

Automorphism Groups of Finite p -Groups

Thesis

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by

HEMANT KALRA
(Regd. No. 900911002)

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THAPAR UNIVERSITY - 147 004 (PUNJAB), INDIA.**

July - 2013

Declaration of Authorship

I hereby declare that the work which is being presented in this thesis entitled "*Automorphism Groups of Finite p -Groups*" submitted by me, for the award of the degree of Doctor of Philosophy in the School of Mathematics and Computer Applications, Thapar University, Patiala, is true and original record of my own independent and original research work carried out under the supervision of Dr. Deepak Gumber, Associate Professor, School of Mathematics and Computer Applications, Thapar University, Patiala, India. The matter embodied in this thesis has not been submitted in part or full to any other university or institute for the award of any degree in India or Abroad and that the ideas and references cited herein have been duly acknowledged.

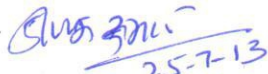
Hemant Kalra
25-07-13
(Hemant Kalra)

Regd. No. 900911002

CERTIFICATE

This is to certify that the thesis "*Automorphism Groups of Finite p -Groups*" which is submitted by Mr. Hemant Kalra, in fulfilment of the requirement for the award of the degree of *Doctor of Philosophy* in the School of Mathematics and Computer Applications, Thapar University, Patiala, is a record of the candidate's own independent and original research work carried out by him under my supervision and guidance. The matter embodied in this thesis has not been submitted in part or full to any University or Institute for the award of any degree.

Attestation by supervisor



(Dr. Deepak Gumber)

Associate Professor

School of Mathematics and Computer Applications

Thapar University

Patiala 147 004

INDIA.

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(Hemant Kalra)

Dedicated
To
My
Family

Abstract

Let G be a group and let $\text{Aut}(G)$ denote the group of all automorphisms of G . An automorphism α of G is called a class-preserving automorphism if for each $x \in G$, there exists an element $g_x \in G$ such that $\alpha(x) = g_x^{-1}xg_x$; and is called an inner automorphism if for all $x \in G$, there exists a fix element $g \in G$ such that $\alpha(x) = g^{-1}xg$. The group $\text{Inn}(G)$ of all inner automorphisms of G is a normal subgroup of the group $\text{Aut}_c(G)$ of all class-preserving automorphisms of G . An automorphism φ of G is called a central automorphism if it commutes with all inner automorphisms of G ; or equivalently $g^{-1}\varphi(g) \in Z(G)$, the center of G , for all $g \in G$. The group of all central automorphisms of G is denoted as $\text{Aut}_z(G)$.

The main objective in this thesis is to study the equality of two automorphism groups of a finite p -group. In chapter 2, we give necessary and sufficient conditions for a finite p -group G such that $\text{Aut}_c(G) = \text{Aut}_z(G)$. We classify all finite p -groups G such that $\text{Aut}_c(G) = \text{Aut}_z(G)$ when $Z(G)$ is cyclic or elementary abelian, or when order of the group is at most p^7 . We complete the classification of all finite p -groups of order $\leq p^5$ for which there exist non-inner class-preserving automorphisms. In chapter 3, we study finite p -groups G for which $\text{Aut}_z(G)$ is of minimal order, that is, $\text{Aut}_z(G) = Z(\text{Inn}(G))$. We give necessary and sufficient conditions on a finite p -group G of coclass upto 4 such that $\text{Aut}_z(G) = Z(\text{Inn}(G))$, and as a consequence, we characterize all finite p -groups G of order at most p^7 for which $\text{Aut}_z(G) = Z(\text{Inn}(G))$. Our results generalize the results obtained by Sharma and Gumber [57].

In 1904, Schur [56] proved that if the index of the center of a group is finite, then its commutator subgroup is also finite. In chapter 4, we give an elementary and different proof of the converse of this theorem using automorphisms. Our results generalize the previous results obtained in this direction by Niroomand [46] and Sury [58]. For a finite group G , the structures of $\text{Inn}(G)$, $\text{Aut}_c(G)$ and $\text{Aut}_z(G)$ are influenced by the structure of the conjugacy classes of G . In the final chapter, we study finite groups in the following three cases: (i) Each non-trivial conjugacy class of G together with the identity element 1 is a subgroup of G , (ii) Union of any two distinct non-trivial conjugacy classes of G together with 1 is a subgroup of G , and (iii) Union of any three distinct non-trivial conjugacy classes of G together with 1 is a subgroup of G .

List of Research Papers

- (1) Kalra, H. and Gumber, D. *On equality of central and class preserving automorphisms of finite p -groups*, Indian J. Pure Appl. Math. **(Accepted)**.
- (2) Gumber, D. and Kalra, H. *Finite p -groups with central automorphism group of minimal order* (Communicated).
- (3) Gumber, D. and Kalra, H. *On the converse of a theorem of Schur*, Arch. Math., **101** (2013), 17-20.
- (4) Kalra, H. and Gumber, D. *A note on conjugacy classes of finite groups*, Proc. Indian Acad. Sci. (Math. Sci.) **(Accepted)**.

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CHAPTER 1

Introduction and Basics

1.1 — Introduction

Let G be an arbitrary group and let G' and $Z(G)$ respectively denote the commutator subgroup and the center of G . Let $\text{Aut}(G)$ denote the full automorphism group of G . An automorphism α of G is called a class-preserving automorphism if for each $x \in G$, there exists an element $g_x \in G$ such that $\alpha(x) = g_x^{-1}xg_x$; and is called an inner automorphism if for all $x \in G$, there exists a fix element $g \in G$ such that $\alpha(x) = g^{-1}xg$. The group $\text{Inn}(G)$ of all inner automorphisms of G is a normal subgroup of the group $\text{Aut}_c(G)$ of all class-preserving automorphisms of G . We denote the group $\text{Aut}_c(G)/\text{Inn}(G)$ of all class-preserving outer automorphisms of G by $\text{Out}_c(G)$. An automorphism φ of G is called a central automorphism if it commutes with all inner automorphisms of G ; or equivalently $g^{-1}\varphi(g) \in Z(G)$ for all $g \in G$. Let $\text{Aut}_z(G)$ denote the group of all central automorphisms of G and let $\text{Aut}_z^z(G)$ denote the group of all those central automorphisms which fix the center of G element-wise. We reserve the letter p to denote an arbitrary prime in the Thesis.

Our main objective in the thesis is to study different automorphism groups of a

finite p -group and to study the structure of a finite p -group whose different automorphism groups are equal. Interest in the equality of two different automorphism groups dates back to 1911 when Burnside [10, Note B] posed the following question: Does there exist a finite group G such that G has a non-inner class preserving automorphism? In 1913, Burnside [11] himself gave a positive answer to his question. He constructed a group G of order p^6 isomorphic to the group W consisting of all 3×3 matrices of the form

$$M = \begin{pmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ z & y & 1 \end{pmatrix}$$

with x, y, z in the field \mathbb{F}_{p^2} of p^2 elements, where p is an odd prime. For this group G , $\text{Aut}_c(G) \neq \text{Inn}(G)$. In 1947, Wall [63] constructed some smaller and simpler groups G for which $\text{Aut}_c(G) \neq \text{Inn}(G)$. The smallest group constructed by Wall is of order 32. In 1968, Sah [55] explored many basic properties of $\text{Aut}_c(G)$ and $\text{Out}_c(G)$ for a finite group G . It is a well known result of Gaschutz [21] that every finite p -group has a non-inner automorphism. In 1980, Heineken [28] constructed finite p -groups G whose all automorphisms are class-preserving. It follows that for all these groups G , $\text{Out}_c(G) \neq 1$. In 1988, Feit and Seitz [19] proved that $\text{Out}_c(G) = 1$ for all simple groups.

A map $f : G \rightarrow G$ is called a cocycle of G if $f(xy) = f(x)xf(y)x^{-1}$ for all $x, y \in G$. A cocycle f of G is called a local coboundary if for each $x \in G$, there exists an element $a_x \in G$ such that $f(x) = a_x^{-1}xa_x x^{-1}$; and is called a global coboundary, or simply a coboundary, if for all $x \in G$, there exists a fix element $a \in G$ such that $f(x) = a^{-1}xax^{-1}$. A group G is said to enjoy *Hasse Principle* if every local coboundary of G is a coboundary. It is known that a finite group

G enjoys *Hasse Principle* if and only if every class-preserving automorphism of G is an inner automorphism [48]. In the years from 1998 to 2000, Ono and Wada [47, 48, 49, 50, 51, 61, 62] in a series of papers showed that various types of groups like free groups of finite rank, symmetric and alternating groups, special and general linear groups over a Euclidian domain D enjoy *Hasse Principle*. In 2000 and 2001, Kumar and Vermani [39, 40] proved that every non-abelian finite p -group having a maximal subgroup which is cyclic enjoys *Hasse Principle*. In particular, they proved that every extraspecial p -group and every group of order p^4 enjoy *Hasse Principle*. In 2002, Kumar and Vermani [41] found those non-abelian finite p -groups of order p^m , $m > 4$, having a normal cyclic subgroup of order p^{m-2} but having no element of order p^{m-1} which enjoy *Hasse Principle*. In 2004, Fuma and Ninomiya [20] showed that every non-abelian group of order p^m having a cyclic subgroup of order p^{m-2} but having no normal cyclic subgroup of order p^{m-2} and no element of order p^{m-1} enjoys *Hasse Principle*.

It follows from [33] that all finite p -groups of order p^5 , where p is an odd prime, are partitioned into ten isoclinism families. In 2008, Yadav [65] proved that if G and H are two finite non-abelian isoclinic groups, then $\text{Aut}_c(G) \simeq \text{Aut}_c(H)$. He then showed that $\text{Out}_c(G) \neq 1$ for the groups $\phi_7(1^5)$ and $\phi_{10}(1^5)$ from seventh and tenth family in [33], and concluded that if G is a finite p -group of order p^5 , where p is an odd prime, then $\text{Out}_c(G) \neq 1$ if and only if G is isoclinic to a group either in the seventh or in the tenth family.

In 2001, Curran and McCaughan [15] gave necessary and sufficient conditions for a finite p -group G such that $\text{Aut}_z(G) = \text{Inn}(G)$. They proved that for any finite

p -group G , $\text{Aut}_z(G) = \text{Inn}(G)$ if and only if $G' = Z(G)$ and $Z(G)$ is cyclic. In 2007, Attar [3] proved that for any finite p -group G , $\text{Aut}_z^z(G) = \text{Inn}(G)$ if and only if either G is abelian or nilpotency class of G is 2 and $Z(G)$ is cyclic. In 2009, Yadav [66] gave necessary and sufficient conditions on a finite p -group G of nilpotency class 2 for which $\text{Aut}_z(G) = \text{Aut}_z^z(G)$. Recently, Attar [4] and Jafari [32] have generalized this result of Yadav. They have given necessary and sufficient conditions on a finite p -group G of arbitrary nilpotency class for which each central automorphism of G fixes the center element-wise.

In 2011, Yadav [67, Problem 6.3] asked to classify all finite p -groups of class 2 for which $\text{Aut}_c(G) = \text{Aut}_z(G)$. In Chapter 2, we give necessary and sufficient conditions on a finite p -group G such that $\text{Aut}_c(G) = \text{Aut}_z(G)$ (Yadav [68] has also obtained a similar result). We also classify all finite p -groups G with cyclic or elementary abelian center such that $\text{Aut}_c(G) = \text{Aut}_z(G)$. It follows from [27] that all finite 2-groups of order 2^5 are partitioned into eight isoclinism families. We study finite 2-groups of order 2^5 for non-inner class-preserving automorphisms and show that only sixth family in [27] has two groups which have non-inner class-preserving automorphisms. This result completes the classification of all finite p -groups of order $\leq p^5$ for which $\text{Out}_c(G) \neq 1$. Using this result, we classify all finite p -groups of order $\leq p^7$ for which $\text{Aut}_c(G) = \text{Aut}_z(G)$. The results of this chapter will appear in [35].

In 2004, Curran [14] studied finite p -groups G for which $\text{Aut}_z(G)$ is of minimal possible order. He proved that for any finite p -group G if $\text{Aut}_z(G) = Z(\text{Inn}(G))$, then $Z(G) \leq G'$. He also proved that if G is a finite p -group of maximal class, then $\text{Aut}_z(G) \neq Z(\text{Inn}(G))$. Observe that if G is a finite p -group of nilpotency class 2, then

$\text{Inn}(G) = Z(\text{Inn}(G))$). It thus follows from the main result of Curran and McCaughan [15] that for any finite p -group G of nilpotency class 2, $\text{Aut}_z(G) = Z(\text{Inn}(G))$ if and only if $G' = Z(G)$ and $Z(G)$ is cyclic. Therefore, to study finite p -groups G for which $\text{Aut}_z(G) = Z(\text{Inn}(G))$, we can assume that nilpotency class of G is bigger than 2 and G is not of maximal class. Recently, Sharma and Gumber [57] have characterized all finite p -groups of order p^5 and all finite p -groups of order p^6 , p an odd prime, such that $\text{Aut}_z(G) = Z(\text{Inn}(G))$.

In Chapter 3, we give necessary and sufficient conditions on a finite p -group with cyclic center such that $\text{Aut}_z(G) = Z(\text{Inn}(G))$. As a consequence of this, we give necessary and sufficient conditions on finite p -groups G of coclass upto 4 for which $\text{Aut}_z(G) = Z(\text{Inn}(G))$ and then characterize finite p -groups G of order $\leq p^7$ such that $\text{Aut}_z(G) = Z(\text{Inn}(G))$. Our proofs are short, elementary and even generalize the main results of Sharma and Gumber [57]. The results of this chapter will appear in [25].

In 1904, Schur [56] proved the result: If the index of the center of a group is finite, then its derived group is also finite. The converse of Schur's theorem is not true in general. For example, it is not true for an infinite extra-special p -group, where p is an odd prime. In 1951, Neumann [45] proved that if G' is finite and G is finitely generated, then $G/Z(G)$ is finite. In 2001, Hilton [31] gave short proofs of Schur's and Neumann's theorems using exact sequences. In 2010, Niroomand [46] generalized the Neumann's theorem to the following: If G' is finite and $G/Z(G)$ is finitely generated by d elements, then

$$|G/Z(G)| \leq |G'|^d. \quad (1.1)$$

In 2010, Sury [58] generalized Niroomand's result to the following: Let G be a group such that the set $K(G)$ of its commutators is finite. Then G' is finite and if $G/Z(G)$ can be generated by d elements, then $|G/Z(G)| \leq |K(G)|^d$.

