorized diagonal Padé approximation method for the numerical solution of Burgers' equation while Korkmaz and Dağ [9] proposed a numerical method for nonlinear Burgers’ equation.

Recently, Korkmaz and Dag [15-22] proposed sinc differential quadrature method, B-spline differential quadrature methods and cosine expansion based differential quadrature method for many nonlinear partial differential equations. Mittal have used polynomial based differential quadrature method for numerical solutions of some two dimensional nonlinear partial differential equations.

In this chapter, Galerkin-finite element method is proposed for the numerical solution of Burgers’ equation. A linear recurrence relationship is found for the numerical solution of resulting system of ordinary differential equations is found vai a Crank-Nocolson approach involving a product approximation. The results show that the proposed method is more accurate.

3.2. Galerkin-Finite Element Method for Numerical Solutions of Burgers’ Equation

The burger's equation

$$\frac{\partial U}{\partial t} - U\frac{\partial U}{\partial x} - v\frac{\partial^2 U}{\partial x^2} = 0 \qquad (3.5)$$

When applying Galerkin's method we minimise the functional

$$\int_{x_0}^{x_N}\left(\frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x} - v\frac{\partial^2 U}{\partial x^2}\right)\phi_i dx = 0 \qquad (3.6)$$

where ϕ_i is the weight function, with respect to nodal variables.

A numerical solutions to the partial differential equation is sought over the region $x_0 \le x \le x_N$ with boundary conditions specified at $x = x_0, x = x_N$ .the region $[x_0, x_N]$ is splitter up into uniformly sized intervals by x_i *such that* $x_0 < x_1 < .... < x_N$ . .A typical finite element of size $h = (x_{m-1} - x_m)$,mapped by, local coordinates η, *where* $x = x_m + \eta h,\ 0 \le \eta \le 1$,makes the integral (3.6) the contribution.

$$\int_0^1\left(\frac{\partial U}{\partial t} + \frac{\hat{U}}{h}\frac{\partial U}{\partial \eta} - \frac{v}{h^2}\frac{\partial^2 U}{\partial \eta^2}\right)\phi_i\, d\eta = 0 \qquad (3.7)$$

where to simplify the integral, $\hat{U}$ is taken to be constant over the element. this leads to

$$\int_0^1\left(\frac{\partial U}{\partial t} + v\frac{\partial U}{\partial \eta} - \ b\frac{\partial^2 U}{\partial \eta^2}\right)\phi_i\, d\eta = 0, \qquad (3.8)$$

where $b = \frac{v}{\eta^2}$ and $v = \frac{\hat{U}}{h}$

and b and v are taken as locally constant over each element. The variation of U over the element

$[x_m, x_{m+1}]$ is expressed as

$$U^e = \sum_{i=1}^{2} P_i u_i \qquad (3.9)$$

where P_1, P_2 are linear spatial basis function and u_1, u_2 are the nodal parameters. With the local coordinate system η defined above the basic functions have the following expressions [18]

$P_1 = 1 - \eta$, $P_2 = \eta$.

For gale kin's method we identify the weight function ϕ_i with basis function P_i giving

$$\int_0^1 \left(\frac{\partial U}{\partial t} + v \frac{\partial U}{\partial \eta} - b \frac{\partial^2 U}{\partial x^2} \right) P_i d\eta = 0 \qquad (3.10)$$

Integrating by parts leads to

$$\int_0^1 \left[\left(\frac{\partial U}{\partial t} + v \frac{\partial U}{\partial \eta} \right) P_i + b \frac{\partial U}{\partial \eta} \frac{\partial P_i}{\partial \eta} \right] d\eta = 0 \qquad (3.11)$$

Now if we substitute for U using equation (3.9) an element's contribution is found in the form

$$\int_0^1 \left[P_i P_j \frac{\partial U_j}{\partial t} + v P_i \frac{\partial P_j}{\partial \eta} u_j + b \frac{\partial P_i}{\partial \eta} \frac{\partial P_j}{\partial \eta} u_j \right] d\eta = 0 \qquad (3.12)$$

In the matrix notation this becomes

$$A^e \frac{\partial u^e}{\partial t} + \left[C^e + bD^e \right] u^e = 0 \qquad (3.13)$$

