A STUDY OF TWO STAGE TRANSPORTATION PROBLEMS

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DEDICATED

TO

GOD, MY PARENTS

AND

MY SUPERVISOR

Certificate

I hereby certify that the work which is being presented in this dissertation entitled ***"A Study of Two Stage Transportation Problems"*** *in partial fulfillment of the requirements for the award of degree of Master of Science, School of Mathematics and Computer Applications, Thapar University, Patiala is an authentic record of my own work carried out under the supervision of Dr. Vikas Sharma.*

The matter presented in this thesis has not been submitted for the award of any other degree of this or any other university.

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This is to certify that the above statement made by the candidate is correct and true to the best of my knowledge.

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Abstract

In this thesis, a class of time minimization transportation problems has been studied where the transportation is done in two stages. Two problems in this regard have been discussed. The First problem focused on the study of two stage bottleneck transportation problem wherein certain situation like storage capacity limitations or lack of maintenance facilities at the production floor the shipment of goods is done in two stages. In the first stage, the minimum requirement of each destination is met and in the second stage the left over the quantities are shipped to destinations. The objective is to find that schedule which minimizes the sum of Stage I and Stage II shipment times. A polynomial time algorithm is discussed to obtain the optimal schedule by investigating the lexicographic feasible solutions of a related standard (TMTP). This problem was first discussed by Sonia and Puri (2002) and later on generalized by Sharma et al. (2010).

Another two stage transportation problem has been discussed in the form of a two stage interval time minimization transportation problem, where total availability of a homogenous product at various sources is known to lie in a specified interval. A polynomial time algorithm is discussed to solve the problem to optimality, where at various steps of the algorithm lexicographic optimal solutions of the restricted versions of a related standard time minimizing transportation problem are examined and finally the global optimal solution is determined. Two different approaches of this problem, one given by Sonia and Puri (2004) and the second approach suggested by Sharma et al. (2008) have been studied.

Contents

Chapter 1

Introduction

1.1 Introduction

Recent period has seen the inception of operation research techniques in every area of life. For example, saving expenditure in shipping the products, or expected return in the environment of risky investments, speed or distance in a physical problem, profit or loss in a business setting etc. Optimization techniques are involved in the procedures used to make a system or design as effective or functional as possible. It involves the use of mathematical models, statistics and algorithm to help decision-making. Researchers use it most often to analyze complex real-world systems, typically with the goal of improving or optimizing performance.

Mathematical Programming is one of its significant branch and has substantial applications in different real life problems. It endeavors to find optimal solutions for a broad range of problems including medical, industry, economics, finance, scientific and engineering problems. The mathematical programming refers to the study of these problems: their mathematical properties, the development and implementation of algorithms to solve these problems and the application of these algorithms to real world problems. The general form of mathematical programming problem is given as below:

$$\text{Optimize } Z = f(X)$$

Subject to

$$g_i(X) \leq 0,\ i = 1,2,...,m$$
$$X \geq 0$$

where $f(X)$ and $g_i(X) \leq 0,\ i = 1,2,...,m$ can be taken to be general functions of the vector $X = (x_1, x_2 ..., x_n) \in R^n$. Thus we have to find a vector X which optimize (minimizes or maximizes) the objective function $f(X)$ subject to the m constraints $g_i(X) \leq 0,\ i = 1,2,...,m$.

Linear programming problem (LPP) is a special class of mathematical programming problems in which objective function and the constraints are linear functions in some unknown variables. In 1947, Dantzig, G.B. formulated the general linear programming and proposed the most popular simplex method for solving this problem. Since the optimal value of the objective of an (LPP) is attained at one of vertex of the feasible region. The main idea behind simplex method is to move from one vertex to another vertex in the improving direction of the objective function, till the one which gives optimal value of the objective is not found.

The general form of linear programming problems can be stated as:

$$\min Z = C^T X$$

Subject to

$$AX = b$$

$$X \geq 0$$

where $X = (x_1, x_2, ..., x_n)^T$, $C = (c_1, c_2, ..., c_n)^T$, $b = (b_1, b_2, ..., b_m)^T$ *and* $A = [a_{ij}]$ is an $(m \times n)$ matrix.

The linear programming model has been applied to a large number of areas including inventory management, scheduling, production, transportation and distribution, finance, agriculture etc. One of the most important and well-structured class of linear programming problems is the class of transportation problems. Transportation Problem (TP) is a logistical problem for organizations especially for manufacturing and transport companies. The transportation problem deals with shipping of a homogeneous commodity from sources to destinations. The objective of transportation problem is to determine the shipping schedule that minimizes total shipping cost while satisfying supply and demand points. To illustrate a typical transportation model, suppose m factories supply a homogeneous product to n warehouses. Let factory i $(i = 1,2,...,m)$ produce a_i units and the warehouse j $(j = 1,2,...,n)$ require b_j units. Suppose the cost of transportation from factory i to warehouse j is c_{ij}. If x_{ij} units be shipped from the factory i to warehouse j then the cost would be $c_{ij}\, x_{ij}$. The objective is to minimize the shipping cost, this problem is widely known as Cost Minimizing Transportation Problem (CMTP). Mathematically this problem can be expressed as follows:

$$\text{Min } z = \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij} x_{ij} \qquad (P_C)$$

Subject to

$$\left.\begin{aligned} &\sum_{j=1}^{n} x_{ij} = a_i\,, \quad i\,(i = 1,2....m) \\ &\sum_{i=1}^{m} x_{ij} = b_j\,, \quad j\,(j = 1,2,...n) \\ &x_{ij} \geq 0, \qquad \forall\, i \; and \; j \end{aligned}\right\} \qquad (1.0)$$

For a feasible solution to exist it is necessary that total supply equals total demand.

$$\text{i.e. } \sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$$

Since transportation problem (P_c) is a LPP. Therefore, an optimal solution of this problem lies at one of the vertex of its feasible region. For details regarding the solution procedure of balanced transportation problem one may refer Hadley (1962). In real life situations, the decision maker may face a situation, where total supply is not equal to total demand, which gives rise to unbalanced transportation problem. This type of problem arises, for example in agencies like Food Corporation of India, which supplies food grains from different warehouses to different distribution centers, where the food grains are available in large amount but the requirement at destinations may be very less.

An important class of transportation problems in terms of its prevalent applications is the Time Minimizing Transportation Problem (TMTP) also called Bottleneck Transportation Problems (BTP). In BTP, the transportation of a homogenous product from sources to destinations is done in parallel and its prime aim is to supply the destinations with required quantity of the product with in a shortest possible time. Each destination can receive its requirement from any number of sources and a source can supply goods to any number of destinations subject to its capacity restrictions. We assume that the parallel transportation is done from all the sources to destinations. Overall, objective is to minimize the maximum time from various sources indexed by $I = \{1,2,...,m\}$ to different destinations indexed by $J = \{1,2,...,n\}$. Thus, the mathematical model for the standard (TMTP) is:

$$Min\,[T(X) = \max_{I \times J}(t_{ij}(x_{ij}))] \qquad (P_T)$$

Subject to

$$\left.\begin{aligned} &\sum_{j=1}^{n} x_{ij} = a_i\,, \quad i\,(i = 1,2....m) \\ &\sum_{i=1}^{m} x_{ij} = b_j\,, \quad j\,(j = 1,2,...n) \\ &x_{ij} \geq 0, \qquad \forall\, i\; and\; j \end{aligned}\right\} \qquad (2.0)$$

where $t_{ij}(x_{ij}), (i, j) \in I \times J$, the shipment time from the source i^{th} to the destination j^{th}, is defined as:

$$t_{ij}(x_{ij}) = t_{ij}(\geq 0) \qquad \text{if } x_{ij} > 0$$

$$= 0 \qquad \text{otherwise}$$

It may be noted that T(X) is also a concave function (Bansal and Puri 1980). Thus, BTP involves minimization of a concave function over a polytope and hence it belongs to the class of Concave Minimization Problem (CMP). For any given feasible solution X= [x_{ij}] satisfying (2.0), the time of transportation is the maximum of $t_{ij}'s$ among the cells in which there are positive allocations i.e. time of transportation is [$\max_{(i,j)} t_{ij} : x_{ij} > 0$]. A feasible solution for which $\max_{\{(i,j) | x_{ij} > 0\}} t_{ij}$ is minimal, will be called optimal. The time of transportation is independent of the amount of commodity shipped from suppliers to consumers while the cost of transportation is dependent on the variation in quantity of commodities shipped. The aim is to minimize the time of transportation.

Unlike the CMTP, a TMTP is not a linear programming problem as the associated objective function is concave in nature. A TMTP is, therefore a non-linear and in particular concave minimization problem. Almost all the techniques for solving TMTP involve solving a CMTP for which strongly polynomial algorithms are known to exist have discussed by Tardos (1985,1986), Orlin (1988) and Kleinschmidt et al. (1995). Therefore it follows that a time minimization transportation problem is also solvable in strongly polynomial time. Solvability in strongly polynomial time means that there exit an algorithm which solves in a number of steps that is bounded by a polynomial function of m and n, the number of sources and destinations respectively. In some situations for example, ambulance services, fire services, military equipment etc. the time of transportation cost attains greater importance than the cost of transportation.

In this thesis, we have discussed algorithms and solution methodologies for various allocation problems viz. Cost Minimizing Transportation Problem (CMTP), Time Minimizing Transportation Problem (TMTP), Imbalanced TMTP and Two Stage Interval TMTP that consists of two stages.

A brief survey of the related literature is presented in the next section and the last section contains a brief summary of the work presented in the thesis.

1.2 Literature Survey

A lot of literature is available to study conventional cost as well as time minimizing transportation problems. Many authors have addressed these problems and many solution techniques have been proposed for TMTP by Hammer (1969), Szwarc (1971), Garfinkel and Rao (1971), Srinivasan et al. (1976), Bracken and McGill (1978), Parkash (1982), Burkard et al. (1991) and Ahuja et al. (1994) and Punnen et al. (1995). The transportation problem was first formulated by Hitchcock in 1941 and later worked by Koopmans in 1948. He used the problem of distribution of a product from several sources to a number of cities at the least cost. Since then this field has caught attention of researchers. Various invariants of CMTP have been studied in the past. G. Appa (1973) discussed some useful variants of the (CMTP). The (CMTP) with mixed constraints was discussed by Brigden (1974), Klingman and Russel (1974). Saroj et al. (1981, 1983) presented a study on a flow constrained cost minimizing transportation problem.

If capacity of each source-destination link is introduced then it is called capacitated cost minimization transportation problem (CCMTP), which falls into the class of capacitated transportation problems (CTP). Capacitated transportation problems have been studied by various authors. Kassay (1981) gave an operator method for solving capacitated transportation problem. Later on, Bit et al. (1993), Zheng et al. (1994), Rachev and Olkin (1999) worked on capacitated transportation problem.

In 1955 Schell stated an extension of the transportation problem. He considered it as a block in which the layer in all directions forms restricted transportation problems. Along with the extensions new computational methods were developed. In 1956 Vidale developed a graphical approach for the solution of the general type of transportation problems. He had suggested a method of successive approximations when the production costs vary with the volume of resource produced. It was extended to problems involving a large number of origins and destinations. Some additional mathematical aspects of the transportation problem and the corresponding simplex method of solution are discussed by Dantzig in 1963.

A zero-one transportation problem is also known as bulk transportation problem (BTP). BTP is studied in literature by Maio and Roveda (1971), Srinivasan and Thompson (1972), Fisher et al. (1986). BTP consists of a set of sources producing a homogenous material with a maximum capacity and a set of destinations whose demand for this material is known and, also this demand is to be met by a single source subject to its capacity.

An important class of mathematical programming is concave minimization problem (CMP). Particular cases of CMP have been studied by Mathur and Puri (1994). A concave minimization problem, in general, can be solved by a branch and bound algorithm as suggested by Locatelli et al. (2001). Generally CMPs are tough problems, but in special instances like concave minimization flow problem, freight transportation problem, production transportation problem and time minimizing transportation problem etc. are polynomially solvable.

An important class of transportation problem belonging to the class of concave minimization problems is the Production Transportation Problem (PTP) which involves an arbitrary fixed number of factories with concave production costs (Hoang et al., 1996; Kuno and Utsunomiya, 2000). Work of Tardos (1986) suggests that the transportation polytope involves a special combinatorial structure that significantly reduces the complexity and it is due to this fact there exists a strongly polynomial time algorithm for special cases of PTP.

Another class of a concave minimization problem is freight transportation problem (FTP) discussed by Klincewicz (1990). In fright transportation problem sources can ship in bulk to one or more intermediate terminals (called consolidation terminals) and at these terminals, shipment from many sources can be consolidated for eventual shipment to the various destinations. Shipment costs are piecewise linear concave functions of the volume shipped and shipping via a consolidation terminal incurs a linear inventory holding cost. Minimum cost solution of direct or indirect shipments were desired.

Time minimizing transportation problem (TMTP) is another special case of CMP. Perhaps, original contribution in the field (TMTP) is due to Hammer (1969). The time minimizing transportation problem has also been studied by Garfinkel and Rao (1971), Szwarc (1971), Bhatia et al. (1977), Ramakrishnan (1977), Sharma and Swarup (1977), Satya Prakash (1982), Issermann (1984), Ahuja (1986), Chandra et al. (1987) and by several other authors. Hammer (1969, 1971) also discussed the lexicographic version of time minimizing transportation problem.

Lexicographic optimal solution (LOS) of TMTP is a feasible solution, say $X \in S$, for which the overall transportation time $T(X)$ is minimized in addition to the minimization of the shipment not only on the longest duration route (source-destination link) but shipments on all other routes of various durations are also minimized. For obtaining an LOS, the set of transportation times on various routes is partitioned into a number of disjoint sets, $B_k,\ k = 1,2,...,s$ where, $B_k = \{(i,j) \in I \times J : t_{ij} = T^K\}$ and $T^1 > T^2 > ... > T^s$

Thus the problem is then converted into a related cost minimizing transportation problem (CMTP) as:

$$\min_{X \in S} \sum_{k=1}^{s} \lambda_k \left(\sum_{\substack{(i,j) \in I \times J \\ t_{ij} = T^K}} x_{ij} \right)$$

where set S is given by

$$S : \left\{ \begin{array}{ll} \sum_{j \in J} x_{ij} = a_i & \forall\, i \in I, \\ \sum_{i \in I} x_{ij} = b_j & \forall\, j \in J, \\ x_{ij} \geq 0 & \forall (i,j) \in I \times J \end{array} \right\}$$

$\lambda_j,\ j = 1,2,...,s$ being positive weights attached to the shipments on the routes where $\lambda_{j+1} >> \lambda_j \quad (j = 1,2,...,s-1)$. An optimal feasible solution of bottleneck transportation problem (BTP) is obtained by solving his associated CMTP, it follows that BTP is also polynomially solvable. The positive weights λ_j , $j = 1,2,...,s$ can be determined as described by Sherali (1982) and Mazzola (1993).