In Chapter 4, we give a very short and different proof of a further generalization of Niroomand's and Sury's results using automorphisms. We prove that if G is an arbitrary group such that $G/Z(G)$ is generated by $x_1Z(G), x_2Z(G), \dots, x_dZ(G)$ and $[x_i, G]$ is finite for each i , $1 \leq i \leq d$, then $|G/Z(G)| \leq \prod_{i=1}^d |[x_i, G]|$ and G' is finite. It follows from our results that if G is a finite group such that equality holds in (1.1), then $G' = K(G)$ and $\text{Inn}(G) = \text{Aut}_c(G)$. It therefore becomes interesting to study groups for which equality holds in (1.1). Yadav [69] also has asked the question for what non-abelian groups equality holds in (1.1). He himself has classified, upto isoclinism, all such groups of nilpotency class 2 with $G/Z(G)$ finite. We prove that if G is a finitely generated nilpotent group of class 2, then $G/Z(G) \simeq G'^{d(G/Z(G))}$ if and only if G' is cyclic and $G/Z(G)$ is homocyclic. We also classify all finite p -groups of maximal class and of coclass 2 for which equality holds in (1.1). Yadav [69] gave some examples of groups G for which equality holds in (1.1). In all these examples G' is finite cyclic. He then posed the following question:

Question: Let G be a finite nilpotent group for which equality holds in (1.1). Is it true that G' is always cyclic?

We give a negative answer to this question by giving an example of a group of order 32 for which equality holds in (1.1), but its commutator subgroup is elementary abelian. The results of this chapter have appeared in [24].

The class-preserving automorphisms, inner automorphisms and central automor-

phisms of a finite group are interrelated and are influenced by the structure of the group, which is further influenced by the structure of its conjugacy classes. For example, if G is minimally generated by x_1, x_2, \dots, x_d , then $|\text{Inn}(G)| \leq |\text{Aut}_c(G)| \leq \prod_{i=1}^d |x_i^G|$. In Chapter 5, we classify all finite groups G in the following three cases:

- (i) Each non-trivial conjugacy class of G together with the identity element 1 is a subgroup of G .
- (ii) Union of any two distinct non-trivial conjugacy classes of G together with 1 is a subgroup of G .
- (iii) Union of any three distinct non-trivial conjugacy classes of G together with 1 is a subgroup of G .

Results of this chapter will appear in [36].

1.2 — Basics

In this section, we give a quick review of some of the basic facts of group theory that are assumed in the foregoing chapters. The definitions and proofs of results presented here can be found in any standard book on group theory. We of course suppose a familiarity of more basic group theoretic terms and concepts like abelian, cyclic, coset, normal subgroup, factor or quotient group, homomorphism, isomorphism, direct product et cetera.

Let G be an arbitrary group and X be a subset of G . The intersection of the family of subgroups of G which contain X is a subgroup of G and is denoted by $\langle X \rangle$. In other words, $\langle X \rangle$ is the smallest subgroup of G which contains X . The subgroup $\langle X \rangle$ is called the subgroup generated by X . If X is non-empty, then $\langle X \rangle$

contains every finite product of the type

$$x_1^{m_1} x_2^{m_2} \dots x_r^{m_r}, \quad r \geq 1, \quad x_i \in X, \quad m_i \in \mathbb{Z},$$

and conversely all such products form a subgroup of G containing X . It follows that $\langle X \rangle$ consists of all such products. A cyclic group is thus generated by a single element. We shall denote a cyclic group of order m by C_m . The **rank** of a group G is the smallest cardinality of a generating set of G and is denoted by $d(G)$. The least common multiple of the orders of the elements of a finite group G is called the **exponent** of G and is denoted by $exp(G)$.

The **commutator** of two elements $a, b \in G$ is the element $[a, b] = a^{-1}b^{-1}ab$ of G and the **commutator subgroup** or the **derived subgroup** G' of G is the subgroup of G generated by all commutators of G . It is easy to see that G' is a normal subgroup of G . If X and Y are two subsets of G , then we define $[X, Y] = \langle [x, y] \mid x \in X, y \in Y \rangle$. Thus $[X, Y]$ is always a subgroup of G . For $x \in G$, $[x, G]$ denotes the set of all commutators $[x, g]$, where $g \in G$. By $K(G)$ we denote the set of all commutators of G . The followings are well known commutator identities

$$[x, yz] = [x, z][x, y][x, y, z]; \quad [xy, z] = [x, z][x, z, y][y, z],$$

where $x, y, z \in G$ and will be frequently used in the thesis without any reference.

A series

$$1 = G_0 \leq G_1 \leq \dots \leq G_l = G$$

of subgroups of G is called a **normal series** if each G_i is a normal subgroup of G .

The normal series above is called a **central series** if for each i , $G_{i+1}/G_i \leq Z(G/G_i)$.

Let $Z_0 = 1$ and let $Z_{i+1}/Z_i = Z(G/Z_i)$ for $i \geq 0$. Observe that Z_1 is the center of G and Z_{i+1}/Z_i , being the center of G/Z_i , is normal in G/Z_i and hence Z_{i+1} is normal in G for all $i \geq 0$. It follows that the series

$$1 = Z_0 \leq Z_1 \leq Z_2 \leq \dots$$

is a central series of G . The subgroup Z_i is called the ***i-th center*** and this series is called the ***upper central series*** of G .

We define subgroups $\gamma_i(\mathbf{G})$, $i \geq 1$, of G by setting

$$\gamma_1(G) = G, \quad \gamma_{i+1}(G) = [\gamma_i(G), G].$$

Observe that $\gamma_2(G) = G'$, each $\gamma_i(G)$ is normal in G and $\gamma_{i+1}(G) \leq \gamma_i(G)$. The series

$$G = \gamma_1(G) \geq \gamma_2(G) \geq \dots \gamma_n(G) \geq \dots$$

is called the ***lower central series*** of G . If the lower central series of a group G terminates in a finite number of steps at 1, and if c is the least natural number such that $\gamma_{c+1}(G) = 1$, then G is called a ***nilpotent group*** of class c . The class of a nilpotent group is denoted by $cl(\mathbf{G})$. Observe that if $cl(G) = 2$, then $G' \leq Z(G)$.

A ***maximal subgroup*** of G is a proper subgroup M such that there is no subgroup H of G with $M < H < G$. The intersection of all the maximal subgroups of G is the ***Frattini subgroup*** $\Phi(\mathbf{G})$ of G . If G has no maximal subgroup, then we set $\Phi(G) = G$. An element a of G is called a ***non-generator*** of G if whenever $G = \langle a, X \rangle$, then $G = \langle X \rangle$. An interesting property of $\Phi(G)$ is that it is exactly the set of all non-generators of G .

Two elements a and b of G are called ***conjugate*** if there exists an element g of

G such that $b = g^{-1}ag$. It is easily seen that “conjugacy” is an equivalence relation on G and therefore it partitions G into equivalence classes. The equivalence class that contains the element a of G is called the **conjugacy class** of a and is denoted as \mathbf{a}^G . Let A be a non-empty subset of G . The set of elements of G which commute with every element of A is called the **centralizer** of A in G , and is denoted as $\mathbf{C}_G(\mathbf{A})$. If $A = \{a\}$ is singleton, then $C_G(\{a\})$ is simply denoted as $C_G(a)$. It is easy to see that $C_G(A)$ is a subgroup of G . For any $a \in G$, $|\mathbf{a}^G| = |G|/|C_G(a)|$.

A finite group G is called a **purely non-abelian** group if it has no non-trivial abelian direct factor. If $Z(G)$ is cyclic or if $Z(G) \leq \Phi(G)$, then G is purely non-abelian.

An isomorphism α of G to itself is called an **automorphism** of G . The set of all automorphisms of G is a group under the usual operation of compositions of mappings. We call this group the full automorphism group of G and denote it by $\mathbf{Aut}(G)$.

Let A be an abelian group and let $\mathbf{Hom}(G, \mathbf{A})$ denote the set of all homomorphisms of G into A . For $f, g \in \mathbf{Hom}(G, A)$, define $fg(x) = f(x)g(x)$. Then $\mathbf{Hom}(G, A)$ becomes an abelian group under this operation. If A, B, C are all finite abelian groups, then $\mathbf{Hom}(A, B \times C) \simeq \mathbf{Hom}(A, B) \times \mathbf{Hom}(A, C)$ and $\mathbf{Hom}(A, B) \simeq \mathbf{Hom}(B, A)$. Also, $\mathbf{Hom}(C_m, C_n) \simeq C_d$, where $d = \gcd(m, n)$.

For a fix prime number p , a group G is called a **p -group** if order of every element of G is a power of p . If G is finite, then G is a p -group if and only if $|G|$, the order of G , is a power of p . The followings are well known facts about p -groups: (i) $G' \leq \Phi(G)$ and (ii) $G/\Phi(G)$ is elementary abelian of rank $d(G)$.

Equality of Central and Class-Preserving Automorphisms

2.1 — Introduction

Let G be a finite group. An automorphism α of G is called a class-preserving automorphism if for all $x \in G$, $\alpha(x) \in x^G$; and an automorphism φ of G is called a central automorphism if $g^{-1}\varphi(g) \in Z(G)$ for all $g \in G$. The set $\text{Aut}_c(G)$ of all class preserving automorphisms and the set $\text{Aut}_z(G)$ of all central automorphisms of G are normal subgroups of $\text{Aut}(G)$.

In section 2.2, we give necessary and sufficient conditions on a finite p -group G such that $\text{Aut}_c(G) = \text{Aut}_z(G)$. As a consequence of this result, we obtain an easy and short proof of the main result of Curran and McCaughan [15]. We also classify all finite p -groups G such that $\text{Aut}_c(G) = \text{Aut}_z(G)$ when $Z(G)$ is cyclic or elementary abelian.

Let X be a finite group and let $\bar{X} = X/Z(X)$. Then commutation in X gives a well defined map $a_X : \bar{X} \times \bar{X} \rightarrow X'$ such that $a_X(xZ(X), yZ(Y)) = [x, y]$ for all $(x, y) \in X \times X$. Two finite groups G and H are called *isoclinic*, as defined by

Hall [26], if there exists an isomorphism ψ of the factor group $\bar{G} = G/Z(G)$ onto $\bar{H} = H/Z(H)$, and an isomorphism θ of the subgroup G' onto H' such that the following diagram is commutative

$$\begin{array}{ccc} \bar{G} \times \bar{G} & \xrightarrow{a_G} & G' \\ \psi \times \psi \downarrow & & \downarrow \theta \\ \bar{H} \times \bar{H} & \xrightarrow{a_H} & H' \end{array}$$

The resulting pair (ψ, θ) is called an *isoclinism* of G onto H . Notice that isoclinism is an equivalence relation among finite groups.

If G and H are two isoclinic groups, then nilpotency class of G is equal to the nilpotency class of H and the terms in the lower central series of G are isomorphic to the respective terms of the lower central series of H . More precisely, one can say that the commutator structure of G is similar to the commutator structure of H . In each isoclinism family of p -groups, there exist groups G of minimal order, say p^m , such that $Z(G) \leq G'$. All such groups are called stem groups of the family. The number m , which is an invariant of the family, is called the rank of the family. Some other invariants are $q_i(G)$ and $r_i(G)$, which are respectively the number of conjugacy classes of G with precisely p^i members and the number of irreducible complex representations of G of degree p^i .

It follows from [27] that all the groups of order 32 are partitioned into 8 isoclinism families. In section 2.3, we show that there are only two groups of order 32 in sixth isoclinism family for which $\text{Aut}_c(G) \neq \text{Inn}(G)$. This result completes the classification of all finite p -groups of order $\leq p^5$ which have non-inner class preserv-

ing automorphisms. We also classify all finite p -groups of order upto p^7 such that $\text{Aut}_c(G) = \text{Aut}_z(G)$. In section 2.4, we find all finite p -groups G of order p^5 and p^6 for which $\text{Aut}_c(G) = \text{Aut}_z(G)$ from the lists of all such p -groups given by Hall and Senior [27] (for $p = 2$) and given by James [33] (for odd primes p).

We next record some definitions and known results which will be used in the rest of the chapter.

Let G be a finite p -group and M be non-trivial proper normal subgroup of G . The pair (G, M) is called a **Camina pair** if $M \subseteq [y, G]$ for all $y \in G - M$; and G is called a **Camina group** if (G, G') is a Camina pair. A finite p -group G is called a **special group** if $Z(G) = G' = \Phi(G)$. In addition, if $|Z(G)| = p$, then G is called an **extra-special group**.

The following well known results of Adney and Yen [1] and Morigi [44] will be referred to as Adney-Yen Lemma and Morigi Lemma respectively in the thesis.

Lemma 2.1.1 ([1, Theorem 1]) *If G is a purely non-abelian group, then there is a one-to-one correspondence between $\text{Aut}_z(G)$ and $\text{Hom}(G/G', Z(G))$.*

Lemma 2.1.2 ([44, Lemma 0.4]) *Let G be a finite nilpotent group of class 2. Then $\exp(G') = \exp(G/Z(G))$ and in the decomposition of $G/Z(G)$ in direct product of cyclic groups, at least two factors of maximal order must occur.*

Lemma 2.1.3 ([42, Lemma 2.1]) *Let G be finite p -group of nilpotency class c and let (G, H) be a Camina pair. Then $H = \gamma_r(G)$ and $H = Z_{c-r+1}(G)$ for some r satisfying $1 < r \leq c$.*

Theorem 2.1.4 ([42, Theorem 2.2]) *Let (G, H) be a Camina pair, let $H = Z(G)$, and let G have class c . Then $Z_r(G)/Z_{r-1}(G)$ has exponent p whenever $1 \leq r \leq c$.*

Lemma 2.1.5 ([14, Lemma 2.8]) *Let A and B be abelian groups with C a proper subgroup or quotient of A , and D a proper subgroup or quotient of B , such that $|A|/|C| = n = |B|/|D|$, for some $n > 1$. Then $\text{Hom}(C, D)$ is isomorphic to a proper subgroup of $\text{Hom}(A, B)$.*

Proposition 2.1.6 ([39, Proposition 2.2]) *Let G be a direct product of its subgroups H and K . Then G enjoys Hasse Principle if and only if both H and K enjoy Hasse Principle.*

Proposition 2.1.7 ([40, Proposition 4.1]) *Let G be a group which can be generated by 2 elements x, y such that every element of G can be written in the form $x^r y^s$, where r, s are integers. Then G enjoys Hasse Principle.*

Proposition 2.1.8 ([29, Proposition 14.4]) *Let G be a finite group having an abelian normal subgroup A with cyclic quotient G/A . Then class-preserving automorphisms of G are inner automorphisms.*

Lemma 2.1.9 ([65, Lemma 2.2]) *Let G be a finite p -group such that $Z(G) \subseteq [x, G]$ for all $x \in G - G'$. Then $|\text{Aut}_c(G)| \geq |\text{Aut}_z(G)||G/Z_2(G)|$.*

Theorem 2.1.10 ([65, Theorem 3.5]) *Let G be a finite p -group of class 2. Let $\{x_1, x_2, \dots, x_d\}$ be a minimal generating set for G such that $[x_i, G]$ is cyclic for each i , $1 \leq i \leq d$. Then $\text{Out}_c(G) = 1$.*

Theorem 2.1.11 ([65, Theorem 5.5]) *Let G be a finite p -group of order p^5 , where p*

is an odd prime. Then $\text{Out}_c(G) \neq 1$ if and only if G is isoclinic to one of the groups G_7 , G_{10} and H .