Where $u^e = (u_1, u_2)^T$ are the relevant nodal parameters. The element matrices is

$$A_{ij}{}^e = \int_0^1 P_i P_j \, d\eta \qquad C_{ij}{}^e = v \int_0^1 P_i \frac{\partial P_j}{\partial \eta} d\eta \qquad D_{ij}{}^e = \int_0^1 \frac{\partial P_i}{\partial \eta} \frac{\partial P_j}{\partial \eta} d\eta$$

And v is given as

$v = \dfrac{u_1}{h}$ is constant over the element.

$$A_{ij}{}^{e} = \int_0^1 P_i\, P_j d\eta$$

$(for\ i = 1 = j)$

$$A_{11} = \int_0^1 P_1 P_1 d\eta = \int_0^1 (1-\eta)(1-\eta)d\eta = -\left((1-\eta)^3/3\right)_0^1 = \frac{1}{3}$$

$(for\ \ i = 1, j = 2\ and\ \ j = 1, i = 2)$

$$A_{12} = A_{21} = \int_0^1 P_1 P_2 d\eta = \int_0^1 (1-\eta)\eta\, d\eta \ = \frac{1}{6}$$

$(for\ i = 2 = j)$

$$A_{22} \ = \int_0^1 P_2 P_2 d\eta = \int_0^1 \eta\eta d\eta = \frac{1}{3}$$

$$A_{ij}{}^{e} = \frac{1}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$C_{ij}{}^{e} = v\int_0^1 P_i \frac{\partial P_j}{\partial \eta} d\eta$$

$for(i = 1 = j)$ $for(i = 1\ \ and\ \ j = 2)$

$$C_{11} = v\int_0^1 (1-\eta)(-1)d\eta = \frac{-v}{2} \qquad C_{12} = v\int_0^1 P_1 \frac{\partial P_2}{\partial \eta} d\eta = v\int_0^1 (1-\eta)d\eta = -\frac{v}{2}$$

$for(\ i = 2\ ,\ j = 1)$ $for(\ i = 2\ =\ j)$

$$C_{21} = v\int_0^1 \eta(-1)d\eta = \frac{-v}{2} \qquad C_{22} = v\int_0^1 \eta d\eta = \frac{v}{2}$$

$$C_{ij}{}^{e} = \frac{v}{2}\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}$$

$$D_{ij}{}^{e} = \int_0^1 \frac{\partial P_i}{\partial \eta}\frac{\partial P_j}{\partial \eta} d\eta$$

$for\ (i = 1 = j)$ $\qquad$ $for(i = 2 = j)$

$$D_{11} = \int_0^1 \frac{\partial P_1}{\partial \eta}\frac{\partial P_2}{\partial \eta} d\eta = \int_0^1 (-1)(-1) d\eta = 1 \qquad D_{22} = \int_0^1 \frac{\partial P_2}{\partial \eta}\frac{\partial P_2}{\partial \eta} d\eta = \int_0^1 1\times 1 d\eta = 1$$

$for(i = 1,\ j = 2\ and\ i = 2,\ j = 1)$

$$D_{12} = D_{21} = \int_0^1 \frac{\partial P_1}{\partial \eta}\frac{\partial P_2}{\partial \eta} d\eta = \int_0^1 (-1)(1) d\eta = -1$$

So,

$$D_{ij}{}^{e} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

By assembling together contributions from all elements we find the matrix equation

$$A\frac{\partial u}{\partial t} + [C + bD]u = 0 \tag{3.14}$$

And $u = (u_0, u_1, ..., u_N)^T$, contains all parameters, a typical member of the equation (3.14) is

For 3 elements $(u_{m-1}\ ,\ u_m,\ u_{m+1}\ )$, we have

$$\frac{1}{6}\begin{bmatrix}2 & 1 & 0\\ 1 & 2+2 & 1\\ 0 & 1 & 2\end{bmatrix}\begin{bmatrix}\dot{u}_{m-1}\\ \dot{u}_m\\ \dot{u}_{m+1}\end{bmatrix}+\left(\frac{1}{2}v\begin{bmatrix}-1 & 1 & 0\\ -1 & 1-1 & 1\\ 0 & -1 & 1\end{bmatrix}+b\begin{bmatrix}1 & -1 & 0\\ -1 & 1+1 & -1\\ 0 & -1 & 1\end{bmatrix}\right)\begin{bmatrix}u_{m-1}\\ u_m\\ u_{m+1}\end{bmatrix}=0$$

