APPLICATIONS of LIE CLASSICAL METHOD to SOME NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS

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under

the guidance of

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to the

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CERTIFICATE

*I hereby certify that the work which is being presented in the thesis entitled "**Applications of Lie Classical Method to Some Nonlinear Partial Differential equations**" which is being submitted for the award of degree of Master of Science, School of Mathematics and Computer Applications, Thapar University, Patiala is an authentic record of my own work carried out under the supervision of Dr.Rajesh Kumar Gupta.*

The matter presented in this thesis has not been submitted for the award of any other degree of this or any other university.

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This is to certify that the above statement made by the candidate is correct and true to the best of my knowledge.

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ABSTRACT

The objective of the thesis entitled, "Applications of *Lie Classical Method Some Nonlinear Partial Differential Equations*", is to obtain the Lie symmetries and the exact solutions of nonlinear partial differential equations (PDEs) or their systems, which represent some of the important physical phenomena. The nonlinear phenomena are encountered in a variety of situations in physics as well as in other natural and applied sciences. Most of these phenomena are governed by nonlinear partial differential equations. The study of these systems of differential equations is often regarded as a difficult and confusing endeavor due various limitations posed by the intrinsic nonlinearity.

The exact solutions of these differential equations which not only play a central role in the theories of these physical phenomena but have also become more and more sought after during the last few years. The mathematical techniques which generate a wide range of solutions and applicable to all type of nonlinear differential equations are few. The group theoretic techniques can be categorized in this class and generally a variety of exact solutions may be furnished in a systematic manner.

The thesis comprises six chapters. The brief outline of the research work presented chapter wise in this thesis is as follows:

In chapter 1, we have described the nonlinear partial differential equations and exact solutions. It contains the various definition of Lie Groups. It also includes the preliminary materials and literature review.

Chapter 2 is an brief description of Lie classical method based on the application to some nonlinear partial differential equations.

Chapter 3 presents symmetry reductions and exact solutions of a Gardner equation. This chapter deals with the classical Lie method to obtain symmetries and reductions. Reduced systems are studied further to generate some exact solutions.

In chapter 4, a form of Fishers equation is studied. For each generator in the optimal system, the Fishers equation is reduced to a system of ordinary differential equations, which is further studied with the aim of deriving certain exact solutions.

In chapter 5, we investigate symmetries and reductions of the Drinfel'd Sokolov Wilson Equation. Corresponding to each basic vector field, the reductions of the system to ordinary differential equations are obtained. These reduced systems are further studied for exact solutions.

In chapter 6, we study the classical Lie symmetries of the Benjamin-Bona-Mahony (BBM) equation which is obtained through the Lie group method of the infinitesimal transformations. Secondly usuing the classical symmetries of the equation, similarity reduction are obtained.

CHAPTER-1

INTRODUCTION

Nonlinear partial differential equations are one of the key area of the interaction of mathematics and other sciences as they govern most physical phenomena. With the development in physics and other sciences nonlinear phenomena have really caught up with us. Virtually all fundamental equations of physics are nonlinear and are difficult to solve. Various standard strategies are adopted to get approximate solutions of nonlinear partial differential equations. But these solutions do not provide much information about the system. Strategies adopted to drive exact solutions to nonlinear equations are being avoided due to complicated and cumbersome calculations. But the strong desire of exact and more general solutions to nonlinear partial differential equations governing nonlinear phenomenon in technological enhancement has made tremendous growth in research interest.

An ordinary differential equation is a relation that contains the functions of only one independent variable, and one or more of its derivatives with respect to that variable. The order of differential equations is given by the maximum number of times the supposed unknown function in it has been differentiated. Ordinary differential equation arises in many differential contexts including geometry etc. Many famous mathematicians have studied differential equations and contributed to the field, including Newton, Clairaut, D Alembert and Euler. Much study has been devoted to the solution of ordinary differential equations. In the case where the equation is linear, it can be solved by analytical methods. Unfortunately, most of the interesting differential equations are non linear and, with a few exceptions, cannot be solved exactly. Approximations solutions are arrived at using computer approximations.

A partial differential equations is a mathematical equation that involves two or more independent variables, an unknown function (dependent on those variables) and partial derivatives of the unknown function with respect to independent variables. The order of a partial differential equation is the order of the highest derivative involved. The study of partial differential equations is a fundamental subject area of mathematics which links important strands of pure mathematics to applied and computational mathematics. Partial differential equations and their solutions exhibit rich and complex structures. Unfortunately, closed analytical expressions for their solutions can be found only in very special circumstances and these are mostly of limited theoretical and practical interest. Thus scientists and mathematicians have naturally led to seeking techniques for the approximation of solutions. A solution (or a particular solution) to a partial differential equation is a function that solves the equation or, in other words, turns it into an identity when substituted into the equation. A solution is called general if it contains all particular solutions of the equation concerned.

The term exact solution is often used for second and higher order nonlinear partial differential equations to denote a particular solution. The exact solutions are also helpful in designing and testing of numerical algorithms. The proposed work will be devoted to obtaining the exact solutions of nonlinear partial differential equations or their systems, which represent some of the important physical systems.

Most of the problems posed by nature, and which are of interest to physicists and mathematicians, are inherently nonlinear and are usually governed by a single or a system of differential equations. Physical examples of linear systems are relatively rare. Nonlinear equations are difficult to solve and the linear approximations used to describe them are often a tacit admission that the underlying equations can not be solved. In fact, the study of nonlinear systems of differential equations is regarded as a difficult and confusing endeavor.

Whilst linear analysis as a mathematical discipline began in the nineteenth century and in the intervening years achieved many spectacular successes throughout sciences, on the other hand the nonlinear equations by virtue of their inherent complexity remained much harder to understand because of their lack of simple superposed solutions. Consequently, numerical methods are being applied to obtain approximate solutions of these equations. But these solutions, however, fail to provide much information about the system.

The study of nonlinear differential equations has not only provided information about the phenomenon but has, in fact, helped in making more precise some of the concepts and theories developed in the last century mathematics. The standard strategies adopted to get the solutions of nonlinear partial differential equations (PDEs) to date are following:

- Linearize the given set of nonlinear equations by invoking certain physical assumption.
- Numerical integration of the equations under appropriate boundary conditions.
- To derive exact solutions of nonlinear equation.

In fact, as the applications of the first two approaches are concerned, a great deal has been contributed. The third approach is being avoided due to cumbersome and complicated calculations. But the strong desire of exact and more general solutions to nonlinear PDEs governing nonlinear phenomenon in technological enhancement and for research purpose made tremendous growth in research interest of last approach. There is much current interest in obtaining exact solutions of nonlinear partial differential equations; these solutions provide information about nonlinear phenomena and various aspects of the physical phenomena. These solutions, often with several important physical parameters, prove useful to discuss and examine the sensitivity of physical phenomena they describe.

Exact solutions for nonlinear equations are rare, and the methods, which can generate families of them, are not only increasingly popular, but increasingly sought. So far, a number of methods have been proposed to construct the exact solutions; the most effective methods include the classical Lie approach [1, 6, 19, 23], the nonclassical approach [2, 13, 27], Steinberg’s symmetry reduction method [7, 8], the direct method [17, 29], the hyperbolic functions expansion methods [17], elliptic functions expansion methods [18,19].

The main tool in our study will be Lie classical method for reduction of partial differential equations (PDEs) and other method including hyperbolic secant-tangent functions expansion method for some exact solutions of ordinary differential equations

(ODEs). Software like Maple can be utilized during the research to drive and to test the authenticity of solutions and for other related purposes.

1.1 Literature Review

The theory of Lie groups is an area of mathematics in which we can see a harmonious interaction between the methods of classical analysis and modern algebra. The theory of Lie groups was developed by the Norwegian mathematician Sophus Lie in the late nineteenth century in connection with his work on systems of differential equations. He developed the theory of "finite and continuous groups". Lie devoted his mathematical career to the development and application of his monumental theory of continuous groups. These groups, now universally known as Lie groups, have a profound impact on all areas of mathematics, both pure and applied, as well as physics, engineering and other mathematically based sciences. Lie groups arise in the study of solutions of differential equations just as a finite groups arise in the study of algebric equatons. The first systematic investigation of the problem of group classification was done by L . V. Ovsiannikov in 1959 for nonlinear heat equation. One of the possible extensions of the Lie theory of invariant solutions was first considered by Bluman and Cole (1969) [5].

Lie’s work systematically related a miscellany of topics in ordinary differential equations (ODEs) including seperable equation, homogeneous equation, and the use of laplace equation. Lie (1881) also indicated that for linear partial differential equations (PDEs), invariance under a Lie group leads directly to superpositions of solutions in terms of transforms.

Lie groups, and hence their infinitesimal generators, can be naturally extended to act on the space of independent variables dependent variable, and derivative of dependent variables up to any finite order. As a consequence, the seemingly intractable nonlinear conditions for group invariance of a given system of differential equations reduce to linear homogeneous equations determining the infinitesimal generators of the group. Since these determining equations form an overdetermined system of linear homogeneous partial differential equations (PDEs) one can usually determine the infinitesimal generators in explicit form. For a given system of differential equations, the setting up of the determining equations is entirely routine. Symbolic manipulation programs exist to set up the determining equations and in some cases explicitly solve them.

1.2 Preliminary Material

We are presenting some of the basic definition essential for techniques to derive the exact solutions of the non linear system.

Definition 1.2.1. A *group* G is a set of elements with a law of composition ϕ between elements satisfying the following axioms:

i) Closure property. For any elements a and b of G, $\phi(a,b)$ is an element of G.

ii) Associative property. For any elements a,b,c of G:

$$\phi(a,\phi(b,c)) = \phi(\phi(a,b),c).$$

iii) Identity element. There exists a unique identity element e of G such that for any element a of G:

$$\phi(a,e) = \phi(e,a) = a .$$

iv) Inverse element. For any element a of G, there exists a unique inverse element a^{-1} in G such that

$$\phi(a,a^{-1}) = \phi(a^{-1},a) = e .$$

Definition 1.2.2 Let $x = (x_1, x_2, x_3, \ldots.., x_n)$ lie in a region $D \subset R^n$ the set of transformations $x^* = X(x;e)$, defined for each x in D and a parameter e in set $S \subset R$ with, $f(e,d)$ defining a law of composition of parameter e and δ in S forms a One-parameter group of transformations on D if the following hold:

(i) For each e in S the transformations are one to one onto D.

(ii) S with the law of composition f forms a group G.

(iii) For each x in D, $x^* = x$ when $e = e_0$ corresponds to identity, that is

$$X(x;e_0) = x .$$

(iv) If $x^* = X(x;e)$, $x^{**} = X(x^*;d)$ then

$$x^{**} = X(x;f(e,d)) .$$

If it satisfies these 4 properties then it is a Group of transformations.

Definition 1.2.3. A one-parameter group of transformations defines a one-parameter lie group of transformations if, in addition to satisfying the property (i) to (iv) of group of transformations, the following hold:

(v) ε is a continuous parameter.

(vi) X is infinitely differentiable with respect to x in D and an analytic function of ε in S.

(vii) $\phi(\varepsilon,\delta)$ is an analytic function of ε and $\delta, \varepsilon \in S,\ \delta \in S$.

Example:

Consider $x^* = x + \varepsilon$

$$y^* = y, \qquad \varepsilon \in R$$

and $\phi(\varepsilon,\delta) = \varepsilon + \delta$.

this forms a one-parameter lie group of transformations.

Definition 1.2.4. Consider a one-parameter lie group of transformations,

$$x^* = X(x;\varepsilon) \qquad (1.2\,a)$$

With the identity $\varepsilon = 0$ and law of composition ϕ .

Expand (1.2 a) about $\varepsilon = 0$

Then we get $x^* = x + \varepsilon\left(\frac{\partial X(x;\varepsilon)}{\partial \varepsilon}\Big|_{\varepsilon=0}\right) + \frac{1}{2}\varepsilon^2\left(\frac{\partial^2 X(x;\varepsilon)}{\partial \varepsilon^2}\Big|_{\varepsilon=0}\right) + \ldots$

$$= x + \varepsilon\left(\frac{\partial X(x;\varepsilon)}{\partial \varepsilon}\Big|_{\varepsilon=0}\right) + o(\varepsilon^2).$$

let $\xi(x) = \left(\frac{\partial X(x;\varepsilon)}{\partial \varepsilon}\Big|_{\varepsilon=0}\right)$.

the transformation $x + \varepsilon\xi(x)$ is called the infinitesimal group of transformation of (1.2 a)

The components of $\xi(x)$ is called infinitesimals of (1.2 a)

Definition 1.2.5. The infinitesimal generator of the one- parameter lie group of transformations (1.2 a) is the operator

$$U = U(x) = \xi(x).\nabla = \sum_{i=1}^{n} \xi_i(x)\frac{\partial}{\partial x_i},$$

where ∇ is the gradient operator

$$\nabla = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots\ldots, \frac{\partial}{\partial x_n}\right).$$

For any differentiable function

$F(x) = F(x_1, x_2, ..., x_n)$, one has

$$UF(x) = \xi(x)\nabla F(x) = \sum_{i=1}^{n} \xi_i(x)\frac{\partial F(x)}{\partial x_i} \quad .$$

Definition 1.2.6. An infinitely differentiable function $F(x)$ is an invariant function of lie group of transformation (1.2 a) if and only if, for any group of transformation (1.2 a)

$$F(x^*) \equiv F(x) \quad .$$

If $F(x)$ is invariant function of (1), then $F(x)$ is called an invariant of (1.2 a) $F(x)$ and is said to be invariant under (1.2 a).

Theorem 1.2.1 $F(x)$ is invariant under a lie group of transformations (1.2 a) if and only

$$UF(x) \equiv 0 \quad .$$

1.2.1. Point Transformations and Prolongations

In later chapters, we will be concerned with the determination of one-parameter Lie groups of transformations admitted by a given system S of differential equations. A one-parameter (ε) Lie group of point transformations is a group of transformations of the form

$$x^* = X(x, u; \varepsilon) \qquad (1.2.1a)$$

$$u^* = U(x, u; \varepsilon), \qquad (1.2.2b)$$

acting on the space of $n+m$ variables

$$x = (x_1, x_2, \ldots, x_n)$$

$$u = (u^1, u^2, \ldots, u^m),$$

where x represents n independent variables and u denotes m dependent variables.

A Lie group of point transformations (1.2.1) admitted by S maps any solution $u = \Theta(x)$ of S into a one-parameter family of solutions $u = \phi(x; \varepsilon)$ of S. Equivalently, a Lie group of point transformations (1.2.1) leaves S invariant in the sense that the form of S is unchanged in terms of the transformed variables (1.2.1) for any solution $u = \Theta(x)$ of S.

Let ∂u denotes the set of nm coordinates corresponding to all first order partial derivatives of u with respect to x:

$$\partial u = \left(\frac{\partial u^1}{\partial x_1}, \frac{\partial u^1}{\partial x_2}, \dots, \frac{\partial u^1}{\partial x_n}, \frac{\partial u^2}{\partial x_1}, \frac{\partial u^2}{\partial x_2}, \cdots \frac{\partial u^2}{\partial x_n}, \cdots \frac{\partial u^m}{\partial x_1}, \frac{\partial u^m}{\partial x_2}, \cdots \frac{\partial u^m}{\partial x_n} \right). \tag{1.2.2}$$

In general, for $k \geq 1$, let $\partial^k u$ denote the set of coordinates

$$u^\mu_{i_1 i_2 \cdots i_k} = \frac{\partial^k u^\mu}{\partial x_{i_1} \partial x_{i_2} \cdots \partial x_{i_k}} ,$$

with $\mu = 1, 2, \dots m$ and $i_j = 1, 2, \dots, n,\ j = 1, 2, \dots, k$ corresponding to all *k*th-order partial derivatives of u with respect to x.

It turns out that the natural transformation of partial derivatives of the dependent variables leads successively to extensions (prolongations) of a one-parameter Lie group of transformations (1.2.1) acting on (x, u)-space to one-parameter Lie groups of transformations acting on $(x, u, \partial u)$-space, $(x, u, \partial u, \partial^2 u)$-space, **...** $(x, u, \partial u, \partial^2 u, \dots \partial^k u)$-space for any $k > 2$. [For a given system S of differential equations, k would be the order of the highest order derivative appearing in S]. Then the infinitesimal transformations of (1.2.1) is naturally extended successively to infinitesimal transformations acting on $(x, u, \partial u, \partial^2 u, \dots, \partial^l u)$-space, $l = 1, 2, \dots, k.$

1.2.2. Multiparameter Lie Groups of Transformations

Consider an r*-parameter Lie group of point transformations*

$$\bar{x}^* = \bar{X}(\bar{x}; \bar{\varepsilon}), \tag{1.2.3}$$

with $\bar{x} = (x_1, x_2, \dots, x_n)$ and parameters $\bar{\varepsilon} = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_r)$. Let the law of composition of parameters be denoted by

$$\varphi(\bar{\varepsilon}, \bar{\delta}) = (\phi_1(\bar{\varepsilon}, \bar{\delta}), \phi_2(\bar{\varepsilon}, \bar{\delta}), \dots, \phi_r(\bar{\varepsilon}, \bar{\delta})),$$

with $\bar{\delta} = (\delta_1, \delta_2, \dots, \delta_r)$ where $\varphi(\bar{\varepsilon}, \bar{\delta})$ satisfies the group axioms with $\bar{\varepsilon} = \bar{0}$ corresponding to the identity $\varepsilon_1 = \varepsilon_2 = \dots = \varepsilon_r = 0$, and $\varphi(\bar{\varepsilon}, \bar{\delta})$ is assumed to be analytic in its domain of definition.

Let the infinitesimal matrix $\Xi(\bar{x})$ be the $r \times n$ matrix with entries

$$\xi_{\alpha j}(\bar{x}) = \left. \frac{\partial \bar{x}^*{}_j}{\partial \varepsilon_\alpha} \right|_{\varepsilon = 0} = \left. \frac{\partial X_j(\bar{x}; \bar{\varepsilon})}{\partial \varepsilon_\alpha} \right|_{\varepsilon = 0}, \quad \alpha = 1, 2, \dots, r, \quad j = 1, 2, \dots, n. \tag{1.2.4}$$

Definition 1.2.7. The *infinitesimal generator* X_α, corresponding to the parameter ε_α of the r-parameter Lie group of transformations (2.1.5), is given by

$$\mathrm{X}_\alpha = \sum_{j=1}^{n} \xi_{\alpha j}(\bar{x})\frac{\partial}{\partial x_j}, \quad \alpha = 1, 2, \dots, r. \tag{1.2.5}$$

Definition 1.2.8. For an r-parameter Lie group of transformations (1.2.3) with infinitesimal generators $\mathrm{X}_\alpha, \alpha = 1, 2, \dots, r$ defined by (1.2.4) and (1.2.5), the *commutator* (Lie Bracket) of X_α and X_β is a first order operator defined by

$$[\mathrm{X}_\alpha, \mathrm{X}_\beta] = \mathrm{X}_\alpha \mathrm{X}_\beta - \mathrm{X}_\beta \mathrm{X}_\alpha = \sum_{i,j=1}^{n}\left[\left(\xi_{\alpha i}(\bar{x})\frac{\partial}{\partial x_i}\right)\left(\xi_{\beta j}(\bar{x})\frac{\partial}{\partial x_j}\right) - \left(\xi_{\beta i}(\bar{x})\frac{\partial}{\partial x_i}\right)\left(\xi_{\alpha j}(\bar{x})\frac{\partial}{\partial x_j}\right)\right]$$

$$= \sum_{j=1}^{n} \eta_j(\bar{x})\frac{\partial}{\partial x_j}, \tag{1.2.5a}$$

where

$$\eta_j(\bar{x}) = \sum_{i=1}^{n}\left(\xi_{\alpha i}(\bar{x})\frac{\partial \xi_{\beta j}(\bar{x})}{\partial x_i} - \xi_{\beta i}(\bar{x})\frac{\partial \xi_{\alpha j}(\bar{x})}{\partial x_i}\right). \tag{1.2.5b}$$

It immediately follows that

$$[\mathrm{X}_\alpha, \mathrm{X}_\beta] = -[\mathrm{X}_\beta, \mathrm{X}_\alpha]. \tag{1.2.6}$$

Theorem 1.2.2 *The commutator of any two infinitesimal generators of an r-parameter Lie group of transformations is also an infinitesimal generator. In particular,*

$$[\mathrm{X}_\alpha, \mathrm{X}_\beta] = \sum_{\gamma=1}^{r} C^{\gamma}_{\alpha\beta}\mathrm{X}_\gamma, \tag{1.2.7}$$

where the coefficients $C^{\gamma}_{\alpha\beta}$ are constants called structure constants, $\alpha, \beta, \gamma = 1, 2, \dots, r$.

Definition 1.2.9. Equations (1.2.7) are called the *commutation relations* of the r-parameter Lie group of transformations (1.2.3) with the infinitesimal generators (1.2.5).

For any three infinitesimal generators X_α, X_β and X_γ, by direct computation one can show that Jacobi's identity holds:

$$[\mathrm{X}_\alpha,[\mathrm{X}_\beta, \mathrm{X}_\gamma]] + [\mathrm{X}_\beta,[\mathrm{X}_\gamma, \mathrm{X}_\alpha]] + [\mathrm{X}_\gamma,[\mathrm{X}_\alpha, \mathrm{X}_\beta]] = 0. \tag{1.2.8}$$

Definition 1.2.10. A *Lie algebra* $\mathscr{L}$ is a vector space over $\Re$ or C with a bilinear bracket operation (the commutator) satisfying the properties (1.2.6), (1.2.8) and, most important, (1.2.7). In particular, the set of infinitesimal generators $\{\mathrm{X}_\alpha\}, \alpha = 1, 2, \ldots, r,$ of an r-parameter Lie group of transformations (1.2.3) forms an r-dimensional Lie algebra over $\Re$.