Lemma 2.1.12 ([7, Lemma 36.5 (a)]) *If a finite p -group G is a two-generated group of class 2, then G' is cyclic.*

2.2 — Main Results

We start with the following theorem which gives necessary and sufficient conditions on a finite p -group G such that $\text{Aut}_c(G) = \text{Aut}_z(G)$.

Theorem 2.2.1 *Let G be a finite p -group. Then $\text{Aut}_c(G) = \text{Aut}_z(G)$ if and only if $\text{Aut}_c(G) \simeq \text{Hom}(G/Z(G), G')$ and $G' = Z(G)$.*

Proof. First suppose that $\text{Aut}_c(G) = \text{Aut}_z(G)$. We show that $Z(G) \leq \Phi(G)$. Assume contrarily that $Z(G)$ is not contained in $\Phi(G)$. We can then choose an element $h \in Z(G) - \Phi(G)$ for some maximal subgroup M of G . Observe that $G = M\langle h \rangle$ and every element $g \in G$ can be written as $g = mh^i$ for some $m \in M$ and $0 \leq i \leq p-1$. Let $z \in Z(G) \cap \Phi(G)$ be of order p . It is easy to check that the map $\mu : G \rightarrow G$ defined by

$$\mu(mh^i) = mh^i z^i, \quad m \in M, \quad 0 \leq i \leq p-1$$

is a central automorphism of G but not a class-preserving automorphism of G . It thus follows that $Z(G) \leq \Phi(G)$. For any commutator $[a, b] \in G$, where $a, b \in G$, we can define an inner automorphism $T_b : G \rightarrow G$ such that $a^{-1}T_b(a) = [a, b] \in Z(G)$. Thus $G' \leq Z(G)$. For any $\mu \in \text{Aut}_z(G)$, the map $\psi_\mu : G/Z(G) \rightarrow G'$ defined as

$$\psi_\mu(bZ(G)) = b^{-1}\mu(b)$$

is a homomorphism. It is easy to see that the map

$$\psi : \text{Aut}_z(G) \rightarrow \text{Hom}(G/Z(G), G')$$

sending μ to ψ_μ is a monomorphism. For any $\tau \in \text{Hom}(G/Z(G), G')$, the map $\mu : G \rightarrow G$ defined as

$$\mu(g) = g\tau(gZ(G)), \quad g \in G$$

is a central automorphism and $\psi(\mu) = \psi_\mu = \tau$. Thus ψ is onto as well and hence

$$\text{Aut}_c(G) \simeq \text{Hom}(G/Z(G), G').$$

But, by Adney-Yen Lemma, $|\text{Aut}_z(G)| = |\text{Aut}_c(G)| = |\text{Hom}(G/G', Z(G))|$ and so

$$|\text{Hom}(G/Z(G), G')| = |\text{Hom}(G/G', Z(G))|. \quad (2.1)$$

Suppose to the contrary that $G' < Z(G)$. Then $G/Z(G)$ is a proper quotient of G/G' and

$$|(G/G')/(G/Z(G))| = |Z(G)/G'| > 1.$$

It thus follows from Lemma 2.1.5 that $\text{Hom}(G/Z(G), G')$ is isomorphic to a proper subgroup of $\text{Hom}(G/G', Z(G))$. This is a contradiction to (2.1) and hence $G' = Z(G)$.

Conversely assume that $\text{Aut}_c(G) \simeq \text{Hom}(G/Z(G), G')$ and $G' = Z(G)$. Observe that since $G' = Z(G)$, $\text{Aut}_c(G) \leq \text{Aut}_z(G)$ and G is purely non-abelian. By Adney-Yen Lemma

$$|\text{Aut}_z(G)| = |\text{Hom}(G/G', Z(G))| = |\text{Hom}(G/Z(G), G')| = |\text{Aut}_c(G)|.$$

This completes the proof. □

As a consequence of Theorem 2.2.1, we next give an easy proof of main result of Curran and McCaughan [15]. The proof of only if part is easy and is similar to as given in [15].

Theorem 2.2.2 *If G is a finite p -group, then $\text{Aut}_z(G) = \text{Inn}(G)$ if and only if $G' = Z(G)$ and $Z(G)$ is cyclic.*

Proof. Suppose $G' = Z(G)$ and $Z(G)$ is cyclic. Then $\text{Inn}(G) \leq \text{Aut}_z(G)$, G is purely non-abelian and $\exp(G/Z(G)) = \exp(G')$ by Morigi Lemma. By Adney-Yen Lemma

$$|\text{Aut}_z(G)| = |\text{Hom}(G/G', Z(G))| = |G/Z(G)| = |\text{Inn}(G)|,$$

and thus $\text{Aut}_z(G) = \text{Inn}(G)$.

Conversely suppose that $\text{Inn}(G) = \text{Aut}_z(G)$. Then $cl(G) = 2$ and hence

$$\text{Inn}(G) = \text{Aut}_c(G) = \text{Aut}_z(G).$$

By Theorem 2.2.1, $G' = Z(G)$ and

$$\text{Hom}(G/Z(G), G') \simeq \text{Aut}_z(G) = \text{Inn}(G) \simeq G/Z(G).$$

Since $\exp(G/Z(G)) = \exp(G')$, it follows that G' is cyclic. □

Theorem 2.2.3 *Let G be a finite non-abelian p -group such that $Z(G)$ is elementary abelian. Then $\text{Aut}_c(G) = \text{Aut}_z(G)$ if and only if G is a Camina p -group of nilpotency class 2.*

Proof. First suppose that G is a Camina p -group of nilpotency class 2. Then $G' = Z(G)$ by Lemma 2.1.3. Let $\varphi \in \text{Aut}_c(G)$ and $g \in G$. Then $g^{-1}\varphi(g) \in G' = Z(G)$. Thus φ is a central automorphism. On the other hand, let ψ be a central

automorphism and let $h \in G$. If $h \in Z(G)$, then $\psi(h) = h$ and if $h \in G - Z(G)$, then $h^{-1}\psi(h) \in Z(G) \subseteq [h, G]$. It thus follows that $\psi(h) = b^{-1}hb$ for some $b \in G$. Therefore ψ is a class preserving automorphism and hence $\text{Aut}_c(G) = \text{Aut}_z(G)$.

Conversely suppose that $\text{Aut}_c(G) = \text{Aut}_z(G)$. By Theorem 2.2.1, $G' = Z(G)$ and $\text{Aut}_c(G) \simeq \text{Hom}(G/Z(G), G')$. It follows that

$$\exp(G/Z(G)) = \exp(G') = \exp(Z(G)) = p.$$

Let $|G| = p^l$, $\{y_1, y_2, \dots, y_r\}$ be a minimal generating set of G and let $|G'| = p^s$.

Then

$$|\text{Aut}_c(G)| = |\text{Hom}(G/Z(G), G')| = p^{s(l-s)} = p^{sr}.$$

Any element $y \in G - G'$ is a part of a minimal generating set $\{y = y_1, y_2, \dots, y_r\}$ for G . If possible, suppose that $[y, G] < G'$. Then

$$|y^G| = |[y, G]| < |G'| = p^s.$$

Thus $|\text{Aut}_c(G)| < p^{sr}$, which is a contradiction and hence G is a Camina p -group of class 2. □

Theorem 2.2.4 *Let G be a finite non-abelian p -group such that $Z(G)$ is cyclic. Then $\text{Aut}_c(G) = \text{Aut}_z(G)$ if and only if $Z(G) = G'$.*

Proof. If $Z(G) = G'$, then $\text{Inn}(G) = \text{Aut}_z(G)$ by Theorem 2.2.2. Let μ be any class-preserving automorphism of G and let $g \in G$. Then $g^{-1}\mu(g) \in G' = Z(G)$. Thus μ is a central automorphism and hence

$$\text{Aut}_c(G) \leq \text{Aut}_z(G) = \text{Inn}(G) \leq \text{Aut}_c(G).$$

Converse follows from Theorem 2.2.1. □

2.3 — Groups of order upto p^7

In this section, we classify all finite p -groups of order p^n ($n \leq 7$) such that $\text{Aut}_c(G) = \text{Aut}_z(G)$. For this, we need next theorem which also completes the classification of all finite p -groups of order $\leq p^5$ such that $\text{Aut}_c(G) \neq \text{Inn}(G)$. In the theorem, we show that out of 51 groups of order 32, there are only 2 groups for which $\text{Aut}_c(G) \neq \text{Inn}(G)$. A list of groups of order 32 is available from Hall and Senior [27], and from Sag and Wamsley [54] with minimal presentations of groups. In [27], groups of order 32 are divided into 8 isoclinism families. We denote the i th family by Φ_i . As mentioned in [54], the groups are listed in same order in both [27] and [54] and so we take the liberty of choosing the presentation of a group from any of the lists. Unless or otherwise stated, we use Sag and Wamsley's list and adopt the same notations for the nomenclature and presentations of the groups. However, we write the i -th group of order 32 in the list as G_i and generators 1,2,3 and 4 respectively as x, y, z and w . Yadav [65] has proved that if G and H are two finite non-abelian isoclinic groups, then $\text{Aut}_c(G) \simeq \text{Aut}_c(H)$. Therefore, it is sufficient to pick only one group from each isoclinism family. We need the following lemma to prove the theorem.

Lemma 2.3.1 *If G is a group of order p^n , $n \geq 3$, and $|Z(G)| = p^{n-2}$, then $\text{Aut}_c(G) = \text{Inn}(G)$.*

Proof. Observe that nilpotency class of G is 2. If x is a non-central element of G , then $|C_G(x)| = p^{n-1}$ and thus $|x^G| = |[x, G]| = p$. If x is a central element of G , then $|x^G| = |[x, G]| = 1$. In any case, $[x, G]$ is cyclic and therefore $\text{Aut}_c(G) = \text{Inn}(G)$ by Theorem 2.1.10. □

Theorem 2.3.2 *For all groups G of order 32, except for the groups G_{44} and G_{45} in the sixth family Φ_6 , $\text{Aut}_c(G) = \text{Inn}(G)$.*

Proof. The first family Φ_1 contains abelian groups G_1 to G_7 and therefore all class preserving automorphisms for these groups are inner. From the second family Φ_2 , we pick

$$G_{11} = \{x^2, y^4, z^2, [x, y, x], [x, y, y], [x, z], [y, z]\}.$$

Clearly z is in the center, and since

$$[x, y^2] = [x, y]^2 = [x^2, y] = 1,$$

it follows that y^2 is in the center and $|[x, y]| = 2$. Also, since z commutes with x and y , it commutes with $[x, y]$ as well. Thus $[x, y]$ is also in the center and hence $|Z(G_{11})| = 8$. Therefore $\text{Aut}_c(G_{11}) = \text{Inn}(G_{11})$ by above lemma.

From third isoclinism family, we take

$$G_{23} = \{x^8, y^2, z^2, [x, y]x^2, [x, z], [y, z]\}.$$

Observe that $G_{23} = H \oplus K$, where

$$H = \langle x, y \mid x^8 = y^2 = 1, yx = x^7y \rangle$$

and

$$K = \langle z \mid z^2 = 1 \rangle.$$

It is easily seen that K is cyclic and therefore $\text{Aut}_c(K) = \text{Inn}(K)$. Also, since every element of H can be written as $x^i y^j$ for suitable i and j , $\text{Aut}_c(H) = \text{Inn}(H)$ by Proposition 2.1.7. Hence $\text{Aut}_c(G_{23}) = \text{Inn}(G_{23})$ by Proposition 2.1.6.

In Φ_4 , consider

$$G_{34} = \{x^4, y^4, z^2, [x, y], [x, z]x^2, [y, z]y^2\}.$$

Let $H_{34} = \langle x, y \rangle$. Then H_{34} is an abelian normal subgroup of order 16, and therefore G_{34}/H_{34} is a cyclic group of order 2. Hence $\text{Aut}_c(G_{34}) = \text{Inn}(G_{34})$ by Proposition 2.1.8.

From fifth family, consider

$$G_{43} = \{w^2, x^2y^{-2}, x^2z^{-2}, [z, y]x^2, [w, x]x^2, [x, y], [x, z], [y, w], [z, w]\}.$$

Since $d(G_{43}) = 4$, $|\Phi(G_{43})| = |(G_{43})'| = 2$. Also, since G_{43} is a stem group, $Z(G_{43}) \leq (G_{43})'$. Thus G_{43} is an extra-special group and hence $\text{Aut}_c(G_{43}) = \text{Inn}(G_{43})$ by [39, Theorem 3.2].

Next consider the group

$$G = G_{44} = \{x^2, z^2, [y, x]y^4, [y, z]y^2, [x, z]\}$$

from Φ_6 . Since

$$yx = xy[y, x] = xyy^{-4} = xy^{-3},$$

and since $[y, z] = y^{-2}$ implies that $zy = y^{-1}z$, it follows that every element of G can be written in the form $x^i y^j z^k$, where $0 \leq i, k \leq 1$, and $0 \leq j \leq 7$; and hence $|y| = 8$. Since G is of nilpotency class 3 and $d(G) = 3$, it follows that $|\Phi(G)| = |G'| = 4$. But $y^2 = [z, y] \in G'$ is of order 4, therefore $G' = \langle y^2 \rangle$. Now G_{44} is a stem group, therefore $Z(G) < G'$. Since

$$[z, y^4] = [z, y^2]^2 = [z, y]^4 = y^8 = 1$$

and

$$[x, y^2] = [x, y]^2 = y^8 = 1,$$

$y^4 \in Z(G)$ and thus $Z(G) = \langle y^4 \rangle$. Therefore $G/Z(G)$ is a class 2 group of order 16.

Thus $Z(G/Z(G))$ is of order 4 and hence $|Z_2(G)| = 8$. We prove that $Z(G) \subseteq [g, G]$

for all $g \in G - G'$. Let $g = x^i y^j z^k \in G - G'$. First suppose that j is even. Then

$g = y^j x^i z^k$. Both i and k cannot be zero, because then $g \in G'$. If $k = 1$, then

$$[g, y^2] = [y^j x^i z, y^2] = [x^i z, y^2] = [z, y^2] = y^4$$

and if $k = 0$, then

$$[g, y] = [y^j x, y] = [x, y] = y^4.$$

Thus $Z(G) \subseteq [g, G]$ for all $g \in G - G'$ in this case. Next suppose that j is odd.

