

# **Methods for Solving Non-cooperative Games with Fuzzy Payoffs**

A Thesis

Submitted in fulfilment of the requirement  
for the award of the Degree of

**DOCTOR OF PHILOSOPHY  
IN  
MATHEMATICS**

SUBMITTED BY

**Tina Verma**

**(Registration no.: 901311003)**



**SCHOOL OF MATHEMATICS  
THAPAR UNIVERSITY PATIALA-147004  
PUNJAB (INDIA)  
January, 2016**

## CERTIFICATE

This is to certify that the thesis entitled, "**Methods for Solving Non-cooperative Games with Fuzzy Payoffs**", submitted by Ms. Tina Verma in the fulfillment of the requirement for the award of the degree of **Doctor of Philosophy** in the School of Mathematics, Thapar University, Patiala, is a record of candidate's own work carried out by her under our supervision and guidance.

The matter embodied in this thesis has not been submitted in part or full for the award of any degree in any other University or Institute.

Attestation by supervisors

  
**Dr. Amit Kumar**  
Associate Professor  
School of Mathematics  
Thapar University  
Patiala-147004  
India

  
**Dr. Srimantoorao S. Appadoo**  
Associate Professor and Head  
Department of Supply Chain Management  
Asper School of Business  
University of Manitoba  
Winnipeg, MB Canada, R3T-5V4

## DECLARATION

It is certified that the research work presented in this thesis entitled “**Methods for Solving Non-cooperative Games with Fuzzy Payoffs**”, submitted for the award of the degree of **Doctor of Philosophy** in the School of Mathematics, Thapar University, Patiala, is an authentic record of my own research work carried out under the supervision of Dr. Amit Kumar (School of Mathematics, Thapar University, Patiala) and Dr. S.S. Appadoo (Department of Supply Chain and Management, University of Manitoba, Canada) and references cited herein have been duly acknowledged.

The matter presented in this thesis has not been previously submitted in part or full to any other University or Institute for the award of degree in India or Abroad.

Tina  
Verma

Tina Verma

Attestation by supervisors



**Dr. Amit Kumar**

Associate Professor  
School of Mathematics  
Thapar University  
Patiala-147004  
India



**Dr. Srimantoorao S. Appadoo**

Associate Professor and Head  
Department of Supply Chain Management  
Asper School of Business  
University of Manitoba  
Winnipeg, MB Canada, R3T-5V4

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---

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*Tina Verma*  
Tina Verma



## A. List of Publications

1. **Verma, Tina**, and Amit Kumar. “A note on “A methodology for matrix games with payoffs of triangular intuitionistic fuzzy number””. *Journal of Intelligent & Fuzzy Systems* 27(4) (2014): 1689-1691. (**JCR 2015 Impact Factor: 1.812**).
2. **Verma, Tina**, and Amit Kumar. “A note on “A lexicographic method for matrix games with payoffs of triangular intuitionistic fuzzy numbers””. *International Journal of Computational Intelligence Systems* 8(4) (2015): 690-700. (**JCR 2015 Impact Factor: 0.574**).
3. **Verma, Tina**, Amit Kumar, and S. S. Appadoo. “Modified difference-index based ranking bilinear programming approach to solving bimatrix games with payoffs of trapezoidal intuitionistic fuzzy numbers”. *Journal of Intelligent & Fuzzy Systems* 29(4) (2015): 1607-1618. (**JCR 2015 Impact Factor: 1.812**).
4. **Verma, Tina**. “A note on “Matrix game with interval data””. *Mathematical Sciences International Research Journal* 4(1) (2015): 18-19.
5. **Verma, Tina**. “A note on “Solving bimatrix games with fuzzy payoffs by introducing Nature as a third player””. *Computer Engineering and Intelligent Systems* 6(2) (2015): 58-61.
6. **Verma, Tina**, Amit Kumar, and Janusz Kacprzyk. “A novel approach to the solution of matrix games with payoffs expressed by trapezoidal intuitionistic fuzzy numbers”. *Journal of Automation, Mobile Robotics & Intelligent Systems* 9(3) (2015): 25-46.
7. **Verma, Tina**, Amit Kumar, and S. S. Appadoo. “Modified fast approach to compute fuzzy values of matrix games with payoffs of triangular fuzzy numbers”. *International Academic Research Journal of Business and Management* 3(9) (2015): 43-52.
8. **Verma, Tina**, and Amit Kumar “Gaurika method for solving matrix games with interval payoffs”. *Omega* (Communicated).
9. **Verma, Tina**, and Amit Kumar “Mehtar method for solving matrix games with fuzzy

payoffs”. *European Journal of Operational Research* (Communicated).

10. **Verma, Tina**, and Amit Kumar “Vaishnavi method for solving constrained matrix games with fuzzy payoffs”. *Fuzzy Optimization and Decision Making* (Communicated).

11. **Verma, Tina**, and Amit Kumar “Ambika methods for solving matrix games with intuitionistic fuzzy payoffs”. *IEEE Transactions on Fuzzy Systems* (Communicated).

## **B. Paper presented in International Conferences**

1. **Verma, Tina**, Amit Kumar, and S. S. Appadoo. “Modified fast approach to compute fuzzy values of matrix games with payoffs of triangular fuzzy numbers” in Third International Conference on Business and Management held on December, 26-27, 2014 organized by Academic Research Conference 2014 in Mumbai, India.

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3. **Verma, Tina**. "A note on “Matrix game with interval data”” in International Conference on Advances in Mathematical Sciences–2015 (ICAMS-2015) held on March 19-21, 2015 organized by Khalsa College Patiala in collaboration with International Multidisciplinary Research Foundation.

## **Abstract**

In this thesis, flaws of some of the existing methods, published in last ten years, for solving matrix games with interval/fuzzy/intuitionistic fuzzy payoffs, constrained matrix games with fuzzy payoffs and bimatrix games with intuitionistic fuzzy payoffs are pointed out. To resolve flaws of the existing methods, new methods are also proposed.

The chapter wise summary of the thesis is as follows:

### **Chapter 1**

In this chapter, flaws of the existing methods [37,68,69,79,84,100] for solving matrix games (or two person zero sum games) with interval payoffs (matrix games in which payoffs are represented by intervals) are pointed out. To resolve these flaws, a new method (named as Gaurika method) is also proposed to obtain the optimal strategies as well as minimum expected gain of Player I and maximum expected loss of Player II for matrix games with interval payoffs. To illustrate the proposed Gaurika method, some existing numerical problems of matrix games with interval payoffs are solved by the proposed Gaurika method.

### **Chapter 2**

In this chapter, flaws of the existing methods [36,65,70,71,83] for solving matrix game with fuzzy payoffs (matrix games in which payoffs are represented as fuzzy numbers) are pointed out. Also, to resolve these flaws, a new method (named as Mehar method) is proposed to obtain the optimal strategies as well as minimum expected gain of Player I and maximum expected loss of Player II for matrix games with fuzzy payoffs. To illustrate the proposed Mehar method, an existing numerical problem of matrix games with fuzzy payoffs are solved by the proposed Mehar method.

### **Chapter 3**

In this chapter, flaws of the existing methods [64,74-76] for solving constrained matrix games with fuzzy payoffs (constrained matrix games in which payoffs are represented by fuzzy

numbers) are pointed out. Also, to resolve these flaws, a new method (named as Vaishnavi method) is proposed to obtain the optimal strategies as well as minimum expected gain of Player I and maximum expected loss of Player II for constrained matrix games with fuzzy payoffs. To illustrate the proposed Vaishnavi method, some existing numerical problems of constrained matrix games with fuzzy payoffs are solved by the proposed Vaishnavi method.

## **Chapter 4**

In this chapter, flaws of the existing methods [78,95-97] for solving matrix games with intuitionistic fuzzy payoffs (matrix games in which payoffs are represented by intuitionistic fuzzy numbers) are pointed out. Also, to resolve these flaws, new methods (named as Ambika method-I, Ambika method-II, Ambika method-III and Ambika method-IV) are proposed to obtain the optimal strategies as well as minimum expected gain of Player I and maximum expected loss of Player II for matrix games with intuitionistic fuzzy payoffs. To illustrate proposed Ambika methods, some existing numerical problems of matrix games with intuitionistic fuzzy payoffs are solved by proposed Ambika methods.

## **Chapter 5**

Li and Yang [80] pointed out that there is no method in the literature for solving such bimatrix games or two person non-zero sum games (matrix games in which gain of one player is not equal to the loss of other player) in which payoffs are represented by intuitionistic fuzzy numbers and proposed a method for the same. In this chapter, it is pointed out that Li and Yang have considered some mathematical incorrect assumptions in their proposed method. For resolving the shortcomings of Li and Yang's method, a new method (named as Mehar method) is proposed. Also, the exact optimal solution of the numerical problem, solved by Li and Yang by their proposed method, is obtained by the proposed Mehar method.

## **Chapter 6**

In this chapter, future work has been suggested.

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# Chapter 1

## Gaurika method for solving matrix games with interval payoffs\*

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In this chapter, flaws of the existing methods [37,68,69,79,84,100] for solving matrix games (or two person zero sum games) with interval payoffs (matrix games in which payoffs are represented by intervals) are pointed out. To resolve these flaws, a new method (named as Gaurika method) is also proposed to obtain the optimal strategies as well as minimum expected gain of Player I and maximum expected loss of Player II of matrix games with interval payoffs. To illustrate the proposed Gaurika method, some existing numerical problems of matrix games with interval payoffs are solved by the proposed Gaurika method.

### 1.1 Matrix games with interval payoffs

Game theory [106] is a mathematical tool to describe strategic interactions among multiple decision makers who behave rationally. It has many applications in broad areas such as strategic welfare, economic or social problems, animal behavior, political voting systems etc. Although, the concept of game theory was started with Von Neumann's study on zero-sum games [124], in which he proved the famous minimax theorem for zero-sum games. It was also basis for the book "Theory of Games and Economic Behavior" by Von Neumann and Morgenstern [125]. But, Game theory was significantly advanced at Princeton University through the work of Nash [98].

Games are roughly be classified into two major categories: Cooperative games and Non-cooperative games [106]. Cooperation of players may be assumed in games, since it often

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\* The contents of this chapter are communicated in *Omega*.

exists in reality, though in many, if not most cases, the existence of non-cooperation is more attractive because it is often more realistic, especially in the presence of competition between players.

In non-cooperative games, important from a conceptual and applications point of view, class of games are matrix games.

In matrix games, there are only two players Player I and Player II, and it can be denoted by an  $m \times n$  matrix  $A$  of real numbers. Thus, such a game is called matrix game. A (mixed) strategy of Player I is a probability distribution  $x$  over the rows of  $A$ , i.e., an element of the set

$$X = \left\{ x = (x_1, x_2, \dots, x_m) \mid \sum_{i=1}^m x_i = 1, x_i \geq 0, i = 1, 2, \dots, m \right\}.$$

Similarly, a strategy of Player II is a probability distribution  $y$  over the columns of  $A$  i.e., an element of the set

$$Y = \left\{ y = (y_1, y_2, \dots, y_n) \mid \sum_{j=1}^n y_j = 1, y_j \geq 0, j = 1, 2, \dots, n \right\}.$$

A strategy  $x$  of Player I is called pure if  $x_i = 1$  for some  $i = 1, 2, \dots, m$ . Similarly, a strategy  $y$  of Player II is called pure if  $y_j = 1$  for some  $j = 1, 2, \dots, n$ . If Player I plays over row  $i$  (i.e., pure strategy  $x = (0, 0, \dots, x_i = 1, 0, \dots, 0)$ ) and Player II plays over column  $j$  (i.e., pure strategy  $y = (0, 0, \dots, y_j = 1, 0, \dots, 0)$ ), then Player I receives  $a_{ij}$  and Player II pays  $a_{ij}$ , where  $a_{ij}$  is the entry in row  $i$  and column  $j$  of matrix  $A$ . If Player I plays strategy  $x$  and Player II

plays strategy  $y$ , then Player I receives the expected payoff  $E(x, y) = \sum_{j=1}^n \left( \sum_{i=1}^m a_{ij} x_i \right) y_j$ .

A position  $(i, j)$  is called saddle point if  $a_{ij} \geq a_{kj} \forall k = 1, 2, \dots, m$  and  $a_{ij} \leq a_{il} \forall l = 1, 2, \dots, n$  i.e., if  $a_{ij}$  is maximal in its column  $j$  and minimal in its row  $i$ . Evidently, if  $(i, j)$  is a saddle point, then  $a_{ij}$  must be the value of the game.

In the classical (or crisp) game theory, usually the payoffs of players are represented by

crisp values i.e. real numbers. But in real life, there is need to represent the players' payoffs by their subjective judgments (or opinions) about competitive situations (or outcomes) instead of real numbers. For example, the decision problem in which two companies try to improve some products sales in some target market may be regarded as a game problem. In this scenario, the payoffs of players (i.e., companies) are represented by the company managers subjective judgments (or opinions) of the product shares in the target market at various situations. Such subjective judgments may be expressed with terms of linguistic variables such as “very large”, “larger”, “medium” and “small” as well as “smaller”. Obviously, these judgments usually involve some fuzziness or uncertainty due to the bounded rationality of players and behavior complexity. One way to describe impreciseness in the payoffs is to represent the payoffs by intervals.

In the literature [5,6,37,68,69,79,84,99,100,112], such matrix games in which payoffs are represented by intervals are named as matrix games with interval payoffs.

**Theorem 1.1 [106].** If  $a_i, i = 1, 2, \dots, n$  are  $n$  real numbers then

$$\text{minimum} \left\{ \sum_{i=1}^n a_i x_i \mid \sum_{i=1}^n x_i = 1, x_i \geq 0, i = 1, 2, \dots, n \right\} = \text{minimum} \{a_i, i = 1, 2, \dots, n\}.$$

**Proof.** Let,  $\text{minimum} \{a_i, i = 1, 2, \dots, n\} = a_k$ . Then,

$$\begin{aligned} a_k &\leq a_i \quad \forall i = 1, 2, \dots, n \\ \Rightarrow a_k x_i &\leq a_i x_i, \quad x_i \geq 0 \quad \forall i = 1, 2, \dots, n \\ \Rightarrow \sum_{i=1}^n a_k x_i &\leq \sum_{i=1}^n a_i x_i \\ \Rightarrow a_k \sum_{i=1}^n x_i &\leq \sum_{i=1}^n a_i x_i \\ \Rightarrow a_k &\leq \sum_{i=1}^n a_i x_i \\ \Rightarrow a_k &\leq \text{minimum} \left\{ \sum_{i=1}^n a_i x_i \mid \sum_{i=1}^n x_i = 1, x_i \geq 0, i = 1, 2, \dots, n \right\} \end{aligned}$$

$$\Rightarrow \text{minimum} \{a_i, i=1, 2, \dots, n\} \leq \text{minimum} \left\{ \sum_{i=1}^n a_i x_i \mid \sum_{i=1}^n x_i = 1, x_i \geq 0, i=1, 2, \dots, n \right\} \quad (1.1)$$

Furthermore,  $a_k = a_1 \times 0 + a_2 \times 0 + \dots + a_k \times 1 + \dots + a_n \times 0$

$$\begin{aligned} a_k &\geq \text{minimum} \left\{ \sum_{i=1}^n a_i x_i \mid \sum_{i=1}^n x_i = 1, x_i \geq 0, i=1, 2, \dots, n \right\} \\ \Rightarrow \text{minimum} \{a_i, i=1, 2, \dots, n\} &\geq \text{minimum} \left\{ \sum_{i=1}^n a_i x_i \mid \sum_{i=1}^n x_i = 1, x_i \geq 0, i=1, 2, \dots, n \right\} \end{aligned} \quad (1.2)$$

From, (1.1) and (1.2)

$$\text{minimum} \left\{ \sum_{i=1}^n a_i x_i \mid \sum_{i=1}^n x_i = 1, x_i \geq 0, i=1, 2, \dots, n \right\} = \text{minimum} \{a_i, i=1, 2, \dots, n\}.$$

**Theorem 1.2 [79].** If  $[a_i^L, a_i^R], i=1, 2, \dots, n$  are  $n$  intervals then,

$$\text{minimum} \left\{ \sum_{i=1}^n [a_i^L, a_i^R] x_i \mid \sum_{i=1}^n x_i = 1, x_i \geq 0, i=1, 2, \dots, n \right\} = \text{minimum} \{[a_i^L, a_i^R], i=1, 2, \dots, n\}.$$

**Proof.** Let,  $\text{minimum} \{[a_i^L, a_i^R], i=1, 2, \dots, n\} = [a_p^L, a_p^R]$ . Then,

$$\begin{aligned} [a_p^L, a_p^R] &\leq [a_i^L, a_i^R] \quad \forall i=1, 2, \dots, n \\ \Rightarrow [a_p^L, a_p^R] x_i &\leq [a_i^L, a_i^R] x_i \quad \forall i=1, 2, \dots, n \\ \Rightarrow \sum_{i=1}^n [a_p^L, a_p^R] x_i &\leq \sum_{i=1}^n [a_i^L, a_i^R] x_i \\ \Rightarrow [a_p^L, a_p^R] \sum_{i=1}^n x_i &\leq \sum_{i=1}^n [a_i^L, a_i^R] x_i \\ \Rightarrow [a_p^L, a_p^R] &\leq \sum_{i=1}^n [a_i^L, a_i^R] x_i, \\ \Rightarrow \text{minimum} \{[a_i^L, a_i^R], i=1, 2, \dots, n\} &\leq \text{minimum} \left\{ \sum_{i=1}^n [a_i^L, a_i^R] x_i \mid \sum_{i=1}^n x_i = 1, x_i \geq 0, i=1, 2, \dots, n \right\} \end{aligned} \quad (1.3)$$

Furthermore,

$$\begin{aligned} [a_p^L, a_p^R] &= [a_1^L, a_1^R] \times 0 + [a_2^L, a_2^R] \times 0 + \dots + [a_p^L, a_p^R] \times 1 + \dots + [a_n^L, a_n^R] \times 0 \\ \Rightarrow [a_p^L, a_p^R] &\geq \text{minimum} \left\{ \sum_{i=1}^n [a_i^L, a_i^R] x_i \mid \sum_{i=1}^n x_i = 1, x_i \geq 0, i=1, 2, \dots, n \right\} \end{aligned}$$

$$\Rightarrow \text{minimum} \left\{ \left[ a_i^L, a_i^R \right], i = 1, 2, \dots, n \right\} \geq \text{minimum} \left\{ \sum_{i=1}^n \left[ a_i^L, a_i^R \right] x_i \mid \sum_{i=1}^n x_i = 1, x_i \geq 0, i = 1, 2, \dots, n \right\} \quad (1.4)$$

From, (1.3) and (1.4) ,

$$\text{minimum} \left\{ \sum_{i=1}^n \left[ a_i^L, a_i^R \right] x_i \mid \sum_{i=1}^n x_i = 1, x_i \geq 0, i = 1, 2, \dots, n \right\} = \text{minimum} \left\{ \left[ a_i^L, a_i^R \right], i = 1, 2, \dots, n \right\}.$$

Similarly, it can be proved that

$$\text{maximum} \left\{ \sum_{i=1}^n \left[ a_i^L, a_i^R \right] x_i \mid \sum_{i=1}^n x_i = 1, x_i \geq 0, i = 1, 2, \dots, n \right\} = \text{maximum} \left\{ \left[ a_i^L, a_i^R \right], i = 1, 2, \dots, n \right\}.$$

## 1.2. Existing mathematical formulation of matrix games with interval payoffs

Let, Player I and Player II be two players and  $A = (a_{ij})_{m \times n}$  be the payoff matrix for Player I. Let,  $S_1 = \{\delta_1, \delta_2, \dots, \delta_m\}$  and  $S_2 = \{\eta_1, \eta_2, \dots, \eta_n\}$  be set of course of actions available to Player I and Player II respectively and  $x_1, x_2, \dots, x_m$  and  $y_1, y_2, \dots, y_n$  be the probabilities for selecting the course of action  $\delta_1, \delta_2, \dots, \delta_m$  and  $\eta_1, \eta_2, \dots, \eta_n$  respectively. Let,

$$X = \left\{ x = (x_1, x_2, \dots, x_m) \mid \sum_{i=1}^m x_i = 1, x_i \geq 0, i = 1, 2, \dots, m \right\} \text{ and}$$

$$Y = \left\{ y = (y_1, y_2, \dots, y_n) \mid \sum_{j=1}^n y_j = 1, y_j \geq 0, j = 1, 2, \dots, n \right\} \text{ be the set of strategies for Player I and}$$

Player II respectively. Then, the optimal value of Problem 1.2.1 and Problem 1.2.2 represents the minimum expected gain of Player I and maximum expected loss of Player II respectively [106]. Furthermore, the optimal solution  $\{x_1, x_2, \dots, x_m\}$  and  $\{y_1, y_2, \dots, y_n\}$  of Problem 1.2.1 and Problem 1.2.2 represents the optimal strategies of Player I and Player II respectively.

### Problem 1.2.1

Maximize  $\{\nu\}$

Subject to

### Problem 1.2.2

Minimize  $\{\omega\}$

Subject to

$$\sum_{i=1}^m a_{ij} x_i \geq \nu, j = 1, 2, \dots, n;$$

$$\sum_{i=1}^m x_i = 1; x_i \geq 0, i = 1, 2, \dots, m.$$

$$\sum_{j=1}^n a_{ij} y_j \leq \omega, i = 1, 2, \dots, m;$$

$$\sum_{j=1}^n y_j = 1; y_j \geq 0, j = 1, 2, \dots, n.$$

On the same direction, in the literature [37,68,69,79,84,100] , it is assumed that if payoffs are represented by intervals  $[a_{ij}^L, a_{ij}^R]$  instead of real numbers  $a_{ij}$  then to find the minimum expected gain of Player I and maximum expected loss of Player II is equivalent to find the optimal value of Problem 1.2.3 and Problem 1.2.4 respectively. Furthermore, the optimal solution  $\{x_1, x_2, \dots, x_m\}$  and  $\{y_1, y_2, \dots, y_n\}$  of Problem 1.2.3 and Problem 1.2.4 represents the optimal strategies of Player I and Player II respectively.

#### Problem 1.2.3

Maximize  $\{\nu^L, \nu^R\}$

Subject to

$$\sum_{i=1}^m [a_{ij}^L, a_{ij}^R] x_i \geq [\nu^L, \nu^R], j = 1, 2, \dots, n;$$

$$\sum_{i=1}^m x_i = 1;$$

$$x_i \geq 0, i = 1, 2, \dots, m.$$

#### Problem 1.2.4

Minimize  $\{\omega^L, \omega^R\}$

Subject to

$$\sum_{j=1}^n [a_{ij}^L, a_{ij}^R] y_j \leq [\omega^L, \omega^R], i = 1, 2, \dots, m;$$

$$\sum_{j=1}^n y_j = 1;$$

$$y_j \geq 0, j = 1, 2, \dots, n.$$

**Remark 1.1.** It can be easily verified that the pair of Problem 1.2.1 and Problem 1.2.2 is a primal-dual pair of crisp linear programming problems. So, the minimum expected gain of Player I will be equal to the maximum expected loss of Player II.

### 1.3. Literature review of matrix game with interval payoffs.

In this section, a brief review of the methods, proposed in the literature in last ten years for solving matrix games with interval payoffs, is presented.

Nayak & Pal [99] proposed a graphical method for solving  $2 \times n$  or  $m \times 2$  matrix games with interval payoffs. This method cannot be used for solving  $m \times n$  matrix games with interval

payoffs.

Liu & Kao [84] transformed the Problem 1.2.3 into Problem 1.3.1 and Problem 1.3.2 for evaluating the right end point  $(\nu^R)$  and left end point  $(\nu^L)$  respectively of the minimum expected gain  $[\nu^L, \nu^R]$  of Player I as well as Problem 1.2.4 into Problem 1.3.3 and Problem 1.3.4 respectively for evaluating the left end point  $(\omega^L)$  and right end point  $(\omega^R)$  of the maximum expected loss  $[\omega^L, \omega^R]$  of Player II. Liu & Kao [84] used the optimal values of  $\nu^L, \nu^R$  and  $\omega^L, \omega^R$  to obtain the interval  $[\nu^L, \nu^R]$  and  $[\omega^L, \omega^R]$ , representing the minimum expected gain of Player I and maximum expected loss of Player II respectively.

**Problem 1.3.1.**

$$\nu^R = \text{Maximize } \{v\}$$

Subject to

$$\begin{aligned} \sum_{i=1}^m p_{ij} &\geq v, \quad j = 1, 2, \dots, n; \\ a_{ij}^L x_i &\leq p_{ij} \leq a_{ij}^R x_i, \quad i = 1, 2, \dots, m, \\ &\quad j = 1, 2, \dots, n; \\ \sum_{i=1}^m x_i &= 1; x_i \geq 0, \quad i = 1, 2, \dots, m. \end{aligned}$$

**Problem 1.3.3,**

$$\omega^L = \text{Minimize } \{u\}$$

Subject to

$$\begin{aligned} \sum_{j=1}^n s_{ij} &\leq u, \quad i = 1, 2, \dots, m; \\ a_{ij}^L y_j &\leq s_{ij} \leq a_{ij}^R y_j, \quad i = 1, 2, \dots, m, \\ &\quad j = 1, 2, \dots, n; \\ \sum_{j=1}^n y_j &= 1; y_j \geq 0, \quad j = 1, 2, \dots, n. \end{aligned}$$

**Problem 1.3.2.**

$$\nu^L = \text{Minimize } \{u\}$$

Subject to

$$\begin{aligned} \sum_{j=1}^n q_{ij} &\leq u, \quad i = 1, 2, \dots, m; \\ a_{ij}^L y_j &\leq q_{ij} \leq a_{ij}^R y_j, \quad i = 1, 2, \dots, m, \\ &\quad j = 1, 2, \dots, n; \\ \sum_{j=1}^n y_j &= 1; y_j \geq 0, \quad j = 1, 2, \dots, n. \end{aligned}$$

**Problem 1.3.4,**

$$\omega^R = \text{Maximize } \{v\}$$

Subject to

$$\begin{aligned} \sum_{i=1}^m t_{ij} &\geq v, \quad j = 1, 2, \dots, n; \\ a_{ij}^L x_i &\leq t_{ij} \leq a_{ij}^R x_i, \quad i = 1, 2, \dots, m, \\ &\quad j = 1, 2, \dots, n; \\ \sum_{i=1}^m x_i &= 1; x_i \geq 0, \quad i = 1, 2, \dots, m. \end{aligned}$$

Liu & Kao [84] themselves pointed out that their method cannot be used to obtain the optimal strategy of matrix games with interval payoffs and to develop a new method for the same may be considered as a future research work.

Collins & Hu [37] proposed a method for finding the optimal solution of matrix games with interval payoffs by assuming the minimum expected gain of Player I and the maximum expected loss of Player II as real numbers. Collins & Hu [37] themselves pointed out that as the payoffs are intervals so the minimum expected gain of Player I and maximum expected loss of Player II will be a linear combination of intervals. Therefore, it is unrealistic to assume the minimum expected gain of Player I and maximum expected loss of Player II as real numbers. Collins & Hu [37] also pointed out that to propose a method to find the optimal solution of matrix games with interval payoffs without considering this unrealistic assumption may be considered as a future research work.

Using Moore's concept of set inclusion [92], Nayak & Pal [100] transformed the Problem 1.2.3 and Problem 1.2.4 into Problem 1.3.5 and Problem 1.3.6 for a fixed parameter  $\alpha = 1$  for evaluating left end point  $\nu^L$  of minimum expected gain  $[\nu^L, \nu^R]$  of Player I and left end point  $\omega^L$  of maximum expected loss  $[\omega^L, \omega^R]$  of Player II respectively as well as the corresponding optimal strategies.

**Problem 1.3.5**

Maximize  $\{\nu^L\}$

Subject to

$$\sum_{i=1}^m a_{ij}^L x_i \geq \nu^L, \quad j = 1, 2, \dots, n;$$

$$\sum_{i=1}^m x_i = 1; x_i \geq 0, \quad i = 1, 2, \dots, m.$$

**Problem 1.3.6**

Minimize  $\{\omega^L\}$

Subject to

$$\sum_{j=1}^n a_{ij}^R y_j \leq \omega^L, \quad i = 1, 2, \dots, m;$$

$$\sum_{j=1}^n y_j = 1; y_j \geq 0, \quad j = 1, 2, \dots, n.$$

Li [69] pointed out that Nayak & Pal [100], have wrongly employed Moore's concept

of set inclusions [92] to transform the Problem 1.2.4 into Problem 1.3.6. Li [69] claimed that using Moore's concept of set inclusions [92], Problem 1.2.4 will be transformed into Problem 1.3.7 instead of Problem 1.3.6.

**Problem 1.3.7**

Minimize  $\{\omega^R\}$

Subject to

$$\sum_{j=1}^n a_{ij}^R y_j \leq \omega^R; \quad i = 1, 2, \dots, m; \quad \sum_{j=1}^n y_j = 1; \quad y_j \geq 0, \quad j = 1, 2, \dots, n.$$

Furthermore, Li [69] pointed out that minimum expected gain of Player I and the corresponding the optimal strategy, obtained by solving Problem 1.3.5, depend only on the left limits/bounds of the interval  $[a_{ij}^L, a_{ij}^R]$  whereas the maximum expected loss of Player II and the corresponding the optimal strategy, obtained by solving Problem 1.3.7, depend only on the right limits/bounds of the interval  $[a_{ij}^L, a_{ij}^R]$ . Li [69] also pointed out that the right end point  $v^R$  of minimum expected gain of Player I cannot be obtained by solving Problem 1.3.5 and left end point  $\omega^L$  of maximum expected loss of Player II cannot be obtained by solving Problem 1.3.7. Therefore, the minimum expected gain of Player I and maximum expected loss of Player II will not be bounded intervals, which shows that the method, proposed by Nayak & Pal [100], cannot be employed to solve a generic  $m \times n$  matrix games with interval payoffs.

To resolve these flaws of the method, proposed by Nayak & Pal [100], Li [69] transformed the Problem 1.2.3 and Problem 1.2.4 into Problem 1.3.8 and Problem 1.3.9 respectively.

