SOME VARIANTS OF TRANSPORTATION PROBLEM

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Submitted by

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INDIA

DEDICATED

TO

GOD, MY PARENTS

AND

MY SUPERVISORS

CERTIFICATE

I hereby certify that the work which is being presented in the thesis entitled "**Some Variants of Transportation Problem**" in partial fulfillment of the requirements for the award of degree of Master of Science, School of Mathematics and Computer Applications (SMCA), Thapar University, Patiala is an authentic record of my own work carried out under the supervision of **Mr. Vikas Sharma** and **Dr. Geeta Kumari**.

The matter presented in this thesis has not been submitted for the award of any other degree of this or any other university.

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This is to certify that the above statement made by the candidate is correct and true to the best of my knowledge.

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ABSTRACT

The transportation theory is a name given to the study of optimal transportation and allocation of resources. This thesis is devoted to discuss some variants of the transportation problem. Constrained transportation problem is presented along with the variants of the transportation problem dealing with the time minimization and bi-criteria objective function.

The present thesis contains four chapters. Chapter one is introductory in nature. This chapter includes basic definitions and concepts used throughout the work. Chapter two presents brief review of the work done in the area of bounded and flow constrained transportation problem. In chapter three, a two-stage interval time minimization problem is studied where minimum amount available at each source is shipped to the destination in the first stage and enough quantity of the product is shipped in second stage to meet the demand at destinations exactly. In chapter four, a special type of bi-criteria transportation problem involving the trade-off between cost of transportation and time of transportation is studied. In chapter two to four, numerical examples are solved in support of above concepts.

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Chapter 1

INTRODUCTION

The transportation problem is an optimization problem with a linear objective function and linear constraints. Suppose a firm produces goods at m different supply centers. Label them $i = 1, 2, ..., m$. The supply produced at supply center i be S_i. The demand for the good is spread out at n different demand centers. Label them $j = 1, 2, ..., n$. The demand at the j^{th} demand center is D_j . The problem of the firm is to get goods from supply centers to demand centers at minimum cost. Assume that the cost of shipping one unit from supply center i to demand center j is $c_{i,j}$ and that shipping cost is linear. If $x_{i,j}$ units be shipped from supply center i to demand center j, then the cost would be $c_{i,j}\ x_{i,j}$.

Mathematically, the general transportation problem stated as a linear programming problem is as follows:

P(1.1) Minimize $z = \sum\limits_{i=1}^{m} \sum\limits_{j=1}^{n} x_{i,j}\ c_{i,j}$

subject to the constraints

$$\begin{aligned}\sum_{j=1}^{n} x_{i,j} &= a_i, \quad i = 1, 2, ..., m \quad \text{(Supply constraints)}\\ \sum_{i=1}^{m} x_{i,j} &= b_j, \quad j = 1, 2, ..., n \quad \text{(Demand constraints)}\end{aligned}$$

and $x_{i,j} \geq 0$ for all i and j.

Feasible Solution: A necessary and sufficient condition for the existence of a feasible solution to the general transportation problem is that

$$\text{Total Supply} = \text{Total Demand}$$

i.e.,
$$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$$

Basic Feasible Solution: The number of basic variables of the general transportation problem at any stage of feasible solution must be $m + n - 1$.

1.1 Duality in Transportation Problem

Since a transportation problem is itself a linear programming problem, there always exist another linear programming based upon the same data. Let the original transportation problem be the primal problem and its associated one be its dual problem. For the general transportation problem, let the primal problem is P(1.1) and the corresponding dual pair for it is as follows:

P(1.2) Maximize $z^* = \sum_{i=1}^{m} u_i\ a_i + \sum_{j=1}^{n} v_j\ b_j$

subject to the constraints

$$u_i + v_j \leq c_{i,j}, \quad i = 1, 2, ..., m \text{ and } j = 1, 2, ..., n$$

$$u_i \text{ and } v_j \text{ unrestricted in sign for all } i,\ j.$$

where $u_i,\ i = 1, 2, ..., m$ and $v_j,\ j = 1, 2, ..., n$ are dual variables.

Finding an optimal solution of a General Transportation Problem Swarup et al.[26]

Step 1

We first find initial basic feasible solution for given problem if the objective is to minimize transportation cost by using any one of the method :

1. North-West Corner Method.

2. Least-Cost Method.

3. Vogel's Approximation Method (or Penalty Method).

Step 2

The optimal solution is obtained by Modified Distribution (MODI) method (or u-v method).

1.2 Unbalanced Transportation Problem

In some transportation problems, the total supply may not be equal to the total demand. This is likely to be true when it is impossible or rather unprofitable to supply all that is required at the destinations or to ship all that is available at the origins. Such problems are called unbalanced transportation problem.

The following cases can arise in unbalanced transportation problem:

(a) Allocation with surplus (Overproduction).

(b) Allocation with shortage (Underproduction).

(c) Allocation with surplus and shortage.

(d) Storage and oversupply.

(a) Allocation with surplus (Overproduction)

Suppose that the total production at the origins equals or exceeds the total requirement at the destinations and that it is required to meet the requirements of all the destinations exactly. In this case, some origins need not ship all of their production. The problem is:

P(1.3) Minimize $z = \sum_{i=1}^{m}\sum_{j=1}^{n} c_{i,j}\, x_{i,j}$

subject to

$$\sum_{j=1}^{n} x_{i,j} \le a_i, \quad i = 1, ..., m,$$

$$\sum_{i=1}^{m} x_{i,j} = b_j, \quad j = 1, ..., n,$$

$$x_{i,j} \ge 0 \text{ for all } i \text{ and } j,$$

where z represents the transportation cost. This transportation problem is feasible if and only if $\sum\limits_{i=1}^{m} a_i \geq \sum\limits_{j=1}^{n} b_j$.

Two cases arise here

(i) $\sum\limits_{i=1}^{m} a_i \ = \ \sum\limits_{j=1}^{n} b_j$

In this case we replace all the inequalities by equations and the resulting problem becomes a balanced transportation problem.

(ii) $\sum\limits_{i=1}^{m} a_i > \sum\limits_{j=1}^{n} b_j$

It represents an unbalanced transportation problem and to restore the structure of a standard transportation model or array, we introduce slack variables in inequalities as follows:

P(1.4) Minimize $z = \sum\limits_{i=1}^{m}\sum\limits_{j=0}^{n} c_{i,j}\ x_{i,j}$

subject to

$$\sum_{j=0}^{n} x_{i,j} = a_i, \quad i = 1, ..., m,$$

$$\sum_{i=1}^{m} x_{i,j} = b_j, \quad j = 0, ..., n,$$

$$x_{i,j} \geq 0 \text{ for all } i \text{ and } j,$$

where $b_0 = \sum\limits_{i=1}^{m} a_i - \sum\limits_{j=1}^{n} b_j$, $c_{i,0} = 0$ for all i, and the slack variable $x_{i,0}$ represents the amount of material left unutilized at the i^{th} origin. So $c_{i,0}x_{i,0}$ in the objective function of the problem P(1.4) can be interpreted as the cost of storing the surplus at the i^{th} origin. In P(1.4), since $c_{i,0} = 0$ for all i, the storage cost at all the origins is taken as zero.

(b) Allocation with shortage (Underproduction)

Consider a transportation problem in which the total requirement at all the destinations equals or exceeds the total supplies at all the origins and in which it is required to exhaust the supplies at all the origins. Thus the problem is:

$$\textbf{P(1.5)} \qquad \text{Minimize } z = \sum_{i=1}^{m}\sum_{j=1}^{n} c_{i,j}\, x_{i,j}$$

subject to

$$\sum_{j=1}^{n} x_{i,j} = a_i, \quad i = 1, ..., m,$$

$$\sum_{i=1}^{m} x_{i,j} \leq b_j, \quad j = 1, ..., n,$$

$$x_{i,j} \geq 0 \text{ for all } i \text{ and } j,$$

Clearly, this problem is feasible if and only if $\sum_{i=1}^{m} a_i \leq \sum_{j=1}^{n} b_j$.

Now, two cases arise:

(i) $\sum_{i=1}^{m} a_i \; = \; \sum_{j=1}^{n} b_j$

Replacing all the inequalities by equations the resulting problem becomes a balanced transportation problem.

(ii) $\sum_{i=1}^{m} a_i \; < \; \sum_{j=1}^{n} b_j$

In this case, the unbalanced transportation problem P(1.5) is equivalent to the balanced transportation problem P(1.1)

$$\textbf{P(1.6)} \qquad \text{Minimize } z = \sum_{i=0}^{m}\sum_{j=1}^{n} c_{i,j}\, x_{i,j}$$

subject to

$$\sum_{j=1}^{n} x_{i,j} = a_i, \quad i = 0, 1, ..., m,$$

$$\sum_{i=0}^{m} x_{i,j} = b_j, \quad j = 1, ..., n,$$

$$x_{i,j} \geq 0 \text{ for all } i \text{ and } j,$$

where $a_0 \; = \; \sum_{i=1}^{n} b_j - \sum_{i=1}^{m} a_i$.

The slack variables $x_{0,j},\ j = 1, ..., n$ represents the unmet demand at the j^{th} destination. In problem P(1.6), at each destination j, no penalty is charged for the unmet demand $x_{0,j}$, i.e., $c_{0,j} = 0$ for all j. Failure to meet the demand results in a loss of revenue. In many practical problems, a penalty cost is charged for not supplying the commodities at the destination. In such situations, we wish to minimize the sum of transportation and penalty cost.

(c) Allocation with surplus and shortage

Consider a transportation situation in which it is not necessary to meet all the demands or to ship all that is available. That is, consider the problem:

P(1.7) Minimize $z = \sum_{i=1}^{m}\sum_{j=1}^{n} c_{i,j}\ x_{i,j}$

subject to

$$\sum_{j=1}^{n} x_{i,j} \le a_i, \quad i = 1, ..., m,$$

$$\sum_{i=1}^{m} x_{i,j} \le b_j, \quad j = 1, ..., n,$$

$$x_{i,j} \ge 0 \text{ for all } i \text{ and } j,$$

If $c_{i,j} \ge 0$ for all i and j, then clearly the optimal solution to this problem is the trivial solution $x_{i,j} = 0$ for all i and j with the minimum transportation cost as zero. That is, the optimal solution says that nothing needs to be shipped. Thus, in order to make the problem meaningful, we associate penalties with the unmet demands at the destinations and associate storage costs with surpluses at the origins and minimize the sum of the transportation, storage and penalty costs.

(d) Storage and Oversupply

Consider the following transportation model corresponding to the situation where the minimum demand requirement at the j^{th} destination is $b_j,\ j = 1, ..., n$ and the supply available at the i^{th} origin is $a_i,\ i = 1, ..., m$:

P(1.8) Minimize $z = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{i,j}\ x_{i,j}$

subject to

$$\sum_{j=1}^{n} x_{i,j} \leq a_i, \quad i = 1, ..., m,$$

$$\sum_{i=1}^{m} x_{i,j} \geq b_j, \quad j = 1, ..., n,$$

$$x_{i,j} \geq 0 \text{ for all } i \text{ and } j.$$

Clearly problem P(1.8) is feasible if and only if $\sum_{i=1}^{m} a_i \geq \sum_{j=1}^{n} b_j$.

Now two cases arise here:

(i) $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$.

Replacing all the inequalities by equations in this problem the resulting problem becomes a balanced transportation problem.

(ii) $\sum_{i=1}^{m} a_i > \sum_{j=1}^{n} b_j$.

and $c_{i,j}$ are real numbers. In this case, $b_0 = \sum_{i=1}^{m} a_i - \sum_{j=1}^{n} b_j$ and for each $i = 1, ..., m,\ c_{i,0} = \min\{0, c_{i,1}, ..., c_{i,n}\}$.

A variant of the standard cost minimizing transportation problem P(1.1) is a transportation problem with upper and lower bounds on source availabilities and destination requirements with total flow. The study of such problem is motivated by dead mileage minimization in transport systems [28]. With the knowledge of the locations of bus-depots and the pick up points of passengers, minimizing the dead mileage in terms of running empty buses between the depots to the pick-up points is also one of the problems of study in this section. The aim is to reduce the cost of covering the dead mileage as also the time consumed. This study is very important aspect of transport infrastructure in a city.