Then $g = y^{j-1} x^i y z^k$. If $k = 1$, then

$$[g, y^2] = [y^{j-1} (x^i y z), y^2] = [x^i (y z), y^2] = [y z, y^2] = [z, y^2] = [z, y]^2 = y^4,$$

and if $k = 0, i = 1$, then

$$[g, y] = [y^{j-1} (x y), y] = [x y, y] = [x, y] = y^4,$$

and finally if $k = 0 = i$, then

$$[g, x] = [y^{j-1} y, x] = [y, x] = y^4.$$

Thus $Z(G) \subseteq [g, G]$ for all $g \in G - G'$ in this case as well. Therefore, by Lemma

2.1.9.

$$|\text{Aut}_c(G)| \geq |\text{Aut}_z(G)| |G/Z_2(G)|.$$

Since $Z(G) < G'$, G is purely non abelian and therefore

$$|\text{Aut}_z(G)| = |\text{Hom}(G/G', Z(G))| = |\text{Hom}(C_2 \times C_2 \times C_2, C_2)| = 8$$

by Adney-Yen Lemma. Thus $|\text{Aut}_c(G)| \geq 2^5 > 2^4 = |\text{Inn}(G)|$.

From seventh family, we take

$$G_{46} = \{x^2, y^4, [x, y, x], [x, y, y, y]\}.$$

Let $u = [x, y]$ and $v = [x, y, y] = [u, y]$. Then u commutes with x and v commutes with y . Observe that $1 = [x^2, y] = [x, y]^2 = u^2$; $[x, y^2] = [x, y]^2[x, y, y] = v$; $1 = [x, y^4] = [x, y^2]^2[x, y^2, y^2] = v^2$. Thus $|u| = |v| = 2$. Since

$$xy^2 = (xy)y = (yxu)y = yx(uy) = yx(yuv) = y(xy)uv = y(yxu)uv = y^2xv,$$

therefore

$$vx = (xy^2xy^2)x = (xy^2x)y^2x = (xy^2x)(xy^2v) = xv.$$

Thus v commutes with x and hence with u as well. It is easy to see that every element of G_{46} can be written in the form $x^i y^j u^k v^l$, where $0 \leq i, k, l \leq 1$ and $0 \leq j \leq 3$. Take

$$H_{46} = \langle x, u, v \rangle.$$

Then H_{46} is an abelian normal subgroup of order 8 such that $G_{46}/H_{46} = \langle yH_{46} \rangle$ is a cyclic group of order 4. Thus $\text{Aut}_c(G_{46}) = \text{Inn}(G_{46})$ by Proposition 2.1.8.

Finally consider

$$G_{51} = \{x^8 y^{-2}, [y, x^7]x^2\}$$

from Φ_8 . The relation $[x^7, y] = x^2$ implies that $yx^9 = x^7y$. Post multiplying by $x^8 = y^2$, we get $yx = x^7y^3 = x^{15}y$. Thus every element of G_{51} can be written as $x^i y^j$, $0 \leq i \leq 15, 0 \leq j \leq 1$ and hence $\text{Aut}_c(G_{51}) = \text{Inn}(G_{51})$ by Proposition 2.1.7. □

Remark: The order of $\text{Aut}_c(G)$ for $G = G_{44}$ in the above theorem is in fact exactly equal to 32. Observe that $C_G(x) = \langle x, y^2, z \rangle$, $C_G(y) = \langle y \rangle$ and $C_G(z) = \langle x, y^4, z \rangle$. Thus $|C_G(x)| = 16$ and $|C_G(y)| = |C_G(z)| = 8$. Hence $|x^G| = 2$ and $|y^G| = |z^G| = 4$. Since any class preserving automorphism preserves the conjugacy classes, there are $|x^G|$, $|y^G|$ and $|z^G|$ choices for the images of x , y and z respectively under it. It follows that $|\text{Aut}_c(G)| \leq |x^G||y^G||z^G| = 32$.

Theorem 2.3.3 *Let G be a non-abelian group of order p^n , $3 \leq n \leq 5$. Then $\text{Aut}_c(G) = \text{Aut}_z(G)$ if and only if $G' = Z(G)$ and $Z(G)$ is cyclic.*

Proof. Suppose $\text{Aut}_c(G) = \text{Aut}_z(G)$. Then $G' = Z(G)$ by Theorem 2.2.1. If $|G| = p^3$ or p^4 , then $\text{Aut}_c(G) = \text{Inn}(G)$ by [39, 40]. If $|G| = p^5$, then since class of G is 2, it follows from Theorem 2.3.2 (for $p = 2$) and from Theorem 2.1.11 (for odd p) that $\text{Aut}_c(G) = \text{Inn}(G)$. Thus $\text{Inn}(G) = \text{Aut}_z(G)$ and hence G' is cyclic by Theorem 2.2.2. The converse follows from Theorem 2.2.4. \square

Theorem 2.3.4 *Let G be a non-abelian group of order p^6 . Then $\text{Aut}_c(G) = \text{Aut}_z(G)$ if and only if either $G' = Z(G)$ and $Z(G)$ is cyclic or G is a Camina p -group of nilpotency class 2.*

Proof. Suppose $\text{Aut}_c(G) = \text{Aut}_z(G)$. Then $G' = Z(G)$ by Theorem 2.2.1. If $Z(G)$ is not cyclic, then we prove that $Z(G)$ is elementary abelian and the result will then follow from Theorem 2.2.3. Now $p^2 \leq |Z(G)| \leq p^4$. If $|Z(G)| = p^2$, then it is trivially elementary abelian. Let $|Z(G)| = p^3$ and let $Z(G) \simeq C_{p^2} \times C_p$. Then $\exp(Z(G)) = \exp(G') = \exp(G/Z(G)) = p^2$ and hence $G/Z(G) \simeq C_{p^2} \times C_p$, which

is a contradiction to Morigi Lemma. If $|Z(G)| = p^4$, then $G/Z(G)$ is elementary abelian. Therefore $\exp(Z(G)) = \exp(G') = \exp(G/Z(G)) = p$ and hence $Z(G)$ is elementary abelian. The converse follows from Theorems 2.2.3 and 2.2.4. \square

Theorem 2.3.5 *Let G be a non-abelian p -group of order p^7 . Then $\text{Aut}_c(G) = \text{Aut}_z(G)$ if and only if either $G' = Z(G)$ and $Z(G)$ is cyclic or G is a Camina p -group of nilpotency class 2.*

Proof. Suppose $\text{Aut}_c(G) = \text{Aut}_z(G)$. Then $G' = Z(G)$ by Theorem 2.2.1. If $Z(G)$ is not cyclic, then as in Theorem 2.3.4, we need only prove that $Z(G)$ is elementary abelian. Now $p^2 \leq |Z(G)| \leq p^5$. The cases $|Z(G)| = p^2$ or p^5 can be handled as in the above theorem. Let $|Z(G)| = p^3$ and let $Z(G) \simeq C_{p^2} \times C_p$. Then $G/Z(G) \simeq C_{p^2} \times C_{p^2}$ by Morigi Lemma. Thus G is a 2-generated class 2 group and hence G' is cyclic by Lemma 2.1.12. This is a contradiction to $G' = Z(G) \simeq C_{p^2} \times C_p$. If $|Z(G)| = p^4$, then $\exp(Z(G)) = p^2$ or p^3 , $|G/Z(G)| = p^3$ and $\exp(G/Z(G)) = p^2$ or p^3 . This is not possible by Morigi Lemma. The converse follows from Theorems 2.2.3 and 2.2.4. \square

2.4 — Application

In this section, we use the classification of all groups of order p^n , $5 \leq n \leq 6$, given by James [33] for odd p and by Hall and Senior [27] for $p = 2$. As an application of our results, we find those groups G of order p^5 and p^6 for which $\text{Aut}_c(G) = \text{Aut}_z(G)$.

Theorem 2.4.1 *Let G be a non-abelian group of order p^5 , where p is an odd prime. Then $\text{Aut}_c(G) = \text{Aut}_z(G)$ if and only if $G \in \Phi_5$.*

Proof. By Theorem 2.3.3, $\text{Aut}_c(G) = \text{Aut}_z(G)$ if and only if $G' = Z(G)$ and $Z(G)$ is cyclic. This happens if and only if $G \in \Phi_5$ by §4.1 of [33]. \square

Lemma 2.4.2 *A non-abelian group G of order p^6 , where p is an odd prime, is a Camina group of class 2 if and only if $G \in \Phi_{15}$.*

Proof. In a Camina p -group of class 2, $G' = Z(G)$ and $|x^G| = |G'|$ for each $x \in G - Z(G)$. This happens if and only if $G \in \Phi_{15}$ by §4.1 of [33]. \square

Theorem 2.4.3 *Let G be a non-abelian group of order p^6 , where p is an odd prime. Then $\text{Aut}_c(G) = \text{Aut}_z(G)$ if and only if $G \in \Phi_{14}$ or Φ_{15} .*

Proof. By Theorem 2.3.4, $\text{Aut}_c(G) = \text{Aut}_z(G)$ if and only if either G is a Camina p -group of nilpotency class 2 or $G' = Z(G)$ and $Z(G)$ is cyclic. Now G is a Camina p -group of class 2 if and only if $G \in \Phi_{15}$ by above lemma, and $G' = Z(G)$ and $Z(G)$ is cyclic if and only if $G \in \Phi_{14}$ by §4.1 of [33]. \square

Theorem 2.4.4 *Let G be a non-abelian group of order 2^5 . Then $\text{Aut}_c(G) = \text{Aut}_z(G)$ if and only if $G \in \Phi_5$.*

Proof. By Theorem 2.3.3, $\text{Aut}_c(G) = \text{Aut}_z(G)$ if and only if $G' = Z(G)$ and $Z(G)$ is cyclic. There are only two families *viz.* Φ_4 and Φ_5 which consist of groups G such that $G' = Z(G)$. Consider the group

$$H = G_{34} = \{x^4, y^4, z^2, [x, y], [x, z]x^2, [y, z]y^2\}$$

from Φ_4 . Observe that

$$[x^2, z] = [x, z]^2 = x^4 = 1 \quad \text{and} \quad [y^2, z] = [y, z]^2 = y^4 = 1.$$

Thus $x^2, y^2 \in Z(H) = H'$ and hence $|H'| \geq 4$. Since $d(H) = 3$, $|\Phi(H)| = 4$ and thus $Z(H) = H' = \Phi(H)$. Now $\exp(H/Z(H)) = \exp(H')$ implies that H' is elementary abelian. As proved in Theorem 2.3.2, the group G_{43} of fifth family Φ_5 is an extra-special group. Since G' is family invariant, all the groups in fourth family have elementary abelian center and all the groups in fifth family have cyclic center. Hence $\text{Aut}_c(G) = \text{Aut}_z(G)$ if and only if $G \in \Phi_5$. \square

In the next lemma, all the presentations for groups are taken from [27]. For the sake of brevity, we only write the relators. And for the same reason, a relator of the form $[\alpha_i, \alpha_j]$ is excluded from the presentation if α_i commutes with α_j . For the group $G_{(i,j)}$, i stands for the group number and j stands for the family number.

Lemma 2.4.5 *A non-abelian group G of order 2^6 is a Camina group of class 2 if and only if $G \in \Phi_{13}$.*

Proof. Since $G' = Z(G)$ for a Camina p -group of class 2, we need only consider the families Φ_9 to Φ_{13} . These families contain stem groups and the structure of conjugacy classes for stem groups is an invariant of the isoclinism family. Therefore, it is sufficient to pick one group from each of these families. First consider

$$G = G_{(144,9)} = \langle \alpha_1^2, \alpha_2^2, \alpha_3^2, \alpha_4^2, \alpha_5^2, \alpha_6^2, [\alpha_4, \alpha_5]\alpha_1^{-1}, [\alpha_4, \alpha_6]\alpha_2^{-1}, [\alpha_5, \alpha_6]\alpha_3^{-1} \rangle.$$

Since $\alpha_1, \alpha_2, \alpha_3 \in Z(G)$, $|Z(G)| = |G'| \geq 8$. Therefore if $x \in G - Z(G)$, then $|x^G| = |[x, G]| \leq 4$ and hence G is not a Camina group. Next consider

$$G = G_{(154,10)} = \langle \alpha_1^2, \alpha_2^2, \alpha_4^2, \alpha_6^2, \alpha_3^2\alpha_1^{-1}, \alpha_5^2\alpha_2^{-1}, [\alpha_3, \alpha_4]\alpha_1^{-1}, [\alpha_5, \alpha_6]\alpha_2^{-1} \rangle.$$

Observe that $\alpha_1, \alpha_2 \in Z(G)$ and hence $|Z(G)| = |G'| \geq 4$. Since

$$\alpha_1, \alpha_2, \alpha_3, \alpha_5, \alpha_6 \in C_G(\alpha_3)$$

and every element of the subgroup generated by $\{\alpha_1, \alpha_2, \alpha_3, \alpha_5, \alpha_6\}$ can be written as $\alpha_1^i \alpha_2^j \alpha_3^k \alpha_5^l \alpha_6^m$, $0 \leq i, j, k, l, m \leq 1$, it follows that $|C_G(\alpha_3)| = 32$. Thus

$$|\alpha_3^G| = |[\alpha_3, G]| = 2 \neq |G'|$$

and hence G is not a Camina group. Next consider

$$G = G_{(169,11)} = \langle \alpha_1^2, \alpha_2^2, \alpha_3^2, \alpha_4^2, \alpha_6^2, \alpha_5^2 \alpha_2^{-1}, [\alpha_4, \alpha_5] \alpha_1^{-1}, [\alpha_3, \alpha_6] \alpha_1^{-1}, [\alpha_5, \alpha_6] \alpha_2^{-1} \rangle.$$

As above, we can show that

$$|\alpha_3^G| = |[\alpha_3, G]| = 2 \neq |G'| \geq 4$$

and hence $G_{(169,11)}$ is also not a Camina group. We next consider

$$G = G_{(180,12)} = \langle \alpha_1^2, \alpha_2^2 \alpha_1^{-1}, \alpha_3^2 \alpha_1^{-1}, \alpha_4^2, \alpha_5^2 \alpha_3^{-1}, \alpha_6^2 \alpha_4^{-1}, [\alpha_4, \alpha_5] \alpha_1^{-1}, \\ [\alpha_3, \alpha_6] \alpha_1^{-1}, [\alpha_5, \alpha_6] \alpha_2^{-1} \rangle.$$

If $|Z(G)| = 16$, then $G/Z(G)$ and hence G' is elementary abelian, which is not so because $\alpha_2 = [\alpha_5, \alpha_6] \in G'$ is of order 4. If $|Z(G)| = 8$, then $Z(G) \simeq C_2^2 \times C_2$ or $Z(G) \simeq C_2 \times C_2 \times C_2$. The first case is not possible by Morigi Lemma and the second one is ruled out because $\alpha_2 = [\alpha_5, \alpha_6] \in G' = Z(G)$ is of order 4. Thus $Z(G)$ is a cyclic group, generated by α_2 , of order 4. Therefore G cannot be a Camina group by Lemma 2.1.3 and Theorem 2.1.4. Finally consider

$$G = G_{(183,13)} = \langle \alpha_1^2, \alpha_2^2, \alpha_3^2 \alpha_1^{-1}, \alpha_4^2 \alpha_2^{-1}, \alpha_5^2, \alpha_6^2, [\alpha_3, \alpha_5] \alpha_1^{-1}, \\ [\alpha_4, \alpha_5] \alpha_2^{-1} \alpha_1^{-1}, [\alpha_3, \alpha_6] \alpha_2^{-1} \alpha_1^{-1}, [\alpha_4, \alpha_6] \alpha_2^{-1} \rangle.$$

