

GEOMETRY OF GENERIC SUBMANIFOLDS

Thesis submitted in partial fulfillment of the requirement for

The award of the degree of

Masters of Science

in

Mathematics and Computing

Submitted by

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JULY 2009

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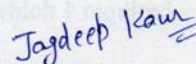
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CERTIFICATE

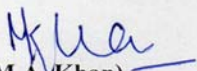
ACKNOWLEDGEMENT


I hereby certify that the work which is being presented in the thesis entitled “**Geometry of Generic submanifolds**” in partial fulfillment of the requirements for the award of degree of Master of Science, School of Mathematics and Computer Applications, Thapar University, Patiala is an authentic record of my own work carried out under the supervision of **Dr. M.A. Khan** and **Dr. Rajesh Kumar Gupta**.

The matter presented in this thesis has not been submitted for the award of any other degree of this or any other university.

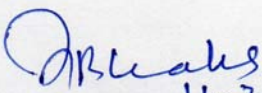

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This is to certify that the above statement made by the candidate is correct and true to the best of our knowledge.


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ACKNOWLEDGEMENT

I feel privileged to express my sincere regards and gratitude to my supervisors Dr. M.A. Khan and Dr. Rajesh Kumar Gupta for their expert guidance, cool temperament, valuable suggestions, support, advice and continuous encouragement throughout the course of my thesis work.

I am highly obliged to Prof. S.S. Bhatia, Head SMCA, Thapar University, Patiala, for their motivation and inspiration that triggered me for thesis work.

I would like to thank all the staff members and my co-students who were always there at the need of the hour and provided with all the help and facilities, which I required for the completion of my thesis.

I am also thankful to the authors whose works I have consulted and quoted in this work.

Last but not the least I would like to thank God for not letting me down at the time of crisis and showing me the silver lining in the dark clouds.

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Place: Patiala

Dated: 14 July, 2009

Preface

To study the geometry of a manifold, it is more convenient to first embed into a known manifold and then study its geometry. This approach gave impetus to study of submanifold which later developed into an independent and fascinating topic of study. The submanifold of an almost Hermitian manifold presents an interesting geometric study as its almost complex structure transform a vector into a vector perpendicular to it. Thus in turn, gives rise to two types of submanifolds, namely invariant and anti-invariant submanifolds. These are also known as holomorphic and totally real submanifold. These submanifolds were extensively studied by many differential geometrs. A Bejancu [1] in 1978 introduced the notion of CR-submanifolds of a Kaehler manifold which generalize the holomorphic and totally real submanifolds, after that B.Y. Chen [3] studied CR-submanifold of Kaehler manifold.

Chapter I is introductory and serves. The purpose of developing the basic concepts keeping in view the pre-requisites of the subsequent chapters.

Chapter II is introductory chapter of CR-submanifold of a Kaehler manifold. This chapter contains some basic results of CR-submanifold of a Kaehler manifold. All the results of this chapter are by B.Y.Chen from his paper “CR-submanifold of a kaehler manifold”.

Chapter III is the second technical chapter which deals with the generic submanifold of a kaehler manifold. Its contents are essentially the results obtained by B.Y. Chen in his paper “Differential geometry of real submanifold of a Kaehler manifold”[4]. The generic submanifold are natural generalization of holomorphic, totally real and CR-submanifold. In this paper B.Y. Chen studied integrability condition of holomorphic and totally real distribution and totally geodesicness of leaves of these distributions.

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Chapter I

Introduction

The purpose of this chapter is to introduce the basic concepts , preliminary notions and some fundamental results ,which we require for the development of the subject in the present dissertation. Thus we have given a brief resume of some of the results in the geometry of almost hermitian manifolds and the allied structure and the geometry of submanifolds of the manifolds. Much though all the results are readily available in review articles and some in standard books e.g. Nomizu and Kobayashi [5] , B.Y. Chen [2]. In this chapter we have also fix up our notions and terminology.

Topological Manifold (1.1.1).A Manifold \overline{M} of n-dimension is a Hausdorff Topological space which is locally Euclidean i.e. $\forall x \in \overline{M} \exists$ a neighbourhood U of $x \in \overline{M}$, and a homeomorphism ϕ of U onto an open set in IR^n . The dimension R^n is called the dimension of manifold.

Chart(1.1.2): Let \overline{M} be a set . A chart on \overline{M} is a pair (ϕ, U) consisting of a subset U of \overline{M} and a 1-1 map ϕ of U onto an open set in R^n .

Two charts (ϕ_1, U_1) and (ϕ_2, U_2) are said to be C^r - related, if either $U_1 \cap U_2 = \phi$ or $\phi_1 \circ \phi_2^{-1}$ and $\phi_2 \circ \phi_1^{-1}$ are C^r functions.

Atlas(1.1.3): The collection of all charts is called atlas.

Differentiable Manifold(1.1.4): Let (U_i, ϕ_i) and (U_j, ϕ_j) be two charts on \overline{M} such that $U_i \cap U_j \neq \phi$. Any point $p \in U_i \cap U_j$ shall have two local coordinate system, $\phi_i(p) = (x^1, \dots, x^n)$, $\phi_j(p) = (y^1, \dots, y^n)$ say the two coordinate system are related as

follows. We require that the mapping $\phi_j \circ \phi_i^{-1} : \phi_i(U_i \cap U_j) \rightarrow \phi_j(U_i \cap U_j)$ be differentiable of class C^k . Then we say that the atlas on \overline{M} is a differentiable of class C^k . If $\alpha(U_i, \phi_i) \in F$ is a maximal family of charts. \overline{M} equipped with a maximal differentiable structure of class C^k is then called differentiable manifold of class C^k of $\phi_j \circ \phi_i^{-1}$ is C^∞ function then \overline{M} is called smooth manifold.

(i) $S^1 = \{(x, y) \in R^2 \mid x^2 + y^2 = 1\}$ is a one – dimensional differentiable manifold.

(ii) S^2 (2 dimension sphere) is a differentiable manifold.

(iii) S^n is n - dimension differentiable manifold.

(iv) R, R^2, \dots, R^n are differentiable manifold.

Tangent vector(1.1.5): A tangent vector X_m at $m \in \overline{M}$ is a real valued function on $C^\infty(m)$

i.e., $X_m : C^\infty(m) \rightarrow IR$ such that

$$(a) X_m(af + bg) = aX_m f + bX_m g \quad a, b \in IR.$$

$$(b) X_m(fg) = f(m)X_m g + g(m)X_m f$$

where $f, g \in C^\infty(m)$ and $a, b \in IR$.

Tangent space(1.1.6): A tangent space $T_m \overline{M}$ to \overline{M} at a point $m \in \overline{M}$ is the set of all tangent vectors of $m \in \overline{M}$.

Vector field(1.1.7): Vector field X on a manifold \overline{M} is a smooth mapping which to each $m \in \overline{M}$ assigns a tangent vector $X_m \in T_m(\overline{M})$. Locally we can express $X = \xi^i \frac{\partial}{\partial x^i}$ where ξ^i are differentiable functions, moreover, for $f \in C^\infty(\overline{M})$. $Xf \in C^\infty(\overline{M})$ i.e.

$$Xf(m) = X_m f \quad m \in \overline{M}.$$

Alternatively, we can define a vector field X on \overline{M} ; a derivation of the algebra $C^\infty(\overline{M})$ i.e.,

$$X : C^\infty(\overline{M}) \rightarrow C^\infty(\overline{M}) \text{ such that}$$

$$(a) X(af + bg) = aXf + bXg$$

$$(b) X(fg) = f(Xg) + g(Xf)$$

for all $f, g \in C^\infty(\overline{M})$ and $a, b \in \mathbb{R}$.

Set of all vector fields is denoted by $D(\overline{M})$.