Proposition. *Let* G *be a Lie group with Lie algebra* $\mathscr{L}$. *For each vector* $v \in \mathscr{L}$, *the adjoint vector ad* v *at* $w \in \mathscr{L}$ *is*

$$ad\ v\big|_w = [w, v] = -[v, w].$$

(For proof and other related details refer to Olver [25]).

The adjoint representation $\mathrm{Ad}\,G$ of the underlying Lie group can be reconstructed either by integrating the system of linear ordinary differential equations

$$\frac{dw}{d\varepsilon} = \mathrm{ad}\ v\big|_w, \quad w(0) = w_0,$$

with solution

$$w(\varepsilon) = Ad(\exp(\varepsilon v))w_0,$$

or, more simply, by summing the Lie series

$$Ad(\exp(\varepsilon v))w_0 = \sum_{n=0}^{\infty} \frac{\varepsilon^n}{n!}\ (\mathrm{ad}\ v)^n (w_0)$$

$$= w_0 - \varepsilon[v, w_0] + \frac{\varepsilon^2}{2}[v, [v, w_0]] - \cdots. \tag{1.2.9}$$

Definition 1.2.11. In Optimal System of Lie Symmetries we define a relation between two invariant solutions to hold true if the first one can be mapped to the other by applying the transformation group generated by a linear combination of the operator .Since these mappings are reflexive, symmetric, transitive, the relation is an equivalence relation which induces a natural partition on the set of all group invariant solutions into equivalence classes. We need only present one solution from each equivalence classes, as the rest may be found by applying appropriate group symmetries; a complete set of such solution is referred to as an "optimal system" of group invariant solution.

The problem of deriving an optimal system of group invariant solutions is equivalent to an optimal system of lie symmetries (or subalgebra is spanned by these operators). The method used here is given by Olver [30] which basically consists of taking the linear combinations of the generator and reducing them to their simplest equivalent form by applying a carefully chosen adjoint transformations.

$$Adj\left[exp(\varepsilon v_i)\right]v_j = v_j - \varepsilon\left[v_i, v_j\right] + \frac{\varepsilon^2}{2}\left[v_i, \left[v_i, v_j\right]\right] + \ldots,$$

where $[v_i, v_j]$ is the usual commutator, given

$$\left[v_i, v_j\right] = v_i v_j - v_j v_i .$$

1.2.3. Extended Infinitesimal Transformations

The situation of *m* dependent variables $u = (u^1, u^2, \ldots, u^m)$ and *n* independent variables $x = (x_1, x_2, \ldots, x_n),\ \ u = u(x),$ with $m \geq 2$, arises in studying systems of differential equations. This leads to consideration of extended transformations from (x,u)-space to $(x, u, \partial u, \partial^2 u, \ldots, \partial^k u)$-space where $\partial^k u$ denotes the components of all *k*th-order partial derivatives of u with respect to x. Consider the *k*th-extended transformation over the $(x, u, \partial u, \partial^2 u, \ldots, \partial^k u)$-space

$$x_i^* = X_i(x,u;\varepsilon) = x_i + \varepsilon \xi_i(x,u) + O(\varepsilon^2) \tag{1.2.10a}$$

$$(u^\mu)^* = U^\mu(x,u;\varepsilon) = u^\mu + \varepsilon \eta^\mu(x,u) + O(\varepsilon^2) \tag{1.2.10b}$$

$$(u_i^\mu)^* = U_i^\mu(x,u,\partial u;\varepsilon) = u_i^\mu + \varepsilon \eta_i^{(1)\mu}(x,u,\partial u) + O(\varepsilon^2) \tag{1.2.10c}$$

$$\vdots$$

$$(u_{i_1 i_2 \ldots i_k}^\mu)^* = U_{i_1 i_2 \ldots i_k}^\mu(x,u,\partial u,\ldots \partial^k u;\varepsilon) = u_{i_1 i_2 \ldots i_k}^\mu + \varepsilon \eta_{i_1 i_2 \ldots i_k}^{(k)\mu}(x,u,\partial u,\ldots,\partial^k u) + O(\varepsilon^2), \tag{1.2.10d}$$

with the extended infinitesimals $\eta_{i_1 i_2 \ldots i_k}^{(k)\mu}$ are given by

$$\eta_i^{(1)\mu} = D_i \eta^\mu - (D_i \xi_j) u_j^\mu , \tag{1.2.11}$$

and

$$\eta_{i_1 i_2 \ldots i_k}^{(k)\mu} = D_{i_k} \eta_{i_1 i_2 \ldots i_{k-1}}^{(k-1)\mu} - (D_{i_k} \xi_j) u_{i_1 i_2 \ldots i_{k-1}}^\mu , \tag{1.2.12}$$

$i_l = 1, 2, \ldots, n$ for $l = 1, 2, \ldots, k$ with $k \geq 2$. D_i is total derivative operator and given by

$$D_i = \frac{\partial}{\partial x_i} + u_i^\mu \frac{\partial}{\partial u^\mu} + u_{ij}^\mu \frac{\partial}{\partial u_j^\mu} + \ldots + u_{i i_1 i_2 \ldots i_n}^\mu \frac{\partial}{\partial u_{i_1 i_2 \ldots i_n}^\mu} + \ldots ,$$

with summation over a repeated index. Here, the *k*th-extended infinitesimal generator is given by

$$\mathrm{X}^{(k)} = \xi_i(x,u)\frac{\partial}{\partial x_i} + \eta^{\mu}(x,u)\frac{\partial}{\partial u^{\mu}} + \eta_i^{(1)\mu}(x,u,\partial u)\frac{\partial}{\partial u_i^{\mu}} + \ldots$$
$$+\eta_{i_1 i_2 \ldots i_k}^{(k)\mu}(x,u,\partial u,\ldots,\partial^k u)\frac{\partial}{\partial u_{i_1 i_2 \ldots i_k}^{\mu}}, \qquad k \ge 1.$$

1.2.4. Invariance for a System of PDEs

Consider a system of N PDEs with m dependent variables $u = (u^1, u^2, \ldots, u^m)$ and n independent variables $x = (x_1, x_2, \ldots, x_n)$, given by

$$F^{\mu}(x,u,\partial u,\partial^2 u,\ldots,\partial^k u) = 0, \ \mu = 1,2,\ldots,N. \tag{1.2.13}$$

Definition 1.2.12. The one-parameter Lie group of point transformations

$$x^* = X(x,u;\varepsilon) \tag{1.2.14a}$$

$$u^* = U(x,u;\varepsilon), \tag{1.2.14b}$$

leaves invariant the system of PDEs (1.2.4), i.e., is a point symmetry admitted by (1.2.13), if and only if its *k*th extension, defined by (1.2.10), leaves invariant the *N* surfaces in $(x,u,\partial u,\partial^2 u,\ldots\ldots\partial^k u)$ -space, defined by (1.2.10).

Theorem 1.2.3. (Infinitesimal Criterion for the Invariance of a System of PDEs). *Let*

$$\mathrm{X} = \xi_i(x,u)\frac{\partial}{\partial x_i} + \eta^{\nu}(x,u)\frac{\partial}{\partial u^{\nu}} \tag{1.2.15}$$

be the infinitesimal generator of the Lie group of point transformations (1.2.14). Let

$$\begin{aligned}\mathrm{X}^{(k)} = \xi_i(x,u)\frac{\partial}{\partial x_i} + \eta^{\nu}(x,u)\frac{\partial}{\partial u^{\nu}} + \eta_i^{(1)\nu}(x,u,\partial u)\frac{\partial}{\partial u_i^{\nu}} + \ldots \\ +\eta_{i_1 i_2 \ldots i_k}^{(k)\nu}(x,u,\partial u,\ldots,\partial^k u)\frac{\partial}{\partial u_{i_1 i_2 \ldots i_k}^{\nu}}\end{aligned} \tag{1.2.16}$$

be the kth-extended infinitesimal generator of (1.2.15), where $\eta_i^{(1)\nu}$ *is given by (1.2.11) and* $\eta_{i_1 i_2 \ldots i_j}^{(j)\mu}$ *by (1.2.11),* $\nu = 1,2,\ldots,m$ *and* $i_j = 1,2,\ldots,n,$ *for* $j = 1,2,\ldots,k,$ *in terms of* $\xi(x,u) = (\xi_1(x,u),\xi_2(x,u),\ldots,\xi_n(x,u)),$ $\eta(x,u) = (\eta^1(x,u),\eta^2(x,u),\ldots,\eta^m(x,u)).$ *Then the one-parameter Lie group of point transformations (2.1.8) is admitted by the system of PDEs (1.2.13) if and only if*

$$\mathrm{X}^{(k)}F^{\sigma}(x,u,\partial u,\partial^2 u,\ldots,\partial^k u)=0, \qquad (1.2.17)$$

when u *satisfies (1.2.13) for each* $\sigma = 1,2,\ldots N$.

1.2.4-1. Determining Equations

Consider a system of PDEs (1.2.13) with each of its PDEs given in a solved form

$$u^{\nu_\mu}_{i_1 i_2\ldots i_{k\mu}} = f^{\mu}(x,u,\partial u,\partial^2 u,\ldots,\partial^k u)=0. \qquad (1.2.18)$$

In terms of some specific l_μth-order partial derivative of u^{ν_μ} for some $\nu_\mu = 1,2,\ldots,m,$ where $f^{\mu}(x,u,\partial u,\partial^2 u,\ldots,\partial^k u)$ does not depend explicitly on any of the components $u^{\nu_\sigma}_{i_1 i_2\ldots i_{k_\mu}}$, $\sigma = 1,2,\ldots,N,$ for each $\mu = 1,2,\ldots,N.$ From theorem 1.2.3., we see that the system of PDEs (1.2.18) admits the point symmetry

$$\mathrm{X} = \xi_i(x,u)\frac{\partial}{\partial x_i} + \eta^{\nu}(x,u)\frac{\partial}{\partial u^{\nu}}, \qquad (1.2.19)$$

with the *k*th extension of (1.2.19) given by (1.2.16), if and only if

$$\begin{aligned}\eta^{(l_\mu)\nu_\mu}_{i_1 i_2\ldots i_{l_\mu}} &= \xi_j\frac{\partial f^{\mu}}{\partial x_j} + \eta^{\nu}\frac{\partial f^{\mu}}{\partial u^{\nu}} + \eta^{(1)\nu}_{j}\frac{\partial f^{\mu}}{\partial u^{\nu}_{j}} + \ldots\ldots \\ &\quad + \eta^{(k)\nu}_{j_1 j_2\cdots j_k}\frac{\partial f^{\mu}}{\partial u^{\nu}_{j_1 j_2\cdots j_k}}, \qquad \mu = 1,2,\ldots,N,\end{aligned} \qquad (1.2.20)$$

with

$$u^{\nu_\sigma}_{i_1 i_2\ldots i_{k\sigma}} = f^{\sigma}(x,u,\partial u,\partial^2 u,\ldots,\partial^k u)=0, \qquad \sigma = 1,2,\ldots,N. \qquad (1.2.21)$$

It is easy to see that $\eta^{(p)\nu}_{j_1 j_2\cdots j_p}$ is a polynomial in the components of $\partial u,\partial^2 u,\ldots,\partial^p u,$ with coefficients that are linear homogeneous in the components of $\xi(x,u),\ \eta(x,u)$ and their derivatives to order *p*. Thus ξ and η appear linearly in (1.2.20). As is the situation for a given scalar PDE, the system of symmetry determining equations (1.2.20-21) leads to a system of linear homogeneous PDEs for ξ and η. First we eliminate the components $u^{\nu_\sigma}_{i_1 i_2\ldots i_{k\sigma}}$ and their differential consequences from (1.2.20) by substitution from (1.2.21) and the differential consequences of (1.2.21), $\sigma = 1,2,\ldots,N.$ Consequently, the components of x,u and the remaining components of $\partial u,\partial^2 u,\ldots,\partial^k u$ that appear in the resulting system of symmetry determining equations

(1.2.20) are themselves independent variables, i.e., they take on arbitrary values. Since the resulting expression for (1.2.20) holds for any values of these independent variables, one obtains a system of linear homogeneous PDEs for ξ and η that constitutes a set of determining equations for the infinitesimal generators X admitted by the given system of PDEs (1.2.13). In particular, if each $f^{\mu}(x,u,\partial u,\partial^2 u,\ldots,\partial^k u),\ \mu=1,2,\ldots,N$, is a polynomial in the components of $\partial u,\partial^2 u,\ldots,\partial^k u$, then the system of symmetry determining equations (1.2.20) yields polynomial equations in the independent components of $\partial u,\partial^2 u,\ldots,\partial^k u$. Consequently, the coefficients of these polynomial equations must vanish separately. This yields the set of linear determining equations for ξ and η. Typically, the numbers of determining equations are far greater than $n+m$, so that the set of determining equations is very overdetermined.

Definition 1.2.13. $u=\Theta(x),$ with components $u^{\nu}=\Theta^{\nu}(x),\ \nu=1,2,\ldots,m,$ is an invariant solution of the system of PDEs (1.2.13) resulting from admitted point symmetry with infinitesimal generator (1.2.15) if and only if:

(i) $u^{\nu}=\Theta^{\nu}(x)$ is an invariant surface of (1.2.15) for each $\nu=1,2,\ldots,m$;

(ii) $u=\Theta(x)$ solves (1.2.13).

It follows that $u=\Theta(x)$ is an invariant of the system of PDEs (1.2.13), resulting from its invariance under the Lie group of point transformations (1.2.14), if and only if $u=\Theta(x)$ satisfies:

(i) $X(u^{\nu}-\Theta^{\nu}(x))=0$ when $u=\Theta(x),\ \nu=1,2,\ldots,m;$ i.e.,

$$\xi_i(x,\Theta(x))\frac{\partial\,\Theta(x)}{\partial\,x_i}=\eta^{\nu}(x,\Theta(x)),\ \nu=1,2,\ldots,m; \qquad (1.2.22)$$

(ii) $F^{\mu}(x,u,\partial u,\partial^2 u,\ldots,\partial^k u)=0$ when $u=\Theta(x),\mu=1,2,\ldots,N,$ i.e.,

$$F^{\mu}(x,\Theta(x),\partial\Theta(x),\partial^2\Theta(x),\ldots,\partial^k\Theta(x))=0,\ \mu=1,2,\ldots,N\,, \qquad (1.2.23)$$

where $\partial^j\Theta(x)$ denotes $\partial^j\Theta^{\nu}(x)/\partial_{x_{i1}}\partial x_{i_2}\ldots\partial x_i$,$\nu=1,2,\ldots,n,$ for $j=1,2,\ldots,k.$

Equations (1.2.22) are the invariant surface conditions for the invariant solutions of the system of PDEs (1.2.23) resulting from its invariance under the point symmetry (1.2.15). As is the situation for the scalar PDE, invariant solutions can be determined by the following procedure:

Invariant form method. Here, we solve the invariant surface conditions (1.2.22) by explicitly solving the corresponding characteristics equations for $u=\Theta(x)$ given by

$$\frac{dx_1}{\xi_1(x,u)}=\frac{dx_2}{\xi_2(x,u)}=\ldots=\frac{dx_n}{\xi_n(x,u)}=\frac{du^1}{\eta^1(x,u)}=\frac{du^2}{\eta^2(x,u)}=\ldots=\frac{du^m}{\eta^m(x,u)}. \qquad (1.2.24)$$

If $y_1(x,u), y_2(x,u),\ldots, y_{n-1}(x,u), v^1(x,u), v^2(x,u),\ldots, v^m(x,u),$ are $n+m-1$ functionally independent constants that arise from solving the system of $n+m-1$ first-order ODEs (1.2.24) with the Jacobian $\partial(v^1,v^2,\ldots,v^m)/\partial(u^1,u^2,\ldots,u^m)\neq 0,$ then the general solution $u=\Theta(x)$ of the system of ODEs (1.2.11) is given implicitly by the invariant form

$$v^\nu(x,u)=\Phi^\nu(y_1(x,u)y_2(x,u),\ldots,y_{n-1}(x,u)), \qquad (1.2.25)$$

where Φ^ν is an arbitrary differentiable function of $y_1(x,u)y_2(x,u),\ldots,y_{n-1}(x,u)$, for $\nu=1,2,\ldots,m.$ Note that $y_1(x,u)y_2(x,u),\ldots,y_{n-1}(x,u)$, $v^1(x,u),v^2(x,u),\ldots,v^m(x,u)$ are $n+m-1$ functionally independent group invariants of (1.2.15) and hence are $n+m-1$ canonical coordinates for the Lie group of points transformations (1.2.14). Let $y_n(x,u)$ be the $(n+m)th$ canonical coordinate satisfying

$$\mathrm{X}y_n=1.$$

If the system of PDEs (1.2.13) is transformed into a system of PDEs in terms of independent variables $y_1,y_2,\ldots,y_n$ and dependent variables $v^1,v^2,\ldots,v^m,$ then the transformed system of PDEs admits the one-parameter Lie group of transformations.

$$y*_i=y_i, i=1,2,\ldots,n-1,$$
$$y*_n=y_n+\varepsilon,$$
$$v*^\nu=v^\nu,\ \nu=1,2,\ldots,m.$$

Thus, the variable y_n does not appear explicitly in the transformed system of ODEs and, hence, the transformed system of PDEs has solutions of the form (1.2.25) consequently, the system of PDEs (1.2.13) has invariant solutions given implicitly by the invariant form (1.2.15). Such solutions are found by solving a reduced system of differential equations with $n-1$ independent variables $y_1,y_2,\ldots,y_{n-1}$ and m dependent variables $v^1,v^2,\ldots,v^m$. The variables $y_1,y_2,\ldots,y_{n-1}$ are commonly called similarity variables. The reduced system of differential equations is found by substituting the invariant form (1.2.25) into the given system of PDEs (1.2.13). We assume that this substitution does not lead to a singular differential equation. Note that if $\frac{\partial \xi}{\partial u} \equiv 0,$ as is typically the case, then $y_i = y_i(x), i = 1, 2, \ldots, n-1.$ If $n = 2,$ then the reduced system of differential equations is a system ODEs.

CHAPTER 2

METHODOLOGY

In this thesis, we deal with the methods of group invariant solutions, based on the theory of continuous group of transformations, better known as ‘Lie groups’, acting on the space of independent and dependent variables of the system. The method is due originally to Sophus Lie who unified and extended the bewildering special methods of integration of differential equations. Through the constructive procedures Lie established that, in the case of ordinary differential equations (ODEs), invariance under one-parameter symmetry group implies that the order of the equation can be reduced by one.

Lie’s work for ordinary differential equations examines in a systematic and comprehensive way a wide spectrum of topics such as integrating factors, separable and homogeneous equations, reduction of order, methods of undetermined coefficients and variation of parameters, Euler equation and homogeneous equations with constant coefficients. Further for linear partial differential equations, Lie has established that the invariance under continuous group of transformations leads directly to superposition of solutions in terms of transformations.

The work put up in this thesis has primarily been based on certain concepts of group symmetry through the applications of the last two techniques mentioned above.

By symmetry group of a single or a system of partial differential equations, we mean a continuous group of transformations acting on the space of independent and dependent variables which leaves the equation(s) invariant. The solutions of partial differential equation(s) are all found by solving a reduced system of differential equations involving fewer independent variables. Thus, in particular, the solutions to a partial differential equation in two independent variables which is invariant under one parameter symmetry group can be found by solving a ‘reduced’ ordinary differential equation.

From the view point of deriving explicit solutions to a system of differential equations, ordinary or partial, group theoretic methods carry a lot of potential yet these methods could not be very popular as the algebraic calculations involved were too complex and cumbersome. However, with the advent of software such Maple and some others, the task has been greatly simplified.

As mentioned earlier the work comprising this thesis is based primarily on the applications of Lie Classical method to some nonlinear partial differential equations. The problems are dealt-with in two phases- in the first, the symmetries of the system under investigation are derived using the Classical Lie method and then in the second phase, after successful deduction of the reduced systems of ordinary differential equations, the efforts are confined to furnish the exact solutions.

2.1 Extended Infinitesimal (Prolongation)

We are working with the applications of the Lie group to differential equations, hence infinitesimal corresponding to derivative are pointed here.

2.1.1 Invariance for a System of PDEs

Consider a kth-order PDE

$$u_{i_1,i_2,\ldots i_k} = f(x,u,\partial u,\partial^2 u,\ldots,\partial^k u), \tag{2.1}$$

where $f(x,u,\partial u,\partial^2 u,\ldots,\partial^k u),$ doesnot depend explicitly on $u_{i_1,i_2,\ldots i_k}$,

we first show how to derive the Lie group of infinitesimal transformations with infinitesimal

$$\begin{aligned} u^* &= U^*(x,t,u;\varepsilon) = u + \varepsilon\eta(x,t,u) + O(\varepsilon^2) \\ x^* &= X^*(x,t,u;\varepsilon) = x + \varepsilon\xi(x,t,u) + O(\varepsilon^2) \\ t^* &= T^*(x,t,u;\varepsilon) = t + \varepsilon\tau(x,t,u) + O(\varepsilon^2). \end{aligned}$$

where η,ξ and τ be the infinitesimal corresponding to u,x and t

On invoking the invariance criterion the following relation from the coefficients of the first order of ε is deduced. Then the symmetry determining is in the form of

$$\eta^{(k)}_{i_1,i_2\ldots i_k} = \xi_j \frac{\partial f}{\partial x_j} + \eta\frac{\partial f}{\partial u} + \eta^{(1)}_j \frac{\partial f}{\partial u_j} + \ldots + \eta^{(k)}_{j_1,j_2\cdots j_k} \frac{\partial f}{\partial u_{j_1,j_2\cdots j_k}},$$

when u satisfies (2.1)

Now find the values of $\eta^x,\eta^t,\eta^{xx},\eta^{xt},\eta^{xxx},\eta^{xxt}$etc

First we need to calculate the auxiliary functions $\frac{\partial x}{\partial x^*},\frac{\partial x}{\partial t^*},\frac{\partial t}{\partial t^*},\frac{\partial t}{\partial x^*}$

By $\frac{\partial x}{\partial x^*}$ we understand that $u = \theta(x,t)$ and that only t^* is held fixed.