Observe that $G' = Z(G) = \langle \alpha_1, \alpha_2 \rangle$ and every element g of G is of the form

$$z\alpha_3^{m_3}\alpha_4^{m_4}\alpha_5^{m_5}\alpha_6^{m_6}, \quad z \in Z(G), \quad 0 \leq m_3, m_4, m_5, m_6 \leq 1.$$

We claim that $Z(G) \subseteq [g, G]$ for all $g \in G - Z(G)$. First suppose that $m_6 = 1$. If $m_5 = 0$, then $[g, \alpha_4] = [\alpha_5^{m_5}\alpha_6, \alpha_4] = \alpha_2$ where as $[g, \alpha_3] = \alpha_1\alpha_2$. And if $m_5 = 1$, then $[g, \alpha_4] = \alpha_1$ where as $[g, \alpha_3] = \alpha_2$. Now suppose that $m_6 = 0$ and $m_5 = 1$. Then $[g, \alpha_3] = \alpha_1$ and $[g, \alpha_4] = \alpha_1\alpha_2$. We next suppose that $m_5 = m_6 = 0$ and $m_4 = 1$. If $m_3 = 0$, then $[g, \alpha_6] = \alpha_2$ where as $[g, \alpha_5] = \alpha_1\alpha_2$. And if $m_3 = 1$, then $[g, \alpha_6] = \alpha_1$ where as $[g, \alpha_5] = \alpha_2$. Finally suppose that $m_4 = m_5 = m_6 = 0$ and $m_3 = 1$. Then $[g, \alpha_5] = \alpha_1$ and $[g, \alpha_6] = \alpha_1\alpha_2$. In every possibility, $Z(G) \subseteq [g, G]$ for all $g \in G - Z(G)$. This proves the claim and the lemma. \square

Theorem 2.4.6 *Let G be a non-abelian group of order 2^6 . Then $\text{Aut}_c(G) = \text{Aut}_z(G)$ if and only if $G \in \Phi_{12}$ or Φ_{13} .*

Proof. First suppose that $\text{Aut}_c(G) = \text{Aut}_z(G)$. By Theorem 2.3.4, either G is a Camina group of class 2 or $G' = Z(G)$ and $Z(G)$ is cyclic. If G is a Camina group of class 2, then $G \in \Phi_{13}$ by above lemma. Assume $G' = Z(G)$ and $Z(G)$ is cyclic. There are only five isoclinism families *viz.* Φ_9 to Φ_{13} which consist of groups G such that $G' = Z(G)$. Consider $G_{144} \in \Phi_9$. By above lemma $|Z(G_{144})| \geq 8$ and by [54], $d(G_{144}) = 3$. Thus $Z(G_{144}) = (G_{144})' = \Phi(G_{144})$. Now $\exp(G_{144}/Z(G_{144})) = \exp((G_{144})')$ implies that $Z(G_{144})$ is elementary abelian of order 8. Next consider $G_{154} \in \Phi_{10}$. By above lemma $|Z(G_{154})| \geq 4$ and by [54], $d(G_{154}) = 4$. Thus $Z(G_{154}) = (G_{154})' = \Phi(G_{154})$. Then $\exp(G_{154}/Z(G_{154})) = \exp((G_{154})')$ implies that $Z(G_{154})$ is elementary abelian of order 4. Next consider $G_{169} \in \Phi_{11}$. By above lemma

$|Z(G_{169})| \geq 4$ and by [54], $d(G_{169}) = 4$. Thus $Z(G_{169}) = (G_{169})' = \Phi(G_{169})$. Therefore $\exp(G_{169}/Z(G_{169})) = \exp((G_{169})')$ implies that $Z(G_{169})$ is elementary abelian of order 4. Finally, if $G \in \Phi_{12}$, then $Z(G)$ is cyclic by Lemma 2.4.5. \square

CHAPTER 3

Central Automorphism Group of Minimal Order

3.1 — Introduction

Let G be a finite p -group. An automorphism α of G is called a central automorphism if it commutes with all inner automorphisms; or equivalently, $x^{-1}\alpha(x) \in Z(G)$ for all $x \in G$. The group $\text{Aut}_z(G)$ of all central automorphisms of G is a normal subgroup of G and fixes the commutator subgroup G' of G element-wise. Observe that the center $Z(\text{Inn}(G))$ of the group of all inner automorphisms of G is always contained in $\text{Aut}_z(G)$. It follows that $Z(\text{Inn}(G)) \leq \text{Aut}_z(G) \leq \text{Aut}(G)$ for any group G . Non-abelian p -groups G for which $\text{Aut}_z(G) = \text{Aut}(G)$ have been well studied (see for example [6, 13, 34]). Curran [14] first considered the other extreme possibility for $\text{Aut}_z(G)$. He studied non-abelian finite p -groups G and gave some necessary conditions on G such that $\text{Aut}_z(G) = Z(\text{Inn}(G))$. Recently, Sharma and Gumber [57] have characterized all finite p -groups G of order p^5 and p^6 (p an odd prime) for which $\text{Aut}_z(G) = Z(\text{Inn}(G))$.

In section 3.2, we give necessary and sufficient conditions on a finite p -group G with cyclic center such that $\text{Aut}_z(G)$ is of minimal order. As a consequence of it,

we obtain necessary and sufficient conditions on finite p -groups G of coclass upto 4 for which $\text{Aut}_z(G) = Z(\text{Inn}(G))$. In section 3.3, as an applications of the results obtained in section 3.2, we characterize all finite p -groups G of order upto p^7 for which $\text{Aut}_z(G) = Z(\text{Inn}(G))$. Our proofs are easy, short and even generalize the main results of Sharma and Gumber [57].

If G is a finite p -group and if $\mathbf{G}^p = \langle x^p | x \in \mathbf{G} \rangle$, then $\Phi(G) = G'G^p$. By C_m^n we denote the n copies of C_m . A finite p -group G of order p^n and of nilpotency class c is said to of coclass $n - c$. A group of **coclass** 1 is of course a group of **maximal class**.

Let G be a finite p -group of maximal class and let $|G| = p^n$. Observe that G/G' is not cyclic and $\gamma_n(G) = 1$. It follows that $|G'| = p^{n-2}$ and $|\gamma_i(G)| = p^{n-i}$ for all $i, 3 \leq i \leq n$. Also, since $G/Z_{n-2}(G)$ cannot be cyclic, $|Z_i(G)| = p^i$ for all $i, 1 \leq i \leq n - 2$. By using similar arguments, it is easy to see that if G is of coclass upto 4, then $p \leq |Z(G)| \leq p^4$, $p^2 \leq |Z_2(G)| \leq p^5$ and $p^{n-5} \leq |G'| \leq p^{n-2}$.

The following results of Curran [14, Corollaries 3.7, 3.8] and Gavioli [22, Lemma 2] will be used quite frequently.

Lemma 3.1.1 *Let G be a finite non-abelian p -group such that $\text{Aut}_z(G) = Z(\text{Inn}(G))$. Then $Z(G) \leq G'$ and $Z(\text{Inn}(G))$ is not cyclic.*

Lemma 3.1.2 *Let G be a (not necessarily finite) group with center $Z(G)$ and second center $Z_2(G)$, A an abelian normal subgroup of G with $Z(G) \leq A \leq Z_2(G)$. Then $A/Z(G)$ embeds in $\text{Hom}(G/A, Z(G))$.*

We next prove an important lemma which is a small but very useful modification of the following lemma of Alperin [2, Lemma 3].

Lemma 3.1.3 *Let A be a central and characteristic subgroup of a group H . The group K of those automorphisms of H which leave A element-wise fixed and which induce the identity automorphism on H/A is naturally isomorphic to the group $\text{Hom}(H/A, A)$ of homomorphisms of H/A into A .*

Lemma 3.1.4 *Let G be any group and Y be a central subgroup of G contained in a normal subgroup X of G . Then the group of all automorphisms of G that induce identity on X and G/Y is isomorphic to $\text{Hom}(G/X, Y)$.*

Proof. Let $\text{Aut}_X^Y(G)$ denote the group of all automorphisms of G that induce identity on both X and G/Y . For any $\mu \in \text{Aut}_X^Y(G)$, the map

$$\psi_\mu : G/X \longrightarrow Y$$

defined as $\psi_\mu(bX) = b^{-1}\mu(b)$ is a homomorphism. The map

$$\psi : \text{Aut}_X^Y(G) \longrightarrow \text{Hom}(G/X, Y)$$

defined as $\psi(\mu) = \psi_\mu$ is a monomorphism. For any $\tau \in \text{Hom}(G/X, Y)$, the map $\mu : G \rightarrow G$ defined as $\mu(g) = g\tau(gX)$ is an automorphism of G inducing identity on both X and G/Y and $\psi(\mu) = \tau$. Thus ψ is onto as well. \square

Let $\text{Aut}_Z^z(G)$ denote the group of all those automorphisms of G which induce identity on both $G/Z(G)$ and $Z(G)$. It immediately follows from above lemma that $\text{Aut}_Z^z(G) \simeq \text{Hom}(G/G'Z(G), Z(G))$. The following main result of Attar [3] is now an easy consequence.

Corollary 3.1.5 *Let G be a finite non-abelian p -group. Then $\text{Aut}_z^z(G) = \text{Inn}(G)$ if and only if nilpotency class of G is 2 and $Z(G)$ is cyclic.*

Proof. Observe that if $\text{Aut}_z^z(G) = \text{Inn}(G)$, then nilpotency class of G is 2. The proof then follows from the simple fact that if A and B are finite abelian p -groups such that $\exp(A) \mid \exp(B)$, then $\text{Hom}(A, B) \simeq A$ if and only if B is cyclic. \square

3.2 — Groups of coclass upto 4

Let

$$G/G' \simeq \prod_{i=1}^n C_{p^{\alpha_i}} = C_{p^{\alpha_1}} \times C_{p^{\alpha_2}} \times \dots \times C_{p^{\alpha_n}}$$

and

$$Z_2(G)/Z(G) \simeq \prod_{i=1}^m C_{p^{\beta_i}} = C_{p^{\beta_1}} \times C_{p^{\beta_2}} \times \dots \times C_{p^{\beta_m}}$$

be the decompositions of G/G' and $Z_2(G)/Z(G)$ into cyclic groups, where for each i , $\alpha_i \geq \alpha_{i+1}$ and $\beta_i \geq \beta_{i+1}$ are positive integers. Observe that if $\text{Aut}_z(G) = Z(\text{Inn}(G))$, then $Z(G) \leq G'$ and hence by Lemma 3.1.4

$$\text{Aut}_z(G) \simeq \text{Hom}(G/G', Z(G)) \simeq Z_2(G)/Z(G).$$

It thus follows that $d(G)d(Z(G)) = d(Z_2(G)/Z(G))$.

Theorem 3.2.1 *Let G be a finite p -group with cyclic center $Z(G) \simeq C_{p^{\gamma_1}}$. Then $\text{Aut}_z(G) = Z(\text{Inn}(G))$ if and only if either G/G' is isomorphic to $Z_2(G)/Z(G)$ or $d(G) = d(Z_2(G)/Z(G))$, $\beta_i = \gamma_1$ for $1 \leq i \leq r$ and $\beta_i = \alpha_i$ for $r+1 \leq i \leq n$, where r , $1 \leq r \leq n$, is the largest such that $\alpha_r \geq \gamma_1$.*

Proof. First suppose that $\text{Aut}_z(G) = Z(\text{Inn}(G))$. Then $m = d(Z_2(G)/Z(G)) = d(G) = n$. Also, since $\exp(Z_2(G)/Z(G)) \leq \exp(Z(G))$ by [52, 5.2.22], therefore $\exp(G/G') \geq \exp(Z_2(G)/Z(G))$. If $\exp(G/G') = \exp(Z_2(G)/Z(G))$, then $G/G' \simeq Z_2(G)/Z(G)$; and if $\exp(G/G') > \exp(Z_2(G)/Z(G))$, then $\exp(Z_2(G)/Z(G)) = \exp(Z(G))$ and hence $\beta_1 = \gamma_1$. Suppose r , $1 \leq r \leq n$, is the largest such that $\alpha_r \geq \gamma_1$. Then

$$\text{Aut}_z(G) \simeq \text{Hom}(\prod_{i=1}^n C_{p^{\alpha_i}}, C_{p^{\gamma_1}}) \simeq C_{p^{\gamma_1}}^r \times \prod_{i=r+1}^n C_{p^{\alpha_i}}.$$

But $\text{Aut}_z(G) \simeq Z_2(G)/Z(G) \simeq \prod_{i=1}^n C_{p^{\beta_i}}$. It follows that $\beta_i = \gamma_1$ for $1 \leq i \leq r$ and $\beta_i = \alpha_i$ for $r+1 \leq i \leq n$.

Conversely, since $Z(G)$ is cyclic, G is purely non-abelian and therefore

$$|\text{Aut}_z(G)| = |\text{Hom}(G/G', Z(G))|$$

by Adney-Yen Lemma. It is now easy to see, using the assumptions, that $|\text{Aut}_z(G)| = |Z(\text{Inn}(G))|$ and hence $\text{Aut}_z(G) = Z(\text{Inn}(G))$. \square

Let G be a finite p -group of order p^n and coclass ≤ 4 such that $\text{Aut}_z(G) = Z(\text{Inn}(G))$. Then $d(G)d(Z(G)) = d(Z_2(G)/Z(G))$ and $p^{n-5} \leq |G'| \leq p^{n-2}$. Also, since $Z(\text{Inn}(G))$ cannot be cyclic, $p \leq |Z(G)| \leq p^3$ and $p^3 \leq |Z_2(G)| \leq p^5$. It is easy to see that $Z(G)$ is cyclic in each case and hence we obtain the following three corollaries.