Sub-Manifold(1.1.8): If \overline{M} and N satisfy the following two conditions, then N is called a submanifold of \overline{M} .

1. N is a subset of \overline{M} .
2. The identity map $i : N \rightarrow \overline{M}$ is an imbedding of N into \overline{M} .

For example $S^1 = \{(x, y) | x^2 + y^2 = 1\}$ is the one-dimensional submanifold of \mathbb{R}^2 .

Riemannian Manifold(1.1.9): If on a C^∞ manifold \overline{M} , there exist a metric g on $T\overline{M}$ satisfying the following conditions

- (a) g is (0,2) tensor. i.e., g is bilinear on $T\overline{M}$.
- (b) $g(X, Y) = g(Y, X) \quad \forall X, Y \in T\overline{M}$.
- (c) $g(X, Y) = 0 \quad \forall X \in T\overline{M} \Rightarrow Y = 0$
- (d) $g(X, X) \geq 0, X \neq 0$

then g is called Riemannian metric and (\overline{M}, g) is called Riemannian manifold.

Structures on manifold

Almost complex structure(1.1.10): An almost complex structure on a real differentiable manifold \overline{M} is a tensor field J which is at every point $P \in \overline{M}$, an endomorphism of the

tangent space $T_p(\overline{M})$, such that $J^2 = -I$ where I denotes the identity transformation. A manifold with a fixed almost complex structure is called an almost complex manifold.

On an almost complex manifold there always exist a Riemannian metric g consistent with the almost complex structure J i.e; satisfying

$$g(JU, JV) = g(U, V) \quad \forall U, V \in T(\overline{M})$$

and here g is called Hermitian metric and an almost complex manifold with a Hermitian metric is called an almost Hermitian manifold.

By the virtue of which g is called a **Hermitian Metric** and an almost complex manifold with a hermitian metric is called an **Almost Hermitian Manifold**.

Example : An endomorphism $J : \mathbb{R}^4 \rightarrow \mathbb{R}^4$ defined by

$$J(X, Y, Z, W) = (-Y, X, -W, Z)$$

is an almost complex structure and \mathbb{R}^4 is called an almost complex manifold.

Example: An endomorphism $J : \mathbb{R}^6 \rightarrow \mathbb{R}^6$ defined by

$$J(X, Y, Z, T, U, V) = (-Y, X, -T, Z, -V, U)$$

is also an almost complex structure and \mathbb{R}^6 is an almost complex manifold.

Nijenhuis tensor (1.1.11)

The **Nijenhuis Tensor** S of J is defined as

$$S(U, V) = [U, V] + J[JU, V] + J[U, JV] - [JU, JV]$$

and we can verify that

$$S(JU, V) = S(U, JV) = -JS(U, V)$$

Proof:

$$\begin{aligned} S(JU, V) &= [JU, V] + J[J^2U, V] + J[JU, JV] - [J^2U, JV] \\ &= [JU, V] - J[U, V] + J[JU, JV] + [U, JV] \end{aligned} \quad (1.1.2)$$

$$\begin{aligned} \text{and } S(U, JV) &= [U, JV] + J[JU, JV] + J[U, J^2V] - [JU, J^2V] \\ &= [U, JV] + J[JU, JV] - J[U, V] + [JU, V] \end{aligned} \quad (1.1.3)$$

and

$$-JS(U, V) = -J\{[U, V] + J[JU, V] + J[U, JV] - [JU, JV]\}$$

$$\begin{aligned}
&= -J\{\nabla_U V - \nabla_V U + \mathcal{N}_{JU} V - \mathcal{N}_{JV} U + \mathcal{N}_{UV} J V - \mathcal{N}_{JU} J U - \nabla_{JU} J V + \nabla_{JV} J U\} \\
&= -\mathcal{N}_{UV} V + \mathcal{N}_{JV} U + \nabla_{JU} V - \nabla_V J U + \nabla_U J V - \nabla_{JV} U + \mathcal{N}_{JU} J V - \mathcal{N}_{JV} J U \\
&= [U, J V] + J[J U, J V] - J[U, V] + [J U, V]
\end{aligned} \tag{1.1.4}$$

From equation (1.1.2) , (1.1.3) and (1.1.4) we get :

$$S(JU, V) = S(U, JV) = -JS(U, V)$$

hence proof is completed.

If we extend the Riemannian Connexion to be a derivation $\bar{\nabla}$ of \bar{M} then we have the following formula :

$$(\bar{\nabla}_U J)V = \bar{\nabla}_U J V - J \bar{\nabla}_U V \tag{1.1.5}$$

Kaehler Manifold (1.1.12). A Hermitian metric on an almost complex manifold is called a **Kaehler metric** if the fundamental two form is closed i.e. , $(\bar{\nabla}_U J)V = 0$. A complex manifold equipped with a Kaehler metric is said to be Kaehler manifold. In other words, an almost complex manifold \bar{M} is Kaehler manifold if

$$(\bar{\nabla}_U J)V = 0$$

for all U, V in $T(\bar{M})$. In this case the connexion $\bar{\nabla}$ on \bar{M} is said to be Kaehlerian connexion.

Submanifold Theory (1.2)

Let M be the submanifold of manifold \bar{M} , then the tangent vectors of $T(\bar{M})$ which are normal to $T(M)$ form the **Normal bundle** $T^\perp(M)$ of M . Hence for every point $p \in M$, the tangent space $T_{f(p)}(\bar{M})$ of \bar{M} admits the following decomposition :

$$T_{f(p)}(\bar{M}) = T_p(M) + T_p^\perp(M)$$

The Riemannian connexion $\bar{\nabla}$ of \bar{M} induces canonically the connexion ∇ and D on $T(M)$ and on the normal bundle $T^\perp(M)$ respectively governed by the Gauss and Weingarten formulae:

Gauss formula:

$$\bar{\nabla}_U V = \nabla_U V + h(U, V) \quad (1.2.1)$$

Weingarten formula:

$$\bar{\nabla}_U \xi = -A_\xi U + D_U \xi \quad (1.2.2)$$

where U, V are tangential vector fields on M and $\xi \in T^\perp(M)$, h and A_ξ are second fundamental forms and are related by

$$g(h(U, V), \xi) = g(A_\xi U, V) \quad (1.2.3)$$

Holomorphic submanifold (1.2.1)[2]. A submanifold M of an almost Hermitian manifold \bar{M} is said to be *holomorphic* or *invariant* if

$$J(T_p(M)) = T_p(M),$$

for all $p \in M$.

Totally Real submanifold (1.2.2)[2]. A submanifold M of an almost Hermitian manifold \bar{M} is said to be *totally real* or *anti-invariant* if

$$J(T_p(M)) \subseteq T_p^\perp(M),$$

for all $p \in M$.

A new class of submanifolds of an Hermitian manifolds was defined and simplified by A. Bejancu and the above classes namely *invariant* and *totally real submaniflod* are particular cases of that class and that class is named as CR- submanifold i.e.

CR-submanifold provides a single setting to study the invariant and anti-invariant submanifolds of an almost Hermitian manifold.

Let \overline{M} be an almost Hermitian manifold with an almost complex structure J and Hermitian metric g , and let M be a submanifold. At each point $p \in M$, let D_p be the **maximal holomorphic subspace** of the tangent space $T_p(M)$ i.e.,

$$D_p = T_p(M) \cap JT_p(M).$$

If the dimension of D_p is same for all $p \in M$ we get a holomorphic distribution D on M .

CR – submanifold (1.2.3)[3]. A Riemannian submanifold is said to be **CR-submanifold** of an almost Hermitian manifold \overline{M} if there exist on M a holomorphic distribution D such that its orthogonal complementary distribution D^\perp is totally real i.e. $JD^\perp \subseteq T_p^\perp(M)$.