In some circumstances, when the due to destination constraint, it is preferable to transport the product in two stages. Sonia et al. (2002, 2004a, 2004b, 2006) and Sharma et al. (2008, 2010) discussed invariants of transportation problem, in which the transportation is done in two stages. Capacitated two-stage time minimization transportation problem has been discussed by Sharma, V., Dahiya, K. and Verma, V. (2010).

1.3 Summary of the Thesis

This section briefly summarizes the research work carried out in Chapter 2 and Chapter 3. In the present dissertation, we have studied in which the overall requirement of a homogeneous product at the various destinations does not match the total availability at the various origins.

Chapter 2 discusses “Two Stage Bottleneck Transportation Problem”. In some circumstances due to storage restrictions, destinations are unable to receive the quantity in excess of their minimum demand. For example, problem of transporting perishable items like medicines and food packets to areas hit by natural calamities or sending military equipments to different posts during war times, where the items are available in abundance but due to shortage of storage capacity at destinations, over supply may not be possible. In this case, shipment in one stage is not possible. Therefore, items are shipped to destinations from origins in two stages. Initially, the minimum demands of the destinations are shipped from origins to the destinations. After consuming part of whole of this initial shipment, they are prepared to receive the excess quantity in the second stage, so in Stage-II, the surplus quantity (if any) at the sources is shipped to the destinations. In both the stages, the transportation of the product from sources to destinations is done in parallel. Overall, the aim is to find the optimal feasible solution for which the sum of the transportation times in both the stages is minimum. Mathematically Stage-I problem can be stated as:

$$\min_{X \in S} \left[T_1(X) \right] \qquad (P_{2.1})$$

where the set of feasible solution is given by

$$S : \begin{cases} \sum_{j=1}^{n} x_{ij} \le a_i \,, \; i \in I \\ \sum_{i=1}^{m} x_{ij} = b_j \,, \; j \in J \\ x_{ij} \ge 0, \qquad \forall \, (i, j) \in I \times J \end{cases}$$

and $T_1(X) = \max_{I \times J} [t_{ij}(x_{ij}) / x_{ij} > 0]$

Corresponding to a feasible solution $X \in S$ of the Stage-I, the set of feasible solutions of Stage-II is given by:

$$S(X):\begin{cases}\sum_{j=1}^{n}\bar{x}_{ij}=a_i',\ i\in I\\ \sum_{i=1}^{m}\bar{x}_{ij}\geq 0,\ j\in J\\ \bar{x}_{ij}\geq 0,\quad \forall(i,j)\in I\times J\end{cases}$$

where a_i' is the quantity of the product available at the i^{th} source on completion of Stage-I. Clearly $\sum_{i\in I}a_i'=\sum_{i\in I}a_i-\sum_{j\in J}b_j$. Thus, Stage-II problem associated with a feasible solution $X\in S$ of the Stage-I problem can be formulated as:

$$\min_{\bar{X}\in S(X)}[T_2(\bar{X})] \qquad (P_{2.2})$$

where,

$$T_2(\bar{X})=\max_{I\times J}[t_{ij}(\bar{x}_{ij})/\bar{x}_{ij}>0]$$

Thus, two stage bottleneck transportation can be stated as:

$$\min_{X\in S}\left(T_1(X)+\left(\min_{\bar{X}\in S(X)}T_2(\bar{X})\right)\right) \qquad (P_{2.3})$$

This problem was originally studied by Sonia and Malhotra (2002). As transportation time in each stage is a concave function, the sum of these will also be a concave function. Thus two stage bottleneck transportation problem is a concave minimization problem over a polytope. This two stage bottleneck transportation is related to an imbalanced TMTP, which is solved by considering an equivalent balanced TMTP. The proposed algorithm involves study of the lexicographic optimal solutions (LOS) of this related standard TMTP and its restricted version, thereby obtaining feasible solutions of Stage-I and Stage-II iteratively such that the sum of the Stage-I and Stage-II shipment time is the least.

Chapter 3 discusses another two stage bottleneck transportation problem. This chapter consists of two sections. Section 3.1, discusses a variant of time minimization transportation problem in the form of two stage interval time minimization transportation problem discussed by Sonia et al. (2004) and an alternate approach has been suggested by Sharma et al. (2008) in section 3.2. The problem consists of two stages. In the first stage minimum amount available at each source is shipped to the destinations and enough quantity of the product is dispatched in second stage so as to meet the demand at destinations exactly. In each stage, aim is to minimize the duration of transportation and overall goal is to minimize the sum of two stage shipment times. In both the stages, transportation of the product from the sources to the destinations is done in parallel. There are many problems where it becomes necessary to study TMTP wherein the products are shipped to the destinations in two stages. Such situations motivated the study of a two-stage interval time minimizing transportation where the total availability of the product at the sources lies in specified intervals.

A practical situation corresponding to this type of problem is the example of the production of maintenance-free sealed industrial batteries. Production is the continuous process depending on the available resources. However, each battery has a certain shelf life and batteries need to be periodically re-charged, else the whole lot becomes dead resulting in the loss of the finished goods. Often due to lack of re-charging facilities on the production floor, each batch of manufactured batteries is transported immediately to the destinations; this corresponds to the first stage. In the second stage, just enough maintenance free-sealed batteries from the sources are shipped in order to satisfy the industrial users' demands at the destinations. Shipment is done in such a way as to minimize the overall transportation time. The mathematical model for this two stage interval TMTP is:

$$\min_{Y \in S_2}\left[T_1(Y) + \left(\min_{Z \in S_2(Y)} T_2(Z)\right)\right] \qquad (P_{3.1c})$$

Subject to

$$S_2 : \begin{cases} \sum_{j \in J} y_{ij} = a_i,\ i \in I \\ \sum_{i \in I} y_{ij} \le b_j,\ j \in J \\ y_{ij} \ge 0, \forall (i,j) \in I \times J \end{cases}$$

where, corresponding to a feasible solution $Y = (y_{ij})$ of Stage-I, the feasible solution for the Stage-II is given as:

$$S_2(Y): \quad \begin{cases} \sum_{j \in J} z_{ij} \le a_i' - a_i, \qquad i \in I \\ \sum_{i \in I} z_{ij} = b_j - \sum_{i \in I} y_{ij},\ j \in J \\ z_{ij} \ge 0,\ \forall (i,j) \in I \times J \end{cases}$$

where $b'_j = \sum_{i \in I} y_{ij},\ j \in J$

The availability at the source i belongs to [a_i, a'_i] and demand at the destination j is b_j.

A two stage interval TMTP, where total availability of the product at the sources lies in specified intervals, is shown to be related to an ordinary interval TMTP, which is further equivalent to a standard TMTP. Feasible solutions of the Stage-I and Stage-II problems are derived from a feasible solution of this standard TMTP. Due to the dependence of Stage-II on Stage-I special types of solutions viz. lexicographic optimal solutions (LOS) of TMTP and its restricted versions are investigated. In all the pairs of Stage-I and Stage-II shipment times, the shipment time of one stage is the minimum corresponding to the other.

The next section 3.2 discusses an alternative approach of two stage interval TMTP in which sources ship all of their hand material to the demand points in the first stage and the second stage shipment covers the demand that is not fulfilled in the first stage. This problem was first discussed by Sonia et al. (2004). In their methodology, two sequences of Stage-I and Stage-II time are generated. One of the sequences consists of generating pairs of the form $(T_1(.), T_2(.) : T_1(.) > T_2(.))$ by solving time minimization transportation problem of the form $P_{L\beta}(T_2(.))$ and cost minimization transportation problem of the form $C_{L\beta}(T_1(.), T_2(.))$where the problem $P_{L\beta}(T_2(.))$ reduces the on hand shipment time for Stage-II, and the problem $C_{L\beta}(T_1(.), T_2(.))$ gives the minimum shipment time for Stage-II corresponding to the Stage-I shipment time obtained from $P_{L\beta}(T_2(.))$ .Similarly the sequence of two stage shipment time of the form $(T_1(.), T_2(.) : T_1(.) < T_2(.))$ is obtained by solving the problems $P_{U\beta}(T_1(.))$ and $C_{U\beta}(T_2(.), T_1(.))$, where these problems play a similar role as played by $P_{L\beta}(T_2(.))$ and $C_{L\beta}(T_1(.), T_2(.))$ with their role for Stage-I and Stage-II reversed. Further it has been established theoretically that the global minimum value of the $(P_{3.1c})$ problem is obtained out from these generated pairs.

The present work aims at reducing the computational complexity of the method discussed by Sonia et al. (2004), thereby suggesting a different approach to solve this problem, which adopts only one sequence of Stage-I and Stage-II problems in contrast to the two way procedure adopted by Sonia et al. (2004). The algorithm developed in section 3.2 generates the sequence of Stage-I and Stage-II shipment times, where at each iteration

Stage-I time strictly decreases and Stage-II time strictly increases. Feasible solutions of Stage-I and Stage-II problems are derived from a feasible solution of the standard TMTP. Moreover, it has also been shown that in the developed algorithm a finite number of cost minimizing transportation problem (CMTP) are solved to generate different pairs of Stage-I and Stage-II shipment times.

Chapter 2

Two Stage Bottleneck Transportation Problem

A study of unbalanced bottleneck transportation problem has been done in this chapter, where the total availability of homogenous product at the sources is more than the minimum requirement of the same at the destinations. Time of transportation, which is independent of the quantity transported, from each source to each destination is known. The problem consists of two stages. In the first stage only that much quantity is send from the sources which meet exactly the minimum requirement of the destinations. After the completion of Stage-I, the left over quantities (if any) at the sources are shipped to the destinations in Stage-II. In both the stages transportation of the product from the sources is done in parallel. A polynomial time algorithm is developed which obtains the optimal schedules for the two stages by investigating the lexicographic feasible solution of a related standard TMTP.

2.1 Mathematical Formulation:

Let $a_i, i \in I$ be the availability of a homogeneous product at the i^{th} source and let $b_j, j \in J$ be the requirement of the same at the j^{th} destination, where $\sum_{i \in I} a_i > \sum_{j \in J} b_j$. Since the total availability is more than the total requirement the shipment is done in two stages. In first stage, the total requirement at destinations is satisfied and surplus amount is supplied in second stage.

The Stage-I problem is formulated as:

$$\min_{X=\{x_{ij}\}\in S} \left\lfloor \max_{I\times J} \left(t_{ij}\left(x_{ij}\right)\right)\right\rfloor \qquad \left(P_{2.1}\right)$$

where the set S is given by

$$S:\begin{cases}\sum_{j=1}^{n} x_{ij} \le a_i\,, & i \in I\\ \sum_{i=1}^{m} x_{ij} = b_j\,, & j \in J\\ x_{ij} \ge 0, & \forall (i,j) \in I \times J\end{cases}$$

Corresponding to a feasible solution of $X = \{x_{ij}\}$ of Stage-I problem, let $S(X)$ be the set of feasible solutions of Stage-II problem, which is stated as below:

$$\min_{\bar{X}=\{\bar{x}_{ij}\} \in S(X)} \left[\max_{I\times J} \left(t_{ij}\left(\bar{x}_{ij}\right)\right)\right] \qquad (P_{2.2})$$

where the set $S(X)$ is given by

$$S(X):\begin{cases}\sum_{j=1}^{n} \bar{x}_{ij} = a'_i\,, & i \in I\\ \sum_{i=1}^{m} \bar{x}_{ij} \ge 0, & j \in J\\ \bar{x}_{ij} \ge 0, & \forall (i,j) \in I \times J\end{cases}$$

where a'_i is the quantity of the product available at the i^{th} source on completion of Stage-I. Clearly $\sum_{i\in I} a'_i = \sum_{i\in I} a_i - \sum_{j\in J} b_j$. We aim at finding the feasible solution of Stage-I and the corresponding optimal time of Stage-II, is such a way that the sum of shipment times of both the stages is least. Thus, a two stage bottleneck transportation problem is defined as:

$$\min_{X=\{x_{ij}\} \in S} \left[\max_{I\times J} \left(t_{ij}\left(x_{ij}\right)\right) + \min_{\bar{X} \in S(X)} \left[\max_{I\times J} \left(t_{ij}\left(\bar{x}_{ij}\right)\right)\right]\right] \qquad (P_{2.3})$$

The set of feasible solutions of the problem $(P_{2.3})$ is union of the sets of feasible solutions of the Stage-I and Stage-II problem. To find a feasible solution of Stage-I problem, we define the following equivalent balanced bottleneck transportation problem:

$$\min_{Y=\{y_{ij}\} \in S'} [\max_{I\times J'} (t'_{ij}(y_{ij}))] \qquad (P^*_{2.1})$$

where the set S' is given by :

$$S' : \begin{cases} \sum_{j \in J'} y_{ij} = a_i & \forall\ i \in I \\ \sum_{i \in I} y_{ij} = b'_j & \forall\ j \in J' \\ y_{ij} \geq 0 & \forall\ (i,j) \in I \times J' \end{cases}$$

and

$$t'_{ij}(y_{ij}) = t'_{ij} \quad if\ y_{ij} > 0,$$
$$= 0 \quad otherwise$$

where,

$$J' = J \cup \{n+1\},$$
$$b'_j = b_j \ \forall\ j \in J$$
$$b'_{n+1} = \sum_{i \in I} a_i - \sum_{j \in J} b_j,$$
$$t'_{ij} = t_{ij} \ \forall (i,j) \in I \times J,$$
$$t'_{in+1} = 0 \ \forall\ i \in I,$$

2.2 Theoretical Development

An OFS of Stage-I problem $(P_{2.1})$ is obtained from an OFS of the related problem $(P^*_{2.1})$, as $(P^*_{2.1})$ is a balanced TMTP and can easily be solved. In theorem 1 and 2 an equivalence has been established between problems $(P_{2.1})$ and $(P^*_{2.1})$.