Suppose that minimum and maximum holding capacity of a depot be a_i and A_i, respectively for all $i \in I$, the index set of depots. Let minimum number b_j and maximum number B_j of buses be needed at j^{th} starting point $j \in J$, are known in addition to l_{ij}, the distance from ith depot to the jth starting point, then mathematical model for problem of minimizing dead mileage is as follows :

P(1.9) Minimize $\sum\limits_{i \in I} \sum\limits_{j \in J} l_{i,j} x_{i,j}$

subject to

$$a_i \leq \sum_{j \in J} x_{i,j} \leq A_i, \quad i \in I,$$

$$b_j \leq \sum_{i \in I} x_{i,j} \leq B_j, \quad j \in J,$$

$$x_{i,j} \geq 0 \text{ and integers}, \ i \in I, j \in J,$$

where $I = \{1, 2, ..., m\}$, $J = \{1, 2, ..., n\}$ and $x_{i,j}$ stands for the number of buses dispatched from ith depot to jth starting point.

This problem is also solved by constructing an equivalent bounded variable transportation problem which has one more row and column as suggested by Charnes and Klingman [8]. In chapter 2, an approach is suggested where an equivalent transportation problem is constructed. An additional constraint, called the flow constraint $\sum\limits_{i \in I} \sum\limits_{j \in J} x_{i,j} = F$ is included in the problem P(1.9), where F is the number of buses available.

In time minimization transportation problem, the time of transporting goods from

m origins to n destinations is minimized, satisfying certain conditions in respect of availabilities at sources and requirements at the destinations.

Thus, a time minimizing transportation problem is :

P(1.10) Minimize $Z = [\text{Max}_(i,j)\ t_{i,j}/x_{i,j} > 0]$

subject to
$$\sum_{j=1}^{n} x_{i,j} = a_i,\ i = 1, 2, ..., m$$
$$\sum_{i=1}^{m} x_{i,j} = b_j,\ j = 1, 2, ..., n$$
$$x_{i,j} \geq 0 \text{ for all } i \text{and} j$$

Here $t_{i,j}$ is the time of transporting goods from i^{th} origin to the j^{th} destination, where the availability at i^{th} origin is a_i and requirement at j^{th} destination is b_j. For any given feasible solution, $X = [x_{i,j}]$ satisfying the above constraints, the time of transportation is the maximum of $t'_{i,j}s$ among the cells in which there are positive allocations, i.e., corresponding to the solution $X = [x_{i,j}]$, the time of transportation is $Z = [\text{max}_(i,j)t_{i,j}/x_{i,j} > 0]$.

The aim is to minimize the time of transportation. Such problems arise, when it is required to transport perishable goods or, during war days, it is required to transport food and armaments in the shortest possible time. Hammer [17] first discussed the time minimization transportation problem in 1969. In literature, some of the algorithms to solve this problem are given by Szwarc [27], Garfinkel et al. [12], Bhatia et al. [3].

A variant of time minimization transportation problem is discussed by Sonia et al.

[25] and Vikas et al. [23] in the form of two stage interval time minimization transportation problem. Two stage interval time minimization transportation problem is one in which minimum amount available at each source is shipped to the destinations in the first stage and the second stage shipment covers the demand that is not fulfilled in first stage. In each stage, aim is to minimize the duration of transportation and the overall goal is to minimize the sum of two stage shipment times. In both the stages, transportation of the product from the sources to the destinations is done in parallel. The algorithm developed in chapter 3 generates the sequence of stage-I and stage-II time, where at each iteration, stage-I time decreases strictly and stage-II time increases.

In some situations for example, ambulance services, fire services etc. the time of transportation cost attains greater importance than the cost of transportation. The method of minimizing the time in such situations has been devised by Bhatia[5]. In such situations, transport cost and transmit time cannot be viewed as two independent problems and there is need to have a trade-off between time and cost. By studying the constrained transportation problem given by Verma and Puri [28] and the time-cost trade-off in bulk transportation problem given by Gupta et al. [15], an algorithm is developed in chapter-4 to find all efficient time-cost trade-off pairs in transportation problem.

Chapter 2

TRANSPORTATION PROBLEM WITH BOUNDS ON AVAILABILITIES AND REQUIREMENTS

2.1 Introduction

In this chapter, the problem of minimizing dead mileage in transport system is discussed with upper and lower bounds on source availabilities and destination requirements. After locating the depots and starting points where from the buses start picking up passengers, minimization of dead mileage from depots to the starting points of bus services by properly allocating the buses to various depots and by assigning the buses optimally to various starting points from various depots helps in reducing the cost of covering the dead mileage.

Let a_i and A_i respectively be the minimum and maximum holding capacity of a depot, $i \in I$, the index set of depots. Suppose minimum number b_j and maximum number B_j of buses be needed at jth starting point, $j \in J$, are known in addition to the distance l_{ij}, from ith depot to the jth starting point, then mathematical model for problem of minimizing dead mileage is as follows :

P(2.1) Minimize $\sum_{i \in I} \sum_{j \in J} l_{i,j} \ x_{i,j}$

subject to
$$a_i \leq \sum_{j \in J} x_{i,j} \leq A_i, \quad i \in I,$$

$$b_j \leq \sum_{i \in I} x_{i,j} \leq B_j, \quad j \in J,$$

$x_{i,j} \geq 0$ and integers, $i \in I, j \in J$,

where $I = \{1, 2, ..., m\}$, $J = \{1, 2, ..., n\}$, and $x_{i,j}$ stands for the number of buses dispatched from ith depot to jth starting point.

In this chapter a method is introduced which can be used to reduce the problem P(2.1) to a standard transportation problem. For this following notations are used:

I' : index set of $2m+1$ rows, i.e., $I' = \{1, 2, ..., m, m+1, ..., 2m, 2m+1\}$

I'' : index set of $2m$ rows, i.e., $I'' = \{1, 2, ..., m, m+1, ..., 2m\}$

J' : index set of $2n+1$ columns, i.e., $J' = \{1, 2, ..., n, n+1, ..., 2n, 2n+1\}$

J'' : index set of $2n$ columns, i.e., $J'' = \{1, 2, ..., n, n+1, ..., 2n\}$

The possibilities which can arise here are:

(I) $\sum_{i \in I} A_i > \sum_{j \in J} B_j$

(II) $\sum_{i \in I} A_i < \sum_{j \in J} B_j$

(III) $\sum_{i \in I} A_i = \sum_{j \in J} B_j$

Case I. If $\sum\limits_{i\in I} A_i > \sum\limits_{j\in J} B_j$, then in order to solve problem P(2.1) an equivalent problem P(2.2) is defined as:

P(2.2) Minimize $\sum\limits_{i\in I''}\sum\limits_{j\in J'} d_{i,j} z_{i,j}$

subject to

$$\sum_{j\in J'} z_{i,j} = a'_i, \quad i \in I'',$$

$$\sum_{i\in I''} z_{i,j} = b'_j, \quad j \in J',$$

$$z_{i,j} \geq 0, i \in I'', \ j \in J'$$

where,

$$a'_i = a_i, i \in I$$

$$a'_{m+i} = A_i - a_i, \ i \in I$$

$$b'_j = b_j, \ j \in J$$

$$b'_{n+j} = B_j - b_j, \ j \in J$$

$$b'_{2n+1} = \sum_{i\in I} A_i - \sum_{j\in J} B_j$$

$$d_{i,j} = l_{i,j}, \ i \in I, \ j \in J$$

$$d_{m+i,j} = l_{i,j}, \ i \in I, \ j \in J$$

$$d_{i,n+j} = l_{i,j}, \ i \in I, \ j \in J$$

$$d_{i,2n+1} = \begin{cases} M, & i \in I \\ 0, & i \in I''\backslash I \end{cases}$$

$$d_{m+i,n+j} = 0, \ i \in I, j \in J.$$

Restricted Feasible Solution : A feasible solution $\{z_{i,j}\}, \ i \in I'', j \in J'$ of problem

P(2.2), where $z_{ij} = 0$ if $d_{ij} = \mathrm{M}$, is called its 'restricted feasible solution'.

Remark 2.1 If problem P(2.2) has no 'restricted feasible solution', then problem P(2.1) is infeasible.

Theorem 2.1 If $\{z_{i,j}\},\ i \in I'',\ j \in J'$ is a 'restricted feasible solution' of problem P(2.2), then there exists $x_{i,j},\ i \in I,\ j \in J$ such that it is a feasible solution to problem P(2.1) and conversely.

Proof: Let $\{z_{i,j}\},\ i \in I'',\ j \in J'$ be a 'restricted feasible solution' of P(2.2). Define $\{x_{i,j}\}, \forall\ i \in I,\ j \in J$ as follows:

$$x_{i,j} = z_{i,j} + z_{m+i,j} + z_{i,n+j}$$

Then clearly, $\forall\ i \in I,\ j \in J,\ x_{i,j} \geq 0$ and integers, since $z_{i,j}, i \in I'',\ j \in J'$ is a feasible solution of problem P(2.2), with $a'_i,\ b'_j,\ i \in I'',\ j \in J'$.

For all $i \in I$, consider

$$\begin{aligned}
\sum_{j \in J} x_{i,j} &= \sum_{j \in J} (z_{i,j} + z_{m+i,j} + z_{i,n+j}) \\
&= \sum_{j \in J} z_{i,j} + \sum_{j \in J} z_{m+i,j} + \sum_{j \in J} z_{i,n+j} + z_{i,2n+1} \text{ (since } z_{i,2n+1} = 0 \text{ as } d_{i,2n+1} = \mathrm{M}) \\
&= \sum_{j \in J'} z_{i,j} + \sum_{j \in J} z_{m+i,j} \qquad (2.1) \\
&\leq \sum_{j \in J'} z_{i,j} + \sum_{j \in J'} z_{m+i,j} \\
&= a_i + A_i - a_i \\
&= A_i \qquad (2.2)
\end{aligned}$$

From (2.1), we also get

$$\begin{aligned}\sum_{j\in J} x_{i,j} &= \sum_{j\in J'} z_{i,j} + \sum_{j\in J} z_{m+i,j}\\ &\geq \sum_{j\in J'} z_{i,j} = a_i \qquad (2.3)\end{aligned}$$

Combining (2.2) and (2.3), we get

$$a_i \leq \sum_{j\in J} x_{i,j} \leq A_i, \quad i \in I.$$

Similarly, for all $j \in J$,

$$\begin{aligned}\sum_{i\in I} x_{i,j} &= \sum_{i\in I} (z_{i,j} + z_{m+i,j} + z_{i,n+j})\\ &= \sum_{i\in I} z_{i,j} + \sum_{i\in I} z_{m+i,j} + \sum_{i\in I} z_{i,n+j}\\ &= \sum_{i\in I} z_{i,j} + \sum_{i\in I''\setminus I} z_{i,j} + \sum_{i\in I} z_{i,n+j}\\ &= \sum_{i\in I''} z_{i,j} + \sum_{i\in I} z_{i,n+j} \qquad (2.4)\\ &\leq \sum_{i\in I''} z_{i,j} + \sum_{i\in I''} z_{i,n+j}\\ &= b_j + B_j - b_j = B_j \qquad (2.5)\end{aligned}$$

Equation (2.4) also implies

$$\begin{aligned}\sum_{i\in I} x_{i,j} &= \sum_{i\in I''} z_{i,j} + \sum_{i\in I} z_{i,n+j}\\ &\geq \sum_{i\in I''} z_{i,j} = b_j, \text{ for all } j \in J \qquad (2.6)\end{aligned}$$

Thus (2.5) and (2.6) imply

$$b_j \leq \sum_{i\in I} x_{i,j} \leq B_j, \quad j \in J$$

Thus, $\{x_{i,j}\},\ i \in I,\ j \in J$ is a feasible solution of problem P(2.1).

Conversely, let $x_{i,j},\ i \in I,\ j \in J$ be feasible solution of problem P(2.1), then

$$a_i \leq \sum_{j\in J} x_{i,j} (= A_i') \leq A_i, \quad i \in I$$

$$b_j \leq \sum_{i\in I} x_{i,j} (= B_j') \leq B_j, \quad j \in J$$

$$x_{i,j} \geq 0, \ i \in I, \ j \in J$$

Construct a problem as:

P(2.3) Minimize $\sum_{i\in I''} \sum_{j\in J''} w_{i,j} \ l''_{i,j}$

subject to

$$\sum_{i\in I''} w_{i,j} = b_j, \text{ for all } j \in J$$

$$\sum_{j\in J''} w_{i,j} = a_i, \text{ for all } i \in I$$

$$\sum_{j\in J} w_{m+i,j} = A_i' - a_i, \text{ for all } i \in I$$

$$\sum_{i\in I} w_{i,n+j} = B_j' - b_j, \text{ for all } j \in J$$

$$w_{i,j} \geq 0, \ \forall \ i \in I'', j \in J''$$

where, $l''_{i,j} = l_{i,j}, \text{ for all } i \in I, \ j \in J$

$l''_{m+i,j} = l_{m+i,j}, \text{ for all } i \in I, \ j \in J$

$l''_{i,n+j} = l_{i,n+j}, \text{ for all } i \in I, \ j \in J$

$l''_{m+i,n+j} = M, \text{ for all } i \in I, \ j \in J$

Define $\{z_{i,j}\}, \ i \in I'', \ j \in J'$ as:

$z_{i,j} = w_{i,j}, \text{ for all } i \in I, \ j \in J$

$z_{m+i,j} = w_{m+i,j}, \text{ for all } i \in I, \ j \in J$

$z_{i,n+j} = w_{i,n+j}, \text{ for all } i \in I, \ j \in J$

$z_{i,2n+1} = 0 \ \forall \ i \in I$

where $\{w_{i,j}\}$ $i \in I'', \ j \in J''$ is a 'restricted feasible solution' of problem P(2.3).