Hence $\frac{\partial x}{\partial x^*} = \frac{\partial}{\partial x^*}\left[x^* - \varepsilon\xi(x,t,\theta) + O(\varepsilon^2)\right]$

$$= 1 - \varepsilon\left[\frac{\partial\xi}{\partial x}\frac{\partial x}{\partial x^*} + \frac{\partial\xi}{\partial u}\frac{\partial\theta}{\partial x}\frac{\partial x}{\partial x^*}\right] + O(\varepsilon^2)$$

$$= 1 - \varepsilon\left[\frac{\partial\xi}{\partial x} + \frac{\partial\xi}{\partial u}\theta_x\right] + O(\varepsilon^2).$$

Similarly $\frac{\partial x}{\partial t^*} = -\varepsilon\left[\frac{\partial\xi}{\partial t} + \frac{\partial\xi}{\partial u}\theta_t\right] + O(\varepsilon^2)$

$$\frac{\partial t}{\partial t^*} = 1 - \varepsilon\left[\frac{\partial\tau}{\partial t} + \frac{\partial\tau}{\partial u}\theta_t\right] + O(\varepsilon^2) \qquad (2.2.1)$$

$$\frac{\partial t}{\partial x^*} = -\varepsilon\left[\frac{\partial\tau}{\partial x} + \frac{\partial\tau}{\partial u}\theta_x\right] + O(\varepsilon^2).$$

Recall that $u^* = U^*(x,t,\theta(x,t);\varepsilon) = \theta(x,t) + \varepsilon\eta(x,t,\theta) + O(\varepsilon^2)$.

Hence we find the extensions are :

First extensions

$$\frac{\partial u^*}{\partial x^*} = \frac{\partial\left[\theta(x,t) + \varepsilon\eta(x,t,\theta)\right]}{\partial x^*} + O(\varepsilon^2)$$

$$= \frac{\partial\left[\theta(x,t) + \varepsilon\eta(x,t,\theta)\right]}{\partial x}\frac{\partial x}{\partial x^*} + \frac{\partial t}{\partial x^*}\theta_t + O(\varepsilon^2). \qquad (2.2.2)$$

Substituting (2.2.1) into (2.2.2) we are led to

$$\frac{\partial u^*}{\partial x^*} = \theta_x + \varepsilon\left[\frac{\partial\eta}{\partial x} + \left(\frac{\partial\eta}{\partial u} - \frac{\partial\xi}{\partial x}\right)\theta_x - \frac{\partial\tau}{\partial x}\theta_t - \frac{\partial\xi}{\partial u}\theta_x^2 - \frac{\partial\tau}{\partial u}\theta_x\theta_t\right] + O(\varepsilon^2). \qquad (2.2.3)$$

Let η^x and η^t denote the infinitesimals of $\frac{\partial u^*}{\partial x^*}$ and $\frac{\partial u^*}{\partial t^*}$ respectively. Then

$$\eta^x = \left[\frac{\partial\eta}{\partial x} + \left(\frac{\partial\eta}{\partial u} - \frac{\partial\xi}{\partial x}\right)\theta_x - \frac{\partial\tau}{\partial x}\theta_t - \frac{\partial\xi}{\partial u}\theta_x^2 - \frac{\partial\tau}{\partial u}\theta_x\theta_t\right], \qquad (2.2.4)$$

and similarly

$$\eta^t = \left[\frac{\partial\eta}{\partial t} + \left(\frac{\partial\eta}{\partial u} - \frac{\partial\tau}{\partial t}\right)\theta_t - \frac{\partial\xi}{\partial t}\theta_x - \frac{\partial\tau}{\partial u}\theta_t^2 - \frac{\partial\xi}{\partial u}\theta_x\theta_t\right]. \qquad (2.2.5)$$

Second extensions

$$\eta^{xx} = \frac{\partial^2\eta}{\partial x^2} + \left(2\frac{\partial^2\eta}{\partial x\partial u} - \frac{\partial^2\xi}{\partial x^2} \right)\theta_x - \frac{\partial^2\tau}{\partial x^2}\theta_t + \left(\frac{\partial^2\eta}{\partial u^2} - \frac{2\partial^2\xi}{\partial x\partial u} \right)\theta_x^2 - 2\frac{\partial^2\tau}{\partial x\partial u}\theta_x\theta_t - \frac{\partial^2\xi}{\partial u^2}$$

$$\theta_x^3 \frac{\partial^2\tau}{\partial u^2}\theta_x^2\theta_t + \left(\frac{\partial\eta}{\partial u} - 2\frac{\partial\xi}{\partial x} \right)\theta_{xx} - 2\frac{\partial\tau}{\partial x}\theta_{xt} - 3\frac{\partial\xi}{\partial u}\theta_{xx}\theta_x - \frac{\partial\tau}{\partial u}\theta_{xx}\theta_t - 2\frac{\partial\tau}{\partial u}\theta_{xt}\theta_t .$$

$$\eta^{xt} = \frac{\partial^2\eta}{\partial x\partial t} + \left(\frac{\partial^2\eta}{\partial x\partial u} - \frac{\partial^2\tau}{\partial t\partial x} \right)\theta_t + \left(\frac{\partial^2\eta}{\partial t\partial u} - \frac{\partial^2\xi}{\partial x\partial t} \right)\theta_x - \frac{\partial^2\tau}{\partial x\partial u}\theta_t^2 + \left(\frac{\partial^2\eta}{\partial u^2} - \frac{\partial^2\xi}{\partial x\partial u} - \frac{\partial^2\tau}{\partial t\partial u} \right)$$

$$\theta_x\theta_t - \frac{\partial^2\xi}{\partial t\partial u}\theta_x^2 - \frac{\partial^2\tau}{\partial u^2}\theta_t^2\theta_x + \frac{\partial^2\xi}{\partial u^2}\theta_x^2\theta_t - \frac{\partial\tau}{\partial x}\theta_{tt} + \left(\frac{\partial\eta}{\partial u} - \frac{\partial\xi}{\partial x} - \frac{\partial\tau}{\partial t} \right)\theta_{xt} - \frac{\partial\xi}{\partial t}\theta_{xx} -$$

$$2\frac{\partial\tau}{\partial u}\theta_t\theta_{xt} - 2\frac{\partial\xi}{\partial u}\theta_{xt}\theta_x - \frac{\partial\tau}{\partial u}\theta_x\theta_{tt} - \frac{\partial\xi}{\partial u}\theta_{xx}\theta_t .$$

$$\eta^{tt} = \frac{\partial^2\eta}{\partial t^2} + \left(2\frac{\partial^2\eta}{\partial t\partial u} - \frac{\partial^2\tau}{\partial t^2} \right)\theta_t - \frac{\partial^2\xi}{\partial t^2}\theta_x + \left(\frac{\partial^2\eta}{\partial u^2} - \frac{2\partial^2\tau}{\partial t\partial u} \right)\theta_t^2 - 2\frac{\partial^2\xi}{\partial t\partial u}\theta_x\theta_t - \frac{\partial^2\tau}{\partial u^2}\theta_t^3 -$$

$$\frac{\partial^2\xi}{\partial u^2}\theta_t^2\theta_x + \left(\frac{\partial\eta}{\partial u} - 2\frac{\partial\tau}{\partial t} \right)\theta_{tt} - 2\frac{\partial\xi}{\partial t}\theta_{xt} - 3\frac{\partial\tau}{\partial u}\theta_{tt}\theta_t - \frac{\partial\xi}{\partial u}\theta_{tt}\theta_x - 2\frac{\partial\xi}{\partial u}\theta_{xt}\theta_t .$$

Third extensions

$$\eta^{xxx} = \frac{\partial^3\eta}{\partial x^3} + \left(3\frac{\partial^3\eta}{\partial x^2\partial u} - \frac{\partial^3\xi}{\partial x^3} \right)\theta_x - \frac{\partial^3\tau}{\partial x^3}\theta_t - 3\frac{\partial^3\tau}{\partial x^2\partial u}\theta_x\theta_t + 3\left(\frac{\partial^3\eta}{\partial u^2\partial x} - \frac{\partial^3\xi}{\partial x^2\partial u} \right)\theta_x^2 -$$

$$3\frac{\partial^3\tau}{\partial x\partial u^2}\theta_x^2\theta_t + \left(\frac{\partial^3\eta}{\partial u^3} - 3\frac{\partial^3\xi}{\partial u^2\partial u} \right)\theta_x^3 - \frac{\partial^3\tau}{\partial u^3}\theta_x^3\theta_t - \frac{\partial^3\xi}{\partial u^3}\theta_x^4 - 3\frac{\partial^2\tau}{\partial x^2}\theta_{xt} + 3\left(\frac{\partial^2\eta}{\partial x\partial u} \right.$$

$$\left. - \frac{\partial^2\xi}{\partial x^2} \right)\theta_{xx} + 3\left(\frac{\partial^2\eta}{\partial u^2} - 3\frac{\partial^2\xi}{\partial x\partial u} \right)\theta_{xx}\theta_x - 3\frac{\partial^2\tau}{\partial x\partial u}\theta_{xx}\theta_t - 6\frac{\partial^2\tau}{\partial x\partial u}\theta_{xt}\theta_x - 6\frac{\partial^2\xi}{\partial u^2}\theta_{xx}$$

$$\theta_x^2 - 3\frac{\partial^2\tau}{\partial u^2}\theta_{xt}\theta_x^2 - 3\frac{\partial^2\tau}{\partial u^2}\theta_{xx}\theta_x\theta_t - 3\frac{\partial\xi}{\partial u}\theta_{xx}^2 - 3\frac{\partial\tau}{\partial u}\theta_{xx}\theta_{xt} - 3\frac{\partial\tau}{\partial x}\theta_{xxt} + \left(\frac{\partial\eta}{\partial u} + \right.$$

$$\left. 3\frac{\partial\xi}{\partial x} \right)\theta_{xx} - 4\frac{\partial\xi}{\partial u}\theta_{xxx}\theta_x - \frac{\partial\tau}{\partial u}\theta_{xxx}\theta_t - 3\frac{\partial\tau}{\partial u}\theta_{xxt}\theta_x .$$

$$\eta^{xxt} = \frac{\partial^3\eta}{\partial x^2\partial t} + \left(\frac{\partial^3\eta}{\partial x^2\partial u} - \frac{\partial^3\tau}{\partial x^2\partial t} \right)\theta_t + \left(2\frac{\partial^3\eta}{\partial x\partial t\partial u} - \frac{\partial^3\xi}{\partial x^2\partial t} \right)\theta_x + \left(\frac{\partial^3\eta}{\partial t\partial u^2} - 2\frac{\partial^3\xi}{\partial x\partial t\partial u} \right)\theta_x^2$$

$$+ \left(2\frac{\partial^3\eta}{\partial xu^2} - \frac{\partial^3\xi}{\partial x^2\partial u} - 2\frac{\partial^3\tau}{\partial x\partial t\partial u} \right)\theta_x\theta_t - \frac{\partial^3\tau}{\partial x^2\partial u}\theta_t^2 - 2\frac{\partial^3\tau}{\partial x\partial u^2}\theta_x\theta_t^2 - \frac{\partial^3\xi}{\partial t\partial u^2}\theta_x^3 +$$

$$\left(\frac{\partial^3\eta}{\partial u^3} - 2\frac{\partial^3\xi}{\partial x\partial u^2} - \frac{\partial^3\tau}{\partial t\partial u^2} \right)\theta_x^2\theta_t - \frac{\partial^3\xi}{\partial u^3}\theta_x^3\theta_t - \frac{\partial^3\tau}{\partial u^3}\theta_x^2\theta_t^2 + \left(\frac{\partial^2\eta}{\partial t\partial u} - 2\frac{\partial^2\xi}{\partial x\partial t} \right)\theta_{xx}$$

$$+ \left(2\frac{\partial^2\eta}{\partial x\partial u} - \frac{\partial^2\xi}{\partial x^2} - 2\frac{\partial^2\tau}{\partial x\partial t} \right)\theta_{xt} - \frac{\partial^2\tau}{\partial x^2}\theta_{tt} - 4\frac{\partial^2\tau}{\partial x\partial u}\theta_{xt}\theta_t - 2\frac{\partial^2\tau}{\partial x\partial u}\theta_{tt}\theta_x - 3\frac{\partial^2\xi}{\partial t\partial u}\theta_{xx}\theta_x$$

$$+\left(\frac{\partial^2\eta}{\partial u^2}-2\frac{\partial^2\xi}{\partial x\partial u}-\frac{\partial^2\tau}{\partial t\partial u}\right)2\theta_{xt}\theta_x+\left(\frac{\partial^2\eta}{\partial u^2}-2\frac{\partial^2\xi}{\partial x\partial u}-\frac{\partial^2\tau}{\partial t\partial u}\right)\theta_{xx}\theta_t-3\frac{\partial^2\xi}{\partial u^2}\theta_{xt}\theta_x^2-$$

$$4\frac{\partial^2\tau}{\partial u^2}\theta_{xt}\theta_x\theta_t-\frac{\partial^2\tau}{\partial u^2}\theta_{tt}\theta_x^2-3\frac{\partial^2\xi}{\partial u^2}\theta_{xx}\theta_x\theta_t-\frac{\partial^2\tau}{\partial u^2}\theta_{xx}\theta_t^2-3\frac{\partial\xi}{\partial u}\theta_{xt}\theta_{xx}-2\frac{\partial\tau}{\partial u}\theta_{xt}^2$$

$$-\frac{\partial\tau}{\partial u}\theta_{xx}\theta_{tt}+\left(\frac{\partial\eta}{\partial u}-2\frac{\partial\xi}{\partial x}-\frac{\partial\tau}{\partial t}\right)\theta_{xxt}-2\frac{\partial\tau}{\partial x}\theta_{xtt}-\frac{\partial\xi}{\partial t}\theta_{xxx}-3\frac{\partial\xi}{\partial u}\theta_{xxt}\theta_x-\frac{\partial\xi}{\partial u}\theta_{xxx}\theta_t$$

$$-2\frac{\partial\tau}{\partial u}\theta_{xxt}\theta_t-2\frac{\partial\tau}{\partial u}\theta_{xtt}\theta_x.$$

For two dependent variables the values of η^x,η^t, $\beta^x,\beta^t,\beta^{xx},\beta^{xxx}$

First we need to calculate the auxiliary functions $\frac{\partial x}{\partial x^*},\frac{\partial x}{\partial t^*},\frac{\partial t}{\partial t^*},\frac{\partial t}{\partial x^*}$

By $\frac{\partial x}{\partial x^*}$ we understand that $u=\theta,\alpha(x,t)$ and that only t^* is held fixed.

Hence $\frac{\partial x}{\partial x^*}=\frac{\partial}{\partial x^*}\left[x^*-\varepsilon\xi(x,t,\theta,\alpha)+O(\varepsilon^2)\right]$

$$=1-\varepsilon\left[\frac{\partial\xi}{\partial x}\frac{\partial x}{\partial x^*}+\frac{\partial\xi}{\partial u}\frac{\partial\theta}{\partial x}\frac{\partial x}{\partial x^*}+\frac{\partial\xi}{\partial w}\frac{\partial w}{\partial x}\frac{\partial x}{\partial x^*}\right]+O(\varepsilon^2)$$

$$=1-\varepsilon\left[\frac{\partial\xi}{\partial x}+\frac{\partial\xi}{\partial u}\theta_x+\frac{\partial\xi}{\partial w}\alpha_x\right]+O(\varepsilon^2)$$

Similarly $\frac{\partial x}{\partial t^*}=-\varepsilon\left[\frac{\partial\xi}{\partial t}+\frac{\partial\xi}{\partial u}\theta_t+\frac{\partial\xi}{\partial w}\alpha_t\right]+O(\varepsilon^2)$

$$\frac{\partial t}{\partial t^*}=1-\varepsilon\left[\frac{\partial\tau}{\partial t}+\frac{\partial\tau}{\partial u}\theta_t+\frac{\partial\tau}{\partial w}\alpha_t\right]+O(\varepsilon^2) \qquad (2.2.1)$$

$$\frac{\partial t}{\partial x^*}=-\varepsilon\left[\frac{\partial\tau}{\partial x}+\frac{\partial\tau}{\partial u}\theta_x+\frac{\partial\tau}{\partial w}\alpha_x\right]+O(\varepsilon^2)$$

Recall that $u^*=U^*(x,t,\theta(x,t);\varepsilon)=\theta(x,t)+\varepsilon\eta(x,t,\theta,\alpha)+O(\varepsilon^2)$

Hence we find the extensions are :

First extensions

$$\frac{\partial u^*}{\partial x^*}=\frac{\partial\left[\theta(x,t)+\varepsilon\eta(x,t,\theta,\alpha)\right]}{\partial x^*}+O(\varepsilon^2)$$

$$=\frac{\partial\left[\theta(x,t)+\varepsilon\eta(x,t,\theta,\alpha)\right]}{\partial x}\frac{\partial x}{\partial x^*}+\frac{\partial t}{\partial x^*}\theta_t+O(\varepsilon^2). \qquad (2.2.2)$$

Substituting (2.2.1) into (2.2.2) we are led to

$$\begin{aligned}\frac{\partial u^*}{\partial x^*} = \theta_x + \varepsilon\bigg[&\frac{\partial\eta}{\partial x} + \left(\frac{\partial\eta}{\partial u} - \frac{\partial\xi}{\partial x}\right)\theta_x - \frac{\partial\tau}{\partial x}\theta_t - \frac{\partial\xi}{\partial u}\theta_x^2 - \frac{\partial\tau}{\partial u}\theta_x\theta_t - \frac{\partial\xi}{\partial w}\alpha_x\theta_x \\ &- \frac{\partial\tau}{\partial w}\alpha_x\bigg) + O(\varepsilon^2). \qquad (2.2.3)\end{aligned}$$

Let η^x and η^t denote the infinitesimals of $\frac{\partial u^*}{\partial x^*}$ and $\frac{\partial u^*}{\partial t^*}$ respectively. Then

$$\begin{aligned}\eta^x = \bigg[&\frac{\partial\eta}{\partial x} + \left(\frac{\partial\eta}{\partial u} - \frac{\partial\xi}{\partial x}\right)\theta_x - \frac{\partial\tau}{\partial x}\theta_t - \frac{\partial\xi}{\partial u}\theta_x^2 - \frac{\partial\tau}{\partial u}\theta_x\theta_t - \frac{\partial\xi}{\partial w}\alpha_x\theta_x + \\ &\frac{\partial\tau}{\partial w}\alpha_x\theta_t + \frac{\partial\eta}{\partial w}\alpha_x\bigg) \qquad (2.2.4)\end{aligned}$$

And similarly

$$\begin{aligned}\eta^t = \bigg[&\frac{\partial\eta}{\partial t} + \left(\frac{\partial\eta}{\partial u} - \frac{\partial\tau}{\partial t}\right)\theta_t - \frac{\partial\xi}{\partial t}\theta_x - \frac{\partial\tau}{\partial u}\theta_t^2 - \frac{\partial\xi}{\partial u}\theta_x\theta_t - \frac{\partial\tau}{\partial w}\alpha_t\theta_t + \\ &\frac{\partial\eta}{\partial w}\alpha_t - \frac{\partial\xi}{\partial w}\alpha_t\theta_x\bigg). \qquad (2.2.5)\end{aligned}$$

Recall that $w^* = W^*(x,t,\alpha(x,t);\varepsilon) = \alpha(x,t) + \varepsilon\eta(x,t,\theta,\alpha) + O(\varepsilon^2).$

Hence we find the extensions are :

First extensions

$$\begin{aligned}\frac{\partial w^*}{\partial x^*} &= \frac{\partial\left[\alpha(x,t) + \varepsilon\eta(x,t,\theta,\alpha)\right]}{\partial x^*} + O(\varepsilon^2) \\ &= \frac{\partial\left[\alpha(x,t) + \varepsilon\eta(x,t,\theta,\alpha)\right]}{\partial x}\frac{\partial x}{\partial x^*} + \frac{\partial t}{\partial x^*}\alpha_t + O(\varepsilon^2). \qquad (2.2.6)\end{aligned}$$

Substituting (2.2.1) into (2.2.2) we are led to

$$\begin{aligned}\frac{\partial w^*}{\partial x^*} = \alpha_x + \varepsilon\bigg[&\frac{\partial\beta}{\partial x} + \left(\frac{\partial\beta}{\partial w} - \frac{\partial\xi}{\partial x}\right)\alpha_x - \frac{\partial\tau}{\partial x}\alpha_t - \frac{\partial\xi}{\partial w}\alpha_x^2 - \frac{\partial\tau}{\partial w}\alpha_x\alpha_t - \frac{\partial\xi}{\partial w}\alpha_x\theta_x \\ &- \frac{\partial\tau}{\partial u}\theta_x\alpha_t + \frac{\partial\beta}{\partial u}\theta_x\bigg] + \frac{\partial\beta}{\partial u}\theta_x + O(\varepsilon^2). \qquad (2.2.7)\end{aligned}$$