Theorem 1. Corresponding to every feasible solution of the problem $(P_{2.1})$, there exists a feasible solution of the problem $(P^*_{2.1})$ and vice versa.

Proof. Let $X = \{x_{ij}\}$ be a feasible solution of $(P_{2.1})$.

Define $\{y_{ij}\}$

$$\left.\begin{aligned} y_{ij} &= x_{ij} & \forall (i,j) \in I \times J, \\ y_{in+1} &= a_i - \sum_{j \in J} x_{ij} & \forall\, i \in I \end{aligned}\right\} \qquad (1)$$

Then, for $i \in I$

$$\sum_{j=1}^{n+1} y_{ij} = \sum_{j=1}^{n} y_{ij} + y_{in+1} = \sum_{j=1}^{n} x_{ij} + \left(a_i - \sum_{j=1}^{n} x_{ij} \right) = a_i$$

For each $j \in J$,

$$\sum_{i=1}^{m} y_{ij} = \sum_{i=1}^{m} x_{ij} = b_j \left[\because x_{ij} = y_{ij} \;\; \forall \; (i, j) \in I \times J \right]$$

and

$$\sum_{i=1}^{m} y_{in+1} = \sum_{i=1}^{m} \left(a_i - \sum_{j=1}^{n} x_{ij} \right) = \sum_{i=1}^{m} a_i - \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij}$$

$$= \sum_{i=1}^{m} a_i - \sum_{j=1}^{n} b_j = b'_{n+1}$$

For each $i \in I$,

$$y_{in+1} = a_i - \sum_{j=1}^{n} y_{ij} = a_i - \sum_{j=1}^{n} x_{ij} \le a_i < \infty$$

and

$$y_{in+1} = a_i - \sum_{j=1}^{n} x_{ij} \ge 0.$$

Therefore, $\{ y_{ij} \}$ defined by the equation (1) is a feasible solution of $\left(P^*_{2.1} \right)$.

Conversely, let $\{ y_{ij} \}_{I \times J'}$ be a feasible solution of $\left(P^*_{2.1} \right)$.

Define $\{ x_{ij} \}$ as follows:

$$x_{ij} = y_{ij} \;\; \forall \; (i, j) \in I \times J \qquad (2)$$

For $i \in I$,

$$\sum_{j=1}^{n+1} y_{ij} = a_i,$$

$$\Rightarrow \sum_{j=1}^{n} y_{ij} = a_i - y_{in+1} \le a_i (\because y_{in+1} \ge 0)$$

$$\Rightarrow \sum_{j=1}^{n} x_{ij} \le a_i \; \forall \, i \in I$$

Also, $\sum_{i=1}^{m} x_{ij} = \sum_{i=1}^{m} y_{ij} = b_j \; \forall \; j \in J$

Therefore, $\{ x_{ij} \}$ defined by (2) is a feasible solution of the problem $(P_{2.1})$.

Theorem 2. Corresponding feasible solutions of the problem $(P_{2.1})$ and $(P_{2.1}^{*})$ yield same value of objective functions.

Proof. Let $Y=\{y_{ij}\}$ be a feasible solution of problem $(P_{2.1}^{*})$ yielding time T_1^{*} and let $X=\{x_{ij}\}$ be the corresponding feasible solution of the problem $(P_{2.1})$ giving value T_1 .

$$
\begin{aligned}
T_1^{*} &= \max_{I\times J'}\left(t'_{ij}(y_{ij}) \middle| y_{ij}>0\right)\\
&= \max\left[\max_{I\times J}\left(t'_{ij}(y_{ij}) \middle| y_{ij}>0\right), \max_{I}\left(t'_{in+1}(y_{in+1}) \middle| y_{in+1}>0\right)\right]\\
&= \max\left[\max_{I\times J}\left(t_{ij}(y_{ij}) \middle| y_{ij}>0\right), 0\right]\\
&= \max_{I\times J}\left(t_{ij}(y_{ij}) \middle| y_{ij}>0\right)\\
&= \max_{I\times J}\left(t_{ij}(x_{ij}) \middle| x_{ij}>0\right)\\
&= T_1
\end{aligned}
$$

Remark. Since there is a one-one correspondence between the set of feasible solution $(P_{2.1})$ and $(P_{2.1}^{*})$ the corresponding feasible solution yield same value of objective function, it follows that $(P_{2.1})$ and $(P_{2.1}^{*})$ are equivalent. Hence an optimal solution of the problem $(P_{2.1}^{*})$ yield minimum time of Stage-I.

Solution Strategy

A feasible solution of the problem $(P_{2.1}^{*})$ provides a feasible solution of the problem $(P_{2.1})$ and conversely. In order to find an optimal feasible solution of the problem $(P_{2.1})$, feasible solution of the problem $(P_{2.1}^{*})$ and corresponding optimal feasible solution of the Stage-II problem are obtained such that $\left(T_1(\cdot)+T_2(\cdot)\right)$ is the least. Instead of solving Stage-I and Stage-II problem separately, a unified approach has been obtained which avoid investigation of a number of undesirable solutions of the problem $(P_{2.1})$. In this regard, specific restricted versions of the problem $(P_{2.1}^{*})$ are introduced.

Restricted versions of $\left(P_{2.1}^{*}\right)$

Let the cells (i, j) in $I \times J$ be partitioned into the disjoint sets corresponding to the distinct time entries of t_{ij} in the time matrix. That is, $M_k = \left\{(i, j) \in I \times J : t_{ij} = T^k\right\}$ where $T^1 > T^2 > .....T^p$.

At each iteration, a restricted version of the problem $\left(P_{2.1}^{*}\right)$ is obtained by abandoning some undesirable routes. If at the k^{th} iteration, value of the objective function $T_2\,(.)$ of Stage-II problem is T_2^k then, set $t_{i,n+1} = M, \forall\, i \in I$ for which $\min_{j \in J} t_{ij} \geq T_2^k$, where M is a very large positive real number. Also $t_{ij} = M \;\forall\; (i, j) \in I \times J$ for which $\left(t_{ij} + T_2^L\right) \geq \min_{h \in \{1,2,......k-1\}}\left(T_1^h + T_2^h\right)$, where, T_2^L represents the value of lower bound on the value of $T_2\,(.)$. This restricted version of $\left(P_{2.1}^{*}\right)$ defined at time T_2^k is denoted by $P_{2.1}^{*}\left(T_2^k\right)$ and its LOS by Y_2^k. The LOS, if M-feasible, of the problem $P_{2.1}^{*}\left(T_2^k\right)$ provides a feasible solution of the problem $\left(P_{2.1}\right)$, with appropriate values viz. $T_1^{k+1}\left(\geq T_1^k\right)$ and $T_2^{k+1}\left(< T_2^k\right)$ of the transportation times for the Stage-I and Stage-II respectively, where an M-feasible solution is one in which $y_{ij} = 0 \;\forall\; (i, j) \in I \times J$ for which $t_{ij} = M$.

Theorem 3. Let LOS of the problem $P_{2.1}^{*}\left(T_2^g\right)$ be an M-feasible solution for $g = 0,1,2,....k+1$, then $T_1\left(Y^{g+1}\right) \geq T_1\left(Y^g\right)$, where Y^g is an LOS of the problem $P_{2.1}^{*}\left(T_2^g\right)$.

Proof. Let $T_1\left(Y^{g+1}\right) = T_1\left(Y^g\right)$. We claim that $T_1^{g+1} \geq T_1^g$.

Suppose, on contrary $T_1^{g+1} < T_1^g$. Therefore, in the problem $P_{2.1}^{*}\left(T_2^{g-1}\right)$, by its definition, $t_{ij} < M \;\forall (i, j) \in I \times J,\; t_{ij} = T_1^{g+1}$ *and* $t_{in+1} = 0$ *for* $i \in I$ for which $\min_j t_{ij} \leq T_2^g < T_2^{g-1}$. Therefore, an LOS of the problem $P_{2.1}^{*}\left(T_2^g\right)$ yielding the optimal time of transportation of Stage-I as T_1^{g+1}, is a basic feasible solution of the problem $P_{2.1}^{*}\left(T_2^{g-1}\right)$. Thus, we have a basic feasible solution of the problem $P_{2.1}^{*}\left(T_2^{g-1}\right)$ with associated time $T_1^{g+1} < T_1^g$. This is a contradiction because T_1^g is the optimal time of transportation of Stage-I for the problem $P_{2.1}^{*}\left(T_2^{g-1}\right)$, yielded by its M-feasible LOS. Hence $T_1^{g+1} \geq T_1^{g}$.

Theorem 4. Let if possible $\hat{T}_1 + \hat{T}_2 = \min_{k\geq 0}\left[T_1^k + T_2^k\right]$, where T_1^k and T_2^k are the times of transportation of Stage-I and Stage-II, respectively corresponding to an M-feasible optimal solution of the problem $P_{2.1}^{*}\left(T_2^{k-1}\right)$, $k \geq 1$, and T_1^0 and T_2^0 are the transportation times corresponding to an LOS of the problem $\left(P_{2.1}^{*}\right)$, then $\hat{T}_1 + \hat{T}_2$ is the optimal value of objective function of the problem $\left(P_{2.3}\right)$.

Proof. Let, if possible $\left(\hat{T}_1 + \hat{T}_2\right)$ be not the optimal value of the objective function of the problem $\left(P_{2.3}\right)$, then there exists a pair $\left(\tilde{T}_1, \tilde{T}_2\right)$ yielded by some feasible solution $\tilde{Y}$ of problem $\left(P_{2.3}\right)$ such that

$$\tilde{T}_1 + \tilde{T}_2 < \hat{T}_1 + \hat{T}_2 \qquad (3)$$

Clearly, $\tilde{T}_2 \geq T_2^L$. Also $\tilde{T}_2 \leq T_2^0$ because if $\tilde{T}_2 > T_2^0$ and $\tilde{T}_1 + \tilde{T}_2 < T_1^0 + T_2^0$ implies $\tilde{T}_1 < T_1^0$, which contradicts the optimality of T_1^0. Thus, $T_2^L \leq \tilde{T}_2 \leq T_2^0$.

If $\left(T_1^r, T_2^r\right)$ is a pair such that either the OFS of the problem $P_{2.1}^{*}\left(T_2^r\right)$ is not an M-feasible solution or $T_2^r = T_2^L$, then $\tilde{T}_2 \geq T_2^L$. Hence, there exists an index $k \geq 1$ and $T_2^k \leq \tilde{T}_2 \leq T_2^0$.

Case-1 $\tilde{T}_2 > T_2^k$

In the problem $P_{2.1}^{*}\left(T_2^{k-1}\right)$ let $B = \left\{(i,j) : t_{ij} + T_2^L < \min_{h=\{0,1,2,.....k-1\}}\left(T_1^h + T_1^h\right)\right\}$.

As an M-feasible LOS of $P_{2.1}^{*}\left(T_2^{k-1}\right)$ yields the corresponding transportation times as T_1^k, therefore, the set $\left\{(i,j) : t_{ij} = T_1^k\right\} \subseteq B$.

Thus, an M-feasible LOS of the problem $P_{2.1}^{*}\left(T_2^{k-1}\right)$ should give the optimal time of transportation of Stage-I as $\tilde{T}_1$, which is not true. Therefore, $\tilde{T}_2 \leq T_2^k$.

Case-2 $\tilde{T}_2 = T_2^k$

As $\left(\tilde{T}_1 + \tilde{T}_2\right) < \left(T_1^k + T_2^k\right)$ in this case also we get $\tilde{T}_1 < T_1^k$, which leads to a contradiction. Thus, there does not exist $\left(\tilde{T}_1, \tilde{T}_2\right)$, satisfying (3).

Computing T_2^L

Find $\min_{I\times J}\{t_{ij}\} = t_{r_1 s_1}$. If $b'_{n+1} \le a_{r_1}$, then $T_2^L = t_{r_1 s_1}$ else, find $\min_{I\times J\setminus(r_1,s_1)} t_{ij} = t_{r_2 s_2}$

If $b'_{n+1} \le a_{r_1} + a_{r_2}$ then $T_2^L = t_{r_2 s_2}$.

Continuing in this way we get,

$$\min_{I\times J\setminus\{(r_1,s_1),(r_2,s_2),\ldots\ldots(r_k,s_k)\}} t_{ij} = t_{r_{k+1} s_{k+1}}$$

If

$$b'_{n+1} \le \sum_{l=1}^{k+1} a_{r_l}$$

and

$$\sum_{l=1}^{k} a_{r_l} < b'_{n+1},$$

then

$$T_2^L = t_{r_{k+1} s_{k+1}}$$

2.3 Algorithm

Initial Step: Obtain an OFS of the problem $\left(P_{2.1}^*\right)$ and note the time of Stage-I and Stage-II, say T_1^0 and T_2^0, respectively. If $T_2^0 = T_2^L$, then stop and go to Terminal Step; else, go to General Step.

General Step: Let the pairs in hand be $\left(T_1^g, T_2^g\right)$ for $g = 1, 2, \ldots., k-1$.

Construct the problem $P_{2.1}^*\left(T_2^{k-1}\right)$ and find its OFS. If this is not M-feasible solution then stop and go to Terminal Step, otherwise read the time T_1^k of Stage-I and T_2^k of Stage-II. If $T_2^k = T_2^L$, stop and go to Terminal Step, else repeat the General Step for next higher value of k.

Terminal Step: Declare $\min_{r \ge 0}\left\{T_1^r + T_2^r\right\}$ as the optimal value of the objective function of the problem $\left(P_{2.3}\right)$.