For $i \in I,\ j \in J$. Define

$$z_{m+i,n+j} = w'_{i,j}, i \in I,\ j \in J$$

$$z_{m+i,2n+1} = w'_{i,n+1}, i \in I$$

where $w'_{i,j},\ i \in I,\ j \in J \cup \{n+1\}$, is any feasible solution of the following standard transportation problem

$$\text{Minimize} \sum_{i \in I} \sum_{j \in J \cup \{n+1\}} l^*_{ij}\ w^*_{ij}$$

subject to

$$\sum_{j \in J \cup \{n+1\}} w^*_{i,j} = A_i - A'_i,\ i \in I$$

$$\sum_{i \in I} w^*_{i,j} = B_j - B'_j,\ \forall\ j \in J$$

$$\sum_{i \in I} w^*_{i,n+1} = \sum_{i \in I} A_i - \sum_{j \in J} B_j$$

$$w^*_{i,j} \geq 0,\ \text{for all } i \in I,\ j \in J \cup \{n+1\}$$

where $l^\star_{i,j} = 0,\ \forall\ i \in I,\ j \in J \cup \{n+1\}$

Then clearly,
$$\sum_{j \in J} z_{m+i,n+j} + z_{m+i,2n+1} = A_i - A'_i,\ \forall\ i \in I \tag{2.7}$$

$$\sum_{i \in I} z_{m+i,n+i} = B_j - B'_j,\ \forall\ j \in J \tag{2.8}$$

$$\sum_{i \in I} z_{m+i,2n+1} = \sum_{i \in I} A_i - \sum_{j \in J} B_j \tag{2.9}$$

For all $i \in I$, consider
$$\begin{aligned}
\sum_{j \in J'} z_{i,j} &= \sum_{j \in J''} z_{i,j} + z_{i,2n+1} \\
&= \sum_{j \in J''} z_{i,j} && \text{(since } z_{i,2n+1} = 0) \\
&= \sum_{j \in J} z_{i,j} + \sum_{j \in J} z_{i,n+j} \\
&= \sum_{j \in J} w_{i,j} + \sum_{j \in J} w_{i,n+j} \\
&= \sum_{j \in J''} w_{i,j} \\
&= a_i
\end{aligned}$$

Thus $\sum_{j\in J'} z_{i,j} = a_i, \ \forall \ i \in I$

For all $i \in I$, consider, $\sum_{j\in J'} z_{m+i,j} = \sum_{j\in J} z_{m+i,j} + \sum_{j\in J} z_{m+i,n+j} + z_{m+i,2n+1}$

$$= A'_i - a_i + A_i - A'_i \qquad \text{(from (2.7))}$$

$$= A_i - a_i$$

Thus, for all $i \in I$, $\sum_{j\in J'} z_{m+i,j} = A_i - a_i$

For all $j \in J$, $\sum_{i\in I''} z_{i,j} = \sum_{i\in I} z_{i,j} + \sum_{i\in I''-I} z_{i,j}$

$$= \sum_{i\in I} z_{i,j} + \sum_{i\in I} z_{m+i,j}$$

$$= \sum_{i\in I} w_{i,j} + \sum_{i\in I} w_{m+i,j}$$

$$= \sum_{i\in I''} w_{i,j}$$

$$= b_j$$

Thus, for all $j \in J$, $\sum_{i\in I''} z_{i,j} = b_j$

For all $j \in J$, $\sum_{i\in I''} z_{i,n+j} = \sum_{i\in I} z_{i,n+j} + \sum_{i\in I''-I} z_{in+j}$

$$= \sum_{i\in I} z_{i,n+j} + \sum_{i\in I} z_{m+i,n+j}$$

$$= B'_j - bj + B_j - B'_j \qquad \text{(by (2.8))}$$

$$= B_j - b_j$$

Thus, $\sum_{i\in I''} z_{i,n+j} = B_j - b_j, \forall \ j \in J$

$$\sum_{i\in I''} z_{i,2n+1} = \sum_{i\in I} z_{i,2n+1} + \sum_{i\in I} z_{m+i,2n+1}$$

$$= \ 0 + \sum_{i\in I} z_{m+i,2n+1} \qquad \text{(as } z_{i,2n+1} = 0 \ \forall \ i \in I\text{)}$$

$$= \sum_{i\in I} A_i - \sum_{j\in J} B_j \qquad \text{(by (2.9))}$$

Clearly, $z_{i,j} \geq 0 \ \forall\ i \in I'',\ j \in J'$.

Thus $\{z_{i,j}\},\ i \in I'',\ j \in J'$ is a 'restricted feasible solution' of a problem P(2.2).

Theorem 2.2 Let $\{x_{i,j}\},\ i \in I,\ j \in J$ be a feasible solution of problem P(2.1) and $\{z_{i,j}\},\ i \in I'',\ j \in J'$, be the corresponding 'restricted feasible solution' of problem P(2.2), then $\sum\limits_{i\in I}\sum\limits_{j\in J} l_{i,j}x_{i,j} = \sum\limits_{i\in I''}\sum\limits_{j\in J'} d_{i,j}z_{i,j}$.

Proof: Consider,

$$\sum_{i\in I''}\sum_{j\in J'} d_{i,j}\ z_{i,j} = \sum_{i\in I''}\sum_{j\in J} d_{i,j}\ z_{i,j} + \sum_{i\in I''}\sum_{j\in J''\setminus J} d_{i,j}\ z_{i,j} + \sum_{i\in I''} d_{i,2n+1}\ z_{i,2n+1}$$

$$= \sum_{i\in I''}\sum_{j\in J} d_{i,j}\ z_{i,j} + \sum_{i\in I''}\sum_{j\in J} d_{i,n+j}\ z_{i,n+j} + \sum_{i\in I} d_{i,2n+1}\ z_{i,2n+1} + \sum_{i\in I''\setminus I} d_{i,2n+1}\ z_{i,2n+1}$$

$$= \sum_{i\in I''}\sum_{j\in J} d_{i,j}\ z_{i,j} + \sum_{i\in I''}\sum_{j\in J} d_{i,n+j}\ z_{i,n+j} + 0 + 0 (\text{since } z_{i,2n+1} = 0\ \forall\ i \in I \text{ and } d_{i,2n+1} = 0,\ i \in I''\setminus I)$$

$$= \sum_{i\in I}\sum_{j\in J} d_{i,j}\ z_{i,j} + \sum_{i\in I''\setminus I}\sum_{j\in J} d_{i,j}\ z_{i,j} + \sum_{i\in I}\sum_{j\in J} d_{i,n+j}\ z_{i,n+j} + \sum_{i\in I''\setminus I}\sum_{j\in J} d_{i,n+j}\ z_{i,n+j}$$

$$= \sum_{i\in I}\sum_{j\in J} d_{i,j}\ z_{i,j} + \sum_{i\in I}\sum_{j\in J} d_{m+i,j}\ z_{m+i,j} + \sum_{i\in I}\sum_{j\in J} d_{i,n+j}\ z_{i,n+j} + \sum_{i\in I}\sum_{j\in J} d_{m+i,n+j}\ z_{m+i,n+j}$$

$$= \sum_{i\in I}\sum_{j\in J} d_{i,j}\ z_{i,j} + \sum_{i\in I}\sum_{j\in J} d_{m+i,j}\ z_{m+i,j} + \sum_{i\in I}\sum_{j\in J} d_{i,n+j}\ z_{i,n+j} + 0\ (\text{since } d_{m+i,n+j} = 0\ \forall\ i \in I,\ j \in J)$$

$$= \sum_{i\in I}\sum_{j\in J} l_{i,j}\ z_{i,j} + \sum_{i\in I}\sum_{j\in J} l_{i,j}\ z_{m+i,j} + \sum_{i\in I}\sum_{j\in J} l_{i,j}\ z_{i,n+j} \qquad (\text{from } (2.2))$$

$$= \sum_{i\in I}\sum_{j\in J} l_{i,j}\ (z_{i,j} + z_{m+i,j} + z_{i,n+j})$$

$$= \sum_{i\in I}\sum_{j\in J} l_{i,j}\ x_{i,j}$$

Theorem 2.3 If $\{x_{i,j}\},\ i \in I,\ j \in J$ is an optimal feasible solution of P(2.1), then the corresponding 'restricted feasible solution' $\{z_{i,j}\},\ i \in I'',\ j \in J'$, of problem P(2.2) is also an optimal feasible solution of problem P(2.2) and conversely.

Proof: Let $\{x_{i,j}\},\ i \in I,\ j \in J$ be an optimal feasible solution of (2.1). On the contrary, let

$\{z_{i,j}\},\ i \in I'', j \in J'$, with $z'_{i,j} \neq z_{i,j}$ for at least one $i \in I'',\ j \in J'$ be a 'restricted feasible solution' of P(2.2), such that it is an optimal solution to P(2.2).

Then $$\sum_{i \in I''} \sum_{j \in J'} d_{i,j}\ z'_{i,j} < \sum_{i \in I''} \sum_{j \in J'} d_{i,j}\ z_{i,j} \tag{2.10}$$

Also $\{z'_{i,j}\}$ being 'restricted feasible solution' of problem P(2.2), by Theorems 2.1 and 2.2, there exist a feasible solution $\{x''_{i,j}\},\ i \in I,\ j \in J$ of problem P(2.1), such that

$$\sum_{i \in I''} \sum_{j \in J'} d_{i,j}\ z'_{i,j} = \sum_{i \in I} \sum_{j \in J} l_{i,j}\ x''_{i,j} \tag{2.11}$$

Moreover, by Theorem 2.2

$$\sum_{i \in I''} \sum_{j \in J'} d_{i,j}\ z_{i,j} = \sum_{i \in I} \sum_{j \in J} l_{i,j}\ x_{i,j} \tag{2.12}$$

Substituting (2.11) and (2.12) in (2.10), we get

$\sum_{i \in I} \sum_{j \in J} l_{i,j}\ x''_{i,j} < \sum_{i \in I} \sum_{j \in J} l_{i,j}\ x_{i,j}$, a contradiction to the assumption that $\{x_{i,j}\},\ i \in I,\ j \in J$ is an optimal feasible solution of problem P(2.1).

Conversely, let $\{z_{i,j}\},\ i \in I'',\ j \in J'$, be a 'restricted feasible solution' of problem P(2.2), such that it is also an optimal solution.

On contrary, let $\{x'_{i,j}\}$, with $x'_{i,j} \neq x_{i,j}$ for at least one $i \in I,\ j \in J$ be an optimal feasible solution of problem P(2.1).

Then $$\sum_{i \in I} \sum_{j \in J} l_{i,j}\ x'_{i,j} < \sum_{i \in I} \sum_{j \in J} l_{i,j}\ x_{i,j} \tag{2.13}$$

Since $\{x'_{i,j}\}$ is a feasible solution of problem P(2.1), thus by Theorems 2.1 and 2.2, there exists $\{z''_{i,j}\}$, a 'restricted feasible solution' of problem P(2.2), such that

$$\sum_{i\in I}\sum_{j\in J} l_{i,j}\ x'_{i,j} = \sum_{i\in I''}\sum_{j\in J'} d_{i,j}\ z''_{i,j} \tag{2.14}$$

Also by Theorem (2.2),

$$\sum_{i\in I}\sum_{j\in J} l_{i,j}\ x_{i,j} = \sum_{i\in I''}\sum_{j\in J'} d_{i,j}\ z_{i,j} \tag{2.15}$$

By substituting (2.14) and (2.15) in (2.13), we get

$$\sum_{i\in I''}\sum_{j\in J'} d_{i,j}\ z''_{i,j} < \sum_{i\in I''}\sum_{j\in J'} d_{i,j}\ z_{i,j},$$

a contradiction to the assumption that $\{z_{i,j}\},\ i \in I'',\ j \in J'$, is an optimal feasible solution of problem P(2.2).

Thus, by Theorems 2.1, 2.2 and 2.3, problems P(2.1) and P(2.2) are equivalent.