**Problem 1.3.8**

$$\text{Maximize} \left\{ v^L, \frac{v^L + v^R}{2} \right\}$$

Subject to

**Problem 1.3.9**

$$\text{Minimize} \left\{ \omega^R, \frac{\omega^L + \omega^R}{2} \right\}$$

Subject to

$$\sum_{i=1}^m a_{ij}^L x_i \geq v^L, \quad j = 1, 2, \dots, n;$$

$$(1-\alpha) \sum_{i=1}^m a_{ij}^R x_i + \alpha \sum_{i=1}^m a_{ij}^L x_i \geq (1-\alpha)v^R + \alpha v^L, \quad j = 1, 2, \dots, n;$$

$$v^R - v^L \geq 0;$$

$$\sum_{i=1}^m x_i = 1;$$

$$x_i \geq 0, \quad i = 1, 2, \dots, m.$$

$$\sum_{j=1}^n a_{ij}^R y_j \leq \omega^R, \quad i = 1, 2, \dots, m;$$

$$\alpha \sum_{j=1}^n a_{ij}^R y_j + (1-\alpha) \sum_{j=1}^n a_{ij}^L y_j \leq \alpha \omega^R + (1-\alpha)\omega^L, \quad i = 1, 2, \dots, m;$$

$$\omega^R - \omega^L \geq 0;$$

$$\sum_{j=1}^n y_j = 1;$$

$$y_j \geq 0, \quad j = 1, 2, \dots, n.$$

Furthermore, Li [69] transformed the Problem 1.3.8 into Problem 1.3.10 and Problem 1.3.11 as well as transformed the Problem 1.3.9 into Problem 1.3.12 and Problem 1.3.13. Finally, Li [69] used the optimal solution of Problem 1.3.11 to obtain the minimum expected gain and optimal strategy for Player I and used optimal solution of Problem 1.3.13 to obtain the maximum expected loss and optimal strategy of Player II.

#### Problem 1.3.10

$$\text{Maximize } \{v^L\}$$

Subject to

$$\sum_{i=1}^m a_{ij}^L x_i \geq v^L, \quad j = 1, 2, \dots, n;$$

$$\sum_{i=1}^m (1-\alpha) a_{ij}^R x_i + \sum_{i=1}^m \alpha a_{ij}^L x_i \geq (1-\alpha)v^R + \alpha v^L, \quad j = 1, 2, \dots, n;$$

$$v^R - v^L \geq 0;$$

$$\sum_{i=1}^m x_i = 1;$$

$$x_i \geq 0, \quad i = 1, 2, \dots, m.$$

#### Problem 1.3.12

$$\text{Minimize } \{\omega^R\}$$

#### Problem 1.3.11

$$\text{Maximize } \left\{ \frac{v^L + v^R}{2} \right\}$$

Subject to

$$\sum_{i=1}^m a_{ij}^L x_i \geq v^L, \quad j = 1, 2, \dots, n;$$

$$\sum_{i=1}^m (1-\alpha) a_{ij}^R x_i + \sum_{i=1}^m \alpha a_{ij}^L x_i \geq (1-\alpha)v^R + \alpha v^L, \quad j = 1, 2, \dots, n;$$

$$v^R - v^L \geq 0;$$

$$v^L \geq \text{Optimal value of Problem 1.3.10}$$

$$\sum_{i=1}^m x_i = 1;$$

$$x_i \geq 0, \quad i = 1, 2, \dots, m.$$

#### Problem 1.3.13

$$\text{Minimize } \left\{ \frac{\omega^L + \omega^R}{2} \right\}$$

Subject to

$$\begin{aligned}
& \sum_{j=1}^n a_{ij}^R y_j \leq \omega^R, \quad i = 1, 2, \dots, m; \\
& \alpha \sum_{j=1}^n a_{ij}^R y_j + (1-\alpha) \sum_{j=1}^n a_{ij}^L y_j \leq \\
& \quad \alpha \omega^R + (1-\alpha) \omega^L, \quad i = 1, 2, \dots, m; \\
& \omega^R - \omega^L \geq 0; \\
& \sum_{j=1}^n y_j = 1; \\
& y_j \geq 0, \quad j = 1, 2, \dots, n.
\end{aligned}$$

Subject to

$$\begin{aligned}
& \sum_{j=1}^n a_{ij}^R y_j \leq \omega^R, \quad i = 1, 2, \dots, m; \\
& \alpha \sum_{j=1}^n a_{ij}^R y_j + (1-\alpha) \sum_{j=1}^n a_{ij}^L y_j \leq \\
& \quad \alpha \omega^R + (1-\alpha) \omega^L, \quad i = 1, 2, \dots, m; \\
& \omega^R - \omega^L \geq 0; \\
& \omega^R \leq \text{Optimal value of Problem 1.3.12}; \\
& \sum_{j=1}^n y_j = 1; \\
& y_j \geq 0, \quad j = 1, 2, \dots, n.
\end{aligned}$$

Li et al. [79] transformed the Problem 1.3.8 into Problem 1.3.14, instead of transforming Problem 1.3.10 and Problem 1.3.11 as well as transformed Problem 1.3.9 into Problem 1.3.15, instead of transforming Problem 1.3.12 and Problem 1.3.13. Finally, Li et al. [79] used optimal solution of Problem 1.3.14 to obtain the minimum expected gain and optimal strategy for Player I and used optimal solution of Problem 1.3.15 to obtain the maximum expected loss and optimal strategy for Player II.

#### Problem 1.3.14

$$\text{Maximize } \left\{ \frac{3v^L + v^R}{4} \right\}$$

Subject to

$$\begin{aligned}
& \sum_{i=1}^m a_{ij}^L x_i \geq v^L, \quad j = 1, 2, \dots, n; \\
& (1-\alpha) \sum_{i=1}^m a_{ij}^R x_i + \alpha \sum_{i=1}^m a_{ij}^L x_i \geq \\
& \quad (1-\alpha) v^R + \alpha v^L, \quad j = 1, 2, \dots, n; \\
& v^R - v^L \geq 0; \\
& \sum_{i=1}^m x_i = 1; \\
& x_i \geq 0, \quad i = 1, 2, \dots, m.
\end{aligned}$$

#### Problem 1.3.15

$$\text{Minimize } \left\{ \frac{\omega^L + 3\omega^R}{4} \right\}$$

Subject to

$$\begin{aligned}
& \sum_{j=1}^n a_{ij}^R y_j \leq \omega^R, \quad i = 1, 2, \dots, m; \\
& \alpha \sum_{j=1}^n a_{ij}^R y_j + (1-\alpha) \sum_{j=1}^n a_{ij}^L y_j \leq \\
& \quad \alpha \omega^R + (1-\alpha) \omega^L, \quad i = 1, 2, \dots, m; \\
& \omega^R - \omega^L \geq 0; \\
& \sum_{j=1}^n y_j = 1; \\
& y_j \geq 0, \quad j = 1, 2, \dots, n.
\end{aligned}$$

Li [68] pointed out that on solving matrix games with interval payoffs, the obtained

minimum expected gain of Player I, represented by interval, should be equal to the maximum expected loss of Player II. While, for the minimum expected gain for Player I and maximum expected loss of Player II, obtained on solving matrix games with interval payoffs by using existing methods [69,79], this condition is not satisfying. To resolve this fundamental error, Li [68] split Problem 1.2.3 into Problem 1.3.16 and Problem 1.3.17 as well as split Problem 1.2.4 into Problem 1.3.18 and Problem 1.3.19. Finally, Li [68] used the optimal value of Problem 1.3.16 and Problem 1.3.17 to obtain the minimum expected gain of Player I and used the optimal value of Problem 1.3.18 and Problem 1.3.19 to obtain maximum expected loss of Player II.

**Problem 1.3.16**

Maximize  $\{\nu^L\}$

Subject to

$$\sum_{i=1}^m a_{ij}^L x_i \geq \nu^L, \quad j = 1, 2, \dots, n;$$

$$\sum_{i=1}^m x_i = 1; x_i \geq 0, \quad i = 1, 2, \dots, m.$$

**Problem 1.3.18**

Minimize  $\{\omega^L\}$

Subject to

$$\sum_{j=1}^n a_{ij}^L y_j \leq \omega^L, \quad i = 1, 2, \dots, m;$$

$$\sum_{j=1}^n y_j = 1; y_j \geq 0, \quad j = 1, 2, \dots, n.$$

**Problem 1.3.17**

Maximize  $\{\nu^R\}$

Subject to

$$\sum_{i=1}^m a_{ij}^R x_i \geq \nu^R, \quad j = 1, 2, \dots, n;$$

$$\sum_{i=1}^m x_i = 1; x_i \geq 0, \quad i = 1, 2, \dots, m.$$

**Problem 1.3.19**

Minimize  $\{\omega^R\}$

Subject to

$$\sum_{j=1}^n a_{ij}^R y_j \leq \omega^R, \quad i = 1, 2, \dots, m;$$

$$\sum_{j=1}^n y_j = 1; y_j \geq 0, \quad j = 1, 2, \dots, n.$$

Akyar and Akyar [5] proposed a graphical method for solving  $2 \times n$  or  $m \times 2$  matrix games with interval payoffs. This method cannot be used for  $m \times n$  matrix games with interval payoffs.

Akyar et al. [6] proposed an iterative method for solving crisp matrix games to matrix

game with interval payoffs. Akyar et al. [6] themselves pointed out that by using this method only approximate solution is obtained while by using linear programming exact solution is obtained.

#### 1.4 Arithmetic operations over intervals

In this section, some arithmetic operations over intervals are presented [92].

Let,  $A = [a^L, a^R]$  and  $B = [b^L, b^R]$  be two intervals. Then,

(i) The addition of two intervals is defined as  $A + B = [a^L + b^L, a^R + b^R]$ .

(ii) The subtraction of two intervals is defined as  $A - B = [a^L - b^R, a^R - b^L]$ .

(iii) If  $\lambda$  is a scalar then  $\lambda A = \lambda [a^L, a^R] = \begin{cases} [\lambda a^L, \lambda a^R] & \text{if } \lambda \geq 0 \\ [\lambda a^R, \lambda a^L] & \text{if } \lambda < 0. \end{cases}$

(iv) The multiplication of two intervals is defined as

$$AB = [\text{minimum}\{a^L b^L, a^L b^R, a^R b^L, a^R b^R\}, \text{maximum}\{a^L b^L, a^L b^R, a^R b^L, a^R b^R\}].$$

#### 1.5 Flaws of the existing methods

In this section, flaws of the existing methods [37,68,69,79,84,100] are pointed out.

1. Liu & Kao [84] considered a matrix game with following payoff matrix for Player I.

$$A = \begin{bmatrix} 10 & -30 & 20 & [-30, -25] & 10 \\ [24, 30] & 30 & [-40, -30] & 30 & [-38, -34] \\ [-3, 3] & [35, 40] & -10 & -30 & 20 \\ -40 & 0 & -20 & [23, 26] & -10 \end{bmatrix},$$

Liu and Kao [84] claimed that the minimum expected gain of Player I and maximum expected loss of Player II for the considered problem can be obtained by solving Problem 1.5.1 [84, Section 4, pp. 1699] and Problem 1.5.2 [84, Section 4, pp. 1699] respectively.

##### **Problem 1.5.1 [84, Section 4, pp. 1699]**

Maximize  $\{[v^L, v^R]\}$

Subject to

$$\begin{aligned}10x_1 + [24, 30]x_2 + [-3, 3]x_3 - 40x_4 &\geq [\nu^L, \nu^R]; \\ -30x_1 + 30x_2 + [35, 40]x_3 &\geq [\nu^L, \nu^R]; \\ 20x_1 + [-40, -30]x_2 - 10x_3 - 20x_4 &\geq [\nu^L, \nu^R]; \\ [-30, -25]x_1 + 30x_2 - 30x_3 + [23, 26]x_4 &\geq [\nu^L, \nu^R]; \\ 10x_1 + [-38, -34]x_2 + 20x_3 - 10x_4 &\geq [\nu^L, \nu^R]; \\ x_1 + x_2 + x_3 + x_4 &= 1; \\ x_1, x_2, x_3, x_4 &\geq 0.\end{aligned}$$

**Problem 1.5.2 [84, Section 4, pp. 1699]**

$$\text{Minimize } \{[\omega^L, \omega^R]\}$$

Subject to

$$\begin{aligned}10y_1 - 30y_2 + 20y_3 + [-30, -25]y_4 + 10y_5 &\leq [\omega^L, \omega^R]; \\ [24, 30]y_1 + 30y_2 + [-40, -30]y_3 + 30y_4 + [-38, -34]y_5 &\leq [\omega^L, \omega^R]; \\ [-3, 3]y_1 + [35, 40]y_2 - 10y_3 - 30y_4 + 20y_5 &\leq [\omega^L, \omega^R]; \\ -40y_1 - 20y_2 + [23, 26]y_4 - 10y_5 &\leq [\omega^L, \omega^R]; \\ y_1 + y_2 + y_3 + y_4 + y_5 &= 1, \\ y_1, y_2, y_3, y_4, y_5 &\geq 0.\end{aligned}$$

In the optimal solution of the Problem 1.5.1 [84, Section 4, pp. 1699], the variables  $x_1, x_2, x_3$  and  $x_4$  should always be real numbers i.e., lower bound and upper bound of variables  $x_1, x_2, x_3$  and  $x_4$  should be equal. However, it is obvious from the optimal values of variables  $x_1, x_2, x_3$  and  $x_4$ , obtained by Liu and Kao [84] which are shown in Table 1.1, that the lower bound and upper bound of the variables  $x_1, x_2, x_3$  and  $x_4$  are not equal i.e., the optimal values of variables  $x_1, x_2, x_3$  and  $x_4$ , obtained by Liu & Kao [84], are not real numbers.

Similarly, in the optimal solution of the Problem 1.5.2 [84, Section 4, pp. 1699] the variables  $y_1, y_2, y_3, y_4$  and  $y_5$  should always be real numbers i.e., lower bound and upper bound

of the variables  $y_1, y_2, y_3, y_4$  and  $y_5$  should be equal. However, it is obvious from the optimal values of variables  $y_1, y_2, y_3, y_4$  and  $y_5$ , obtained by Liu and Kao [84] which are shown in Table 1.2, that the lower bound and upper bound of the variables  $y_1, y_2, y_3, y_4$  and  $y_5$  are not equal i.e., the optimal values of variables  $y_1, y_2, y_3, y_4$  and  $y_5$  obtained Liu & Kao [84], are not real numbers.

Hence, neither the optimal solution, obtained by Liu and Kao [84] which are shown in Table 1.1, is the optimal solution of Problem 1.5.1 [84, Section 4, pp. 1699] nor the optimal solution, obtained by Liu and Kao [84] which are shown in Table 1.2, is the optimal solution of the Problem 1.5.2 [84, Section 4, pp. 1699].

**Table 1.1** Lower bound and upper bound of variables [84, pp.1699]

| Lower bound ( $x_i^L$ ) | Upper bound ( $x_i^R$ ) |
|-------------------------|-------------------------|
| $x_1^L = 0.4328$        | $x_1^R = 0.4583$        |
| $x_2^L = 0.1517$        | $x_2^R = 0.2186$        |
| $x_3^L = 0.1086$        | $x_3^R = 0.1114$        |
| $x_4^L = 0.3069$        | $x_4^R = 0.2117$        |

**Table 1.2** Lower bound and upper bound of variables [84, pp.1699]

| Lower bound ( $y_j^L$ ) | Upper bound ( $y_j^R$ ) |
|-------------------------|-------------------------|
| $y_1^L = 0.0175$        | $y_1^R = 0$             |
| $y_2^L = 0.2312$        | $y_2^R = 0.1915$        |
| $y_3^L = 0.4933$        | $y_3^R = 0.4362$        |
| $y_4^L = 0.2580$        | $y_4^R = 0.2697$        |

|             |                  |
|-------------|------------------|
| $y_5^L = 0$ | $y_5^R = 0.1026$ |
|-------------|------------------|

2. Li [69] and Li et al. [79] considered that if  $[a^L, a^R]$  and  $[b^L, b^R]$  are two intervals then for

a particular value of  $\alpha \in [0, 1)$ ,

(i)  $[a^L, a^R] \leq [b^L, b^R]$  i.e.,  $\text{minimum}\{[a^L, a^R], [b^L, b^R]\} = [a^L, a^R]$  iff  $a^R \leq b^R$  and

$$(1-\alpha)a^L + \alpha a^R \leq (1-\alpha)b^L + \alpha b^R.$$

(ii)  $[a^L, a^R] \geq [b^L, b^R]$  i.e.,  $\text{maximum}\{[a^L, a^R], [b^L, b^R]\} = [a^L, a^R]$  iff  $a^L \geq b^L$  and

$$\alpha a^L + (1-\alpha)a^R \geq \alpha b^L + (1-\alpha)b^R.$$

Using this relation, Li [69,79] claimed that to find the minimum expected gain of Player

I i.e.,  $[\nu^L, \nu^R] = \text{minimum}\left\{\sum_{i=1}^m [a_{ij}^L, a_{ij}^R] x_i, j=1, 2, \dots, n \mid \sum_{i=1}^m x_i = 1, x_i \geq 0, i=1, 2, \dots, m\right\}$  and to

find the maximum expected loss of Player II i.e.,

$$[\omega^L, \omega^R] = \text{maximum}\left\{\sum_{j=1}^n [a_{ij}^L, a_{ij}^R] y_j, i=1, 2, \dots, m \mid \sum_{j=1}^n y_j = 1; y_j \geq 0, j=1, 2, \dots, n\right\}$$
 is

equivalent to find the optimal solution of Problem 1.2.3 and Problem 1.2.4 respectively.

Furthermore, using the relation,  $[a^L, a^R] \geq [b^L, b^R] \Rightarrow a^L \geq b^L$  and

$$\alpha a^L + (1-\alpha)a^R \geq \alpha b^L + (1-\alpha)b^R \text{ and } [a^L, a^R] \leq [b^L, b^R] \Rightarrow a^R \leq b^R \text{ and}$$

$$(1-\alpha)a^L + \alpha a^R \leq (1-\alpha)b^L + \alpha b^R, \text{ Li [69] and Li et al. [79] transformed the Problem 1.2.3}$$

into Problem 1.5.3.

### Problem 1.5.3

Maximize  $\{[\nu^L, \nu^R]\}$

Subject to

$$\sum_{i=1}^m a_{ij}^L x_i \geq \nu^L, j=1, 2, \dots, n;$$

$$\alpha \sum_{i=1}^m a_{ij}^L x_i + (1-\alpha) \sum_{i=1}^m a_{ij}^R x_i \geq \alpha v^L + (1-\alpha) v^R, j=1,2,\dots,n;$$

$$\sum_{i=1}^m x_i = 1;$$

$$x_i \geq 0, i=1,2,\dots,m.$$

Now, if  $S = \{x_i, i=1,2,\dots,m; v^L, v^R\}$  is the set of all feasible solutions of Problem 1.5.3

then a feasible solution  $\{x_i^*, i=1,2,\dots,m, v^{L^*}, v^{R^*}\}$  will be an optimal solution of Problem 1.5.3

if  $v^{L^*} \leq v^L \forall v^L \in S$  and  $\alpha v^{L^*} + (1-\alpha) v^{R^*} \leq \alpha v^L + (1-\alpha) v^R \forall v^L, v^R \in S$ .

However, if  $v_1^{L^*}, v_1^{R^*}$  and  $v_2^{L^*}, v_2^{R^*}$  are two feasible solutions of Problem 1.5.3 such that

$$v_1^{L^*} \geq v_2^{L^*} \text{ but } \alpha v_1^{L^*} + (1-\alpha) v_1^{R^*} \leq \alpha v_2^{L^*} + (1-\alpha) v_2^{R^*} \text{ or } v_1^{L^*} \leq v_2^{L^*} \text{ but}$$

$$\alpha v_1^{L^*} + (1-\alpha) v_1^{R^*} \geq \alpha v_2^{L^*} + (1-\alpha) v_2^{R^*} \text{ Then, both } [v_1^{L^*}, v_1^{R^*}] \text{ and } [v_2^{L^*}, v_2^{R^*}] \text{ will be the optimal}$$

solution of Problem 1.5.3. For example, if  $[1,5]$  and  $[2,3]$  are feasible solutions of Problem

1.5.3 at  $\alpha = \frac{1}{2}$  then neither  $[1,5] > [2,3]$  nor  $[2,3] > [1,5]$ . So, both  $[1,5]$  and  $[2,3]$  will represent

maximum  $\{[2,3], [1,5]\}$  and hence both  $[2,3]$  and  $[1,5]$  will be optimal value of Problem 1.5.3.

To obtain the unique optimal value of Problem 1.5.3, Li [69] and Li et al. [79] used the following methods:

2(a) Li [69] used the following method to find the optimal solution  $\{x_i^*, i=1,2,\dots,m, v^{L^*}, v^{R^*}\}$

from the set of all the feasible solutions  $S = \{x_i, i=1,2,\dots,m; v^L, v^R\}$ .

Find minimum  $\{v^L \mid v^L \in S\}$ .

**Case 1:** If minimum  $\{v^L\}$  occurs corresponding to only one feasible solution then that feasible solution is optimal solution of Problem 1.5.3.

**Case 2:** If minimum  $\{v^L\}$  occurs corresponding to two or more than two feasible solutions then

find

$$\text{minimum} \left\{ \frac{v^L + v^R}{2}; v^L, v^R \text{ are those feasible solutions corresponding to which } \text{minimum} \{v^L\} \text{ exists} \right\}.$$

The feasible solutions corresponding to which minimum value of  $\frac{v^L + v^R}{2}$  will exists, will be the optimal solution of Problem 1.5.3.

It is obvious that Li [69] has transformed the constraints of Problem 1.2.3 into constraints of Problem 1.5.3 by using the relation  $[a^L, a^R] \geq [b^L, b^R] \Rightarrow a^L \geq b^L$  and  $\alpha a^L + (1-\alpha)a^R \geq \alpha b^L + (1-\alpha)b^R$ . While, Li [69] has used the relation  $[a^L, a^R] \geq [b^L, b^R]$

$\Rightarrow a^L > b^L$  or if  $a^L = b^L$  then  $\frac{a^L + a^R}{2} > \frac{b^L + b^R}{2}$  for choosing the optimal solution of Problem 1.5.3 from all the possible feasible solutions of Problem 1.5.3 i.e., Li [69] has used two different methods simultaneously for solving Problem 1.5.3, which is mathematically incorrect.

Similarly, it can be easily verified that Li [69] has used two different methods  $[a^L, a^R] \leq [b^L, b^R] \Rightarrow (1-\alpha)a^L + \alpha a^R \leq (1-\alpha)b^L + \alpha b^R$  and  $a^R \leq b^R$  as well as  $[a^L, a^R] \leq [b^L, b^R] \Rightarrow a^R < b^R$  or if  $a^R = b^R$  then  $\frac{a^L + a^R}{2} < \frac{b^L + b^R}{2}$  simultaneously for solving Problem 1.2.4, which is mathematically incorrect.

2(b) Li et al. [79] used the following method to find the optimal solution  $\{x_i^*, i=1, 2, \dots, m, v^{L*}, v^{R*}\}$  from the set of all feasible solutions  $S = \{x_i, i=1, 2, \dots, m; v^L, v^R\}$ .

**Step 1:** Find  $\text{maximum} \left\{ \frac{3v^L + v^R}{4} \mid v^L, v^R \in S \right\}$ .

**Step 2:** All the feasible solution  $\{x_i, i=1, 2, \dots, m; v^L, v^R\}$  corresponding to which value of

$$\frac{3v^L + v^R}{4} \text{ will be equal to } \text{maximum} \left\{ \frac{3v^L + v^R}{4} \mid v^L, v^R \in S \right\} \text{ will be optimal solution of}$$

### Problem 1.5.3.

It is obvious that Li et al. [79] has transformed the constraints of Problem 1.2.3 into constraints of Problem 1.5.3 by using the relation  $[a^L, a^R] \geq [b^L, b^R] \Rightarrow a^L \geq b^L$  and  $\alpha a^L + (1-\alpha)a^R \geq \alpha b^L + (1-\alpha)b^R$ . While, Li et al. [79] has used the relation  $[a^L, a^R] \geq [b^L, b^R] \Rightarrow \frac{3a^L + a^R}{4} \geq \frac{3b^L + b^R}{4}$  for choosing optimal solution of Problem 1.5.3 from all possible feasible solutions of Problem 1.5.3 i.e., Li et al. [79] has used two different methods simultaneously for solving Problem 1.5.3, which is mathematically incorrect.

Similarly, it can be easily verified that Li et al. [79] has used two different methods  $[a^L, a^R] \leq [b^L, b^R] \Rightarrow (1-\alpha)a^L + \alpha a^R \leq (1-\alpha)b^L + \alpha b^R$  and  $a^R \leq b^R$  as well as  $[a^L, a^R] \leq [b^L, b^R] \Rightarrow \frac{a^L + 3a^R}{4} \leq \frac{b^L + 3b^R}{4}$  simultaneously for solving Problem 1.5.3, which is mathematically incorrect.

3. It is obvious that the Problem 1.3.16 and the Problem 1.3.17 are independent problems.

Therefore, on solving the Problem 1.3.16 and the Problem 1.3.17 different optimal strategies are obtained. Hence, the optimal strategy of the Problem 1.2.3 cannot be obtained with the help of optimal strategies of the Problem 1.3.16 and the Problem 1.3.17. Similarly, the optimal strategy of the Problem 1.2.4 cannot be obtained with the help of optimal strategies of the Problem 1.3.18 and Problem 1.3.19. However, in the existing method [68], the minimum expected gain of Player I and maximum expected loss of Player II are obtained by transforming Problem 1.2.3 into Problem 1.3.16 and Problem 1.3.17 and Problem 1.2.4 into Problem 1.3.18 and Problem 1.3.19, which is mathematically incorrect.

### 1.6 Invalidity of existing mathematical formulation of matrix games with interval payoffs

In all the existing methods [37,68,69,79,84,100], Problem 1.2.3 and Problem 1.2.4 are solved by different ways to find the minimum expected gain of Player I and maximum expected

loss of Player II respectively. In this section, firstly the method, followed in the existing methods [37,68,69,79,84,100], to obtain Problem 1.2.3 and Problem 1.2.4 is presented. Then, it is shown that the authors have considered several mathematically incorrect assumptions to obtain Problem 1.2.3 and Problem 1.2.4 and hence, Problem 1.2.3 and Problem 1.2.4 as well as existing methods [37,68,69,79,84,100], based on Problem 1.2.3 and Problem 1.2.4, are not valid.

### 1.6.1 Existing method to obtain mathematical formulation of matrix games with interval payoffs

Let, Player I and Player II be two players and  $A = (a_{ij})_{m \times n}$  be the payoff matrix for Player I. Let,  $S_1 = \{\delta_1, \delta_2, \dots, \delta_m\}$  and  $S_2 = \{\eta_1, \eta_2, \dots, \eta_n\}$  be set of course of actions available to Player I and Player II respectively and  $x_1, x_2, \dots, x_m$  and  $y_1, y_2, \dots, y_n$  be the probabilities for selecting the course of action  $\delta_1, \delta_2, \dots, \delta_m$  and  $\eta_1, \eta_2, \dots, \eta_n$  respectively. Let,

$$X = \left\{ x = (x_1, x_2, \dots, x_m) \mid \sum_{i=1}^m x_i = 1, x_i \geq 0, i = 1, 2, \dots, m \right\} \text{ and}$$

$$Y = \left\{ y = (y_1, y_2, \dots, y_n) \mid \sum_{j=1}^n y_j = 1, y_j \geq 0, j = 1, 2, \dots, n \right\} \text{ be the set of strategies for Player I and}$$

Player II respectively. Then, to find minimum expected gain of Player I there is need to find

$$(y_1^*, y_2^*, \dots, y_n^*) \in Y \text{ so that the value of } \sum_{j=1}^n \left( \sum_{i=1}^m [a_{ij}^L, a_{ij}^R] x_i \right) y_j^* \text{ is minimum}$$

$\forall (x_1, x_2, \dots, x_m) \in X$  and to find maximum expected loss of Player II there is need to find

$$(x_1^*, x_2^*, \dots, x_m^*) \in X \text{ so that the value of } \sum_{j=1}^n \left( \sum_{i=1}^m [a_{ij}^L, a_{ij}^R] x_i^* \right) y_j \text{ is maximum } \forall (y_1, y_2, \dots, y_n) \in Y$$

i.e., to find minimum expected gain of Player I there is need to find the optimal solution

$\{y_j^*, j = 1, 2, \dots, n\}$  of Problem 1.6.1 and to find maximum expected loss of Player II, there is

need to find the optimal solution  $\{x_i^*, i = 1, 2, \dots, m\}$  of Problem 1.6.2.

**Problem 1.6.1**

$$\text{Minimize } \left\{ \sum_{j=1}^n \left( \sum_{i=1}^m x_i [a_{ij}^L, a_{ij}^R] \right) y_j \right\}$$

Subject to

$$\sum_{i=1}^m x_i = 1; \sum_{j=1}^n y_j = 1;$$

$$x_i \geq 0, i = 1, 2, \dots, m; y_j \geq 0, j = 1, 2, \dots, n.$$

**Problem 1.6.2**

$$\text{Maximize } \left\{ \sum_{i=1}^m \left( \sum_{j=1}^n [a_{ij}^L, a_{ij}^R] y_j \right) x_i \right\}$$

Subject to

$$\sum_{i=1}^m x_i = 1; \sum_{j=1}^n y_j = 1;$$

$$x_i \geq 0, i = 1, 2, \dots, m; y_j \geq 0, j = 1, 2, \dots, n.$$

In the literature [37,68,69,79,84,100], Problem 1.6.1 and Problem 1.6.2 have been transformed into Problem 1.6.3 and Problem 1.6.4 respectively and hence, can be transformed into Problem 1.6.5 and Problem 1.6.6 respectively.