2.4 Example:

Consider the following 6×4 imbalanced bottleneck transportation problem:

	D_1	D_2	D_3	D_4	a_i
S_1	5	6	4	3	30
S_2	7	9	12	10	40
S_3	2	8	7	4	45
S_4	11	5	9	8	25
S_5	6	10	5	3	50
S_6	12	4	2	10	20
b_j	50	40	30	40	

The partition of various time routes is given as:

$T_1 = 12 > T_2 = 11 > T_3 = 10 > T_4 = 9 > T_5 = 8 > T_6 = 7 > T_7 = 6 > T_8 = 5 > T_9 = 4 > T_{10} = 3 > T_{11} = 2$

Here $T_2^L = 2$.

Add a new dummy destination with demand $\sum_i a_i - \sum_j b_j$ and $t_{i5} = 0$ in order to get the following balanced TMTP.

	D_1	D_2	D_3	D_4	D_5	a_i
S_1	5	6	4	3	0	30
S_2	7	9	12	10	0	40
S_3	2	8	7	4	0	45
S_4	11	5	9	8	0	25
S_5	6	10	5	3	0	50
S_6	12	4	2	10	0	20
b_j	50	40	30	40	50	

Initial Step. An OBFS of the problem $(P_{2.1}^{*})$ gives $T_1^0 = 5$ and corresponding $T_2^0 = 7$ given in the following table:

	D_1	D_2	D_3	D_4	D_5	a_i
S_1	5 (5)	6	4 (25)	3	0	30
S_2	7	9	12	10	0 (40)	40
S_3	2 (45)	8	7	4	0	45
S_4	11	5 (20)	9	8	0(5)	25
S_5	6	10	5 (5)	3 (40)	0 (5)	50
S_6	12	4 (20)	2	10	0	20
b_j	50	40	30	40	50	

So the current value of $T_1^0 + T_2^0 = 12$. Since $T_2^0 > T_2^L$. Go to General Step of the algorithm.

General Step.

Iteration 1. Construct the problem $P_{2.1}^{*}(T_2^0)$ its OBFS yields value $T_1^1 = 7$ and corresponding $T_2^1 = 5$ given in the following table:

	D_1	D_2	D_3	D_4	D_5	a_i
S_1	5	6	4 (30)	3 (0)	0	30
S_2	7 (40)	9	M	M	M	40
S_3	2 (10)	8	7	4	0 (35)	45
S_4	11	5 (20)	9	8	0 (5)	25
S_5	6	10	5 (5)	3 (40)	0 (10)	50
S_6	12	4 (20)	2	10	0	20
b_j	50	40	30	40	50	

So the current value of $T_1^1 + T_2^1 = 12$ and the current value of $T_1 + T_2$ is $\left(T_1^0 + T_2^0, T_1^1 + T_2^1\right) = 12$. As $T_2^1 > T_2^L$, solve $P_{2.1}^*\left(T_2^1\right)$.

Iteration 2. $P_{2.1}^*\left(T_2^1\right)$ yields time $\left(T_1^2, T_2^2\right) = (7,3)$ given in the following table:

	D_1	D_2	D_3	D_4	D_5	a_i
S_1	5	6	4 (25)	3 (5)	0	30
S_2	7 (40)	9	M	M	M	40
S_3	2 (10)	8	7	4	0 (35)	45
S_4	M	5 (25)	9	8	M	25
S_5	6	M	5	3 (35)	0 (15)	50
S_6	M	4 (15)	2 (5)	M	0	20
b_j	50	40	30	40	50	

Iteration 3. $P_{2.1}^*\left(T_2^2\right)$ yields time $\left(T_1^3, T_2^3\right) = (7,2)$ given in the following table

	D_1	D_2	D_3	D_4	D_5	a_i
S_1	5 (10)	6	4 (20)	3	M	30
S_2	7 (40)	9	M	M	M	40
S_3	2 (0)	8	7	4	0 (45)	45
S_4	M	5 (25)	9	8	M	25
S_5	6	M	5 (10)	3 (40)	M	50
S_6	M	4 (15)	2	M	0 (5)	20
b_j	50	40	30	40	50	

Since $T_2^3 = T_2^L$, stop and go to Terminal Step.

Terminal Step. The optimal value of the objective function $(P_{2.3})$ is given by $\min_{h=0,1,..3}\left(T_1^h + T_2^h\right) = 9$. Hence the optimal value of the problem $(P_{2.3})$ corresponds to pair $(7,2)$.

Concluding Remark

- At each iteration, we are solving only one problem and the OFS of the Stage-II problem is obtained by just allocating the surplus amount in each row to the minimum and then next minimum time route till all the surplus amount in each row is exhausted.

Chapter 3

Bottleneck Interval Transportation Problem

This chapter discusses two stage interval time minimization transportation problems in which the total availability is known to lie in a specified interval. Two different approaches are discussed in this regard. The present problem consists of two stages, second stage being dependent upon the first stage. In the first stage, the sources ship all of their on-hand material to the demand points, while a second-stage delivery covers the demand that is not fulfilled in the first stage. It is assumed that in both the stages transportation of the product is done in parallel. In each stage, the objective is to minimize the shipment time. Overall goal is to find a solution that minimizes the sum of first and second stage shipment time.

3.1 Two Stage Bottleneck Transportation Problem

3.1.1 Mathematical Formulation of Two Stage Interval TMTP:

Suppose a_i and a_i' , $i \in I$ denote the respectively the minimum and maximum availability of a homogenous product at the source i and b_j, $j \in J$ the demand of the same at the destination j, where $\sum_{i \in I} a_i < \sum_{j \in J} b_j < \sum_{i \in I} a_i'$

In the first stage of two stage interval time minimization transportation problem the quantity $a_i < (a_i')$ is transported from each source and after its completion, enough quantity of the product is dispatched in the second stage so as to exactly satisfy the demand points at the destination. The Stage-I is thus formulated as:

$$\min_{Y \in S_2} [T_1(Y)] \qquad (P_{3.1a})$$

where, $T_1(Y) = \max_{I \times J} (t_{ij}(y_{ij}))$ and the set S_2 is given by

$$S_2 : \begin{cases} \sum_{j \in J} y_{ij} = a_i, & i \in I \\ \sum_{i \in I} y_{ij} \le b_j, & j \in J \\ y_{ij} \ge 0, & \forall (i, j) \in I \times J \end{cases}$$

Corresponding to a feasible solution $Y = (y_{ij})$ of the stage-I problem, let S_2 (Y) be the set of feasible solutions of the Stage-II problem which is stated as:

$$\min_{Z \in S_2(Y)} [T_2(Z)] \qquad (P_{3.1b})$$

where, $T_2(Z) = \max_{I \times J}(t_{ij}(z_{ij}))$ and

$$S_2(Y) : \begin{cases} \sum_{j \in J} z_{ij} \le a'_i - a_i & i \in I \\ \sum_{i \in I} z_{ij} = b_j - b'_j & j \in J \\ z_{ij} \ge 0, & \forall (i, j) \in I \times J \end{cases}$$

and $b'_j = \sum_{i \in I} y_{ij}, \ j \in J$

The aim is to find that feasible shipment schedule for the Stage–I transportation corresponding to which the optimal feasible shipment schedule for the Stage–II transportation is such that the sum of the associated shipment times in the two stages is the least.

Thus, a two stage interval bottleneck transportation problem can be stated as:

$$\min_{Y \in S_2} \left[T_1(Y) + \left(\min_{Z \in S_2(Y)} T_2(Z) \right) \right] \qquad (P_{3.1c})$$

Closely related to the problem $(P_{3.1c})$ is the interval bottleneck transportation problem $(P_{3.1d})$ defined as:

$$\min_{X \in S'_2} [T(X)] = \min_{X \in S'_2} \left[\max_{I \times J} (t_{ij}(x_{ij})) \right] \qquad (P_{3.1d})$$

where,

$$S'_2 : \begin{cases} a_i \le \sum_{j \in J} x_{ij} \le a'_i, & i \in I \\ \sum_{i \in I} x_{ij} = b_j, & j \in J \\ x_{ij} \ge 0, & \forall (i, j) \in I \times J \end{cases}$$

Clearly, a feasible solution of the problem $(P_{3.1c})$ provides a feasible solution to the problem $(P_{3.1d})$ and conversely. Associated with problem $(P_{3.1d})$ a balanced transportation problem is defined as:

$$\min_{X\in\hat{S}_2}\left[\hat{T}(X)\right]=\min_{X\in\hat{S}_2}\left[\max_{\hat{I}\times\hat{J}}\left(\hat{t}_{ij}\left(x_{ij}\right)\right)\right] \qquad (P_{3.1e})$$

where,

$$\hat{S}_2:\begin{cases}\sum_{j\in\hat{J}}x_{ij}=\hat{a}_i, & i\in\hat{I}\\ \sum_{i\in\hat{I}}x_{ij}=\hat{b}_j, & j\in\hat{J}\\ x_{ij}\geq 0, & \forall\,(i,j)\in\hat{I}\times\hat{J}\end{cases}$$

where,

$$\begin{aligned}
&\hat{I}=\{1,2,\ldots,m,m+1,\ldots,2m\}\\
&\hat{J}=J\cup\{n+1\}\\
&\hat{a}_i=a_i,\ i=1,2,\ldots,m\\
&\hat{a}_{m+i}=a'_i-a_i,\ i=1,2,\ldots,m\\
&\hat{b}_j=b_j,\ j\in J\\
&\hat{b}_{n+1}=\sum_{i\in I}a'_i-\sum_{j\in J}b_j\\
&\hat{t}_{ij}=t_{ij},\ i=1,2,\ldots,m,\ j\in J\\
&\hat{t}_{m+i,j}=t_{ij},\ i=1,2,\ldots,m,\ j\in J\\
&\hat{t}_{i,n+1}=M\,(>>0),\ i=1,2,\ldots,m\\
&\hat{t}_{m+i,n+1}=0,\ i=1,2,\ldots,m
\end{aligned}$$

An LOS of the problem $(P_{3.1e})$ will provide the overall minimum shipment time for $(P_{3.1e})$.

3.1.2 Theoretical Development

As the shipment times in Stage-I and Stage-II are concave functions, two stage interval BTP aims at minimizing a concave function over a polytope. Hence $(P_{3.1c})$ is also a concave minimization problem. As global minimizer of a CMP over a polytope is attainable at an extreme point of the polytope, it is desirable to investigate only its extreme points.

M-feasible solution: A feasible solution $X=(x_{ij})$ of the problem $(P_{3.1e})$ is said to be M-feasible solution (MFS) if $x_{ij}=0\ \forall\,(i,j):\,t_{ij}=M$.

It may be noted that if $X=(x_{ij})_{\hat{I}\times\hat{J}}$ is an M-feasible solution of problem $(P_{3.1e})$, then $Y = \left(y_{ij}\right)_{I \times J}$ and $Z = \left(z_{ij}\right)_{I \times J}$ are respectively feasible solutions of Stage-I and Stage-II problems where, $y_{ij} = x_{ij} \forall (i,j) \in I \times J$ and $z_{ij} = x_{m+i,j} \forall (i,j) \in I \times J$.

Further if $\left(\min_{Z \in S_2(Y)} T_2(Z) = T_2\left(\hat{Z}\right) \right)$ then a feasible solution of the problem $\left(P_{3.1c}\right)$ consists of the feasible solution Y for Stage-I and feasible solution $\hat{Z}$ for Stage-II.

The next two theorems establish equivalence between the problem $\left(P_{3.1d}\right)$ and the problem $\left(P_{3.1e}\right)$ over the set of its M-feasible solutions.

Theorem 3.1.1: An M-feasible solution of the problem $\left(P_{3.1e}\right)$ corresponds to a feasible solution of the problem $\left(P_{3.1d}\right)$ and vice-versa.

Proof: Let $Y = \left(y_{ij}\right)$ be a feasible solution of the problem $\left(P_{3.1e}\right)$. For each $i \in I$, define $x_{ij} = y_{ij} + y_{m+i,j} \quad \forall j \in J$. It can be easily establish that $X = \left(x_{ij}\right)$ is a feasible solution of the problem $\left(P_{3.1d}\right)$.

Conversely, let $X = \left(x_{ij}\right),\ (i,j) \in (I \times J)$ be a feasible solution of the problem $\left(P_{3.1d}\right)$. Let $\sum_{j \in J} x_{ij} = \bar{a}_i$, $i \in I$. Clearly $a_i \le \bar{a}_i \le a_i'$. Therefore,

$$\sum_{j \in J} x_{ij} = \bar{a}_i, \quad i \in I$$

$$\sum_{i \in I} x_{ij} = b_j, \quad j \in J$$

$$x_{ij} \ge 0, \qquad \forall (i,j) \in (I \times J)$$

Obviously, $\sum_{i \in I} \bar{a}_i = \sum_{j \in J} b_j$

Consider the following imbalanced BTP:

$$\min T(Z) = \min \left[\max_{I \times J} \left(t_{ij}\left(z_{ij}\right) \right) \right]$$

Subject to

$$\sum_{j \in J} z_{ij} = a_i,\ i \in I$$

$$\sum_{i \in I} z_{ij} \le b_j,\ j \in J$$

$$z_{ij} \ge 0, \qquad \forall (i,j) \in (I \times J)$$

Let $Z = \left(z_{ij}\right)$ be an optimal feasible solution of this problem and let

$\sum_{i \in I} z_{ij} = \bar{b}_j,\ j \in J$ .Now, set

$$\left.\begin{aligned} y_{ij} &= z_{ij}, \quad \forall (i, j) \in (I \times J) \\ y_{i,n+1} &= 0, \quad i \in I \end{aligned}\right\} \tag{3.2}$$

Next, consider the balanced BTP defined as follows:

$$\min T(W) = \min\left[\max_{I \times J}\left(t_{ij}\left(w_{ij}\right)\right)\right]$$

Subject to,

$$\sum_{j \in J} w_{ij} = \bar{a}_i - a_i, \quad i \in I$$
$$\sum_{i \in I} w_{ij} = b_j - \bar{b}_j, \quad j \in J$$
$$w_{ij} \geq 0, \quad \forall (i, j) \in (I \times J)$$

If $W = \left(w_{ij}\right)$ is an optimal solution of the above problem, then set

$$\left.\begin{aligned} y_{m+i,j} &= w_{ij}, \quad \forall (i, j) \in (I \times J) \\ y_{m+i,n+1} &= a_i' - \bar{a}_i, \quad i \in I \end{aligned}\right\} \tag{3.3}$$

It can be easily seen that $Y = \left(y_{ij}\right), \left(i, j\right) \in \hat{I} \times \hat{J}$ as defined by (3.2) and (3.3) above, is an M-feasible solution of the problem $\left(P_{3.1e}\right)$.