Case II: If $\sum_{i\in I} A_i < \sum_{j\in J} B_j$, then another related standard transportation problem P(2.4) is defined as:

P(2.4) Minimize $\sum_{i\in I'}\sum_{j\in J''} d_{i,j}\ z_{i,j}$

subject to

$$\sum_{j\in J''} z_{i,j} = a'_i,$$

$$\sum_{i\in I'} z_{i,j} = b'_j,$$

$$z_{i,j} \geq\ 0,\ \text{for all } i \in I',\ j \in J''$$

where

$a'_i = a_i,\ i \in I$

$a'_{m+i} = A_i - a_i,\ i \in I$

$a'_{2m+1} = \sum_{j\in J} B_j - \sum_{i\in I} A_i$

$b'_j = b_j,\ j \in J$

$b'_{n+j} = B_j - b_j,\ j \in J$

$d_{i,j} = l_{i,j},\ i \in I,\ j \in J$

$d_{m+i,j} = l_{i,j},\ i \in I,\ j \in J$

$d_{i,n+j} = l_{i,j},\ i \in I,\ j \in J$

$$d_{2n+1,j} = \begin{cases} M, & j \in J \\ 0, & J \in J''\backslash J \end{cases}$$

$d_{m+i,n+j} = \ 0,\ i \in I,\ j \in J.$

Case III: When $\sum_{i\in I} A_i = \sum_{j\in J} B_j$, then the equivalent standard transportation problem is defined as:

P(2.5) Minimize $\sum_{i\in I''}\sum_{j\in J''} d_{i,j}\ z_{i,j}$

subject to

$$\sum_{i\in I''} z_{ij} = b'_j,\ j \in J''$$

$$\sum_{j\in J''} z_{ij} = a'_i,\ i \in I''$$

$$z_{i,j} \geq \ 0,\ \text{forall } i \in I'',\ j \in J''$$

If in P(2.1), the total number of buses available is known to be F, then an extra linear constraint introduced in P(2.1) yields a transportation problem P(2.6), which is stated as below:

P(2.6) Minimize $\sum_{i\in I}\sum_{j\in J} l_{i,j}\ x_{i,j}$

subject to

$$a_i \leq \sum_{j\in J} x_{i,j} \leq A_i,\quad i \in I$$

$$b_j \leq \sum_{i\in I} x_{i,j} \leq B_j,\quad j \in J$$

$$\sum_{i\in I}\sum_{j\in J}\ x_{i,j} = F$$

$$x_{i,j} \geq 0 \text{ and integers},\ i \in I,\ j \in J.$$

The constraint $\sum_{i\in I}\sum_{j\in J}\ x_{i,j} = F$ is referred to as flow constraint.

To solve P(2.6) a related standard transportation problem P(2.7) is constructed which is defined as:

P(2.7) Minimize $\sum_{i \in I'} \sum_{j \in J'} l'_{i,j} \; y_{i,j}$

subject to

$$\sum_{j \in J'} y_{i,j} = a'_i, \quad i \in I'$$

$$\sum_{i \in I'} y_{i,j} = b'_j, \quad j \in J'$$

$$y_{i,j} \geq 0, \quad i \in I', \; j \in J'$$

where

$a'_i = a_i, \; i \in I$

$a'_{m+i} = A_i - a_i, \; i \in I$

$a'_{2m+1} = \sum_{j \in J} B_j - F$

$b'_j = b_j, \; j \in J$

$b'_{n+j} = B_j - b_j, \; j \in J$

$b'_{2n+1} = \sum_{i \in I} A_i - F$

$l'_{i,j} = l_{i,j}, \quad i \in I, \; j \in J$

$l'_{m+i,j} = l_{i,j}, \quad i \in I, \; j \in J$

$l'_{i,n+j} = l_{i,j}, \quad i \in I, \ j \in J$

$$l'_{i,2n+1} = \begin{cases} M, & i \in I \\ 0, & i = m+1, ..., 2m \end{cases}$$

$$l'_{i,2m+1} = \begin{cases} M, & j \in J \\ 0, & j = n+1, ..., 2n \end{cases}$$

$l''_{2m+1,2n+1} = M$

$l'_{m+i,n+j} = M, \quad i \in I \ j \in J.$

Theorem 2.4 If $\{y_{i,j}\}, \ i \in I', \ j \in J'$ is an restricted feasible solution of P(2.7), then there exists $\{x_{i,j}\}, \ i \in I, \ j \in J$, such that it is a feasible solution of P(2.6) and conversely.

Proof. Define, $\quad x_{i,j} = y_{i,j} + y_{m+i,j} + y_{i,n+j}, \ i \in I, \ j \in J.$

Clearly $\ x_{i,j} \geq 0, \ i \in I, \ j \in J.$

Consider, $$\sum_{j \in J} x_{i,j} = \sum_{j \in J} y_{i,j} + \sum_{j \in J} y_{m+i,j} + \sum_{j \in J} y_{i,n+j}$$
$$= \sum_{j \in J''} y_{i,j} + \sum_{j \in J} y_{m+i,j}$$
$$\leq \sum_{j \in J'} y_{i,j} + \sum_{j \in J'} y_{m+i,j}$$
$$= a_i + (A_i - a_i)$$
$$= A_i, \ i \in I.$$

Hence, $$\sum_{j \in J} x_{i,j} \leq A_i, \ i \in I. \tag{2.16}$$

Consider, $$\sum_{j \in J} x_{i,j} = \sum_{j \in J} y_{i,j} + \sum_{j \in J} y_{m+i,j} + \sum_{j \in J} y_{i,n+j}$$
$$= \sum_{j \in J} y_{i,j} + \sum_{j \in J} y_{i,n+j} + y_{i,2n+1} + \sum_{j \in J} y_{m+i,j}$$
$$= \sum_{j \in J'} y_{i,j} + \sum_{j \in J} y_{m+i,j}$$

$$\geq \sum_{j\in J'} y_{i,j}$$

$$= a_i.$$

Hence $\sum\limits_{j\in J} x_{i,j} \geq a_i,\ i \in I.$ (2.17)

So (2.16) and (2.17) imply that $a_i \leq \sum\limits_{j\in J} x_{i,j} \leq A_i,\ i \in I.$

Similarly, it can be shown that $b_j \leq \sum\limits_{i\in I} x_{i,j} \leq B_j,\ j \in J.$

Now we claim that $\sum\limits_{i\in I}\sum\limits_{j\in J} x_{i,j} = F$

$$\begin{aligned}
F &= \sum_{i\in I''}\sum_{j\in J'} y_{i,j} - \sum_{i\in I''} y_{i,2n+1} \\
&= \sum_{i\in I''}\sum_{j\in J''} y_{i,j} + \sum_{i\in I''} y_{i,2n+1} - \sum_{i\in I''} y_{i,2n+1} \\
&= \sum_{i\in I''}\sum_{j\in J''} y_{i,j} \\
&= \sum_{i\in I''}\sum_{j\in J} y_{i,j} + \sum_{i\in I''}\sum_{j\in J''\backslash J} y_{i,j} \\
&= \sum_{i\in I''}\sum_{j\in J} y_{i,j} + \sum_{i\in I}\sum_{j\in J''\backslash J} y_{i,j} + \sum_{i\in I''\backslash I}\sum_{j\in J''} y_{i,j} \\
&= \sum_{i\in I}\sum_{j\in J} y_{i,j} + \sum_{i\in I''\backslash I}\sum_{j\in J} y_{i,j} + \sum_{i\in I}\sum_{j\in J''\backslash J} y_{i,j} + \sum_{i\in I''\backslash I}\sum_{j\in J''\backslash J} y_{ij} \\
&= \sum_{i\in I}\sum_{j\in J} y_{i,j} + \sum_{i\in I}\sum_{j\in J} y_{m+i,j} + \sum_{i\in I}\sum_{j\in J} y_{i,n+j} + \sum_{i\in I}\sum_{j\in J} y_{m+i,n+j} \\
&= \sum_{i\in I}\sum_{j\in J} y_{i,j} + \sum_{i\in I}\sum_{j\in J} y_{m+i,j} + \sum_{i\in I}\sum_{j\in J} y_{i,n+j} \\
&= \sum_{i\in I}\sum_{j\in J} x_{i,j}.
\end{aligned}$$

Hence $\sum\limits_{i\in I}\sum\limits_{j\in J} x_{i,j} = F.$

Conversely, let $\{x_{i,j}\},\ i \in I,\ j \in J$, be a feasible solution of problem P(2.6). Then

$$a_i \leq \sum_{j\in J} x_{i,j}(= A'_i) \leq A_i,\ i \in I$$

$$b_j \leq \sum_{i\in I} x_{i,j}(= B'_j) \leq B_j,\ j \in J$$

$$x_{ij} \geq 0,\ i \in I,\ j \in J.$$

In order to prove the result, construct a new problem as:

P(2.8) Minimize $\sum_{i\in I''} \sum_{j\in J''} l''_{i,j}\ u_{i,j}$

subject to

$$\sum_{j\in J''} u_{i,j} = a_i,\ i \in I$$

$$\sum_{j\in J''} u_{m+i,j} = A'_i - a_i,\ i \in I$$

$$\sum_{i\in I''} u_{i,j} = b\ j \in J$$

$$\sum_{i\in I''} u_{i,n+j} = B'_j - b_j,\ j \in J$$

$$u_{i,j} \geq 0,\ i \in I'',\ j \in J''$$

where

$$l''_{i,j} = l'_{i,j},\ i \in I'',\ j \in J''.$$

Let $\{u_{i,j}\},\ i \in I''\ j \in J''$ be a feasible solution of P(2.8). Then $y_{i,j} = u_{i,j},\ i \in I''\ j \in J''$.

Let $y_{i,2n+1} = 0,\ i \in I.$

$$y_{m+i,2n+1} = A_i - A'_i,\ i \in I$$

$$y_{2m+1,j} = 0,\ j \in J$$

$$y_{2m+1,n+j} = B_j - B'_j,\ j \in J$$

$$y_{2m+1,2n+1} = 0.$$

For $i \in I$, $\sum_{j\in J''} u_{i,j} = a_i.$

Hence $\sum_{j\in J''} y_{i,j} = a_i.$

Since $y_{i,2n+1} = 0,\ i \in I$, so $\sum\limits_{j \in J'} y_{i,j} = \sum\limits_{j \in J''} y_{i,j} + y_{i,2n+1} = a_i = a'_i,\ i \in I$

This implies that $\sum\limits_{j \in J''} y_{m+i,j} = A'_i - a_i,\ i \in I.$

$$\begin{aligned}\sum_{j \in J'} y_{m+i,j} = \sum_{j \in J''} & y_{m+i,j} + y_{m+i,2n+1} \\ &= A'_i - a_i + A_i - A'_i \\ &= A_i - a_i,\ i \in I.\end{aligned}$$

For $i = 2m+1$, $\sum\limits_{j \in J'} y_{2m+1,j} = \sum\limits_{j \in J} y_{2m+1,j} + \sum\limits_{j \in J} y_{2m+1,n+j} + y_{2m+1,2n+1}$

$$\begin{aligned}&= 0 + \sum_{j \in J} y_{2m+1,n+j} (\text{since } y_{2m+1,j} = 0,\ j \in J \text{ and } y_{2m+1,2n+1} = 0) \\ &= \sum_{j \in J} B_j - \sum_{j \in J} B'_j \\ &= \sum_{j \in J} B_j - \sum_{j \in J} [\sum_{i \in I} x_{i,j}] \\ &= \sum_{j \in J} B_j - F.\end{aligned}$$

Similarly it can be show that column constraints of P(2.7) are satisfied. Hence $\{y_{i,j}\},\ i \in I',\ j \in J'$, is a feasible solution of P(2.7).

Theorem 2.5 If $\{x_{i,j}\},\ i \in I\ j \in J$ is an optimal feasible solution of P(2.6), then the corresponding restricted feasible solution $\{y_{i,j}\},\ i \in I',\ j \in J'$ of P(2.7) is also an optimal solution of P(2.7).