Let β^xdenote the infinitesimals of $\frac{\partial w^*}{\partial x^*}$respectively. Then

$$\begin{aligned}\beta^x = \bigg[&\frac{\partial\beta}{\partial x} + \left(\frac{\partial\beta}{\partial w} - \frac{\partial\xi}{\partial x}\right)\alpha_x - \frac{\partial\tau}{\partial x}\alpha_t - \frac{\partial\xi}{\partial w}\alpha_x^2 - \frac{\partial\tau}{\partial w}\alpha_x\alpha_t - \frac{\partial\xi}{\partial u}\alpha_x\theta_x - \\ &\frac{\partial\tau}{\partial u}\alpha_t\theta_x + \frac{\partial\beta}{\partial u}\theta_x\bigg], \qquad (2.2.8)\end{aligned}$$

and similarly

$$\beta^{t}=\left[\frac{\partial\beta}{\partial t}+\left(\frac{\partial\beta}{\partial w}-\frac{\partial\tau}{\partial t}\right)\alpha_{t}-\frac{\partial\xi}{\partial t}\alpha_{x}-\frac{\partial\tau}{\partial w}\alpha_{t}^{2}-\frac{\partial\xi}{\partial w}\alpha_{x}\alpha_{t}-\frac{\partial\tau}{\partial u}\alpha_{t}\theta_{t}-\frac{\partial\xi}{\partial u}\right.$$

$$\left.\alpha_{x}\theta_{t}+\frac{\partial\beta}{\partial u}\theta_{t}\right]. \tag{2.2.9}$$

Second extension

$$\beta^{xx}=\frac{\partial^{2}\beta}{\partial x^{2}}+\left(2\frac{\partial^{2}\beta}{\partial x\partial w}-\frac{\partial^{2}\xi}{\partial x^{2}}\right)\alpha_{x}-\frac{\partial^{2}\tau}{\partial x^{2}}\alpha_{t}+\left(\frac{\partial^{2}\beta}{\partial w^{2}}-\frac{2\partial^{2}\xi}{\partial x\partial w}\right)\alpha_{x}^{2}-2\frac{\partial^{2}\tau}{\partial x\partial w}\alpha_{x}\alpha_{t}-\frac{\partial^{2}\xi}{\partial u^{2}}\alpha_{x}^{3}$$

$$-\frac{\partial^{2}\tau}{\partial w^{2}}\alpha_{x}^{2}\alpha_{t}+\left(\frac{\partial\beta}{\partial w}-2\frac{\partial\xi}{\partial x}\right)\alpha_{xx}-2\frac{\partial\tau}{\partial x}\alpha_{xt}-3\frac{\partial\xi}{\partial u}\alpha_{xx}\alpha_{x}-\frac{\partial\tau}{\partial w}\alpha_{xx}\alpha_{t}-2\frac{\partial\tau}{\partial w}\alpha_{xt}\alpha_{x}-$$

$$2\frac{\partial\xi}{\partial u}\theta_{x}\alpha_{xx}-2\frac{\partial\tau}{\partial u}\alpha_{xt}\theta_{x}+\left(2\frac{\partial^{2}\beta}{\partial u\partial w}-2\frac{\partial^{2}\xi}{\partial x\partial u}\right)\theta_{x}\alpha_{x}-\frac{\partial^{2}\xi}{\partial u^{2}}\theta_{x}^{2}\alpha_{x}-2\frac{\partial^{2}\xi}{\partial u\partial w}\alpha_{x}^{2}\theta_{x}-$$

$$\frac{\partial\xi}{\partial u}\alpha_{x}\theta_{xx}-2\frac{\partial^{2}\tau}{\partial x\partial u}\theta_{x}\alpha_{t}-\frac{\partial^{2}\tau}{\partial u^{2}}\theta_{x}^{2}\alpha_{t}-2\frac{\partial^{2}\tau}{\partial u\partial w}\theta_{x}\alpha_{x}\alpha_{t}-\frac{\partial\tau}{\partial u}\theta_{xx}\alpha_{t}-\frac{\partial^{2}\beta}{\partial u^{2}}\theta_{x}^{2}+\frac{\partial^{2}\beta}{\partial x\partial u}$$

$$\theta_{x}+\frac{\partial\beta}{\partial u}\theta_{xx}.$$

Third extension

$$\beta^{xxx}=\frac{\partial^{3}\beta}{\partial x^{3}}+\left(3\frac{\partial^{3}\beta}{\partial x^{2}\partial w}-\frac{\partial^{3}\xi}{\partial x^{3}}\right)\alpha_{x}-\frac{\partial^{3}\tau}{\partial x^{3}}\alpha_{t}-3\frac{\partial^{3}\tau}{\partial x^{2}\partial w}\alpha_{x}\alpha_{t}+3\left(\frac{\partial^{3}\beta}{\partial x\partial w^{2}}-\frac{\partial^{3}\xi}{\partial x^{2}\partial w}\right)\alpha_{x}^{2}-$$

$$3\frac{\partial^{3}\tau}{\partial x\partial w^{2}}\alpha_{x}^{2}\alpha_{t}+\left(\frac{\partial^{3}\beta}{\partial w^{3}}-3\frac{\partial^{3}\xi}{\partial x\partial w^{2}}\right)\alpha_{x}^{3}-\frac{\partial^{3}\tau}{\partial w^{3}}\alpha_{x}^{3}\alpha_{t}-\frac{\partial^{3}\xi}{\partial w^{3}}\alpha_{x}^{4}-3\frac{\partial^{2}\tau}{\partial x^{2}}\alpha_{xt}+3\left(\frac{\partial^{2}\beta}{\partial x\partial w}\right.$$

$$\left.-\frac{\partial^{2}\xi}{\partial x^{2}}\right)\alpha_{xx}+3\left(\frac{\partial^{2}\beta}{\partial w^{2}}-3\frac{\partial^{2}\xi}{\partial x\partial w}\right)\alpha_{xx}\alpha_{x}-3\frac{\partial^{2}\tau}{\partial x\partial w}\alpha_{xx}\alpha_{t}-6\frac{\partial^{2}\tau}{\partial x\partial w}\alpha_{xt}\alpha_{x}-6\frac{\partial^{2}\xi}{\partial w^{2}}\alpha_{xx}$$

$$\alpha_{x}^{2}-3\frac{\partial^{2}\tau}{\partial w^{2}}\alpha_{xt}\alpha_{x}^{2}-3\frac{\partial^{2}\tau}{\partial w^{2}}\alpha_{xx}\alpha_{x}\alpha_{t}-3\frac{\partial\xi}{\partial w}\alpha_{xx}^{2}-3\frac{\partial\tau}{\partial w}\alpha_{xx}\alpha_{xt}-3\frac{\partial\tau}{\partial x}\alpha_{xxt}+\left(\frac{\partial\beta}{\partial w}-\right.$$

$$\left.3\frac{\partial\xi}{\partial x}\right)\alpha_{xxx}-4\frac{\partial\xi}{\partial w}\alpha_{xxx}\alpha_{x}-\frac{\partial\tau}{\partial w}\alpha_{xxx}\alpha_{t}-3\frac{\partial\tau}{\partial w}\alpha_{xxt}\alpha_{x}-3\frac{\partial\xi}{\partial u}\alpha_{xxx}\theta_{x}-3\frac{\partial\tau}{\partial u}\theta_{x}\alpha_{xxt}+$$

$$3\frac{\partial^{3}\beta}{\partial x^{2}\partial u}\theta_{x}+\left(6\frac{\partial^{3}\beta}{\partial x\partial u\partial w}-3\frac{\partial^{3}\xi}{\partial x^{2}\partial u}\right)\alpha_{x}\theta_{x}-3\frac{\partial^{3}\tau}{\partial x^{2}\partial u}\theta_{x}\alpha_{t}+\left(3\frac{\partial^{3}\beta}{\partial u\partial w^{2}}-6\frac{\partial^{3}\xi}{\partial x\partial u\partial w}\right)$$

$$\theta_{x}\,\alpha_{x}^{2}-6\frac{\partial^{3}\tau}{\partial x\partial u\partial w}\theta_{x}\alpha_{x}\alpha_{t}-3\frac{\partial^{3}\xi}{\partial u\partial w^{2}}\theta_{x}\,\alpha_{x}^{3}-3\frac{\partial^{3}\tau}{\partial u\partial w^{2}}\theta_{x}\,\alpha_{x}^{2}\alpha_{t}+\left(3\frac{\partial^{2}\beta}{\partial u\partial w}-6\frac{\partial^{2}\xi}{\partial u\partial x}\right)$$

$$\begin{aligned}
&\theta_x\alpha_{xx}-6\frac{\partial^2\tau}{\partial u\partial x}\theta_x\alpha_{xt}-\left(5\frac{\partial^2\xi}{\partial u\partial w}-4\frac{\partial^3\xi}{\partial u\partial w^2}\right)\theta_x\alpha_x\alpha_{xx}-3\frac{\partial^2\tau}{\partial u\partial w}\theta_x\alpha_{xx}\alpha_t-6\frac{\partial^2\tau}{\partial u\partial w}\\
&\alpha_x^2-3\frac{\partial^2\tau}{\partial w^2}\alpha_{xt}\alpha_x^2-3\frac{\partial^2\tau}{\partial w^2}\alpha_{xx}\alpha_x\alpha_t-3\frac{\partial\xi}{\partial w}\alpha_{xx}^2-3\frac{\partial\tau}{\partial w}\alpha_{xx}\alpha_{xt}-3\frac{\partial\tau}{\partial x}\alpha_{xxt}+\left(\frac{\partial\beta}{\partial w}-\right.\\
&\theta_x\alpha_x\alpha_{xt}-2\frac{\partial^2\xi}{\partial u^2}\alpha_{xx}\theta_x^2-3\frac{\partial\xi}{\partial u}\theta_{xx}\alpha_{xx}-3\frac{\partial^2\tau}{\partial u^2}\theta_x^2\alpha_{xt}-3\frac{\partial\tau}{\partial u}\theta_{xx}\alpha_{xt}+3\left(\frac{\partial^2\beta}{\partial u\partial w}-\frac{\partial^2\xi}{\partial u\partial w}\right)\\
&\theta_{xx}\alpha_x+\left(3\frac{\partial^3\xi}{\partial u\partial u^2}+3\frac{\partial^3\beta}{\partial u^2\partial w}\right)\theta_x^2\alpha_x-3\frac{\partial^3\xi}{\partial u^3}\theta_x^3\alpha_x-3\frac{\partial^3\xi}{\partial u^2\partial w}\theta_x^2\alpha_x^2-3\frac{\partial^2\xi}{\partial u^2}\theta_x\,\theta_{xx}\alpha_x-\\
&\frac{\partial^2\xi}{\partial u^2}\theta_x^2\alpha_{xx}-3\frac{\partial^2\xi}{\partial u\partial w}\,\theta_{xx}\alpha_x^2-\frac{\partial\xi}{\partial u}\theta_{xxx}\alpha_x+3\frac{\partial^3\tau}{\partial u\partial u^2}\theta_x^2\alpha_t-\frac{\partial^2\tau}{\partial x\partial u}\theta_{xxx}\alpha_t-\frac{\partial^3\tau}{\partial u^3}\theta_x^3\alpha_t-\\
&3\frac{\partial^3\xi}{\partial u^2\partial w}\theta_x^2\alpha_x\alpha_t-3\frac{\partial^2\tau}{\partial u^2}\theta_x\theta_{xx}\alpha_t-3\frac{\partial^2\tau}{\partial u\partial w}\theta_{xx}\alpha_x\alpha_t-\frac{\partial\tau}{\partial u}\theta_{xxx}\alpha_t+3\frac{\partial^3\beta}{\partial x\partial u^2}\theta_x^2+3\frac{\partial^2\beta}{\partial x\partial u}\\
&\theta_{xx}+3\frac{\partial^2\beta}{\partial u^2}\theta_x\theta_{xx}+\frac{\partial\beta}{\partial u}\theta_{xxx}+\frac{\partial^3\beta}{\partial u^3}\theta_x^3.
\end{aligned}$$

2.2. Classical Lie Method: An Algorithmic Overview

The classical method essentially consists of finding symmetry reductions of PDEs with the help of determining equations obtained under the condition of invariance (1.2.17) of the system of PDEs. More specifically, when a given system of PDEs (1.2.13) is subjected to invariance under one-parameter Lie group of transformations (1.2.14), one arrives at an over determined linear system of equations for the group infinitesimals. These infinitesimals of the transformations help us obtain the reductions of the system. The symmetries and reductions reported in chapters 3, 4, 5 and 6 are based on the application of this method.

The stepwise procedure is as follows:

Consider a system of *N* PDEs with *m* dependent variables $u=(u^1,u^2,\ldots,u^m)$ and *n* independent variables $x=(x_1,x_2,\ldots,x_n),$ given by

$$F^{\mu}(x,u,\partial u,\partial^2 u,\ldots,\partial^k u)=0,\ \mu=1,2,\ldots,N. \qquad (2.2.1)$$

1) Let the one-parameter Lie group of point transformations (1.2.14a, b) leaves invariant the system of PDEs (2.2.1).
2) Apply the prolonged operator $\mathrm{X}^{(k)}$ given by (1.2.16) to each equation of the system (2.2.1) and require that

$$\mathrm{X}^{(k)}F^{\mu}\Big|_{F^{\nu}=0}=0 \qquad \mu,\nu=1,2,\ldots,N\,. \qquad (2.2.2)$$

The meaning of the condition (2.2.2) is that $\mathrm{X}^{(k)}$ vanishes on the solution set of the originally given system (2.2.1). Precisely, this condition assures that $u(x)$ is solution of (2.2.1) whenever $u^{*}(x^{*})$ is one.

3) Following the procedure as mentioned in section (1.2.4-1), a system of linear PDEs for ξ and η that constitutes a set of determining equations for the infinitesimal generator X admitted by the given system of PDEs (2.2.1) is obtained.

4) The solutions of the determining equations will lead to the explicit forms of ξ and η.

5) Construct the corresponding characteristics equations (1.2.24) and obtain u in terms of $n-1$ new independent variables.

6) Rewrite the system (2.2.1) in these new coordinates to get the reduced form of the system.

CHAPTER 3

GARDNER EQUATION

The mathematical theory of the nonlinear evolution equations, starting from the Korteweg-de Vries (KdV) equation and the modified Korteweg-de Vries (mKdV) equation is an area of research for the past few decades [27, 30, 31, 32]. The Korteweg - de Vries equation (KdV) is a well-known model for description of nonlinear long internal waves in the ocean. Its coefficients are defined by vertical density and currents stratification. In a shelf zone the coefficient of quadratic nonlinearity may tend to zero, and the nonlinear model should be modified. In particular, The Gardner’s equation, that is also known as the mixed KDV-mKdV equation. The Gardner equation, which differs from the KdV by presence of an additional term of cubic nonlinearity may be used as the generalized model.

This Gardner equation shows up, particularly, in the context of internal gravity waves in a density-stratified ocean. This is commonly described by the KdV equations and its versions with small nonlinearity. However, there are situations when waves with strong nonlinearity is experienced, as in the case of Coastal Ocean Probe Experiment during 1995 in the Oregon Bay, the problem of creating an adequate theoretical model was deemed necessary. This lead to the study of Gardner equation [15].

3.1 Symmetries

Consider the Gardner equation

$$u_t = 6(u+\varepsilon^2u^2)u_x - u_{xxx} = 0. \tag{3.1}$$

Let the group of infinitesimal transformations be defined as

$$\begin{aligned} u^* &= u + \varepsilon\eta(x,t,u) + O(\varepsilon^2) \\ x^* &= x + \varepsilon\xi(x,t,u) + O(\varepsilon^2) \\ t^* &= t + \varepsilon\tau(x,t,u) + O(\varepsilon^2). \end{aligned}$$

On invoking the invariance criterion the following relation from these coefficient of the first order of ε is deduced,

$$\eta^t - 6u\eta^x - 6\eta u_x - 6\alpha^2u^2\eta^x - 12\alpha^2 uu_x\eta - \eta^{xxx} = 0, \tag{3.2}$$

where $\eta^t, \eta^x, \eta^{xxx}$ are infinitesimal generator corresponding to u_t, u_x, u_{xxx},

Substitute the values of $\eta^t, \eta^x, \eta^{xxx}$ in (3.2) we get

$$\bigg(\frac{\partial\eta}{\partial t} - 6u\frac{\partial\eta}{\partial x} - 6\alpha^2u^2\frac{\partial\eta}{\partial x} - \frac{\partial^3\eta}{\partial x^3}\bigg) + \bigg[\bigg(\frac{\partial\eta}{\partial u} - \frac{\partial\tau}{\partial t} + 6u\frac{\partial\tau}{\partial x} + \frac{\partial^3\tau}{\partial x^3} + 6\alpha^2u^2\frac{\partial\tau}{\partial x}\bigg)6(u+\alpha^2u^2) - \frac{\partial\xi}{\partial t}$$

$$-6u\bigg(\frac{\partial\eta}{\partial u} - \frac{\partial\xi}{\partial x}\bigg) - 6\eta - 6\alpha^2u^2\bigg(\frac{\partial\eta}{\partial u} - \frac{\partial\xi}{\partial x}\bigg) - 12\alpha^2u\eta - 3\frac{\partial^3\eta}{\partial x^2\partial u} + \frac{\partial^3\xi}{\partial x^3}\bigg]\theta_x + \bigg[-36(u+\alpha^2u^2)^2\frac{\partial\tau}{\partial u}$$

$$-6(u+\alpha^2u^2)^2\frac{\partial\xi}{\partial u} + 6u\frac{\partial\xi}{\partial u} + 36(u+\alpha^2u^2)^2\frac{\partial\tau}{\partial u} + 6\alpha^2u^2\frac{\partial\xi}{\partial u} + 6(u+\alpha^2u^2)\bigg(6\alpha^2u^2\frac{\partial\tau}{\partial u} + 3\frac{\partial^3\tau}{\partial u\partial x^2}\bigg)$$

$$-3\bigg(\frac{\partial^3\eta}{\partial u^2\partial x} - \frac{\partial^3\xi}{\partial u\partial x^2}\bigg)\bigg]\theta_x^2 + \bigg(\frac{\partial\eta}{\partial u} - \frac{\partial\tau}{\partial t} + 6u\frac{\partial\tau}{\partial x} + \frac{\partial^3\tau}{\partial x^3} + 6\alpha^2u^2\frac{\partial\tau}{\partial x} - \frac{\partial\eta}{\partial u} + 3\frac{\partial\xi}{\partial x}\bigg)\theta_{xxx} + (-12(u+\alpha^2$$

$$u^2\frac{\partial\tau}{\partial u} + (5-6u)\frac{\partial\xi}{\partial u} + 6\alpha^2u^2\frac{\partial\tau}{\partial u} + 6(u+\alpha^2u^2)\frac{\partial\tau}{\partial u} + 3\frac{\partial^3\tau}{\partial x^2\partial u}\bigg)\theta_x\theta_{xxx} + 3\frac{\partial^3\tau}{\partial u^2\partial x}\theta_x^2\theta_{xxx} + 3\frac{\partial^2\tau}{\partial x^2}\theta_{xt}$$

$$+\bigg[18(u+\alpha^2u^2)\frac{\partial^3\tau}{\partial u^2\partial x} - \bigg(\frac{\partial^3\eta}{\partial u^3} - 3\frac{\partial^3\xi}{\partial u^2\partial x}\bigg)\bigg]\theta_x^3 - 3\bigg(\frac{\partial^2\eta}{\partial x\partial u} - \frac{\partial^2\xi}{\partial x^2}\bigg)\theta_{xx} + -\bigg[3\bigg(\frac{\partial^2\eta}{\partial u^2} - 3\frac{\partial^2\xi}{\partial x\partial u}\bigg) + 18$$

$$(u+\alpha^2u^2)\frac{\partial^2\tau}{\partial x\partial u}\bigg]\theta_x\theta_{xx} - 3\frac{\partial\xi}{\partial u}\theta_{xx}^2 + 6\frac{\partial^2\tau}{\partial x\partial u}\theta_x\theta_{xt} + \bigg(6\frac{\partial^2\xi}{\partial u^2} + 18(u+\alpha^2u^2)\frac{\partial^2\tau}{\partial u^2}\bigg)\theta_{xx}\theta_x^2 + 3\frac{\partial^2\tau}{\partial x\partial u}$$

$$\theta_{xx}\theta_{xxx} + 3\frac{\partial^2\tau}{\partial u^2}\theta_{xt}\theta_x^2 + 3\frac{\partial^2\tau}{\partial u^2}\theta_x\theta_{xx}\theta_{xxx} + 3\frac{\partial\tau}{\partial x}\theta_{xxt} + 3\frac{\partial\tau}{\partial u}\theta_{xx}\theta_{xt} + 3\frac{\partial\tau}{\partial x}\theta_{xxt}\theta_x + \frac{\partial^3\tau}{\partial u^3}\theta_x^3\theta_{xxx} +$$

$$\bigg(6(u+\alpha^2u^2)\frac{\partial^3\tau}{\partial u^3} + \frac{\partial^3\xi}{\partial u^3}\bigg)\theta_x^4 = 0.$$