Theorem 3.1.2: The value of the objective function of the problem $\left(P_{3.1e}\right)$ at an M-feasible is same as the value of objective function of the problem $\left(P_{3.1d}\right)$ at the corresponding feasible solution.

Proof: Let $Y = \left(y_{ij}\right)$ be an M-feasible solution of the problem with $\left(P_{3.1e}\right)$ the value of its objective function as T_β. Let $X = \left(x_{ij}\right)$ be the corresponding feasible solution of the problem $\left(P_{3.1d}\right)$ giving T_α as the value of its objective function. Then, $T_\beta = \max_{\hat{I} \times \hat{J}}\left(\hat{t}_{ij}\left(y_{ij}\right)\right)$

$$= \left\{\max_{I \times J}\left(t_{ij}\left(y_{ij}\right)\right), \max_{I}\left(t_{i,n+1}\left(y_{i,n+1}\right)\right), \max_{I \times J}\left(t_{m+i,j}\left(y_{m+i,j}\right)\right), \max_{J}\left(t_{m+i,n+1}\left(y_{m+i,n+1}\right)\right)\right\}$$

$$= \left\{\max_{I \times J}\left(t_{ij}\left(y_{ij}\right)\right), \max_{I \times J}\left(t_{m+i,j}\left(y_{m+i,j}\right)\right)\right\}$$

$$= \left\{ \max_{I \times J} \left(t_{ij} \left(x_{ij} \right) \right) \right\}$$

Since $x_{ij} = y_{ij} + y_{m+i,j}$ and $t_{m+i,j} = t_{ij}$, $(i, j) \in I \times J$

$$= T_{\alpha}$$

3.1.3 Solution Strategy

The solution strategy for the two stage interval TMTP depends upon the following types of standard TMTP's and CMTP's.

The possibility of Stage-II shipment time less than the on hand shipment time, say T^{l-1}, is examined by studying the bottleneck transportation problem $P_{L\beta}\left(T^{l-1}\right)$ derived from the problem $\left(P_{3.1e}\right)$ by abandoning the routes $(m+i, j)$ for which $t_{m+i,j} \geq T^{l-1}, (i, j) \in I \times J$ i.e., setting $t_{m+i,j} = M(>>0), (i, j) \in I \times J$ for which $t_{m+i,j} \geq T^{l-1}$. If optimal solution of $P_{L\beta}\left(T^{l-1}\right)$ is an MFS, then the new Stage-I shipment time is more than the on hand Stage-I shipment time and the corresponding Stage-II shipment time is less than the on hand Stage-II shipment time.

Suppose an optimal solution of $P_{L\beta}\left(T^{l-1}\right)$ is M-feasible and let Stage-I and Stage-II shipment times corresponding to this optimal solution be T^k and T^l respectively. To find the least possible Stage-II shipment time corresponding to the Stage-I shipment time T^k, the following cost minimizing transportation problem (call it $C_{L\beta}(T^k, T^l)$) is solved.

$$\min_{X \in \hat{S}_2} \sum_{I \times J} c_{ij}\, x_{ij} \qquad C_{L\beta}(T^k, T^l)$$

where,

$$c_{ij} = M,\ (i, j) \in I \times J,\ \textit{for which} \quad t_{ij} > T^k$$
$$= 0,\ (i, j) \in I \times J,\ \textit{for which} \quad t_{ij} \leq T^k$$

$$c_{m+i,j} = M,\ (i, j) \in I \times J,\ \textit{for which} \quad t_{m+i,j} > T^l$$
$$= \lambda_l,\ (i, j) \in I \times J,\ \textit{for which} \quad t_{m+i,j} = T^l$$
$$= \lambda_{l+1},\ (i, j) \in I \times J,\ \textit{for which} \quad t_{m+i,j} = T^{l+1}$$
$$\vdots \qquad \vdots$$
$$= \lambda_s,\ (i, j) \in I \times J,\ \textit{for which} \quad t_{m+i,j} = T^s$$

and $\lambda_{j+1} >> \lambda_j,\ j = 1,2,...s-1$. As an M-feasible optimal solution of the problem $P_{L\beta}\left(T^{l-1}\right)$ is a feasible solution of $C_{L\beta}(T^k, T^l)$ and as $\sum_{j \in J} b_j > \sum_{i \in I} a_i$, it follows that optimal value $C_{L\beta}(T^k, T^l)$ in will be non-zero. Corresponding to Stage-I shipment time T^k the optimal feasible solution of $C_{L\beta}(T^k, T^l)$ provides the least Stage-II shipment time which is less than or equal to T^l. Thus, OBFS of $C_{L\beta}(T^k, T^l)$ provides a feasible solution of the problem $\left(P_{3.1c}\right)$. It may be observed that the problems $P_{L\beta}\left(T^{l-1}\right)$ and $C_{L\beta}(T^k, T^l)$ for various values of l help in generating the pairs of the type $\left((T_1(.), T_2(.)) : T_1(.) > T_2(.)\right)$ for the Stage-I and Stage-II shipment times.

Similarly to generate the pairs of the type $\left((T_1(.), T_2(.)) : T_1(.) < T_2(.)\right)$, bottleneck transportation problems of the type $P_{U\beta}\left(T^j\right)$ and CMTPs of the type $C_{U\beta}(T^k, T^l)$ are studied, where $P_{U\beta}\left(T^j\right)$ is the BTP derived from the problem $\left(P_{3.1e}\right)$ by abandoning the routes $(i, j) \in I \times J$ for which $t_{ij} \geq T^j,\ (i, j) \in I \times J$ and is defined as:

$$\min_{X \in \hat{S}_2} \sum_{I \times J} c_{ij} x_{ij} \qquad\qquad C_{U\beta}(T^k, T^l)$$

where,

$$
\begin{aligned}
c_{m+i,j} &= M, \ (i,j)\in I\times J, \ \textit{for which} \quad t_{m+i,j} > T^{l} \\
&= 0, \quad (i,j)\in I\times J, \ \textit{for which} \quad t_{m+i,j} \le T^{l} \\
c_{ij} \quad &= M, \ (i,j)\in I\times J, \ \textit{for which} \ t_{ij} > T^{k} \\
&= \lambda_k, \ (i,j)\in I\times J, \ \textit{for which} \ t_{ij} = T^{k} \\
&= \lambda_{k+1}, (i,j)\in I\times J, \ \textit{for which} \ t_{ij} = T^{k+1} \\
&\quad \vdots \quad \vdots \\
&= \lambda_s, \ (i,j)\in I\times J, \ \textit{for which} \ t_{ij} = T^{s}
\end{aligned}
$$

and $\lambda_{j+1} >> \lambda_j, j = k, k+1, ... s-1$.

A brief outline of the solution strategy for finding the global solution for the problem $(P_{3.1c})$ is as follows. After finding an optimal solution of the problem $(P_{3.1e})$ the corresponding value of Stage-I shipment time and Stage-II shipment time are obtained. Let Stage-I shipment time be more than or equal to the Stage –II shipment time. First Stage–II shipment time is reduced to its minimum corresponding to the Stage-I shipment time by solving appropriate CMTP of type $C_{L\beta}(;)$. Further, an appropriate restricted BTP depending upon the Stage-II shipment time is solved to obtain a new pair of Stage-I and Stage-II shipment times which is further refined by solving appropriate CMTP of the aforementioned type. This process of solving CMTP and BTP alternately goes on till the termination occurs.

Various claims are established in the following theorems for a sound mathematical foundation for the procedure to find the optimal solution for the two stage interval TMTP $(P_{3.1c})$.

Theorem 3.1.3: In a pair $\left(T^{r-p_k}, T^{r+q_k}\right)$ corresponding to the OBFS of the problem $CP_{L\beta}\left(T^{r-p_k}, T^{r+q_k^0}\right)$, T^{r-p_k} is the minimum Stage-I shipment time corresponding to time T^{r+q_k} of stage-II shipment, where T^{r-p_k} and $T^{r+q_k^0}$ are the Stage-I and Stage-II shipment times respectively to the M-feasible LOS of $P_{L\beta}\left(T^{r+q_{k-1}}\right)$.

Proof: To prove that the Stage-I shipment time T^{r-p_k} is the minimum corresponding to the Stage-II shipment time T^{r+q_k}, assume the contrary. Suppose that T^{r-p_k} is not the minimum shipment time for Stage-I corresponding to time

T^{r+q_k} of Stage-II shipment. This implies that there exists a solution, say $\hat{X}$, of the problem (P_b) such that $T_1(\hat{X}) = T^{r-\hat{p}} < T^{r-p_k}$ and $T_2(\hat{X}) = T^{r+q_k}$. Clearly $T^r \leq T_1(\hat{X}) < T^{r-p_k}$ as $T^r\left(= T^{r-p_0}\right)$ is the minimum shipment time for Stage-I yielded by the LOS of the problem $(P_{3.1e})$. M-feasible LOS of $P_{L\beta}\left(T^{r+q_{k-1}}\right)$ yields the Stage-I shipment time T^{r-p_k}, which is also overall shipment time for this TMTP. Also by definition $P_{Lb}\left(T^{r+q_{k-1}}\right)$ it follows that $\hat{X}$ is its feasible solution.

By assumption $\hat{X}$ yields Stage-I shipment time $T^{r-\hat{p}}\left(< T^{r-p_k}\right)$ and Stage-II shipment time T^{r+q_k}. Thus $\hat{X}$ yields overall shipment time for the time minimizing transportation problem $P_{Lb}\left(T^{r+q_{k-1}}\right)$ smaller than the one yielded by its LOS, which cannot be true. Hence T^{r-p_k} is the minimum Stage-I shipment time corresponding to the Stage-II shipment time T^{r+q_k}.

Theorem 3.1.4: In a pair $\left(T^{r+\tilde{q}_k}, T^{r-\tilde{p}_k}\right)$ corresponding to the OBFS of the problem $CP_{U\beta}\left(T^{r+\tilde{q}_k^0}, T^{r-\tilde{p}_k}\right)$, $T^{r-\tilde{p}_k}$ is the minimum Stage-II shipment time corresponding to time $T^{r+\tilde{q}_k}$ of Stage-I shipment, where $T^{r+\tilde{q}_k^0}$ and $T^{r-\tilde{p}_k}$ are the Stage-I and Stage-II shipment times corresponding to the M-feasible LOS of $P_{U\beta}\left(T^{r+\tilde{q}_{k-1}}\right)$.

Proof: The proof is similar to the proof of the above theorem.

Theorem 3.1.5: If LOS, say $\underline{X}^{r+q_{K+1}^0}$, of the time minimizing transportation problem $P_{Lb}\left(T^{r+q_k}\right)$ is not an MFS, then $T_1\left(\underline{X}^{r+q_{K+1}^0}\right) + T_2\left(\underline{X}^{r+q_{K+1}^0}\right) \geq \min_{j=0,1,2...k}\left\{T^{r-p_j} + T^{r+q_j}\right\}$ each of problems $P_{L\beta}\left(T^{r+q_j}\right) \forall\, j = 1,2...,k-1$ has M-feasible LOS.

Proof: As LOS of $P_{Lb}\left(T^{r+q_k}\right)$ is not an M-feasible solution; the Stage-II shipment time cannot be further reduced below T^{r+q_k}. Currently best value of the sum of the shipment times in Stage-I and Stage-II is $\min_{j=0,1,2...k}\left\{T^{r-p_j} + T^{r+q_j}\right\}$. At non M-feasible LOS of $P_{Lb}\left(T^{r+q_k}\right)$ either (i) the Stage-I shipment time is one of the first $(k+1)$ recorded times: $T^{r-p_0}, T^{r-p_1}......, T^{r-p_k}$ in which case the corresponding minimum

Stage-II shipment time is already known or (ii) the Stage-I shipment time is none of the first $(k+1)$ recorded times but it lies in the interval $\left[T^{r-p_0}, T^{r-q_0}\right]$, in which case one of the recorded Stage-I and Stage-II shipment times would yield a smaller value of the sum of the shipment times or (iii) the Stage-I shipment is more than T^{r-p_k} in which case is $\min_{j=0,1,2...k}\left\{T^{r-p_j}+T^{r+q_j}\right\}$ will be smaller than the sum of the Stage-I and Stage-II shipment times at the current non M-feasible LOS of $P_{Lb}\left(T^{r+q_k}\right)$. Hence the sum of the Stage-I and Stage-II shipment times corresponding to a non M-feasible LOS of the problem $P_{Lb}\left(T^{r+q_k}\right)$ is not less than the current best value of the sum. That is,

$$T_1\left(\underset{-}{X}^{r+q_{k+1}^0}\right)+T_2\left(\underset{-}{X}^{r+q_{k+1}^0}\right)\geq \min_{j=0,1,2...k}\left\{T^{r-p_j}+T^{r+q_j}\right\}$$

Remark 3.1.1: If LOS of the problem $P_{Lb}\left(T^{r+q_j}\right)$ is not an MFS, then no further restricted version of the problem $(P_{3.1e})$, namely $P_{L\beta}\left(T^{r+q_j}\right)\forall\, j\geq k$ can provide a solution of the problem $(P_{3.1c})$ yielding value better than $\min_{j=0,1,2...k}\left\{T^{r-p_j}+T^{r+q_j}\right\}$

Remark 3.1.2: Let pairs in hand of Stage-I and Stage-II shipment times be $\left\{T^{r-p_j}+T^{r+q_j}\right\}$, $j=0,1,2...k$. Let LOS of the problem $P_{Lb}\left(T^{r+q_k}\right)$ be an MFS. Then, Stage-I shipment time corresponding to this M-feasible LOS is more than T^{r-p_k} since for a given Stage-I shipment time less than or equal to T^{r-p_k}. (Recall that M-feasible solution LOS of $P_{Lb}\left(T^{r+q_k}\right)$ yields Stage-II shipment time less than T^{r+q_k}).