Proof. If possible, let $\{y_{i,j}\},\ i \in I',\ j \in J'$, be a restricted feasible solution of P(2.7) such that it is optimal to P(2.7), then

$$\sum_{i \in I'} \sum_{j \in J'} l'_{i,j}\ y'_{i,j} < \sum_{i \in I'} \sum_{j \in J'} l'_{i,j}\ y_{i,j}.$$

Since $\{y_{i,j}\},\ i \in I',\ j \in J'$ is a restricted feasible solution of P(2.7), there exists $\{x''_{i,j}\},\ i \in I,\ j \in J$, a corresponding feasible solution of P(2.6) such that

$$\sum_{i\in I}\sum_{j\in J} l_{i,j}\ x''_{i,j} < \sum_{i\in I'}\sum_{j\in J'} l'_{i,j}\ y'_{i,j} = \sum_{i\in I}\sum_{j\in J} l_{i,j}\ x_{i,j}.$$

Thus

$$\sum_{i\in I}\sum_{j\in J} l_{i,j}\ x''_{i,j} < \sum_{i\in I}\sum_{j\in J} l_{i,j}\ x_{i,j}.$$

which implies that $\{x''_{i,j}\},\ i \in I,\ j \in J$ is an optimal feasible solution of P(2.6), a contradiction to the assumption that $\{x_{i,j}\},\ i \in I,\ j \in J$ is optimal to P(2.6). Hence $\{y_{i,j}\},\ i \in I',\ j \in J'$ is an optimal restricted feasible solution of P(2.7).

Note: Let F_k denote the total flow obtained on solving P(2.1). Let Z_k and Z are the optimal values of objective function in P(2.1) and P(2.6) respectively. If F in P(2.6) is such that $F < F_k$, then $Z > Z_k$ for if this is not so, then $Z < Z_k$, which in turn implies that Z is the optimal value of the objective function in P(2.1), which is not true. Thus $Z > Z_k$. So whenever $F < F_k$ it follows that $Z > Z_k$, which leads to paradox. On the other hand if $F > F_k$, again $Z > Z_k$, and then there is no paradox.

NUMERICAL EXAMPLE

Example 2.1

$$\begin{aligned} \text{Minimize}\quad & \sum_{i=1}^{3}\sum_{j=1}^{2} l_{ij}\ x_{ij} \\ \text{subject to}\quad & 3 \le \sum_{j=1}^{2} x_{1j} \le 11 \\ & 3 \le \sum_{j=1}^{2} x_{2j} \le 10 \\ & 9 \le \sum_{j=1}^{3} x_{3j} \le 14 \\ & 6 \le \sum_{i=1}^{3} x_{i1} \le 15 \\ & 7 \le \sum_{i=1}^{3} x_{i2} \le 12 \end{aligned}$$

$$\sum_{i=1}^{3}\sum_{j=1}^{2} x_{ij} = 15$$

where $x_{ij} \geq 0,\ i = 1, 2, 3$ and $j = 1, 2$.

This problem is of type P(2.6), which is equivalent to P(2.7). Tabular form of the above problem is:

	D_1	D_2	D_3	D_4	D_5
S_1	3	6	3	6	M
S_2	2	4	2	4	M
S_3	5	8	5	8	M
S_4	3	6	M	M	0
S_5	2	4	M	M	0
S_6	5	8	M	M	0
S_7	M	M	0	0	M

The table representing optimal solution is :

	D_1	D_2	D_3	D_4	D_5
S_1	3 **(3)**	6	3	6	M
S_2	2	4 **(3)**	2	4	M
S_3	5 **(3)**	8 **(4)**	5 **(2)**	8	M **(0)**
S_4	3	6	M	M	0 **(8)**
S_5	2	4	M	M	0 **(7)**
S_6	5	8	M	M	0 **(5)**
S_7	M	M	0 **(7)**	0 **(5)**	M

where figures in () show allocations. Optimal solution of original problem is

	D_1	D_2
S_1	3 (3)	6
S_2	2	4 (3)
S_3	5 (5)	8 (4)

Hence for $F = 15$, cost $Z = 78$.

If there is no flow constraint, i.e., $\sum_{i=1}^{3}\sum_{j=1}^{2} x_{i,j} = 15$ is not there, then the problem is similar to P(2.1) and can be solved by solving the equivalent problem P(2.2), whose solution in the tabular form would be

	D_1	D_2	D_3	D_4	D_5
S_1	3	6	3	6	M
S_2	2	4	2	4	M
S_3	5	8	5	8	M
S_4	3	6	0	0	0
S_5	2	4	0	0	0
S_6	5	8	0	0	0

	D_1	D_2	D_3	D_4	D_5
S_1	3 (1)	6	3 (2)	6	M
S_2	2	4 (3)	2	4	M
S_3	5 (5)	8 (4)	5	8	M
S_4	3	6	0	0 (0)	0 (8)
S_5	2	4	0 (2)	0 (5)	0
S_6	5	8	0 (5)	0	0

Then, the optimal solution to original problem without flow constraint is :

	D_1	D_2
S_1	3 **(3)**	6
S_2	2	4 **(3)**
S_3	5 **(5)**	8 **(4)**

Here also, cost $Z = 78$.

Example 2.2

$$\begin{aligned}
\text{Minimize} \quad & \sum_{i=1}^{3}\sum_{j=1}^{2} l_{ij}\ x_{ij} \\
\text{subject to} \quad & 3 \le \sum_{j=1}^{2}\ x_{1j} \le 14 \\
& 4 \le \sum_{j=1}^{2}\ x_{2j} \le 15 \\
& 6 \le \sum_{j=1}^{3}\ x_{3j} \le 15 \\
& 2 \le \sum_{i=1}^{3}\ x_{i1} \le 15 \\
& 5 \le \sum_{i=1}^{3}\ x_{i2} \le 14 \\
& \sum_{i=1}^{3}\sum_{j=1}^{2}\ x_{ij} = 13
\end{aligned}$$

where $x_{ij} \ge 0,\ i = 1, 2, 3$ and $j = 1, 2$.

Its optimal table is :

	D_1	D_2	D_3	D_4	D_5
S_1	2	5 **(1)**	2 **(2)**	5	M
S_2	4	1 **(4)**	4	1	M
S_3	3 **(2)**	6	3 **(4)**	6	M **(0)**
S_4	2	5	M	M	0 **(11)**
S_5	4	1	M	M	0 **(11)**
S_6	3	6	M	M	0 **(9)**
S_7	M	M	0 **(7)**	0 **(9)**	M

Optimal solution of original problem is

	D_1	D_2
S_1	2 **(2)**	5 **(1)**
S_2	4	1 **(4)**
S_3	3 **(6)**	6

Hence for $F = 13$, cost $Z = 31$.

If there is no flow constraint, i.e., $\sum\limits_{i=1}^{3}\sum\limits_{j=1}^{2} x_{ij} = 13$ is not there, then the problem is similar to P(2.1) and can be solved by solving the equivalent problem P(2.2), whose critical solution in the present case would be

	D_1	D_2	D_3	D_4	D_5
S_1	2 **(2)**	5	2 **(1)**	5	M
S_2	4	1 **(4)**	4	1	M
S_3	3	6	3 **(6)**	6	M
S_4	2	5	0 **(6)**	0 **(5)**	0
S_5	4	1 **(1)**	0	0 **(4)**	0 **(6)**
S_6	3	6	0	0	0 **(9)**

The optimal solution to original problem without flow constraint is

	D_1	D_2
S_1	2 **(3)**	5
S_2	4	1 **(5)**
S_3	3 **(6)**	6

Here flow F_k is 14 and cost $Z_k = 29$. It should be noted that solution to original problem in presence of flow constraint yields paradoxical pair $(F = 13,\ Z = 31)$.

CONCLUSION

Constrained transportation problem developed by Verma and Puri [28] with the bounds specified on the source availabilities and destination requirements is discussed. With the inclusion of flow constraint the transportation model becomes more realistic. In the presence of flow of commodities the paradoxical solution is obtained for the example considered in example 2.2.

Chapter 3

Two - Stage Interval Time Minimization Transportation Problem

3.1 Introduction

The Two Stage Time Minimizing Transportation Problem is one in which the total availability of a homogeneous product at the sources is more than the minimum requirement at the destinations. In many real life situations due to storage constraints, destinations are unable to receive the quantity in excess of their minimum initial demand. Only after consuming part or whole of the initial shipment, they are ready to receive the excess quantity in the second stage. In order to handle such situations, the product is transported to the destinations in two stages. In the first stage, just enough of the product is shipped so that the minimum requirements at the destinations are satisfied and surplus quantities (if any) at the sources are shipped to the destination according to time considerations. Consider for example, the production of maintenance - free sealed industrial batteries. Production is a continuous process depending on the available resources. However each battery has a certain shelf-life and needs to be periodically re-charged. Often due to lack of re-charging facilities on the production floor, each batch of manufactured battery is transported immediately to the demand points. This corresponds to the first stage. In the second stage enough batteries are shipped from the sources in order to satisfy the industrial user's demand at destinations. In both stages the transportation of the product from sources to destination is done in parallel with the aim of minimizing the sum of transportation times in the two stages. This objective is a concave function but the problem is different from an earlier investigated one involving the minimization of the sum of a finite number of independent concave functions in the sense that the present objective function involves the sum of two dependent concave functions.

Sonia et al. [25] discussed the two stage interval time minimization transportation problem. Mathematical formulation of the problem considered by them is as follows : let a_i and $A_i, i \in I$ denote respectively the minimum and maximum availability of a homogeneous product at the source i and $b_j, j \in J$ the demand of the same at the destination j, where

$$\sum_{i \in I} a_i < \sum_{j \in J} b_j < \sum_{i \in I} A_i$$

In the first stage of the two stage interval Time Minimization Transportation Problem (TMTP), the quantity a_i $(< A_i)$ is shipped from each source $i,\ i \in I$ and after the completion, enough quantity of the product is dispatched in second stage so as to exactly satisfy the demand b_j at the destination $j,\ j \in J$. The stage I problem is formulated as:

$$\min_{Y=\{y_{ij}\} \in S'} [\max_{I \times J}(t_{ij}(y_{ij}))] = \min_{Y \in S'}[T_1(Y)]$$

where the set S' is given as

$$S' = \begin{cases} \sum\limits_{j\in J} y_{ij} = a_i \ \text{ for all } i \in I, \\ \sum\limits_{i\in I} y_{ij} \leq b_j \ \text{ for all } j \in J, \\ y_{ij} \geq 0 \qquad \text{for all } (i,j) \in I \times J. \end{cases}$$

Corresponding to a feasible solution $Y = y_{ij}$ of stage I problem, let $S'(Y)$ be the set of feasible solutions of stage II problem which is as follows:

$$\min_{Z=\{z_{ij}\}\in S'(Y)} [\max_{I\times J}(t_{ij}(z_{ij}))] = \min_{Z\in S'(Y)} [T_2(Z)]$$

where the set $S'(Y)$ is given as

$$S'(Y) = \begin{cases} \sum\limits_{j\in J} z_{ij} \leq A_i - a_i \ \text{ for all } i \in I, \\ \sum\limits_{i\in I} z_{ij} = b_j - B_j \ \text{ for all } j \in J, \\ z_{ij} \geq 0 \qquad \text{for all } (i,j) \in I \times J. \end{cases}$$

and $B_j = \sum\limits_{i\in I} y_{ij}, \ j \in J.$

Thus a two stage time minimization transportation problem can be defined as:

P(3.1) $\min\limits_{Y=\{y_{ij}\}\in S'} [(T_1(Y))] + \min\limits_{Z\in S'(Y)} [(T_2(Z))]$

Closely related to the problem P(3.1) is the interval time minimizing transportation problem P(3.2) defined as:

P(3.2) $\min\limits_{X\in S} [T(X)] = \min\limits_{X\in S} [\max\limits_{I\times J}(t_{ij}(x_{ij}))]$

where

$$S = \begin{cases} a_i \leq \sum\limits_{j\in J} x_{ij} \leq A_i \ \text{ for all } \ i \in I, \\ \sum\limits_{i\in I} x_{ij} = b_j \ \text{ for all } \ (j \in J), \\ x_{ij} \geq 0 \qquad \text{for all } (i,j) \in I \times J. \end{cases}$$

Clearly a feasible solution of P(3.1) provides a feasible solution to the problem P(3.2) and conversely. Associated with the problem P(3.2) a balanced transportation problem is defined as:

P(3.3) $\min_{X\in\hat{S}}[\hat{T}(X)] = \min_{X\in\hat{S}}[\max_{\hat{I}\times\hat{J}}(\hat{t}_{ij}(x_{ij}))]$

where

$$\hat{S} \;=\; \begin{cases} \sum_{j\in J} x_{ij} = \hat{a}_i \ \text{ for all } i \in \hat{I}, \\ \sum_{i\in I} x_{ij} = \hat{b}_j \ \text{ for all } j \in \hat{J}, \\ x_{ij} \geq 0 \qquad \text{for all } (i,j) \in \hat{I} \times \hat{J}. \end{cases}$$

where, $\hat{I} = \{1, 2, ..., m, m+1, ..., 2m\}$,

$\hat{J} = J \cup \{n+1\}$,

$\hat{a}_i = a_i, i \in I$,

$\hat{a}_{m+i} = A_i - a_i, i \in I$,

$\hat{b}_j = b_j \quad \text{for all} j \in J$,

$\hat{b}_{n+1} = \sum_{i\in I} A_i - \sum_{j\in J} b_j$,

$\hat{t}_{ij} = t_{ij} \quad \text{for all } (i,j) \in I \times J.$

$\hat{t}_{m+i,j} = t_{ij} \quad \text{for all } (i,j) \in I \times J$,

$\hat{t}_{i,n+1} = M \quad \text{for all } i \in I$,

where M is a very large positive number,

$\hat{t}_{m+i,n+1} = 0 \ \text{ for all } i \in I.$

3.2 Theoretical Development

The shipment time in Stage-1 and Stage-2 are concave functions, so the two stage interval time minimization transportation problem aims at minimizing a concave function over a polytope. Hence P(3.1) is also a concave minimization problem. As the global minimum of a concave minimization problem is attained at an extreme point only, it is desirable to investigate only its extreme points. Let the set of transportation time on various routes is partitioned into a number of disjoint sets, $B_h, h = 1, 2, ..., s$, where $B_h = \{(i, j) \in I \times J : t_{ij} = t^h\}$ and $t^j > t^{j+1}$ for all $j = 1, 2, ..., s-1$. Positive weights say $\lambda_{s-h+1}, h = 1, 2, ..., s$ are attached to these sets where, $\lambda_{j+1} >> \lambda_j$ for all$j = 1, 2, ...s-1$. This yields a standard Cost Minimization Transportation Problem (CMTP):

$$\min \sum_{h=1}^{s} \lambda_h \Big(\sum_{(i,j) \in B_h} x_{ij} \Big),$$

where $X = \{x_{ij}\}$ belongs to the transportation polytope over which original (TMTP) is being studied. To find an optimal feasible solution (OFS) of the Stage II problem, define the following (CMTP):

P(3.4)
$$\min_{\hat{S}} \sum_{\hat{I} \times \hat{J}} c_{ij} x_{ij},$$

where

$$c_{i,n+1} = M \quad \text{for} \quad \text{all } i \in I,$$

$$c_{m+i,n+1} = 0 \quad \text{for} \quad \text{all } i \in I,$$

$$c_{ij} = 0 \quad \text{for} \quad \text{all } (i,j) \in I \times J,$$

$$c_{m+i,j} = \lambda_{s-h+1};\ t_{m+i,j} = t^h, \quad \text{for} \quad \text{all } (i,j) \in B_h \quad \text{and} \quad h = 1, 2, ..., s.$$

Let at any given time of Stage I and Stage II say, T_1^{k-1}, T_2^{k-1} respectively, where $T_1^{k-1}, T_2^{k-1} \in \{t_1, t_2, ..., t_s\}$,

$k \in \{1, 2, ..., s+1\}$. The restricted version of the problem P(3.4) denoted by P$(3.4)_k$, $k \geq 1$ is defined as:

P$(3.4)_k$ $$\min_{\hat{S}} \sum_{i \in \hat{I}} \sum_{j \in \hat{J}} c_{ij} x_{ij}$$

where

$$c_{ij} = M \quad if \quad t_{ij} \geq T^{k-1}, \quad \text{for} \quad \text{all } (i,j) \in I \times J,$$

$$= 0 \quad if \quad t_{ij} < T^{k-1}, \quad \text{for} \quad \text{all } (i,j) \in I \times J,$$

$$c_{i,n+1} = M \quad \text{for} \quad \text{all} \quad i \in I,$$

$$c_{m+i,n+1} = 0 \quad \text{for} \quad \text{all} \quad i \in I,$$

$$c_{m+i,j} = \lambda_{s-h+1};\ t_{m+i,j} = t^h,\ (i,j) \in B_h \text{ and } h = 1, 2, ..., s.$$

An (OFS) of the problem P(3.4) is denoted by Y^0 with corresponding stage I time T_1^0 and the stage II time by T_2^0 and let Y^k be an (OFS) of P$(3.4)_k$ yielding corresponding time of stage I and stage II as T_1^k and T_2^k respectively.

Theorem 3.1 T_2^k is the minimum time of stage II corresponding to any given time of stage I in the problem P$(3.4)_k$.

Proof : Let if possible there exist a pair T_1, T_2 yielded by some feasible solution $Y = \{y_{ij}\}$ of P$(3.4)_k$ such that $T_2 < T_2^k$ and $T_1 < T_1^{k-1}$ where $T_2 = t_p$ and $T_2^k = t_q$ for some $p, q \in \{1, 2, ..., s\}$.

Since $T_2 < T_2^k$, therefore $p > q$, which implies $s - p + 1 < s - q + 1$.

Therefore

$$\begin{aligned} Z(Y) &= \sum_{\hat{I}\times\hat{J}} c_{ij}\ y_{ij} = \sum_{h=1}^{s} \lambda_{s-h+1}(\sum_{(i,j)\in B_h} y_{ij}) \\ &= \sum_{h=p}^{s} \lambda_{s-h+1}(\sum_{(i,j)\in B_h} y_{ij}). \end{aligned}$$

Also, $Z(Y^k) = \sum_{\hat{I}\times\hat{J}} c_{ij}\ y^k_{ij} = \sum_{h=1}^{s} \lambda_{s-h+1}(\sum_{(i,j)\in B_h} y^k_{ij})$

$$= \sum_{h=q}^{s} \lambda_{s-h+1}(\sum_{(i,j)\in B_h} y^k_{ij}).$$

Since $\lambda_{i+1} >> \lambda_i, i = 1, 2, ..., s-1$, which implies

$$\sum_{h=1}^{s} \lambda_{s-h+1}(\sum_{(i,j)\in B_h} y_{ij}) < \sum_{h=q}^{s} \lambda_{s-h+1}(\sum_{(i,j)\in B_h} y_{ij})$$

$$\Rightarrow \quad Z(Y) < Z(Y^k).$$

But this contradict the optimality of Y^k, therefore $T_2^k \leq T_2$.

Corollary 3.1 P(3.4) gives optimal time of stage II.

Remark 3.1 From P$(3.4)_k$ we conclude that $T_1^0 > T_1^1 > ... > T_1^l$ and further it has also been observed that $T_2^0 \leq T_2^1 \leq ... \leq T_2^l$. If possible, suppose there exist $T_2^{k+1} < T_2^k$ for some k, $Z_k = Z(Y^k)$ and $Z_{k+1} = Z(Y^{k+1})$. As $T_2^{k+1} < T_2^k$, Y^{k+1} is a feasible solution of P$(3.4)_k$ with $Z_{k+1} < Z_k$, a contradiction to the fact that Y^k is an (OFS) of P$(3.4)_k$.

Remark 3.2 Since optimal time of stage-1 problem is T_1^l, (OBFS) of P$(3.4)_{l+1}$ is not M-feasible.

Remark 3.3 Let $T_1^0 = t^r$ for some $r \in \{1, 2, ..., s\}$ then the maximum number of iterations required to solve this problem is $s - r + 1$.

Theorem 3.2 Let the generated pairs of stage I and stage II time be $(T_1^k, T_2^k),\ k \geq 0$. Then the optimal value of the problem P(3.1) is given by $\min\limits_{\{h=0,1,...,l\}} [T_1^h, T_2^h]$.

Proof : Let if possible there exist a pair (Y_1, Y_2) yielding stage I time and stage II shipment time (T_1, T_2) such that $T_1 + T_2 < \min_{\{h=0,1,...,l\}} [T_1^h, T_2^h]$. Since $T_1^0 > T_1^1 > ... > T_1^l$ and $T_2^0 \leq T_2^1 \leq ... \leq T_2^l$, then the following cases arise:

Case 1. $T_1 > T_1^0$ (3.1)

By construction of P(3.4), (Y_1, Y_2) is a feasible solution of P(3.4). Since T_2^0 is the optimal time for P(3.4), therefore $T_2^0 \leq T_2$. (3.2)

Combining (3.1) and (3.2), we get, $T_1 + T_2 > T_1^0 + T_2^0$.

$$\Rightarrow \quad T_1 + T_2 > \min_{\{h=0,1,...,l\}} [T_1^h, T_2^h].$$

Case 2. $T_1 < T_1^l$

Since $T_1 < T_1^l$, (Y_1, Y_2) is an M - feasible solution of $P(3.4)_l$, which is a contradiction as this problem is not M-feasible.

Case 3. $T_1 \in [T_1^0, T_1^l]$

In this case, either $T_1 = T_1^k$ for some $k = 0, 1, ..., l$ or $T_1 \in (T_1^k, T_1^{k-1})$, since $T_1^{k-1} >$

T_1^k.

(i) If $T_1 = T_1^0$, then by construction of P(3.4), (Y_1, Y_2) is a feasible solution of P(3.4).

$\Rightarrow \quad T_2 \geq T_2^0$, [since T_2^0 is the optimal of stage II in P(3.4)]

$\Rightarrow \; T_1 + T_2 \geq T_1^0 + T_2^0$,

$\Rightarrow \quad T_1 + T_2 \geq \min\limits_{\{h=0,1,...,l\}} [T_1^h, T_2^h]$.

Similarly for the case when $T_1 = T_1^k, \; k \in \{1, 2, ..., l\}$ it can be shown that $T_1 + T_2 \geq T_1^k + T_2^k \geq \min\limits_{\{h=0,1,...,l\}} [T_1^h, T_2^h]$.

(ii) $T_1 \in (T_1^k, T_1^{k-1})$.

Then (Y_1, Y_2) is a feasible solution of $\text{P}(3.4)_k$ (since $T_1 < T_1^{k-1}$). Also $T_2 \geq T_2^k$ and $T_1 > T_1^k$,

$\Rightarrow \quad T_1 + T_2 > T_1^k + T_2^k$,

$\Rightarrow \quad T_1 + T_2 > \min\limits_{\{h=0,1,...,l\}} [T_1^h, T_2^h]$.

Therefore there does not exist a feasible solution $Y = (Y_1, Y_2)$ of $\text{P}(3.4)_k$ yielding time less than $\min\limits_{\{h=0,1,...,l\}} [T_1^h, T_2^h]$. Thus the optimal value of P(3.1) is given by $\min\limits_{\{h=0,1,...,l\}} [T_1^h, T_2^h]$.

3.3 The Procedure

Initial Step. Find an (OBFS) of P(3.4) and thus obtain the corresponding times T_1^0 and T_2^0 of stage I and stage II respectively.

General Step. If $k \geq 1$ at a given pair (T_1^{k-1}, T_2^{k-1}) of stage I and stage II times, solve the problem P$(3.4)_k$. From the (OBFS) of P$(3.4)_k$ construct the pairs (T_1^{k+1}, T_2^{k+1}).

Terminal Step. If (OBFS) of problem P$(3.4)_k$ is not M-feasible, then stop. The optimal value of (3.1) is given by $\min_{\{h=0,1,\ldots,k\}} [T_1^h, T_2^h]$.

3.4 Example

Consider the two stage interval time minimization transportation problem given as:

	D_1	D_2	D_3	D_4	D_5	D_6	a_i	A_i
S_1	26	23	59	38	19	20	6	8
S_2	40	48	20	19	23	59	15	29
S_3	26	38	48	20	19	40	12	18
b_j	6	9	3	14	10	5		

The partition of various time routes is given by:

$t^1(=59) > t^2(=48) > t^3(=40) > t^4(=38) > t^5(=26) > t^6(=23) > t^7(=20) > t^8(=19)$.

Here $t^s = t^8 = 19$ and so $s = 8$.