**Problem 1.6.3**

$$\text{Minimize } \left\{ \sum_{i=1}^m [a_{ij}^L, a_{ij}^R] x_i \right\}$$

Subject to

$$\sum_{i=1}^m x_i = 1; x_i \geq 0, i = 1, 2, \dots, m.$$

**Problem 1.6.4**

$$\text{Maximize } \left\{ \sum_{j=1}^n [a_{ij}^L, a_{ij}^R] y_j \right\}$$

Subject to

$$\sum_{j=1}^n y_j = 1; y_j \geq 0, j = 1, 2, \dots, n.$$

**Problem 1.6.5**

$$\text{Maximize } \{[\nu^L, \nu^R]\}$$

Subject to

$$\sum_{i=1}^m [a_{ij}^L, a_{ij}^R] x_i \geq [\nu^L, \nu^R], j = 1, 2, \dots, n;$$

$$\sum_{i=1}^m x_i = 1; x_i \geq 0, i = 1, 2, \dots, m.$$

**Problem 1.6.6**

$$\text{Minimize } \{[\omega^L, \omega^R]\}$$

Subject to

$$\sum_{j=1}^n [a_{ij}^L, a_{ij}^R] y_j \leq [\omega^L, \omega^R], i = 1, 2, \dots, m;$$

$$\sum_{j=1}^n y_j = 1; y_j \geq 0, j = 1, 2, \dots, n.$$

In the literature [37,68,69,79,84,100], the following method has been used to transform Problem 1.6.1 and Problem 1.6.2 into Problems 1.6.3 and Problem 1.6.4 and hence into

Problem 1.6.5 and Problem 1.6.6 respectively.

In the literature [37,68,69,79,84,100], it is proved that if  $[a_i^L, a_i^R], i = 1, 2, \dots, n$  are  $n$  intervals then  $\text{minimum} \left\{ \sum_{i=1}^n [a_i^L, a_i^R] x_i \mid \sum_{i=1}^n x_i = 1; x_i \geq 0, i = 1, 2, \dots, n \right\} = \text{minimum} \{ [a_i^L, a_i^R], i = 1, 2, \dots, n \}$ . Using this result, the objective function  $\text{minimum} \left\{ \sum_{j=1}^n \left( \sum_{i=1}^m x_i [a_{ij}^L, a_{ij}^R] \right) y_j \right\}$  where  $\sum_{j=1}^n y_j = 1; y_j \geq 0, j = 1, 2, \dots, n$ , of Problem 1.6.1 can be transformed into the objective function  $\text{minimum} \left\{ \sum_{i=1}^m [a_{ij}^L, a_{ij}^R] x_i \right\}$  of Problem 1.6.3 and the objective function  $\text{maximum} \left\{ \sum_{i=1}^m \left( \sum_{j=1}^n [a_{ij}^L, a_{ij}^R] y_j \right) x_i \right\}$  where  $\sum_{i=1}^m x_i = 1; x_i \geq 0, i = 1, 2, \dots, m$ , of Problem 1.6.2 can be transformed into the objective function  $\text{maximum} \left\{ \sum_{j=1}^n [a_{ij}^L, a_{ij}^R] y_j \right\}$  of Problem 1.6.4.

### 1.6.2 Mathematically incorrect assumptions considered in the existing method

If  $[1, 5]$  and  $[3, 4]$  are two intervals then on using the comparing method used in the existing methods [37,68,69,79,84,100],  $\text{minimum} \{ [1, 5], [3, 4] \} = [1, 5]$  and  $[3, 4]$ . While,  $\text{minimum} \{ [1, 5]x_1 + [3, 4]x_2 \mid x_1 + x_2 = 1, x_1 \geq 0, x_2 \geq 0 \} = \lambda[1, 5] + (1 - \lambda)[3, 4], \lambda \in [0, 1]$ . Hence,

$$\text{minimum} \left\{ \sum_{i=1}^n [a_i^L, a_i^R] x_i \mid \sum_{i=1}^n x_i = 1, x_i \geq 0, i = 1, 2, \dots, n \right\} \neq \text{minimum} \{ [a_i^L, a_i^R], i = 1, 2, \dots, n \}.$$

This clearly indicates that in the existing proof of the result

$$\text{minimum} \left\{ \sum_{i=1}^n [a_i^L, a_i^R] x_i \mid \sum_{i=1}^n x_i = 1, x_i \geq 0, i = 1, 2, \dots, n \right\} = \text{minimum} \{ [a_i^L, a_i^R], i = 1, 2, \dots, n \},$$

presented in Theorem 1.2, and hence, in existing method to transform the Problem 1.6.1 and

Problem 1.6.2 into Problem 1.6.3 and Problem 1.6.4 respectively, some mathematically incorrect assumptions have been considered.

In this section, these mathematically incorrect assumptions are pointed out.

(i) In the existing proof [79], presented in Theorem 1.2, it is assumed that  $\text{minimum}\{[a_i^L, a_i^R], i = 1, 2, \dots, n\}$  will be unique interval  $[a_p^L, a_p^R]$  such that

$[a_p^L, a_p^R] \leq [a_i^L, a_i^R] \forall i = 1, 2, \dots, n$ . However, this assumption is not valid. e.g., for the intervals

$[a^L, a^R] = [3, 4]$  and  $[b^L, b^R] = [1, 5]$ , the condition  $a^R < b^R$  and

$(1-\alpha)a^L + \alpha a^R > (1-\alpha)b^L + \alpha b^R$  is satisfying at  $\alpha = \frac{1}{2}$ . So, according to existing methods

[37,68,69,79,84,100] both the intervals [1,5] and [3,4] will represent  $\text{minimum}\{[1,5],[3,4]\}$ .

Hence, to assume that  $\text{minimum}\{[a_i^L, a_i^R], i = 1, 2, \dots, n\}$  will be a unique interval  $[a_p^L, a_p^R]$  is mathematically invalid.

(ii) In the existing proof, presented in Theorem 1.2, it is assumed that if  $[a_p^L, a_p^R]$  is an element of the set  $\{[a_i^L, a_i^R], i = 1, 2, \dots, n\}$  then

$[a_p^L, a_p^R] \leq \text{minimum}\left\{\sum_{i=1}^n [a_i^L, a_i^R] x_i \mid \sum_{i=1}^n x_i = 1, x_i \geq 0, i = 1, 2, \dots, n\right\}$ . However, this assumption

is not valid. e.g. [3,4] is an element of the set  $\{[1,5],[3,4]\}$  but [3,4] is neither smaller nor equal

to [1,5] which is obtained by substituting  $x_1 = 0, x_2 = 1$  in  $[1,5]x_1 + [3,4]x_2$ .

## 1.7 Minimum and maximum of intervals

The methods, used for comparing intervals in the existing methods are not discussed in the published papers [37,68,69,79,84,100] in detailed manner. So, in this section, the methods to find the minimum and maximum of intervals, used in the existing methods [37,68,69,79,84,100], are presented in detailed manner.

### 1.7.1 Minimum of intervals

If  $\{[a_i^L, a_i^R], i = 1, 2, \dots, n\}$  are  $n$  intervals then  $\text{minimum}\{[a_i^L, a_i^R], i = 1, 2, \dots, n\}$ ,

obtained by using the approaches used in all the existing methods [37,68,69,79,84,100], will be either two intervals  $[x^L, x^R]$  and  $[y^L, y^R]$  or will be a unique interval  $[x^L, x^R] = [y^L, y^R]$ .

The intervals  $[x^L, x^R]$  and  $[y^L, y^R]$  can be obtained as follows:

**Step 1:** Choose a real number  $\alpha \in [0, 1)$ .

**Step 2:** Find  $\text{minimum}\{(1-\alpha)a_i^L + \alpha a_i^R, i = 1, 2, \dots, n\}$  for chosen value of  $\alpha$ .

**Case 1:** If  $\text{minimum}\{(1-\alpha)a_i^L + \alpha a_i^R, i = 1, 2, \dots, n\}$  exist corresponding to unique value of  $i$  (say  $p$ ) then  $x^L = a_p^L$  and  $x^R = a_p^R$ .

**Case 2:** If  $\text{minimum}\{(1-\alpha)a_i^L + \alpha a_i^R, i = 1, 2, \dots, n\}$  exist corresponding to more than one values of  $i$  (say  $i = 1, 2, \dots, K$ ) then find  $\text{minimum}\{a_i^R, i = 1, 2, \dots, K\}$ . Let,  $\text{minimum}\{a_i^R, i = 1, 2, \dots, K\}$  exist corresponding to  $i = q$  then  $x^L = a_q^L$  and  $x^R = a_q^R$ .

**Step 3:** Find  $\text{minimum}\{a_i^R, i = 1, 2, \dots, n\}$ .

**Case 1:** If  $\text{minimum}\{a_i^R, i = 1, 2, \dots, n\}$  exist corresponding to unique value of  $i$  (say  $p$ ) then  $y^L = a_p^L$  and  $y^R = a_p^R$ .

**Case 2:** If  $\text{minimum}\{a_i^R, i = 1, 2, \dots, n\}$  exist corresponding to more than one value of  $i$  (say  $i = 1, 2, \dots, K$ ) then find  $\text{minimum}\{(1-\alpha)a_i^L + \alpha a_i^R, i = 1, 2, \dots, K\}$ . Let,  $\text{minimum}\{(1-\alpha)a_i^L + \alpha a_i^R, i = 1, 2, \dots, K\}$  exist corresponding to  $i = q$  then  $y^L = a_q^L$  and  $y^R = a_q^R$ .

### 1.7.2 Maximum of intervals

If  $\{[a_i^L, a_i^R], i=1, 2, \dots, n\}$  are  $n$  intervals then  $\text{maximum}\{[a_i^L, a_i^R], i=1, 2, \dots, n\}$ , obtained by using the approaches used in all the existing methods [37,68,69,79,84,100], will be either two intervals  $[x^L, x^R]$  and  $[y^L, y^R]$  or will be a unique interval  $[x^L, x^R] = [y^L, y^R]$ .

The intervals  $[x^L, x^R]$  and  $[y^L, y^R]$  can be obtained as follows:

**Step 1:** Choose a real number  $\alpha \in [0, 1)$ .

**Step 2:** Find  $\text{maximum}\{\alpha a_i^L + (1-\alpha)a_i^R, i=1, 2, \dots, n\}$  for chosen value of  $\alpha$ .

**Case 1:** If  $\text{maximum}\{\alpha a_i^L + (1-\alpha)a_i^R, i=1, 2, \dots, n\}$  exist corresponding to unique value of  $i$  (say  $p$ ) then  $x^L = a_p^L$  and  $x^R = a_p^R$ .

**Case 2:** If  $\text{maximum}\{\alpha a_i^L + (1-\alpha)a_i^R, i=1, 2, \dots, n\}$  exist corresponding to more than one values of  $i$  (say  $i=1, 2, \dots, K$ ) then find  $\text{maximum}\{a_i^L, i=1, 2, \dots, K\}$ . Let,  $\text{maximum}\{a_i^L, i=1, 2, \dots, K\}$  exist corresponding to  $i=q$  then  $x^L = a_q^L$  and  $x^R = a_q^R$ .

**Step 3:** Find  $\text{maximum}\{a_i^L, i=1, 2, \dots, n\}$ .

**Case 1:** If  $\text{maximum}\{a_i^L, i=1, 2, \dots, n\}$  exist corresponding to unique value of  $i$  (say  $p$ ) then  $y^L = a_p^L$  and  $y^R = a_p^R$ .

**Case 2:** If  $\text{maximum}\{a_i^L, i=1, 2, \dots, n\}$  exist corresponding to more than one value of  $i$  (say  $i=1, 2, \dots, K$ ) then find  $\text{maximum}\{\alpha a_i^L + (1-\alpha)a_i^R, i=1, 2, \dots, K\}$ . Let,  $\text{maximum}\{\alpha a_i^L + (1-\alpha)a_i^R, i=1, 2, \dots, K\}$  exist corresponding to  $i=q$  then  $y^L = a_q^L$  and  $y^R = a_q^R$ .

## 1.8 Proposed Gaurika method

It is obvious from Section 1.6 that it is not correct to use Problem 1.2.3 and Problem 1.2.4 to obtain the minimum expected gain of Player I and maximum expected loss of Player II as Problem 1.2.3 and Problem 1.2.4 are obtained by considering some mathematically incorrect assumptions. Hence, it is not mathematically correct to use any of the existing methods [37,68,69,79,84,100] in which the minimum expected gain of Player I and maximum expected loss of Player II is obtained with the help of optimal solution of Problem 1.2.3 and Problem 1.2.4 respectively.

Furthermore, in the existing methods [37,68,69,79,84,100], it is assumed that if  $[a^L, a^R]$  and  $[b^L, b^R]$  are two intervals then for a chosen value of  $\alpha \in [0,1)$   $[a^L, a^R] \geq [b^L, b^R]$  if  $a^L \geq b^L$  and  $\alpha a^L + (1-\alpha)a^R \geq \alpha b^L + (1-\alpha)b^R$  and  $[a^L, a^R] \leq [b^L, b^R]$  if  $a^R \leq b^R$  and  $(1-\alpha)a^L + \alpha a^R \leq (1-\alpha)b^L + \alpha b^R$ .

In this section, a new method (named as Gaurika method), on the basis of this comparison method, is proposed to find minimum expected gain of Player I, maximum expected loss of Player II and their corresponding optimal strategies.

### 1.8.1 Minimum expected gain of Player I

Using the comparing method,  $[a^L, a^R] \leq [b^L, b^R] \Rightarrow a^R \leq b^R$  and  $(1-\alpha)a^L + \alpha a^R \leq (1-\alpha)b^L + \alpha b^R$ , the minimum expected gain of Player I and corresponding optimal strategies can be obtained as follows:

**Step 1:** Find  $(y_1^*, y_2^*, \dots, y_n^*) \in Y$  such that value of  $\sum_{j=1}^n \left( \sum_{i=1}^m [a_{ij}^L, a_{ij}^R] x_i \right) y_j$  is minimum for all

$(x_1, x_2, \dots, x_m) \in X$  i.e., find the optimal solution  $\{y_j^*, j = 1, 2, \dots, n\}$  of Problem 1.8.1.

#### Problem 1.8.1

Minimize  $\left\{ \sum_{j=1}^n \sum_{i=1}^m [a_{ij}^L, a_{ij}^R] x_i y_j \right\}$

Subject to

$$\sum_{i=1}^m x_i = 1; \sum_{j=1}^n y_j = 1; x_i \geq 0, i = 1, 2, \dots, m; y_j \geq 0, j = 1, 2, \dots, n.$$

**Step 2:** Using the property,  $\sum_{i=1}^n \lambda [a_{ij}^L, a_{ij}^R] = \sum_{i=1}^n [\lambda a_{ij}^L, \lambda a_{ij}^R], \lambda \geq 0$ , the Problem 1.8.1 can be transformed into Problem 1.8.2.

**Problem 1.8.2**

$$\text{Minimize } \left\{ \sum_{j=1}^n \sum_{i=1}^m [a_{ij}^L x_i y_j, a_{ij}^R x_i y_j] \right\}$$

Subject to

$$\sum_{i=1}^m x_i = 1; \sum_{j=1}^n y_j = 1; x_i \geq 0, i = 1, 2, \dots, m; y_j \geq 0, j = 1, 2, \dots, n.$$

**Step 3:** According to comparison method, to find the optimal solution  $\{y_1^*, y_2^*, \dots, y_n^*\}$  of

Problem 1.8.2 such that value of  $\sum_{j=1}^n \sum_{i=1}^m [a_{ij}^L x_i y_j, a_{ij}^R x_i y_j]$  is minimum for all  $(x_1, x_2, \dots, x_m) \in X$

is equivalent to find  $\{y_1^*, y_2^*, \dots, y_n^*\}$  such that value of  $(1-\alpha) \sum_{j=1}^n \sum_{i=1}^m a_{ij}^L x_i y_j + \alpha \sum_{j=1}^n \sum_{i=1}^m a_{ij}^R x_i y_j$  and

$\sum_{j=1}^n \sum_{i=1}^m a_{ij}^R x_i y_j$  is minimum for all  $(x_1, x_2, \dots, x_m) \in X$ . If it is not possible by find

$\{y_1^*, y_2^*, \dots, y_n^*\}$  for which the value of  $(1-\alpha) \sum_{j=1}^n \sum_{i=1}^m a_{ij}^L x_i y_j + \alpha \sum_{j=1}^n \sum_{i=1}^m a_{ij}^R x_i y_j$  and  $\sum_{j=1}^n \sum_{i=1}^m a_{ij}^R x_i y_j$  is

minimum then find such  $\{y_{j_1}^*, j = 1, 2, \dots, n\}$  for which the value of

$(1-\alpha) \sum_{j=1}^n \sum_{i=1}^m a_{ij}^L x_{i1} y_{j1} + \alpha \sum_{j=1}^n \sum_{i=1}^m a_{ij}^R x_{i1} y_{j1}$  is minimum but value of  $\sum_{j=1}^n \sum_{i=1}^m a_{ij}^R x_{i1} y_{j1}$  is not minimum for

all  $(x_{11}, x_{21}, \dots, x_{m1}) \in X$  and to find such  $\{y_{j_2}^*, j = 1, 2, \dots, n\}$  for which the value of

$(1-\alpha)\sum_{j=1}^n\sum_{i=1}^ma_{ij}^Lx_{i2}y_{j2}+\alpha\sum_{j=1}^n\sum_{i=1}^ma_{ij}^Rx_{i2}y_{j2}$  is not minimum but value of  $\sum_{j=1}^n\sum_{i=1}^ma_{ij}^Rx_{i2}y_{j2}$  is minimum

for all  $(x_{12}, x_{22}, \dots, x_{m2}) \in X$  i.e., to find optimal solution  $\{y_{j1}^*, j=1, 2, \dots, n\}$  and  $\{y_{j2}^*, j=1, 2, \dots, n\}$  of Problem 1.8.3 and Problem 1.8.4 respectively.

### Problem 1.8.3

$$\text{Minimize } \left\{ (1-\alpha)\sum_{j=1}^n\sum_{i=1}^ma_{ij}^Lx_{i1}y_{j1} + \alpha\sum_{j=1}^n\sum_{i=1}^ma_{ij}^Rx_{i1}y_{j1} \right\}$$

Subject to

$$\sum_{i=1}^mx_{i1}=1; \sum_{j=1}^ny_{j1}=1; x_{i1} \geq 0, i=1, 2, \dots, m; y_{j1} \geq 0, j=1, 2, \dots, n.$$

### Problem 1.8.4

$$\text{Minimize } \left\{ \sum_{j=1}^n\sum_{i=1}^ma_{ij}^Rx_{i2}y_{j2} \right\}$$

Subject to

$$\sum_{i=1}^mx_{i2}=1; \sum_{j=1}^ny_{j2}=1; x_{i2} \geq 0, i=1, 2, \dots, m; y_{j2} \geq 0, j=1, 2, \dots, n.$$

**Step 4:** Since, in Problem 1.8.3 and Problem 1.8.4 only  $y_{j1}$  and  $y_{j2}$  respectively have been considered as decision variables. So, Problem 1.8.3 and Problem 1.8.4 are linear programming problem and hence, the optimal value of Problem 1.8.3 and Problem 1.8.4 will be equal to optimal value of its corresponding dual problem i.e., Problem 1.8.5 and Problem 1.8.6 respectively.

### Problem 1.8.5

$$\text{Maximize } \{v_1\}$$

Subject to

### Problem 1.8.6

$$\text{Maximize } \{v_2\}$$

Subject to

$$(1-\alpha)\sum_{i=1}^m a_{ij}^L x_{i1} + \alpha\sum_{i=1}^m a_{ij}^R x_{i1} \geq v_1, j=1,2,\dots,n;$$

$$\sum_{i=1}^m x_{i1} = 1; x_{i1} \geq 0, i=1,2,\dots,m.$$

$$\sum_{i=1}^m a_{ij}^R x_{i2} \geq v_2, j=1,2,\dots,n;$$

$$\sum_{i=1}^m x_{i2} = 1; x_{i2} \geq 0, i=1,2,\dots,m.$$

**Step 5:** Substitute the value of  $\{x_{i1}^*, i=1,2,\dots,m\}$  and  $\{x_{i2}^*, i=1,2,\dots,m\}$  of Problem 1.8.5 and Problem 1.8.6 in Problem 1.8.3 and Problem 1.8.4 respectively and find all the alternative basic optimal solution  $\{y_{j1}^{k*}, j=1,2,\dots,n, k=1,2,\dots,l\}$  and  $\{y_{j2}^{q*}, j=1,2,\dots,n, q=1,2,\dots,h\}$  of Problem 1.8.3 and Problem 1.8.4 respectively.

**Step 6:** Find

$$\text{minimum} \left\{ \sum_{j=1}^n \sum_{i=1}^m x_{i1}^* [a_{ij}^L, a_{ij}^R] y_{j1}^{k*}, \sum_{j=1}^n \sum_{i=1}^m x_{i2}^* [a_{ij}^L, a_{ij}^R] y_{j2}^{q*}, k=1,2,\dots,l; q=1,2,\dots,h \right\}.$$

**Step 7:** All the intervals which will be

$$\text{minimum} \left\{ \sum_{j=1}^n \sum_{i=1}^m x_{i1}^* [a_{ij}^L, a_{ij}^R] y_{j1}^{k*}, \sum_{j=1}^n \sum_{i=1}^m x_{i2}^* [a_{ij}^L, a_{ij}^R] y_{j2}^{q*}, k=1,2,\dots,l; q=1,2,\dots,h \right\}, \text{ will}$$

represent minimum expected gain of Player I and the optimal strategies for all such minimum

which will be obtained corresponding to  $\sum_{j=1}^n \sum_{i=1}^m x_{i1}^* [a_{ij}^L, a_{ij}^R] y_{j1}^{k*}, k=1,2,\dots,l$  will be

$\{x_{i1}^*, i=1,2,\dots,m\}$  and for all such minimum which will be obtained corresponding to

$$\sum_{j=1}^n \sum_{i=1}^m x_{i1}^* [a_{ij}^L, a_{ij}^R] y_{j1}^{q*}, q=1,2,\dots,h \text{ will be } \{x_{i2}^*, i=1,2,\dots,m\}.$$

## 1.8.2 Maximum expected loss of Player II

Using the comparing method,  $[a^L, a^R] \geq [b^L, b^R] \Rightarrow a^L \geq b^L$  and

$\alpha a^L + (1-\alpha)a^R \geq \alpha b^L + (1-\alpha)b^R$ , the maximum expected loss of Player II and corresponding optimal strategies can be obtained as follows:

**Step 1:** Find  $(x_1^*, x_2^*, \dots, x_m^*) \in X$  such that value of  $\sum_{j=1}^n \left( \sum_{i=1}^m [a_{ij}^L, a_{ij}^R] x_i \right) y_j$  is maximum for all

$(y_1, y_2, \dots, y_n) \in Y$  i.e., find the solution  $\{x_i^*, i = 1, 2, \dots, m\}$  of Problem 1.8.7.

**Problem 1.8.7**

$$\text{Maximize} \left\{ \sum_{i=1}^m \sum_{j=1}^n [a_{ij}^L, a_{ij}^R] y_j x_i \right\}$$

Subject to

$$\sum_{i=1}^m x_i = 1; \sum_{j=1}^n y_j = 1; x_i \geq 0, i = 1, 2, \dots, m; y_j \geq 0, j = 1, 2, \dots, n.$$

**Step 2:** Using the property,  $\sum_{i=1}^n \lambda [a_{ij}^L, a_{ij}^R] = \sum_{i=1}^n [\lambda a_{ij}^L, \lambda a_{ij}^R], \lambda \geq 0$ , the Problem 1.8.7 can be

transformed into Problem 1.8.8.

**Problem 1.8.8**

$$\text{Maximize} \left\{ \sum_{i=1}^m \sum_{j=1}^n [a_{ij}^L x_i y_j, a_{ij}^R x_i y_j] \right\}$$

Subject to

$$\sum_{i=1}^m x_i = 1; \sum_{j=1}^n y_j = 1; x_i \geq 0, i = 1, 2, \dots, m; y_j \geq 0, j = 1, 2, \dots, n.$$

**Step 3:** According to comparing method, to find the optimal solution  $\{x_1^*, x_2^*, \dots, x_m^*\}$  of Problem

1.8.8 such that value of  $\sum_{i=1}^m \sum_{j=1}^n [a_{ij}^L x_i y_j, a_{ij}^R x_i y_j]$  is maximum for all  $(y_1, y_2, \dots, y_n) \in Y$  is

equivalent to find  $\{x_1^*, x_2^*, \dots, x_m^*\}$  such that value of  $\alpha \sum_{i=1}^m \sum_{j=1}^n a_{ij}^L x_i y_j + (1-\alpha) \sum_{i=1}^m \sum_{j=1}^n a_{ij}^R x_i y_j$  and

$\sum_{i=1}^m \sum_{j=1}^n a_{ij}^L x_i y_j$  is maximum for all  $(y_1, y_2, \dots, y_m) \in Y$  or if it is not possible to find

$\{x_1^*, x_2^*, \dots, x_m^*\}$  for which the value of  $\alpha \sum_{i=1}^m \sum_{j=1}^n a_{ij}^L x_i y_j + (1-\alpha) \sum_{i=1}^m \sum_{j=1}^n a_{ij}^R x_i y_j$  and  $\sum_{i=1}^m \sum_{j=1}^n a_{ij}^L x_i y_j$  is

maximum then find such  $\{x_{i1}^*, i = 1, 2, \dots, m\}$  for which the value of

$\alpha \sum_{i=1}^m \sum_{j=1}^n a_{ij}^L x_{i1} y_{j1} + (1-\alpha) \sum_{i=1}^m \sum_{j=1}^n a_{ij}^R x_{i1} y_{j1}$  is maximum but value of  $\sum_{j=1}^n \sum_{i=1}^m a_{ij}^L x_{i1} y_{j1}$  is not maximum

for all  $(y_{11}, y_{21}, \dots, y_{n1}) \in Y$  and to find such  $\{x_{i2}^*, i=1, 2, \dots, m\}$  for which the value of

$\alpha \sum_{i=1}^m \sum_{j=1}^n a_{ij}^L x_{i2} y_{j2} + (1-\alpha) \sum_{i=1}^m \sum_{j=1}^n a_{ij}^R x_{i2} y_{j2}$  is not maximum but value of  $\sum_{j=1}^n \sum_{i=1}^m a_{ij}^L x_{i2} y_{j2}$  is

maximum for all  $(y_{12}, y_{22}, \dots, y_{n2}) \in Y$  i.e., to find optimal solution  $\{x_{i1}^*, i=1, 2, \dots, m\}$  and

$\{x_{i2}^*, i=1, 2, \dots, m\}$  of Problem 1.8.9 and Problem 1.8.10 respectively.

#### Problem 1.8.9

Maximize  $\left\{ \alpha \sum_{i=1}^m \sum_{j=1}^n a_{ij}^L x_{i1} y_{j1} + (1-\alpha) \sum_{i=1}^m \sum_{j=1}^n a_{ij}^R x_{i1} y_{j1} \right\}$

Subject to

$$\sum_{i=1}^m x_{i1} = 1; \sum_{j=1}^n y_{j1} = 1; x_{i1} \geq 0, i=1, 2, \dots, m; y_{j1} \geq 0, j=1, 2, \dots, n.$$

#### Problem 1.8.10

Maximize  $\left\{ \sum_{i=1}^m \sum_{j=1}^n a_{ij}^L x_{i2} y_{j2} \right\}$

Subject to

$$\sum_{i=1}^m x_{i2} = 1; \sum_{j=1}^n y_{j2} = 1; x_{i2} \geq 0, i=1, 2, \dots, m; y_{j2} \geq 0, j=1, 2, \dots, n.$$

**Step 4:** Since, in Problem 1.8.9 and Problem 1.8.10 only  $x_{i1}$  and  $x_{i2}$  respectively have been considered as decision variables. So, Problem 1.8.9 and Problem 1.8.10 are linear programming problem and hence, the optimal value of Problem 1.8.9 and Problem 1.8.10 will be equal to optimal value of its corresponding dual problem i.e., Problem 1.8.11 and Problem 1.8.12 respectively.

#### Problem 1.8.11

#### Problem 1.8.12

Minimize  $\{\omega_1\}$

Maximize  $\{\omega_2\}$

Subject to

Subject to

$$\alpha \sum_{j=1}^n a_{ij}^L y_{j1} + (1-\alpha) \sum_{j=1}^n a_{ij}^R y_{j1} \leq \omega_1, i = 1, 2, \dots, m;$$

$$\sum_{j=1}^n a_{ij}^L y_{j2} \leq \omega_2, i = 1, 2, \dots, m;$$

$$\sum_{j=1}^n y_{j1} = 1; y_{j1} \geq 0, j = 1, 2, \dots, n.$$

$$\sum_{j=1}^n y_{j2} = 1; y_{j2} \geq 0, j = 1, 2, \dots, n.$$

**Step 5:** Substitute the value of  $\{y_{j1}^*, j = 1, 2, \dots, n\}$  and  $\{y_{j2}^*, j = 1, 2, \dots, n\}$  of Problem 1.8.11

and Problem 1.8.12 in Problem 1.8.9 and Problem 1.8.10 respectively and find all the

alternative basic optimal solution  $\{x_{i1}^{k*}, i = 1, 2, \dots, m, k = 1, 2, \dots, l\}$  and

$\{x_{i1}^{q*}, i = 1, 2, \dots, m, q = 1, 2, \dots, h\}$  of Problem 1.8.9 and Problem 1.8.10 respectively.