Theorem 3.1.6: If LOS of the problem $P_{Ub}\left(T^{r+q_0}\right)$, say $\overline{X}^{r+\tilde{q}_0}$, is not an MFS, then

$$T_1\left(\overline{X}^{r+\tilde{q}_0^0}\right)+T_2\left(\overline{X}^{r+\tilde{q}_0^0}\right)\geq \min_{j\geq 0}\left\{T^{r-p_j}+T^{r+q_j}\right\}$$

Proof: As $\overline{X}^{r+\tilde{q}_0^0}$ is not an MFS, we have $\bar{x}_{ij}^{r+\tilde{q}_0^0}>0$ for some $(i,j)\in I\times J$ for which $t_{ij}\geq T^{r+q_0}$, *i.e.* $T_1\left(\overline{X}^{r+\tilde{q}_0^0}\right)\geq T^{r+q_0}$. Also we have $T_2(\overline{X}^{r+\tilde{q}_0^0})\geq T^r=T^{r-p_0}$ (since $p_0=0$).

Therefore,

$$T_1(\bar{X}^{r+\tilde{q}_0^0}) + T_2(\bar{X}^{r+\tilde{q}_0^0}) \geq T^{r+q_0} + T^{r-p_0} \geq \min_{j\geq 0}\left\{T^{r-p_j} + T^{r+q_j}\right\}.$$

Hence the result.

Remark 3.1.3: If LOS of the problem $P_{Ub}\left(T^{r+q_0}\right)$ is not an MFS, then the restricted versions $P_{U\beta}\left(T^{r+\tilde{q}_j}\right)\ \forall\ j \geq 0$ of the problem $\left(P_{3.1e}\right)$ cannot provide optimal solution of the problem $\left(P_{3.1c}\right)$. This also implies that there does not exist a feasible solution of the problem $\left(P_{3.1c}\right)$ having Stage-I time less than T^{r+q_0}.

Theorem 3.1.7: If LOS, say $\bar{X}^{r+\tilde{q}_0^{k+1}}$, of $P_{Ub}\left(T^{r+\tilde{q}_k}\right)$ is not an MFS, then $T_1\left(\bar{X}^{r+\tilde{q}_{k+1}^0}\right) + T_2\left(\bar{X}^{r+\tilde{q}_{k+1}^0}\right) \geq \min_{j=0,1,\ldots,k}\left(T^{r+\tilde{q}_j} + T^{r-\tilde{p}_j}\right)$ where, each of the problem $P_{U\beta}\left(T^{r+\tilde{q}_j}\right)\ \forall\ j = 1,2,\ldots,k-1$ has an M-feasible LOS.

Proof: The proof is similar to the the proof of theorem 3.1.5.

Remark 3.1.4: If LOS of the problem $P_{U\beta}\left(T^{r+\tilde{q}_k}\right)$ is not an MFS, then no further restricted version of $\left(P_{3.1e}\right)$, namely, $P_{U\beta}\left(T^{r+\tilde{q}_j}\right)\ \forall\ j \geq k$ can provide a solution of the problem $\left(P_{3.1c}\right)$ yielding value better than $\min_{j=0,1,2\ldots k}\left(T^{r+\tilde{q}_j} + T^{r-\tilde{p}_j}\right)$.

Remark 3.1.5: If LOS of the problem $P_{Ub}\left(T^{r+\tilde{q}_j}\right)$ is an MFS for all $j = 1, 2, \ldots k$ then as $T_1\left(\bar{X}^{r+\tilde{q}_{k+1}^0}\right) < T^{r+\tilde{q}_k}$, it follows that $T_2\left(\bar{X}^{r+\tilde{q}_{k+1}^0}\right) > T^{r-\tilde{p}_k}$ since if $T_2\left(\bar{X}^{r+\tilde{q}_{k+1}^0}\right) \leq T^{r-\tilde{p}_k}$ then corresponding minimum Stage-I time will be greater than or equal to $T^{r+\tilde{q}_k}$.

The next theorem proves that proposed solution methodology indeed obtains the global optimal solution of the two stage interval TMTP $\left(P_{3.1c}\right)$.

Theorem 3.1.8: If the generated pairs of Stage-I and Stage-II shipment times are (T^{r-p_k}, T^{r+q_k}), $k \geq 0$ and $\left(T^{r+\tilde{q}_k}, T^{r-\tilde{p}_k}\right)$, $k \geq 0$ then, the optimal value of the objective function of the problem $\left(P_{3.1c}\right)$ is,

$$\min\left\{\min_{k\geq 0}\left(T^{r-p_k} + T^{r+q_k}\right), \min_{k\geq 0}\left(T^{r+\tilde{q}_k} + T^{r-\tilde{p}_k}\right)\right\}.$$

Proof: If the theorem is not to be true, then there must exist a feasible solution, say X_G, of the problem $(P_{3.1c})$ such that the corresponding Stage-I and Stage-II shipment times (call them $T_1(X_G)$ and $T_2(X_G)$ respectively) are such that

$$T_1(X_G)+T_2(X_G)<\min\left\{\begin{array}{l}\min\limits_{k\geq 0}\left(T^{r-p_k}+T^{r+q_k}\right)\\ \min\limits_{k\geq 0}\left(T^{r+\tilde{q}_k}+T^{r-\tilde{p}_k}\right)\end{array}\right\}$$

As X_G is a feasible solution of the problem $(P_{3.1c})$, it follows that $T_2(X_G)$ is the minimum Stage-II shipment corresponding to the Stage-I shipment time $T_1(X_G)$. Therefore, X_G is an M-feasible solution of the problem $(P_{3.1e})$.

The above inequality implies that $T_1(X_G)+T_2(X_G)<T^{r-p_0}+T^{r+q_0}$. Without loss of generality, assume that $T_1(X_G)\geq T_2(X_G)$. As $T^{r-p_0}\left(=T^r\right)$ is the optimal transportation time for the time minimizing transportation problem $(P_{3.1e})$, it follows that $T_1(X_G)\geq T^{r-p_0}$ and hence $T_2(X_G)<T^{r+q_0}$. Therefore, there exists an index, say d (a positive integer not less than 1), such that

$$T^{r+q_d}<T_2(X_G)<T^{r+q_{d-1}}<T^{r+q_0}.$$

This implies that X_G is an M-feasible solution of the problem $P_{L\beta}\left(T^{r+q_{d-1}}\right)$. M-feasible LOS of $P_{L\beta}\left(T^{r+q_{d-1}}\right)$ yields Stage-I shipment time T^{r-p_d}. By hypothesis, $T_1(X_G)+T_2(X_G)<T^{r-p_d}+T^{r+q_d}$

As $T_2(X_G)>T^{r+q_d}$, we have $T_1(X_G)<T^{r-p_d}$. This implies that X_G is a solution better than the M-feasible LOS of $P_{L\beta}\left(T^{r+q_{d-1}}\right)$, which is not true. Hence there does not exist any feasible solution of $(P_{3.1c})$ yielding sum of Stage-I and Stage-II times less than $\min\left\{\min\limits_{k\geq 0}\left(T^{r-p_k}+T^{r+q_k}\right), \min\limits_{k\geq 0}\left(T^{r+\tilde{q}_k}+T^{r-\tilde{p}_k}\right)\right\}$.

3.1.4 Algorithm

Step1 Obtain an LOS of the problem $(P_{3.1e})$. Note the corresponding Stage-I time as $T^r=T^{r-p_0}$ and Stage-II time as $T^{r+q_0^0}$. Solve cost minimizing transportation problem $CP_{Lb}\left(T^{r-p_0}, T^{r+q_0^0}\right)$ to find the minimum Stage-II shipment time, say T^{r+q^0}, of Stage-II corresponding to the time T^{r-p_0} of Stage-I shipment. Record this pair as $\left(T^{r-p_0}, T^{r+q_0}\right)$.

If $T^{r-p_0} = T^1$ or $T^{r+q_0} = T^s$, then stop and go to Step 3. Else, go to Step 2.

Step 2 $(k \geq 1)$ Construct the problem $P_{Lb}\left(T^{r+q_{k-1}}\right)$ and find its LOS. If it is not an MFS, then go to Step 3. Else, solve cost minimizing transportation problem $CP_{Lb}\left(T^{r-q_k}, T^{r+q_k^0}\right)$ to find the minimum Stage-II shipment time T^{r+q_k} corresponding to the time T^{r-p_k} of Stage-I shipment. Record the pair $\left(T^{r-q_k}, T^{r+q_k}\right)$.

If $T^{r-p_k} = T^1$ or $T^{r+q_k} = T^s$, then stop and go to Step 3. Else, execute Step 2 for next higher value of k.

Step 3 Construct the problem $P_{Ub}\left(T^{r+q_0}\right)$ and obtain its LOS. If it is not an M-feasible solution, then go to Step 5. Else, note the Stage-I shipment time as $T^{r+\tilde{q}_0^0}$ and Stage-II time as $T^{r-\tilde{p}_0}$. To find the minimum Stage-I shipment time, $T^{r+\tilde{q}_0^0}$, corresponding to time $T^{r-\tilde{p}_0}$ of Stage-II shipment, solve the cost minimizing transportation problem $CP_{Ub}\left(T^{r+\tilde{q}_0^0}, T^{r-\tilde{p}_0}\right)$. Record the pair $\left(T^{r+\tilde{q}_0}, T^{r-\tilde{p}_0}\right)$.

If $T^{r+\tilde{q}_0} = T^s$ or $T^{r-\tilde{p}_0} = T^1$, then stop and go to Step 5. Else, go to Step 4.

Step 4 $(k \geq 1)$ Construct the problem $P_{Ub}\left(T^{r+\tilde{q}_{k-1}}\right)$ and find its LOS. If it is not an M-feasible solution, then go to Step 5. Else, note the Stage-I shipment time as $T^{r+\tilde{q}_k^0}$ and Stage-II shipment as $T^{r-\tilde{p}_k}$.

Solve the cost minimizing transportation problem $CP_{Ub}\left(T^{r+\tilde{q}_k^0}, T^{r-\tilde{p}_k^0}\right)$ to find the minimum Stage-I shipment time $T^{r+\tilde{q}_k}$ corresponding to Stage-II shipment time $T^{r-\tilde{p}_k}$. Record the pair $\left(T^{r+\tilde{q}_k}, T^{r-\tilde{p}_k}\right)$.

If, $T^{r-\tilde{p}_k} = T^1$ or $T^{r+\tilde{q}_k} = T^s$, then stop and go to Step 5. Else, repeat this step for next higher value of k.

Step 5 Find $\min\left\{\min_{k\geq 0}\left(T^{r-p_k}+T^{r+q_k}\right),\min_{k\geq 0}\left(T^{r+\tilde{q}_k}+T^{r-\tilde{p}_k}\right)\right\}$. This will be the optimal value of the objective function of the problem $(P_{3.1c})$.

3.1.5 Example

Suppose the following 3×6 Two Stage Interval TMTP given as below:

	D_1	D_2	D_3	D_4	D_5	D_6	a_i	a'_i
S_1	26	23	59	38	19	20	6	8
S_2	40	48	20	19	23	59	15	29
S_3	26	38	48	20	19	40	12	18
b_j	6	9	3	14	10	5		

The partition of various time routes is given by:

$t^1(=59)>t^2(=48)>t^3(=40)>t^4(=38)>t^5(=26)>t^6(=23)>t^7(=20)>t^8(=19)$.

Here $t^s=t^8=19 \therefore s=8.$

The set of transportation time on various routes is partitioned into a number of disjoint sets which is defined as below:

$t_{13},t_{26}=t^1=B_1=59$

$t_{22},t_{33}=t^2=B_2=48$

$t_{21},t_{36}=t^3=B_3=40$

$t_{14},t_{32}=t^4=B_4=38$

$t_{11},t_{31}=t^5=B_5=26$

$t_{12},t_{25}=t^6=B_6=23$

$t_{16},t_{23},t_{34}=t^7=B_7=20$

$t_{15},t_{24},t_{35}=t^8=B_8=19$

The corresponding problem $(P_{3.1e})$ is given in the following table:

	D_1	D_2	D_3	D_4	D_5	D_6	D_7	a_i
S_1	26	23	59	38	19	20	M	6
S_2	40	48	20	19	23	59	M	15
S_3	26	38	48	20	19	40	M	12
S_4	26	23	59	38	19	20	0	2
S_5	40	48	20	19	23	59	0	14
S_6	26	38	48	20	19	40	0	6
b_j	6	9	3	14	10	5	8	

	D_1	D_2	D_3	D_4	D_5	D_6	D_7
S_1	26	23 (3)	59	38	19	20 (3)	M
S_2	40	48	20 (3)	19 (2)	23(10)	59	M
S_3	26 (6)	38	48	20 (6)	19	40	M
S_4	26	23	59	38	19	20 (2)	0
S_5	40	48	20	19 (6)	23	59	0 (8)
S_6	26	38 (6)	48	20	19 (0)	40	0

An optimal solution of the problem $(P_{3.1e})$ yields the Stage-I time 26 and Stage-II time as 38.Thus, $T^r = T^4 = 38, r = 4$.

To obtain the minimum Stage-I shipment time corresponding to Stage-II shipment time 38, the cost minimizing transportation problem $CP_{Ub} = (26, 38)$ is solved. Its optimal solution yields the pair $(26, 38)$.

Hence the first recorded pair of Stage-I and Stage-II time is $(26, 38)$.To obtain a new pair, the time minimizing transportation problem $P_{Ub}(26)$ is solved. Its optimal solution is M-feasible and yields a pair $(23, 40)$ of the Stage-I and Stage-II shipment times .The optimal solution of the cost minimizing transportation problem $CP_{Ub}(23, 40)$ is given as

	D_1	D_2	D_3	D_4	D_5	D_6	D_7
S_1	M	$/_6$(6)	M	M	$/_8$	$/_7$	M
S_2	M	M	$/_7$(3)	$/_8$(2)	$/_6$(10)	M	M
S_3	M	M	M	$/_7$(12)	$/_8$	M	M
S_4	0	0	M	0	0	0 (2)	0
S_5	0 (6)	M	0	0 (0)	0	M	0 (8)
S_6	0 (0)	0 (3)	M	0	0	0 (3)	0

The optimal solution of the $CP_{Ub}(23,40)$ gives the pair (23,40). Hence the second recorded pair of Stage-I and Stage-II shipment times is (23,40).