The partition of positive weights is given as:

$t^1 \;=\; B_1 \;=\; 59 \;=\; (1,3),(2,6)$

$t^2 \;=\; B_2 \;=\; 48 \;=\; (2,2),(3,3)$

$t^3 \;=\; B_3 \;=\; 40 \;=\; (2,1),(3,6)$

$t^4 \;=\; B_4 \;=\; 38 \;=\; (1,4),(3,2)$

$t^5 \;=\; B_5 \;=\; 26 \;=\; (1,1),(3,1)$

$t^6 \;=\; B_6 \;=\; 23 \;=\; (1,2),(2,5)$

$t^7 \;=\; B_7 \;=\; 20 \;=\; (1,6),(2,3),(3,4)$

$t^8 \;=\; B_8 \;=\; 19 \;=\; (1,5),(2,4),(3,5)$

$t_{m+1,3} \;=\; t^1$, here $h = 1$, $c_{m+1,3} \;=\; \lambda_{8-1+1} \;=\; \lambda_8$

$t_{m+2,2} \;=\; t^2$, here $h = 2$, $c_{m+2,2} \;=\; \lambda_{8-2+1} \;=\; \lambda_7$

$t_{m+2,1} \;=\; t^3$, here $h = 3$, $c_{m+2,1} \;=\; \lambda_{8-3+1} \;=\; \lambda_6$

$t_{m+1,4} \;=\; t^4$, here $h = 4$, $c_{m+1,4} \;=\; \lambda_{8-4+1} \;=\; \lambda_5$

$t_{m+1,1} \;=\; t^5$, here $h = 5$, $c_{m+1,1} \;=\; \lambda_{8-5+1} \;=\; \lambda_4$

$t_{m+1,2} \;=\; t^6$, here $h = 6$, $c_{m+1,2} \;=\; \lambda_{8-6+1} \;=\; \lambda_3$

$t_{m+1,6} \;=\; t^7$, here $h = 7$, $c_{m+1,6} \;=\; \lambda_{8-7+1} \;=\; \lambda_2$

$t_{m+1,5} \;=\; t^8$, here $h = 8$, $c_{m+1,5} \;=\; \lambda_{8-8+1} \;=\; \lambda_1$

The corresponding problem P(3.3) is given in the following table:

	D_1	D_2	D_3	D_4	D_5	D_6	D_7	a'_i
S_1	26	23	59	38	19	20	M	6
S_2	40	48	20	19	23	59	M	15
S_3	26	38	48	20	19	40	M	12
S_1	26	23	59	38	19	20	0	2
S_2	40	48	20	19	23	59	0	14
S_3	26	38	48	20	19	40	0	6
b'_j	6	9	3	14	10	5	8	

An optimal table of the problem P(3.4) is:

	D_1	D_2	D_3	D_4	D_5	D_6	D_7
S_1	0	$0^{\mathbf{(1)}}$	M	0	0	$0^{\mathbf{(5)}}$	M
S_2	$0^{\mathbf{(2)}}$	M	$0^{\mathbf{(3)}}$	$0^{\mathbf{(8)}}$	$0^{\mathbf{(2)}}$	M	M
S_3	$0^{\mathbf{(4)}}$	$0^{\mathbf{(8)}}$	M	0	0	0	M
S_4	λ_4	λ_3	λ_8	λ_5	$\lambda_1^{\mathbf{(2)}}$	λ_2	0
S_5	λ_6	λ_7	λ_2	$\lambda_1^{\mathbf{(6)}}$	λ_3	λ_8	$0^{\mathbf{(8)}}$
S_6	λ_4	λ_5	λ_7	λ_2	$\lambda_1^{\mathbf{(6)}}$	λ_6	0

An optimal basic feasible solution of the problem P(3.4) yields stage-I time as 40 and stage-II time as 19, where 19 is the optimal time of stage-II.

Next pair is obtained from the problem P$(3.4)_1$ in which routes having time 40 at stage-I is blocked. The optimal table of the problem P$(3.4)_1$ is given as:

	D_1	D_2	D_3	D_4	D_5	D_6	D_7
S_1	0	$0^{\mathbf{(3)}}$	M	0	0	$0^{\mathbf{(3)}}$	M
S_2	M	M	$0^{\mathbf{(3)}}$	$0^{\mathbf{(2)}}$	$0^{\mathbf{(10)}}$	M	M
S_3	$0^{\mathbf{(6)}}$	$0^{\mathbf{(6)}}$	M	0	0	M	M
S_4	λ_4	λ_3	λ_8	λ_5	$\lambda_1^{\mathbf{(0)}}$	$\lambda_2^{\mathbf{(2)}}$	0
S_5	λ_6	λ_7	λ_2	$\lambda_1^{\mathbf{(12)}}$	λ_3	λ_8	$0^{\mathbf{(2)}}$
S_6	λ_4	λ_5	λ_7	λ_2	λ_1	λ_6	$0^{\mathbf{(6)}}$

An optimal basic feasible solution of the problem $P(3.4)_1$ yields stage-I time as 38 and stage-II time as 20, where 20 is the minimum time for stage-II corresponding to the stage-I time 38.

Next pair is obtained from the problem $P(3.4)_2$ in which routes having time 38 at stage-I is blocked. The optimal table of the problem $P(3.4)_2$ is given as:

	D_1	D_2	D_3	D_4	D_5	D_6	D_7
S_1	0	$0^{\mathbf{(3)}}$	M	M	0	$0^{\mathbf{(3)}}$	M
S_2	M	M	$0^{\mathbf{(3)}}$	$0^{\mathbf{(8)}}$	$0^{\mathbf{(4)}}$	M	M
S_3	$0^{\mathbf{(6)}}$	M	M	0	$0^{\mathbf{(6)}}$	M	M
S_4	λ_4	λ_3	λ_8	λ_5	λ_1	$\lambda_2^{\mathbf{(2)}}$	0
S_5	λ_6	λ_7	λ_2	$\lambda_1^{\mathbf{(6)}}$	λ_3	λ_8	$0^{\mathbf{(8)}}$
S_6	λ_4	$\lambda_5^{\mathbf{(6)}}$	λ_7	λ_2	λ_1	λ_6	$0^{\mathbf{(0)}}$

An optimal basic feasible solution of the problem $P(3.4)_2$ yields stage-I time as 26 and stage-II time as 38.

Proceeding in the same way, next pair is obtained from the problem $P(3.4)_3$ in which routes having time 26 at stage-I is blocked. The optimal table of the problem $P(3.4)_3$

is given as:

	D_1	D_2	D_3	D_4	D_5	D_6	D_7
S_1	M	$0^{\mathbf{(3)}}$	M	M	0	$0^{\mathbf{(3)}}$	M
S_2	M	M	$0^{\mathbf{(3)}}$	$0^{\mathbf{(2)}}$	$0^{\mathbf{(10)}}$	M	M
S_3	M	M	M	$0^{\mathbf{(12)}}$	0	M	M
S_4	λ_4	λ_3	λ_8	λ_5	λ_1	$\lambda_2{}^{\mathbf{(2)}}$	0
S_5	$\lambda_6{}^{\mathbf{(6)}}$	λ_7	λ_2	$\lambda_1{}^{\mathbf{(0)}}$	λ_3	λ_8	$0^{\mathbf{(8)}}$
S_6	$\lambda_4{}^{\mathbf{(0)}}$	$\lambda_5{}^{\mathbf{(6)}}$	λ_7	λ_2	λ_1	λ_6	0

An optimal basic feasible solution of the problem $P(3.4)_3$ yields stage-I time as 23 and stage-II time as 40.

Algorithm terminates here as problem $P(3.4)_4$ is no more M-feasible.

Thus $\min\{40+19, 38+20, 26+38, 23+40\} = 58$. Hence the optimal value of the problem $P(3.1)$ corresponds to the pair $(38, 20)$.

CONCLUSION

Algorithm developed by Sharma et al. [23] generates a sequence of stage-I and stage-II time, where at each iteration stage-I time decreases strictly and stage-II time increases. The aim is to minimize the sum of two stage shipment times.

Chapter 4

Modified Time - Cost Trade - Off Relations In Transportation Problem

4.1 Introduction

In the classical transportation problem of linear programming, the traditional objective is one of minimizing the total cost. However, in certain situations such as in the transportation of perishable items, mission oriented military supplies, ambulance services and fire services the time for completing the transportation to all of the demand points attains greater importance than the cost of transportation. In such situations, transport cost and transmit time cannot be viewed as two independent problems and there is need to have a trade-off between time and cost. By studying the constrained transportation problem given by Verma and Puri [28] and the time-cost trade-off in bulk transportation problem given by Gupta et al. [15], an algorithm is developed in this present chapter to find all efficient time-cost trade-off pairs in the transportation problem.

4.2 Theoretical Development

To obtain time-cost trade-off relations in the transportation problem the following bicriteria problem is considered:

P(4.1) $\mathrm{Min}_{X \in S}\ (Z(X), T(X))$

where

$$S \;=\; \begin{cases} a_i \le \sum\limits_{j \in J} x_{i,j} \le A_i, \quad \text{for all } i \in I, \\ \\ \sum\limits_{i \in I} x_{i,j} \le b_j, \quad \text{for all } j \in J, \end{cases}$$

$X \;=\; \{x_{i,j}\} \;\in R^{mn}$,

$I \;=\; \{1, 2, ..., m\}$, index set of sources,

$J \;=\; \{1, 2, ..., n\}$, index set of destinations,

a_i = minimum availability at the ith sources, $i \in I$,

A_i = maximum availability at the ith sources, $i \in I$,

b_j = requirement at the jth destination, $j \in J$,

$Z(X) \;=\; \sum\limits_{I} \sum\limits_{J} c_{i,j}\ x_{i,j}$, $c_{i,j}$ being the time transportation cost of transportation goods from ith source to the jth destination,

$T(X) \;=\; \mathrm{Max}_{(i,j) \in I \times J}\ [t_{i,j}\ x_{i,j}]$, $t_{i,j}$ being the time of transporting goods from ith source to the jth destination.

Here b_j is defined as $a_i \le b_j \le A_i$ and $c_{i,j}$ and $t_{i,j}$ are independent of the quantity transported.

To solve problem P(4.1), an equivalent standard cost minimization transportation problem (CMTP) is defined as :

P(4.2)

$$\text{Min} \sum_{i\in\hat{I}} \sum_{j\in\hat{J}} d_{i,j}\; x_{i,j}$$

subject to

$$\sum_{j\in\hat{J}} x_{i,j} \;=\; \hat{a}_i, \quad i\in\hat{I}$$

$$\sum_{i\in\hat{I}} x_{i,j} \;=\; \hat{b}_j, \quad j\in\hat{J}$$

$$x_{i,j} \geq 0,\; i\in\hat{I},\; j\in\hat{J}.$$

where, $\hat{I} = \{1, 2, ..., m, m+1, ..., 2m\}$

$\hat{J} = \{1, 2, ..., n, n+1\}$

$\hat{a}_i = a_i, \quad i\in I$

$\hat{a}_{m+i} = A_i - a_i, \quad i\in I$

$\hat{b}_j = b_j, \quad j\in J$

$\hat{b}_{n+1} = \sum_{i\in I} A_i \;-\; \sum_{j\in J} b_j$

$d_{i,j} = c_{i,j}, \quad i\in I,\; j\in J$

$d_{m+i,j} = c_{i,j}, \quad i\in I,\; j\in J$

$d_{i,n+1} = M, \quad i\in I$

$d_{m+i,n+1} = 0, \quad i\in I$

Definition 4.1 Efficient Point: An $\hat{X}\in S$ is called an efficient point if there does not exist $X\in S$ such that

$$(Z(X), T(X)) < (Z(\hat{X}), T(\hat{X})).$$

If $\hat{X}$ is an efficient point, then $(Z(\hat{X}), T(\hat{X}))$ is called an efficient pair.

Definition 4.2 Dominated Point: $\hat{X} \in S$ is said to be dominated by $X \in S$ if $(Z(X), T(X)) \leq (Z(\hat{X}), T(\hat{X}))$.

Definition 4.3 Feasible Pair: A pair (Z, T) is called a feasible pair if there exist $X \in S$ such that $Z(X) = Z$ and $T(X) = T$.

Definition 4.4 Z_T : Optimal objective function value in CMTP(T).

Cost minimizing transportation problem at time T, denoted by CMTP(T), is defined as:

P(4.3) $$\text{Minimize}_{X \in S} \sum_{\hat{I}} \sum_{\hat{J}} c'_{i,j}\, x_{i,j} = Z_T(X)$$

where $$\begin{aligned} c'_{i,j} &= c_{i,j}, \quad \text{if } t_{i,j} < T \\ &= M, \quad \text{if } t_{i,j} \geqq T \end{aligned}$$

M being an infinitely large positive real number.

Remark 4.1 If Z_{T_1} and Z_{T_2} are respectively the finite optimal values in CMTP(T_1) and CMTP(T_2) and $T_1 > T_2$, then $Z_{T_1} \leqq Z_{T_2}$, because optimal solution of CMTP(T_2) is a feasible solution of CMTP(T_1).

Lemma 4.1 If (Z, T) is a feasible pair yielded by an $X \in S$ and $Z_T = Z$, then (Z, T) is not an efficient pair.

Proof: Consider $(Z(X), T(X)) = (Z, T)$.

Let the optimal solution of problem P(4.3) be Y.

Therefore, $Z(Y) = Z_T$ and $T(Y) < T$.

If $Z(Y) = Z(X)$, then $(Z(Y), T(Y))$ dominates $(Z(X), T(X))$ and hence (Z, T) is not an efficient pair.

Theorem 4.1 If (Z, T) is an efficient pair, then $Z_T > Z$, Z_T being finite optimal value in CMTP(T).

Proof: Consider the CMTP P(4.3).