**Step 6:** Find

$$\text{maximum} \left\{ \sum_{i=1}^m \sum_{j=1}^n x_{i1}^{k*} [a_{ij}^L, a_{ij}^R] y_{j1}^*, \sum_{i=1}^m \sum_{j=1}^n x_{i2}^{q*} [a_{ij}^L, a_{ij}^R] y_{j2}^*, k = 1, 2, \dots, l; q = 1, 2, \dots, h \right\}.$$

**Step 7:** All the intervals which will be

$$\text{maximum} \left\{ \sum_{i=1}^m \sum_{j=1}^n x_{i1}^{k*} [a_{ij}^L, a_{ij}^R] y_{j1}^*, \sum_{i=1}^m \sum_{j=1}^n x_{i2}^{q*} [a_{ij}^L, a_{ij}^R] y_{j2}^*, k = 1, 2, \dots, l; q = 1, 2, \dots, h \right\}, \text{ will}$$

represent maximum expected loss of Player II and the optimal strategies for all such

maximum which will be obtained corresponding to  $\sum_{i=1}^m \sum_{j=1}^n x_{i1}^{k*} [a_{ij}^L, a_{ij}^R] y_{j1}^*, k = 1, 2, \dots, l$  will be

$\{y_{j1}^*, j = 1, 2, \dots, n\}$  and for all such maximum which will be obtained corresponding to

$$\sum_{i=1}^m \sum_{j=1}^n x_{i2}^{q*} [a_{ij}^L, a_{ij}^R] y_{j2}^*, q = 1, 2, \dots, h \text{ will be } \{y_{j2}^*, j = 1, 2, \dots, n\}.$$

## 1.9 Numerical examples

In this section, some existing numerical examples have been solved by proposed Gaurika method.

### 1.9.1 Existing numerical example considered by Nayak & Pal

In this section, matrix games with interval payoffs  $A = \begin{bmatrix} [-3, -1] & [4, 6] \\ [6, 8] & [-7, -5] \end{bmatrix}$ , chosen by

Nayak and Pal [100], is solved by the proposed Gaurika method.

### 1.9.1.1 Minimum expected gain of Player I

Using the proposed Gaurika method minimum expected gain of Player I and corresponding optimal strategies, can be obtained as follows:

**Step 1:** Find  $(y_1^*, y_2^*) \in Y$  such that value of  $([-3, -1]x_1y_1 + [6, 8]x_2y_1 + [4, 6]x_1y_2 + [-7, -5]x_2y_2)$

is minimum for all  $(x_1, x_2) \in X$  i.e., find the optimal solution of Problem 1.9.1.

#### Problem 1.9.1

Minimize  $\{[-3, -1]x_1y_1 + [6, 8]x_2y_1 + [4, 6]x_1y_2 + [-7, -5]x_2y_2\}$

Subject to

$$x_1 + x_2 = 1; y_1 + y_2 = 1; x_1, x_2 \geq 0; y_1, y_2 \geq 0.$$

**Step 2:** Using the property,  $\sum_{i=1}^n \lambda [a_{ij}^L, a_{ij}^R] = \sum_{i=1}^n [\lambda a_{ij}^L, \lambda a_{ij}^R], \lambda \geq 0$  and

$\sum_{i=1}^n [a_i^L, a_i^R] = \left[ \sum_{i=1}^n a_i^L, \sum_{i=1}^n a_i^R \right]$  the Problem 1.9.1 can be transformed into Problem 1.9.2.

#### Problem 1.9.2

Minimize  $\{[-3x_1y_1 + 6x_2y_1 + 4x_1y_2 - 7x_2y_2, -1x_1y_1 + 8x_2y_1 + 6x_1y_2 - 5x_2y_2]\}$

Subject to

$$x_1 + x_2 = 1; y_1 + y_2 = 1; x_1, x_2 \geq 0; y_1, y_2 \geq 0.$$

**Step 3:** According to comparing method, to find the optimal solution  $\{y_1^*, y_2^*\}$  of Problem 1.9.2

such that value of  $[-3x_1y_1 + 6x_2y_1 + 4x_1y_2 - 7x_2y_2, -1x_1y_1 + 8x_2y_1 + 6x_1y_2 - 5x_2y_2]$  is minimum

for all  $(x_1, x_2) \in X$  is equivalent to find  $\{y_1^*, y_2^*\}$  such that value of

$(1-\alpha)(-3x_1y_1+6x_2y_1+4x_1y_2-7x_2y_2)+\alpha(-1x_1y_1+8x_2y_1+6x_1y_2-5x_2y_2)$  and

$(-1x_1y_1+8x_2y_1+6x_1y_2-5x_2y_2)$  is minimum for all  $(x_1, x_2) \in X$  or if it is not possible to find

$\{y_1^*, y_2^*\}$  for which the value of

$(1-\alpha)(-3x_{11}y_{11}+6x_{21}y_{11}+4x_{11}y_{21}-7x_{21}y_{21})+\alpha(-1x_{11}y_{11}+8x_{21}y_{11}+6x_{11}y_{21}-5x_{21}y_{21})$  and

$(-1x_{11}y_{11}+8x_{21}y_{11}+6x_{11}y_{21}-5x_{21}y_{21})$  is minimum then find such  $\{y_{j1}^*, j=1, 2\}$  for which the

value of  $(1-\alpha)(-3x_{11}y_{11}+6x_{21}y_{11}+4x_{11}y_{21}-7x_{21}y_{21})+\alpha(-1x_{11}y_{11}+8x_{21}y_{11}+6x_{11}y_{21}-5x_{21}y_{21})$  is

minimum but value of  $(-1x_{11}y_{11}+8x_{21}y_{11}+6x_{11}y_{21}-5x_{21}y_{21})$  is not minimum for all

$(x_{11}, x_{21}) \in X$  and to find such  $\{y_{j2}^*, j=1, 2\}$  for which the value of

$(1-\alpha)(-3x_{12}y_{12}+6x_{22}y_{12}+4x_{12}y_{22}-7x_{22}y_{22})+\alpha(-1x_{12}y_{12}+8x_{22}y_{12}+6x_{12}y_{22}-5x_{22}y_{22})$  is not

minimum but value of  $(-1x_{12}y_{12}+8x_{22}y_{12}+6x_{12}y_{22}-5x_{22}y_{22})$  is minimum for all  $(x_{12}, x_{22}) \in X$

i.e., find optimal solution  $\{y_{j1}^*, j=1, 2\}$  and  $\{y_{j2}^*, j=1, 2\}$  of Problem 1.9.3 and Problem 1.9.4

respectively.

### Problem 1.9.3

Minimize

$$\{(1-\alpha)(-3x_{11}y_{11}+6x_{21}y_{11}+4x_{11}y_{21}-7x_{21}y_{21})+\alpha(-1x_{11}y_{11}+8x_{21}y_{11}+6x_{11}y_{21}-5x_{21}y_{21})\}$$

Subject to

$$x_{11} + x_{21} = 1; y_{11} + y_{21} = 1;$$

$$x_{11}, x_{21} \geq 0; y_{11}, y_{21} \geq 0.$$

### Problem 1.9.4

Minimize  $\{-1x_{12}y_{12}+8x_{22}y_{12}+6x_{12}y_{22}-5x_{22}y_{22}\}$

Subject to

$$x_{12} + x_{22} = 1; y_{12} + y_{22} = 1;$$

$$x_{12}, x_{22} \geq 0; y_{12}, y_{22} \geq 0.$$

**Step 4:** Since, in Problem 1.9.3 and Problem 1.9.4 only  $y_{11}, y_{21}$  and  $y_{12}, y_{22}$  respectively have been considered as decision variables. So, Problem 1.9.3 and Problem 1.9.4 are linear programming problem and hence, the optimal value of Problem 1.9.3 and Problem 1.9.4 will be equal to optimal value of its corresponding dual problem i.e., Problem 1.9.5 and Problem 1.9.6 respectively.

**Problem 1.9.5**

Maximize  $\{v_1\}$

Subject to

$$(1-\alpha)(-3x_{11} + 6x_{21}) + \alpha(-1x_{11} + 8x_{21}) \geq v_1;$$

$$(1-\alpha)(4x_{11} - 7x_{21}) + \alpha(6x_{11} - 5x_{21}) \geq v_1;$$

$$x_{11} + x_{21} = 1; x_{11}, x_{21} \geq 0.$$

**Problem 1.9.6**

Maximize  $\{v_2\}$

Subject to

$$-1x_{12} + 8x_{22} \geq v_2;$$

$$6x_{12} - 5x_{22} \geq v_2;$$

$$x_{12} + x_{22} = 1; x_{12}, x_{22} \geq 0.$$

**Step 5:** The optimal solution of Problem 1.9.5 for all  $\alpha \in [0,1)$  is  $\left\{x_{11} = \frac{13}{20}, x_{21} = \frac{7}{20}\right\}$  and the

optimal solution of Problem 1.9.6 is  $\left\{x_{12} = \frac{13}{20}, x_{22} = \frac{7}{20}\right\}$ . On solving Problem 1.9.3 and

Problem 1.9.4 after substituting these values  $\left\{x_{11} = \frac{13}{20}, x_{21} = \frac{7}{20}\right\}$  and  $\left\{x_{12} = \frac{13}{20}, x_{22} = \frac{7}{20}\right\}$

respectively in Problem 1.9.3 and Problem 1.9.4, the obtained alternative basic optimal

solutions of Problem 1.9.3 and Problem 1.9.4 are  $\{y_{11}^1 = 0, y_{21}^1 = 1\}, \{y_{11}^2 = 1, y_{21}^2 = 0\}$  and

$\{y_{12}^1 = 0, y_{22}^1 = 1\}, \{y_{12}^2 = 1, y_{22}^2 = 0\}$  respectively.

**Step 6:** Now, minimum  $\{-3, -1\}x_{11}y_{11}^1 + [4, 6]x_{11}y_{21}^1 + [6, 8]x_{21}y_{11}^1 + [-7, -5]x_{21}y_{21}^1, [-3, -1]x_{11}y_{12}^1$   
 $+ [4, 6]x_{11}y_{22}^1 + [6, 8]x_{21}y_{12}^1 + [-7, -5]x_{21}y_{22}^1, [-3, -1]x_{12}y_{12}^1 + [4, 6]x_{12}y_{22}^1 + [6, 8]x_{22}y_{12}^1 + [-7, -5]x_{22}y_{22}^1,$

$$[-3, -1]x_{12}y_{12}^2 + [4, 6]x_{12}y_{22}^2 + [6, 8]x_{22}y_{12}^2 + [-7, -5]x_{22}y_{22}^2 \} = \text{minimum} \left\{ \left[ \frac{3}{20}, \frac{43}{20} \right], \left[ \frac{3}{20}, \frac{43}{20} \right], \left[ \frac{3}{20}, \frac{43}{20} \right] \right\}$$

$$\left[ \frac{3}{20}, \frac{43}{20} \right] \} = \left[ \frac{3}{20}, \frac{43}{20} \right].$$

**Step 7:** Since, minimum, obtained in Step 6, is  $\left[ \frac{3}{20}, \frac{43}{20} \right]$ . So, the minimum expected gain of

Player I is  $\left[ \frac{3}{20}, \frac{43}{20} \right]$  and the optimal strategies for Player I is  $\left\{ x_{11} = \frac{13}{20}, x_{21} = \frac{7}{20} \right\} =$

$$\left\{ x_{12} = \frac{13}{20}, x_{22} = \frac{7}{20} \right\}.$$

### 1.9.1.2 Maximum expected loss of Player II

Using the proposed Gaurika method maximum expected loss of Player II and corresponding optimal strategies, can be obtained as follows:

**Step 1:** Find  $(x_1^*, x_2^*) \in X$  such that value of  $([-3, -1]x_1y_1 + [6, 8]x_2y_1 + [4, 6]x_1y_2 + [-7, -5]x_2y_2)$

is maximum for all  $(y_1^*, y_2^*) \in Y$  i.e., find the optimal solution of Problem 1.9.7.

#### Problem 1.9.7

Maximize  $\{[-3, -1]x_1y_1 + [6, 8]x_2y_1 + [4, 6]x_1y_2 + [-7, -5]x_2y_2\}$

Subject to

$$x_1 + x_2 = 1; y_1 + y_2 = 1; x_1, x_2 \geq 0; y_1, y_2 \geq 0.$$

**Step 2:** Using the property,  $\sum_{i=1}^n \lambda [a_{ij}^L, a_{ij}^R] = \sum_{i=1}^n [\lambda a_{ij}^L, \lambda a_{ij}^R], \lambda \geq 0$  and

$\sum_{i=1}^n [a_i^L, a_i^R] = \left[ \sum_{i=1}^n a_i^L, \sum_{i=1}^n a_i^R \right]$ , the Problem 1.9.7 can be transformed into Problem 1.9.8.

#### Problem 1.9.8

Maximize  $\{[-3x_1y_1 + 6x_2y_1 + 4x_1y_2 - 7x_2y_2, -1x_1y_1 + 8x_2y_1 + 6x_1y_2 - 5x_2y_2]\}$

Subject to

$$x_1 + x_2 = 1; y_1 + y_2 = 1; x_1, x_2 \geq 0; y_1, y_2 \geq 0.$$

**Step 3:** According to comparing method, to find the optimal solution  $\{x_1^*, x_2^*\}$  of Problem 1.9.8

such that value of  $[-3x_1y_1 + 6x_2y_1 + 4x_1y_2 - 7x_2y_2, -1x_1y_1 + 8x_2y_1 + 6x_1y_2 - 5x_2y_2]$  is maximum

for all  $(y_1, y_2) \in Y$  is equivalent to find  $\{x_1^*, x_2^*\}$  such that value of

$$\alpha(-3x_1y_1 + 6x_2y_1 + 4x_1y_2 - 7x_2y_2) + (1-\alpha)(-1x_1y_1 + 8x_2y_1 + 6x_1y_2 - 5x_2y_2) \quad \text{and} \quad (-3x_1y_1 +$$

$$6x_2y_1 + 4x_1y_2 - 7x_2y_2) \text{ is maximum for all } (y_1, y_2) \in Y \text{ or if it is not possible to find } \{x_1^*, x_2^*\}$$

for which the value of  $\alpha(-3x_{11}y_{11} + 6x_{21}y_{11} + 4x_{11}y_{21} - 7x_{21}y_{21}) +$

$$(1-\alpha)(-1x_{11}y_{11} + 8x_{21}y_{11} + 6x_{11}y_{21} - 5x_{21}y_{21}) \quad \text{and} \quad (-3x_{11}y_{11} + 6x_{21}y_{11} + 4x_{11}y_{21} - 7x_{21}y_{21})$$

is maximum then find such  $\{x_{i1}^*, i=1,2\}$  for which the value of

$$\alpha(-3x_{11}y_{11} + 6x_{21}y_{11} + 4x_{11}y_{21} - 7x_{21}y_{21}) + (1-\alpha)(-1x_{11}y_{11} + 8x_{21}y_{11} + 6x_{11}y_{21} - 5x_{21}y_{21})$$

is maximum but value of  $(-3x_{11}y_{11} + 6x_{21}y_{11} + 4x_{11}y_{21} - 7x_{21}y_{21})$  is not maximum for all

$$(y_{11}, y_{21}) \in Y \quad \text{and to find such } \{x_{i2}^*, i=1,2\} \text{ for which the value of}$$

$$\alpha(-3x_{12}y_{12} + 6x_{22}y_{12} + 4x_{12}y_{22} - 7x_{22}y_{22}) + (1-\alpha)(-1x_{12}y_{12} + 8x_{22}y_{12} + 6x_{12}y_{22} - 5x_{22}y_{22})$$

is not maximum but value of  $(-3x_{12}y_{12} + 6x_{22}y_{12} + 4x_{12}y_{22} - 7x_{22}y_{22})$  is maximum for all  $(y_{12}, y_{22}) \in Y$

i.e., to find optimal solution  $\{x_{i1}^*, i=1,2\}$  and  $\{x_{i2}^*, i=1,2\}$  of Problem 1.9.9 and Problem 1.9.10

respectively.

### Problem 1.9.9

Maximize

$$\left\{ \alpha(-3x_{11}y_{11} + 6x_{21}y_{11} + 4x_{11}y_{21} - 7x_{21}y_{21}) + (1-\alpha)(-1x_{11}y_{11} + 8x_{21}y_{11} + 6x_{11}y_{21} - 5x_{21}y_{21}) \right\}$$

Subject to

$$x_{11} + x_{21} = 1; y_{11} + y_{21} = 1;$$

$$x_{11}, x_{21} \geq 0; y_{11}, y_{21} \geq 0.$$

**Problem 1.9.10**

$$\text{Maximize } \{-3x_{12}y_{12} + 6x_{22}y_{12} + 4x_{12}y_{22} - 7x_{22}y_{22}\}$$

Subject to

$$x_{12} + x_{22} = 1; y_{12} + y_{22} = 1;$$

$$x_{12}, x_{22} \geq 0; y_{12}, y_{22} \geq 0.$$

**Step 4:** Since, in Problem 1.9.9 and Problem 1.9.10 only  $x_{11}, x_{21}$  and  $x_{12}, x_{22}$  respectively have been considered as decision variables. So, Problem 1.9.9 and Problem 1.9.10 are linear programming problem and hence, the optimal value of Problem 1.9.9 and Problem 1.9.10 will be equal to optimal value of its corresponding dual problem i.e., Problem 1.9.11 and Problem 1.9.12 respectively.

**Problem 1.9.11**

$$\text{Minimize } \{\omega_1\}$$

Subject to

$$\alpha(-3y_{11} + 4y_{21}) + (1-\alpha)(-1y_{11} + 6y_{21}) \leq \omega_1;$$

$$\alpha(6y_{11} - 7y_{21}) + (1-\alpha)(8y_{11} - 5y_{21}) \leq \omega_1;$$

$$y_{11} + y_{21} = 1; y_{11}, y_{21} \geq 0.$$

**Problem 1.9.12**

$$\text{Minimize } \{\omega_2\}$$

Subject to

$$-3y_{12} + 4y_{22} \leq \omega_2;$$

$$6y_{12} - 7y_{22} \leq \omega_2;$$

$$y_{12} + y_{22} = 1; y_{12}, y_{22} \geq 0.$$

**Step 5:** The optimal solution of Problem 1.9.11 for all  $\alpha \in [0,1]$  is  $\left\{ y_{11} = \frac{11}{20}, y_{21} = \frac{9}{20} \right\}$  and

the optimal solution of Problem 1.9.12 is  $\left\{ y_{12} = \frac{11}{20}, y_{22} = \frac{9}{20} \right\}$ . On solving Problem 1.9.9 and

Problem 1.9.10 after substituting these values  $\left\{y_{11} = \frac{11}{20}, y_{21} = \frac{9}{20}\right\}$  and  $\left\{y_{12} = \frac{11}{20}, y_{22} = \frac{9}{20}\right\}$

respectively in Problem 1.9.9 and Problem 1.9.10, the obtained alternative basic optimal solutions of Problem 1.9.9 and Problem 1.9.10 are  $\{x_{11}^1 = 0, x_{21}^1 = 1\}, \{x_{11}^2 = 1, x_{21}^2 = 0\}$  and  $\{x_{12}^1 = 0, x_{22}^1 = 1\}, \{x_{12}^2 = 1, x_{22}^2 = 0\}$  respectively.

**Step 6:** Now,  $\text{maximum}\{[-3, -1]x_{11}^1y_{11} + [4, 6]x_{11}^1y_{21} + [6, 8]x_{21}^1y_{11} + [-7, -5]x_{21}^1y_{21}, [-3, -1]x_{11}^2y_{11}$   
 $+ [4, 6]x_{11}^2y_{21} + [6, 8]x_{21}^2y_{11} + [-7, -5]x_{21}^2y_{21}, [-3, -1]x_{12}^1y_{12} + [4, 6]x_{12}^1y_{22} + [6, 8]x_{22}^1y_{12} + [-7, -5]x_{22}^1y_{22},$   
 $[-3, -1]x_{12}^2y_{12} + [4, 6]x_{12}^2y_{22} + [6, 8]x_{22}^2y_{12} + [-7, -5]x_{22}^2y_{22}\} = \text{maximum}\left\{\left[\frac{3}{20}, \frac{43}{20}\right], \left[\frac{3}{20}, \frac{43}{20}\right], \left[\frac{3}{20}, \frac{43}{20}\right]\right.$   
 $\left.\left[\frac{3}{20}, \frac{43}{20}\right]\right\} = \left[\frac{3}{20}, \frac{43}{20}\right].$

**Step 7:** Since, maximum, obtained in Step 6, is  $\left[\frac{3}{20}, \frac{43}{20}\right]$ . So, the maximum expected loss of

Player II is  $\left[\frac{3}{20}, \frac{43}{20}\right]$  and the optimal strategies for Player II are

$$\left\{y_{11} = \frac{11}{20}, y_{21} = \frac{9}{20}\right\} = \left\{y_{12} = \frac{11}{20}, y_{22} = \frac{9}{20}\right\}.$$

## 1.9.2 Existing numerical example considered by Li et al.

In this section, matrix games with interval payoffs  $A = \begin{bmatrix} [175, 190] & [120, 158] \\ [80, 100] & [180, 190] \end{bmatrix}$ ,

considered by Li et al. [79], is solved by the proposed Gaurika method.

### 1.9.2.1 Minimum expected gain of Player I

Using the proposed Gaurika method minimum expected gain of Player I and corresponding optimal strategies, can be obtained as follows:

**Step 1:** Find  $(y_1^*, y_2^*) \in Y$  such that value of  $([175, 190]x_1y_1 + [80, 100]x_2y_1 + [120, 158]x_1y_2 +$

$[180,190]x_2y_2$ ) is minimum for all  $(x_1^*, x_2^*) \in X$  i.e., find the optimal solution of Problem 1.9.13.

**Problem 1.9.13**

Minimize  $\{[175,190]x_1y_1 + [80,100]x_2y_1 + [120,158]x_1y_2 + [180,190]x_2y_2\}$

Subject to

$$x_1 + x_2 = 1; y_1 + y_2 = 1; x_1, x_2 \geq 0; y_1, y_2 \geq 0.$$

**Step 2:** Using the property,  $\sum_{i=1}^n \lambda [a_{ij}^L, a_{ij}^R] = \sum_{i=1}^n [\lambda a_{ij}^L, \lambda a_{ij}^R], \lambda \geq 0$  and

$\sum_{i=1}^n [a_i^L, a_i^R] = \left[ \sum_{i=1}^n a_i^L, \sum_{i=1}^n a_i^R \right]$ , the Problem 1.9.13 can be transformed into Problem 1.9.14.

**Problem 1.9.14**

Minimize  $\{[175x_1y_1 + 80x_2y_1 + 120x_1y_2 + 180x_2y_2, 190x_1y_1 + 100x_2y_1 + 158x_1y_2 + 190x_2y_2]\}$

Subject to

$$x_1 + x_2 = 1; y_1 + y_2 = 1; x_1, x_2 \geq 0; y_1, y_2 \geq 0.$$

**Step 3:** According to comparing method, to find the optimal solution  $\{y_1^*, y_2^*\}$  of Problem

1.9.14 such that value of  $[175x_1y_1 + 80x_2y_1 + 120x_1y_2 + 180x_2y_2, 190x_1y_1 + 100x_2y_1 + 158x_1y_2 +$

$190x_2y_2]$  is minimum for all  $(x_1, x_2) \in X$  is equivalent to find  $\{y_1^*, y_2^*\}$  such that value of

$(1-\alpha)(175x_1y_1 + 80x_2y_1 + 120x_1y_2 + 180x_2y_2) + \alpha(190x_1y_1 + 100x_2y_1 + 158x_1y_2 + 190x_2y_2)$  and

$(190x_1y_1 + 100x_2y_1 + 158x_1y_2 + 190x_2y_2)$  is minimum for all  $(x_1, x_2) \in X$  or if it is not possible

to find  $\{y_1^*, y_2^*\}$  for which the value of  $(1-\alpha)(175x_1y_1 + 80x_2y_1 + 120x_1y_2 + 180x_2y_2) +$

$\alpha(190x_1y_1 + 100x_2y_1 + 158x_1y_2 + 190x_2y_2)$  and  $(190x_1y_1 + 100x_2y_1 + 158x_1y_2 + 190x_2y_2)$  are

minimum then find such  $\{y_{j1}^*, j=1,2\}$  for which the value of

$$(1-\alpha)(175x_{11}y_{11}+80x_{21}y_{11}+120x_{11}y_{21}+180x_{21}y_{21})+\alpha(190x_{11}y_{11}+100x_{21}y_{11}+158x_{11}y_{21}+190x_{21}y_{21})$$

is minimum but value of  $(190x_{11}y_{11}+100x_{21}y_{11}+158x_{11}y_{21}+190x_{21}y_{21})$  is not minimum for all

$(x_{11}, x_{21}) \in X$  and to find such  $\{y_{j2}^*, j=1,2\}$  for which the value of

$$(1-\alpha)(175x_{12}y_{12}+80x_{22}y_{12}+120x_{12}y_{22}+180x_{22}y_{22})+\alpha(190x_{12}y_{12}+100x_{22}y_{12}+158x_{12}y_{22}+190x_{22}y_{22})$$

is not minimum but value of  $(190x_{12}y_{12}+100x_{22}y_{12}+158x_{12}y_{22}+190x_{22}y_{22})$  is minimum for

all  $(x_{12}, x_{22}) \in X$  i.e., to find optimal solution  $\{y_{j1}^*, j=1,2\}$  and  $\{y_{j2}^*, j=1,2\}$  of Problem

1.9.15 and Problem 1.9.16 respectively.

#### **Problem 1.9.15**

Minimize

$$\{(1-\alpha)(175x_{11}y_{11}+80x_{21}y_{11}+120x_{11}y_{21}+180x_{21}y_{21})+\alpha(190x_{11}y_{11}+100x_{21}y_{11}+158x_{11}y_{21}+190x_{21}y_{21})\}$$

Subject to

$$x_{11} + x_{21} = 1; y_{11} + y_{21} = 1; x_{11}, x_{21} \geq 0; y_{11}, y_{21} \geq 0.$$

#### **Problem 1.9.16**

$$\text{Minimize } \{(190x_{12}y_{12}+100x_{22}y_{12}+158x_{12}y_{22}+190x_{22}y_{22})\}$$

Subject to

$$x_{12} + x_{22} = 1; y_{12} + y_{22} = 1; x_{12}, x_{22} \geq 0; y_{12}, y_{22} \geq 0.$$

**Step 4:** Since, in Problem 1.9.15 and Problem 1.9.16 only  $y_{11}, y_{21}$  and  $y_{12}, y_{22}$  respectively have been considered as decision variables. So, Problem 1.9.15 and Problem 1.9.16 are linear programming problem and hence, the optimal value of Problem 1.9.15 and Problem 1.9.16 will be equal to optimal value of its corresponding dual problem i.e., Problem 1.9.17 and Problem 1.9.18 respectively.

**Problem 1.9.17**Maximize  $\{v_1\}$ 

Subject to

$$\begin{aligned} (1-\alpha)(175x_{11} + 80x_{21}) + \alpha(190x_{11} + 100x_{21}) &\geq v_1; \\ (1-\alpha)(120x_{11} + 180x_{21}) + \alpha(158x_{11} + 190x_{21}) &\geq v_1; \\ x_{11} + x_{21} &= 1; x_{11}, x_{21} \geq 0. \end{aligned}$$

**Problem 1.9.18**Maximize  $\{v_2\}$ 

Subject to

$$\begin{aligned} 190x_{12} + 100x_{22} &\geq v_2; \\ 158x_{12} + 190x_{22} &\geq v_2; \\ x_{12} + x_{22} &= 1; x_{12}, x_{22} \geq 0. \end{aligned}$$

**Step 5:** The optimal solution of Problem 1.9.17 and Problem 1.9.18 for  $\alpha = 0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.55, 0.56, 0.58, 0.6, 0.7, 0.8$  and  $0.9$  are shown in second and fourth column of Table 1.3.

On solving the Problem 1.9.15 after substituting the optimal solution, shown in second column of Table 1.3, the obtained alternative basic optimal solutions of Problem 1.9.15 for all chosen values of  $\alpha$  are  $\{y_{11}^1 = 0, y_{21}^1 = 1\}, \{y_{11}^2 = 1, y_{21}^2 = 0\}$  and on solving the Problem 1.9.16 after substituting the optimal solution, shown in fourth column of Table 1.3, the obtained alternative basic optimal solutions of Problem 1.9.16 for all chosen values of  $\alpha$  are  $\{y_{12}^1 = 0, y_{22}^1 = 1\}, \{y_{12}^2 = 1, y_{22}^2 = 0\}$ .

**Step 6:** Now, minimum  $\{[175, 190]x_{11}^\alpha y_{11}^1 + [120, 158]x_{11}^\alpha y_{21}^1 + [80, 100]x_{21}^\alpha y_{11}^1 + [180, 190]x_{21}^\alpha y_{21}^1, [175, 190]x_{11}^\alpha y_{11}^2 + [120, 158]x_{11}^\alpha y_{21}^2 + [80, 100]x_{21}^\alpha y_{11}^2 + [180, 190]x_{21}^\alpha y_{21}^2, [175, 190]x_{12}^\alpha y_{12}^1 + [120, 158]x_{12}^\alpha y_{22}^1 + [80, 100]x_{22}^\alpha y_{12}^1 + [180, 190]x_{22}^\alpha y_{22}^1\}$  at  $\alpha = 0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.55, 0.56, 0.58, 0.6, 0.7, 0.8$  and  $0.9$  are shown in Table 1.3.

**Step 7:** The intervals, shown in third column of Table 1.3, represents the minimum expected gain of Player I corresponding to the optimal strategies for Player I, shown in second column of Table 1.3 and the intervals, shown in fifth column of Table 1.3, represents the minimum expected gain of Player I corresponding to the optimal strategies for Player I are shown in fourth column of Table 1.3.