Next, the bottleneck transportation problem $P_{Ub}(23)$. Its optimal solution is not a M-feasible solution and hence no pairs with Stage-I shipment time less than Stage-II shipment time exist.

Next the TMTP $P_{Lb}(26)$ is solved. Its optimal solution is an M-feasible and yields Stage-I and Stage-II shipment time pair as (38,23). Then, we solve cost minimizing transportation problem $CP_{Lb}(38,23)$ to obtain the minimum Stage-II shipment time corresponding to the shipment time 38 of Stage-I, which is given as:

	D_1	D_2	D_3	D_4	D_5	D_6	D_7
S_1	0	0 (1)	M	0 (0)	0	0 (5)	M
S_2	M	M	0 (3)	0 (12)	0	M	M
S_3	0 (6)	0 (6)	M	0	0	M	M
S_4	M	$/_6$(2)	M	M	$/_8$	$/_7$	0
S_5	M	M	$/_7$	$/_8$	$/_6$(10)	M	0 (4)
S_6	M	M	M	$/_7$(2)	$/_8$	M	0 (4)

Optimal solution of $CP_{Lb}(38,23)$ generates the pair (38,20).Hence the 3rd recorded pair of the Stage-I and Stage-II shipment is (38,20).

Further optimal solution of $P_{Lb}(20)$ yields the pair (40,19) that is refined by solving cost minimization transportation problem $CP_{Lb}(40,19)$ is given as:

	D_1	D_2	D_3	D_4	D_5	D_6	D_7
S_1	0	0 (1)	M	0	0	0 (5)	M
S_2	0 (2)	M	0 (3)	0 (8)	0 (2)	M	M
S_3	0 (4)	0 (8)	M	0	0	0	M
S_4	M	M	M	M	$/_8$ (2)	M	0
S_5	M	M	M	$/_8$(6)	M	M	0 (8)
S_6	M	M	M	M	$/_8$(6)	M	0

The optimal solution of $CP_{Lb}(40,19)$ returns the pair (40,19). Since Stage-II time has reached $T^s = 19$, we stop here.

Hence the fourth recorded pair of the Stage-I and Stage-II times is (40,19). Now, Min {26+38, 23+40, 38+20, 40+19} = 58. Hence for the optimal solution for the optimal solution of the problem $(P_{2.1})$, the Stage-I shipment time is 38 and the Stage-II shipment time is 20 yielding the sum of Stage-I and Stage-II shipments as 58. Stage-I and Stage-II shipment schedule for this optimal solution are given in the following table.

	D_1	D_2	D_3	D_4	D_5	D_6	D_7	a_i
S_1	26	23 (1)	59	38 (0)	19	20 (5)	M	6
S_2	40	48	20 (3)	19 (12)	23	59	M	15
S_3	26 (6)	38 (6)	48	20	19	40	M	12
S_4	26	23 (2)	59	38	19	20	0	2
S_5	40	48	20	19	23 (10)	59	0 (4)	14
S_6	26	38	48	20 (2)	19	40	0 (4)	6
b_j	6	9	3	14	10	5	8	

3.2 An Alternate Approach of Two Stage Interval Bottleneck Transportation Problem

This section discusses the different method of two stage interval bottleneck transportation problem, in which the total availability of a homogeneous product is known to lie in a specified interval. This problem was first considered by Sonia et al. (2004) where in first stage the sources ship all of their on-hand material to the demand points and the second stage shipment covers the demand that is not fulfilled in the first stage. In each stage, aim is to minimize the duration of transportation and overall goal is to minimize the sum of two stage shipment times. In their methodology, two sequences of Stage-I and Stage-II time are generated. One of the sequences consists of generating pairs of the form $(T_1(.), T_2(.) : T_1(.) > T_2(.))$ by solving time minimization transportation problem of the form $P_{L\beta}(T_2(.))$ and cost minimization transportation problem of the form $C_{L\beta}(T_1(.), T_2(.))$where the problem $P_{L\beta}(T_2(.))$ reduces the on hand shipment time for Stage-II, and the problem $C_{L\beta}(T_1(.), T_2(.))$ gives the minimum shipment time for Stage-II corresponding to the Stage-I shipment time obtained from $P_{L\beta}(T_2(.))$. Similarly the sequence of two stage shipment time of the form $(T_1(.), T_2(.) : T_1(.) < T_2(.))$ is obtained by solving the problems $P_{U\beta}(T_1(.))$ and

$C_{U\beta}(T_2(.),T_1(.))$, where these problems play a similar role as played by $P_{L\beta}(T_2(.))$ and $C_{L\beta}(T_1(.),T_2(.))$ with their role for Stage-I and Stage-II reversed. Further it has been established theoretically that the global minimum value of the $(P_{3.1c})$ problem is obtained out from these generated pairs.

The present work aims at reducing the computational complexity of the method discussed by Sonia et al. (2004), thereby suggesting a different approach to solve this problem, which adopts only one sequence of Stage-I and Stage-II problems in contrast to the two way procedure adopted by Sonia et al. (2004). The algorithm developed in section 3.2 generates the sequence of Stage-I and Stage-II shipment times, where at each iteration Stage-I time strictly decreases and Stage-II time strictly increases. Feasible solutions of Stage-I and Stage-II problems are derived from a feasible solution of the standard TMTP.

Moreover, it has also been shown that in the developed algorithm a finite number of cost minimizing transportation problem (CMTP) are solved to generate different pairs of Stage-I and Stage-II shipment times.

3.2.1 Mathematical Formulation

Let a_i and a'_i, $i \in I$ denote respectively the minimum and maximum availability of a homogeneous product at the source i and $b_j, j \in J$ the demand of the same at destination j, where, $\sum_{i\in I} a_i < \sum_{j\in J} b_j < \sum_{i\in I} a'_i$

In the first stage of the two stage interval (TMTP), the quality $a_i\,(< a'_i)$ is shipped from each source $i,\ i \in I$ and after the completion; enough quality of the product is dispatched in second stage so as to exactly satisfy the demand b_j at the destination $j,\ j \in J$ .The Stage-I problem is thus formulated as:

$$\min_{Y=\{y_{ij}\}\in S'} \left[\max_{I\times J}\left(t_{ij}\left(y_{ij}\right)\right)\right] = \min_{Y\in S'}\left[T_1(Y)\right] \qquad (P_{3.2a})$$

where the set S' is given by

$$S':\begin{cases} \sum_{j\in J} y_{ij} = a_i & \forall\, i \in I \\ \sum_{i\in I} y_{ij} \le b_j & \forall\, j \in J \\ y_{ij} \ge 0 & \forall\,(i,j)\in I\times J \end{cases}$$

Corresponding to a feasible solution $Y = \{y_{ij}\}$ of Stage-I problem, let $S'(Y)$ be the set of feasible solutions of Stage-II problem which is stated below:

$$\min_{Z=\{z_{ij}\}\in S'(Y)} \left[\max_{I\times J}\left(t_{ij}\left(z_{ij}\right)\right)\right] = \min_{Z\in S'(Y)} \left[T_2(Z)\right] \qquad \left(P_{3.2b}\right)$$

where the set $S'(Y)$ is given by

$$S'(Y): \begin{cases} \sum_{j\in J} z_{ij} = a_i' - a_i & \forall\, i \in I \\ \sum_{i\in I} z_{ij} = b_j - b_j{}' & \forall\, j \in J \\ z_{ij} \geq 0, & \forall\,(i, j) \in I\times J \end{cases}$$

and $b_j{}' = \sum_{i\in I} y_{ij},\ j \in J$.

Thus a two stage time minimization transportation problem can be defined as:

$$\min_{Y=\{y_{ij}\}\in S'} \left[\left(T_1(Y)\right) + \min_{Z\in S'(Y)} \left[\left(T_2(Z)\right)\right]\right] \qquad \left(P_{3.2c}\right)$$

Closely related to the problem $\left(P_{3.2c}\right)$ is the time minimizing transportation problem $\left(P_{3.2d}\right)$ defined as:

$$\min_{X\in S} \left[T(X)\right] = \min_{X\in S} \left[\max_{I\times J}\left(t_{ij}\left(x_{ij}\right)\right)\right] \qquad \left(P_{3.2d}\right)$$

where

$$S: \begin{cases} a_i \leq \sum_{j\in J} x_{ij} \leq a_i' & \forall\, i \in I \\ \sum_{i\in I} x_{ij} = b_j & \forall\, j \in J \\ x_{ij} \geq 0 & \forall\,(i, j) \in I\times J \end{cases}$$

Clearly a feasible solution of $\left(P_{3.2c}\right)$ provides a feasible solution to the problem $\left(P_{3.2d}\right)$ and conversely. Associated with the problem $\left(P_{3.2d}\right)$ a balanced transportation problem is defined as:

$$\min_{X\in \hat{S}} \left[\hat{T}(X)\right] = \min_{X\in \hat{S}} \left[\max_{\hat{I}\times\hat{J}}\left(\hat{t}_{ij}\left(x_{ij}\right)\right)\right] \qquad \left(P_{3.2e}\right)$$

where,

$$\hat{S}:\begin{cases}\sum_{j\in J} x_{ij} = \hat{a}_i & \forall\, i\in\hat{I}\\ \sum_{i\in I} x_{ij} = \hat{b}_j & \forall\, j\in\hat{J}\\ x_{ij}\ge 0 & \forall (i,j)\in\hat{I}\times\hat{J}\end{cases}$$

where,

$$\hat{I} = \{1,2,,.....m, m+1,.....2m\},$$
$$\hat{J} = J\cup\{n+1\},$$
$$\hat{a}_i = a_i,\ i\in I$$
$$\hat{a}_{m+i} = a_i' - a_i,\ i\in I,$$
$$\hat{b}_j = b_j\ \ \forall\, j\in J,$$
$$\hat{b}_{n+1} = \sum_{i\in I} a_i' - \sum_{j\in J} b_j,$$
$$\hat{t}_{ij} = t_{ij}\ \forall\ (i,j)\in I\times J,$$
$$\hat{t}_{m+i,j} = t_{ij}\ \forall\ (i,j)\in I\times J,$$
$$\hat{t}_{i,n+1} = M\ \ \forall\ i\in I,$$

where M is a very large positive number,

$$\hat{t}_{m+i,n+1} = 0\ \ \forall i\in I$$

3.2.2 Theoretical Development

As shipment time in Stage-I and Stage-II are concave functions, two stage interval time minimization transportation problem aims at minimizing a concave function over a polytope. Hence $(P_{3.2c})$ is also a concave minimization problem. As the global minimum of a concave minimization problem is attained at the extreme point only, it is desirable to investigate only its extreme points. Let the set of transportation time on various routes is partitioned into a number of disjoint sets, $B_h = h = 1,2,...,s$ where $B_h = \{(i,j)\in I\times J : t_{ij} = t^h\}$.

and $t_{ij} = t^j > t^{j+1}\ \forall\, j = 1,2,...,s-1$. Positive weights say $\lambda_{s-h+1},\ h = 1,2,...,s$ are attached to these sets where, $\lambda_{j+1} >> \lambda_j\ \forall\, j = 1,2,...,s-1$. This yields a standard cost minimization transportation problem as:

$$\min \sum_{h=1}^{s} \lambda_h \left(\sum_{(i,j)\in B_h} x_{ij} \right)$$

where $X = \{x_{ij}\}$ belongs to the transportation polytope over which original (TMTP) is being studied. To find an (OFS) of the Stage-II problem, we define the following problem (CMTP) as:

$$\min_{\hat{s}} \sum_{\hat{I}\times\hat{J}} c_{ij} x_{ij} \qquad (CP)$$

where

$$\begin{aligned} &C_{i,n+1} = M && \forall\, i \in I \\ &C_{m+i,n+1} = 0 && \forall i \in I \\ &C_{ij} = 0 && \forall\, (i,j) \in I \times J \\ &C_{m+i,j} = \lambda_{s-h+1};\ t_{m+i,j} = t^h,\ \forall\, (i,j) \in B_h \text{ and } h = 1,2,\dots,s \end{aligned}$$

Let at any given time of Stage-I and Stage-II say, T_1^{k-1}, T_2^{k-1} respectively, where $T_1^{k-1}, T_2^{k-1} \in \{t_1, t_2, \dots\dots t_s\}, k \in \{1,2,\dots\dots,s+1\}$. The restricted version of the problem (CP) denoted by $(CP_k),\ K \geq 1$ is defined below:

$$\min_{\hat{s}} \sum_{i\in\hat{I}} \sum_{j\in\hat{J}} c_{ij} x_{ij} \qquad (CP_k)$$

where,

$$\begin{aligned} &c_{ij} = M \text{ if } t_{ij} \geq T^{k-1}, (i,j) \in I \times J \\ &\quad = 0 \text{ if } t_{ij} < T^{k-1}, (i,j) \in I \times J \\ &c_{i,n+1} = M\ \forall\ i \in I, \\ &c_{m+i,n+1} = 0\ \forall\ i \in I, \\ &c_{m+i,j} = \lambda_{s-h+1};\ t_{m+i,j} = t^h, (i,j) \in B_h \\ &\&\, h = 1,2,\dots\dots s. \end{aligned}$$

An (OFS) of the problem (CP) is denoted by Y^0 with corresponding Stage-I time T_1^0 and Stage-II time by T_2^0 and let Y^k be an OFS of (CP_k) yielding corresponding time of Stage-I and Stage-II as T_1^k and T_2^k respectively.

Theorem 3.2.1 T_2^k is the minimum time of Stage –II corresponding to any given time of Stage-I in the problem (CP_k).

Proof: Let if possible there exist a pair (T_1, T_2) yielded by some feasible solution $Y = \{y_{ij}\}$ of (CP_k) such that $T_2 < T_2^k$ and $T_1 < T_1^{k-1}$ where $T_2 = t_p$ and $T_2^k = t_q$ for some $p, q \in \{1,2,.....,s\}$. Since $T_2 < T_2^{\,k}$, therefore $p > q$, which implies $s - p + 1 < s - q + 1$.