For example: Let IR^4 be an almost Hermitian manifold with an almost complex structure $J : IR^6 \rightarrow IR^6$ defined by

$$J(x, y, z, t, u, v) = (-y, x, -t, z, -v, u)$$

Now,

$$J(x, y, 0, 0, 0, 0) = (-y, x, 0, 0, 0, 0)$$

and

$$J(0, 0, z, 0, 0, 0) = (0, 0, 0, z, 0, 0)$$

Now we get XY - plane as a holomorphic distribution D and Z -plane as a totally real distribution D^\perp , so IR^3 is CR- submanifold of an almost Hermitian manifold IR^6 .

Remark: We observe from the above definition that the dimension of D is always even and JD^\perp being a sub bundle of $T^\perp(M)$, the normal bundle splits as,

$$T^\perp(M) = JD^\perp \oplus \nu$$

where ν is complement of JD^\perp in $T^\perp(M)$ and here ν is invariant under J .

Verification: To prove that ν is invariant under J .

Let $\xi \in \nu$ and $Z \in D^\perp$ then $JZ \in JD^\perp$, thus

$$g(J\xi, JZ) = g(\xi, Z) = 0.$$

Hence $J\xi \in \nu$, which proves that ν is invariant under J .

Definition (1.2.4)[3]. A CR- submanifold M is said to be *proper* if neither D nor $D^\perp = \{0\}$. Obviously if $D = \{0\}$ then M is *totally real submanifold* and if $D^\perp = \{0\}$, then M is *holomorphic submanifold*.

Definition (1.2.5)[3]. A CR- submanifold M is called *anti-holomorphic* submanifold if $JD_p^\perp = T_p^\perp(M)$ for all $p \in M$.

CR- product (1.3)

A CR- submanifold M is called a CR- product if it is locally a Riemannian product of a holomorphic submanifold M^T and a totally real submanifold M^\perp .

From the above definition for a CR- product submanifold, the leaves of D and D^\perp are totally geodesic in M and vice - versa. The leaves of a distribution D on a manifold M are totally geodesic in M if and only if $\nabla_X Y \in D$ for all $X, Y \in D$. Thus in the setting of CR- submanifold of an almost Hermitian manifold, the leaves of D are totally geodesic in M if and only if

$$\nabla_X Y \in D \tag{1.3.1}$$

for all $X, Y \in D$. Which is equivalent to the conditions

$$\nabla_X W \in D^\perp \tag{1.3.2}$$

for all $X \in D$ and $W \in D^\perp$.

Let $\nabla_X Y \in D \quad \forall X, Y \in D$, we have

$$g(\nabla_X Y, Z) = 0 \quad \text{for } Z \in D^\perp$$

$$\nabla_X (g(Y, Z)) - g(Y, \nabla_X Z) - (\nabla_X g)(Y, Z) = 0$$

$$\{ \nabla_X (g(Z, W)) = (\nabla_X g)(Z, W) + g(\nabla_X Z, W) + g(Z, \nabla_X W) \}$$

Now, we get

$$-g(Y, \nabla_X Z) = 0$$

$$g(Y, \nabla_X Z) = 0$$

therefore,

$$\nabla_X Z \in D^\perp.$$

Hence both are equal.

Similarly for the totally geodesicness of the leaves of D^\perp , the conditions

$$\nabla_Z W \in D^\perp \quad (1.3.3)$$

and

$$\nabla_Z X \in D \quad (1.3.4)$$

for X in D and $Z, W \in D^\perp$ are equivalent.

Let $\nabla_Z W \in D^\perp \quad \forall Z, W \in D^\perp$, we get

$$\begin{aligned} g(\nabla_Z W, X) &= 0 \quad \text{for } X \in D \\ \nabla_Z(g(W, X)) - g(W, \nabla_Z X) - (\nabla_X g)(W, X) &= 0 \\ -g(W, \nabla_Z X) &= 0 \\ g(W, \nabla_Z X) &= 0 \end{aligned}$$

thus ,

$$\nabla_Z X \in D.$$

Hence both are equivalent.

From the definition (1.3), a CR- submanifold is a CR- product if and only if the leaves of D and D^\perp are totally geodesic in M . Hence combining (1.3.1) and (1.3.4), we conclude that a CR- submanifold of an almost Hermitian manifold is a CR- product if and only if

$$\nabla_U X \in D \quad (1.3.5)$$

for all $U \in T(M)$ and $X \in D$.

Similarly, combining equations (1.3.2) and (1.3.3), it is concluded that a CR- submanifold is CR-product if and only if

$$\nabla_U Z \in D^\perp \quad (1.3.6)$$

for all $U \in T(M)$ and $Z \in D^\perp$. Condition (1.3.5) and (1.3.6) are equivalent because

$$g(\nabla_U X, Z) = 0 \Leftrightarrow g(X, \nabla_U Z) = 0.$$

Let $g(\nabla_U X, Z) = 0$, then we get

$$\begin{aligned} \nabla_U(g(X, Z)) - g(X, \nabla_U Z) - (\nabla_U g)(Z, W) &= 0 \\ -g(X, \nabla_U Z) &= 0 \\ g(X, \nabla_U Z) &= 0. \end{aligned}$$

Conversely , let $g(X, \nabla_U Z) = 0$

$$\nabla_U(g(X, Z)) - g(\nabla_U X, Z) - (\nabla_U g)(X, Z) = 0$$

$$-g(\nabla_U X, Z) = 0$$

hence ,

$$g(\nabla_U X, Z) = 0$$

For any vector field U tangent to M , we put

$$JU = PU + FU \tag{1.3.7}$$

where PU and FU are the tangential and normal components of JU respectively. Then P is an endomorphism of the tangent bundle $T(M)$ and F is a normal bundle valued one form on $T(M)$. It is easy to observe that $PU \in D$ and $FU \in JD^\perp$.

Similarly for any vector ξ normal to M , if we put

$$J\xi = t\xi + f\xi \tag{1.3.8}$$

with $t\xi$ and $f\xi$ as tangential and normal components of $J\xi$ respectively then f can be treated as endomorphism of the normal bundle $T^\perp(M)$ and t , a tangent bundle valued one form on $T^\perp(M)$.

some observations:

(1) For any $U \in T(M)$, $PU \in D$ and $FU \in JD^\perp$.

(2) For any $\xi \in T^\perp(M)$, $t\xi \in D^\perp$ and $f\xi \in \nu$.

Verification:

(1) For $U \in T(M)$ and $Z \in D^\perp$

$$g(PU, Z) = g(JU, Z)$$

$$g(PU, Z) = -g(U, JZ)$$

As $JZ \in T^\perp(M)$, we get

$$g(PU, Z) = 0$$

thus ,

$$PU \in D .$$

For $\xi \in \nu$,

$$g(FU, \xi) = g(JU, \xi)$$

$$g(FU, \xi) = -g(U, J\xi)$$

$$g(FU, \xi) = 0.$$

Therefore ,

$$FU \in JD^\perp$$

(2) For $\xi \in T^\perp(M)$ and $X \in D$

$$g(t\xi, X) = g(J\xi, X)$$

$$g(t\xi, X) = -g(\xi, JX)$$

$$g(t\xi, X) = 0$$

so ,

$$t\xi \in D^\perp.$$

For $\xi \in T^\perp(M)$ and $Z \in D^\perp$ then $JZ \in JD^\perp$,

$$g(JZ, f\xi) = g(JZ, J\xi)$$

$$g(JZ, f\xi) = g(Z, \xi)$$

$$g(JZ, f\xi) = 0.$$

Therefore $f\xi \in \nu$, hence proved.