Now equate the coefficients of $\theta_{xx}^2, \theta_{xt}, \theta_{xxt}, \theta_x$, const, etc equal to zero and we obtain

$$\frac{\partial\xi}{\partial u} = \frac{\partial\tau}{\partial x} = \frac{\partial\tau}{\partial u} = 0 \tag{3.3}$$

$$\frac{\partial\eta}{\partial t} - 6u\frac{\partial\eta}{\partial x} - 6\alpha^2u^2\frac{\partial\eta}{\partial x} - \frac{\partial^3\eta}{\partial x^3} = 0 \tag{3.4}$$

$$6(u+\alpha^2u^2)\bigg(\frac{\partial\eta}{\partial u} - \frac{\partial\tau}{\partial t}\bigg) - \frac{\partial\xi}{\partial t} - 6u\bigg(\frac{\partial\eta}{\partial u} - \frac{\partial\xi}{\partial x}\bigg) - 6\eta - 6\alpha^2u^2$$

$$\bigg(\frac{\partial\eta}{\partial u} - \frac{\partial\xi}{\partial x}\bigg) - 12\alpha^2u\eta - 3\frac{\partial^3\eta}{\partial x^2\partial u} + \frac{\partial^3\xi}{\partial x^3} = 0 \tag{3.5}$$

$$-3\frac{\partial^3\eta}{\partial u^2\partial x} = 0 \tag{3.6}$$

$$-\frac{\partial\tau}{\partial t} + 3\frac{\partial\xi}{\partial x} = 0 \tag{3.7}$$

$$\frac{\partial^3\eta}{\partial u^3} = 0 \tag{3.8}$$

$$-3\left(\frac{\partial^2\eta}{\partial x\partial u}-\frac{\partial^2\xi}{\partial x^2}\right)=0 \tag{3.9}$$

$$-3\frac{\partial^2\eta}{\partial u^2}=0. \tag{3.10}$$

From equations (3.3) and (3.10) we get

$$\xi=h(x,t)\ ,\ \tau=j(t)\ ,\ \eta=f(x,t)u+g(x,t). \tag{3.11}$$

Put the value of ξ, τ and η in above coefficients (3.3)-(3.10)

$$\begin{aligned}&uf_t(x,t)+g_t(x,t)-6u^2f_x(x,t)-6ug_x(x,t)-6\alpha^2u^3f_x(x,t)-6\alpha^2u^2g_x(x,t)\\&-uf_{xxx}(x,t)-g_{xxx}(x,t)=0\end{aligned} \tag{3.12}$$

$$\begin{aligned}&6(u+\alpha^2u^2)(f(x,t)-j_t(t))-h_t(x,t)-6u(f(x,t)-h_x(x,t))-6uf(x,t)-6g(x,t)\\&-6\alpha^2u^2(f(x,t)-h_x(x,t))-12\alpha^2u^2f(x,t)-12\alpha^2ug(x,t)-3f_{xx}(x,t)+h_{xxx}(x,t)=0\end{aligned} \tag{3.13}$$

$$-j_t(t)+3h_x(x,t)=0 \tag{3.14}$$

$$-3(f_x(x,t)-h_{xx}(x,t))=0. \tag{3.15}$$

By equating the coefficient of u we get

$$f_t(x,t)-6g_x(x,t)-f_{xxx}(x,t)=0 \tag{3.16}$$

$$6(f(x,t)-j_t(t)-f(x,t)+h_x(x,t)-6f(x,t))-12\alpha^2g(x,t)=0. \tag{3.17}$$

By equating the coefficient of u^2

$$-6f_x(x,t)-6\alpha^2g_x(x,t)=0 \tag{3.18}$$

$$6\alpha^2(f(x,t)-j_t(t)-f(x,t)+h_x(x,t))-12\alpha^2f(x,t)=0. \tag{3.19}$$

On equating the coefficient of u^3

We get $f_x(x,t)=0.$ (3.20)

By equating the coefficients of without $u(x,t)$ and $u^2(x,t)$

$$g_t(x,t)-g_{xxx}(x,t)=0 \tag{3.21}$$

$$-h_t(x,t)-6g(x,t)-3f_{xx}(x,t)+h_{xxx}(x,t)=0 \tag{3.22}$$

$$-j_t(t)+3h_x(x,t)=0 \tag{3.23}$$

$$-3(f_x(x,t)-h_{xx}(x,t))=0. \tag{3.24}$$

Now we get $f_x(x,t)=0.$

$$g_x(x,t)=0. \tag{3.25}$$

From equation (3.18) we get

From equation (3.21) we get

$$g_t(x,t) = 0. \tag{3.26}$$

Therefore from equation (3.25) and (3.26) we get

$$g = \text{constant}$$

$$g = a \text{ say.}$$

From equation (3.19) and (3.17)

$$-j_t(t) + h_x(x,t) = 2f(x,t) \tag{3.27}$$

$$-j_t(t) + h_x(x,t) - f(x,t) = 2\alpha^2 g(x,t). \tag{3.28}$$

Solving equation (3.27), (3.28)

$$f = 2\alpha^2 a.$$

Therefore from equation (3.22)

$$h_t(x,t) = -6a$$

$$h = -6at + k(x). \tag{3.29}$$

From (3.19), (3.23) we get

$$-j_t(t) + h_x(x,t) = 4\alpha^2 a \tag{3.30}$$

$$-j_t(t) + 3h_x(x,t) = 0. \tag{3.31}$$

After solving these equations (3.30)-(3.31) we get

$$h = -2\alpha^2 ax + n(t). \tag{3.32}$$

Therefore from (3.29) and (3.32) we get

$$h = -2\alpha^2 ax - 6at + b,$$

where a, b are arbitrary constants.

From (3.23) we get

$$j(t) = -6\alpha^2 a$$

$$j = -6\alpha^2 at + c,$$

where a, c are arbitrary constants

Therefore the value of f, g, h and j is

$$f = 2\alpha^2 a \ , \ g = a \ , \ j = -6\alpha^2 at + b$$

$$h = -2\alpha^2 ax - 6at + b.$$

Now put the value f, g, h and j in equation (3.11)

Then the solution of the determining equations is given by

$$\begin{aligned} &\xi(x,t) = -2\alpha^2 ax - 6at + b \\ &\tau(x,t) = -6\alpha^2 at + b \\ &\eta(x,t) = (2\alpha^2 a\ )u + a, \end{aligned} \tag{3.33}$$

where a,b,c are three arbitrary parameters. Hence, the point symmetry generators admitted by the Gardner equation (3.1) are

$$\begin{aligned} &V_1 = -2\alpha^2 x\frac{\partial}{\partial x} - 6t\frac{\partial}{\partial x} - 6\alpha^2 t\frac{\partial}{\partial t} + (2\alpha^2 u + 1)\frac{\partial}{\partial u} \\ &V_2 = \frac{\partial}{\partial x}, \quad V_3 = \frac{\partial}{\partial t}. \end{aligned} \tag{3.34}$$

3.2 Optimal System

The commutator table-3.35 and adjoint table-3.36 for Lie algebra (3.34) can easily be constructed as follows:

The commutator (Lie bracket) of V_α and V_β is first order operator defined by

$$\left[V_\alpha, V_\beta\right] = V_\alpha V_\beta - V_\beta V_\alpha.$$

Commutator Table-3.35

	V_1	V_2	V_3
V_1	0	$2\alpha^2 V_2$	$6(V_2 + \alpha^2 V_3)$
V_2	$-2\alpha^2 V_2$	0	0
V_3	$6(V_2 + \alpha^2 V_3)$	0	0

Formula of adjoint table is

$$Adj\left[exp(\varepsilon V_i)\right]V_j = V_j - \varepsilon\left[V_i, V_j\right] + \frac{\varepsilon^2}{2}\left[V_i, \left[V_i, V_j\right]\right] + \ldots,$$

Adjoint Table-3.36

Adj	V_1	V_2	V_3
V_1	V_1	$V_2(1 - 2\alpha^2 \varepsilon e^{\frac{-\varepsilon}{2}})$	$V_3 e^{-6\alpha^2\varepsilon} - 6\varepsilon V_2 e^{-8\alpha^2\varepsilon}$
V_2	$V_1 + 2\varepsilon\alpha^2 V_2$	V_2	V_3
V_3	$V_1 + 6\varepsilon V_2 + 6\alpha^2\varepsilon V_3$	V_2	V_3

We deduce an optimal system of sub algebras with their corresponding generators as follows:

(1) V_1
(2) $V_2 + \mu V_3$
(3) V_3 ,

where μ is arbitrary constant.

3.3 Reductions and Exact Solutions

In this section, corresponding to each generator in the optimal system of sub algebra, the reductions of PDEs (3.1) into ODEs in terms of similarity variable ξ and the new dependent variables- F, are obtained using the auxiliary equations (3.33). Some exact solutions of each of each reduced system are then attempted.

Generator (1)

The generator (1) is V_1 .

The corresponding characteristic equations are given by,

$$\frac{dx}{-2\alpha^2 x - 6t} = \frac{dt}{-6\alpha^2 t} = \frac{du}{2\alpha^2 u + 1}.$$

Thus the generator (1) in the optimal system defines the similarity variable and similarity solution as follows

$$\xi = t^{\frac{-1}{3}} x - \frac{3}{2\alpha^2} t^{\frac{2}{3}}$$

$$u(x,t) = \frac{1}{2\alpha^2}(t^{\frac{-1}{3}} F(\xi) - 1).$$

Using the similarity variable, the forms of the similarity solution, the system of partial differential equation (PDE) (3.1) reduces to the followings system of Ordinary differential equation (ODE)

$$6\alpha^2 F''' + 2\xi F'\alpha^2 + 2F\alpha^2 + 9F^2 F' = 0.$$

In this case we are able to find only reduction.

Generator (2)

The generator (2) is $V_2 + \mu V_3$.

The corresponding characteristic equations are given by,

$$\frac{dx}{1} = \frac{dt}{\mu} = \frac{du}{0}.$$

Thus the generator (2) in the optimal system defines the similarity variable and similarity solution as follows:

$$\xi = \mu x - t$$
$$u(x,t) = F(\xi).$$

Using the similarity variable, the forms of the similarity solution, the system of partial differential equation (PDE) (3.1) reduces to the following system of Ordinary Differential Equation (ODE):

$$F' + 6\mu F F' F''' + \alpha^2 F^2 F' + \mu^3 F''' = 0.$$

Using maple we obtain the exact solutions

$$F(\xi) = -\frac{1}{2\alpha^2} - \frac{1}{2}\frac{1}{\mu\alpha^2}\left(\sqrt{2}\sqrt{\mu(-2\alpha^2+3\mu)}\sec h\left(C_1 - \frac{1}{2}\frac{\sqrt{2}\sqrt{\mu(-2\alpha^2+3\mu)}\xi}{\mu^2\alpha}\right)\right),$$

$$F(\xi) = -\frac{1}{2\alpha^2} + \frac{1}{2}\frac{1}{\mu\alpha^2}\left(\sqrt{2}\sqrt{\mu(-2\alpha^2+3\mu)}\sec h\left(C_1 - \frac{1}{2}\frac{\sqrt{2}\sqrt{\mu(-2\alpha^2+3\mu)}\xi}{\mu^2\alpha}\right)\right),$$

$$F(\xi) = -\frac{1}{2\alpha^2} - \frac{1}{2}\frac{1}{\mu\alpha^2}\left(\sqrt{2}\sqrt{\mu(-2\alpha^2+3\mu)}\sec h\left(C_1 + \frac{1}{2}\frac{\sqrt{2}\sqrt{\mu(-2\alpha^2+3\mu)}\xi}{\mu^2\alpha}\right)\right),$$

$$F(\xi) = -\frac{1}{2\alpha^2} + \frac{1}{2}\frac{1}{\mu\alpha^2}\left(\sqrt{2}\sqrt{\mu(-2\alpha^2+3\mu)}\sec h\left(C_1 + \frac{1}{2}\frac{\sqrt{2}\sqrt{\mu(-2\alpha^2+3\mu)}\xi}{\mu^2\alpha}\right)\right),$$

$$F(\xi) = -\frac{1}{2\alpha^2} - \frac{1}{2}\frac{1}{\mu\alpha^2}\left(\sqrt{\mu(-2\alpha^2+3\mu)}\tanh\left(C_1 - \frac{1}{2}\frac{\sqrt{-\mu(-2\alpha^2+3\mu)}\xi}{\mu^2\alpha}\right)\right),$$

$$F(\xi) = -\frac{1}{2\alpha^2} + \frac{1}{2}\frac{1}{\mu\alpha^2}\left(\sqrt{\mu(-2\alpha^2+3\mu)}\tanh\left(C_1 - \frac{1}{2}\frac{\sqrt{-\mu(-2\alpha^2+3\mu)}\xi}{\mu^2\alpha}\right)\right),$$

$$F(\xi) = -\frac{1}{2\alpha^2} - \frac{1}{2}\frac{1}{\mu\alpha^2}\left(\sqrt{\mu(-2\alpha^2+3\mu)}\tanh\left(C_1 + \frac{1}{2}\frac{\sqrt{-\mu(-2\alpha^2+3\mu)}\xi}{\mu^2\alpha}\right)\right),$$

$$F(\xi) = -\frac{1}{2\alpha^2} + \frac{1}{2}\frac{1}{\mu\alpha^2}\left(\sqrt{\mu(-2\alpha^2+3\mu)}\tanh\left(C_1 + \frac{1}{2}\frac{\sqrt{-\mu(-2\alpha^2+3\mu)}\xi}{\mu^2\alpha}\right)\right),$$

Thus, the following solution of the equation (3.1) is obtained:

$$u(x,t)=-\frac{1}{2\alpha^{2}}-\frac{1}{2}\frac{1}{\mu\alpha^{2}}\left(\sqrt{2}\sqrt{\mu(-2\alpha^{2}+3\mu)}\sec h\left(C_{1}-\frac{1}{2}\frac{\sqrt{2}\sqrt{\mu(-2\alpha^{2}+3\mu)}}{\mu^{2}\alpha}\right.\right.$$

$$(\mu x-t)), \tag{3.3.1}$$

$$u(x,t)=-\frac{1}{2\alpha^{2}}+\frac{1}{2}\frac{1}{\mu\alpha^{2}}\left(\sqrt{2}\sqrt{\mu(-2\alpha^{2}+3\mu)}\sec h\left(C_{1}-\frac{1}{2}\frac{\sqrt{2}\sqrt{\mu(-2\alpha^{2}+3\mu)}}{\mu^{2}\alpha}\right.\right.$$

$$(\mu x-t)), \tag{3.3.2}$$

$$u(x,t)=-\frac{1}{2\alpha^{2}}-\frac{1}{2}\frac{1}{\mu\alpha^{2}}\left(\sqrt{2}\sqrt{\mu(-2\alpha^{2}+3\mu)}\sec h\left(C_{1}+\frac{1}{2}\frac{\sqrt{2}\sqrt{\mu(-2\alpha^{2}+3\mu)}(\mu x-t)}{\mu^{2}\alpha}\right)\right),$$

$$u(x,t)=-\frac{1}{2\alpha^{2}}+\frac{1}{2}\frac{1}{\mu\alpha^{2}}\left(\sqrt{2}\sqrt{\mu(-2\alpha^{2}+3\mu)}\sec h\left(C_{1}+\frac{1}{2}\frac{\sqrt{2}\sqrt{\mu(-2\alpha^{2}+3\mu)}(\mu x-t)}{\mu^{2}\alpha}\right)\right),$$

$$u(x,t)=-\frac{1}{2\alpha^{2}}-\frac{1}{2}\frac{1}{\mu\alpha^{2}}\left(\sqrt{\mu(-2\alpha^{2}+3\mu)}\tanh\left(C_{1}-\frac{1}{2}\frac{\sqrt{-\mu(-2\alpha^{2}+3\mu)}(\mu x-t)}{\mu^{2}\alpha}\right)\right),$$

$$u(x,t)=-\frac{1}{2\alpha^{2}}+\frac{1}{2}\frac{1}{\mu\alpha^{2}}\left(\sqrt{\mu(-2\alpha^{2}+3\mu)}\tanh\left(C_{1}-\frac{1}{2}\frac{\sqrt{-\mu(-2\alpha^{2}+3\mu)}(\mu x-t)}{\mu^{2}\alpha}\right)\right),$$

$$u(x,t)=-\frac{1}{2\alpha^{2}}-\frac{1}{2}\frac{1}{\mu\alpha^{2}}\left(\sqrt{\mu(-2\alpha^{2}+3\mu)}\tanh\left(C_{1}+\frac{1}{2}\frac{\sqrt{-\mu(-2\alpha^{2}+3\mu)}(\mu x-t)}{\mu^{2}\alpha}\right)\right),$$

$$u(x,t)=-\frac{1}{2\alpha^{2}}+\frac{1}{2}\frac{1}{\mu\alpha^{2}}\left(\sqrt{\mu(-2\alpha^{2}+3\mu)}\tanh\left(C_{1}+\frac{1}{2}\frac{\sqrt{-\mu(-2\alpha^{2}+3\mu)}(\mu x-t)}{\mu^{2}\alpha}\right)\right),$$

Generator (3)

The generator (3) is V_3.

The corresponding characteristic equations are given by,

$$\frac{dx}{0}=\frac{dt}{1}=\frac{du}{0}.$$

Thus the generator (2) in the optimal system defines the similarity variable and similarity solution as follows

$$\xi = x$$
$$u(x,t) = F(\xi).$$

Using the similarity variable, the forms of the similarity solution, the system of partial differential equation (PDE) (3.1) reduces to the following system of Ordinary Differential equation (ODE):

$$6FF'+6\alpha^{2}F^{2}F'+F'''=0.$$

After solving this equation we get

$$F(\xi)=-\frac{1}{2\alpha^2}-\frac{1}{2}\frac{\sqrt{6}\sec h\left(-C_1+\frac{1}{2}\frac{\sqrt{6}\xi}{\alpha}\right)}{\alpha^2},\ F(\xi)=-\frac{1}{2\alpha^2}+\frac{1}{2}$$

$$\frac{\sqrt{6}\sec h\left(-C_1+\frac{1}{2}\frac{\sqrt{6}\xi}{\alpha}\right)}{\alpha^2},\ F(\xi)=-\frac{1}{2\alpha^2}-\frac{1}{2}\frac{\sqrt{6}\sec h\left(C_1+\frac{1}{2}\frac{\sqrt{6}\xi}{\alpha}\right)}{\alpha^2},$$

$$F(\xi)=-\frac{1}{2\alpha^2}+\frac{1}{2}\frac{\sqrt{6}\sec h\left(C_1+\frac{1}{2}\frac{\sqrt{6}\xi}{\alpha}\right)}{\alpha^2},\ \ F(\xi)=-\frac{1}{2\alpha^2}-\frac{1}{2}$$

$$\frac{\sqrt{3}\tan h\left(-C_1+\frac{\frac{1}{2}i\sqrt{3}\xi}{\alpha}\right)}{\alpha^2},\ \ F(\xi)=-\frac{1}{2\alpha^2}+\frac{1}{2}\frac{\sqrt{3}\tan h\left(-C_1+\frac{\frac{1}{2}i\sqrt{3}\xi}{\alpha}\right)}{\alpha^2},$$

$$F(\xi)=-\frac{1}{2\alpha^2}-\frac{1}{2}\frac{\sqrt{3}\tan h\left(C_1+\frac{\frac{1}{2}i\sqrt{3}\xi}{\alpha}\right)}{\alpha^2},\ \ F(\xi)=-\frac{1}{2\alpha^2}+\frac{1}{2}$$

$$\frac{\sqrt{3}\tan h\left(C_1+\frac{\frac{1}{2}i\sqrt{3}\xi}{\alpha}\right)}{\alpha^2}.$$

Thus, the following solution of the equation (3.1) is obtained:

$$u(x,t)=-\frac{1}{2\alpha^2}-\frac{1}{2}\frac{\sqrt{6}\sec h\left(-C_1+\frac{1}{2}\frac{\sqrt{6}x}{\alpha}\right)}{\alpha^2},\ u(x,t)=-\frac{1}{2\alpha^2}+\frac{1}{2}$$

$$\frac{\sqrt{6}\sec h\left(-C_1+\frac{1}{2}\frac{\sqrt{6}x}{\alpha}\right)}{\alpha^2},\ \ u(x,t)=-\frac{1}{2\alpha^2}-\frac{1}{2}\frac{\sqrt{6}\sec h\left(C_1+\frac{1}{2}\frac{\sqrt{6}x}{\alpha}\right)}{\alpha^2},$$

$$u(x,t)=-\frac{1}{2\alpha^2}+\frac{1}{2}\frac{\sqrt{6}\sec h\left(C_1+\frac{1}{2}\frac{\sqrt{6}x}{\alpha}\right)}{\alpha^2},\ \ u(x,t)=-\frac{1}{2\alpha^2}-\frac{1}{2}$$

$$\frac{\sqrt{3}\tan h\left(-C_1+\frac{\frac{1}{2}i\sqrt{3}x}{\alpha}\right)}{\alpha^2},\ u(x,t)=-\frac{1}{2\alpha^2}+\frac{1}{2}\frac{\sqrt{3}\tan h\left(-C_1+\frac{\frac{1}{2}i\sqrt{3}x}{\alpha}\right)}{\alpha^2},$$

$$u(x,t)=-\frac{1}{2\alpha^2}-\frac{1}{2}\frac{\sqrt{3}\tan h\left(C_1+\frac{\frac{1}{2}i\sqrt{3}x}{\alpha}\right)}{\alpha^2},\ u(x,t)=-\frac{1}{2\alpha^2}+\frac{1}{2}$$

$$\frac{\sqrt{3}\tan h\left(C_1+\frac{\frac{1}{2}i\sqrt{3}x}{\alpha}\right)}{\alpha^2},$$

where C_1 is constant.