**Table 1.3** Minimum expected gain of Player I

| $\alpha$ | $\{x_1^*, x_2^*\}$                                    | $[v_1^{L^*}, v_1^{R^*}]$                                 | $\{x_1^*, x_2^*\}$                            | $[v_2^{L^*}, v_2^{R^*}]$                         |
|----------|-------------------------------------------------------|----------------------------------------------------------|-----------------------------------------------|--------------------------------------------------|
| 0        | $\left\{\frac{20}{31}, \frac{11}{31}\right\}$         | $\left[\frac{4380}{31}, \frac{4900}{31}\right]$          | $\left\{\frac{45}{61}, \frac{16}{61}\right\}$ | $\left[\frac{8280}{61}, \frac{10150}{61}\right]$ |
| 0.1      | $\left\{\frac{990}{1517}, \frac{527}{1517}\right\}$   | $\left[\frac{215410}{1517}, \frac{240800}{1517}\right]$  | $\left\{\frac{45}{61}, \frac{16}{61}\right\}$ | $\left[\frac{8280}{61}, \frac{10150}{61}\right]$ |
| 0.2      | $\left\{\frac{35}{53}, \frac{18}{53}\right\}$         | $\left[\frac{7565}{53}, \frac{8450}{53}\right]$          | $\left\{\frac{45}{61}, \frac{16}{61}\right\}$ | $\left[\frac{8280}{61}, \frac{10150}{61}\right]$ |
| 0.3      | $\left\{\frac{970}{1451}, \frac{481}{1451}\right\}$   | $\left[\frac{208230}{1451}, \frac{232400}{1451}\right]$  | $\left\{\frac{45}{61}, \frac{16}{61}\right\}$ | $\left[\frac{8280}{61}, \frac{10150}{61}\right]$ |
| 0.4      | $\left\{\frac{480}{709}, \frac{229}{709}\right\}$     | $\left[\frac{102320}{709}, \frac{114100}{709}\right]$    | $\left\{\frac{45}{61}, \frac{16}{61}\right\}$ | $\left[\frac{8280}{61}, \frac{10150}{61}\right]$ |
| 0.5      | $\left\{\frac{190}{277}, \frac{87}{277}\right\}$      | $\left[\frac{40210}{277}, \frac{44800}{277}\right]$      | $\left\{\frac{45}{61}, \frac{16}{61}\right\}$ | $\left[\frac{8280}{61}, \frac{10150}{61}\right]$ |
| 0.55     | $\left\{\frac{270}{391}, \frac{121}{391}\right\}$     | $\left[\frac{56930}{391}, \frac{63400}{391}\right]$      | $\left\{\frac{45}{61}, \frac{16}{61}\right\}$ | $\left[\frac{8280}{61}, \frac{10150}{61}\right]$ |
| 0.56     | $\left\{\frac{2360}{3413}, \frac{1053}{3413}\right\}$ | $\left[\frac{497240}{3413}, \frac{553700}{3413}\right]$  | $\left\{\frac{45}{61}, \frac{16}{61}\right\}$ | $\left[\frac{8280}{61}, \frac{10150}{61}\right]$ |
| 0.58     | $\left\{\frac{4710}{6793}, \frac{2083}{6793}\right\}$ | $\left[\frac{990890}{6793}, \frac{1103200}{6793}\right]$ | $\left\{\frac{45}{61}, \frac{16}{61}\right\}$ | $\left[\frac{8280}{61}, \frac{10150}{61}\right]$ |
| 0.6      | $\left\{\frac{235}{338}, \frac{103}{338}\right\}$     | $\left[\frac{49365}{338}, \frac{27475}{169}\right]$      | $\left\{\frac{45}{61}, \frac{16}{61}\right\}$ | $\left[\frac{8280}{61}, \frac{10150}{61}\right]$ |
| 0.7      | $\left\{\frac{930}{1319}, \frac{389}{1319}\right\}$   | $\left[\frac{193870}{1319}, \frac{215600}{1319}\right]$  | $\left\{\frac{45}{61}, \frac{16}{61}\right\}$ | $\left[\frac{8280}{61}, \frac{10150}{61}\right]$ |
| 0.8      | $\left\{\frac{460}{643}, \frac{183}{643}\right\}$     | $\left[\frac{95140}{643}, \frac{105700}{643}\right]$     | $\left\{\frac{45}{61}, \frac{16}{61}\right\}$ | $\left[\frac{8280}{61}, \frac{10150}{61}\right]$ |

|     |                                                    |                                                       |                                                 |                                                    |
|-----|----------------------------------------------------|-------------------------------------------------------|-------------------------------------------------|----------------------------------------------------|
| 0.9 | $\left\{ \frac{130}{179}, \frac{49}{179} \right\}$ | $\left[ \frac{26670}{179}, \frac{29600}{179} \right]$ | $\left\{ \frac{45}{61}, \frac{16}{61} \right\}$ | $\left[ \frac{8280}{61}, \frac{10150}{61} \right]$ |
|-----|----------------------------------------------------|-------------------------------------------------------|-------------------------------------------------|----------------------------------------------------|

### 1.9.2.2 Maximum expected loss of Player II

Using the proposed Gaurika method maximum expected loss of Player II and corresponding optimal strategies, can be obtained as follows:

**Step 1:** Find  $(x_1^*, x_2^*) \in X$  such that value of  $([175, 190]x_1y_1 + [80, 100]x_2y_1 + [120, 158]x_1y_2 + [180, 190]x_2y_2)$  is maximum for all  $(y_1^*, y_2^*) \in Y$  i.e., find the optimal solution of Problem

1.9.19.

#### Problem 1.9.19

Maximize  $\{[175, 190]x_1y_1 + [80, 100]x_2y_1 + [120, 158]x_1y_2 + [180, 190]x_2y_2\}$

Subject to

$$x_1 + x_2 = 1; y_1 + y_2 = 1; x_1, x_2 \geq 0; y_1, y_2 \geq 0.$$

**Step 2:** Using the property,  $\sum_{i=1}^n \lambda [a_{ij}^L, a_{ij}^R] = \sum_{i=1}^n [\lambda a_{ij}^L, \lambda a_{ij}^R], \lambda \geq 0$  and

$\sum_{i=1}^n [a_i^L, a_i^R] = \left[ \sum_{i=1}^n a_i^L, \sum_{i=1}^n a_i^R \right]$ , the Problem 1.9.19 can be transformed into Problem 1.9.20.

#### Problem 1.9.20

Maximize  $\{[175x_1y_1 + 80x_2y_1 + 120x_1y_2 + 180x_2y_2, 190x_1y_1 + 100x_2y_1 + 158x_1y_2 + 190x_2y_2]\}$

Subject to

$$x_1 + x_2 = 1; y_1 + y_2 = 1; x_1, x_2 \geq 0; y_1, y_2 \geq 0.$$

**Step 3:** According to comparing method, to find the optimal solution  $\{x_1^*, x_2^*\}$  of Problem 1.9.20

such that value of  $[175x_1y_1 + 80x_2y_1 + 120x_1y_2 + 180x_2y_2, 190x_1y_1 + 100x_2y_1 + 158x_1y_2 +$

$190x_2y_2]$  is maximum for all  $(y_1, y_2) \in Y$  is equivalent to find  $\{x_1^*, x_2^*\}$  such that value of  $\alpha(175x_1y_1 + 80x_2y_1 + 120x_1y_2 + 180x_2y_2) + (1-\alpha)(190x_1y_1 + 100x_2y_1 + 158x_1y_2 + 190x_2y_2)$  and  $(175x_1y_1 + 80x_2y_1 + 120x_1y_2 + 180x_2y_2)$  is maximum for all  $(y_1, y_2) \in Y$  or if it is not possible to find  $\{x_1^*, x_2^*\}$  for which the value of  $\alpha(175x_1y_1 + 80x_2y_1 + 120x_1y_2 + 180x_2y_2) + (1-\alpha)(190x_1y_1 + 100x_2y_1 + 158x_1y_2 + 190x_2y_2)$  and  $(175x_1y_1 + 80x_2y_1 + 120x_1y_2 + 180x_2y_2)$  is maximum then find such  $\{x_{i1}^*, i=1, 2\}$  for which the value of  $\alpha(175x_{11}y_{11} + 80x_{21}y_{11} + 120x_{11}y_{21} + 180x_{21}y_{12}) + (1-\alpha)(190x_{11}y_{11} + 100x_{21}y_{11} + 158x_{11}y_{21} + 190x_{21}y_{21})$  is maximum but value of  $(175x_{11}y_{11} + 80x_{21}y_{11} + 120x_{11}y_{21} + 180x_{21}y_{21})$  is not maximum for all  $(y_{11}, y_{21}) \in Y$  and to find such  $\{x_{i2}^*, i=1, 2\}$  for which the value of  $\alpha(175x_{12}y_{12} + 80x_{22}y_{12} + 120x_{12}y_{22} + 180x_{22}y_{22}) + (1-\alpha)(190x_{12}y_{12} + 100x_{22}y_{12} + 158x_{12}y_{22} + 190x_{22}y_{22})$  is not maximum but value of  $(175x_{12}y_{12} + 80x_{22}y_{12} + 120x_{12}y_{22} + 180x_{22}y_{22})$  is maximum for all  $(y_{12}, y_{22}) \in Y$  i.e., to find optimal solution  $\{x_{i1}^*, i=1, 2\}$  and  $\{x_{i2}^*, i=1, 2\}$  of Problem 1.9.21 and Problem 1.9.22 respectively.

### Problem 1.9.21

Maximize

$$\{\alpha(175x_{11}y_{11} + 80x_{21}y_{11} + 120x_{11}y_{21} + 180x_{21}y_{12}) + (1-\alpha)(190x_{11}y_{11} + 100x_{21}y_{11} + 158x_{11}y_{21} + 190x_{21}y_{21})\}$$

Subject to

$$x_{11} + x_{21} = 1; y_{11} + y_{21} = 1; x_{11}, x_{21} \geq 0; y_{11}, y_{21} \geq 0.$$

### Problem 1.9.22

$$\text{Maximize } \{175x_{12}y_{12} + 80x_{22}y_{12} + 120x_{12}y_{22} + 180x_{22}y_{22}\}$$

Subject to

$$x_{12} + x_{22} = 1; y_{12} + y_{22} = 1; x_{12}, x_{22} \geq 0; y_{12}, y_{22} \geq 0.$$

**Step 4:** Since, in Problem 1.9.21 and Problem 1.9.22 only  $x_{11}, x_{21}$  and  $x_{12}, x_{22}$  respectively have been considered as decision variables. So, Problem 1.9.21 and Problem 1.9.22 are linear programming problem and hence, the optimal value of Problem 1.9.21 and Problem 1.9.22 will be equal to optimal value of its corresponding dual problem i.e., Problem 1.9.23 and Problem 1.9.24 respectively.

**Problem 1.9.23**

Minimize  $\{\omega_1\}$

Subject to

$$\begin{aligned} \alpha(175y_{11} + 120y_{21}) + (1-\alpha)(190y_{11} + 158y_{21}) &\leq \omega_1; \\ \alpha(80y_{11} + 180y_{21}) + (1-\alpha)(100y_{11} + 190y_{21}) &\leq \omega_1; \\ y_{11} + y_{21} &= 1; y_{11}, y_{21} \geq 0. \end{aligned}$$

**Problem 1.9.24**

Minimize  $\{\omega_2\}$

Subject to

$$\begin{aligned} 175y_{12} + 120y_{22} &\leq \omega_2; \\ 80y_{12} + 180y_{22} &\leq \omega_2; \\ y_{12} + y_{22} &= 1; y_{12}, y_{22} \geq 0. \end{aligned}$$

**Step 5:** The optimal solution of Problem 1.9.23 and Problem 1.9.24 for  $\alpha = 0, 0.1, 0.2, 0.3, 0.4, 0.55, 0.56, 0.58, 0.6, 0.7, 0.8$  and  $0.9$  are shown in second and fourth column of Table 1.4. On solving the Problem 1.9.21 after substituting the optimal solution, shown in second column of Table 1.4, the alternative basic optimal solutions of Problem 1.9.15 for all chosen values of  $\alpha$  are  $\{x_{11}^1 = 0, x_{21}^1 = 1\}, \{x_{11}^2 = 1, x_{21}^2 = 0\}$  and on solving the Problem 1.9.22 after substituting the optimal solutions, shown in fourth column of Table 1.4, the alternative basic optimal solutions of Problem 1.9.24 for all chosen values of  $\alpha$  are  $\{x_{12}^1 = 0, x_{22}^1 = 1\}, \{x_{12}^2 = 1, x_{22}^2 = 0\}$ .

**Step 6:** Now, maximum  $\{[175, 190]x_{11}^1 y_{11}^\alpha + [120, 158]x_{11}^1 y_{21}^\alpha + [80, 100]x_{21}^1 y_{11}^\alpha + [180, 190]x_{21}^1 y_{21}^\alpha, [175, 190]x_{11}^2 y_{11}^\alpha + [120, 158]x_{11}^2 y_{21}^\alpha + [80, 100]x_{21}^2 y_{11}^\alpha + [180, 190]x_{21}^2 y_{21}^\alpha, [175, 190]x_{12}^2 y_{12}^\alpha + [120, 158]x_{12}^2 y_{22}^\alpha + [80, 100]x_{22}^2 y_{12}^\alpha + [180, 190]x_{22}^2 y_{22}^\alpha\}$  at  $\alpha = 0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.55, 0.56, 0.58, 0.6, 0.7, 0.8$  and  $0.9$  are shown in Table 1.4.

**Step 7:** The intervals, shown in third column of Table 1.4, represents the maximum expected loss of Player II corresponding to the optimal strategies for Player II, shown in second column of Table 1.4 and the intervals, shown in fifth column of Table 1.4, represents the maximum expected loss of Player II corresponding to the optimal strategies for Player II are shown in fourth column of Table 1.4.

**Table 1.4** Maximum expected loss of Player II

| $\alpha$ | $\{y_1^*, y_2^*\}$                                      | $\left[ \omega_1^{L^*}, \omega_1^{R^*} \right]$             | $\{y_1^*, y_2^*\}$                              | $\left[ \omega_2^{L^*}, \omega_2^{R^*} \right]$   |
|----------|---------------------------------------------------------|-------------------------------------------------------------|-------------------------------------------------|---------------------------------------------------|
| 0        | $\left\{ \frac{16}{61}, \frac{45}{61} \right\}$         | $\left[ \frac{9380}{61}, \frac{10150}{61} \right]$          | $\left\{ \frac{12}{31}, \frac{19}{31} \right\}$ | $\left[ \frac{4380}{31}, \frac{5282}{31} \right]$ |
| 0.1      | $\left\{ \frac{348}{1253}, \frac{905}{1253} \right\}$   | $\left[ \frac{190740}{1253}, \frac{206750}{1253} \right]$   | $\left\{ \frac{12}{31}, \frac{19}{31} \right\}$ | $\left[ \frac{4380}{31}, \frac{5282}{31} \right]$ |
| 0.2      | $\left\{ \frac{188}{643}, \frac{455}{643} \right\}$     | $\left[ \frac{96940}{643}, \frac{105250}{643} \right]$      | $\left\{ \frac{12}{31}, \frac{19}{31} \right\}$ | $\left[ \frac{4380}{31}, \frac{5282}{31} \right]$ |
| 0.3      | $\left\{ \frac{404}{1319}, \frac{915}{1319} \right\}$   | $\left[ \frac{197020}{1319}, \frac{214250}{1319} \right]$   | $\left\{ \frac{12}{31}, \frac{19}{31} \right\}$ | $\left[ \frac{4380}{31}, \frac{5282}{31} \right]$ |
| 0.4      | $\left\{ \frac{54}{169}, \frac{115}{169} \right\}$      | $\left[ \frac{25020}{169}, \frac{27250}{169} \right]$       | $\left\{ \frac{12}{31}, \frac{19}{31} \right\}$ | $\left[ \frac{4380}{31}, \frac{5282}{31} \right]$ |
| 0.5      | $\left\{ \frac{92}{277}, \frac{185}{277} \right\}$      | $\left[ \frac{40660}{277}, \frac{44350}{277} \right]$       | $\left\{ \frac{12}{31}, \frac{19}{31} \right\}$ | $\left[ \frac{4380}{31}, \frac{5282}{31} \right]$ |
| 0.55     | $\left\{ \frac{948}{2803}, \frac{1855}{2803} \right\}$  | $\left[ \frac{409740}{2803}, \frac{447250}{2803} \right]$   | $\left\{ \frac{12}{31}, \frac{19}{31} \right\}$ | $\left[ \frac{4380}{31}, \frac{5282}{31} \right]$ |
| 0.56     | $\left\{ \frac{149}{439}, \frac{290}{439} \right\}$     | $\left[ \frac{64120}{439}, \frac{70000}{439} \right]$       | $\left\{ \frac{12}{31}, \frac{19}{31} \right\}$ | $\left[ \frac{4380}{31}, \frac{5282}{31} \right]$ |
| 0.58     | $\left\{ \frac{2412}{7057}, \frac{4645}{7057} \right\}$ | $\left[ \frac{1029060}{7057}, \frac{1123750}{7057} \right]$ | $\left\{ \frac{12}{31}, \frac{19}{31} \right\}$ | $\left[ \frac{4380}{31}, \frac{5282}{31} \right]$ |

|     |                                                       |                                                           |                                                 |                                                   |
|-----|-------------------------------------------------------|-----------------------------------------------------------|-------------------------------------------------|---------------------------------------------------|
| 0.6 | $\left\{ \frac{244}{709}, \frac{465}{709} \right\}$   | $\left[ \frac{103220}{709}, \frac{112750}{709} \right]$   | $\left\{ \frac{12}{31}, \frac{19}{31} \right\}$ | $\left[ \frac{4380}{31}, \frac{5282}{31} \right]$ |
| 0.7 | $\left\{ \frac{516}{1451}, \frac{935}{1451} \right\}$ | $\left[ \frac{209580}{1451}, \frac{229250}{1451} \right]$ | $\left\{ \frac{12}{31}, \frac{19}{31} \right\}$ | $\left[ \frac{4380}{31}, \frac{5282}{31} \right]$ |
| 0.8 | $\left\{ \frac{136}{371}, \frac{235}{371} \right\}$   | $\left[ \frac{53180}{371}, \frac{58250}{371} \right]$     | $\left\{ \frac{12}{31}, \frac{19}{31} \right\}$ | $\left[ \frac{4380}{31}, \frac{5282}{31} \right]$ |
| 0.9 | $\left\{ \frac{572}{1517}, \frac{945}{1517} \right\}$ | $\left[ \frac{215860}{1517}, \frac{236750}{1517} \right]$ | $\left\{ \frac{12}{31}, \frac{19}{31} \right\}$ | $\left[ \frac{4380}{31}, \frac{5282}{31} \right]$ |

### 1.10 Conclusion

On the basis of present study, it can be concluded that some mathematically incorrect assumptions have been considered in the existing methods [37,68,69,79,84,100] for solving matrix games with interval payoffs. Therefore, it is not genuine to use these methods. Furthermore, to resolve flaws of the existing methods [37,68,69,79,84,100], a new method (named as Gaurika method) is proposed for solving matrix games with interval payoffs.

# Chapter 2

## Mehar method for solving matrix games with fuzzy payoffs<sup>†</sup>

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In this chapter, flaws of the existing methods [36,65,70,71,83] for solving matrix game with fuzzy payoffs (matrix games in which payoffs are represented as fuzzy numbers) are pointed out. To resolve these flaws, a new method (named as Mehar method) is also proposed to obtain the optimal strategies as well as minimum expected gain of Player I and maximum expected loss of Player II for matrix games with fuzzy payoffs. To illustrate the proposed Mehar method, the existing numerical problems of matrix games with fuzzy payoffs are solved by the proposed Mehar method.

### 2.1 Matrix games with fuzzy payoffs

In an interval the membership value of every element belongs to the set is equal so it is not realistic to represent the payoffs by intervals e.g., Let, 'A' grade has been assigned to those students who secured marks between 90 to 100 in an examination, so the range of grade 'A' if represented in the interval will be [90,100], which indicates that the student who secured 90.5 marks and who secured 99.5 marks are equally efficient. But in general, it is not so. The student who secured 99.5 marks is more efficient than the student who secured 90.5 marks.

Zadeh [128] proposed an interesting generalization of interval called fuzzy sets to capture this aspect of human behavior. In the literature [15,28-30,36,43,65,70,71,73,83] fuzzy numbers have been used to represent the payoffs and such matrix games in which payoffs are represented in matrix games by fuzzy numbers are named as matrix games with fuzzy payoffs.

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<sup>†</sup> The contents of this chapter are communicated in *European Journal of Operational Research*.

## 2.2 Preliminaries

In this section, some basic definitions, arithmetic operations of fuzzy numbers and method for comparing fuzzy numbers is presented [42].

### 2.2.1 Some basic definitions

In this section, some basic definitions are reviewed [42].

**Definition 2.1** Let,  $X$  be a classical set of objects. Then, the set of ordered pairs

$\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) : x \in X\}$ , where  $\mu_{\tilde{A}} : X \rightarrow [0, 1]$ , is called a fuzzy set in  $X$ . The evaluation function  $\mu_{\tilde{A}}$  is called the membership function.

**Definition 2.2** Let,  $\tilde{A}$  be a fuzzy set in  $X$  and  $\alpha \in (0, 1]$  be a real number. Then,  $\alpha$ -cut of the fuzzy set  $\tilde{A}$  is the crisp set  $A(\alpha) = \{x \in X : \mu_{\tilde{A}}(x) \geq \alpha\}$ .

**Definition 2.3** Let,  $\tilde{A}$  be a fuzzy set in  $X$ . Then, the support of  $\tilde{A}$  is the crisp set given by  $\{x \in X : \mu_{\tilde{A}}(x) > 0\}$ .

**Definition 2.4** Let,  $\tilde{A}$  be a fuzzy set in  $X$ . Then, the fuzzy set  $\tilde{A}$  is said to be normal if  $\sup_{x \in X} \mu_{\tilde{A}}(x) = 1$ .

**Definition 2.5** A fuzzy set  $\tilde{A}$  in set of real numbers  $\mathbb{R}$  is called a fuzzy number if it satisfies the following conditions:

- (i)  $\tilde{A}$  is normal,
- (ii)  $A(\alpha)$  is a closed interval for every  $\alpha \in (0, 1]$ ,
- (iii) The support of  $\tilde{A}$  is bounded.

**Definition 2.6** A fuzzy number  $\tilde{A} = (a^L(0), a(1), a^R(0))$  is called a triangular fuzzy number if its membership function  $\mu_{\tilde{A}}$  is given by

$$\mu_{\tilde{A}}(x) = \begin{cases} \frac{x - a^L(0)}{a(1) - a^L(0)}, & a^L(0) \leq x < a(1), \\ \frac{x - a^R(0)}{a(1) - a^R(0)}, & a(1) \leq x \leq a^R(0), \\ 0, & x < a^L(0), x > a^R(0). \end{cases}$$

Furthermore, the  $\alpha$ -cut of the triangular fuzzy number  $\tilde{A}$  is the closed interval

$$A(\alpha) = \left[ (a(1) - a^L(0))\alpha + a^L(0), (a(1) - a^R(0))\alpha + a^R(0) \right].$$

**Definition 2.7** A fuzzy number  $\tilde{A} = (a^L(0), a^L(1), a^R(1), a^R(0))$  is called a trapezoidal fuzzy number if its membership function  $\mu_{\tilde{A}}$  is given by

$$\mu_{\tilde{A}}(x) = \begin{cases} \frac{x - a^L(0)}{a^L(1) - a^L(0)}, & a^L(0) \leq x < a^L(1), \\ 1, & a^L(1) \leq x \leq a^R(1), \\ \frac{x - a^R(0)}{a^R(1) - a^R(0)}, & a^R(1) < x \leq a^R(0), \\ 0, & x < a^L(0), x > a^R(0). \end{cases}$$

Furthermore, the  $\alpha$ -cut of the trapezoidal fuzzy number  $\tilde{A}$  is the closed interval

$$A^\alpha = \left[ (a^L(1) - a^L(0))\alpha + a^L(0), (a^R(1) - a^R(0))\alpha + a^R(0) \right].$$

### 2.2.2 Arithmetic operations of trapezoidal fuzzy numbers

In this section, some arithmetic operations of two trapezoidal fuzzy numbers, defined on universal set of real numbers  $\mathbb{R}$ , are presented [55,56].

(i) Let,  $\tilde{A}_1 = (a_1^L(0), a_1^L(1), a_1^R(1), a_1^R(0))$  and  $\tilde{A}_2 = (a_2^L(0), a_2^L(1), a_2^R(1), a_2^R(0))$  be two trapezoidal fuzzy numbers. Then,  $\tilde{A}_1 + \tilde{A}_2 = (a_1^L(0) + a_2^L(0), a_1^L(1) + a_2^L(1), a_1^R(1) + a_2^R(1), a_1^R(0) + a_2^R(0))$ .

(ii) Let,  $\tilde{A}_1 = (a_1^L(0), a_1^L(1), a_1^R(1), a_1^R(0))$  be a trapezoidal fuzzy number and  $\lambda$  be a real number.

$$\text{Then, } \lambda \tilde{A}_1 = \begin{cases} (\lambda a_1^L(0), \lambda a_1^L(1), \lambda a_1^R(1), \lambda a_1^R(0)), & \lambda \geq 0, \\ (\lambda a_1^R(0), \lambda a_1^R(1), \lambda a_1^L(1), \lambda a_1^L(0)), & \lambda < 0. \end{cases}$$

(iii) Let,  $\tilde{A}_1 = (a_1^L(0), a_1^L(1), a_1^R(1), a_1^R(0))$  and  $\tilde{A}_2 = (a_2^L(0), a_2^L(1), a_2^R(1), a_2^R(0))$  be two trapezoidal fuzzy numbers. Then,  $\tilde{A}_1 - \tilde{A}_2 = (a_1^L(0) - a_2^R(0), a_1^L(1) - a_2^R(1), a_1^R(1) - a_2^L(1), a_1^R(0) - a_2^L(0))$ .

### 2.2.3 Comparison of fuzzy numbers

If  $a$  and  $b$  are two distinct real numbers then it can be easily verified that  $a > b$  or  $a < b$ .

However, if  $\tilde{A}_1$  and  $\tilde{A}_2$  are two fuzzy numbers then there is no unique way to verify  $\tilde{A}_1 \succ \tilde{A}_2$  or  $\tilde{A}_1 \prec \tilde{A}_2$ . In this section, the method, used by the authors [36,65,70,71,83] to compare two trapezoidal fuzzy numbers, is presented.

Let,  $\tilde{A}_1 = (a_1^L(0), a_1^L(1), a_1^R(1), a_1^R(0))$  and  $\tilde{A}_2 = (a_2^L(0), a_2^L(1), a_2^R(1), a_2^R(0))$  be two trapezoidal fuzzy numbers. Then,

$$(i) \tilde{A}_1 \succeq \tilde{A}_2 \Leftrightarrow a_1^L(0) \geq a_2^L(0), a_1^L(1) \geq a_2^L(1), a_1^R(1) \geq a_2^R(1), a_1^R(0) \geq a_2^R(0).$$

$$(ii) \tilde{A}_1 = \tilde{A}_2 \Leftrightarrow a_1^L(0) = a_2^L(0), a_1^L(1) = a_2^L(1), a_1^R(1) = a_2^R(1), a_1^R(0) = a_2^R(0).$$

### 2.3 Existing mathematical formulation of matrix games with fuzzy payoffs

In the literature, Problem 2.3.1 and Problem 2.3.2 have been used to obtain the minimum expected gain and maximum expected loss of Player II as well as their corresponding optimal strategies of Player II. These problem have been obtained by using the same procedure, discussed in Section 1.6.1 of Chapter 1, just by replacing the intervals  $[a_{ij}^L, a_{ij}^R]$ ,  $[\nu^L, \nu^R]$  and  $[\omega^L, \omega^R]$  with fuzzy numbers  $\tilde{a}_{ij}, \tilde{\nu}$  and  $\tilde{\omega}$  respectively.

#### Problem 2.3.1

Maximize  $\{\tilde{\nu}\}$

Subject to

#### Problem 2.3.2

Minimize  $\{\tilde{\omega}\}$

Subject to

$$\begin{aligned}\sum_{i=1}^m \tilde{a}_{ij} x_i &\succeq \tilde{v}, \quad j=1, 2, \dots, n; \\ \sum_{i=1}^m x_i &= 1; x_i \geq 0, \quad i=1, 2, \dots, m.\end{aligned}$$

$$\begin{aligned}\sum_{j=1}^n \tilde{a}_{ij} y_j &\preceq \tilde{w}, \quad i=1, 2, \dots, m; \\ \sum_{j=1}^n y_j &= 1; y_j \geq 0, \quad j=1, 2, \dots, n.\end{aligned}$$

## 2.4 Literature review of matrix games with fuzzy payoffs

In this section, a brief review of the methods, proposed in the literature in last ten years for solving matrix games with triangular/trapezoidal fuzzy payoffs, is presented.

Liu & Kao [83] firstly transformed the Problems 2.3.1 and Problem 2.3.2 into Problem 2.4.1 and Problem 2.4.2 respectively, by replacing the trapezoidal fuzzy numbers  $\tilde{a}_{ij}, \tilde{v}$  and  $\tilde{w}$  with its  $\alpha$ -cuts  $[a_{ij}^L(\alpha), a_{ij}^R(\alpha)]$ ,  $[\nu^L(\alpha), \nu^R(\alpha)]$  and  $[\omega^L(\alpha), \omega^R(\alpha)]$  respectively.

### Problem 2.4.1

Maximize  $\{[\nu^L(\alpha), \nu^R(\alpha)]\}$

Subject to

$$\begin{aligned}\sum_{i=1}^m [a_{ij}^L(\alpha), a_{ij}^R(\alpha)] x_i &\geq \\ &[\nu^L(\alpha), \nu^R(\alpha)], \quad j=1, 2, \dots, n; \\ \sum_{i=1}^m x_i &= 1; \\ x_i &\geq 0, \quad i=1, 2, \dots, m.\end{aligned}$$

### Problem 2.4.2

Minimize  $\{[\omega^L(\alpha), \omega^R(\alpha)]\}$

Subject to

$$\begin{aligned}\sum_{j=1}^n [a_{ij}^L(\alpha), a_{ij}^R(\alpha)] y_j &\leq \\ &[\omega^L(\alpha), \omega^R(\alpha)], \quad i=1, 2, \dots, m; \\ \sum_{j=1}^n y_j &= 1; \\ y_j &\geq 0, \quad j=1, 2, \dots, n.\end{aligned}$$

Then, Liu & Kao [83] used the existing method [84], discussed in Section 1.2 of Chapter 1, to obtain the optimal solution  $\{x_i, i=1, 2, \dots, m; \nu^L(\alpha), \nu^R(\alpha)\}$  and  $\{y_j, j=1, 2, \dots, n; \omega^L(\alpha), \omega^R(\alpha)\}$  of Problem 2.4.1 and Problem 2.4.2 respectively corresponding to different value of  $\alpha$ . Finally, Liu & Kao [83] used the optimal values  $\nu^L(\alpha), \nu^R(\alpha)$  and  $\omega^L(\alpha), \omega^R(\alpha)$  to obtain the fuzzy numbers  $\tilde{v}$  and  $\tilde{w}$ , representing the minimum expected gain of Player I and maximum expected loss of Player II respectively.