Corollary 3.2.2 *Let G be a finite p -group of coclass 2. Then $\text{Aut}_z(G) = Z(\text{Inn}(G))$ if and only if $Z(G) \simeq C_p$ and $d(G) = d(Z_2(G)/Z(G)) = 2$.*

Proof. First suppose that $\text{Aut}_z(G) = Z(\text{Inn}(G))$. Then $|Z(G)| = p$, $|Z_2(G)| = p^3$ and $Z(\text{Inn}(G)) \simeq C_p \times C_p$. It follows by theorem that either $G/G' \simeq Z_2(G)/Z(G)$ or $d(G) = d(Z_2(G)/Z(G))$. Thus $d(G) = d(Z_2(G)/Z(G)) = 2$. Conversely, suppose that $Z(G) \simeq C_p$ and $d(G) = d(Z_2(G)/Z(G)) = 2$. Then $\gamma_1 = 1$ and $Z_2(G)/Z(G) \simeq C_p \times C_p$. Thus $\alpha_i \geq \gamma_1$ and $\beta_i = \gamma_1 = 1$ for all i . The result now follows by theorem. \square

Corollary 3.2.3 *Let G be a finite p -group of coclass 3. Then $\text{Aut}_z(G) = Z(\text{Inn}(G))$ if and only if either $Z(G) \simeq C_p$ and $d(G) = d(Z_2(G)/Z(G)) = 2, 3$ or $Z(G) \simeq C_{p^2}$ and $Z_2(G)/Z(G) \simeq G/G'$.*

Proof. The proof is similar as in above corollary. \square

Corollary 3.2.4 *Let G be a finite p -group of coclass 4. Then $\text{Aut}_z(G) = Z(\text{Inn}(G))$ if and only if one of the followings holds:*

- (a) $Z(G) \simeq C_p$ and $d(G) = d(Z_2(G)/Z(G)) = 2, 3, 4$,
- (b) $Z(G) \simeq C_{p^2}$ and either (i) $Z_2(G)/Z(G) \simeq G/G'$ or (ii) $Z_2(G)/Z(G) \simeq C_{p^2} \times C_p$ and $G/G' \simeq C_{p^3} \times C_p$ or (iii) $Z_2(G)/Z(G) \simeq C_{p^2} \times C_p$ and $G/G' \simeq C_{p^4} \times C_p$,
- (c) $Z(G) \simeq C_{p^3}$ and $Z_2(G)/Z(G) \simeq G/G'$.

Proof. It is not very hard to see that if any of the three conditions hold, then $\text{Aut}_z(G) = Z(\text{Inn}(G))$. Conversely, we prove only part (b), because the other two can be proved using the arguments as in Corollary 3.2.2. Observe that $p^4 \leq |Z_2(G)| \leq p^5$. If $|Z_2(G)| = p^4$, then $G/G' \simeq Z_2(G)/Z(G)$ by theorem. Next suppose that $|Z_2(G)| = p^5$. If $Z_2(G)/Z(G) \simeq C_p^3$, then $G/G' \simeq Z_2(G)/Z(G)$ by theorem; and if $Z_2(G)/Z(G) \simeq C_{p^2} \times C_p$, then $\beta_1 = \gamma_1 = 2$ and $\beta_2 < \gamma_1$, and hence either $G/G' \simeq Z_2(G)/Z(G)$ or $G/G' \simeq C_{p^4} \times C_p$ or $G/G' \simeq C_{p^3} \times C_p$. \square

3.3 — Groups of order upto p^7

As a consequence of the results proved in last section, we now characterize all finite p -groups G of order upto p^7 for which $\text{Aut}_z(G) = Z(\text{Inn}(G))$. Proofs are easy, short and even generalize the main results of Sharma and Gumber [57]. The next theorem follows immediately from Corollary 3.2.2.

Theorem 3.3.1 *Let G be a finite p -group of order p^5 and nilpotency class 3. Then $\text{Aut}_z(G) = Z(\text{Inn}(G))$ if and only if $Z(G) \simeq C_p$ and $d(G) = d(Z_2(G)/Z(G)) = 2$.*

Theorem 3.3.2 *Let G be a finite p -group of order p^6 and $cl(G) = 3$ or 4. Then $\text{Aut}_z(G) = Z(\text{Inn}(G))$ if and only if $Z(G) \simeq C_p$ and $d(G) = d(Z_2(G)/Z(G)) = 2$.*

Proof. If $cl(G) = 4$, then the result follows by Corollary 3.2.2. Therefore suppose that $cl(G) = 3$. Then either $Z(G) \simeq C_{p^2}$ and $Z_2(G)/Z(G) \simeq G/G'$ or $Z(G) \simeq C_p$ and $d(G) = d(Z_2(G)/Z(G)) = 2, 3$ by Corollary 3.2.3. We rule out two possibilities to get the result. First suppose that $Z(G) \simeq C_{p^2}$. Then $G/Z(G)$ is a group of order p^4 and nilpotency class 2. It follows that $|Z_2(G)| = p^4$ and $|G'| = p^3$, and hence G/G' and $Z_2(G)/Z(G)$ cannot be isomorphic. Next suppose that $|Z(G)| = p$ and $d(G) = d(Z_2(G)/Z(G)) = 3$. We show that $Z_2(G)$ is abelian. Since $d(G) = cl(G) = 3$, $|G'| = p^3$ or p^2 . If $|G'| = p^3$, then since $G' \leq Z(Z_2(G))$, $Z_2(G)$ is abelian. Therefore suppose that $|G'| = p^2$. Since $G/Z_2(G)$ is elementary abelian, $\Phi(G) \leq Z_2(G)$ and hence $G' \leq C_G(Z_2(G)) \leq C_G(\Phi(G))$. Thus $\Phi(G)$ is abelian. Let $Z_2(G) = \langle h, \Phi(G) \rangle$. Then, since h centralizes $\Phi(G)$, $Z_2(G)$ is abelian. It now follows by Lemma 3.1.2 that $Z_2(G)/Z(G)$ embeds in $\text{Hom}(G/Z_2(G), Z(G)) \simeq C_p \times C_p$, which is not possible. This completes the proof. \square

Theorem 3.3.3 *Let G be a finite p -group of order p^7 . Then $\text{Aut}_z(G) = Z(\text{Inn}(G))$*

if and only if one of the followings holds:

(i) $cl(G) = 3, Z(G) \simeq C_p$ and $d(G) = d(Z_2(G)/Z(G)) = 2, 3, 4,$

(ii) $cl(G) = 4$ and either $Z(G) \simeq C_p, d(G) = d(Z_2(G)/Z(G)) = 2, 3$ or $Z(G) \simeq C_{p^2}$ and $Z_2(G)/Z(G) \simeq G/G',$

(iii) $cl(G) = 5, Z(G) \simeq C_p$ and $d(G) = d(Z_2(G)/Z(G)) = 2.$

Proof. If G is of nilpotency class 4 (resp. 5), then the result follows from Corollaries 3.2.3 (resp. 3.2.2). Therefore suppose that $cl(G) = 3$. Then, by Corollary 3.2.4, we get possibilities (a), (b) and (c) for G . To get the final result, we rule out the possibilities (b) and (c). First suppose that $Z(G) \simeq C_{p^3}$ and $Z_2(G)/Z(G) \simeq G/G'$. Then, as in above theorem, G/G' is not isomorphic to $Z_2(G)/Z(G)$. Now suppose that $Z(G) \simeq C_{p^2}$ and $Z_2(G)/Z(G) \simeq G/G' \simeq C_p \times C_p$. Then $|G'| = p^5 > p^4 = |Z_2(G)|$, which is a contradiction to $cl(G) = 3$. Next suppose that $Z_2(G)/Z(G) \simeq G/G' \simeq C_p \times C_p \times C_p$. Since $G' \leq Z(Z_2(G))$, $Z_2(G)$ is abelian and we get a contradiction by Lemma 3.1.2. Finally suppose that $Z_2(G)/Z(G) \simeq C_p^2 \times C_p$. Then $Z_2(G) = \langle x, y, Z(G) \rangle$, where x and y have orders p and p^2 modulo $Z(G)$. Since $x^p \in Z(G)$ and $y \in \Phi(G) = Z_2(G)$, $Z_2(G)$ is abelian and again we get a contradiction by Lemma 3.1.2. This proves the theorem. \square

Converse of a Theorem of Schur

4.1 — Introduction

A well known theorem of Schur [56] states that if the index of the center of a group is finite, then its derived group is also finite. The converse of Schur's theorem is not true in general. Newman [45] in 1951, Hilton [31] in 2001, and Niroomand [46] and Sury [58] in 2010 have given proofs of converse of Schur theorem with different assumptions.

Quite unexpectedly, we here prove a further generalization of the converse of Schur's theorem in a very elementary and different way using automorphisms. Some other related results are also proved in the sequel.

An element of finite order in a group is called a *torsion element* and that of infinite order is called a *torsion-free element*. A group is called a *torsion group* if all its elements are torsion; and is called *torsion-free* if none of its element, other than the identity, is torsion. A group which contains both torsion and torsion-free elements is called a *mixed group*. Following Bachmuth [5], we call an automorphism α of G an *IA-automorphism* if it induces the identity

automorphism on the abelianized group G/G' ; or equivalently $x^{-1}\alpha(x) \in G'$ for all $x \in G$. All IA-automorphisms of G form a normal subgroup $\mathbf{IA}(G)$ of $\text{Aut}(G)$. Let $\mathbf{IA}(G)^*$ denote the group of all those IA-automorphisms of G which fix $Z(G)$ element-wise. Observe that

$$\text{Inn}(G) \leq \text{Aut}_c(G) \leq \mathbf{IA}(G)^* \leq \mathbf{IA}(G).$$

Theorem 4.1.1 ([8, Theorem 1.5]) *Let G be a finite p -group. If G/G' is an abelian group with two invariants λ, μ , where $\lambda \geq \mu$, then $G'/\gamma_3(G)$ is a cyclic group of order at most p^μ .*

4.2 — Main Results

Let G be an arbitrary group such that G' is finite and $G/Z(G)$ is minimally generated by

$$x_1Z(G), x_2Z(G), \dots, x_dZ(G).$$

If $\alpha \in \mathbf{IA}(G)^*$, then α fixes $Z(G)$ element-wise and $\alpha(x_i) = x_i y_i$, where $y_i \in G'$ for each $i, 1 \leq i \leq d$. It thus follows that $\mathbf{IA}(G)^*$ is finite and

$$|\mathbf{IA}(G)^*| \leq |G'|^d.$$

We have thus proved the following:

Theorem 4.2.1 *Let G be an arbitrary group such that $d(G/Z(G))$ and G' are finite.*

Then $|\mathbf{IA}(G)^| \leq |G'|^{d(G/Z(G))}$.*

In particular, since $\text{Inn}(G) \leq \mathbf{IA}(G)^*$, we obtain the following main theorem of Niroomand [46].

Corollary 4.2.2 *Let G be an arbitrary group such that $d = d(G/Z(G))$ and G' are finite. Then*

$$|G/Z(G)| \leq |G'|^d. \quad (4.1)$$

Now, rather than G' , suppose that $[x_i, G]$ is finite for all $i, 1 \leq i \leq d$. Let $a \in G$, and let α be the inner automorphism of G defined by conjugation by a . Then

$$\alpha(x_i) = a^{-1}x_i a = x_i[x_i, a].$$

Since

$$|[x_i, G]| = |x_i^G| \leq |K(G)|,$$

we have thus generalized the main theorem of Sury [58] and proved Theorem A of Yadav [69] in the following theorem.

Theorem 4.2.3 *Let G be an arbitrary group such that $G/Z(G)$ is finitely generated by $x_1Z(G), x_2Z(G), \dots, x_dZ(G)$ and $[x_i, G]$ is finite for all $i, 1 \leq i \leq d$. Then $|G/Z(G)| \leq \prod_{i=1}^d |[x_i, G]|$ and G' is finite.*

Theorem 4.2.4 *Let G be an arbitrary group. Then $\text{Inn}(G)$ is finite if and only if $\text{IA}(G)^*$ is finite. In particular, $\text{Inn}(G)$ is finite if and only if $\text{Aut}_c(G)$ is finite.*

Proof. If $\text{Inn}(G)$ is finite, then G' is finite by Schur's theorem. It then follows from Theorem 4.2.1 that $\text{IA}(G)^*$ is finite. Converse is obvious. \square

Niroomand [46] shows that the inequality (4.1) is sharp by considering the quaternion group Q_8 . But it is clear from Theorem 4.2.3 that the bound obtained by Niroomand is not the best possible, and the equality in the case of $G = Q_8$ results from

$G' = [x, G]$ for all $x \in G - \Phi(G)$. We observe from inequality (4.1) and Theorem 4.2.3 that

$$|G/Z(G)| \leq \prod_{i=1}^d |[x_i, G]| \leq |K(G)|^d \leq |G'|^d.$$

It follows that if G is an arbitrary group such that $G/Z(G)$ is finitely generated, G' is finite, and equality holds in (4.1), then $G' = K(G)$. In particular, if G is a finite group for which equality holds in (4.1), then $G' = K(G)$ and

$$\text{Inn}(G) = \text{Aut}_c(G) = \text{IA}(G)^*.$$

It now becomes interesting to classify the groups for which equality holds in (4.1). Of course it holds for all abelian groups. Yadav [69] also has asked the question for what non-abelian groups equality holds in (4.1). He himself has classified, upto isoclinism, all such nilpotent groups G of class 2 with $G/Z(G)$ finite. We here prove the following results.

Theorem 4.2.5 *Let G be a finitely generated nilpotent group of class 2. Then $G/Z(G) \cong G'^{d(G/Z(G))}$ if and only if G' is cyclic and $G/Z(G)$ is homocyclic.*

Proof. Observe that exponents of G' and $G/Z(G)$ are same. Let $G/Z(G) \simeq A \times \mathbb{Z}^a$ and $G' \simeq C \times \mathbb{Z}^c$, where A and C are respective torsion parts and a and c are respective torsion-free ranks of $G/Z(G)$ and G' . First suppose that G' is cyclic and $G/Z(G)$ is homocyclic. If G' is torsion-free, then $G/Z(G)$ is also torsion-free. It follows that

$$G/Z(G) \simeq \mathbb{Z}^{d(G/Z(G))} \simeq G'^{d(G/Z(G))}.$$

In case G' is finite, then $G/Z(G)$ is finite and hence $G/Z(G) \simeq G'^{d(G/Z(G))}$.