CHAPTER-4

FISHER’S EQUATION

In mathematics, Fisher’s equation is also known as the Fisher Kolmogorov equation and the Fisher –KPP equation, named after R. A. Fisher and A. N. Kolmogorov [9, 10] is the partial differential equation

$$\frac{\partial u}{\partial t}=u(1-u)+\frac{\partial^2 u}{\partial x^2}.$$

There are many methods to solve Fisher’s equation, but each method can only lead to a special solution. A method namely, the exp-function method, is employed to solve th Fisher’s equation by Xin-Wei Zhou. Aronson and Weinberger’s work on the Fisher type nonlinear diffusion equation [3] has shown the existence of distinct selection mechanism, that is the solution $u(x,t)$ of the Fisher equation in (1+1) dimensions,

$$u_t=u_{xx}+u(1-u), \qquad (a)$$

converges to a local traveling wave with a definite speed from a wide class of initial data. Further it is known that equation (a) has a traveling wave solution called a cline [26] which is nothing but a wave traveling in the x-direction $c \geq c_{\min}=2.$

4.1 Essential Vector Field and Optimal System

Consider the Fisher’s equation

$$u_t=u(1-u)-u_{xx}=0. \qquad (4.1)$$

Let the group of infinitesimal transformations be defined as

$$u^*=u+\varepsilon\eta(x,t,u)+O(\varepsilon^2)$$
$$x^*=x+\varepsilon\xi(x,t,u)+O(\varepsilon^2)$$
$$t^*=t+\varepsilon\tau(x,t,u)+O(\varepsilon^2).$$

On invoking the invariance criterion the following relation from the coefficients of the first order of ε is deduced,

$$\eta^t=\eta-2u\eta+\eta^{xx}, \qquad (4.2)$$

where η^t, η^{xx} are infinitesimal generator corresponding to u_t, u_{xx}

Substitute the values of η^t, η^{xx} in (4.2) we get,

$$\left(\frac{\partial\eta}{\partial t}-\eta+2u\eta-\frac{\partial^2\eta}{\partial x^2}\right)+\left(\frac{\partial\eta}{\partial u}-\frac{\partial^2\tau}{\partial x^2}-\frac{\partial\tau}{\partial t}\right)\theta+\left(\frac{\partial\eta}{\partial u}-\frac{\partial\tau}{\partial t}-\frac{\partial\tau}{\partial u}-\frac{\partial^2\tau}{\partial x^2}\right)\theta^2+\left(-\frac{\partial\tau}{\partial t}-\frac{\partial^2\tau}{\partial x^2}+\theta_{xx}\right.$$
$$\left.+2\frac{\partial\xi}{\partial x}\right)\theta_{xx}+\left(-\frac{\partial\xi}{\partial t}-2\frac{\partial^2\eta}{\partial x\partial u}+\frac{\partial^2\xi}{\partial x^2}\right)\theta_x+2\frac{\partial\tau}{\partial u}\theta^3-\frac{\partial\tau}{\partial u}\theta\theta_{xx}+\frac{\partial\tau}{\partial u}\theta^2\theta_{xx}+\left(-\frac{\partial\xi}{\partial u}+2\frac{\partial^2\tau}{\partial x\partial u}\right)\theta\theta_x$$
$$+\left(-\frac{\partial\xi}{\partial u}-2\frac{\partial^2\tau}{\partial x\partial u}\right)\theta^2\theta_x+\left(-\frac{\partial\xi}{\partial u}+2\frac{\partial^2\tau}{\partial x\partial u}+3\frac{\partial\xi}{\partial u}\right)\theta_x\theta_{xx}+\left(-\frac{\partial^2\eta}{\partial u^2}+2\frac{\partial^2\xi}{\partial x\partial u}\right)\theta_x^2+\frac{\partial^2\xi}{\partial u^2}\theta_x^3+\frac{\partial^2\tau}{\partial u^2}$$
$$\theta\theta_x^2-\frac{\partial^2\tau}{\partial u^2}\theta^2\theta_x^2+\frac{\partial^2\tau}{\partial u^2}\theta_x^2\theta_{xx}+2\frac{\partial\tau}{\partial x}\theta_{xt}+2\frac{\partial\tau}{\partial u}\theta_{xt}\theta_x=0$$

Equate the coefficient of $\theta, \theta^2, \theta_x, \theta_{xx}$ etc and so on we get,

$$\frac{\partial\tau}{\partial u}=\frac{\partial\tau}{\partial x}=\frac{\partial\xi}{\partial u}=0 \tag{4.3}$$

$$\left(\frac{\partial\eta}{\partial t}-\eta+2u\eta-\frac{\partial^2\eta}{\partial x^2}\right)=0 \tag{4.4}$$

$$\left(\frac{\partial\eta}{\partial u}-\frac{\partial\tau}{\partial t}\right)=0 \tag{4.5}$$

$$\left(-\frac{\partial\tau}{\partial t}+2\frac{\partial\xi}{\partial x}\right)=0 \tag{4.6}$$

$$\left(-\frac{\partial\xi}{\partial t}-2\frac{\partial^2\eta}{\partial x\partial u}+\frac{\partial^2\xi}{\partial x^2}\right)=0 \tag{4.7}$$

$$\left(-\frac{\partial^2\eta}{\partial u^2}+2\frac{\partial^2\xi}{\partial x\partial u}\right)=0 \tag{4.8}$$

$$\frac{\partial^2\xi}{\partial u^2}=0 \tag{4.9}$$

From equation (4.3) we get

$$\tau = j(t),\ \xi = h(x,t) \tag{4.10}$$

From equation (4.8) we get

$$\frac{\partial^2\eta}{\partial u^2}=0$$
$$\eta = f(x,t)u+g(x,t) \tag{4.11}$$

Now put the value of τ, ξ, η in above coefficients say (4.4)-(4.9)

$$u_t f_t(x,t)+g_t(x,t)-uf(x,t)-g(x,t)-2u^2 f(x,t)$$
$$-2ug(x,t)-u_{xx} f_{xx}(x,t)-g_{xx}(x,t)=0 \tag{4.12}$$

$$f(x,t) - j_t(t) = 0 \tag{4.13}$$

$$j_t(t) + 2h_x(x,t) = 0 \tag{4.14}$$

$$h_t(x,t) - 2f_x(x,t) + h_{xx}(x,t) = 0. \tag{4.15}$$

By equating the coefficient of u^2 in equation (4.12) we get

$$f(x,t) = 0. \tag{4.16}$$

On equating the coefficient of u in equation (4.12)

$$-f(x,t) - 2g(x,t) = 0.$$

From equation (4.16)

$$g(x,t) = 0. \tag{4.17}$$

From equation (4.13)

$$j_t(t) = 0$$
$$j = a,$$

where a is constant.

From equation (4.14) we get

$$2h_x(x,t) = 0 \tag{4.18}$$
$$h = b(t).$$

From equation (4.15)

$$h_t(x,t) = 0.$$

From equation (4.18)

h is constant say $h = c.$

Therefore $f = g = 0,\ j = a,\ h = c.$

Put the value of f, g, h, j in above coefficients (4.10)-(4.11)

Then the solution of the determining equations is given by

$$\xi(x,t) = a$$

$$\tau(x,t) = c, \tag{4.19}$$

where a, c are two arbitrary parameters. Hence, the point symmetry generators admitted by the Fisher’s equation (4.1) are

$$V_1 = \frac{\partial}{\partial x}, \quad V_2 = \frac{\partial}{\partial t}. \tag{4.20}$$

Optimal system

The commutator and adjoint resresentation table Lie algebra (4.20) can easily constructed as follows:

Commutator Table-4.21

	V_1	V_2
V_1	0	0
V_2	0	0

Adjoint Table-4.22

Adj	V_1	V_2
V_1	V_1	V_2
V_2	V_1	V_2

We deduce an optimal system of sub algabras with their corresponding essential vector field:

(1) $\mu V_1 + V_2$

(2) V_1, (4.23)

where μ is arbitrary constant.

4.2 Reductions of Optimal System and Exact Solutions

In this section, we use the method of characteristics to determine the invariants and reduced ODEs corresponding to each subalgebra given in (4.23). Symmetry variables and the invariants of the subalgebras of the Lie algebra is described as below. The result of this can be summarized as follows, where ξ is similarity variable, $F(\xi)$ is invariant function related to u, and have to be determined using the reduced ODEs.

Essential Vector Field (1)

The Essential Vector Field (1) is $\mu V_1 + V_2$.

The corresponding characteristic equations are given by,

$$\frac{dx}{\mu} = \frac{dt}{1} = \frac{du}{0}.$$

Thus the Essential Vector Field (1) in the optimal system defines the similarity variable and similarity solution as follows:

$$\xi = \mu x - t$$
$$u(x,t) = F(\xi).$$

Using the similarity variable, the forms of the similarity solution, the system of partial differential equation (PDE) (4.1) reduces to the followings system of Ordinary differential equation (ODE):

$$F - F^2 + F' + \mu^2 F'' = 0.$$

In this case we able to find only reduction.

Essential Vector Field (2)

The Essential Vector Field (2) is V_1.

The corresponding characteristic equations are given by,

$$\frac{dx}{1} = \frac{dt}{0} = \frac{du}{0}.$$

Thus the Essential Vector Field (2) in the optimal system defines the similarity variable and similarity solution as follows:

$$\xi = x$$
$$u(x,t) = F(\xi).$$

Using the similarity variable, the forms of the similarity solution, the system of partial differential equation (PDE) (4.1) reduces to the followings system of ordinary differential equation (ODE):

$$F(1-F) + F'' = 0.$$

Using maple we obtain the exact solutions

$$F(\xi) = 1 - \frac{3}{2}\sec h\left(-C_1 + \frac{1}{2}\xi\right)^2, \quad F(\xi) = 1 - \frac{3}{2}\sec h\left(C_1 + \frac{1}{2}\xi\right)^2,$$

$$F(\xi) = \frac{3}{2}\sec h\left(-C_1 + \frac{1}{2}i\xi\right)^2, \quad F(\xi) = \frac{3}{2}\sec h\left(C_1 + \frac{1}{2}i\xi\right)^2,$$

$$F(\xi) = \frac{1}{2} + \frac{3}{2}\tan h\left(-C_1 + \frac{1}{2}\xi\right)^2, \quad F(\xi) = -\frac{1}{2} + \frac{3}{2}\tan h\left(C_1 + \frac{1}{2}\xi\right)^2,$$

$$F(\xi) = \frac{3}{2} - \frac{3}{2}\tan h\left(-C_1 + \frac{1}{2}\xi\right)^2, \quad F(\xi) = \frac{3}{2} - \frac{3}{2}\tan h\left(C_1 + \frac{1}{2}\xi\right)^2.$$

Thus the solution of equation (4.1) is as follows:

$$u(x,t) = 1 - \frac{3}{2}\sec h\left(-C_1 + \frac{1}{2}x\right)^2,\ u(x,t) = 1 - \frac{3}{2}\sec h\left(C_1 + \frac{1}{2}x\right)^2,$$

$$u(x,t) = \frac{3}{2}\sec h\left(-C_1 + \frac{1}{2}ix\right)^2,\ u(x,t) = \frac{3}{2}\sec h\left(C_1 + \frac{1}{2}ix\right)^2,$$

$$u(x,t) = \frac{1}{2} + \frac{3}{2}\tan h\left(-C_1 + \frac{1}{2}x\right)^2,\ u(x,t) = -\frac{1}{2} + \frac{3}{2}\tan h\left(C_1 + \frac{1}{2}x\right)^2,$$

$$u(x,t) = \frac{3}{2} - \frac{3}{2}\tan h\left(-C_1 + \frac{1}{2}x\right)^2,\ u(x,t) = \frac{3}{2} - \frac{3}{2}\tan h\left(C_1 + \frac{1}{2}x\right)^2,$$

where C_1 is constant.

CHAPTER-5

DRINFEL'D SOKOLOV WILSON EQUATION

Drinfeld Sokolov Wilson System:

$$u_t = 3ww_x$$
$$w_t = 2w_{xxx} + 2uw_x + u_x w.$$

Zhao Xue-Qin and Zhi Hong-Yan [33] solved the Drinfeld sokolov Wilson equation by F-expansion method. It is also solved for Darboux Transformation and Explicit Solutions by Gen Xian-Guo and Wu Li-hua [20]. Analytical solutions of Drinfeld equation also found by Erik Sweet and Robert A.van Gorder [16].

5.1 Symmetries

Consider the Drinfel'd-Sokolov-Wilson equation

$$\begin{aligned} u_t &= 3ww_x \\ w_t &= 2w_{xxx} + 2uw_x + u_x w. \end{aligned} \tag{5.1}$$

Let the group of infinitesimal transformations be defined as

$$u^* = u + \varepsilon\eta(x,t,u) + O(\varepsilon^2)$$
$$x^* = x + \varepsilon\xi(x,t,u) + O(\varepsilon^2)$$
$$t^* = t + \varepsilon\tau(x,t,u) + O(\varepsilon^2).$$

On invoking the invariance criterion the following relation from the coefficients of the first order of ε is deduced,

$$\eta^t - 3w\beta^x - 3w_x\beta = 0$$
$$\beta^t - 2\beta^{xxx} - 2u\beta^x - 2\eta w_x - \eta^x w - u_x\beta = 0,$$

First we solve 1st determining equation i.e.

$$\eta_t - 3w\beta_x - 3w_x\beta = 0, \tag{5.2}$$

where $\eta^t, \eta^x, \beta^t, \beta^x, \beta^{xxx}$ are infinitesimal generator corresponding to $u_t, u_x, w_t, w_x, w_{xxx}$

Substitute the values of $\eta^t, \eta^x, \beta^t, \beta^x, \beta^{xxx}$ in (5.2) we get

$$\frac{\partial\eta}{\partial t}-3w\frac{\partial\beta}{\partial x}+\left(3w\left(\frac{\partial\eta}{\partial u}-\frac{\partial\tau}{\partial t}-\frac{\partial\beta}{\partial w}+\frac{\partial\xi}{\partial x}\right)-3\beta\right)\alpha_x+\left(-\frac{\partial\xi}{\partial t}-3w\frac{\partial\beta}{\partial u}\right)\theta_x+\left(-9\frac{\partial\tau}{\partial u}\right.$$

$$\left.w^2+3w\frac{\partial\xi}{\partial w}\right)\alpha_x^2-2w\frac{\partial\tau}{\partial w}\alpha_x\alpha_t+\left(\frac{\partial\eta}{\partial w}+3w\frac{\partial\tau}{\partial x}\right)\alpha_t+\left(-\frac{\partial\xi}{\partial w}+3w\frac{\partial\tau}{\partial u}\right)\alpha_t\theta_x=0$$

Equate the coefficient of $\alpha_t,\alpha_x,\theta_x,\alpha_x^2,\alpha_t\theta_x$ equal to zero and we obtain

$$\frac{\partial\eta}{\partial t}-3w\frac{\partial\beta}{\partial x}=0 \tag{5.3}$$

$$\left(3w\left(\frac{\partial\eta}{\partial u}-\frac{\partial\tau}{\partial t}-\frac{\partial\beta}{\partial w}+\frac{\partial\xi}{\partial x}\right)-3\beta\right)=0 \tag{5.4}$$

$$\left(-\frac{\partial\xi}{\partial t}-3w\frac{\partial\beta}{\partial u}\right)=0 \tag{5.5}$$

$$\left(-9\frac{\partial\tau}{\partial u}w^2+3w\frac{\partial\xi}{\partial w}\right)=0 \tag{5.6}$$

$$-2w\frac{\partial\tau}{\partial w}=0 \tag{5.7}$$

$$\left(\frac{\partial\eta}{\partial w}+3w\frac{\partial\tau}{\partial x}\right)=0 \tag{5.8}$$

$$\left(-\frac{\partial\xi}{\partial w}+3w\frac{\partial\tau}{\partial u}\right)=0. \tag{5.9}$$

Now we solve the symmetry determining equation

$$\beta^t-2\beta^{xxx}-2u\beta^x-2\eta w_x-\eta^x w-u_x\beta=0.$$

$$\left(\frac{\partial\beta}{\partial t}-2\frac{\partial^3\beta}{\partial x^3}-2u\frac{\partial\beta}{\partial x}-w\frac{\partial\eta}{\partial x}\right)+\left(2u\left(\frac{\partial\beta}{\partial w}-\frac{\partial\tau}{\partial t}\right)-6\frac{\partial^3\beta}{\partial x^2\partial w}+2\frac{\partial^3\xi}{\partial x^3}+4u\frac{\partial^3\tau}{\partial x^3}-2u\frac{\partial\beta}{\partial w}+2u\frac{\partial\xi}{\partial x}\right.$$

$$\left.-2\eta-w\frac{\partial\eta}{\partial w}+4u\frac{\partial\tau}{\partial x}-\frac{\partial\xi}{\partial t}\right)\alpha_x+\left(-2\frac{\partial\tau}{\partial t}+4\frac{\partial^3\tau}{\partial x^3}+6\frac{\partial\xi}{\partial x}+4u\frac{\partial\tau}{\partial x}\right)\alpha_{xxx}+\left[-4u^2\frac{\partial\tau}{\partial w}-2u\frac{\partial\xi}{\partial w}+\right.$$

$$\left.12u\frac{\partial^3\tau}{\partial x^3\partial w}-6\left(\frac{\partial^3\beta}{\partial x\partial w^2}-\frac{\partial^3\xi}{\partial x^2\partial w}\right)+2u\frac{\partial\xi}{\partial w}+4u^2\frac{\partial\tau}{\partial w}\right]\alpha_x^2+\left[-6\frac{\partial\xi}{\partial w}+12\frac{\partial^3\tau}{\partial x^2\partial w}+8\right]\alpha_x\alpha_{xxx}+$$

$$\left[w\left(\frac{\partial\beta}{\partial w}-\frac{\partial\tau}{\partial t}\right)+2w\frac{\partial^2\tau}{\partial x^3}-2\left(\frac{\partial^3\beta}{\partial x^2\partial u}+\frac{\partial^3\beta}{\partial u^2\partial x}\right)-2uw\frac{\partial\tau}{\partial x}-2u\frac{\partial\beta}{\partial u}-w\left(\frac{\partial\eta}{\partial u}-\frac{\partial\xi}{\partial x}\right)\right]\theta_x+\left[-w^2\right.$$

$$\left.\frac{\partial\tau}{\partial w}+6w\frac{\partial^3\tau}{\partial x^2\partial u}-2\left(3\frac{\partial^3\beta}{\partial u^2\partial x}\right)+2uw\frac{\partial\tau}{\partial u}+w\frac{\partial\xi}{\partial u}\right]\theta_x^2+\left[-4uw\frac{\partial\tau}{\partial w}-w\frac{\partial\xi}{\partial w}-2\left(6\frac{\partial^3\beta}{\partial x\partial w\partial u}-3\right.\right.$$

$$\left.\left.\frac{\partial^3\xi}{\partial x^2\partial u}\right)+12u\frac{\partial^3\tau}{\partial x^2\partial u}+2uw\frac{\partial\tau}{\partial w}+u\frac{\partial\xi}{\partial u}+4u^2\frac{\partial\tau}{\partial u}+w\frac{\partial\xi}{\partial w}+6w\frac{\partial^3\tau}{\partial x^2\partial w}\right]\alpha_x\theta_x+\left[-4w\frac{\partial\tau}{\partial w}+2w\right.$$

$$\left.\frac{\partial\tau}{\partial w}+6\frac{\partial\xi}{\partial u}+24\frac{\partial^3\tau}{\partial x^2\partial u}+4u\frac{\partial\tau}{\partial u}\right]\theta_x\alpha_{xxx}+\left(\frac{\partial\beta}{\partial u}+w\frac{\partial\tau}{\partial x}\right)\theta_t+\left[-2u\frac{\partial\tau}{\partial u}+w\frac{\partial\tau}{\partial w}-\frac{\partial\xi}{\partial u}\right]\alpha_x\theta_t+$$

$$12u\frac{\partial^3\tau}{\partial w^2\partial x} - 2\left(\frac{\partial^3\beta}{\partial w^3} - 3\frac{\partial^3\xi}{\partial w^2\partial x}\right)\alpha_x^3 - 2\frac{\partial\tau}{\partial u}\alpha_{xxx}\theta_t + 12\frac{\partial^3\xi}{\partial w^2\partial x}\alpha_{xxx}\alpha_x^2 + \Bigg[6w\frac{\partial^3\tau}{\partial w^2\partial x} - 2\Bigg(3\frac{\partial^3\beta}{\partial w^2\partial u}$$

$$-6\frac{\partial^3\xi}{\partial x\partial w\partial u}\Bigg) + 24u\frac{\partial^3\tau}{\partial x\partial w\partial u}\Bigg]\alpha_x^2\theta_x + 2\frac{\partial^3\tau}{\partial w^3}\alpha_x^3\alpha_{xxx} + \left(4u\frac{\partial^3\tau}{\partial w^3} + 2\frac{\partial^3\xi}{\partial w^3}\right)\alpha_x^4 + \Bigg(\Bigg[4u\frac{\partial^3\tau}{\partial w^3} + 2$$