Since, on solving matrix games with fuzzy payoffs by using the existing method [84],

optimal strategies of Player I and Player II are not obtained. Therefore, it is not genuine to use Liu & Kao' [83] method to find the solution of matrix games with fuzzy payoffs.

Li [65] transformed the Problem 2.3.1 and Problem 2.3.2 into Problem 2.4.3 and Problem 2.4.4 respectively.

**Problem 2.4.3**

$$\text{Maximize } \left\{ \frac{\nu^L(0) + \nu^R(0)}{2} \right\}$$

Subject to

$$\sum_{i=1}^m a_{ij}^L(0)x_i \geq \nu^L(0), \quad j = 1, 2, \dots, n;$$

$$\sum_{i=1}^m a_{ij}(1)x_i \geq \nu(1), \quad j = 1, 2, \dots, n;$$

$$\sum_{i=1}^m a_{ij}^R(0)x_i \geq \nu^R(0), \quad j = 1, 2, \dots, n;$$

$$\nu^L(0) \geq \nu^{L^0}(0);$$

$$\nu^R(0) \geq \nu^{R^0}(0);$$

$$\sum_{i=1}^m x_i = 1;$$

$$x_i \geq 0, \quad i = 1, 2, \dots, m.$$

**Problem 2.4.4**

$$\text{Minimize } \left\{ \frac{\omega^R(0) + \omega^L(0)}{2} \right\}$$

Subject to

$$\sum_{j=1}^n a_{ij}^L(0)y_j \leq \omega^L(0), \quad i = 1, 2, \dots, m;$$

$$\sum_{j=1}^n a_{ij}(1)y_j \leq \omega(1), \quad i = 1, 2, \dots, m;$$

$$\sum_{j=1}^n a_{ij}^R(0)y_j \leq \omega^R(0), \quad i = 1, 2, \dots, m;$$

$$\omega^L(0) \leq \omega^{L^0}(0);$$

$$\omega^R(0) \leq \omega^{R^0}(0);$$

$$\sum_{j=1}^n y_j = 1;$$

$$y_j \geq 0, \quad j = 1, 2, \dots, n.$$

where,  $\nu^{L^0}(0), \nu^0(1)$  and  $\nu^{R^0}(0)$  are the optimal solution of Problem 2.4.5 and  $\omega^{L^0}(0), \omega^0(1)$

and  $\omega^{R^0}(0)$  are the optimal solution of Problem 2.4.6.

**Problem 2.4.5**

$$\text{Maximize } \{\nu(1)\}$$

Subject to

**Problem 2.4.6**

$$\text{Minimize } \{\omega(1)\}$$

Subject to

$$\begin{aligned}
\sum_{i=1}^m a_{ij}^L(0)x_i &\geq v^L(0), \quad j = 1, 2, \dots, n; \\
\sum_{i=1}^m a_{ij}(1)x_i &\geq v(1), \quad j = 1, 2, \dots, n; \\
\sum_{i=1}^m a_{ij}^R(0)x_i &\geq v^R(0), \quad j = 1, 2, \dots, n; \\
\sum_{i=1}^m x_i &= 1; x_i \geq 0, \quad i = 1, 2, \dots, m.
\end{aligned}$$

$$\begin{aligned}
\sum_{j=1}^n a_{ij}^L(0)y_j &\leq \omega^L(0), \quad i = 1, 2, \dots, m; \\
\sum_{j=1}^n a_{ij}(1)y_j &\leq \omega(1), \quad i = 1, 2, \dots, m; \\
\sum_{j=1}^n a_{ij}^R(0)y_j &\leq \omega^R(0), \quad i = 1, 2, \dots, m; \\
\sum_{j=1}^n y_j &= 1; y_j \geq 0, \quad j = 1, 2, \dots, n.
\end{aligned}$$

Li [65] claimed that the optimal solution  $\{x_i, i = 1, 2, \dots, m\}$  of Problem 2.4.3 will be the optimal strategies of Player I and using the optimal solution  $\{v^L(0), v(1), v^R(0)\}$  of Problem 2.4.3, the triangular fuzzy number  $(v^L(0), v(1), v^R(0))$ , representing the minimum expected gain of Player I, can be obtained. Also, the optimal solution  $\{y_j, j = 1, 2, \dots, n\}$  of Problem 2.4.4 will be the optimal strategies of Player II and using the optimal solution  $\{\omega^L(0), \omega(1), \omega^R(0)\}$  of Problem 2.4.4, the triangular fuzzy number  $(\omega^L(0), \omega(1), \omega^R(0))$ , representing the maximum expected loss of Player II, can be obtained.

Clemente et al. [36] transformed the Problem 2.3.1 and Problem 2.3.2 into Problem 2.4.7 and Problem 2.4.8 respectively.

#### Problem 2.4.7

Maximize  $\{(v^L(0), v(1), v^R(0))\}$

Subject to

$$\begin{aligned}
\sum_{i=1}^m a_{ij}^L(0)x_i &\geq v^L(0), \quad j = 1, 2, \dots, n; \\
\sum_{i=1}^m a_{ij}(1)x_i &\geq v(1), \quad j = 1, 2, \dots, n; \\
\sum_{i=1}^m a_{ij}^R(0)x_i &\geq v^R(0), \quad j = 1, 2, \dots, n; \\
\sum_{i=1}^m x_i &= 1; x_i \geq 0, \quad i = 1, 2, \dots, m.
\end{aligned}$$

#### Problem 2.4.8

Minimize  $\{(\omega^L(0), \omega(1), \omega^R(0))\}$

Subject to

$$\begin{aligned}
\sum_{j=1}^n a_{ij}^L(0)y_j &\leq \omega^L(0), \quad i = 1, 2, \dots, m; \\
\sum_{j=1}^n a_{ij}(1)y_j &\leq \omega(1); \\
\sum_{j=1}^n a_{ij}^R(0)y_j &\leq \omega^R(0), \quad i = 1, 2, \dots, m; \\
\sum_{j=1}^n y_j &= 1; y_j \geq 0, \quad j = 1, 2, \dots, n.
\end{aligned}$$

Clemente et al. [36] claimed that the Pareto optimal solution  $\{x_i, i = 1, 2, \dots, m\}$  of Problem 2.4.7 represents the Pareto optimal security strategy and the corresponding triangular fuzzy number  $(\nu^L(0), \nu(1), \nu^R(0))$  represents the security level (minimum expected gain of Player I). Also, the Pareto optimal solution  $\{y_j, j = 1, 2, \dots, n\}$  of Problem 2.4.8 represents the Pareto optimal security strategy and the corresponding triangular fuzzy number  $(\omega^L(0), \omega(1), \omega^R(0))$  represents the security level (maximum expected loss of Player II).

Li [70] pointed out that on solving matrix games with fuzzy payoffs, the obtained minimum expected gain of Player I, represented by a fuzzy number, should be equal to the obtained maximum expected loss of Player II. While, for the minimum expected gain of Player I and maximum expected loss of Player II, obtained on solving matrix games with fuzzy payoffs by using the existing methods [15,65,70,83], this condition is not satisfying. To resolve this fundamental error of the existing methods [15,65,83], Li [70] split Problem 2.3.1 into three independent problems, Problem 2.4.9, Problem 2.4.10 and Problem 2.4.11 as well as split Problem 2.3.2 into three independent problems, Problem 2.4.12, Problem 2.4.13 and Problem 2.4.14.

**Problem 2.4.9**

Maximize  $\{\nu^L(0)\}$

Subject to

$$\begin{aligned} \sum_{i=1}^m a_{ij}^L(0)x_i &\geq \nu^L(0), \quad j = 1, 2, \dots, n; \\ \sum_{i=1}^m x_i &= 1; \\ x_i &\geq 0, \quad i = 1, 2, \dots, m. \end{aligned}$$

**Problem 2.4.11**

Maximize  $\{\nu^R(0)\}$

**Problem 2.4.10**

Maximize  $\{\nu(1)\}$

Subject to

$$\begin{aligned} \sum_{i=1}^m a_{ij}(1)x_i &\geq \nu(1), \quad j = 1, 2, \dots, n; \\ \sum_{i=1}^m x_i &= 1; \\ x_i &\geq 0, \quad i = 1, 2, \dots, m. \end{aligned}$$

**Problem 2.4.12**

Minimize  $\{\omega^L(0)\}$

Subject to

$$\sum_{i=1}^m a_{ij}^R(0)x_i \geq v^R(0), j = 1, 2, \dots, n;$$

$$\sum_{i=1}^m x_i = 1; x_i \geq 0, i = 1, 2, \dots, m.$$

**Problem 2.4.13**

Minimize  $\{\omega(1)\}$

Subject to

$$\sum_{j=1}^n a_{ij}(1)y_j \leq \omega(1), i = 1, 2, \dots, m;$$

$$\sum_{j=1}^n y_j = 1; y_j \geq 0, j = 1, 2, \dots, n.$$

Subject to

$$\sum_{j=1}^n a_{ij}^L(0)y_j \leq \omega^L(0), i = 1, 2, \dots, m;$$

$$\sum_{j=1}^n y_j = 1; y_j \geq 0, j = 1, 2, \dots, n.$$

**Problem 2.4.14**

Minimize  $\{\omega^R(0)\}$

Subject to

$$\sum_{j=1}^n a_{ij}^R(0)y_j \leq \omega^R(0), i = 1, 2, \dots, m;$$

$$\sum_{j=1}^n y_j = 1; y_j \geq 0, j = 1, 2, \dots, n.$$

Li [70] claimed that if  $v^{L*}(0), v^*(1), v^{R*}(0), \omega^{L*}(0), \omega^*(1)$  and  $\omega^{R*}(0)$  are the optimal values of Problem 2.4.9, Problem 2.4.10, Problem 2.4.11, Problem 2.4.12, Problem 2.4.13 and Problem 2.4.14 respectively then  $(v^{L*}(0), v^*(1), v^{R*}(0))$  and  $(\omega^{L*}(0), \omega^*(1), \omega^{R*}(0))$  represents the minimum expected gain of Player I and the maximum expected loss of Player II respectively.

Dutta and Gupta [43] extended the existing models [85] by representing payoffs as trapezoidal fuzzy numbers instead of triangular fuzzy numbers as well as proposed the existence of equilibrium strategies for these models. To find the solution of matrix games with trapezoidal fuzzy numbers, Dutta and Gupta [43] firstly transformed a matrix games with fuzzy payoffs  $(a_{ij}^L(0), a_{ij}^L(1), a_{ij}^R(1), a_{ij}^R(0))$  into a parametric bi-matrix games with  $(1-\lambda)a_{ij}^L(0) + \lambda a_{ij}^L(1)$  as gains of Player I and  $(1-\mu)a_{ij}^R(1) + \mu a_{ij}^R(0)$  as losses of Player II, where  $\lambda, \mu \in [0, 1]$ . Then, Dutta and Gupta [43] obtained the Pareto Nash equilibrium strategies  $\{x_i, i = 1, 2, \dots, m\}$  and  $\{y_j, j = 1, 2, \dots, n\}$  of Player I and Player II by solving the inequalities

$$\sum_{j=1}^n \left[ (1-\lambda) a_{pj}^L(0) + \lambda a_{pj}^L(1) \right] y_j \leq \sum_{i=1}^m \sum_{j=1}^n x_i \left[ (1-\lambda) a_{ij}^L(0) + \lambda a_{ij}^L(1) \right], p=1,2,\dots,m,$$

$$\sum_{i=1}^m \sum_{j=1}^n x_i \left[ (1-\mu) a_{ij}^R(1) + \mu a_{ij}^R(0) \right] y_j \leq \sum_{i=1}^m x_i \left[ (1-\mu) a_{qi}^R(1) + \mu a_{qi}^R(0) \right] y_j, q=1,2,\dots,n \quad \text{and} \quad \text{the}$$

expected payoffs by using the expressions  $\sum_{i=1}^m \sum_{j=1}^n x_i \left[ (1-\lambda) a_{ij}^L(0) + \lambda a_{ij}^R(0) \right]$  and

$$\sum_{i=1}^m \sum_{j=1}^n x_i \left[ (1-\mu) a_{ij}^L(0) + \mu a_{ij}^R(0) \right] y_j.$$

Li [71] used the existing method [70] for solving such matrix games in which payoffs are represented by trapezoidal fuzzy numbers.

## 2.5 Flaws of the existing methods

In this section, flaws of the existing methods [36,65,70,71,83] are pointed out.

1. Liu and Kao [83] solved the Problem 2.5.1 [83, Section 4, pp. 899] and Problem 2.5.2 [83, Section 4, pp. 900] to illustrate their proposed method.

### Problem 2.5.1 [83, Section 4, pp. 899]

Maximize  $(\tilde{v})$

Subject to

$$\begin{aligned} (26, 27, 28, 30)x_1 - 28x_2 + (-27, -25, -25, -22)x_3 + 30x_4 &\geq \tilde{v}, \\ -10x_1 + 20x_2 + (10, 11, 12, 14)x_3 + (-38, -36, -35, -34)x_4 &\geq \tilde{v}, \\ -20x_1 - 10x_2 + 20x_3 + (-34, -32, -32, -30)x_4 &\geq \tilde{v}, \\ (-3, 0, 0, 3)x_1 + (32, 34, 35, 37)x_2 - 30x_3 + (32, 33, 34, 36)x_4 &\geq \tilde{v}, \\ x_1 + x_2 + x_3 + x_4 &= 1, \\ x_1, x_2, x_3, x_4 &\geq 0, \tilde{v} \text{ unrestricted in sign.} \end{aligned}$$

### Problem 2.5.2 [83, Section 4, pp. 900]

Minimize  $(\tilde{u})$

Subject to

$$(26, 27, 28, 30)y_1 - 10y_2 - 20y_3 + (-3, 0, 0, 3)y_4 \leq \tilde{u},$$

$$\begin{aligned}
& -28y_1 + 20y_2 - 10y_3 + (32, 34, 35, 37)y_4 \leq \tilde{u}, \\
& (-27, -25, -25, -22)y_1 + (10, 11, 12, 14)y_2 + 20y_3 - 30y_4 \leq \tilde{u}, \\
& 30y_1 + (-38, -36, -35, -34)y_2 + (-34, -32, -32, -30)y_3 + (32, 33, 34, 36)y_4 \leq \tilde{u}, \\
& y_1 + y_2 + y_3 + y_4 = 1, y_1, y_2, y_3, y_4 \geq 0, \tilde{u} \text{ unrestricted in sign.}
\end{aligned}$$

In the optimal solution of the Problem 2.5.1 [83, Section 4, pp. 899], the variables  $x_1, x_2, x_3$  and  $x_4$  should always be real numbers i.e., lower bound and upper bound of the  $\alpha$  - cut of variables  $x_1, x_2, x_3$  and  $x_4$  corresponding to different values of  $\alpha$  - should be equal. However, it is obvious from the optimal values of variables  $x_1, x_2, x_3$  and  $x_4$  obtained by Liu and Kao [83] which are shown in Table 2.1, that the lower bound and upper bound of the  $\alpha$  - cut of variables  $x_1, x_2, x_3$  and  $x_4$  corresponding to  $\alpha = 0$  and  $\alpha = 1$  are not equal i.e., the optimal values of variables  $x_1, x_2, x_3$  and  $x_4$ , obtained by Liu & Kao [83], are not real numbers.

Similarly, in the optimal solution of the Problem 2.5.2 [83, Section 4, pp. 900] the variables  $y_1, y_2, y_3$  and  $y_4$  should always be real numbers i.e., lower bound and upper bound of the  $\alpha$  -cut of variables  $y_1, y_2, y_3$  and  $y_4$  corresponding to different values of  $\alpha$  should be equal. However, it is obvious from the optimal values of variables  $y_1, y_2, y_3$  and  $y_4$ , obtained by Liu and Kao [83] which are shown in Table 2.2, that the lower bound and upper bound of the  $\alpha$  -cut of variables  $y_1, y_2, y_3$  and  $y_4$  corresponding to  $\alpha = 0$  and  $\alpha = 1$  are not equal i.e., the optimal values of variables  $y_1, y_2, y_3$  and  $y_4$ , obtained Liu & Kao [83], are not real numbers.

Hence, neither the optimal solution, obtained by Liu and Kao [83] which is shown in Table 2.1, is the optimal solution of Problem 2.5.1 [83, Section 4, pp. 899] nor the optimal solution, obtained by Liu and Kao [83] which is shown in Table 2.2, is the optimal solution of the Problem 2.5.2 [83, Section 4, pp. 900].

**Table 2.1** Lower bound and upper bound of the  $\alpha$  – cut of variables [83, pp. 901]

| Lower bound of $\alpha$ – cut $(x_i^L(\alpha))$ | Upper bound of $\alpha$ – cut $(x_i^R(\alpha))$ |
|-------------------------------------------------|-------------------------------------------------|
| $x_1^L(0) = 0.2045$                             | $x_1^R(0) = 0.1331$                             |
| $x_2^L(0) = 0.1144$                             | $x_2^R(0) = 0.0925$                             |
| $x_3^L(0) = 0.4589$                             | $x_3^R(0) = 0.5045$                             |
| $x_4^L(0) = 0.2222$                             | $x_4^R(0) = 0.2699$                             |
| $x_1^L(1) = 0.1890$                             | $x_1^R(1) = 0.1690$                             |
| $x_2^L(1) = 0.1092$                             | $x_2^R(1) = 0.0977$                             |
| $x_3^L(1) = 0.4713$                             | $x_3^R(1) = 0.4843$                             |
| $x_4^L(1) = 0.2305$                             | $x_4^R(1) = 0.2490$                             |

**Table 2.5.2** Lower bound and upper bound of the  $\alpha$  – cut of variables [83, pp. 901]

| Lower bound of $\alpha$ – cut $(y_j^L(\alpha))$ | Upper bound of $\alpha$ – cut $(y_j^R(\alpha))$ |
|-------------------------------------------------|-------------------------------------------------|
| $y_1^L(0) = 0.2621$                             | $y_1^R(0) = 0.2514$                             |
| $y_2^L(0) = 0.0750$                             | $y_2^R(0) = 0.1047$                             |
| $y_3^L(0) = 0.4521$                             | $y_3^R(0) = 0.4357$                             |
| $y_4^L(0) = 0.2108$                             | $y_4^R(0) = 0.2082$                             |
| $y_1^L(1) = 0.2580$                             | $y_1^R(1) = 0.2568$                             |
| $y_2^L(1) = 0.00887$                            | $y_2^R(1) = 0.0880$                             |
| $y_3^L(1) = 0.4451$                             | $y_3^R(1) = 0.4476$                             |
| $y_4^L(1) = 0.2082$                             | $y_4^R(1) = 0.2076$                             |

2. Clemente et al. [36] have used the relation,  $(a^L(0), a(1), a^R(0)) \preceq (b^L(0), b(1), b^R(0)) \Rightarrow a^L(0) \leq b^L(0), a(1) \leq b(1), a^R(0) \leq b^R(0)$  for transforming the fuzzy constraints of Problem 2.3.1 into crisp constraints of Problem 2.4.7. Therefore, the same relation should be used to find the optimal solution of Problem 2.4.7. However, Clemente et al. [36] have claimed that the Problem 2.4.7 is a multi-objective problem and can be solved by any of the existing methods. As there are several methods for solving multi-objective linear programming problem and it is not necessary that for the optimal solution  $(v^{L*}(0), v^*(1), v^{R*}(0))$  of Problem 2.4.7, obtained by all the existing methods the condition  $(v^{L*}(0), v^*(1), v^{R*}(0)) \succeq (v^L(0), v(1), v^R(0))$ , where  $v^L(0), v(1), v^R(0)$  is a feasible solution of Problem 2.4.7, will be satisfied. Also, as the constraints  $v^L(0) \leq v(1) \leq v^R(0)$  are not considered in Problem 2.4.7 so for the optimal solution of Problem 2.4.7, the condition  $v^L(0) \leq v(1) \leq v^R(0)$  may or may not be satisfied i.e., it is not always possible to obtain a triangular fuzzy number, representing the optimal value of Problem 2.4.7, by using the optimal solution of Problem 2.4.7. e.g., it can be verified that  $v^L(0) = \frac{3725}{24}, v(1) = 0, v^R(0) = 0, x_1 = \frac{19}{24}, x_2 = \frac{5}{24}$  is an efficient solution of existing Problem 2.5.3. However, in the efficient solution the inequalities  $v(1) - v^L(0) \geq 0$  and  $v^R(0) - v(1) \geq 0$  are not satisfying i.e., the value of game  $\tilde{v} = (v^L(0), v(1), v^R(0)) = \left(\frac{3725}{24}, 0, 0\right)$  is not a triangular fuzzy number.

### Problem 2.5.3

Maximize  $\{(v^L(0), v(1), v^R(0))\}$

Subject to

$$175x_1 + 80x_2 \geq v^L(0); \quad 150x_1 + 175x_2 \geq v^R(0);$$

$$\begin{aligned} 190x_1 + 100x_2 &\geq v^R(0); & 158x_1 + 190x_2 &\geq v^R(0); \\ 180x_1 + 90x_2 &\geq v(1); & 156x_1 + 180x_2 &\geq v(1); & x_1 + x_2 &= 1; x_1, x_2 \geq 0. \end{aligned}$$

3. Li [70] considered that if  $(a^L(0), a(1), a^R(0))$  and  $(b^L(0), b(1), b^R(0))$  are two triangular fuzzy numbers then  $(a^L(0), a(1), a^R(0)) \preceq (b^L(0), b(1), b^R(0))$  i.e.,

$$\text{minimum} \left\{ (a^L(0), a(1), a^R(0)), (b^L(0), b(1), b^R(0)) \right\} = (a^L(0), a(1), a^R(0)) \quad \text{iff} \quad a^L(0) \leq b^L(0),$$

$$a(1) \leq b(1) \text{ and } a^R(0) \leq b^R(0).$$

Using this relation, Li [70] claimed that to find the minimum expected gain of Player I i.e.,

$$(v^L(0), v(1), v^R(0)) = \text{minimum} \left\{ \sum_{i=1}^m (a_{ij}^L(0), a_{ij}(1), a_{ij}^R(0)) x_i, j = 1, 2, \dots, n \mid \sum_{i=1}^m x_i = 1, x_i \geq 0, i = 1, 2, \dots, m \right\}$$

and to find the maximum expected loss of Player II i.e.,

$$(\omega^L(0), \omega(1), \omega^R(0)) = \text{maximum} \left\{ \sum_{j=1}^n (a_{ij}^L(0), a_{ij}(1), a_{ij}^R(0)) y_j, i = 1, 2, \dots, m \mid \sum_{j=1}^n y_j = 1, y_j \geq 0, j = 1, 2, \dots, n \right\}$$

is equivalent to find the optimal solution of Problem 2.3.1 and Problem 2.3.2 respectively.

Furthermore, using the relation,

$$(a^L(0), a(1), a^R(0)) \succeq (b^L(0), b(1), b^R(0)) \Rightarrow a^L(0) \geq b^L(0), a(1) \geq b(1), a^R(0) \geq b^R(0), \text{ Li [70]}$$

transformed the Problem 2.3.1 into Problem 2.5.4.

#### Problem 2.5.4

$$\text{Maximize } \{(v^L(0), v(1), v^R(0))\}$$

Subject to

$$\begin{aligned} \sum_{i=1}^m a_{ij}^L(0) x_i &\geq v^L(0), j = 1, 2, \dots, n; & \sum_{i=1}^m a_{ij}(1) x_i &\geq v(1), j = 1, 2, \dots, n; \\ \sum_{i=1}^m a_{ij}^R(0) x_i &\geq v^R(0), j = 1, 2, \dots, n; & \sum_{i=1}^m x_i &= 1; x_i \geq 0, i = 1, 2, \dots, m. \end{aligned}$$

Now, if  $S = \{x_i, i = 1, 2, \dots, m; v^L(0), v(1), v^R(0)\}$  is the set of all feasible solutions of

Problem 2.5.4 then a feasible solution  $\{x_i^*, i = 1, 2, \dots, m, v^L(0), v^*(1), v^R(0)\}$  will be an optimal solution of Problem 2.5.4 if  $v^L(0) \leq v^L(0), v^*(1) \leq v(1), v^R(0) \leq v^R(0) \forall v^L(0), v(1), v^R(0) \in S$ .

However, if  $v_1^L(0), v_1(1), v_1^R(0)$  and  $v_2^L(0), v_2(1), v_2^R(0)$  are two feasible solutions of Problem 2.5.4 such that  $v_1^L(0) \leq v_2^L(0), v_1(1) \leq v_2(1)$  and  $v_1^R(0) \leq v_2^R(0)$  are not satisfied. Then both  $(v_1^L(0), v_1(1), v_1^R(0))$  and  $(v_2^L(0), v_2(1), v_2^R(0))$  will be optimal solution of Problem 2.5.4. For example, if  $(1, 4, 5)$  and  $(2, 3, 4)$  are feasible solutions of Problem 2.5.4 then neither  $(1, 4, 5) \succ (2, 3, 4)$  nor  $(2, 3, 4) \succ (1, 4, 5)$ . So, both  $(1, 4, 5)$  and  $(2, 3, 4)$  will represent maximum  $\{(2, 3, 4), (1, 4, 5)\}$  and hence,  $(1, 4, 5)$  and  $(2, 3, 4)$  will be optimal solution of Problem 2.5.4.

To tackle this problem, Li [65] used the following method to find the optimal solution  $\{x_i^*, i = 1, 2, \dots, m, v^L(0), v^*(1), v^R(0)\}$  from the set of all the feasible solutions  $S = \{x_i, i = 1, 2, \dots, m; v^L(0), v(1), v^R(0)\}$ .

Find maximum  $\{v(1) \mid v(1) \in S\}$ .

**Case 1:** If maximum  $\{v(1)\}$  occurs corresponding to only one feasible solution then that feasible solution is optimal solution of Problem 2.5.4.

**Case 2:** If maximum  $\{v(1)\}$  occurs corresponding to two or more than two feasible solutions then find

$$\text{maximum} \left\{ \frac{v^L(0) + v^R(0)}{2}; v^L(0), v^R(0) \text{ are those feasible solutions corresponding to} \right.$$

which maximum  $\{v(1)\}$  exists  $\left. \right\}$ . The feasible solutions corresponding to which maximum value of  $\frac{v^L(0) + v^R(0)}{2}$  will exist, will be the optimal solution of Problem 2.5.4.

It is obvious that Li [65] has transformed the constraints of Problem 2.3.1 into constraints of Problem 2.5.4 by using the relation  $(a^L(0), a(1), a^R(0)) \succeq (b^L(0), b(1), b^R(0)) \Rightarrow a^L(0) \geq b^L(0), a(1) \geq b(1), a^R(0) \geq b^R(0)$ . While, Li [65] has used the relation  $(a^L(0), a(1), a^R(0)) \succeq (b^L(0), b(1), b^R(0)) \Rightarrow a(1) \geq b(1)$  or if  $a(1) = b(1)$  then  $\frac{a^L(0) + a^R(0)}{2} > \frac{b^L(0) + b^R(0)}{2}$  for choosing the optimal solution of Problem 2.5.4 from all the possible feasible solutions of Problem 2.5.4 i.e., Li [65] has used two different methods simultaneously for solving Problem 2.3.1, which is mathematically incorrect.

Similarly, it can be easily verified that Li [65] has used two different methods  $(a^L(0), a(1), a^R(0)) \preceq (b^L(0), b(1), b^R(0)) \Rightarrow a^L(0) \leq b^L(0), a(1) \leq b(1), a^R(0) \leq b^R(0)$  as well as  $(a^L(0), a(1), a^R(0)) \preceq (b^L(0), b(1), b^R(0)) \Rightarrow a(1) < b(1)$  or if  $a(1) = b(1)$  then  $\frac{a^L(0) + a^R(0)}{2} < \frac{b^L(0) + b^R(0)}{2}$  simultaneously for solving Problem 2.3.2, which is mathematically incorrect.

## 2.6 Invalidity of existing mathematical formulations of matrix games with fuzzy payoffs

In Section 1.6 of Chapter 1, it has been proved that

$$\text{minimum} \left\{ \sum_{i=1}^n [a_i^L, a_i^R] x_i \mid \sum_{i=1}^n x_i = 1, x_i \geq 0, i = 1, 2, \dots, n \right\} \neq \text{minimum} \left\{ [a_i^L, a_i^R], i = 1, 2, \dots, n \right\}.$$

Similarly, it can be easily verified and proved that for the comparing method used in the existing methods [36,65,70,71,83],  $\text{minimum} \left\{ \sum_{i=1}^n \tilde{a}_i x_i \mid \sum_{i=1}^n x_i = 1, x_i \geq 0, i = 1, 2, \dots, n \right\} \neq \text{minimum} \{ \tilde{a}_i, i = 1, 2, \dots, n \}$ . While, to obtain the mathematical formulation Problem 2.3.1 and Problem 2.3.2, the mathematically incorrect assumptions

$$\text{minimum} \left\{ \sum_{j=1}^n \left( \sum_{i=1}^m \tilde{a}_{ij} x_i \right) y_j \mid \sum_{i=1}^m x_i = 1, x_i \geq 0, i = 1, 2, \dots, m, \sum_{j=1}^n y_j = 1, y_j \geq 0, j = 1, 2, \dots, n \right\}$$

$$\begin{aligned}
&= \text{minimum} \left\{ \sum_{i=1}^m \tilde{a}_{ij} x_i \mid \sum_{i=1}^m x_i = 1, x_i \geq 0, i = 1, 2, \dots, m \right\} \text{ and} \\
&\text{maximum} \left\{ \sum_{i=1}^m \left( \sum_{j=1}^n \tilde{a}_{ij} y_j \right) x_i \mid \sum_{i=1}^m x_i = 1, x_i \geq 0, i = 1, 2, \dots, m, \sum_{j=1}^n y_j = 1, y_j \geq 0, j = 1, 2, \dots, n \right\} \\
&= \text{maximum} \left\{ \sum_{j=1}^n \tilde{a}_{ij} y_j \mid \sum_{i=1}^m x_i = 1, x_i \geq 0, i = 1, 2, \dots, m \right\} \text{ respectively have been considered.}
\end{aligned}$$

Therefore, the Problem 2.3.1 and Problem 2.3.2 are not valid and hence, the existing methods [36,65,70,71,83 ] in which the minimum expected gain of Player I and maximum loss of Player II are obtained by solving these problems are also not valid.