Therefore,

$$Z(Y) = \sum_{\hat{I}\times\hat{J}} c_{ij} y_{ij} = \sum_{h=1}^{s} \lambda_{s-h+1}\left(\sum_{(i,j)\in B_h} y_{ij}\right)$$

$$= \sum_{h=p}^{s} \lambda_{s-h+1}\left(\sum_{(i,j)\in B_h} y_{ij}\right)$$

also

$$Z(Y) = \sum_{\hat{I}\times\hat{J}} c_{ij} y_{ij}^{k} = \sum_{h=1}^{s} \lambda_{s-h+1}\left(\sum_{(i,j)\in B_h} y_{ij}^{k}\right)$$

$$= \sum_{h=q}^{s} \lambda_{s-h+1}\left(\sum_{(i,j)\in B_h} y_{ij}^{k}\right)$$

Since $\lambda_{i+1} >> \lambda_i, i = 1,2,.....,s-1$

$$\Rightarrow \sum_{h=p}^{s} \lambda_{s-h+1}\left(\sum_{(i,j)\in B_h} y_{ij}\right) < \sum_{h=q}^{s} \lambda_{s-h+1}\left(\sum_{(i,j)\in B_h} y_{ij}\right)$$

$$\Rightarrow Z(Y) < Z(Y^k)$$

But this contradicts the optimality of Y^k, therefore $T_2^k \le T_2$.

Remark 3.2.1 $(CP\ )$ gives optimal time of Stage-II.

Remark 3.2.2 By construction of (CP_k), it is observed that $T_1^0 > T_1^1 > ....T_1^l$ and $T_2^0 \le T_2^1 \le .... \le T_2^l$.

Proof: The first sequence is clear from the construction of problem (CP_k). Let if possible $T_2^{k+1} < T_2^k$, for some k. Let $Z_k = Z(Y^k), Z_{k+1} = Z(Y^{k+1})$. Since $T_2^{k+1} < T_2^k$ by reasoning used in the proof of Theorem 3.2.1, we see that $Z_{k+1} < Z_k$. Since $T_2^{k+1} < T_2^{\,k}$, implies Y^{k+1} is a feasible solution of (CP_k) with $Z_{k+1} < Z_k$, a contradiction to the fact that Y^k is an (OFS) of (CP_k).

Remark 3.2.3 Since optimal time of Stage-I problem is T_1^l, (OBFS) of (CP_{l+1}) is not M-feasible.

Remark 3.2.4 Let $T_1^0 = t^r$ for some $r \in \{1,2,......s\}$ then the maximum number of iterations required to solve this problem is $s - r + 1$.

Theorem 3.2.2 Let the generated pairs of Stage-I and Stage-II time be $\left(T_1^k, T_2^k\right), k \geq 0$. Then the optimal value of the problem $(P_{3.2c})$ is given by $\min_{\{h=0,1,2......l\}} \left[T_1^h, T_2^h\right]$.

Proof: Let if possible, there exists a pair (Y_1, Y_2), yielding Stage-I time and Stage-II shipment time $\left(T_1, T_2\right)$ such that $T_1 + T_2 < \min_{\{h=0,1,2......l\}} \left[T_1^h, T_2^h\right]$. Since $T_1^0 > T_1^1 > .... > T_1^l$ and $T_2^0 \leq T_2^1 \leq ......T_2^l$, then the following case arises:

Case1: $T_1 > T_1^0$ (1)

By construction of (CP), (Y_1, Y_2), is a feasible solution of (CP). Since T_2^0 is optimal time for (CP), therefore

$$T_2^0 \leq T_2 \tag{2}$$

Combining (1) and (2), we get,

$$T_1 + T_2 > T_1^0 + T_2^0$$

$$\Rightarrow T_1 + T_2 > \min_{\{h=0,1,2.....l\}} \left[T_1^h, T_2^h\right]$$

Case 2: $T_1 < T_1^l$

Since $T_1 < T_1^l$, (Y_1, Y_2) is an M-feasible solution of (CP_l), which is a contradiction as this problem is not M-feasible.

Case 3: $T_1 \in \left[T_1^0, T_1^l\right]$

In this case, either $T_1 = T_1^k$ for some $k = 0,1,2,.....,l$ or $T_1 \in \left(T_1^k, T_1^{k-1}\right)\left[\because T_1^{k-1} > T_1^k\right]$.

(i). If $T_1 = T_1^0$, then by construction of (CP), (Y_1, Y_2) is a feasible solution of (CP). This implies, $T_2 \geq T_2^0$ because $\left(T_2^0\right)$ is the optimal time of Stage-II in (CP).

$$\Rightarrow T_1 + T_2 \geq T_1^0 + T_2^0$$

$$\Rightarrow T_1 + T_2 \geq \min_{\{h=0,1,2.....l\}} \left[T_1^h + T_2^h\right]$$

Similarly for the case when $T_1 = T_1^k$, $k \in (1,2,.....,l)$ it can be shown that $T_1 + T_2 \geq T_1^k + T_2^k \geq \min_{\{h=0,1,2.....l\}} \left[T_1^h + T_2^h\right]$

(ii). If $T_1 \in \left(T_1^k, T_1^{k-1}\right)$, then (Y_1, Y_2) is a feasible solution of (CP_k) $\left[\because T_1^k < T_1^{k-1}\right]$.

Also $T_2 \geq T_2^k$, and $T_1 > T_1^k$,

$$\Rightarrow T_1 + T_2 > T_1^k + T_2^k$$
$$\Rightarrow T_1 + T_2 > \min_{\{h=0,1,2.....l\}}\left[T_1^h + T_2^h\right]$$

Therefore, there does not exist a feasible solution $Y = (Y_1, Y_2)$ of (CP_k) yielding time less than $\min_{\{h=0,1,2.....l\}}\left[T_1^h + T_2^h\right]$. Thus, the optimal value of $(P_{3.2c})$ is given by $\min_{\{h=0,1,2.....l\}}\left[T_1^h + T_2^h\right]$.

3.2.3 The Procedure

Initial Step. Find an (OBFS) of (CP) and thus obtain the corresponding times T_1^0 and T_2^0 of Stage-I and Stage-II respectively.

General Step. If $k \geq 1$ at a given pair $\left(T_1^{k-1}, T_1^{k-1}\right)$ of Stage-I and Stage-II times, solve the problem (CP_k) . From the (OBFS) of (CP_k) construct the pairs $\left(T_1^{k+1}, T_2^{k+1}\right)$.

Terminal Step. If (OBFS) of problem (CP_k) is not M-feasible, then Stop. The optimal value of $(P_{3.2c})$ is given by $\min_{\{h=0,1,2.....k\}}\left[T_1^h + T_2^h\right]$.

3.2.4 Example

Consider the two stage interval time minimization problem given as:

	D_1	D_2	D_3	D_4	D_5	D_6	a_i	a_i'
S_1	26	23	59	38	19	20	6	8
S_2	40	48	20	19	23	59	15	29
S_3	26	38	48	20	19	40	12	18
b_j	6	9	3	14	10	5		

The partition of various time routes is given by:

$$t^1(=59) > t^2(=48) > t^3(=40) > t^4(=38) > t^5(=26) > t^6(=23) > t^7(=20) > t^8(=19) \ .$$

Here $t^s = t^8 = 19$.

\ $s = 8$.

The set of transportation time on various routes is partitioned into a number of disjoint sets which is defined as below,

$$t_{13}, t_{26} = t^1 = B_1 = 59$$
$$t_{22}, t_{33} = t^2 = B_2 = 48$$
$$t_{21}, t_{36} = t^3 = B_3 = 40$$
$$t_{14}, t_{32} = t^4 = B_4 = 38$$
$$t_{11}, t_{31} = t^5 = B_5 = 26$$
$$t_{12}, t_{25} = t^6 = B_6 = 23$$
$$t_{16}, t_{23}, t_{34} = t^7 = B_7 = 20$$
$$t_{15}, t_{24}, t_{35} = t^8 = B_8 = 19$$

The corresponding problem $(P_{3.2e})$ is given in the following table:

	D_1	D_2	D_3	D_4	D_5	D_6	D_7	$\hat{a}_i$
S_1	26	23	59	38	19	20	M	6
S_2	40	48	20	19	23	59	M	15
S_3	26	38	48	20	19	40	M	12
S_4	26	23	59	38	19	20	0	2
S_5	40	48	20	19	23	59	0	14
S_6	26	38	48	20	19	40	0	6
$\hat{b}_j$	6	9	3	14	10	5	8	

$$t_{m+1,3}, t_{m+2,6} = t^1 \; here \quad h = 1, c_{m+1,3}, c_{m+2,6} = \lambda_{8-1+1} = \lambda_8$$
$$t_{m+2,2}, t_{m+3,3} = t^2 \; here \quad h = 2, c_{m+2,2}, c_{m+3,3} = \lambda_{8-2+1} = \lambda_7$$
$$t_{m+2,1}, t_{m+3,6} = t^3 \; here \quad h = 3, \; c_{m+2,1}, c_{m+3,6} = \lambda_{8-3+1} = \lambda_6$$
$$t_{m+1,4}, t_{m+3,2} = t^4 \; here \quad h = 4, \; c_{m+1,4}, c_{m+3,2} = \lambda_{8-4+1} = \lambda_5$$
$$t_{m+1,1}, t_{m+3,1} = t^5 \; here \quad h = 5, \; c_{m+1,1}, c_{m+3,1} = \lambda_{8-5+1} = \lambda_4$$
$$t_{m+1,2}, t_{m+2,5} = t^6 \; here \quad h = 6, \; c_{m+1,2}, c_{m+2,5} = \lambda_{8-6+1} = \lambda_3$$
$$t_{m+1,6}, t_{m+2,3}, t_{m+3,4} = t^7 \; here \quad h = 7, \; c_{m+1,6}, c_{m+2,3}, c_{m+3,4} = \lambda_{8-7+1} = \lambda_2$$
$$t_{m+1,5}, t_{m+2,4}, t_{m+3,5} = t^8 \; here \quad h = 8, \; c_{m+1,5}, c_{m+2,4}, c_{m+3,5} = \lambda_{8-8+1} = \lambda_1$$

Now find the optimal basic feasible solution (OBFS) of the problem (CP)

	D_1	D_2	D_3	D_4	D_5	D_6	D_7
S_1	0	0 (1)	0	0	0	0 (5)	M
S_2	0 (2)	0	0 (3)	0 (6)	0 (4)	0	M
S_3	0 (4)	0 (8)	0	0	0	0	M
S_4	λ_4	λ_3	λ_8	λ_5	λ_1	λ_2	0 (2)
S_5	λ_6	λ_7	λ_2	λ_1(8)	λ_3	λ_8	0 (6)
S_6	λ_4	λ_5	λ_7	λ_2	λ_1 (6)	λ_6	0

The optimal basic feasible solution (OBFS) of the problem (CP) gives Stage-I time as $T_1^0 = 40$ and Stage-II time as $T_2^0 = 19$, where 19 is the optimal time of Stage-II. Next pair is obtained by solving the (CP_1) in which routes having time 40 at Stage-I is blocked. The optimal table of the problem (CP_1) is given as:

	D_1	D_2	D_3	D_4	D_5	D_6	D_7
S_1	0	0 (3)	M	0	0	0 (3)	M
S_2	M	M	0 (3)	0 (8)	0 (4)	M	M
S_3	0 (6)	0 (6)	M	0	0	M	M
S_4	l_4	l_3	l_8	l_5	l_1	l_2(2)	0 (0)
S_5	l_6	l_7	l_2	l_1(6)	l_3	l_8	0 (8)
S_6	l_4	l_5	l_7	l_2	l_1 (6)	l_6	0

The optimal basic feasible solution (OBFS) of the problem (CP_1) gives Stage-I time as $T_1^1 = 38$ and Stage-II time as $T_2^1 = 20$, where 20 is the minimum time of Stage-II corresponding to the Stage-I time 38. Next pair is obtained by solving the (CP_2) in which routes having time 38 at Stage-I is blocked. The optimal table of the problem $(CP_2)$is given as:

	D_1	D_2	D_3	D_4	D_5	D_6	D_7
S_1	0	0 (3)	M	M	0	0 (3)	M
S_2	M	M	0 (3)	0 (2)	0 (10)	M	M
S_3	0 (6)	M	M	0 (6)	0	M	M
S_4	l_4	l_3	l_8	l_5	l_1	l_2(2)	0
S_5	l_6	l_7(0)	l_2	l_1(6)	l_3	l_8	0 (8)
S_6	l_4	l_5(6)	l_7	l_2	l_1 (6)	l_6	0

The optimal basic feasible solution (OBFS) of the problem (CP_2) gives Stage-I time as $T_1^2 = 26$ and Stage-II time as $T_2^2 = 38$, where 38 is the optimal time of Stage-II.

Proceeding in the same way, next pair is obtained by solving the (CP_3) in which routes having time 26 at Stage-I is blocked. The optimal table of the problem (CP_3) is given as:

	D_1	D_2	D_3	D_4	D_5	D_6	D_7
S_1	M	0 (3)	M	M	0	0 (3)	M
S_2	M	M	0 (3)	0 (2)	0 (10)	M	M
S_3	M	M	M	0 (12)	0	M	M
S_4	$/_4$	$/_3$	$/_8$	$/_5$	$/_1$	$/_2$(2)	0
S_5	$/_6$(6)	$/_7$	$/_2$	$/_1$(0)	$/_3$	$/_8$	0 (8)
S_6	$/_4$(0)	$/_5$(6)	$/_7$	$/_2$	$/_1$ (6)	$/_6$	0

The optimal basic feasible solution (OBFS) of the problem (CP_3) gives Stage-I time as $T_1^3 = 23$ and Stage-II time as $T_2^3 = 40$, where 40 is the optimal time of Stage-II.

Algorithms terminate here as $(CP_4)$is no more M-feasible solution .Thus

Min {40+19, 38+20, 26+38, 23+40} = 58.

Hence the optimal value of the problem $(P_{3.2c})$corresponds to the pair (38, 20}.

CONCLUSION

Algorithm developed by Sharma et al. [2008] tries to reduce the computational complexity as only one sequence of Stage-I and Stage-II at each iteration Stage-I time decreases and Stage-II time increases in contrast to the two way procedure discussed by Sonia et al. [2008] and problems such $(P_{L\beta})$ and $(CP_{L\beta})$ are avoided as there is no need to reduce the Stage-II time separately corresponding to the given time Stage-I.

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