Conversely suppose that $G/Z(G) \simeq G'^{d(G/Z(G))}$. It is sufficient to prove that G' is cyclic. If G' is finite, then $G/Z(G)$ is finite and hence G' is cyclic. If G' is torsion-free, then $G/Z(G)$ is torsion-free. Thus $G' \simeq \mathbb{Z}^c$ and $G/Z(G) \simeq \mathbb{Z}^a$. Then $\mathbb{Z}^a \simeq \mathbb{Z}^{ac}$ implies that $G' \cong \mathbb{Z}$ is cyclic. Finally suppose that $G' \simeq C \times \mathbb{Z}^c$ is mixed. Then by assumption

$$A \times \mathbb{Z}^a \simeq (C \times \mathbb{Z}^c)^{d(G/Z(G))}.$$

This is not possible because $a < d(G/Z(G))$ and $cd(G/Z(G)) \geq d(G/Z(G))$. \square

Theorem 4.2.6 *Let G be a finite non-abelian p -group of maximal class. Then equality holds in (4.1) if and only if $|G| = p^3$.*

Proof. Observe that $|Z(G)| = p$ and $d(G) = 2$. If $|G| = p^n$, then $|G'| = p^{n-2}$. Suppose that equality holds in (4.1). Then $p^{n-1} = p^{2n-4}$ and hence $n = 3$. Converse is trivial. \square

Theorem 4.2.7 *Let G be a finite non-abelian p -group of coclass 2 for which equality holds in (4.1). Then $|G| = p^4$ or p^5 .*

Proof. Suppose that $|G| = p^n$. Then $p \leq |Z(G)| \leq p^2$ and $p^{n-3} \leq |G'| \leq p^{n-2}$. It follows by assumption that $\text{IA}(G)^* = \text{Inn}(G)$. Observe that if $Z(G) \leq G'$, then $\text{Aut}_z^z(G) \leq \text{IA}(G)^* = \text{Inn}(G)$ and thus $\text{Aut}_z^z(G) = Z(\text{Inn}(G))$. Since $d(G) \geq 2$, $Z(\text{Inn}(G))$ cannot be cyclic. First suppose that $|Z(G)| = p$. Then $|Z_2(G)| = p^3$ and $|\text{Aut}_z^z(G)| = |Z(\text{Inn}(G))| = p^2$. Thus $d(G) = d(G/Z(G)) = 2$. If $|G'| = p^{n-2}$, then by assumption $n = 3$, which is not possible; and if $|G'| = p^{n-3}$, then $n = 5$. Next suppose that $|Z(G)| = p^2$. Then $Z(G)$ is not contained in G' because then

$Z(\text{Inn}(G))$ is cyclic. If $|G'| = p^{n-2}$, then $G' = \Phi(G)$ and thus $d(G/Z(G)) = 1$, a contradiction. Therefore $|G'| = p^{n-3}$, $d(G/Z(G)) = 2$, and hence $n = 4$. \square

We end up the section by giving a negative answer to the following question posed by Yadav [69, Question 2]: Let G be a finite nilpotent group for which equality holds in (4.1). Is it true that G' is cyclic? Our group is of order 32 with elementary abelian commutator subgroup.

Example. Consider the group

$$G = \langle x, y | x^2 y^{-4} = [x, y, x] = [x, y, y] y^{-4} = 1 \rangle$$

which is of nilpotency class 3. Since $d(G) = d(G/Z(G)) = 2$, $Z(G) \leq \Phi(G)$. If $u = [x, y]$, then every element of G is of the form

$$x^i y^j u^k, \quad 0 \leq i, k \leq 1, \quad 0 \leq j \leq 7$$

and $|x| = 4$, $|y| = 8$, $|u| = 2$. Since $|\Phi(G)| = 8$, $\Phi(G) = \langle y^2, u \rangle$. It is easy to see that $Z(G) = \langle y^4 \rangle$ and $|Z(G)| = 2$. If $|G'| = 8$, then $|\gamma_3(G)| = 4$ by Theorem 4.1.1, which is a contradiction to $\gamma_3(G) \leq Z(G)$. Thus $|G'| = 4$ and hence

$$|G/Z(G)| = 2^4 = |G'|^2.$$

But, since $y^4, u \in G'$ are of order 2, G' is elementary abelian. \square

Conjugacy Classes of Finite Groups

5.1 — Introduction

The influence of arithmetic structure of conjugacy classes of a finite group G , like conjugacy class sizes, the number of conjugacy classes or the number of conjugacy class sizes, on the structure of G is an extensively studied problem in group theory. Many authors have studied the influence of some other kind of behaviours of conjugacy classes on the structure of the group. For example Dade and Yadav [16] have classified all finite groups in which the product of any two non-inverse conjugacy classes is again a conjugacy class; Herzog and Schonheim [30] have shown that if G is a finite group of odd order having exactly two non-central conjugacy classes of each order, then $|G| = 21$. For more details one can see the references [9, 37, 64] and the excellent survey article by Camina and Camina [12]. Throughout this chapter, unless or otherwise stated, all groups are finite.

We say that a group G satisfies (i) , $i \geq 1$, if union of any i distinct non-trivial conjugacy classes of G together with 1 is a subgroup of G . In this chapter, we classify all finite groups satisfying (i) for $1 \leq i \leq 3$.

An element x of a group G is called a **real element** if there exists another element g of G such that $x^g = x^{-1}$. In other words, x is conjugate to x^{-1} . The set of all real elements of G is denoted by $\mathbf{Re}(G)$. A group G is called a **rational group** if every element x of G is conjugate to x^m for some natural number m co-prime to $|x|$. An element of order 2 of a group G is called an **involution**. The unique largest nilpotent normal subgroup $\mathbf{F}(G)$ of a group G is called the **Fitting subgroup** of G . It is of course the product of all nilpotent normal subgroups of G . Let G be a group and let $p \in \Pi(G)$, the set of all primes dividing $|G|$. Let P_1, P_2, \dots, P_k be all the Sylow p -subgroups of G . The intersection $\mathbf{O}_p(G)$ of all P_i is a normal subgroup of G and is called the **p -radical** of G . An automorphism α of a group G is called a **fixed-point-free** automorphism if it leaves fixed only the identity element of G .

Proposition 5.1.1 ([18, Lemma 2.4(iv)]) *If N is a normal subgroup of a group G such that $|G/N|$ is odd, then $\mathbf{Re}(G) = \mathbf{Re}(N)$.*

Proposition 5.1.2 ([43, Proposition 1]) *Factor group of a rational group is rational.*

Proposition 5.1.3 ([43, Corollary 2]) *If G is an abelian rational group, then G is an elementary abelian 2-group.*

Proposition 5.1.4 ([17, Proposition 1]) *Let G be a solvable rational group of even order with exactly one conjugacy class of involutions. Then a Sylow 2-subgroup of G is either elementary abelian or isomorphic to the quaternion group of order 8.*

Proposition 5.1.5 ([38, Corollary 16A]) *If H is a Sylow 2-subgroup of a rational group G , then $C_G(H) = Z(H)$.*

Proposition 5.1.6 ([38, Proposition 21]) *Let G be a rational group with an abelian Sylow 2-subgroup H . Then H is elementary abelian, G splits over G' with H as complement, and G' is a 3-group.*

Proposition 5.1.7 ([38, Proposition 35]) *Let G be a rational group with Q_8 , the quaternion group of order 8, as a Sylow 2-subgroup. Then G contains a normal elementary abelian p -group E_p , where $p = 3$ or 5 , such that $G = Q_8E_p$.*

Theorem 5.1.8 ([53, Theorem 10.24]) *Let p_1, p_2, \dots, p_n be the distinct primes dividing the order of a group G . Then $F(G) = O_{p_1}(G)O_{p_2}(G) \dots O_{p_n}(G)$.*

Theorem 5.1.9 ([23, Theorem 6.1.3]) *If G is a solvable group, then $C_G(F(G)) \leq F(G)$.*

Theorem 5.1.10 ([23, Theorem 5.3.16]) *Let P be a p -group and let Q be a non-cyclic abelian q -group of automorphisms of P , q a prime distinct from p . Then*

$$P = \prod_{x \in Q - \{1\}} C_P(x).$$

In particular, P is generated by its subgroups $C_P(x)$ for x in $Q - \{1\}$.

Theorem 5.1.11 ([23, Theorem 10.1.4]) *If α is a fixed-point-free automorphism of order 2 of a group G , then G is abelian and $\alpha(x) = x^{-1}$ for all x in G .*

Theorem 5.1.12 ([59, Theorem 4.4.4]) *The following three conditions on a p -group are equivalent:*

- (i) *Every abelian subgroup is cyclic.*
- (ii) *There is exactly one subgroup of order p .*
- (iii) *The group G is either cyclic or a generalized quaternion group Q_g .*

5.2 — Classifications

It is not very hard to see that an abelian group G satisfies (1) if and only if G is an elementary abelian 2-group. The restriction (1) is of course very harsh on non-abelian groups G , and as such we do not get any non-abelian group with this restriction. We prove the statement in the following theorem.

Theorem 5.2.1 *There is no non-abelian group G satisfying (1).*

Proof. Suppose that G is a non-abelian group satisfying (1). It is easily seen that $|G|$ is even. Let $x \in G - Z(G)$. Since $|x^G|$ and $|x^G| + 1$ both divide the order of G , G cannot be a 2-group. Let $x \in G$ be such that $|x| = mn$, where $m, n \geq 2$. Then two different ordered elements x and x^m are in the same conjugacy class x^G , a contradiction. Thus each element of G is of prime order. This immediately implies that no two elements of G of different order can commute with each other. Let $x \in G$ and let $|x| = p$, a prime. We prove that $C_G(x)$ is a Sylow p -subgroup of G . Suppose that $|G| = p^m t$, where p does not divide t . If $y \in C_G(x)$, then $|y| = p$ and thus $|C_G(x)| = p^n$, where $1 \leq n \leq m$. If $|C_G(x)| < p^m$, then $|x^G| = p^l t$, where $l \geq 1$. Thus p divides $|x^G|$ and $|x^G| + 1$, a contradiction and hence $C_G(x)$ is a Sylow p -subgroup of G . If $|1 \cup x^G| = p^k$, where $k \leq m$, then $p^k - 1 = t$. Let p_1, p_2 be two different primes which divide the order of G and let $|G| = p_1^{m_1} p_2^{m_2} s$, where p_1 and p_2 do not divide s and $s \geq 1$. We choose elements $x_1, x_2 \in G$ such that $|x_1| = p_1$ and $|x_2| = p_2$. Then as before, we get the equations

$$p_1^{m_1} - 1 = p_2^{m_2} s$$

and

$$p_2^{n_2} - 1 = p_1^{m_1} s,$$

where $n_1 \leq m_1$ and $n_2 \leq m_2$. Thus

$$p_1^{n_1} - 1 = p_2^{m_2} s = p_2^{m_2 - n_2} s(p_1^{m_1} s + 1) \geq p_1^{m_1} + 1 > p_1^{n_1} - 1,$$

a final contradiction. □

It is easy to see that if G is isomorphic to C_3 , the cyclic group of order 3, then G satisfies (2). Conversely suppose that G is an abelian group satisfying (2). Of course then $|G| \geq 3$. Let $x, y \in G - \{1\}$ be two distinct elements. Then $\{1, x, y\}$ is a subgroup of G such that $y = x^2$. If $z \in G - \{1, x\}$, then $\{1, x, z\}$ is a subgroup of G such that $z = x^2 = y$. Thus G is a cyclic group of order 3. The following theorem settles the non-abelian case.

Theorem 5.2.2 *A non-abelian group G satisfies (2) if and only if $G \simeq S_3$.*

Proof. If $G \simeq S_3$, then G satisfies (2). Conversely suppose that G is a non-abelian group satisfying (2). Assume $|G| = p^n$, where p is an odd prime. Then $|Z(G)| \geq 3$. By preceding paragraph $|Z(G)| = 3$ and hence $|G| = 3^n$. Choose x a non-central and x_1 a central element of G . Let

$$H_0 = 1 \cup x^G \cup x_1^G.$$

Then $|H_0| = 1 + 1 + 3^a$, where $1 \leq a < n$, is not divisible by 3, a contradiction.

Thus G cannot be a p -group.

Let $y \in G$ be of composite order. We can write $|y| = pm$, where p is a prime and $m \geq 2$. Choose an element $y_1 \in G$ of order q , where q is a prime different from p . Then

$$y^m \in H_1 = 1 \cup y^G \cup y_1^G,$$

because $H_1 \leq G$. But on the other hand, neither y^m belongs to y^G nor to y_1^G . It thus follows that every element of G is of prime order.

Suppose $|G|$ is odd and let $p \neq q \in \Pi(G)$. Let x_2 and y_2 be two elements of G such that $|x_2| = p$ and $|y_2| = q$. Since $|x_2^G|$ is odd, x_2^{-1} cannot be in x_2^G and hence x_2^{-1} is not in the subgroup

$$H_2 = 1 \cup x_2^G \cup y_2^G.$$

This is a contradiction and thus $|G|$ is even.

We claim that $|G|$ is divisible by exactly two distinct primes. On the contrary assume that

$$|G| = p^l q^m r^n s,$$

where p, q, r are distinct primes, $l, m, n, s \geq 1$ and $(p^l q^m r^n, s) = 1$. First suppose that all same ordered elements in G are conjugate. Let $x_3, y_3, z_3 \in G$ be of order p, q and r respectively. Since every element of G is of prime order, $C_G(x_3)$ is a p -group and therefore $|x_3^G| = p^{l_1} q^m r^n s$, where $0 \leq l_1 < l$. Similarly $|y_3^G| = p^l q^{m_1} r^n s$ and $|z_3^G| = p^l q^m r^{n_1} s$, where $0 \leq m_1 < m$ and $0 \leq n_1 < n$. Let

$$H_3 = 1 \cup x_3^G \cup y_3^G, \quad K_3 = 1 \cup y_3^G \cup z_3^G \quad \text{and} \quad L_3 = 1 \cup z_3^G \cup x_3^G.$$

Then $H_3 \cap K_3 = 1 \cup y_3^G$ is a Sylow q -subgroup of G and $H_3 \cap L_3 = 1 \cup x_3^G$ is a Sylow

p -subgroup of G . Therefore

$$q^m = p^l q^{m_1} r^n s + 1 \quad \text{and} \quad p^l = p^{l_1} q^m r^n s + 1.$$

It follows that

$$q^m = p^l q^{m_1} r^n s + 1 = (p^{l_1} q^m r^n s + 1) q^{m_1} r^n s + 1 > q^m,$$

a contradiction. We therefore suppose that there exists a prime $p \in \Pi(G)$ such that all the elements of order p are not conjugate to each other. Let x_4, y_4 be two non-conjugate elements of order p and let $z_4, w_4 \in G$ be of order q and r respectively.

Then

$$|x_4^G| = p^{l_2} q^{m_2} r^{n_2} s, \quad |y_4^G| = p^{l_3} q^{m_3} r^{n_3} s, \quad |z_4^G| = p^{l_4} q^{m_4} r^{n_4} s \quad \text{and} \quad |w_4^G| = p^{l_5} q^{m_5} r^{n_5} s,$$

where $0 \leq l_2, l_3 < l$, $0 \leq m_2 < m$ and $0 \leq n_2 < n$. Let

$$H_4 = 1 \cup x_4^G \cup z_4^G \quad \text{and} \quad K_4 = 1 \cup y_4^G \cup w_4^G.$$

Then H_4 and K_4 are disjoint normal subgroups of G and

$$|H_4 K_4| = |H_4| |K_4| > |G|.$$

This is a contradiction and hence the claim.