The optimal solution $\hat{X}$ of this problem is such that $Z(\hat{X}) = Z$ and $T(\hat{X}) = T$, otherwise (Z, T) would be dominated. Thus optimal value in P(4.3) is Z. The optimal solution Y of CMTP(T) is also a feasible solution of P(4.3).

Therefore, $Z(Y) \geqq Z$.

$Z(Y) \neq Z$ because then $(Z(Y), T(Y))$ dominates (Z, T), which contradicts the efficiency of (Z, T).

Therefore, $Z(Y) > Z$, i.e., $Z_T > Z$.

Theorem 4.2 Consider (Z, T) as a feasible pair. If $Z_T > Z$, then (Z, T) is an efficient pair (Z_T being finite).

Proof: Z_T is the optimal value of the objective function in CMTP(T).

Suppose any feasible solution $X \in S$ gives feasible pair (Z_1, T_1) such that $T_1 < T$

where

$Z(X) = Z_1,\ T(X) = T_1.$

As $T_1 < T$, then X will be a feasible solution of CMTP(T).

Therefore, $Z(X_1) \geqq Z_T$.

$Z_1 > Z$ (since $Z_T > Z$)

so $(Z_1, T_1) \nleq (Z, T)$.

That is, there does not exist $X \in S$ such that $(Z(X), T(X)) \leq (Z, T)$.

Hence (Z, T) is an efficient pair.

Theorem 4.3 (Z, T) is an efficient pair $Z_T = \lambda M\ +\ v$; $(\lambda > 0)$ and $v \in R$. Then

(i) there doesn't exist any efficient pair $(\hat{Z}, \hat{T})$ with $\hat{T} < T$, and

(ii) $T = Min_{X \in S}\ T(X)$.

Proof: (i) $Z_T = \lambda M\ +\ v$ which means CMTP(T) has no feasible solution $\hat{X}$ with $T(\hat{X}) < T$ and $Z(X)$ is finite. Hence there doesn't exist any efficient pair $(\hat{Z}, \hat{T})$ with $\hat{T}, T$.

(ii) As (Z, T) is a feasible pair, then there exists $X \in S$ such that $T(X) = T$.

From (i) we can say that there doesn't exist any $\hat{X} \in S$ such that $T(\hat{X}) < T$.

Therefore $T = Min_{X \in S}\ T(X)$.

4.3 Algorithm

1. Finding first efficient pair:

Find optimal solution X_0 of $\text{Min}_{X \in S} Z(X)$. Compute $Z(X_0) = Z_0$ and $T(X_0) = T_0$.

Solve CMTP(T_0), let its optimal solution be Y. $Z(Y) \geqq Z_0$.

If $Z(Y) > Z_0$, (Z_0, T_0) is the first efficient pair and X_0 is the first efficient point. [Theorem 4.2]

If $Z(Y) = Z_0$, (Z_0, T_0) is dominated by $(Z(Y), T(Y))$ (by Lemma 4.1)

Solve CMTP$(T(Y))$ and let it's optimal solution be $\hat{Y}$.

If $Z(\hat{Y}) = Z(Y)$, $(Z(\hat{Y}), T(\hat{Y}))$ dominates $(Z(Y), T(Y))$.

Solve CMTP$(T(\hat{Y}))$ and proceeding in same way find first efficient pair (Z_1, T_1) and the corresponding efficient point.

2. To find $(K+1)$th efficient pair $(K \geq 1)$ $(Z_i, T_i),\ i = 1, ..., K$ are K efficient pairs.

Let optimal solution of CMTP(T_K) be Y. As (Z_K, T_K) is the Kth efficient pair, $Z_{T_K} \equiv Z(Y) > Z_K$ and $T(Y) < T_K$.

Therefore, optimal solution of CMTP(T_K) yields a feasible point $Y \in S$ giving the pair $(Z(Y), T(Y))$.

Find optimal solution $\hat{Y}$ of CMTP$(T(Y))$. If $Z(\hat{Y}) > Z(Y))$, then $(Z(Y), T(Y))$ is the $(K+1)$th efficient pair and Y is the corresponding efficient point. [Theorem 4.2].

If $Z(\hat{Y}) = Z(Y)$, $(Z(Y), T(Y))$ is dominated by $(Z(\hat{Y}), T(\hat{Y}))$. To examine whether

$(Z(\hat{Y}), T(\hat{Y}))$ is $(K+1)$th efficient pair, solve CMTP$(T(\hat{Y}))$ to obtain its optimal solution $\bar{Y}$.

If $Z(\bar{Y}) > Z(\hat{Y})$, then $(Z(\hat{Y}), T(\hat{Y}))$ is designated as $(K+1)$th efficient pair and Y as $(K+1)$th efficient point, otherwise $(Z(\hat{Y}), T(\hat{Y}))$ gets dominated by $(Z(\bar{Y}), T(\bar{Y}))$.

Continuing in this process we obtain $(K+1)$th efficient pair.

4.4 Termination

Let $(Z_i, T_i),\ i = 1, ..., K$ be the K efficient pairs obtained up till now. Solve CMTP(T_K). If $Z_{T_K} = \lambda M + v,\ (\lambda > 0)$ and $v \in R$, then no more efficient pairs exist.

Theorem 4.4 Algorithm records all the efficient pairs. That is, no efficient pair is missed by the algorithm.

Proof: Let, if possible, (Z, T) be an efficient pair not recorded by the algorithm. That is, there exist two consecutive recorded efficient pairs say (Z_p, T_p) and (Z_{p+1}, T_{p+1}), such that $Z_p < Z < Z_{p+1}$ and $T_p > T > T_{p+1}$.

Let optimal solution of CMTP(T_p) be Y. As (Z_p, T_p) is efficient, $Z(Y) > Z_p$ [from Theorem 4.1].

Also, by definition of CMTP(T_p), $T(Y) < T_p$.

Let optimal solution of CMTP$(T(Y))$ be $\hat{Y}$.

(a) If $Z(\hat{Y}) > Z(Y)$, then $(Z(Y), T(Y))$ is recorded as $(p+1)th$ efficient pair. [from Theorem 4.2]

That is, $(Z(Y), T(Y)) = (Z_{p+1}, T_{p+1})$. The feasible solution $X \in S$ yielding efficient pair (Z, T) will also be a feasible solution of CMTP(T_p). As Y is optimal solution of CMTP(T_p), $\quad Z \ = \ Z(X) \geqq Z(Y)$.

As $T > T_{p+1},\ (Z, T) \geq (Z(Y), T(Y))$.

That is, $(Z, T) \geq (Z_{p+1}, T_{p+1})$, this means (Z, T) is not an efficient pair, which is a contradiction.

(b) When $Z(\hat{Y}) = Z(Y)$, then $(Z(\hat{Y}), T(\hat{Y}))$ dominates $(Z(Y), T(Y))$. As (Z, T) is an efficient pair, $(Z(Y), T(Y))$ is different from (Z, T). Clearly, $T_p > T(Y)$.

If possible, let $T(Y) \leqq T,\ Y$ being optimal solution of CMTP(T_p) and X is its feasible solution.

We have, $Z(Y) \leqq Z(X) \ = \ Z$.

Therefore, $(Z(Y), T(Y))$ dominates (Z, T), which is not true.

Hence, $T(Y) > T$. Therefore, $T_p > T(Y) > T > T_{p+1}$.

Let $\bar{Y}$ be an optimal solution of CMTP($T(\hat{Y})$).

If $Z(\bar{Y}) > Z(\hat{Y})$, then $(Z(\hat{Y}), T(\hat{Y}))$ would be recorded as $(p+1)$th efficient pair.

[by Theorem4.2]

That is, $(Z(\hat{Y}), T(\hat{Y})) \ = \ (Z_{p+1}, T_{p+1})$.

As $T < T(Y)$, X will be a feasible solution of CMTP$(T(Y))$ whose optimal solution is taken as $\hat{Y}$.

So, $Z(X) \geqq Z(\hat{Y}) \ = \ Z_{p+1}$.

Therefore, $(Z(X), T(X)) \geqq (Z_{p+1}, T_{p+1})$ [since $T(X) = T > T_{p+1}$].

This means, (Z, T) is again dominated, which is a contradiction. Hence, no efficient point is omitted by the algorithm.

4.5 Example

$\text{Min}_{X\in S}\ (Z(X), T(X))$

where the set S is :

$5 \leq \sum_{j=1}^{4}\ x_{1j} \leq 9$

$3 \leq \sum_{j=1}^{4}\ x_{2j} \leq 10$

$6 \leq \sum_{j=1}^{4}\ x_{3j} \leq 15$

$\sum_{i=1}^{3}\ x_{i1} \leq 8$

$\sum_{i=1}^{3}\ x_{i2} \leq 6$

$\sum_{i=1}^{3}\ x_{i3} \leq 7$

$\sum_{i=1}^{3}\ x_{i4} \leq 9$

$Z(X) = \sum_{I} \sum_{J} c_{i,j}\ x_{i,j},\ \ c_{i,j}$ being the time transportation cost of transportation goods from ith source to the jth destination,

$T(X) = \text{Max}_{(i,j)\in I\times J}\ [t_{i,j}\ x_{i,j}],\ t_{i,j}$ being the time of transporting goods from ith source to the jth destination,

The tabular form of the problem is :

	D_1	D_2	D_3	D_4
S_1	10 5	9 6	11 3	7 2
S_2	11 2	10 3	13 5	14 6
S_3	8 4	6 5	9 8	10 3

In the table allocations in the left corner represents cost and the allocations at the bottom represents time.

The tabular form of the above problem in the form of CMTP P(4.2) is :

	D_1	D_2	D_3	D_4	D_5
S_1	10	9	11	7	M
S_2	11	10	13	14	M
S_3	8	6	9	10	M
S_4	10	9	11	7	0
S_5	11	10	13	14	0
S_6	8	6	9	10	0

The optimal table of the given problem in the form of $P(4.2)$ is :

	D_1	D_2	D_3	D_4	D_5
S_1	10 5	9 6	11 3	7 **(5)** 2	M
S_2	11 **(3)** 2	10 3	13 5	14 6	M
S_3	8 **(2)** 4	6 **(4)** 5	9 8	10 3	M
S_4	10 **(0)** 5	9 6	11 3	7 **(4)** 2	0
S_5	11 **(3)** 2	10 3	13 5	14 6	0 **(4)**
S_6	8 4	6 **(2)** 5	9 **(7)** 8	10 3	0

In this table allocations are represented in **()**. From the optimal table we get cost $Z_0 = 250$ and time $T_0 = \text{Max}\{2, 4, 5, 8\} = 8$.

We get $(Z_0, T_0) = (250, 8)$.

As we get time $T_0 = 8$ so we block the routes having time $T_0 = 8$ and optimal table of CMTP(8) is:

	D_1	D_2	D_3	D_4	D_5
S_1	10	9	11	7 **(5)**	M
S_2	11	10	13 **(3)**	14	M
S_3	8	6 **(6)**	M	10 **(0)**	M
S_4	10	9	11	7 **(4)**	0
S_5	11	10	13 **(4)**	14	0 **(3)**
S_6	8 **(8)**	6 **(0)**	M	10	0 **(1)**

Now we get cost $Z_1 = 254$ and time $T_1 = \text{Max}\{2, 5, 3, 4\} = 5$.

Since $Z_1 > Z_0$, so $(Z_1, T_1) = (254, 5)$ is the first efficient pair.

Now by blocking the routes having time $T_1 = 5$. The optimal table of CMTP(5) is :

	D_1	D_2	D_3	D_4	D_5
S_1	M	M	11 **(5)**	7	M
S_2	11	10	M	M	M
S_3	8	M	M	10 **(6)**	M
S_4	M	M	11 **(2)**	7 **(2)**	0
S_5	11	10 **(3)**	M	M	0 **(4)**
S_6	8 **(8)**	M	M	10 **(1)**	0 **(0)**

From the optimal table of CMTP(5) we get cost $Z_2 = 285$ and time $T_2 = \text{Max}\{3, 2, 4\} = 4$.

Since $Z_2 > Z_1$, so $(Z_2, T_2) = (285, 4)$ is the second efficient pair.

Now by blocking the routes having time $T_2 = 4$ the problem CMTP(4) yields no efficient pair as from the optimal table of CMTP(4) we get cost $Z_{T_2} = \lambda M + v$.[from Theorem 4.3]

Therefore, there does not exist any efficient pair with time $T < T_2 = 4$.

Thus, our algorithm terminates here with time-cost trade-off pairs $(250, 8)$, $(254, 5)$, $(285, 4)$.

CONCLUSION

A modified time-cost trade-off relations for general transportation problem is presented. In this problem it is considered that upper and lower bounds are specified on the source availabilities. Three time-cost trade-off pairs are obtained for the example considered here.

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