$$\frac{1}{6}\left[\dot{u}_{m-1}+4\dot{u}_m+4\dot{u}_{m+1}\right]+\left[\frac{-1}{2}v_{m-1}+\frac{1}{2}v_{m-1}-\frac{1}{2}v_m+\frac{1}{2}v_m\right]+\left[-b+2b-b\right]\begin{bmatrix}u_{m-1}\\ u_m\\ u_{m+1}\end{bmatrix}=0$$

$$\frac{\partial}{\partial t}\left[\frac{1}{6}u_{m-1}+\frac{2}{3}u_m+\frac{1}{6}u_{m+1}\right]=\left(\frac{1}{2}v_{m-1}+b\right)u_{m-1}-\left[\frac{1}{2}\left(v_{m-1}-v_m\right)+2b\right]u_m-\left(\frac{1}{2}v_m-b\right)u_{m+1}$$

We can use Crank-Nicolson approach in order to find a numerical solution for this ordinary differential equation. Taking a time center as $t=\left(n+\frac{1}{2}\right)\Delta t$, We can write

$$\frac{\partial u_m}{\partial t}=\frac{1}{\Delta t}\left(u_m^{\,n+1}-u_m^{\,n}\right),$$

$$u_m=\frac{1}{2}\left(u_m^{\,n+1}+u_m^{\,n}\right)$$

Hence we find the recurrence relationship

$$\left(\frac{1}{6}-\frac{b\Delta t}{2}-\frac{\Delta t}{4}v_{m-1}\right)u_{m-1}^{\,n+1}+\left(\frac{2}{3}+b\Delta t+\frac{\Delta t}{4}\left[v_{m-1}-v_m\right]\right)u_m^{\,n+1}+\left(\frac{1}{6}-\frac{b\Delta t}{2}+\frac{\Delta t}{4}v_m\right)u_{m+1}^{\,n+1}$$
$$=\left(\frac{1}{6}+\frac{b\Delta t}{2}+\frac{\Delta t}{4}v_{m-1}\right)u_{m-1}^{\,n}+\left(\frac{2}{3}-b\Delta t-\frac{\Delta t}{4}\left[v_{m-1}-v_m\right]\right)u_m^{\,n}+\left(\frac{1}{6}+\frac{b\Delta t}{2}-\frac{\Delta t}{4}v_m\right)u_{m+1}^{\,n}$$

The boundary conditions $U(x_0,t)=0$ and $U(x_N,t)=0$ demands $u_0=0$ and $u_N=0$

The above set of quasi-linear equation has matrix which is tri-diagonal in form so that a solution applying the Thomas algorithm is feasible.

3.3 Numerical Experiments

In order to demonstrate the adaptability and the accuracy of the present method, we consider some test example available in the literature. The exact solutions of these examples are also available in the literature which is obtained by Hopf-Cole transformation. The numerical solutions generated by proposed method are compared with exact solution at the different nodal points.

Example1: Consider Burger's equation (3.1) with initial condition

$$u(x,0)=\sin \pi x \quad 0 < x < 1 \tag{3.15}$$

and homogeneous boundary conditions

$$u(0,t)=u(1,t)=0 \qquad 0 \le t \le T$$

The analytic solution to this problem can be expressed as an infinite series

$$U(x,t) = \frac{2\pi v \sum_{n=1}^{\infty} A_n \exp\left(-n^2\pi^2 vt\right) n \sin(n\pi x)}{A_0 + \sum_{n=1}^{\infty} A_n \cos(n\pi x) \exp\left(-n^2\pi^2 vt\right)} \tag{3.16}$$

where

$$A_0 = \int_0^1 \exp\left(\frac{-1}{2\pi v}\left(1-\cos(\pi x)\right)\right) dx \qquad , A_n = 2\int_0^1 \exp\left(\frac{-1}{2\pi v}\left(1-\cos(\pi x)\right)\right) dx \tag{3.17}$$

The numerical solutions of the Example are presented in the Tables 1-2 and Figures 1-3. Table 1 shows the comparison of numerical and exact solutions at $\nu = 1.0$ and at different times. The Table shows as we decrease step length the numerical solutions converges to the exact solutions. Similarly, Table 2 shows the comparison of numerical and exact solutions at

$\nu = 0.1,\ 0.01$ and at different times. The Figures 1-3 show the physical behaviour of the problem at ν and different times.