Let $|G| = 2^a p^b$, where p is an odd prime and $a, b \geq 1$. Let $x \in G$ be of order p , $y \in G$ be of order 2, and

$$H_5 = 1 \cup x^G \cup y^G.$$

Then x^n is conjugate to x for $1 \leq n \leq p-1$. Thus G is a rational group and hence G splits over G' with a Sylow 2-subgroup H as complement and G' is a Sylow 3-subgroup of G by Prop. 5.1.6. For $1 \neq h \in H$, the mapping

$$\tau_h : G' \rightarrow G'$$

sending g to $h^{-1}gh$ is a fixed-point-free automorphism of order 2. Therefore G' is abelian and $\tau_h(g) = g^{-1}$ for all $g \in G'$ by Theorem 5.1.11. It then follows that $|g^G| = 2$ for each $g \in G'$ and thus $|G| = 2 \cdot 3^b$. If $|G'| > 3$, then we can find $g_1, g_2 \in G'$ such that $\langle g_1 \rangle \neq \langle g_2 \rangle$. Then

$$K_5 = 1 \cup g_1^G \cup g_2^G$$

is a subgroup of G of order 5, a contradiction. Thus $|G'| = 3$ and hence $G \simeq S_3$. \square

Observe that if G satisfies (3), then $|G|$ is even. It is easily seen that if

$$G \simeq C_4 \text{ or } G \simeq C_2 \times C_2,$$

then G satisfies (3). Conversely suppose that G is an abelian group satisfying (3).

Let p be an odd prime dividing the order of G and let $x_1 \in G$ be of order p . Let $x_2 \in G$ be of order 2 and let

$$H_1 = 1 \cup x_1^G \cup x_2^G \cup (x_1^2)^G.$$

Then H_1 is a subgroup of G containing x_1, x_2 but not x_1x_2 . Thus G is a 2-group of order bigger than 3. Suppose $d(G) \geq 3$ and let y_1, y_2, y_3 be three generators of G .

Let

$$H_2 = 1 \cup y_1^G \cup y_2^G \cup y_3^G.$$

Then H_2 is a subgroup of G containing y_1, y_2 but not y_1y_2 . Thus $d(G) \leq 2$. Suppose $d(G) = 1$ and let $G = \langle x \rangle$. Let

$$H = 1 \cup x^G \cup (x^2)^G \cup (x^3)^G.$$

Then H is a subgroup of G of order 4 containing G . Thus G is a cyclic group of order 4. Now suppose $d(G) = 2$ and let $G = \langle y, z \rangle$. If $|y| > 2$, then $|y| \geq 4$. Let

$$K = 1 \cup y^G \cup (y^2)^G \cup z^G.$$

Then K is a subgroup of G containing y but not y^3 . Thus $|y| = 2$ and similarly $|z| = 2$. Therefore $G \simeq C_2 \times C_2$. For non-abelian groups, we have the following theorem.

Theorem 5.2.3 *A non-abelian group G satisfies (3) if and only if either G is isomorphic to D_5 or to A_4 , where D_5 is the dihedral group of order 10.*

Proof. It is easily seen that if G is isomorphic to D_5 or to A_4 , then G satisfies (3). For the converse part, we proceed in a number of steps.

Step 1. G cannot be a 2-group.

Proof. Contrary assume that G is a 2-group. Suppose G has 3 or more non-central conjugacy classes and let x_1^G , x_2^G and x_3^G be three of them. Then

$$H = 1 \cup x_1^G \cup x_2^G \cup x_3^G$$

is a subgroup of G of odd order, which is not possible. Thus G has at most two non-central conjugacy classes. Suppose $|Z(G)| > 2$ and let

$$z_1 \neq z_2 \in Z(G) - 1, w \in G - Z(G).$$

Then

$$K = 1 \cup z_1^G \cup z_2^G \cup w^G$$

is a subgroup of G of odd order, which is not possible. Therefore $|Z(G)| = 2$ and

$$G = 1 \cup x_1^G \cup x_2^G \cup z_1^G,$$

where $x_1, x_2 \in G - Z(G)$ and $z_1 \in Z(G) - 1$. We show that $Z(G) = G'$. If not, then either

$$G' = 1 \cup x_1^G \cup z_1^G \text{ or } G' = 1 \cup x_2^G \cup z_1^G.$$

In either of the cases, G' is a maximal subgroup of G . This implies that G/G' and hence G is cyclic, which is not so. Thus $Z(G) = G'$. Therefore $|x_1^G| = |x_2^G| = 2$ and $|G| = 6$, a contradiction. Hence G cannot be a 2-group.

Step 2. $|Z(G)| = 1$.

Proof. Assume $|Z(G)| > 1$ and let p be the largest prime dividing $|Z(G)|$. Suppose $p \geq 3$. Let $x \in Z(G)$ be of order p , $y \in G$ be of order 2 and let

$$H = 1 \cup x^G \cup (x^2)^G \cup y^G.$$

Then H is a subgroup of G containing x and y but not containing xy , a contradiction.

Thus $p = 2$ and $Z(G)$ is a 2-group. Let S be a Sylow 2-subgroup of G containing $Z(G)$. Let $z \in Z(G)$ be of order 2, $w \neq z \in S$ and $u \in G$ be of order q , where $q \in \Pi(G)$ is an odd prime. Then

$$K = 1 \cup z^G \cup w^G \cup u^G$$

is a subgroup of G not containing zu . Thus $S = Z(G)$ is the unique Sylow 2-subgroup of G of order 2. Therefore $|G/Z(G)|$ is odd and by Prop. 5.1.1, $Re(G) = Re(Z(G))$.

Let

$$L = 1 \cup u^G \cup z^G \cup (uz)^G.$$

Now $u^{-1} \in L$ is of order q , therefore $u^{-1} \in u^G$. Thus u is a real element of $G - Z(G)$, which is a contradiction to $Re(G) = Re(Z(G))$. Hence $|Z(G)| = 1$.

Step 3. $|G|$ is divisible by exactly 2 distinct primes.

Proof. Suppose $|G|$ is divisible by at least three distinct primes. We show that order of every element of G is a prime. Let $x_1 \in G$ be of order $p_1 l$ and let $x_2, x_3 \in G$ be of orders p_2 and p_3 respectively, where p_1, p_2, p_3 are distinct primes and $l \geq 2$. Then

$$H = 1 \cup x_1^G \cup x_2^G \cup x_3^G$$

is a subgroup of G containing x_1 but not x_1^l . This is a contradiction and thus every element of G is of prime order. Let $x \in G$ be arbitrary and let $|x| = p$, a prime. Let $y_1, y_2 \in G$ be of orders q_1, q_2 respectively, where $q_1 \neq q_2$ are primes different from p . Let

$$K = 1 \cup x^G \cup y_1^G \cup y_2^G.$$

Then x^m is conjugate to x whenever m is relatively prime to p . Thus G is a rational group with elementary abelian Sylow 2-subgroup and therefore G is a $\{2, 3\}$ group by Prop. 5.1.6. This is a contradiction to $|\Pi(G)| \geq 3$ and hence $|G|$ is divisible by exactly 2 distinct primes.

Step 4. Order of every element of G is prime.

Proof. By step 3, we can assume that

$$|G| = 2^a p^b,$$

where p is an odd prime and $a, b \geq 1$. Let $x \in G$ be arbitrary and let $|x| = m$. If G contains an element of composite order, then we can choose two non-conjugate

elements x_2, x_3 of orders different from m . Let

$$H_1 = 1 \cup x^G \cup x_2^G \cup x_3^G.$$

Then x^r is conjugate to x whenever r is relatively prime to m . Thus G is a rational group and hence G/G' is an elementary abelian 2-group by Prop. 5.1.2 and Prop. 5.1.3. Let $|G/G'| = 2^n$, where $n \leq a$, and let H be a Sylow 2-subgroup of G . Then $|G'| = 2^{a-n}p^b$, $|G' \cap H| = 2^{a-n}$ and

$$|G'H| = \frac{|G'| |H|}{|G' \cap H|} = 2^a p^b = |G|.$$

It follows that $G = G'H$.

First suppose that H is abelian. Then G splits over G' with H as complement and G' is a Sylow 3-subgroup of G by Prop. 5.1.6. Let h_1 and h_2 be two elements of H and let $h_1 = h_2^g$ for some $g \in G$. Then

$$h_2^{-1}h_1^g \in H \cap G' = 1$$

and thus $h_1 = h_2$. Therefore distinct elements of H are not conjugate in G . Let $|H| > 2$, $h_3 \in H$ be of order 2, and let $h_4 (\neq h_3)$ be a non-trivial element of H . Let $x_4 \in G$ be a 3-element and let

$$K = 1 \cup h_3^G \cup h_4^G \cup x_4^G.$$

Then K is a subgroup of G containing h_3 and h_4 but not h_3h_4 , a contradiction. Thus $|H| = 2$ and hence $|G| = 2 \cdot 3^b$. Let h be the non-trivial element of H and let $\tau_h : G' \rightarrow G'$ be defined by $\tau_h(x) = h^{-1}xh$. It is easy to see that τ_h is an automorphism of G' of order 2. Suppose $y \in G'$ is such that $\tau_h(y) = y$. Then

$$y \in C_G(H) = Z(H) = H,$$

by Prop. 5.1.5. Thus $y \in G' \cap H = 1$ and hence τ_h is a fixed-point-free automorphism of G' . It then follows from Theorem 5.1.11 that G' is abelian. Suppose G' contains an element, say w , of order bigger than 3. Since G is rational and $\langle w \rangle = \langle w^k \rangle$ for more than two values of k , $|w^G| > 2$. On the other hand since G' is abelian, $G' \leq C_G(w)$ and thus $|w^G| = 2$. A contradiction and thus G' is an elementary abelian 3-group. If $|G'| > 3$, then we can find $y_1, y_2 \in G'$ such that $\langle y_1 \rangle \neq \langle y_2 \rangle$. Let

$$K_1 = 1 \cup y_1^G \cup y_2^G \cup h^G.$$

Then $y_1, y_2 \in K_1$ but $y_1 y_2 \notin K_1$, because

$$y_1^G = \{y_1, y_1^{-1}\} \text{ and } y_2^G = \{y_2, y_2^{-1}\}.$$

Thus $|G'| = 3$ and $G \simeq S_3$, a contradiction because S_3 has only 2 non-trivial conjugacy classes.

Now suppose that H is non-abelian. Let P be a Sylow p -subgroup of G . Let h_5, h_6 be two non-conjugate elements of $Z(H)$ and let

$$K_2 = 1 \cup h_5^G \cup h_6^G \cup z_1^G,$$

where $z_1 \in Z(P)$. Since $|K_2|$ is odd, K_2 is a p -group containing elements of order 2, which is not possible. Thus all the elements in $Z(H)$ are conjugate and hence $Z(H)$ is an elementary abelian 2-group. We show that all involutions of H are conjugate. Suppose there exists an involution h in H which is not conjugate to some element of $Z(H)$. Let $h_7 \in H$ be of order bigger than 2. Let

$$K_3 = 1 \cup h^G \cup h_7^G \cup z_1^G.$$

Then K_3 is an odd ordered subgroup of G containing even ordered elements. A contradiction and thus all involutions of H and hence of G are conjugate. Thus

$H \simeq Q_8$ by Prop. 5.1.4 and $G \simeq Q_8 E_p$ by Prop. 5.1.7, where $p = 3$ or 5 . Let $e_1, e_2 \in E_p - 1$ be two non-conjugate elements and let $h_8 \in Q_8$ be of order 4. Let

$$K_4 = 1 \cup e_1^G \cup e_2^G \cup h_8^G.$$

Then K_4 is an odd ordered subgroup of G containing even ordered elements. A contradiction and thus all non-trivial elements of E_p are conjugate. Since $|E_p| = p^b$, $(p^b - 1)|8$ and thus $|G| = 24, 40$ or 72 . Using GAP [60] it can be seen that the groups with GAP IDs

$$[24, 12], [72, 15], [72, 39], [72, 40], [72, 41], [72, 43] \text{ and } [72, 44]$$

are the only groups with trivial center. None of these groups satisfy the hypothesis.

This is a contradiction and hence every element of G is of prime order.

Step 5. $G \simeq D_5$ or A_4 .

Proof. By steps 3 and 4,

$$|G| = 2^a p^b, \quad a, b \geq 1$$

and order of every element of G is either 2 or p . Let H be a Sylow 2-subgroup and P be a Sylow p -subgroup of G . First suppose that $a = 1$. For $h \in H$, the map $\lambda_h : P \rightarrow P$ defined by $\lambda_h(x) = h^{-1}xh$ is an automorphism of P of order 2. If λ_h fixes some element, say $x_1 \in P$, then G contains an element x_1h of composite order, which is not so. Thus λ_h is a fixed-point-free automorphism of P of order 2. Therefore P is abelian by Theorem 5.1.11. By hypothesis, all the elements of P are not conjugate. Let $y_1, y_2 \in P$ be two non-conjugate elements. Since P is abelian, $|y_1^G| = |y_2^G| = 2$. If

$$K = 1 \cup y_1^G \cup y_2^G \cup h^G,$$

then $|K| = p^b + 5$ is more than half of $|G|$. Thus $p^b + 5 = 2p^b$ and hence $p = 5$, $b = 1$ and $G \simeq D_5$.

Now suppose that $a > 1$. Since every element of G is of prime order,

$$F(G) = O_2(G) \text{ or } F(G) = O_p(G).$$

Suppose $F(G) = O_p(G)$. Let S be an abelian non-cyclic subgroup of H . Since G is solvable, $C_G(F(G)) \leq F(G)$ by Theorem 5.1.9. Thus $C_{(F(G))}(x) = 1$ for every $1 \neq x \in S$, because otherwise G contains an element of composite order. It thus follows from Theorem 5.1.10 that

$$F(G) = \langle C_{(F(G))}(x) \mid x \in S, x \neq 1 \rangle = 1,$$

a contradiction and hence every abelian subgroup of H is cyclic. Thus either H is cyclic or $H \simeq Q_8$ by Theorem 5.1.12. Since H is elementary abelian, $H \simeq C_2$ and hence $a = 1$, which is not so. Therefore $F(G) = O_2(G)$. If $F(G) = O_2(G) < H$, then since H is elementary abelian,

$$H \leq C_G(F(G)) \leq F(G) < H,$$

a contradiction. Thus $F(G) = O_2(G) = H$ and hence H is a normal subgroup of G .

Using similar arguments, as above used on H , we can see that P is cyclic of order p .

Therefore $|G| = 2^a p$, $a > 1$. Suppose $h_1, h_2 \in H$ are non-conjugate and let $y_1 \in P$.

Let

$$K = 1 \cup h_1^G \cup h_2^G \cup y_1^G.$$

Then K is subgroup of G of odd order containing an element of order 2, a contradiction. Thus all the involutions in G are conjugate and hence $|x^G| = 2^a - 1$, where

x is an involution. On the other hand since $H = C_G(x)$, $|x^G| = p$. By hypothesis, G contains at least two non-trivial non-conjugate elements y_1, y_2 of order p . Let

$$K_1 = 1 \cup x^G \cup y_1^G \cup y_2^G.$$

Then

$$2^a - 1 + 2^a + 2^a + 1 = 2^n p, \quad n \leq a.$$

It follows that $p = 3$ and $a = n$, and since $2^a - 1 = p = 3$, $a = 2$. Thus $|G| = 12$. There are three non-abelian groups of order 12 *viz.* A_4 , D_6 and $C_3 \rtimes C_4$, and it can be easily seen that only A_4 satisfies the hypothesis. Thus $G \simeq A_4$. \square

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