## 2.7 Proposed Mehar method

In the existing methods [36,65,70,71,83], it is assumed that if  $(a^L(0), a^L(1), a^R(1), a^R(0))$  and  $(b^L(0), b^L(1), b^R(1), b^R(0))$  are two trapezoidal fuzzy numbers then  $(a^L(0), a^L(1), a^R(1), a^R(0)) \succeq (b^L(0), b^L(1), b^R(1), b^R(0))$  if  $a^L(0) \geq b^L(0), a^L(1) \geq b^L(1), a^R(1) \geq b^R(1)$  and  $a^R(0) \geq b^R(0)$  and  $(a^L(0), a^L(1), a^R(1), a^R(0)) \preceq (b^L(0), b^L(1), b^R(1), b^R(0))$  if  $a^L(0) \leq b^L(0), a^L(1) \leq b^L(1), a^R(1) \leq b^R(1)$  and  $a^R(0) \leq b^R(0)$ .

In this section, a new method (named as Mehar method), on the basis of this comparing method, is proposed to find minimum expected gain of Player I and maximum expected loss of Player II and their corresponding optimal strategies.

### 2.7.1 Minimum expected gain of Player I

Using the comparing method,  $(a^L(0), a^L(1), a^R(1), a^R(0)) \preceq (b^L(0), b^L(1), b^R(1), b^R(0)) \Rightarrow a^L(0) \leq b^L(0), a^L(1) \leq b^L(1), a^R(1) \leq b^R(1)$  and  $a^R(0) \leq b^R(0)$ , the minimum expected gain of Player I and corresponding optimal strategies can be obtained as follows:

**Step 1:** Find  $(y_1^*, y_2^*, \dots, y_n^*) \in Y$  such that value of  $\sum_{j=1}^n \left( \sum_{i=1}^m (a_{ij}^L(0), a_{ij}^L(1), a_{ij}^R(1), a_{ij}^R(0)) x_i \right) y_j$  is

minimum for all  $(x_1, x_2, \dots, x_m) \in X$  i.e., find the solution  $\{y_j^*, j=1, 2, \dots, n\}$  of Problem 2.7.1.

### Problem 2.7.1

$$\text{Minimize } \left\{ \sum_{j=1}^n \left( \sum_{i=1}^m (a_{ij}^L(0), a_{ij}^L(1), a_{ij}^R(1), a_{ij}^R(0)) x_i \right) y_j \right\}$$

Subject to

$$\sum_{i=1}^m x_i = 1; \sum_{j=1}^n y_j = 1; x_i \geq 0, i=1, 2, \dots, m, y_j \geq 0, j=1, 2, \dots, n.$$

**Step 2:** Using the property,  $\sum_{i=1}^n \lambda (a_i^L(0), a_i^L(1), a_i^R(1), a_i^R(0)) =$

$$\left( \sum_{i=1}^n a_i^L(0) \lambda, \sum_{i=1}^n a_i^L(1) \lambda, \sum_{i=1}^n a_i^R(1) \lambda, \sum_{i=1}^n a_i^R(0) \lambda \right), \lambda \geq 0, \text{ the Problem 2.7.1 can be transformed}$$

into Problem 2.7.2.

### Problem 2.7.2

$$\text{Minimize } \left\{ \sum_{j=1}^n \sum_{i=1}^m (a_{ij}^L(0) x_i y_j, a_{ij}^L(1) x_i y_j, a_{ij}^R(1) x_i y_j, a_{ij}^R(0) x_i y_j) \right\}$$

Subject to

$$\sum_{i=1}^m x_i = 1; \sum_{j=1}^n y_j = 1;$$

$$x_i \geq 0, i=1, 2, \dots, m, y_j \geq 0, j=1, 2, \dots, n.$$

**Step 3:** According to comparing method, to find the optimal solution  $\{y_1^*, y_2^*, \dots, y_n^*\}$  of Problem

2.7.2 such that value of  $\sum_{j=1}^n \sum_{i=1}^m (a_{ij}^L(0) x_i y_j, a_{ij}^L(1) x_i y_j, a_{ij}^R(1) x_i y_j, a_{ij}^R(0) x_i y_j)$  is minimum for all

$(x_1, x_2, \dots, x_m) \in X$  is equivalent to find  $\{y_1^*, y_2^*, \dots, y_n^*\}$  such that value of

$\sum_{j=1}^n \sum_{i=1}^m a_{ij}^L(0) x_i y_j, \sum_{j=1}^n \sum_{i=1}^m a_{ij}^L(1) x_i y_j, \sum_{j=1}^n \sum_{i=1}^m a_{ij}^R(1) x_i y_j$  and  $\sum_{j=1}^n \sum_{i=1}^m a_{ij}^R(0) x_i y_j$  is minimum for all

$(x_1, x_2, \dots, x_m) \in X$  or if it is not possible to find  $\{y_1^*, y_2^*, \dots, y_n^*\}$  for which the value of

$\sum_{j=1}^n \sum_{i=1}^m a_{ij}^L(0)x_i y_j, \sum_{j=1}^n \sum_{i=1}^m a_{ij}^L(1)x_i y_j, \sum_{j=1}^n \sum_{i=1}^m a_{ij}^R(1)x_i y_j$  and  $\sum_{j=1}^n \sum_{i=1}^m a_{ij}^R(0)x_i y_j$  is minimum then find

such  $\{y_{j_1}^*, j = 1, 2, \dots, n\}$  for which the value of  $\sum_{j=1}^n \sum_{i=1}^m a_{ij}^L(0)x_{i1} y_{j1}$  is minimum but value of

$\sum_{j=1}^n \sum_{i=1}^m a_{ij}^L(1)x_{i1} y_{j1}$  and/or  $\sum_{j=1}^n \sum_{i=1}^m a_{ij}^R(1)x_{i1} y_{j1}$  and/or  $\sum_{j=1}^n \sum_{i=1}^m a_{ij}^R(0)x_{i1} y_{j1}$  are not minimum for all

$(x_{11}, x_{21}, \dots, x_{m1}) \in X$  and find such  $\{y_{j_2}^*, j = 1, 2, \dots, n\}$  for which the value of  $\sum_{j=1}^n \sum_{i=1}^m a_{ij}^L(1)x_{i2} y_{j2}$

is minimum but value of  $\sum_{j=1}^n \sum_{i=1}^m a_{ij}^L(0)x_{i2} y_{j2}$  and/or  $\sum_{j=1}^n \sum_{i=1}^m a_{ij}^R(1)x_{i2} y_{j2}$  and/or  $\sum_{j=1}^n \sum_{i=1}^m a_{ij}^R(0)x_{i2} y_{j2}$

is not minimum for all  $(x_{12}, x_{22}, \dots, x_{m2}) \in X$  and to find such  $\{y_{j_3}^*, j = 1, 2, \dots, n\}$  for which the

value of  $\sum_{j=1}^n \sum_{i=1}^m a_{ij}^R(1)x_{i3} y_{j3}$  is minimum but value of

$\sum_{j=1}^n \sum_{i=1}^m a_{ij}^L(0)x_{i3} y_{j3}$  and/or  $\sum_{j=1}^n \sum_{i=1}^m a_{ij}^L(1)x_{i3} y_{j3}$  and/or  $\sum_{j=1}^n \sum_{i=1}^m a_{ij}^R(0)x_{i3} y_{j3}$  is not minimum for all

$(x_{13}, x_{23}, \dots, x_{m3}) \in X$  and find such  $\{y_{j_4}^*, j = 1, 2, \dots, n\}$  for which the value of  $\sum_{j=1}^n \sum_{i=1}^m a_{ij}^R(0)x_{i4} y_{j4}$

is minimum but value of  $\sum_{j=1}^n \sum_{i=1}^m a_{ij}^L(0)x_{i4} y_{j4}$  and/or  $\sum_{j=1}^n \sum_{i=1}^m a_{ij}^L(1)x_{i4} y_{j4}$  and/or  $\sum_{j=1}^n \sum_{i=1}^m a_{ij}^R(1)x_{i4} y_{j4}$

is not minimum for all  $(x_{14}, x_{24}, \dots, x_{m4}) \in X$  i.e., to find optimal solution  $\{y_{j_1}^*, j = 1, 2, \dots, n\}$ ,

$\{y_{j_2}^*, j = 1, 2, \dots, n\}$ ,  $\{y_{j_3}^*, j = 1, 2, \dots, n\}$  and  $\{y_{j_4}^*, j = 1, 2, \dots, n\}$  of Problem 2.7.3 and Problem

2.7.4, Problem 2.7.5 and Problem 2.7.6 respectively.

### Problem 2.7.3

$$\text{Minimize} \left\{ \sum_{j=1}^n \sum_{i=1}^m a_{ij}^L(0)x_{i1} y_{j1} \right\}$$

Subject to

### Problem 2.7.4

$$\text{Minimize} \left\{ \sum_{j=1}^n \sum_{i=1}^m a_{ij}^L(1)x_{i2} y_{j2} \right\}$$

Subject to

$$\sum_{i=1}^m x_{i1} = 1; \sum_{j=1}^n y_{j1} = 1;$$

$$x_{i1} \geq 0, i = 1, 2, \dots, m, y_{j1} \geq 0, j = 1, 2, \dots, n.$$

**Problem 2.7.5**

$$\text{Minimize } \left\{ \sum_{j=1}^n \sum_{i=1}^m a_{ij}^R(1) x_{i3} y_{j3} \right\}$$

Subject to

$$\sum_{i=1}^m x_{i3} = 1; \sum_{j=1}^n y_{j3} = 1;$$

$$x_{i3} \geq 0, i = 1, 2, \dots, m, y_{j3} \geq 0, j = 1, 2, \dots, n.$$

$$\sum_{i=1}^m x_{i2} = 1; \sum_{j=1}^n y_{j2} = 1;$$

$$x_i \geq 0, i = 1, 2, \dots, m, y_{j2} \geq 0, j = 1, 2, \dots, n.$$

**Problem 2.7.6**

$$\text{Minimize } \left\{ \sum_{j=1}^n \sum_{i=1}^m a_{ij}^R(0) x_{i4} y_{j4} \right\}$$

Subject to

$$\sum_{i=1}^m x_{i4} = 1; \sum_{j=1}^n y_{j4} = 1;$$

$$x_{i4} \geq 0, i = 1, 2, \dots, m, y_{j4} \geq 0, j = 1, 2, \dots, n.$$

**Step 4:** Since, in Problem 2.7.3, Problem 2.7.4, Problem 2.7.5 and Problem 2.7.6 only  $y_{j1}, y_{j2}, y_{j3}$  and  $y_{j4}$  respectively have been considered as decision variables. So, Problem 2.7.3, Problem 2.7.4, Problem 2.7.5 and Problem 2.7.6 are linear programming problems and hence, the optimal value of Problem 2.7.3, Problem 2.7.4, Problem 2.7.5 and Problem 2.7.6 will be equal to optimal value of its corresponding dual problem i.e., Problem 2.7.7, Problem 2.7.8, Problem 2.7.9 and Problem 2.7.10 respectively.

**Problem 2.7.7**

$$\text{Maximize } \{v_1\}$$

Subject to

$$\sum_{i=1}^m a_{ij}^L(0) x_{i1} \geq v_1, j = 1, 2, \dots, n;$$

$$\sum_{i=1}^m x_{i1} = 1; x_{i1} \geq 0, i = 1, 2, \dots, m.$$

**Problem 2.7.9**

$$\text{Maximize } \{v_3\}$$

Subject to

**Problem 2.7.8**

$$\text{Maximize } \{v_2\}$$

Subject to

$$\sum_{i=1}^m a_{ij}^L(1) x_{i2} \geq v_2, j = 1, 2, \dots, n;$$

$$\sum_{i=1}^m x_{i2} = 1; x_{i2} \geq 0, i = 1, 2, \dots, m.$$

**Problem 2.7.10**

$$\text{Maximize } \{v_4\}$$

Subject to

$$\sum_{i=1}^m a_{ij}^R(1)x_{i3} \geq v_3, j=1,2,\dots,n;$$

$$\sum_{i=1}^m x_{i3} = 1; x_{i3} \geq 0, i=1,2,\dots,m.$$

$$\sum_{i=1}^m a_{ij}^R(0)x_{i4} \geq v_4, j=1,2,\dots,n;$$

$$\sum_{i=1}^m x_{i4} = 1; x_{i4} \geq 0, i=1,2,\dots,m.$$

**Step 5:** Substitute the value of  $\{x_{i1}^*, i=1,2,\dots,m\}, \{x_{i2}^*, i=1,2,\dots,m\}, \{x_{i3}^*, i=1,2,\dots,m\}$  and  $\{x_{i4}^*, i=1,2,\dots,m\}$  of Problem 2.7.7, Problem 2.7.8, Problem 2.7.9 and Problem 2.7.10 in Problem 2.7.3, Problem 2.7.4, Problem 2.7.5 and Problem 2.7.6 respectively and find all the alternative basic optimal solutions  $\{y_{j1}^{k*}, j=1,2,\dots,n, k=1,2,\dots,l\}, \{y_{j2}^{r*}, j=1,2,\dots,n, r=1,2,\dots,u\}, \{y_{j3}^{s*}, j=1,2,\dots,n, s=1,2,\dots,e\}$  and  $\{y_{j4}^{q*}, j=1,2,\dots,n, q=1,2,\dots,h\}$  of Problem 2.7.3, Problem 2.7.4, Problem 2.7.5 and Problem 2.7.6 respectively.

**Step 6:** Find

$$\text{minimum} \left\{ \begin{array}{l} \sum_{j=1}^n \sum_{i=1}^m (a_{ij}^L(0)x_{i1}^*y_{j1}^{k*}, a_{ij}^L(1)x_{i1}^*y_{j1}^{k*}, a_{ij}^R(1)x_{i1}^*y_{j1}^{k*}, a_{ij}^R(0)x_{i1}^*y_{j1}^{k*}); k=1,2,\dots,l, \\ \sum_{j=1}^n \sum_{i=1}^m (a_{ij}^L(0)x_{i2}^*y_{j2}^{r*}, a_{ij}^L(1)x_{i2}^*y_{j2}^{r*}, a_{ij}^R(1)x_{i2}^*y_{j2}^{r*}, a_{ij}^R(0)x_{i2}^*y_{j2}^{r*}); r=1,2,\dots,u, \\ \sum_{j=1}^n \sum_{i=1}^m (a_{ij}^L(0)x_{i3}^*y_{j3}^{s*}, a_{ij}^L(1)x_{i3}^*y_{j3}^{s*}, a_{ij}^R(1)x_{i3}^*y_{j3}^{s*}, a_{ij}^R(0)x_{i3}^*y_{j3}^{s*}); s=1,2,\dots,e, \\ \sum_{j=1}^n \sum_{i=1}^m (a_{ij}^L(0)x_{i4}^*y_{j4}^{q*}, a_{ij}^L(1)x_{i4}^*y_{j4}^{q*}, a_{ij}^R(1)x_{i4}^*y_{j4}^{q*}, a_{ij}^R(0)x_{i4}^*y_{j4}^{q*}); q=1,2,\dots,h \end{array} \right\}$$

**Step 7:** All the trapezoidal fuzzy numbers which will be minimum, will represent minimum expected gain of Player I and the optimal strategies for all such minimum which will be obtained corresponding to  $\sum_{j=1}^n \sum_{i=1}^m (a_{ij}^L(0)x_{i1}^*y_{j1}^{k*}, a_{ij}^L(1)x_{i1}^*y_{j1}^{k*}, a_{ij}^R(1)x_{i1}^*y_{j1}^{k*}, a_{ij}^R(0)x_{i1}^*y_{j1}^{k*}); k=1,2,\dots,l$  will be  $\{x_{i1}^*, i=1,2,\dots,m\}$ , for all such minimum which will be obtained corresponding to

$$\sum_{j=1}^n \sum_{i=1}^m (a_{ij}^L(0)x_{i2}^*y_{j2}^{r*}, a_{ij}^L(1)x_{i2}^*y_{j2}^{r*}, a_{ij}^R(1)x_{i2}^*y_{j2}^{r*}, a_{ij}^R(0)x_{i2}^*y_{j2}^{r*}); r=1,2,\dots,u, \text{ will be}$$

$\{x_{i2}^*, i=1, 2, \dots, m\}$ , for all such minimum which will be obtained corresponding to

$$\sum_{j=1}^n \sum_{i=1}^m (a_{ij}^L(0)x_{i3}^*y_{j3}^{s*}, a_{ij}^L(1)x_{i3}^*y_{j3}^{s*}, a_{i3}^R(1)x_{i3}^*y_{j3}^{s*}, a_{ij}^R(0)x_{i3}^*y_{j3}^{s*}); s=1, 2, \dots, e, \text{ will be}$$

$\{x_{i3}^*, i=1, 2, \dots, m\}$  and for all such minimum which will be obtained corresponding to

$$\sum_{j=1}^n \sum_{i=1}^m (a_{ij}^L(0)x_{i4}^*y_{j4}^{q*}, a_{ij}^L(1)x_{i4}^*y_{j4}^{q*}, a_{ij}^R(1)x_{i4}^*y_{j4}^{q*}, a_{ij}^R(0)x_{i4}^*y_{j4}^{q*}); q=1, 2, \dots, h \text{ will be}$$

$$\{x_{i4}^*, i=1, 2, \dots, m\}.$$

## 2.7.2 Maximum expected loss of Player II

Using the comparing method,  $(a^L(0), a^L(1), a^R(1), a^R(0)) \succeq (b^L(0), b^L(1), b^R(1), b^R(0))$

$\Rightarrow a^L(0) \geq b^L(0), a^L(1) \geq b^L(1), a^R(1) \geq b^R(1)$  and  $a^R(0) \geq b^R(0)$ , the maximum expected loss of Player II and corresponding optimal strategies can be obtained as follows:

**Step 1:** Find  $(x_1^*, x_2^*, \dots, x_m^*) \in X$  such that value of  $\sum_{j=1}^n \sum_{i=1}^m (a_{ij}^L(0), a_{ij}^L(1), a_{ij}^R(1), a_{ij}^R(0)) x_i y_j$  is

maximum for all  $(y_1, y_2, \dots, y_n) \in Y$  i.e., find the solution  $\{x_i^*, i=1, 2, \dots, m\}$  of Problem 2.7.11.

### Problem 2.7.11

$$\text{Maximize } \left\{ \sum_{j=1}^n \sum_{i=1}^m (a_{ij}^L(0), a_{ij}^L(1), a_{ij}^R(1), a_{ij}^R(0)) x_i y_j \right\}$$

Subject to

$$\sum_{i=1}^m x_i = 1; \sum_{j=1}^n y_j = 1;$$

$$x_i \geq 0, i=1, 2, \dots, m, y_j \geq 0, j=1, 2, \dots, n.$$

**Step 2:** Using the property,  $\sum_{i=1}^n \lambda (a_i^L(0), a_i^L(1), a_i^R(1), a_i^R(0)) =$

$$\left( \sum_{i=1}^n a_i^L(0) \lambda, \sum_{i=1}^n a_i^L(1) \lambda, \sum_{i=1}^n a_i^R(1) \lambda, \sum_{i=1}^n a_i^R(0) \lambda \right), \lambda \geq 0, \text{ the Problem 2.7.11 can be transformed}$$

into Problem 2.7.12.

**Problem 2.7.12**

$$\text{Maximize } \left\{ \sum_{j=1}^n \sum_{i=1}^m \left( a_{ij}^L(0)x_i y_j, a_{ij}^L(1)x_i y_j, a_{ij}^R(1)x_i y_j, a_{ij}^R(0)x_i y_j \right) \right\}$$

Subject to

$$\sum_{i=1}^m x_i = 1; \sum_{j=1}^n y_j = 1; x_i \geq 0, i = 1, 2, \dots, m, y_j \geq 0, j = 1, 2, \dots, n.$$

**Step 3:** According to comparing method, to find the optimal solution  $\{x_1^*, x_2^*, \dots, x_n^*\}$  of Problem

2.7.12 such that value of  $\sum_{j=1}^n \sum_{i=1}^m \left( a_{ij}^L(0)x_i y_j, a_{ij}^L(1)x_i y_j, a_{ij}^R(1)x_i y_j, a_{ij}^R(0)x_i y_j \right)$  is maximum for all

$(y_1, y_2, \dots, y_n) \in Y$  is equivalent to find  $\{x_1^*, x_2^*, \dots, x_m^*\}$  such that value of

$$\sum_{j=1}^n \sum_{i=1}^m a_{ij}^L(0)x_i y_j, \sum_{j=1}^n \sum_{i=1}^m a_{ij}^L(1)x_i y_j, \sum_{j=1}^n \sum_{i=1}^m a_{ij}^R(1)x_i y_j \text{ and } \sum_{j=1}^n \sum_{i=1}^m a_{ij}^R(0)x_i y_j \text{ is maximum for all}$$

$(y_1, y_2, \dots, y_n) \in Y$  or if it is not possible to find  $\{x_1^*, x_2^*, \dots, x_m^*\}$  for which the value of

$$\sum_{j=1}^n \sum_{i=1}^m a_{ij}^L(0)x_i y_j, \sum_{j=1}^n \sum_{i=1}^m a_{ij}^L(1)x_i y_j, \sum_{j=1}^n \sum_{i=1}^m a_{ij}^R(1)x_i y_j \text{ and } \sum_{j=1}^n \sum_{i=1}^m a_{ij}^R(0)x_i y_j \text{ is maximum then find}$$

such  $\{x_{i1}^*, i = 1, 2, \dots, m\}$  for which the value of  $\sum_{j=1}^n \sum_{i=1}^m a_{ij}^L(0)x_{i1} y_{j1}$  is maximum but value of

$$\sum_{j=1}^n \sum_{i=1}^m a_{ij}^L(1)x_{i1} y_{j1} \text{ and/or } \sum_{j=1}^n \sum_{i=1}^m a_{ij}^R(1)x_{i1} y_{j1} \text{ and/or } \sum_{j=1}^n \sum_{i=1}^m a_{ij}^R(0)x_{i1} y_{j1} \text{ is not maximum for all}$$

$(y_{11}, y_{21}, \dots, y_{n1}) \in Y$  and find such  $\{x_{i2}^*, i = 1, 2, \dots, m\}$  for which the value of  $\sum_{j=1}^n \sum_{i=1}^m a_{ij}^L(1)x_{i2} y_{j2}$

is maximum but value of  $\sum_{j=1}^n \sum_{i=1}^m a_{ij}^L(0)x_{i2} y_{j2}$  and/or  $\sum_{j=1}^n \sum_{i=1}^m a_{ij}^R(1)x_{i2} y_{j2}$  and/or  $\sum_{j=1}^n \sum_{i=1}^m a_{ij}^R(0)x_{i2} y_{j2}$

is not maximum for all  $(y_{12}, y_{22}, \dots, y_{n2}) \in Y$  and to find such  $\{x_{i3}^*, i = 1, 2, \dots, m\}$  for which the

value of  $\sum_{j=1}^n \sum_{i=1}^m a_{ij}^R(1)x_{i3}y_{j3}$  is maximum but value of

$\sum_{j=1}^n \sum_{i=1}^m a_{ij}^L(0)x_{i3}y_{j3}$  and/or  $\sum_{j=1}^n \sum_{i=1}^m a_{ij}^L(1)x_{i3}y_{j3}$  and/or  $\sum_{j=1}^n \sum_{i=1}^m a_{ij}^R(0)x_{i3}y_{j3}$  is not maximum for all

$(y_{13}, y_{23}, \dots, y_{n3}) \in Y$  or find such  $\{x_{i4}^*, i=1, 2, \dots, m\}$  for which the value of  $\sum_{j=1}^n \sum_{i=1}^m a_{ij}^R(0)x_{i4}y_{j4}$  is

maximum but value of  $\sum_{j=1}^n \sum_{i=1}^m a_{ij}^L(0)x_{i4}y_{j4}$  and/or  $\sum_{j=1}^n \sum_{i=1}^m a_{ij}^L(1)x_{i4}y_{j4}$  and/or  $\sum_{j=1}^n \sum_{i=1}^m a_{ij}^R(1)x_{i4}y_{j4}$  is

not maximum for all  $(y_{14}, y_{24}, \dots, y_{n4}) \in Y$  i.e., to find optimal solution  $\{x_{i1}^*, i=1, 2, \dots, m\}$ ,

$\{x_{i2}^*, i=1, 2, \dots, m\}$ ,  $\{x_{i3}^*, i=1, 2, \dots, m\}$  and  $\{x_{i4}^*, i=1, 2, \dots, m\}$  of Problem 2.7.13 and Problem

2.7.14, Problem 2.7.15 and Problem 2.7.16 respectively.

#### Problem 2.7.13

$$\text{Maximize } \left\{ \sum_{j=1}^n \sum_{i=1}^m a_{ij}^L(0)x_{i1}y_{j1} \right\}$$

Subject to

$$\sum_{i=1}^m x_{i1} = 1; \sum_{j=1}^n y_{j1} = 1;$$

$$x_{i1} \geq 0, i=1, 2, \dots, m, y_{j1} \geq 0, j=1, 2, \dots, n.$$

#### Problem 2.7.15

$$\text{Maximize } \left\{ \sum_{j=1}^n \sum_{i=1}^m a_{ij}^R(1)x_{i3}y_{j3} \right\}$$

Subject to

$$\sum_{i=1}^m x_{i3} = 1; \sum_{j=1}^n y_{j3} = 1;$$

$$x_{i3} \geq 0, i=1, 2, \dots, m, y_{j3} \geq 0, j=1, 2, \dots, n.$$

#### Problem 2.7.14

$$\text{Maximize } \left\{ \sum_{j=1}^n \sum_{i=1}^m a_{ij}^L(1)x_{i2}y_{j2} \right\}$$

Subject to

$$\sum_{i=1}^m x_{i2} = 1; \sum_{j=1}^n y_{j2} = 1;$$

$$x_i \geq 0, i=1, 2, \dots, m, y_{j2} \geq 0, j=1, 2, \dots, n.$$

#### Problem 2.7.16

$$\text{Maximize } \left\{ \sum_{j=1}^n \sum_{i=1}^m a_{ij}^R(0)x_{i4}y_{j4} \right\}$$

Subject to

$$\sum_{i=1}^m x_{i4} = 1; \sum_{j=1}^n y_{j4} = 1;$$

$$x_{i4} \geq 0, i=1, 2, \dots, m, y_{j4} \geq 0, j=1, 2, \dots, n.$$

**Step 4:** Since, in Problem 2.7.13, Problem 2.7.14, Problem 2.7.15 and Problem 2.7.16 only

$x_{i1}, x_{i2}, x_{i3}$  and  $x_{i4}$  respectively have been considered as decision variables. So, Problem 2.7.13, Problem 2.7.14, Problem 2.7.15 and Problem 2.7.16 are linear programming problems and hence, the optimal value of Problem 2.7.13, Problem 2.7.14, Problem 2.7.15 and Problem 2.7.16 will be equal to optimal value of its corresponding dual problem i.e., Problem 2.7.17, Problem 2.7.18, Problem 2.7.19 and Problem 2.7.20 respectively.

**Problem 2.7.17**

Minimize  $\{\omega_1\}$

Subject to

$$\sum_{j=1}^n a_{ij}^L(0)y_{j1} \leq \omega_1, i = 1, 2, \dots, m;$$

$$\sum_{j=1}^n y_{j1} = 1; y_{j1} \geq 0, j = 1, 2, \dots, n.$$

**Problem 2.7.19**

Minimize  $\{\omega_3\}$

Subject to

$$\sum_{j=1}^n a_{ij}^L(0)y_{j3} \leq \omega_3, i = 1, 2, \dots, m;$$

$$\sum_{j=1}^n y_{j3} = 1; y_{j3} \geq 0, j = 1, 2, \dots, n.$$

**Problem 2.7.18**

Minimize  $\{\omega_2\}$

Subject to

$$\sum_{j=1}^n a_{ij}^L(0)y_{j2} \leq \omega_2, i = 1, 2, \dots, m;$$

$$\sum_{j=1}^n y_{j2} = 1; y_{j2} \geq 0, j = 1, 2, \dots, n.$$

**Problem 2.7.20**

Minimize  $\{\omega_4\}$

Subject to

$$\sum_{j=1}^n a_{ij}^L(0)y_{j4} \leq \omega_4, i = 1, 2, \dots, m;$$

$$\sum_{j=1}^n y_{j4} = 1; y_{j4} \geq 0, j = 1, 2, \dots, n.$$

**Step 5:** Substitute the value of  $\{y_{j1}^*, j = 1, 2, \dots, n\}, \{y_{j2}^*, j = 1, 2, \dots, n\}, \{y_{j3}^*, j = 1, 2, \dots, n\}$  and  $\{y_{j4}^*, j = 1, 2, \dots, n\}$  of Problem 2.7.17, Problem 2.7.18, Problem 2.7.19 and Problem 2.7.20 in Problem 2.7.13, Problem 2.7.14, Problem 2.7.15 and Problem 2.7.16 respectively and find all the alternative basic optimal solutions  $\{x_{i1}^k, i = 1, 2, \dots, m, k = 1, 2, \dots, l\}, \{x_{i2}^{r*}, i = 1, 2, \dots, m, r = 1, 2, \dots, u\}, \{x_{i3}^{s*}, i = 1, 2, \dots, m, s = 1, 2, \dots, e\}$  and  $\{x_{i4}^{q*}, i = 1, 2, \dots, m, q = 1, 2, \dots, h\}$  of Problem 2.7.13, Problem 2.7.14, Problem 2.7.15 and

Problem 2.7.16 respectively.