Example 2: Consider Burger's equation (3.1) with initial condition

$$u(x,0) = 4x(1-x) \qquad 0 < x < 1 \tag{3.18}$$

and boundary condition

$$u(0,t) = 0 = u(1,t) \qquad 0 \le t \le T \tag{3.19}$$

The exact solution of example is obtained by Half-sole transformation and given by

$$U(x,t) = \frac{2\pi\nu\sum_{n=1}^{\infty} A_n \exp(-n^2\pi^2\nu t) n \sin(n\pi x)}{A_0 + \sum_{n=1}^{\infty} A_n \cos(n\pi x)\exp(-n^2\pi^2\nu t)} \tag{3.20}$$

where

$$A_0 = \int_0^1 \exp\left(\frac{-1}{3\nu}(3x^2 - 2x^3)\right)dx \qquad A_n = \int_0^1 \exp\left(\frac{-1}{3\nu}(3x^2 - 2x^3)\right)dx \tag{3.21}$$

The numerical solutions of the Example are presented in the Tables 3-4 and Figures 4-6. Table 3 shows the comparison of numerical and exact solutions at $\nu = 1.0$ and $t = 0.1$ The Table shows numerical solutions are good in agreement with the exact solution. Similarly, Table 4 shows the comparison of numerical and exact solutions at $\nu = 0.1,\ 0.01$ and at different times. The Figures 4-6 show the physical behaviour of the problem at ν and different times.

3.4 Conclusion

A numerical algorithm for the solution of the burger's equation based on Galerkin method employing linear finite elements is developed. The performance of this algorithm is investigated b comparing solutions to two well known problems with data available in literature. The new method produces highly accurate numerical solutions for burger’s equation even for small value of viscosity coefficient. The method does, in fact, produce more accurate results then many of the other methods.

Table 1: Comparison of exact and analytic solutions of Example 1 at different time and x for $\nu = 1.0$

x	t	Present Method			Exact
		$h = 0.25$	$h = 0.125$	$h = 0.0625$	
0.25	0.05	0.4159	0.4155	0.4141	0.4131
	0.10	0.2524	0.2551	0.2546	0.2536
	0.15	0.1527	0.1570	0.1572	0.1566
	0.20	0.0918	0.0963	0.0967	0.0964
0.5	0.05	0.6045	0.6098	0.6100	0.6091
	0.10	0.3649	0.3724	0.3728	0.3716
	0.15	0.2190	0.2268	0.2276	0.2268
	0.20	0.1310	0.1379	0.1389	0.1385
0.75	0.05	0.4477	0.4533	0.4530	0.4502
	0.10	0.2668	0.2739	0.2743	0.2726
	0.15	0.1581	0.1646	0.1652	0.1644
	0.20	0.0938	0.0991	0.0998	0.0994

Table 2: Comparison of the numerical solution with the exact solution Example 1 at different time and x for $\nu = 0.1,\ 0.01$.

x	t	$\nu = 0.1$		$\nu = 0.01$	
		Computed solution	Exact Solution	Computed solution	Exact Solution
0.25	0.4	0.30881	0.62540	0.34229	0.34191
	0.6	0.24069	0.24074	0.26902	0.26896
	1.0	0.16254	0.16256	0.18817	0.18819
0.5	0.4	0.56955	0.56963	0.66797	0.66071
	0.6	0.44714	0.44721	0.53211	0.52942
	1.0	0.29188	0.29192	0.37500	0.37442
0.75	0.4	0.62540	0.62544	0.93680	0.91026
	0.6	0.48715	0.48721	0.77724	0.76724
	1.0	0.28744	0.28747	0.55833	0.55605

Table 3: Comparison between exact and numerical solutions of Example 2.for $\nu = 1.0$ at t=0.1

x	Present method	Exact solution
0.1	0.11271	0.11289
0.2	0.21600	0.21625
0.3	0.30023	0.30097
0.4	0.35824	0.35886
0.5	0.38311	0.38342
0.6	0.37016	0.37066
0.7	0.31899	0.32007
0.8	0.23511	0.23537
0.9	0.12410	0.12472

Table 4: Comparison with exact and existing numerical methods of Example 2 at different times and *x*.