Chapter II

Some basic results for CR-submanifold of a Kaehler manifold

In this chapter we have taken some basic results of CR-submanifold of a Kaehler manifold. All the results of this chapter are due to B.Y. Chen[3]. The notations and symbols are kept same throughout the chapter e.g. M denotes a CR- submanifold of a Kaehler manifold \bar{M} , vectors U, V etc. are taken as vectors tangential to M ; X, Y are vectors in the holomorphic distribution D on M and Z, W etc. denote the vectors in the totally real distribution D^\perp on M . In the decomposition

$$JU = PU + FU$$

and

$$J\xi = t\xi + f\xi$$

it has been observed that

$$PU \in D, FU \in JD^\perp$$

for all U in $T(M)$ and $t\xi \in D^\perp$ and $f\xi \in \nu$.

For all $\xi \in T^\perp(M)$, ν is the orthogonal complement of JD^\perp and is invariant under J , i.e. ,

$$T^\perp(M) = JD^\perp \oplus \nu, \quad JD^\perp \perp \nu$$

Some basic results (2.1).

Consider $\bar{\nabla}_U JZ$ for any $U \in T(M)$ and $Z \in D^\perp$. Using Gauss and Weingarten equations and the fact that $\bar{\nabla}_U J = 0$ we get

$$J\nabla_U Z + Jh(U, Z) = -A_{JZ} + D_U JZ \tag{2.1.1}$$

Verification: We have ,

$$\bar{\nabla}_U JZ = (\bar{\nabla}_U J)Z + J\bar{\nabla}_U Z$$

using Weingarten and Gauss formula , we get

$$-A_{JZ}U + D_U JZ = J(\nabla_U Z + h(U, Z)) \quad \{ \because \bar{\nabla}_U J = 0 \}$$

$$\mathcal{N}_U Z + Jh(U, Z) = -A_{JZ}U + D_U JZ$$

hence proved.

Lemma (2.1.1)[3]. Let M be a CR- submanifold of a Kaehler manifold \overline{M} , then

$$g(\nabla_U Z, X) = g(JA_{JZ}U, X) \quad (2.1.2)$$

$$A_{JZ}W = A_{JW}Z \quad (2.1.3)$$

$$A_{J\xi}X = -A_{\xi}JX \quad (2.1.4)$$

Lemma(2.1.2)[3]. Let M be a CR- submanifold of Kaehler manifold \overline{M} .Then for any $Z, W \in D^\perp$, we have

$$D_W JZ - D_Z JW \in JD^\perp \quad (2.1.5)$$

Lemma (2.1.3)[3]. Let M be CR- submanifold of a Kaehler manifold \overline{M} . Then D is integrable if and only if

$$g(h(X, JY), JZ) = g(h(JX, Y), JZ) \quad (2.1.6)$$

for any vector field $X, Y \in D$ and $Z \in D^\perp$.

Lemma (2.1.4)[3]. For a CR- submanifold M in a Kaehler manifold \overline{M} , the leaf M^\perp of D^\perp is totally geodesic in M if and only if

$$g(h(D, D^\perp), JD^\perp) = 0 \quad (2.1.7)$$

Lemma (2.1.5)[3]. The leaves of the holomorphic distribution D on a CR- submanifold M of a Kaehler manifold \overline{M} are totally geodesic in M if and only if

$$g(h(D, D), JD^\perp) = 0 \quad (2.1.8)$$

Lemma (2.1.6)[3]. If (2.1.7) holds and D is integrable then for any $X \in D$ and $\xi \in JD^\perp$,

$$A_\xi JX = -JA_\xi X \quad (2.1.9)$$

CR- product in Kaehler manifold (2.2)

A CR- submanifold M of a Kaehler manifold is called a CR- product if it is locally the Riemannian product of a holomorphic submanifold M^T and a totally real submanifold M^\perp of M .

Now making use of Lemma (2.1.4) and Lemma (2.1.5), it follows that a CR- submanifold with integrable distribution, is CR- product if and only if

$$A_{JD^\perp} D = 0$$

Verification: From lemma (2.1.4) and (2.1.5), we have

$$g(h(D, D^\perp), JD^\perp) = 0$$

and
$$g(h(D, D), JD^\perp) = 0.$$

From the above equations, we get

$$g(A_{JD^\perp} D, D^\perp) = 0$$

and
$$g(A_{JD^\perp} D, D) = 0 \quad \{\text{using equation (1.2.3)}\}$$

so, we conclude that
$$A_{JD^\perp} D = 0.$$

Hence proved.

Theorem (2.2.1)[3]. A CR-submanifold M of a Kaehler manifold \overline{M} is a CR- product if and only if $\nabla P = 0$.

CR- submanifold with $\nabla F=0$ (2.3)

On a CR- submanifold M in a Kaehler manifold \overline{M} , there is a canonical normal bundle valued one form F on $T(M)$ and a tangent bundle valued one form t on $T^\perp(M)$. In this section we shall classify CR- submanifold with parallel F .

Lemma (2.3.1)[3]. For any vector field U, V tangent to M and ξ normal to M , we have

$$(\nabla_U t)\xi = A_{f\xi}U - PA_\xi U \quad (2.3.1)$$

$$(\nabla_U f)\xi = -FA_\xi U - h(U, t\xi) \quad (2.3.2)$$

$$(\nabla_U F)V = fh(U, V) - h(U, PV) \quad (2.3.3)$$

Proposition (2.3.1)[3]. Let M be a CR- manifold of a Kaehler manifold \overline{M} . Then $\nabla F = 0$ if and only if $\nabla t = 0$.

Chapter III

Generic submanifold of a Kaehler manifold

Given a real submanifold M of a complex manifold \overline{M} , the tangent bundle $T(M)$ of M is not invariant under the complex structure J of \overline{M} , it is therefore, natural to explore the holomorphic distribution and its complementary distribution on M . The generic submanifolds are the outcome of this approach of study. B.Y. Chen [4] introduced these submanifolds and has studied them in the setting of Kaehler manifolds. CR-submanifolds are a special case of these submanifolds. This class of submanifolds in the Kaehler manifolds is studied in this chapter.

Let M be a real submanifold of a Kaehler manifold \overline{M} , with an almost complex structure J . Define

$$D_x = T_x(M) \cap JT_x(M), \quad x \in M,$$

the maximal complex subspace of the tangent space $T_x(\overline{M})$. If the dimension of D_x is constant along M and it defines a differentiable distribution on M , then M is said to be a **generic submanifold**. The distribution D is then called a **holomorphic distribution**. A generic submanifold M in a Kaehler manifold is a **purely real** (respectively, **complex**) if $D_x = \{0\}$ (respectively, $D_x = T_x(M)$). Obviously, if the orthogonal complementary distribution D^\perp of D in $T(M)$ is totally real i.e., $JD^\perp \subset T_x^\perp(M)$. Then M is a CR-submanifold of \overline{M} i.e. CR-submanifold is a special case of generic submanifold. A generic submanifold is said to be **proper** if it is neither a complex submanifold nor a purely real submanifold.

Some properties (3.1)

Let ν_x be the vector space of the holomorphic normal vector of \overline{M} at x i.e.,

$$\nu_x = T_x^\perp(M) \cap JT_x^\perp(M),$$

then ν_x defines a differentiable vector sub-bundle of $T^\perp(M)$. We have the following observation :

- (1) $D_x \perp D_x^\perp$
- (2) $PD_x^\perp \subseteq D_x^\perp$
- (3) $D_x^\perp \cap JD_x^\perp = \{0\}$
- (4) $T^\perp(M) = FD^\perp \oplus \nu$
- (5) $t(T^\perp(M)) = D^\perp$
- (6) $g(FD^\perp, \nu) = 0$

Verification:

(1) Clearly D_x and D_x^\perp are orthogonal to each other, so it follows from the definition.

(2) For $X \in D$ and $Z \in D^\perp$, we know D is invariant under J so $JX \in D$,

$$\begin{aligned} \text{then} \quad 0 &= g(JX, Z) = -g(X, JZ) = -g(X, PZ + FZ) \\ &= -g(X, PZ) - g(X, FZ) = -g(X, PZ) \end{aligned}$$

$$\text{so, we get} \quad g(X, PZ) = 0 \text{ i.e., } PZ \perp X.$$

$$\text{Therefore,} \quad PZ \in D^\perp \quad \forall \quad Z \in D^\perp,$$

this verifies the assertion.