$$\frac{\partial^3\xi}{\partial w^3}\Bigg)\alpha_x^4 + 2w\frac{\partial^3\tau}{\partial w^3} + 6\frac{\partial^3\xi}{\partial w^2\partial u} + 12u\frac{\partial^3\tau}{\partial w^2\partial u}\Bigg]\alpha_x^3\theta_x + 6\frac{\partial^2\tau}{\partial x^2}\alpha_{xt} - 6\left(\frac{\partial^2\beta}{\partial u\partial x} - \frac{\partial^2\xi}{\partial x^2}\right)\alpha_{xx} + \Bigg[-6\Bigg(\frac{\partial^2\beta}{\partial w^2}$$

$$-3\frac{\partial^2\xi}{\partial x\partial w}\Bigg) + 12u\frac{\partial^2\xi}{\partial x\partial w}\Bigg]\alpha_{xx}\alpha_{xxx} + \left[6w\frac{\partial^2\tau}{\partial x\partial w} - \left(6\frac{\partial^2\beta}{\partial u\partial w} - 12\frac{\partial^2\xi}{\partial x\partial u}\right)\right]\theta_x\alpha_{xx} + 6\frac{\partial\tau}{\partial u}\theta_x\alpha_{xxt} + 12\frac{\partial^2\tau}{\partial x\partial w}$$

$$\alpha_{xt}\alpha_x + 6\frac{\partial^2\tau}{\partial w^2}\alpha_{xt}\alpha_x^2 + \left(12\frac{\partial^2\xi}{\partial w^2} + 12u\frac{\partial^2\tau}{\partial w^2}\right)\alpha_{xx}\alpha_x^2 + 6\frac{\partial\tau}{\partial w}\alpha_{xx}\alpha_{xt} + 6\frac{\partial\tau}{\partial w}\alpha_{xxt}\alpha_x + 24u\frac{\partial^3\tau}{\partial x\partial u\partial w}\alpha_x$$

$$\alpha_{xxx}\theta_x + 12w\frac{\partial^3\tau}{\partial x\partial u\partial w}\theta_x^2\alpha_x + \left(6w\frac{\partial^3\tau}{\partial u\partial w^2} + 6\frac{\partial^3\xi}{\partial x\partial u^2} + 12u\frac{\partial^3\tau}{\partial w\partial u^2}\right)\theta_x^2\alpha_x^2 + \left(\frac{\partial\beta}{\partial u}\right)\theta_{xxx} + 12\frac{\partial^2\tau}{\partial w^2}$$

$$\alpha_x\alpha_{xx}\alpha_{xxx} + \left(6w\frac{\partial^2\tau}{\partial w^2} + 2\left(5\frac{\partial^2\xi}{\partial u\partial w} - 4\frac{\partial^3\xi}{\partial u\partial w^2}\right) + 12u\frac{\partial^2\tau}{\partial u\partial w}\right)\alpha_x\alpha_{xx}\theta_x + 6\frac{\partial\xi}{\partial w}\alpha_{xx}^2 + 6\frac{\partial\tau}{\partial x}\alpha_{xxt} + 12$$

$$\frac{\partial^3\tau}{\partial u\partial w^2}\alpha_x^2\alpha_{xxx}\theta_x + 12\frac{\partial^2\tau}{\partial u\partial x}\theta_x\alpha_{xt} + 12\frac{\partial^2\tau}{\partial u\partial w}\alpha_{xx}\alpha_{xxx}\theta_x + \left(6w\frac{\partial^2\tau}{\partial u\partial w} + 6\frac{\partial^2\xi}{\partial u^2}\right)\theta_x^2\alpha_{xx} + 12\frac{\partial^2\tau}{\partial u\partial x}\alpha_x$$

$$\alpha_{xt}\theta_x + 6\frac{\partial^2\tau}{\partial u^2}\theta_x^2\alpha_{xt} + \left[-6\frac{\partial^3\xi}{\partial x\partial u^2} + 12u\frac{\partial^3\tau}{\partial u\partial u^2} + 6w\frac{\partial^3\tau}{\partial w\partial u^2} - 3\frac{\partial^3\beta}{\partial w\partial u^2}\right]\theta_x^2\alpha_x + \Bigg[-6\left(\frac{\partial^2\beta}{\partial w\partial u} - \frac{\partial^2\xi}{\partial x\partial u}\right)$$

$$+12u\frac{\partial^2\tau}{\partial u\partial x}\Bigg]\alpha_x\theta_{xx} + \left[2\left(3\frac{\partial^2\xi}{\partial u\partial w}\right) + 12u\frac{\partial^2\tau}{\partial u\partial w}\right]\theta_{xx}\alpha_x^2 + 12\frac{\partial^3\tau}{\partial x\partial u^2}\theta_x^2\alpha_{xx} + 12\frac{\partial^2\tau}{\partial x\partial u}\theta_{xx}\alpha_{xtx} +$$

$$\left[6\frac{\partial^2\xi}{\partial u^2} + 12u\frac{\partial^2\tau}{\partial u\partial x} + 6w\frac{\partial^2\tau}{\partial u\partial w}\right]\theta_{xx}\alpha_x\theta_x + \left[12w\frac{\partial^2\tau}{\partial u\partial x} - 6\frac{\partial^2\beta}{\partial u^2}\right]\theta_{xx}\theta_x + \left[6w\frac{\partial^3\tau}{\partial x\partial u^2} + \frac{\partial^3\beta}{\partial u^3}\right]\theta_x^3 +$$

$$\left[4u\frac{\partial^3\tau}{\partial u^3} + 2\frac{\partial^3\xi}{\partial u^3}\right]\theta_x^3\alpha_x + 6w\frac{\partial^2\tau}{\partial u^2}\theta_x^2\theta_{xx} + 6\frac{\partial\tau}{\partial u}\theta_{xx}\alpha_{xt} + 6\frac{\partial\xi}{\partial u}\theta_{xx}\alpha_{xxx} + 2\frac{\partial^3\tau}{\partial u^3}\theta_x^3\alpha_{xxx} + \Bigg(2\frac{\partial\xi}{\partial u} +$$

$$4u\frac{\partial\tau}{\partial u}\Bigg)\theta_{xxx}\alpha_x + 6\frac{\partial^2\tau}{\partial u^2}\theta_x\theta_{xx}\alpha_{xxx} + 2w\frac{\partial^3\tau}{\partial u^3}\theta_x^4 + 12\frac{\partial^3\tau}{\partial u^2\partial w}\theta_x^2\alpha_x\alpha_{xxx} + 12\frac{\partial^3\tau}{\partial u^2\partial w}\theta_{xx}\alpha_x\alpha_{xxx} + 4\frac{\partial\tau}{\partial u}$$

$$\theta_{xxx}\alpha_{xxx} - 6\frac{\partial^2\beta}{\partial x\partial u}\theta_{xx} = 0.$$

Now equate the coefficients of $\alpha_x, \alpha_{xt}, \theta_{xxx}\alpha_{xxx}, \theta_{xx}, \theta_x^2\alpha_x\alpha_{xxx}$ and equal to zero we get,

$$\frac{\partial\tau}{\partial u} = \frac{\partial\tau}{\partial w} = \frac{\partial\tau}{\partial x} = 0 \tag{5.10}$$

$$\frac{\partial\xi}{\partial w} = \frac{\partial\xi}{\partial u} = 0 \tag{5.11}$$

$$\frac{\partial\beta}{\partial u} = 0 \tag{5.12}$$

$$\left(\frac{\partial\beta}{\partial t} - 2\frac{\partial^3\beta}{\partial x^3} - 2u\frac{\partial\beta}{\partial x} - w\frac{\partial\eta}{\partial x}\right) = 0 \tag{5.13}$$

$$\left(-\frac{\partial \tau}{\partial t}-6\frac{\partial^3\beta}{\partial x^2\partial w}+2\frac{\partial^3\xi}{\partial x^3}-2u\frac{\partial\beta}{\partial w}+2u\frac{\partial\xi}{\partial x}-2\eta-w\frac{\partial\eta}{\partial w}-\frac{\partial\xi}{\partial t}\right)=0 \tag{5.14}$$

$$\left(-2\frac{\partial\tau}{\partial t}+6\frac{\partial\xi}{\partial x}\right)=0 \tag{5.15}$$

$$-6\frac{\partial^3\beta}{\partial x\partial w^2}=0 \tag{5.16}$$

$$\left[w\left(\frac{\partial\beta}{\partial w}-\frac{\partial\tau}{\partial t}\right)-\left(\frac{\partial\eta}{\partial u}-\frac{\partial\xi}{\partial x}\right)-\beta=0 \right. \tag{5.17}$$

$$-2\frac{\partial^3\beta}{\partial w^3}=0 \tag{5.18}$$

$$6\frac{\partial^2\xi}{\partial x^2}=0 \tag{5.19}$$

$$-6\frac{\partial^2\beta}{\partial w^2}=0. \tag{5.20}$$

From (5.5), (5.8) we get

$$\frac{\partial\xi}{\partial t}=0\ ,\ \frac{\partial\eta}{\partial w}=0 \tag{5.21}$$

Also from (5.10), (5.11), (5.12) and (5.21) we get

$$\tau=j(t)\ ,\ \xi=h(x)\ ,\ \eta=k(x,t,u)\ ,\ \beta=wf(x,t)+g(x,t). \tag{5.22}$$

Put the value of τ,ξ,η and β in equations (5.13)-(5.21) and (5.3)-(5.9) we get

$$\begin{aligned}&wf_t(x,t)+g_t(x,t)-2wf_{xxx}(x,t)-2g_{xxx}(x,t)-2uwf_x(x,t)-2ug_x(x,t)\\&-wk_x(x,t,u)=0\end{aligned} \tag{5.23}$$

$$2uj_t(t)-6f_{xx}(x,t)+2h_{xxx}(x)+2uh_x(x)-2k(x,t,u)=0 \tag{5.24}$$

$$w(f(x,t)-j_t(t))-wf(x,t)-g-w(k_u(x,t,u)-h_x(x))=0 \tag{5.25}$$

$$-2j_t(t)+6h_x(x)=0 \tag{5.26}$$

$$6h_{xx}(x)=0 \tag{5.27}$$

$$k_t(x,t,u)-3w^2f_x(x,t)-3wg_x(x,t)=0 \tag{5.28}$$

$$3w(k_u(x,t,u)-j_t(t)-f(x,t)+h_x(x))-3wf(x,t)-3g(x,t)=0. \tag{5.29}$$

By equating the coefficient of u in equation (5.23), (5.24)

$$-2g_x=0 \tag{5.30}$$

$$2j_t+2h_x=0. \tag{5.31}$$

On equating the coefficient of w in (5.23), (5.25) and (5.29)

$$f_t(x,t) - 2f_{xxx}(x,t) - k_x(x,t,u) = 0 \tag{5.32}$$

$$f(x,t) - j_t(t)) - f(x,t) - k_u(x,t,u) - h_x(x) = 0 \tag{5.33}$$

$$3(k_u(x,t,u) - j_t(t) - f(x,t) + h_x(x) - f(x,t)) = 0. \tag{5.34}$$

Equating the coefficient of w^2 in equation (5.28) we get

$$f_x(x,t) = 0.$$

By equating the coefficients of without u and w in equation (5.23)-(5.29) we get

$$g_t(x,t) - 2g_{xxx}(x,t) = 0 \tag{5.35}$$

$$-6f_{xx}(x,t) + 2h_{xxx}(x) - 2k(x,t,u) = 0 \tag{5.36}$$

$$-g(x,t) = 0 \tag{5.37}$$

$$-2j_t(t) - 6h_x(x) = 0 \tag{5.38}$$

$$-6(k_{xu} - h_{xx}) = 0 \tag{5.39}$$

$$k_t(x,t,u) = 0 \tag{5.40}$$

$$3g(x,t) = 0. \tag{5.41}$$

Now we get $f_x(x,t) = 0$, $k_t(x,t,u) = 0, g(x,t) = 0.$

From equation (5.31) and (5.38) we get

$$2j_t + 2h_x = 0 \tag{5.42}$$

$$-2j_t + 6h_x = 0. \tag{5.43}$$

After solving the equations from (5.42) and (5.43)

$$h = a\ ,\ j = b.$$

From equation (5.36) we get $k = 0.$

Therefore the value of h, j, k, f and g is

$$h = a\ ,\ j = b\ ,\ k = f = g = 0.$$

Then put the value of h, j, k, f and g in (5.22)

Then the solution of the determining equations is given by

$$\xi = a\,,\ \tau = b, \tag{5.44}$$

where a, b are two arbitrary parameters. Hence, the point symmetry generators admitted by the Drinfeld Sokolov Wilson equation (5.1) are

$$V_1 = \frac{\partial}{\partial x}\,,\ V_2 = \frac{\partial}{\partial t}\ . \tag{5.45}$$

5.2 Optimal System

The commutator table-5.46 and adjoint table-5.47 for Lie algebra (5.44) can easily constructed as follows:

The commutator (Lie bracket) of X_α and X_β is first order operator defined by

$$\left[X_\alpha, X_\beta\right] = X_\alpha X_\beta - X_\beta X_\alpha .$$

Commutator Table-5.46

	V_1	V_2
V_1	0	0
V_2	0	0

Formula of adjoint table is

$$Adj\left[exp(\varepsilon V_i)\right]V_j = V_j - \varepsilon\left[V_i, V_j\right] + \frac{\varepsilon^2}{2}\left[V_i,\left[V_i, V_j\right]\right] + \ldots,$$

Adjoint Table-5.47

Adj	V_1	V_2
V_1	V_1	V_2
V_2	V_1	V_2

We deduce an optimal system of sub algebras with their corresponding generators as follows:

(1) $V_1 + \mu V_2$

(2) V_2 , (5.48)

where μ is arbitrary constant.

5.3 Reductions to ODEs and Exact Solutions

In this section, we use the method of characteristics to determine the invariants and reduced ODEs corresponding to each subalgebra given in (5.48). Symmetry variables and the invariants of the subalgebras of the Lie algebra is described as below. The result of this can be summarized as follows, where ξ is similarity variable, $F(\xi)$ is invariant function related to u, and have to be determined using the reduced ODEs.

Vector Field(1)

The Vector Field (1) is $V_1 + \mu V_2$.

The corresponding characteristic equations are given by

$$\frac{dx}{1}=\frac{dt}{\mu}=\frac{du}{0}=\frac{dw}{0}.$$

Thus the Vector Field (1) in the optimal system defines the similarity variable and similarity solution as follows:

$$\xi = \mu x - t$$
$$u(x,t) = F(\xi)$$
$$w(x,t) = G(\xi).$$

The reduced equation for the subalgabra is,

$$-F' - 3\mu GG' = 0$$
$$-G' - 2\mu^3 G''' - 2\mu FG' - \mu F'G = 0,$$

After solving with the help of maple software we get,

$$\{F(\xi) = -\frac{1}{2}\frac{2\mu^3 C_2^2 + 1}{\mu} + 3\mu^2 C_2^2 \sec h(C_1 + C_2\xi)^2,\ G(\xi) = \sqrt{-2\mu}C_2$$
$$\sec h(C_1 + C_2\xi)\},\ \ \{F(\xi) = -\frac{1}{2}\frac{2\mu^3 C_2^2 + 1}{\mu} + 3\mu^2 C_2^2 \sec h(C_1 + C_2\xi)^2,$$
$$G(\xi) = -\sqrt{-2\mu}C_2 \sec h(C_1 + C_2\xi)\},\ \ \{F(\xi) = -\frac{1}{2}\frac{4\mu^3 C_2^2 + 1}{\mu} - 3\mu^2 C_2^2$$
$$\tanh(C_1 + C_2\xi)^2, G(\xi) = \sqrt{2}\sqrt{\mu}C_2 \tan h(C_1 + C_2\xi)\},\ \ \{F(\xi) = -\frac{1}{2}\frac{4\mu^3 C_2^2 + 1}{\mu}$$
$$-3\mu^2 C_2^2 \tanh(C_1 + C_2\xi)^2, G(\xi) = -\sqrt{2}\sqrt{\mu}C_2 \tan h(C_1 + C_2\xi)\}$$

Thus, the solution of the equation (5.1) is

$$\{u(x,t) = -\frac{1}{2}\frac{2\mu^3 C_2^2 + 1}{\mu} + 3\mu^2 C_2^2 \sec h(C_1 + C_2(\mu x - t))^2,\ w(x,t) = \sqrt{-2\mu}C_2$$
$$\sec h(C_1 + C_2(\mu x - t))\}, \tag{5.3.1}$$
$$\{u(x,t) = -\frac{1}{2}\frac{2\mu^3 C_2^2 + 1}{\mu} + 3\mu^2 C_2^2 \sec h(C_1 + C_2$$
$$\{u(x,t) = -\frac{1}{2}\frac{2\mu^3 C_2^2 + 1}{\mu} + 3\mu^2 C_2^2 \sec h(C_1 + C_2(\mu x - t))^2,\ w(x,t) = -\sqrt{-2\mu}C_2$$
$$\sec h(C_1 + C_2(\mu x - t))\}, \tag{5.3.2}$$
$$\{u(x,t) = -\frac{1}{2}\frac{4\mu^3 C_2^2 + 1}{\mu}$$
$$-3\mu^2 C_2^2 \tanh(C_1 + C_2(\mu x - t)))^2, w(x,t) = \sqrt{2}\sqrt{\mu}C_2 \tan h(C_1 + C_2(\mu x - t)))\},$$

$$\left\{u(x,t) = -\frac{1}{2}\frac{4\mu^3C_2^2+1}{\mu} - 3\mu^2C_2^2\tanh(C_1+C_2(\mu x-t)))^2,\right.$$
$$\left.w(x,t) = -\sqrt{2}\sqrt{\mu}C_2\tan h(C_1+C_2(\mu x-t)))\right\}$$

Vector Field (2)

The Vector Field is V_2.

The corresponding characteristic equations are given by,

$$\frac{dx}{1} = \frac{dt}{0} = \frac{du}{0} = \frac{dw}{0}.$$

Thus the Vector Field (2) in the optimal system defines the similarity variable and similarity solution as follows:

$$\xi = t$$
$$u(x,t) = F(\xi)$$
$$w(x,t) = G(\xi).$$

The reduced equation for the subalgabra is ,

$$F' = 0$$
$$G' = 0.$$

By solving we get $F = a, G = b$.

Exact solution is $u(x,t) = a, w(x,t) = b$,

where a, b are arbitrary constant.

CHAPTER-6

BENJAMIN-BONA-MAHONY EQUATION

The Benjamin-Bona-Mahony (BBM) equation (which is sometimes called the regularised long wave (RLW) equation)

$$u_t + u_x + uu_x - u_{xxt} = 0. \qquad (6.1a)$$

was proposed by Benjamin et al (1972) as an alternative to the celebrated Korteweg-De Vries (KDV) equation (Korteweg and De Vries 1895)

$$u_t + u_x + uu_x + u_{xxx} = 0. \qquad (6.2b)$$

was originally derived as approximation for surface water was originally derived as approximation for surface water waves in a uniform channel [4, 14].

Both (6.1a) and (6.2b) also cover cases of the following types: surface waves of long wavelength in liquids, acoustic- gravity waves in compressible fluids, hydromagnetic waves in cold plasma, acoustic waves in anharmonic crystals etc. The wide applicability of these equations in the main reason why, during the last decades, they have attracted so much attention from mathematicians.

The main mathematical difference between KDV and BBM models can be most readily appreciated by comparing the dispersion relation for the respective linearized equations. It can be be easily seen that these relations are comparable only for small wave numbers (i.e., long waves) and they generate drastically different responses to short waves (which are irrelevant to its role as a physical model). This is one of the reasons why, whereas existence and regularity theory for the KDV equation is difficult the theory of the BBM equation is comparatively simple. The computing is also much easier for (6.1a) than for (6.2b).

The applications of Lie transformations group theory for the construction of solutions of nonlinear partial differential equations(PDEs) is the one of the most active fields of research in the theory of nonlinear (PDEs) and applications. The fundamental basis of the technique is that when a differential equations is invariant under a Lie group of transformations, a reduction transformations exist. Most of the required theory and description of the method can be found in [6, 21, 30, 34].

In this we find the Lie symmetry algebra of the BBM equation and present the optimal systems of one dimensional subalgebras of Lie symmetry algebra. We use these subalgebras to perform similarity reductions and to obtain the similarity solutions.