		$\nu = 0.1$		$\nu = 0.01$	
x	t	Present method	Exact solution	Present method	Exact solution
0.25	0.4	0.31748	0.31752	0.36212	0.36226
	0.6	0.24600	0.24614	0.28189	0.28204
	0.8	0.19912	0.19956	0.23001	0.23045
	1.0	0.16513	0.16560	0.19470	0.19469
	3.0	0.02734	0.02775	0.07600	0.07613
0.50	0.4	0.58414	0.58454	0.68350	0.68368
	0.6	0.45723	0.45798	0.54861	0.54832
	0.8	0.36710	0.36740	0.45323	0.45371
	1.0	0.29800	0.29834	0.38532	0.38568
	3.0	0.04045	0.04106	0.15220	0.15218
0.75	0.4	0.64562	0.64562	0.92001	0.92050
	0.6	0.50215	0.50268	0.78211	0.78299
	0.8	0.38515	0.38534	0.66223	0.66272
	1.0	0.29523	0.29586	0.56910	0.56932
	3.0	0.03021	0.03044	0.22678	0.22774

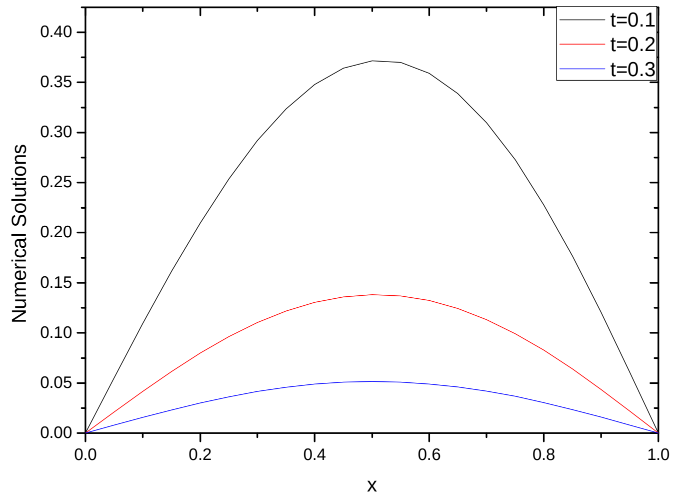

Figure 1: Numerical Solution of Example 1 at different times t and values of $\nu = 1.0$ and $\Delta t = 0.0001$.

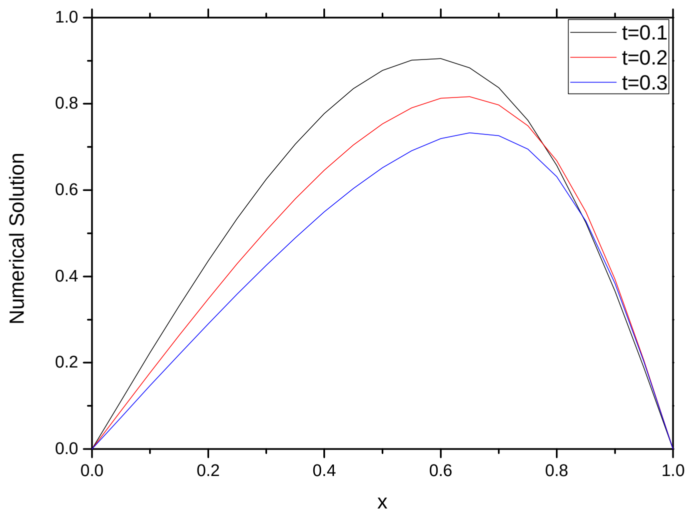

Figure 2: Numerical Solution of Example 1 at different times t and values of $\nu = 0.1$ and $\Delta t = 0.0001$.

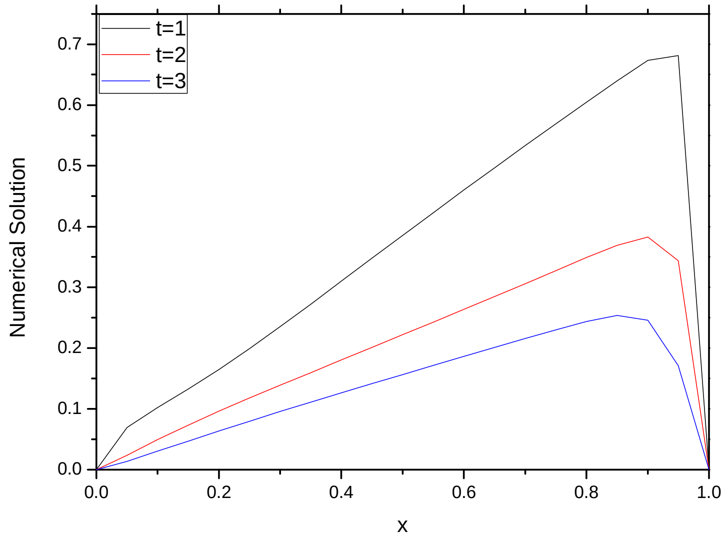

Figure 3: Numerical Solution of Example 1 at different times t and values of $\nu = 0.01$ and $\Delta t = 0.0001$.