(3) Let $Z \in D_x^\perp \cap JD_x^\perp$ then $Z \in D_x^\perp$ and $Z \in JD_x^\perp$.

$$\text{Now, } Z \in JD_x^\perp \text{ then } JZ \in J(JD_x^\perp) \text{ i.e., } JZ \in D_x^\perp$$

we have $Z \in D_x^\perp$ and $JZ \in D_x^\perp$ which shows that Z is invariant under J . But D^\perp is totally real, therefore $Z = 0$

$$\text{i.e.,} \quad D_x^\perp \cap JD_x^\perp = \{0\}$$

hence proved.

(4) From the definition we get that the normal bundle $T^\perp(M)$ is divided into two parts FD^\perp and ν , so the result follows from definition.

(5) For $\xi \in T^\perp(M)$ and $X \in D$

$$g(t\xi, X) = g(J\xi, X) = -g(\xi, JX) = 0$$

so, $t\xi \in D^\perp$ therefore $t(T^\perp(M)) \in D^\perp$,

on the other hand $D^\perp \subseteq t(T^\perp(M))$ (obviously true),

then $D^\perp = t(T^\perp(M))$ hence proved.

(6) By definition , FD^\perp is the orthogonal complementary distribution of ν , then

$$g(FD^\perp, \nu) = 0$$

the result follows from the definition.

Some basic results (3.2)

Throughout the chapter , we will assume M to be a generic submanifold of a Kaehler manifold \bar{M} . As \bar{M} is Kaehler ,

$$\bar{\nabla}_U J V = J \bar{\nabla}_U V$$

for all $U, V \in T(M)$, by using (1.3.7) equation and Gauss formula , we get

$$\begin{aligned} \bar{\nabla}_U (P V + F V) &= J(\nabla_U V + h(U, V)) \\ \bar{\nabla}_U P V + \bar{\nabla}_U F V &= J \nabla_U V + J h(U, V) \end{aligned}$$

Now using the equations (1.3.7) , (1.3.8) and Gauss and Weingarten's formula , we get

$$\nabla_U P V + h(U, P V) - A_{F V} U + D_U F V = P \nabla_U V + F \nabla_U V + f h(U, V) + t h(U, V)$$

equating the tangential parts of the equation , we get

$$\begin{aligned} \nabla_U P V - A_{F V} U &= P \nabla_U V + t h(U, V) \\ \nabla_U P V - P \nabla_U V &= t h(U, V) + A_{F V} U \\ (\nabla_U P) V &= t h(U, V) + A_{F V} U \end{aligned} \tag{3.2.1}$$

now equating the normal parts of the equation , we get

$$\begin{aligned} h(U, P V) + D_U F V &= F \nabla_U V + f h(U, V) \\ D_U F V - F \nabla_U V &= f h(U, V) - h(U, P V) \\ (\nabla_U F) V &= f h(U, V) - h(U, P V) \end{aligned} \tag{3.2.2}$$

now for $\xi \in T^\perp(M)$, we have

$$\bar{\nabla}_U J \xi = J \bar{\nabla}_U \xi$$

using equation (1.3.8) and Weingarten fomula , we get

$$\bar{\nabla}_U(t\xi + f\xi) = J(-A_\xi U + D_U \xi)$$

$$\bar{\nabla}_U t\xi + \bar{\nabla}_U f\xi = -JA_\xi U + JD_U \xi$$

now using equations (1.3.7) , (1.3.8) and Gauss and Weingarten formula , we get

$$\nabla_U t\xi + h(U, t\xi) - A_{f\xi} U + D_U f\xi = -PA_\xi U - FA_\xi U + tD_U \xi + fD_U \xi$$

equating the tangential parts , we get

$$\nabla_U t\xi - A_{f\xi} U = -PA_\xi U + tD_U \xi$$

$$\nabla_U t\xi - tD_U \xi = -PA_\xi U + A_{f\xi} U$$

$$(\nabla_U t)\xi = -PA_\xi U + A_{f\xi} U \quad (3.2.3)$$

now equating the normal parts , we get

$$h(U, t\xi) + D_U f\xi = -FA_\xi U + fD_U \xi$$

$$D_U f\xi - fD_U \xi = -FA_\xi U - h(U, t\xi)$$

$$(\nabla_U f)\xi = -FA_\xi U - h(U, t\xi) \quad (3.2.4)$$

Some integrability conditions (3.3)

As \bar{M} is Kaehler , $\bar{\nabla}_U J V = J \bar{\nabla}_U V$ for all $U, V \in T(\bar{M})$. In particular , for $X \in D$ and $U \in T(M)$,

$$\bar{\nabla}_U J X = J \bar{\nabla}_U X$$

using the Gauss formula , we get

$$\nabla_U J X + h(U, J X) = J \nabla_U X + J h(U, X)$$

$$h(U, J X) = J \nabla_U X + J h(U, X) - \nabla_U J X$$

$$\begin{aligned} g(h(U, J X), \xi) &= g(J \nabla_U X + J h(U, X) - \nabla_U J X, \xi) \quad \forall \xi \in \nu \\ &= g(J \nabla_U X, \xi) + g(J h(U, X), \xi) - g(\nabla_U J X, \xi) \\ &= g(J h(U, X), \xi) \end{aligned}$$

so , we get

$$g(h(U, J X), \xi) = g(J h(U, X), \xi) \quad (3.3.1)$$

Now , for the integrability of the holomprphic distribution D , we have

Proposition (3.3.1)[4]. Let M be a generic submanifold in a Kaehler manifold \bar{M} , then the distribution D is integrable if and only if

$$g(h(X, JY), FZ) = g(h(JX, Y), FZ) \quad (3.3.2)$$

for any vectors $X, Y \in D$ and $Z \in D^\perp$.

Proof: Firstly assume that D is integrable . Take

$$h(X, JY) - h(JX, Y) = (\bar{\nabla}_X JY - \nabla_X JY) - (\bar{\nabla}_Y JX - \nabla_Y JX) \text{ \{using equation (1.2.1)\}}$$

as \bar{M} is a Kaehlerian , the right hand side of the above equation will become

$$h(X, JY) - h(JX, Y) = J(\bar{\nabla}_X Y - \bar{\nabla}_Y X) + (\nabla_Y JX - \nabla_X JY)$$

$$h(X, JY) - h(JX, Y) = J[X, Y] + \nabla_Y JX - \nabla_X JY$$

D is holomorphic distribution of $T(M)$ and D is integrable , then for any $X, Y \in D$, the right hand side of the above equation lies in $T(M)$, whereas the left hand side is a vector field normal to M , thus

$$h(X, JY) - h(JX, Y) = 0$$

$$h(X, JY) = h(JX, Y)$$

hence ,

$$g(h(X, JY), FZ) = g(h(JX, Y), FZ)$$

Conversely, let $g(h(X, JY), FZ) = g(h(JX, Y), FZ)$ for all $X, Y \in D$ and $Z \in D^\perp$.

$$g(h(X, JY) - h(JX, Y), FZ) = 0$$

then ,

$$h(X, JY) - h(JX, Y) \in \nu \quad (3.3.3)$$

For $\xi \in \nu$,

$$g(h(X, JY) - h(JX, Y), \xi) = g(h(X, JY), \xi) - g(h(JX, Y), \xi)$$

using equation (3.3.1) , we get

$$= g(Jh(X, Y), \xi) - g(Jh(X, Y), \xi)$$

$$= 0$$

thus ,

$$h(X, JY) - h(JX, Y) \in FD^\perp \quad (3.3.4)$$

from equation (3.3.3) and (3.3.4) , we get

$$h(X, JY) - h(JX, Y) = 0$$

we have

$$h(X, JY) - h(JX, Y) = J[X, Y] + \nabla_Y JX - \nabla_X JY$$

$$J[X, Y] + \nabla_Y JX - \nabla_X JY = 0$$

$$J[X, Y] = \nabla_X JY - \nabla_Y JX,$$

for any $X, Y \in T(M)$, $\nabla_X JY - \nabla_Y JX$ is tangent to M , $[X, Y]$ lies in D . Therefore D is integrable.