6.1 Lie Symmetries

Consider the Benjamin Bona Mahoney equation

$$u_t + u_x + uu_x - u_{xxt} = 0. \tag{6.1}$$

Let the group of infinitesimal transformations be defined as

$$u^* = u + \varepsilon\eta(x,t,u) + O(\varepsilon^2)$$
$$x^* = x + \varepsilon\xi(x,t,u) + O(\varepsilon^2)$$
$$t^* = t + \varepsilon\tau(x,t,u) + O(\varepsilon^2).$$

On invoking the invariance criterion the following relation from the coefficients of the first order of ε is deduced.

$$\eta^t + \eta^x + u\eta^x + \eta u_x - \eta^{xxt} = 0\ , \tag{6.2}$$

where $\eta^t, \eta^x, \eta^{xxt}$ are infinitesimal generator corresponding to u_t, u_x, u_{xxt}
substitute the values $\eta^t, \eta^x, \eta^{xxt}$ in (6.2) we get

$$\left(\frac{\partial\eta}{\partial t} - 6u\frac{\partial\eta}{\partial x} - 6\alpha^2u^2\frac{\partial\eta}{\partial x} - \frac{\partial^3\eta}{\partial x^3}\right) + \left(-(1+u)\frac{\partial\eta}{\partial u} + (1+u)\frac{\partial\tau}{\partial t} - \frac{\partial\xi}{\partial t} + \frac{\partial\eta}{\partial u} - (1+u)\frac{\partial\xi}{\partial x} + (1+u)\right.$$

$$\left.\frac{\partial\tau}{\partial x} + u\frac{\partial\eta}{\partial u} - \frac{\partial\xi}{\partial x}u + (u+u^2)\frac{\partial\tau}{\partial x} + (1+u)\frac{\partial^3\eta}{\partial x^2\partial t} - (1+u)\frac{\partial^3\tau}{\partial x^2\partial t} - 2\frac{\partial^3\eta}{\partial x\partial t\partial u} + \frac{\partial^3\eta}{\partial x^2\partial t} + \eta\right)\theta_x +$$

$$\left(\frac{\partial\eta}{\partial u} - \frac{\partial\tau}{\partial t} - (1+u)\frac{\partial\tau}{\partial x} - \frac{\partial^3\eta}{\partial x^2\partial u} + \frac{\partial^3\tau}{\partial x^2\partial t} - \frac{\partial\eta}{\partial u} + 2\frac{\partial\xi}{\partial x} + \frac{\partial\tau}{\partial t}\right)\theta_{xxt} + \left(\frac{\partial^3\eta}{\partial u^2\partial t} + 2\frac{\partial^3\xi}{\partial x\partial t\partial u} + 2(1+u)\right.$$

$$\left.\frac{\partial^3\eta}{\partial u^2\partial x} - (1+u)\frac{\partial^3\xi}{\partial x^2\partial u} - 2(1+u)\frac{\partial^3\tau}{\partial x\partial t\partial u} + (1+u)^2\frac{\partial^3\tau}{\partial x^2\partial u}\right)\theta_x^2 + \left(-(1+u)\frac{\partial\tau}{\partial u} - \frac{\partial\xi}{\partial u} - 2\frac{\partial^3\eta}{\partial u^2\partial x} + \right.$$

$$\left.\frac{\partial^3\xi}{\partial x^2\partial u} + 2\frac{\partial^3\tau}{\partial x\partial t\partial u} - 2(1+u)\frac{\partial^3\tau}{\partial x^2\partial u} + 3\frac{\partial\xi}{\partial u} - 2(1+u)\frac{\partial\tau}{\partial u}\right)\theta_x\theta_{xxt} + \left(\frac{\partial^3\tau}{\partial x^2\partial u} + 2\frac{\partial\tau}{\partial u}\right)\theta_{xxt}^2 +$$

$$\left(2(1+u)^2\frac{\partial^3\tau}{\partial u^2\partial x} + \frac{\partial^3\xi}{\partial u^2\partial t} + (1+u)\frac{\partial^3\eta}{\partial u^3} - 2(1+u)\frac{\partial^3\xi}{\partial u^2\partial x} - (1+u)\frac{\partial^3\tau}{\partial u^2\partial t}\right)\theta_x^3 + \left(-4(1+u)\frac{\partial^3\tau}{\partial u^2\partial x}\right.$$

$$
-\frac{\partial^3\eta}{\partial u^2}+2\frac{\partial^3\xi}{\partial u^2\partial x}+\frac{\partial^3\tau}{\partial u^2\partial t}\Bigg)\theta_x^2\theta_{xxt}+\frac{\partial^2\tau}{\partial u^2}\theta_x^2\theta_{tt}+\Bigg(3\frac{\partial^2\xi}{\partial u^2}-2(1+u)\frac{\partial^2\tau}{\partial u^2}\Bigg)\theta_x\theta_{xx}\theta_{xxt}+\Bigg(-(1+u)\frac{\partial^3\xi}{\partial u^3}
$$

$$
+(1+u)^2\frac{\partial^3\tau}{\partial u^3}\Bigg)\theta_x^4+\Bigg(\frac{\partial^3\xi}{\partial u^3}-2(1+u)\frac{\partial^3\tau}{\partial u^3}\Bigg)\theta_x^3\theta_{xxt}+\Bigg(-2\frac{\partial^2\eta}{\partial u\partial t}+\frac{\partial^2\xi}{\partial x\partial t}\Bigg)\theta_{xx}+\Bigg(-2\frac{\partial^2\eta}{\partial x\partial u}+\frac{\partial^2\xi}{\partial x^2}+
$$

$$
+2\frac{\partial^2\tau}{\partial x\partial t}\Bigg)\theta_{xt}+\frac{\partial^2\tau}{\partial x^2}\theta_{tt}+\Bigg(-4(1+u)\frac{\partial^2\tau}{\partial x\partial u}-2\frac{\partial^2\eta}{\partial u^2}+4\frac{\partial^2\xi}{\partial x\partial u}+2\frac{\partial^2\tau}{\partial u\partial t}\Bigg)+4\frac{\partial^2\tau}{\partial x\partial u}\theta_{xt}\theta_{xxt}+\Bigg(2\frac{\partial^2\tau}{\partial x\partial u}
$$

$$
+2\frac{\partial\tau}{\partial u}\Bigg)\theta_x\theta_{tt}+\Bigg(3\frac{\partial^2\xi}{\partial t\partial u}+(1+u)\frac{\partial^2\eta}{\partial u^2}-2(1+u)\frac{\partial^2\xi}{\partial x\partial u}-(1+u)\frac{\partial^2\tau}{\partial u\partial t}\Bigg)\theta_x\theta_{xx}+\Bigg(-\frac{\partial^2\eta}{\partial u^2}+2\frac{\partial^2\xi}{\partial x\partial u}+
$$

$$
\frac{\partial^2\tau}{\partial u\partial t}\Bigg)\theta_{xx}\theta_{xt}+\Bigg(3\frac{\partial^2\xi}{\partial u^2}-4(1+u)\frac{\partial^2\tau}{\partial u^2}\Bigg)\theta_{xt}\theta_x^2+3\frac{\partial\xi}{\partial u}\theta_{xt}\theta_{xx}+\Bigg(\frac{\partial^2\tau}{\partial u^2}+2\frac{\partial^3\tau}{\partial u^2\partial x}\Bigg)\theta_{xxt}^2\theta_x+2\frac{\partial\tau}{\partial u}\theta_{xt}^2+
$$

$$
\frac{\partial\tau}{\partial u}\theta_{tt}\theta_{xx}+2\frac{\partial\tau}{\partial x}\theta_{xtt}+\frac{\partial\xi}{\partial t}\theta_{xxx}+\Bigg(-(1+u)\frac{\partial\xi}{\partial u}\Bigg)\theta_{xxx}\theta_{xxt}+\Bigg(-3(1+u)\frac{\partial^2\xi}{\partial u^2}+(1+u)^2\frac{\partial^2\tau}{\partial u^2}\Bigg)\theta_x^2\theta_{xx}+
$$

$$
4\frac{\partial^2\tau}{\partial u^2}\theta_x\theta_{xt}\theta_{xxt}=0.
$$

Now equate the coefficients of $\theta_{xxx},\theta_{xxt},\theta_{xx}\theta_{tt},\theta_{xtt}$, const etc equal to zero and we obtain

$$
\frac{\partial\xi}{\partial u}=0,\ \frac{\partial\xi}{\partial t}=0,\frac{\partial\tau}{\partial u}=0,\frac{\partial\tau}{\partial x}=0 \tag{6.3}
$$

$$
(1+u)\frac{\partial\tau}{\partial t}-(1+u)\frac{\partial\xi}{\partial x}+(1+u)\frac{\partial^3\eta}{\partial x^2\partial u}-2\frac{\partial^3\eta}{\partial x\partial t\partial u}+\eta=0 \tag{6.4}
$$

$$
-\frac{\partial\tau}{\partial t}-\frac{\partial^3\eta}{\partial x^2\partial u}+2\frac{\partial\xi}{\partial x}+\frac{\partial\tau}{\partial t}=0 \tag{6.5}
$$

$$
\frac{\partial^3\eta}{\partial u^2\partial t}+2(1+u)\frac{\partial^3\eta}{\partial u^2\partial x}=0 \tag{6.6}
$$

$$
-2\frac{\partial^3\eta}{\partial u^2\partial x}=0 \tag{6.7}
$$

$$
(1+u)\frac{\partial^3\eta}{\partial u^3}=0 \tag{6.8}
$$

$$
-\frac{\partial^3\eta}{\partial u^3}=0 \tag{6.9}
$$

$$
\frac{\partial^2\eta}{\partial u\partial t}=0 \tag{6.10}
$$

$$
\frac{\partial^2\eta}{\partial x\partial u}+\frac{\partial^2\xi}{\partial x^2}=0 \tag{6.11}
$$

$$
-2\frac{\partial^2\eta}{\partial u^2}=0 \tag{6.12}
$$

$$
(1+u)\frac{\partial^2\eta}{\partial u^2}=0. \tag{6.13}
$$

$$\frac{\partial^2\eta}{\partial u^2}=0 \tag{6.14}$$

$$\frac{\partial\eta}{\partial t}-6u\frac{\partial\eta}{\partial x}-6\alpha^2u^2\frac{\partial\eta}{\partial x}-\frac{\partial^3\eta}{\partial x^2\partial t}=0. \tag{6.15}$$

From (6.3) and (6.14) we get

$$\xi=h(x)\ ,\ \tau=l(t)\ ,\ \eta=f(x,t)u+g(x,t).$$

Put the value of η,ξ,τ in above coefficient from (6.3)-(6.15)

$$(1+u)l_t(t)-(1+u)h_x(x)+(1+u)f_{xx}(x,t)-2f_{xt}(x,t)+f(x,t)u+g(x,t)=0 \tag{6.16}$$

$$-f_{xx}(x,t)+2h_x(x)=0 \tag{6.17}$$

$$-f_t(x,t)=0 \tag{6.18}$$

$$-f_x(x,t)+2h_{xx}(x)=0 \tag{6.19}$$

$$f_t(x,t)+(1+u)f_x(x,t)-f_{xxt}(x,t)=0. \tag{6.20}$$

By equating the coefficients of u

From equation (6.16) and (6.20) we get

$$l_t(t)-h_x(x)+f_{xx}(x,t)+f(x,t)u=0 \tag{6.21}$$

$$f_x(x,t)=0. \tag{6.22}$$

By equating the coefficients of without u :

From equation (6.16) we get

$$l_t(t)-h_x(x)+f_{xx}(x,t)-f_{xt}(x,t)+g(x,t)=0. \tag{6.23}$$

From (6.18), and (6.17) we get

$$f=b$$

$$h=a.$$

From (6.21) and (6.23)

$$g=b$$

$$l=-bt+c.$$

Put the value of f,g,h and l in above coefficients (6.16)-(6.20)

Then the solution of the determining equations is given by

$$\begin{aligned}\xi(x,t)&=a\\ \tau(x,t)&=-bt+c\\ \eta(x,t)&=b(u+1),\end{aligned} \tag{6.24}$$

where $a,b,c$are three arbitrary parameters.

Hence, the Lie algebra admitted by the Benjamin Bona Mahoney equation (6.1) is

$$V_1=\frac{\partial}{\partial x},\ V_2=-t\frac{\partial}{\partial t}+(1+u)\frac{\partial}{\partial u},V_3=\frac{\partial}{\partial t}\ . \tag{6.25}$$

Optimal system

The commutator and adjoint resresentation table Lie algebra (6.25) can easily constructed as follows:

Commutator Table-6.26

	V_1	V_2	V_3
V_1	0	0	0
V_2	0	0	V_3
V_3	0	$-V_3$	0

Adjoint Table-6.24

Adj	V_1	V_2	V_3
V_1	V_1	V_2	V_3
V_2	V_1	V_2	$V_3 e^{-\varepsilon}$
V_3	V_1	$V_2 + \varepsilon V_3$	V_3

We deduce an optimal system of sub algabras with their corresponding essential vector field:

(1) $V_1 + \mu V_2$
(2) V_2
(3) V_3.

where $\mu$is arbitrary constant.

6.2 Reductions of Optimal System and Exact Solutions

In this section, corresponding to each generator in the optimal system of sub algebra, the reductions of PDEs (6.1) into ODEs in terms of similarity variable ξ and the new dependent variables- F, are obtained using the auxiliary equations (6.24). Some exact solutions of each of each reduced system are then attempted.

Essential vector field (1)

The Essential vector field (1) is $V_1 + \mu V_2$.

The corresponding characteristic equations are given by,

$$\frac{dx}{1}=\frac{dt}{-mt}=\frac{du}{m(1+u)}.$$

The Essential vector field (1) in the optimal system defines the similarity variable and similarity solution as follows:

$$\xi = t\exp(mx)$$
$$u(x,t)=\frac{F(\xi)}{t}-1.$$

Using the similarity variable, the forms of the similarity solution, the system of partial differential equation (PDE) (2.1) reduces to the following system of Ordinary differential equation (ODE):

$$\xi F^{'} - F + m\xi FF^{'} + 2m^2\xi^2 F^{''} + m^2\xi^3 F^{'''} = 0,$$

In this case we are able to find only reduction.

Essential vector field (2)

The Essential vector field (2) is V_2.

The corresponding characteristic equations are given by,

$$\frac{dx}{0}=\frac{dt}{-t}=\frac{du}{m(1+u)}.$$

Thus the Essential vector field (2) in the optimal system defines the similarity variable and similarity solution as follows:

$$\xi = x$$
$$u(x,t)=\frac{F(\xi)}{t}-1.$$

Using the similarity variable, the forms of the similarity solution, the system of partial differential equation (PDE) (6.1) reduces to the following system of Ordinary differential equation (ODE):

$$-F + FF^{'} + F^{''} = 0.$$

In this case we are able to find only reduction.

Essential vector field (3)

The Essential vector field (3) is V_3.

The corresponding characteristic equations are given by,

$$\frac{dx}{0} = \frac{dt}{1} = \frac{du}{0}.$$

Thus the Essential vector field (3) in the optimal system defines the similarity variable and similarity solution as follows:

$$\xi = x$$
$$u(x,t) = F(\xi).$$

Using the similarity variable, the forms of the similarity solution, the system of partial differential equation (PDE) (6.1) reduces to the following system of Ordinary differential equation (ODE):

$$F' - FF' = 0,$$

By solving this equation we get $F = a$.

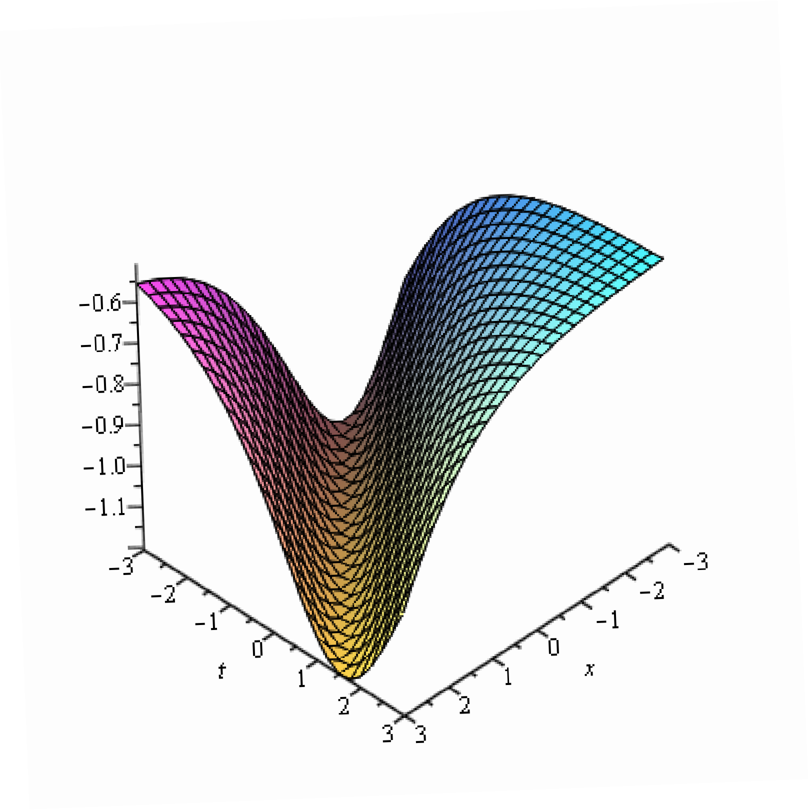
Thus the exact solution is $u(x,t) = a$,

where a is constant.

CONCLUSION

The thesis deals with the Lie Classical Method application to some nonlinear partial differential equations representing some interesting physical systems via Gardner equation, Fisher equation, Drinfel'd Sokolov Wilson equation and Benjamin-Bona-Mahony equation. To determine the admissible symmetries, we referred to as the classical Lie approach. After obtaining the point symmetries admitted by a system under investigation, a formal approach of identifying an optimal system of Lie sub algebras has been adopted with help of the adjoint action of the Lie algebra. The basic generators contained in the optimal system have then been exploited to achieve the desired reduction of PDEs to ODEs. The resulting ODEs have been examined subsequently for various types of exact solutions. The various exact solutions are in hyperbolic functions. Some plots to have an idea about the nature of the solutions are obtained in the thesis.

Figure 1



Section (3.3) for solution (3.3.1) with $c_1 = 1, \mu = 1, \alpha = 1, x = -3..3, t = -3..3$.

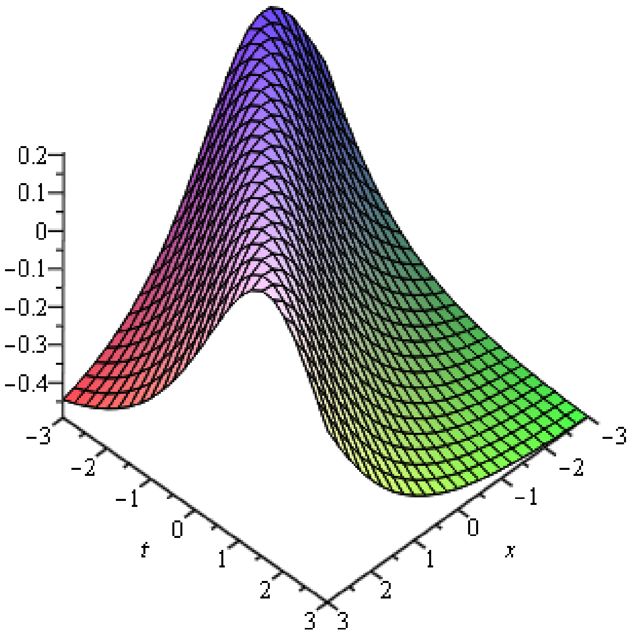


Figure 2

Section (3.3) for solution (3.3.2) with $c_1 = 1, \mu = 1, \alpha = 1, x = -3..3, t = -3..3$.

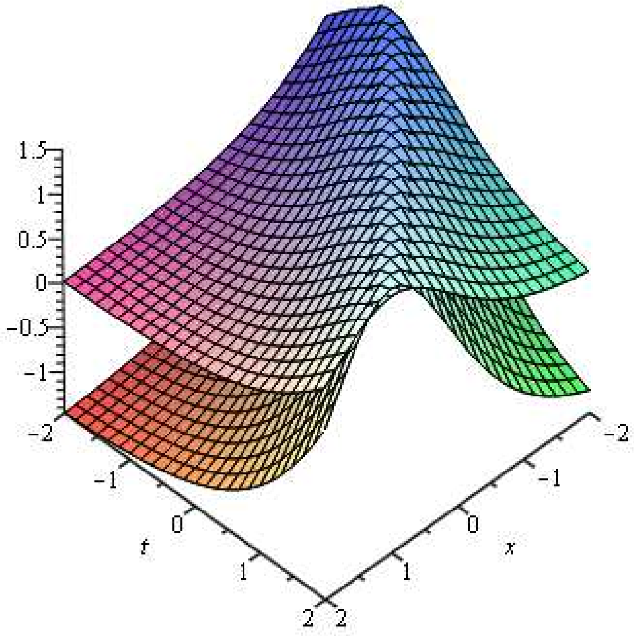


Figure 3

Section (5.3) for solution (5.3.1) with $c_1 = 1, \mu = 1, \alpha = 1, x = -2..2, t = -2..2$.

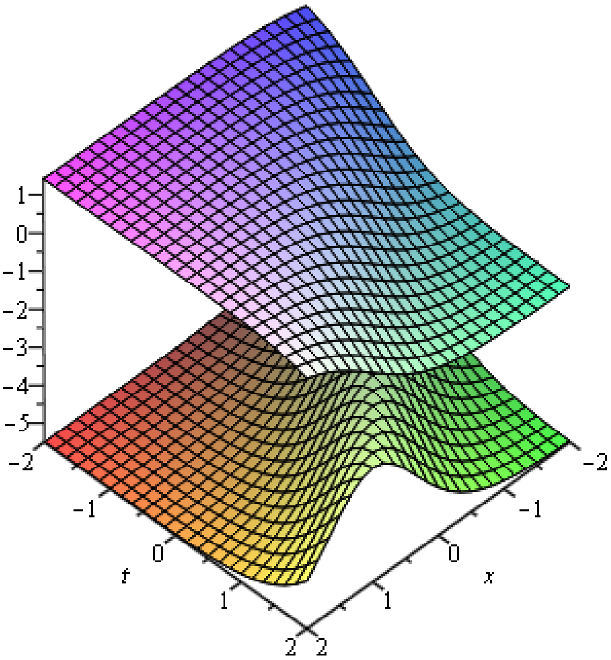


Figure 4

Section (5.3) for solution (5.3.2) with $c_1 = 1, \mu = 1, \alpha = 1, x = -2..2, t = -2..2$.

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