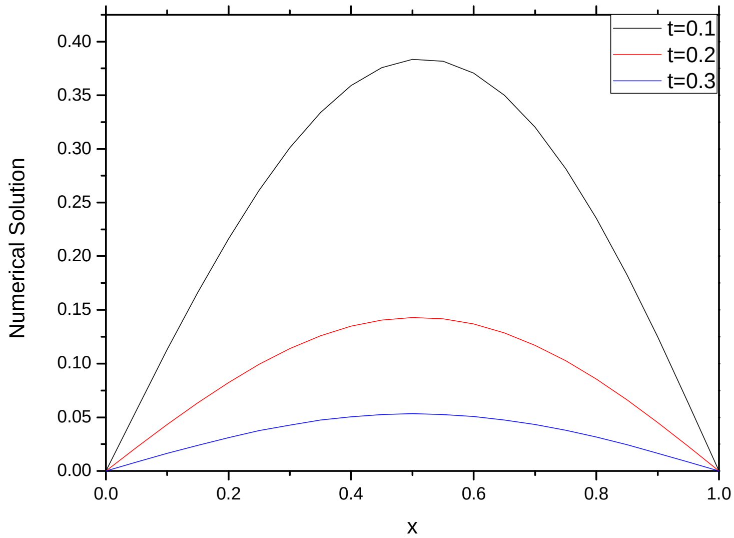

Figure 4: Numerical Solution of Example 2 at different times t and values of $\nu = 1.0$ and $\Delta t = 0.0001$

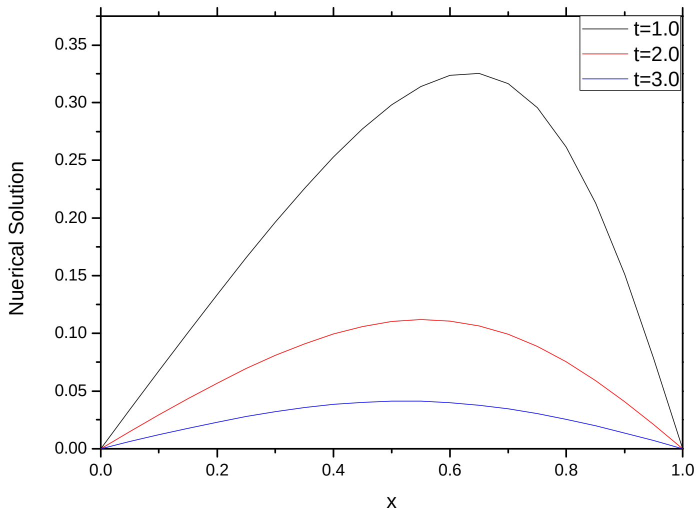

Figure 5: Numerical Solution of Example 2 at different times t and values of $\nu = 0.1$ and $\Delta t = 0.0001$

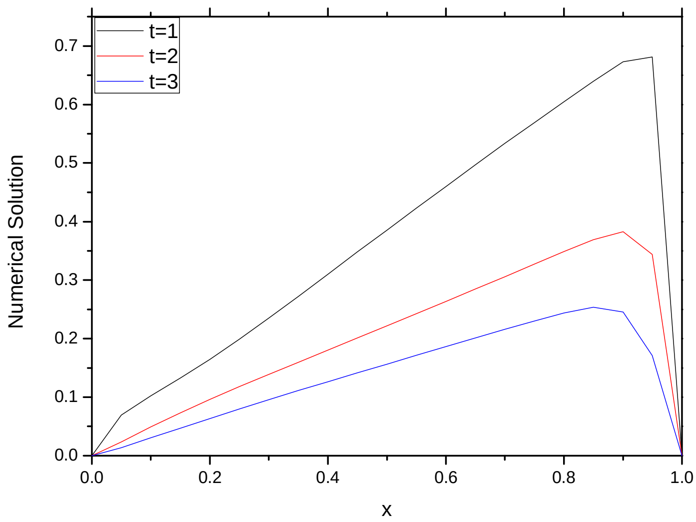

Figure 6: Numerical Solution of Example 2 at different times t and values of $\nu = 0.01$ and $\Delta t = 0.0001$.

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