Proposition (3.3.2)[4]. Let M be a generic submanifold in a Kaehler manifold \overline{M} , if D is integrable then

$$A_{FZ}JX = -JA_{FZ}X$$

for any vector field X in D and Z in D^\perp .

Proof: For any $X \in D$, $U \in T(M)$ and $Z \in D^\perp$, by equation (1.2.3), we get

$$\begin{aligned} g(A_{FZ}JX, U) &= g(h(JX, U), FZ) \\ &= g(h(X, JU), FZ) \end{aligned}$$

by using equation (1.2.3) the above relation becomes

$$\begin{aligned} g(A_{FZ}JX, U) &= g(A_{FZ}X, JU) \\ g(A_{FZ}JX, U) &= -g(JA_{FZ}X, U) \\ g(A_{FZ}JX - JA_{FZ}X, U) &= 0 \end{aligned}$$

hence,

$$A_{FZ}JX = -JA_{FZ}X \quad (3.3.5)$$

For the integrability of the purely real distribution, we have

Proposition (3.3.3)[4]. Let M be generic submanifold in a Kaehler manifold \overline{M} . Then the purely real distribution D^\perp is integrable if and only if

$$\nabla_Z PW - \nabla_W PZ + A_{FZ}W - A_{FW}Z \in D^\perp$$

for any vector field Z, W in D^\perp .

Proof: As \overline{M} is a Kaehler, for any Z, W in D^\perp , we have

$$J\overline{\nabla}_Z W = \overline{\nabla}_Z JW$$

Now , using Gauss formula and equation (1.3.7) , we get

$$J(\nabla_Z W + h(Z, W)) = \bar{\nabla}_Z(PW + FW)$$

$$J\nabla_Z W + Jh(Z, W) = \bar{\nabla}_Z PW + \bar{\nabla}_Z FW$$

simplifying right hand side of equation using Gauss and Weingarten formula

$$J\nabla_Z W + Jh(Z, W) = \nabla_Z PW + h(Z, PW) - A_{FW}Z + D_Z FW$$

operating the almost complex structure J on both sides of the equation ,

$$-\nabla_Z W - h(Z, W) = J\nabla_Z PW + Jh(Z, PW) - JA_{FW}Z + JD_Z FW$$

using equation (1.3.7) and (1.3.8) ,we get

$$\begin{aligned} -\nabla_Z W - h(Z, W) = & P\nabla_Z PW + F\nabla_Z PW + th(Z, PW) + fh(Z, PW) - PA_{FW}Z - FA_{FW}Z \\ & + tD_Z FW + fD_Z FW \end{aligned}$$

equating the tangential parts

$$-\nabla_Z W = P\nabla_Z PW + th(Z, PW) - PA_{FW}Z + tD_Z FW$$

$$\nabla_Z W = PA_{FW}Z - P\nabla_Z PW - th(Z, PW) - tD_Z FW \quad (3.3.6)$$

interchanging the role of Z and W in (3.3.6) , we get

$$\nabla_W Z = PA_{FZ}W - P\nabla_W PZ - th(W, PZ) - tD_W FZ \quad (3.3.7)$$

subtracting (3.3.6) and (3.3.7) we have

$$\begin{aligned} \nabla_Z W - \nabla_W Z = & P\{A_{FW}Z - A_{FZ}W + \nabla_W PZ - \nabla_Z PW\} + t\{h(W, PZ) \\ & - h(Z, PW) + D_W FZ - D_Z FW\} \end{aligned}$$

$$\begin{aligned} [Z, W] = & P\{A_{FW}Z - A_{FZ}W + \nabla_W PZ - \nabla_Z PW\} + t\{h(W, PZ) \\ & - h(Z, PW) + D_W FZ - D_Z FW\} \end{aligned}$$

since $t(T^\perp(M)) \in D^\perp$,

Now $[Z, W] \in D^\perp$ if and only if $A_{FW}Z - A_{FZ}W + \nabla_W PZ - \nabla_Z PW \in D^\perp$.

Hence proved.

Lemma (3.3.1)[4]. Let M be a generic submaifold in a Kaehler manifold \bar{M} . If D is integrable and its leaves are totally geodesic in M , then

$$g(A_{FD^\perp}D, D) = 0$$

Proof: Under the hypothesis $\nabla_X Z \in D^\perp \quad \forall X \in D$ and $Z \in D^\perp$ in view of equation (1.3.2) . So for any vector field $Y \in D$ we have ,

$$\begin{aligned}
0 &= g(\nabla_X Z, JY) = g(\bar{\nabla}_X Z - h(X, Z), JY) \quad \{\text{using Gauss formula}\} \\
&= g(\bar{\nabla}_X Z, JY) - g(h(X, Z), JY) \\
&= g(\bar{\nabla}_X Z, JY) = -g(J\bar{\nabla}_X Z, Y) \\
&= -g(\bar{\nabla}_X JZ, Y) \quad \{\because \bar{M} \text{ is a kaehler manifold}\}
\end{aligned}$$

now using equation (1.3.7) , we get

$$\begin{aligned}
0 &= -g(\bar{\nabla}_X PZ, Y) - g(\bar{\nabla}_X FZ, Y) \\
&= -g(\bar{\nabla}_X FZ, Y) \\
&= -g(-A_{FZ}X + D_X FZ, Y) \\
&= -g(-A_{FZ}X, Y) - g(D_X FZ, Y) \\
&= g(A_{FZ}X, Y)
\end{aligned}$$

so ,

$$g(A_{FZ}X, Y) = 0.$$

Therefore ,

$$g(A_{FD^\perp}D, D) = 0 ,$$

which proves the result.

Converse of the above Lemma is not true in generic submanifold of a Kaehler manifold.

Note: $g(A_{FZ}X, Y) = 0$

by using Weingarten formula

$$\begin{aligned}
g(-\bar{\nabla}_X FZ + D_X FZ, Y) &= 0 \\
g(-\bar{\nabla}_X FZ, Y) + g(D_X FZ, Y) &= 0 \\
-g(\bar{\nabla}_X FZ, Y) &= 0 ,
\end{aligned}$$

using equation (1.3.7) , we get

$$\begin{aligned}
g(\bar{\nabla}_X (JZ - PZ), Y) &= 0 \\
g(\bar{\nabla}_X JZ, Y) - g(\bar{\nabla}_X PZ, Y) &= 0 \\
-g(JZ, \bar{\nabla}_X Y) + g(PZ, \bar{\nabla}_X Y) &= 0
\end{aligned}$$

$$g(Z, J\bar{\nabla}_X Y) + g(PZ, \bar{\nabla}_X Y) = 0$$

as \bar{M} is a Kaehler manifold , so above relation becomes

$$g(Z, \bar{\nabla}_X JY) + g(PZ, \bar{\nabla}_X Y) = 0$$

using Gauss equation , we get

$$\begin{aligned} g(Z, \nabla_X JY + h(X, JY)) + g(PZ, \nabla_X Y + h(X, Y)) &= 0 \\ g(Z, \nabla_X JY) + g(Z, h(X, JY)) + g(PZ, \nabla_X Y) + g(PZ, h(X, Y)) &= 0 \\ g(Z, \nabla_X JY) + g(PZ, \nabla_X Y) &= 0, \end{aligned}$$

which does not imply that $\nabla_X JY \in D$. Whereas in the case of CR- submanifold $PZ=0 \forall Z \in D^\perp$ and therefore the converse of the Lemma follows in that setting . i.e. the above Lemma is infact necessary and sufficient for the case of CR- submanifold of a Kaehler manifold . Whereas for the generic submanifold Lemma gives a necessary condition . For generic submanifold PZ is not zero in general for $Z \in D^\perp$.

Lemma (3.3.2)[4]. Let M be a generic submanifold in a Kaehler manifold \bar{M} . If D^\perp is integrable and its leaves are totally geodesic in \bar{M} , then

$$g(A_{FD^\perp} D, D^\perp) = 0$$

Proof: Under the hypothesis , we have

$$\begin{aligned} 0 &= g(\nabla_Z X, W) \quad \forall X \in D \text{ and } Z \in D^\perp \\ &= g(\bar{\nabla}_Z X - h(Z, X), W) \quad \{\text{using the Gauss formula}\} \\ &= g(\bar{\nabla}_Z X, W) - g(h(Z, X), W) \\ &= g(\bar{\nabla}_Z X, W) \end{aligned}$$

by using (1.1.1), we get

$$g(J\bar{\nabla}_Z X, JW) = 0$$

as \bar{M} is Kaehler manifold , so

$$g(\bar{\nabla}_Z JX, JW) = 0$$

using equation (1.3.7) the above equation will become

$$g(\bar{\nabla}_Z JX, PW + FW) = 0$$

$$g(\bar{\nabla}_Z JX, PW) + g(\bar{\nabla}_Z JX, FW) = 0$$

$$g(\bar{\nabla}_Z JX, FW) = 0$$

now using gauss formula , we will get

$$g(\nabla_Z JX + h(Z, JX), FW) = 0$$

$$g(\nabla_Z JX, FW) + g(h(Z, JX), FW) = 0$$

$$g(h(Z, JX), FW) = 0$$

using the equation (1.2.3), we get

$$g(A_{FW} JX, Z) = 0 ,$$

this proves the Lemma.

Proposition (3.3.4)[4]. Let M be a generic submanifold of Kaehler manifold \bar{M} .

If $g(h(D, D^\perp), FD^\perp) = 0$ or $\{g(A_{FD^\perp} D, D^\perp) = 0\}$, then

$$\nabla_X PZ = A_{FZ} X + P\nabla_X Z$$

for all $X \in D$ and $Z \in D^\perp$.

Proof: By Gauss equation ,

$$\bar{\nabla}_X PZ = \nabla_X PZ + h(X, PZ)$$

also, $\bar{\nabla}_X PZ = \bar{\nabla}_X JZ - \bar{\nabla}_X FZ$ {using equation (1.3.7)}

as \bar{M} is Kaehler manifold,

$$\bar{\nabla}_X PZ = J\bar{\nabla}_X Z - \bar{\nabla}_X FZ$$

using Weingarten formula and Gauss formula right hand side will be

$$= J(\nabla_X Z + h(X, Z)) + A_{FZ} X - D_X FZ$$

by using equations (1.3.7) and (1.3.8), we will get

$$\bar{\nabla}_X PZ = P\nabla_X Z + F\nabla_X Z + th(X, Z) + fh(X, Z) + A_{FZ} X - D_X FZ$$

applying the Gauss formula on the left hand side of above equation

$$\nabla_X PZ + h(X, PZ) = P\nabla_X Z + F\nabla_X Z + th(X, Z) + fh(X, Z) + A_{FZ} X - D_X FZ$$

equating the tangential parts of above equation

$$\nabla_X PZ = P\nabla_X Z + th(X, Z) + A_{FZ}X$$

further , by hypothesis

$$g(h(D, D^\perp), FD^\perp) = 0 \text{ i.e. } g(h(X, Z), FZ) = 0 \quad \forall X \in D, Z \in D^\perp, h(X, Z) \in \nu,$$

$$\text{i.e. ,} \quad th(X, Z) = 0$$

therefore , we get

$$\nabla_X PZ = P\nabla_X Z + A_{FZ}X$$

hence Lemma is proved.

Generic submanifold with parallel canonical structure (3.4)

The endomorphism $P : TM \rightarrow TM$ is said to be *parallel* if $\nabla_U P = 0$ for any vector U

tangent to M . Since the manifold \bar{M} is Kaehler

$$\bar{\nabla}_U J V = J \bar{\nabla}_U V$$

using the equations (1.3.7) and Gauss formula

$$\bar{\nabla}_U P V + \bar{\nabla}_U F V = J(\nabla_U V + h(U, V))$$

using the Gauss equation, Weingarten equation , (1.3.7) and (1.3.8) , we have

$$\begin{aligned} \nabla_U P V + h(U, P V) - A_{FV} U + D_U F V &= P \nabla_U V + F \nabla_U V \\ &+ th(U, V) + fh(U, V) \end{aligned} \quad (3.4.1)$$

equating the tangential parts

$$\nabla_U P V - A_{FV} U = P \nabla_U V + th(U, V)$$

$$\nabla_U P V - P \nabla_U V = A_{FV} U + th(U, V)$$

by equation (1.1.5), we have

$$(\nabla_U P)V = A_{FV} U + th(U, V)$$

$$(\nabla_U P)V - A_{FV} U - th(U, V) = 0$$

for any vector field U, V, W tangent to M , we have

$$\begin{aligned} g((\nabla_U P)V, W) &= g(A_{FV} U + th(U, V), W) \\ &= g(A_{FV} U, W) + g(th(U, V), W) \end{aligned}$$

$$\begin{aligned}
&= g(A_{FV}U, W) + g(Jh(U, V), W) \\
&= g(A_{FV}U, W) - g(h(U, V), FW)
\end{aligned}$$

by equation (1.2.3) the right hand side of above equation will become

$$\begin{aligned}
&= g(h(U, W), FV) - g(h(V, U), FW) \\
&= g(h(W, U), FV) - g(h(V, U), FW)
\end{aligned}$$

again using equation (1.2.3) , the right hand side becomes

$$= g(A_{FV}W, U) - g(A_{FW}V, U)$$

hence ,
$$g((\nabla_U P)V, W) = g(A_{FV}W - A_{FW}V, U) \tag{3.4.2}$$

Lemma (3.4.1)[4]. Let M be a generic subamifold of a Kaehler manifold \overline{M} , then P is parallel if and only if

$$A_{FU}V = A_{FV}U \tag{3.4.3}$$

for any U, V tangent to M .

Proof: Firstly assume that P is parallel , then $\nabla_U P = 0$ for U is tangent to M . By equation (3.4.2)

$$g(A_{FV}W - A_{FW}V, U) = g((\nabla_U P)V, W) = 0$$

then ,
$$g(A_{FV}W - A_{FW}V, U) = 0$$
 , for all U is tangent to M .

therefore ,
$$A_{FV}W - A_{FW}V = 0$$

$$A_{FV}W = A_{FW}V$$

which can also be written as ,

$$A_{FU}V = A_{FV}U$$

for any U, V tangent to M .

conversely, let , $A_{FU}V = A_{FV}U$ for any U, V tangent to M and for any V, W tangent to M ,

$$A_{FV}W = A_{FW}V ,$$

by the above equation (3.4.2) , we will get

$$g((\nabla_U P)V, W) = 0$$

thus,

$$\nabla_U P = 0$$

Lemma is proved.

Proposition (3.4.1)[4]. Let M be a generic subamnifold in a Kaehler manifold \overline{M} . If $\nabla P = 0$, then

(1) the holomorphic distribution D is integrable ,

(2) $A_{FU}X = 0$

Proof: If P is parallel then by Lemma (3.4.1) , we have

$$A_{FU}V = A_{FV}U$$

for U, V tangent to M .

In particular for $X \in D$, we will get

$$A_{FU}X = A_{FX}U$$

so ,

$$A_{FU}X = 0$$

Part (2) is proved.

For any X, Y in D and Z in D^\perp , therefore by proposition (3.3.1), D is integrable . This proves the proposition.

The normal bundle valued one-form F is said to be **parallel** if $\nabla F = 0$. On equating the normal parts of equation (3.4.1) we get

$$h(U, PV) + D_U FV = F\nabla_U V + fh(U, V)$$

$$D_U FV - F\nabla_U V = -h(U, PV) + fh(U, V)$$

$$(D_U F)V = -h(U, PV) + fh(U, V) \tag{3.4.4}$$

For $U, V \in T(M)$, take any vector field $\xi \in T^\perp(M)$,

$$\begin{aligned} g((\nabla_U F)V, \xi) &= g(fh(U, V) - h(U, PV), \xi) \\ &= g(fh(U, V), \xi) - g(h(U, PV), \xi) \\ &= -g(h(U, V), f\xi) - g(h(PV, U), \xi) \\ &= -g(h(V, U), f\xi) - g(A_\xi PV, U) \end{aligned}$$

$$\begin{aligned}
&= -g(A_{f\xi}V, U) - g(A_\xi PV, U) \\
&= -g(A_{f\xi}V + A_\xi PV, U)
\end{aligned}$$

so,
$$g((\nabla_U F)V, \xi) = -g(A_{f\xi}V + A_\xi PV, U) \quad (3.4.5)$$

Lemma (3.4.2)[4]. Let M be generic submanifold in a Kaehler manifold \overline{M} . Then $\nabla F = 0$ if and only if

$$A_{f\xi}U = -A_\xi PU \quad (3.4.6)$$

for any vector U tangent to M and ξ normal to M .

Proof: Firstly assume that F is parallel i.e. $\nabla F = 0$, then by above equation (3.4.5), we get

$$-g(A_{f\xi}V + A_\xi PV, U) = 0$$

where U is any vector tangent to M , then

$$A_{f\xi}V + A_\xi PV = 0$$

$$A_{f\xi}V = -A_\xi PV$$

for any vector V tangent to M and ξ normal to M .

Conversely, take $A_{f\xi}V = -A_\xi PV$ for any vector V tangent to M and ξ normal to M , then

$$A_{f\xi}V + A_\xi PV = 0,$$

using equation (3.4.5), we get

$$g((\nabla_U F)V, \xi) = -g(A_{f\xi}V + A_\xi PV, U) = 0$$

$$g((\nabla_U F)V, \xi) = 0$$

$$\nabla_U F = 0,$$

which proves the Lemma.

Proposition (3.4.2)[4]. Let M be a generic submanifold in a Kaehler manifold \overline{M} , If F is parallel, then

- (1) $A_{FU}X = 0$ and $\nabla_U X \in D$
- (2) D is integrable and its leaves are totally geodesic in M .
- (3) FD^\perp and ν both are parallel in the normal bundle.
- (4) $(\nabla_X P)Z = (\nabla_U P)X = 0$

for any vector fields X in D , and Z in D^\perp and U, V in $T(M)$.

Proof: Using the fact

$$(\overline{\nabla}_U J)X = \overline{\nabla}_U JX - J\overline{\nabla}_U X \quad \forall U \in T(M), X \in D$$

as \overline{M} is Kaehler manifold, so

$$\overline{\nabla}_U JX - J\overline{\nabla}_U X = 0$$

using Gauss formula, we get

$$\nabla_U JX + h(U, JX) - J\nabla_U X - Jh(U, X) = 0$$

using the equations (1.3.7) and (1.3.8), we have

$$\nabla_U JX + h(U, JX) - P\nabla_U X - F\nabla_U X - th(U, X) - fh(U, X) = 0$$

given $\nabla F = 0$, then the above equation will become

$$\nabla_U JX - P\nabla_U X - F\nabla_U X - th(U, X) = 0 \quad \{\text{using (3.4.4)}\}$$

equating the normal parts

$$F\nabla_U X = 0$$

$$\nabla_U X \in D$$

further $A_{FU}X = 0$ by lemma (3.4.2), this proves (1).

Now to prove that D is integrable and its leaves are totally geodesic in M . By part (1) $\nabla_U X \in D$ for all X in D and U in $T(M)$, So the leaves of D are totally geodesic in M .

by part (1), in particular for $Z \in D^\perp$, we get

$$A_{FZ}X = 0$$

for $Y \in D$, we have

$$g(A_{FZ}X, Y) = 0$$

$$g(h(X, Y), FZ) = 0$$

by proposition (3.3.1), D is integrable.

For (3) Gauss equation gives

$$g(h(U, V), J\xi) = g(\bar{\nabla}_U V - \nabla_U V, J\xi)$$

for all $U, V \in T(M)$ and $\xi \in \nu$,

$$\begin{aligned} g(h(U, V), J\xi) &= g(\bar{\nabla}_U V, J\xi) - g(\nabla_U V, J\xi) \\ &= g(\bar{\nabla}_U V, J\xi) \\ &= -g(J\bar{\nabla}_U V, \xi) \end{aligned}$$

using equation (1.1.5), we get

$$\begin{aligned} g(h(U, V), J\xi) &= -g(\bar{\nabla}_U JV - (\bar{\nabla}_U J)V, \xi) \\ &= -g(\bar{\nabla}_U JV, \xi) \quad \{\text{using Kaehler condition}\} \end{aligned}$$

making use of equation (1.3.7)

$$\begin{aligned} g(h(U, V), J\xi) &= -g(\bar{\nabla}_U (PV + FV), \xi) \\ &= -g(\bar{\nabla}_U (PV + FV), \xi) \\ &= -g(\bar{\nabla}_U PV, \xi) - g(\bar{\nabla}_U FV, \xi) \end{aligned}$$

using equation (1.2.3), Gauss formula and Weingarten formula, we get

$$\begin{aligned} g(A_{J\xi}V, U) &= -g(\nabla_U PV + h(U, PV), \xi) - g(-A_{FV}U + D_U FV, \xi) \\ g(A_{J\xi}V, U) &= -g(h(U, PV), \xi) - g(D_U FV, \xi) \end{aligned}$$

again using equation (1.2.3), we have

$$\begin{aligned} g(A_{J\xi}V, U) &= -g(A_\xi PV, U) - g(D_U FV, \xi) \\ g(A_{J\xi}V + A_\xi PV, U) &= -g(D_U FV, \xi) \end{aligned}$$

using Lemma (3.4.2), we get

$$g(D_U FV, \xi) = 0$$

therefore,

$$g(D_U \xi, FV) = 0 \quad \forall U, V \in T(M) \text{ and } \xi \in \nu.$$

hence,

$$D_U \xi \in \nu \text{ i.e., } \nu \text{ is parallel in } T^\perp(M).$$

It further implies that $D_U FV \in FD^\perp$ i.e. FD^\perp is also parallel in $T^\perp(M)$, which proves the (3) part .

From the equation (3.4.2), we have

$$\begin{aligned} g((\nabla_X P)Z, U) &= g(A_{FZ}U - A_{FU}Z, X) \\ &= g(A_{FZ}U, X) - g(A_{FU}Z, X) \\ &= 0 \end{aligned}$$

as $A_{FU}X = 0$ by part (1) therefore ,

$$g((\nabla_X P)Z, U) = 0$$

for all U in $T(M)$. That shows $(\nabla_X P)Z = 0$

now again using (3.4.2), we get

$$g((\nabla_U P)X, V) = g(A_{FX}V - A_{FV}X, U)$$

as $FX = 0$ for $X \in D$ and $A_{FV}X = 0$ by part (2) , the right hand side of the above equation is zero , hence

$$(\nabla_U P)X = 0 ,$$

which proves the proposition.

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