**Step 6:** Find

$$\text{maximum} \left\{ \begin{aligned} &\sum_{j=1}^n \sum_{i=1}^m \left( a_{ij}^L(0) x_{i1}^{k*} y_{j1}^*, a_{ij}^L(1) x_{i1}^{k*} y_{j1}^*, a_{ij}^R(1) x_{i1}^{k*} y_{j1}^*, a_{ij}^R(0) x_{i1}^{k*} y_{j1}^* \right); k=1, 2, \dots, l, \\ &\sum_{j=1}^n \sum_{i=1}^m \left( a_{ij}^L(0) x_{i2}^{r*} y_{j2}^*, a_{ij}^L(1) x_{i2}^{r*} y_{j2}^*, a_{ij}^R(1) x_{i2}^{r*} y_{j2}^*, a_{ij}^R(0) x_{i2}^{r*} y_{j2}^* \right); r=1, 2, \dots, u, \\ &\sum_{j=1}^n \sum_{i=1}^m \left( a_{ij}^L(0) x_{i3}^{s*} y_{j3}^*, a_{ij}^L(1) x_{i3}^{s*} y_{j3}^*, a_{ij}^R(1) x_{i3}^{s*} y_{j3}^*, a_{ij}^R(0) x_{i3}^{s*} y_{j3}^* \right); s=1, 2, \dots, e, \\ &\sum_{j=1}^n \sum_{i=1}^m \left( a_{ij}^L(0) x_{i4}^{q*} y_{j4}^*, a_{ij}^L(1) x_{i4}^{q*} y_{j4}^*, a_{ij}^R(1) x_{i4}^{q*} y_{j4}^*, a_{ij}^R(0) x_{i4}^{q*} y_{j4}^* \right); q=1, 2, \dots, h \end{aligned} \right\}$$

**Step 7:** All the trapezoidal fuzzy numbers which will be maximum, will represent maximum expected loss of Player II and the optimal strategies for all such maximum which will be

obtained corresponding to  $\sum_{j=1}^n \sum_{i=1}^m \left( a_{ij}^L(0) x_{i1}^{k*} y_{j1}^*, a_{ij}^L(1) x_{i1}^{k*} y_{j1}^*, a_{ij}^R(1) x_{i1}^{k*} y_{j1}^*, a_{ij}^R(0) x_{i1}^{k*} y_{j1}^* \right);$

$k=1, 2, \dots, l$  will be  $\{y_{j1}^*, j=1, 2, \dots, n\}$ , for all such maximum which will be obtained

corresponding to  $\sum_{j=1}^n \sum_{i=1}^m \left( a_{ij}^L(0) x_{i2}^{r*} y_{j2}^*, a_{ij}^L(1) x_{i2}^{r*} y_{j2}^*, a_{ij}^R(1) x_{i2}^{r*} y_{j2}^*, a_{ij}^R(0) x_{i2}^{r*} y_{j2}^* \right); r=1, 2, \dots, u$  will

be  $\{y_{j2}^*, j=1, 2, \dots, n\}$ , for all such maximum which will be obtained corresponding to

$\sum_{j=1}^n \sum_{i=1}^m \left( a_{ij}^L(0) x_{i3}^{s*} y_{j3}^*, a_{ij}^L(1) x_{i3}^{s*} y_{j3}^*, a_{ij}^R(1) x_{i3}^{s*} y_{j3}^*, a_{ij}^R(0) x_{i3}^{s*} y_{j3}^* \right); s=1, 2, \dots, e$  will be

$\{y_{j3}^*, j=1, 2, \dots, n\}$  and for all such maximum which will be obtained corresponding to

$\sum_{j=1}^n \sum_{i=1}^m \left( a_{ij}^L(0) x_{i4}^{q*} y_{j4}^*, a_{ij}^L(1) x_{i4}^{q*} y_{j4}^*, a_{ij}^R(1) x_{i4}^{q*} y_{j4}^*, a_{ij}^R(0) x_{i4}^{q*} y_{j4}^* \right); q=1, 2, \dots, h$  will be

$\{y_{j4}^*, j=1, 2, \dots, n\}$ .

## 2.8 Numerical example

In this section, matrix games with fuzzy payoffs  $\tilde{A} = \begin{bmatrix} (175, 180, 190) & (150, 156, 158) \\ (80, 90, 100) & (175, 180, 190) \end{bmatrix}$ ,

chosen by Li [70], is solved by the proposed Mehar method.

### 2.8.1 Minimum expected gain of Player I

Using the proposed Mehar method, the minimum expected gain of Player I and corresponding optimal strategies can be obtained as follows:

**Step 1:** Find  $(y_1^*, y_2^*) \in Y$  such that value of  $((175, 180, 190)x_1y_1 + (150, 156, 158)x_1y_2 + (80, 90, 100)x_2y_1 + (175, 180, 190)x_2y_2)$  is minimum for all  $(x_1, x_2) \in X$  i.e., find the solution  $\{y_j^*, j = 1, 2\}$  of Problem 2.8.1.

#### Problem 2.8.1

Minimize  $\{(175, 180, 190)x_1y_1 + (150, 156, 158)x_1y_2 + (80, 90, 100)x_2y_1 + (175, 180, 190)x_2y_2\}$

Subject to

$$x_1 + x_2 = 1; y_1 + y_2 = 1; x_1, x_2 \geq 0; y_1, y_2 \geq 0.$$

**Step 2:** Using the property,  $\sum_{i=1}^n \lambda(a_i^L(0), a_i^L(1), a_i^R(1), a_i^R(0)) =$

$$\left( \sum_{i=1}^n a_i^L(0)\lambda, \sum_{i=1}^n a_i^L(1)\lambda, \sum_{i=1}^n a_i^R(1)\lambda, \sum_{i=1}^n a_i^R(0)\lambda \right), \lambda \geq 0 \text{ and}$$

$$\sum_{i=1}^n (a_i^L(0), a_i^L(1), a_i^R(1), a_i^R(0)) = \left( \sum_{i=1}^n a_i^L(0), \sum_{i=1}^n a_i^L(1), \sum_{i=1}^n a_i^R(1), \sum_{i=1}^n a_i^R(0) \right), \text{ the Problem 2.8.1}$$

can be transformed into Problem 2.8.2.

#### Problem 2.8.2

$$\text{Minimize } \left\{ \begin{pmatrix} 175x_1y_1 + 150x_1y_2 + 80x_2y_1 + 175x_2y_2, 180x_1y_1 + 156x_1y_2 + 90x_2y_1 + 180x_2y_2, \\ 190x_1y_1 + 158x_1y_2 + 100x_2y_1 + 190x_2y_2 \end{pmatrix} \right\}$$

Subject to

$$x_1 + x_2 = 1; y_1 + y_2 = 1;$$

$$x_1, x_2 \geq 0; y_1, y_2 \geq 0.$$

**Step 3:** Find optimal solution  $\{y_{j1}^*, j=1,2\}$ ,  $\{y_{j2}^*, j=1,2\}$  and  $\{y_{j3}^*, j=1,2\}$  of Problem 2.8.3, Problem 2.8.4 and Problem 2.8.5 respectively.

**Problem 2.8.3**

$$\text{Minimize } \{175x_{11}y_{11} + 150x_{11}y_{21} + 80x_{21}y_{11} + 175x_{21}y_{21}\}$$

Subject to

$$x_{11} + x_{21} = 1; y_{11} + y_{21} = 1;$$

$$x_{11}, x_{21} \geq 0; y_{11}, y_{21} \geq 0.$$

**Problem 2.8.4**

$$\text{Minimize } \{180x_{12}y_{12} + 156x_{12}y_{22} + 90x_{22}y_{12} + 180x_{22}y_{22}\}$$

Subject to

$$x_{12} + x_{22} = 1; y_{12} + y_{22} = 1;$$

$$x_{12}, x_{22} \geq 0; y_{12}, y_{22} \geq 0.$$

**Problem 2.8.5**

$$\text{Minimize } \{190x_{13}y_{13} + 158x_{13}y_{23} + 100x_{23}y_{13} + 190x_{23}y_{23}\}$$

Subject to

$$x_{13} + x_{23} = 1; y_{13} + y_{23} = 1;$$

$$x_{13}, x_{23} \geq 0; y_{13}, y_{23} \geq 0.$$

**Step 4:** Since, in Problem 2.8.3, Problem 2.8.4 and Problem 2.8.5 only  $y_{j1}, y_{j2}$  and  $y_{j3}$  respectively have been considered as decision variables. So, Problem 2.8.3, Problem 2.8.4 and Problem 2.8.5 are linear programming problems and hence, the optimal value of Problem 2.8.3, Problem 2.8.4 and Problem 2.8.5 will be equal to optimal value of its corresponding i.e., dual problem i.e., Problem 2.8.6, Problem 2.8.7 and Problem 2.8.8 respectively.

**Problem 2.8.6**

$$\text{Maximize } \{v_1\}$$

**Problem 2.8.7**

$$\text{Maximize } \{v_2\}$$

Subject to

$$\begin{aligned} 175x_{11} + 80x_{21} &\geq \nu_1; \\ 150x_{11} + 175x_{21} &\geq \nu_1; \\ x_{11} + x_{21} &= 1; x_{11} \geq 0, x_{21} \geq 0. \end{aligned}$$

### Problem 2.8.8

Maximize  $\{\nu_3\}$

Subject to

$$\begin{aligned} 190x_{13} + 100x_{23} &\geq \nu_3; \\ 158x_{13} + 190x_{23} &\geq \nu_3; \\ x_{13} + x_{23} &= 1; x_{13} \geq 0, x_{23} \geq 0. \end{aligned}$$

**Step 5:** Substituting the value of  $\left\{x_{11}^* = \frac{19}{24}, x_{21}^* = \frac{5}{24}\right\}, \left\{x_{12}^* = \frac{15}{19}, x_{22}^* = \frac{4}{19}\right\}$  and

$\left\{x_{13}^* = \frac{45}{61}, x_{23}^* = \frac{16}{61}\right\}$  of Problem 2.8.6, Problem 2.8.7 and Problem 2.8.8 in Problem 2.8.3,

Problem 2.8.4 and Problem 2.8.5 respectively, the obtained alternative basic optimal solutions

of Problem 2.8.3, Problem 2.8.4 and Problem 2.8.5 are  $\{y_{11}^{1*} = 1, y_{21}^{1*} = 0\}, \{y_{11}^{2*} = 0, y_{21}^{2*} = 1\};$

$\{y_{12}^{2*} = 1, y_{22}^{2*} = 0\}, \{y_{12}^{2*} = 0, y_{22}^{2*} = 1\}$  and  $\{y_{13}^{1*} = 1, y_{23}^{1*} = 0\}, \{y_{13}^{2*} = 0, y_{23}^{2*} = 1\}$  respectively.

**Step 6:** Now,

$$\begin{aligned} &\text{minimum} \left\{ \left( \frac{3725}{24}, \frac{645}{4}, \frac{685}{4} \right), \left( \frac{3725}{24}, 161, \frac{494}{3} \right), \left( 155, \frac{3060}{19}, \frac{3250}{19} \right), \left( \frac{2950}{19}, \frac{3060}{19}, \frac{3130}{19} \right), \right. \\ &\quad \left. \left( \frac{9155}{61}, \frac{9540}{61}, \frac{10150}{61} \right), \left( \frac{9550}{61}, \frac{9900}{61}, \frac{10150}{61} \right) \right\} \\ &= \left( \frac{3725}{24}, 161, \frac{494}{3} \right), \left( \frac{2950}{19}, \frac{3060}{19}, \frac{3130}{19} \right), \left( \frac{9155}{61}, \frac{9540}{61}, \frac{10150}{61} \right). \end{aligned}$$

**Step 7:** The triangular fuzzy numbers  $\left( \frac{3725}{24}, 161, \frac{494}{3} \right), \left( \frac{2950}{19}, \frac{3060}{19}, \frac{3130}{19} \right)$  and

$\left(\frac{9155}{61}, \frac{9540}{61}, \frac{10150}{61}\right)$  represent minimum expected gain of Player I. The optimal strategies for corresponding to  $\left(\frac{3725}{24}, 161, \frac{494}{3}\right)$  are  $\left\{x_{11}^* = \frac{19}{24}, x_{21}^* = \frac{5}{24}\right\}$ , for all such minimum which is obtained corresponding to  $\left(\frac{2950}{19}, \frac{3060}{19}, \frac{3130}{19}\right)$  are  $\left\{x_{12}^* = \frac{15}{19}, x_{22}^* = \frac{4}{19}\right\}$  and corresponding to  $\left(\frac{9155}{61}, \frac{9540}{61}, \frac{10150}{61}\right)$  are  $\left\{x_{13}^* = \frac{45}{61}, x_{23}^* = \frac{16}{61}\right\}$ .

## 2.8.2 Maximum expected loss of Player II

The maximum expected loss of Player II and corresponding optimal strategies can be obtained as follows:

**Step 1:** Find  $(x_1^*, x_2^*) \in X$  such that value of  $((175, 180, 190)x_1y_1 + (150, 156, 158)x_1y_2 + (80, 90, 100)x_2y_1 + (175, 180, 190)x_2y_2)$  is maximum for all  $(y_1, y_2) \in Y$  i.e., find the solution  $\{x_i^*, i = 1, 2\}$  of Problem 2.8.9.

### Problem 2.8.9

Maximize  $\{(175, 180, 190)x_1y_1 + (150, 156, 158)x_1y_2 + (80, 90, 100)x_2y_1 + (175, 180, 190)x_2y_2\}$

Subject to

$$x_1 + x_2 = 1; y_1 + y_2 = 1;$$

$$x_1, x_2 \geq 0; y_1, y_2 \geq 0.$$

**Step 2:** Using the property,  $\sum_{i=1}^n \lambda (a_i^L(0), a_i^L(1), a_i^R(1), a_i^R(0)) =$

$\left(\sum_{i=1}^n a_i^L(0)\lambda, \sum_{i=1}^n a_i^L(1)\lambda, \sum_{i=1}^n a_i^R(1)\lambda, \sum_{i=1}^n a_i^R(0)\lambda\right), \lambda \geq 0$ , the Problem 2.8.9 can be transformed

into Problem 2.8.10.

**Problem 2.8.10**

$$\text{Maximize } \left\{ \begin{array}{l} 175x_1y_1 + 150x_1y_2 + 80x_2y_1 + 175x_2y_2, 180x_1y_1 + 156x_1y_2 + 90x_2y_1 + 180x_2y_2, \\ 190x_1y_1 + 158x_1y_2 + 100x_2y_1 + 190x_2y_2 \end{array} \right\}$$

Subject to

$$\begin{aligned} x_1 + x_2 &= 1; y_1 + y_2 = 1; \\ x_1, x_2 &\geq 0; y_1, y_2 \geq 0. \end{aligned}$$

**Step 3:** Find optimal solution  $\{x_{i1}^*, i=1,2\}$ ,  $\{x_{i2}^*, i=1,2\}$  and  $\{x_{i3}^*, i=1,2\}$  of Problem 2.8.11,

Problem 2.8.12 and Problem 2.8.13 respectively.

**Problem 2.8.11**

$$\text{Maximize } \{175x_{11}y_{11} + 150x_{11}y_{21} + 80x_{21}y_{11} + 175x_{21}y_{21}\}$$

Subject to

$$\begin{aligned} x_{11} + x_{21} &= 1; y_{11} + y_{21} = 1; \\ x_{11}, x_{21} &\geq 0; y_{11}, y_{21} \geq 0. \end{aligned}$$

**Problem 2.8.12**

$$\text{Maximize } \{180x_{12}y_{12} + 156x_{12}y_{22} + 90x_{22}y_{12} + 180x_{22}y_{22}\}$$

Subject to

$$\begin{aligned} x_{12} + x_{22} &= 1; y_{12} + y_{22} = 1; \\ x_{12}, x_{22} &\geq 0; y_{12}, y_{22} \geq 0. \end{aligned}$$

**Problem 2.8.13**

$$\text{Maximize } \{190x_{13}y_{13} + 158x_{13}y_{23} + 100x_{23}y_{13} + 190x_{23}y_{23}\}$$

Subject to

$$\begin{aligned} x_{13} + x_{23} &= 1; y_{13} + y_{23} = 1; \\ x_{13}, x_{23} &\geq 0; y_{13}, y_{23} \geq 0. \end{aligned}$$

**Step 4:** Since, in Problem 2.8.11, Problem 2.8.12 and Problem 2.8.13 only  $x_{i1}, x_{i2}$  and  $x_{i3}$  respectively have been considered as decision variables. So, Problem 2.8.11, Problem 2.8.12

and Problem 2.8.13 are linear programming problems and hence, the optimal value of Problem 2.8.11, Problem 2.8.12 and Problem 2.8.13 will be equal to optimal value of its corresponding dual problem i.e., Problem 2.8.14, Problem 2.8.15 and Problem 2.8.16 respectively.

**Problem 2.8.14**

$$\text{Minimize } \{\omega_1\}$$

Subject to

$$175y_{11} + 150y_{21} \leq \omega_1;$$

$$80y_{11} + 175y_{21} \leq \omega_1;$$

$$y_{11} + y_{21} = 1; y_{11} \geq 0, y_{21} \geq 0.$$

**Problem 2.8.15**

$$\text{Minimize } \{\omega_2\}$$

Subject to

$$180y_{12} + 156y_{22} \leq \omega_2;$$

$$90y_{12} + 180y_{22} \leq \omega_2;$$

$$y_{12} + y_{22} = 1; y_{12} \geq 0, y_{22} \geq 0.$$

**Problem 2.8.16**

$$\text{Minimize } \{\omega_3\}$$

Subject to

$$190y_{13} + 158y_{23} \leq \omega_3;$$

$$100y_{13} + 190y_{23} \leq \omega_3;$$

$$y_{13} + y_{23} = 1; y_{13} \geq 0, y_{23} \geq 0.$$

**Step 5:** Substituting the value of  $\left\{y_{11}^* = \frac{5}{24}, y_{21}^* = \frac{19}{24}\right\}, \left\{y_{12}^* = \frac{4}{19}, y_{22}^* = \frac{15}{19}\right\}$  and

$\left\{y_{13}^* = \frac{16}{61}, y_{23}^* = \frac{45}{61}\right\}$  of Problem 2.8.14, Problem 2.8.15 and Problem 2.8.16 in Problem 2.8.10,

Problem 2.8.11 and Problem 2.8.12 respectively, the obtained alternative basic optimal solutions of Problem 2.8.11, Problem 2.8.12 and Problem 2.8.13 are

$$\{x_{11}^{1*} = 1, x_{21}^{1*} = 0\}, \{x_{11}^{2*} = 0, x_{21}^{2*} = 1\}; \{x_{12}^{2*} = 1, x_{22}^{2*} = 0\}, \{x_{12}^{2*} = 0, x_{22}^{2*} = 1\} \quad \text{and} \quad \{x_{13}^{1*} = 1, x_{23}^{1*} = 0\},$$

$$\{y = x_{13}^{2*} = 0, x_{23}^{2*} = 1\} \text{ respectively.}$$

**Step 6:** Now,

$$\begin{aligned} & \text{maximum} \left\{ \left( \frac{3725}{24}, \frac{645}{4}, \frac{685}{4} \right), \left( \frac{3725}{24}, 161, \frac{494}{3} \right), \left( 155, \frac{3060}{19}, \frac{3250}{19} \right), \left( \frac{2950}{19}, \frac{3060}{19}, \frac{3130}{19} \right), \right. \\ & \left. \left( \frac{9155}{61}, \frac{9540}{61}, \frac{10150}{61} \right), \left( \frac{9550}{61}, \frac{9900}{61}, \frac{10150}{61} \right) \right\} \\ & = \left( \frac{3725}{24}, \frac{645}{4}, \frac{685}{4} \right), \left( 155, \frac{3060}{19}, \frac{3250}{19} \right), \left( \frac{9550}{61}, \frac{9900}{61}, \frac{10150}{61} \right). \end{aligned}$$

**Step 7:** The triangular fuzzy numbers  $\left( \frac{3725}{24}, \frac{645}{4}, \frac{685}{4} \right)$ ,  $\left( 155, \frac{3060}{19}, \frac{3250}{19} \right)$  and

$\left( \frac{9550}{61}, \frac{9900}{61}, \frac{10150}{61} \right)$  will represent maximum expected loss of Player II. The optimal

strategies corresponding to  $\left( \frac{3725}{24}, \frac{645}{4}, \frac{685}{4} \right)$  are  $\left\{ y_{11}^* = \frac{5}{24}, y_{21}^* = \frac{19}{24} \right\}$ , corresponding to

$\left( 155, \frac{3060}{19}, \frac{3250}{19} \right)$  are  $\left\{ y_{12}^* = \frac{4}{19}, y_{22}^* = \frac{15}{19} \right\}$  and corresponding to  $\left( \frac{9550}{61}, \frac{9900}{61}, \frac{10150}{61} \right)$  are

$$\left\{ y_{13}^* = \frac{16}{61}, y_{23}^* = \frac{45}{61} \right\}.$$

## 2.9 Conclusion

On the basis of present study, it can be concluded that some mathematically incorrect assumptions have been considered in the existing methods [36,65,70,71,83] for solving matrix games with fuzzy payoffs. Therefore, it is not correct to use these methods. Furthermore, to resolve flaws of the existing methods [36,65,70,71,83], a new method (named as Mehar method) is proposed for solving matrix games with fuzzy payoffs.



# Chapter 3

## Vaishnavi method for solving constrained matrix games with fuzzy payoffs<sup>‡</sup>

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In this chapter, flaws of the existing methods [64,74-76] for solving constrained matrix games with fuzzy payoffs (constrained matrix games in which payoffs are represented by fuzzy numbers) are pointed out. Also, to resolve these flaws, a new method (named as Vaishnavi method) is proposed to obtain the optimal strategies as well as minimum expected gain of Player I and maximum expected loss of Player II for constrained matrix games with fuzzy payoffs. To illustrate the proposed Vaishnavi method, some existing numerical problems of constrained matrix games with fuzzy payoffs are solved by the proposed Vaishnavi method.

### 3.1 Constrained matrix games with fuzzy payoffs

Charnes [34] pointed out that in two person zero sum games (matrix games), it is assumed that for strategies  $\{x_i, i = 1, 2, \dots, m\}$  and  $\{y_j, j = 1, 2, \dots, n\}$  of Player I and Player II respectively only the following constraints should be satisfied.

$$(i) \ x_i \geq 0, i = 1, 2, \dots, m \quad (ii) \ \sum_{i=1}^m x_i = 1 \quad (iii) \ y_j \geq 0, j = 1, 2, \dots, n \quad (iv) \ \sum_{j=1}^n y_j = 1$$

However, there are certain matrix games in real life where the strategies of the players are constrained to satisfy general linear inequalities rather than the only above mentioned constraints. To handle such real life problem [40], Charnes [34] extended the matrix games into constrained matrix games and then later in some what more generally by Kawaguchi and Maruyama [57]. Li and Cheng [74] pointed out that there is no method in the literature to solve

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<sup>‡</sup> The contents of this chapter are communicated in *Fuzzy Optimization and Decision Making*.

such constrained matrix games in which payoffs are represented by fuzzy numbers and proposed a method for solving constrained matrix games with fuzzy payoffs. Later, Li [75,76] with other co-authors proposed different methods for solving constrained matrix games with fuzzy payoffs.

### 3.2 Existing mathematical formulation of constrained matrix games with fuzzy payoffs

In the literature [106], the mathematical formulation of constrained matrix games with crisp payoffs is obtained by replacing the constraint set of strategies

$$X = \left\{ x_i, i = 1, 2, \dots, m \mid x_i \geq 0, i = 1, 2, \dots, m; \sum_{i=1}^m x_i = 1 \right\} \text{ for Player I with}$$

$$X = \left\{ x_i, i = 1, 2, \dots, m \mid x_i \geq 0, i = 1, 2, \dots, m; \sum_{i=1}^m x_i \leq 1, -\sum_{i=1}^m x_i \leq -1, \sum_{i=1}^m b_{il} x_i \leq c_l, l = 1, 2, \dots, p \right\} \text{ and}$$

$$Y = \left\{ y_j, j = 1, 2, \dots, n \mid y_j \geq 0, j = 1, 2, \dots, n; \sum_{j=1}^n y_j = 1 \right\} \text{ for Player II with}$$

$$Y = \left\{ y_j, j = 1, 2, \dots, n \mid y_j \geq 0, j = 1, 2, \dots, n; \sum_{j=1}^n y_j \geq 1, -\sum_{j=1}^n y_j \leq -1, \sum_{j=1}^n e_{kj} y_j \geq d_k, k = 1, 2, \dots, q \right\}$$

i.e., the minimum expected gain of Player I, the maximum expected loss of Player II and their corresponding strategies for constrained matrix games with crisp payoffs can be obtained by solving Problem 3.2.1 and Problem 3.2.2.

#### Problem 3.2.1

$$\text{Minimize } \left\{ \sum_{j=1}^n \sum_{i=1}^m x_i a_{ij} y_j \right\}$$

Subject to

$$\begin{aligned} \sum_{i=1}^m b_{il} x_i &\leq c_l, l = 1, 2, \dots, p; \\ \sum_{j=1}^n e_{kj} y_j &\geq d_k, k = 1, 2, \dots, q; \\ \sum_{i=1}^m x_i &\leq 1; \quad -\sum_{i=1}^m x_i \leq -1; \end{aligned}$$

#### Problem 3.2.2

$$\text{Maximize } \left\{ \sum_{i=1}^m \sum_{j=1}^n x_i a_{ij} y_j \right\}$$

Subject to

$$\begin{aligned} \sum_{i=1}^m b_{il} x_i &\leq c_l, l = 1, 2, \dots, p; \\ \sum_{j=1}^n e_{kj} y_j &\geq d_k, k = 1, 2, \dots, q; \\ \sum_{i=1}^m x_i &\leq 1; \quad -\sum_{i=1}^m x_i \leq -1; \end{aligned}$$

$$\sum_{j=1}^n y_j \geq 1; \quad -\sum_{j=1}^n y_j \geq -1;$$

$$x_i \geq 0, i = 1, 2, \dots, m; \quad y_j \geq 0, j = 1, 2, \dots, n.$$

$$\sum_{j=1}^n y_j \geq 1; \quad -\sum_{j=1}^n y_j \geq -1;$$

$$x_i \geq 0, i = 1, 2, \dots, m; \quad y_j \geq 0, j = 1, 2, \dots, n.$$

Furthermore, in the literature [106], it is pointed out that as in the Problem 3.2.1 only  $y_j$  has been considered as decision variables and in Problem 3.2.2 only  $x_i$  has been considered as decision variables so, Problem 3.2.1 and Problem 3.2.2 are linear programming problems. Therefore, the optimal value of Problem 3.2.1 and Problem 3.2.2 is same as the optimal values of its corresponding dual problem i.e., Problem 3.2.3 and Problem 3.2.4 respectively.

### Problem 3.2.3

$$\text{Maximize } \left\{ \sum_{k=1}^q d_k z_k + z_{q+1} - z_{q+2} \right\}$$

Subject to

$$\sum_{k=1}^q e_{kj} z_k + z_{q+1} - z_{q+2} \leq \sum_{i=1}^m a_{ij} x_i, j = 1, 2, \dots, n;$$

$$\sum_{i=1}^m b_{il} x_i \leq c_l, l = 1, 2, \dots, p;$$

$$\sum_{i=1}^m x_i \leq 1; \quad -\sum_{i=1}^m x_i \leq -1;$$

$$x_i \geq 0, i = 1, 2, \dots, m;$$

$$z_k \geq 0, k = 1, 2, \dots, q + 2.$$

### Problem 3.2.4

$$\text{Minimize } \left\{ \sum_{l=1}^p c_l t_l + t_{p+1} - t_{p+2} \right\}$$

Subject to

$$\sum_{l=1}^p b_{il} t_l + t_{p+1} - t_{p+2} \geq \sum_{j=1}^n a_{ij} y_j, i = 1, 2, \dots, m$$

$$\sum_{j=1}^n e_{kj} y_j \geq d_k, k = 1, 2, \dots, q;$$

$$\sum_{j=1}^n y_j \geq 1; \quad -\sum_{j=1}^n y_j \geq -1;$$

$$t_l \geq 0, l = 1, 2, \dots, p + 2;$$

$$y_j \geq 0, j = 1, 2, \dots, n.$$

On the same direction, Li and Cheng [74] used the following method to obtain the mathematical formulation of constrained matrix games with fuzzy payoffs.

**Step 1:** Replacing the parameters  $a_{ij}, b_{il}, e_{kj}, d_k$  and  $c_l$  with trapezoidal fuzzy numbers

$$(a_{ij}^L(0), a_{ij}^L(1), a_{ij}^R(1), a_{ij}^R(0)), (b_{il}^L(0), b_{il}^L(1), b_{il}^R(1), b_{il}^R(0)), (e_{kj}^L(0), e_{kj}^L(1), e_{kj}^R(1), e_{kj}^R(0)),$$

$$(d_k^L(0), d_k^L(1), d_k^R(1), d_k^R(0)) \text{ and } (c_l^L(0), c_l^L(1), c_l^R(1), c_l^R(0)) \text{ respectively, Problem 3.2.1 and}$$

Problem 3.2.2 are transformed into Problem 3.2.5 and Problem 3.2.6 respectively.

### Problem 3.2.5

$$\text{Minimize } \left\{ \sum_{j=1}^n \left( \sum_{i=1}^m (a_{ij}^L(0), a_{ij}^L(1), a_{ij}^R(1), a_{ij}^R(0)) x_i \right) y_j \right\}$$

Subject to

$$\begin{aligned} \sum_{i=1}^m (b_{il}^L(0), b_{il}^L(1), b_{il}^R(1), b_{il}^R(0)) x_i &\preceq (c_l^L(0), c_l^L(1), c_l^R(1), c_l^R(0)), \quad l = 1, 2, \dots, p; \\ \sum_{j=1}^n (e_{kj}^L(0), e_{kj}^L(1), e_{kj}^R(1), e_{kj}^R(0)) y_j &\succeq (d_k^L(0), d_k^L(1), d_k^R(1), d_k^R(0)), \quad k = 1, 2, \dots, q; \\ \sum_{i=1}^m x_i &\leq 1; \quad -\sum_{i=1}^m x_i \leq -1; \quad \sum_{j=1}^n y_j \geq 1; \quad -\sum_{j=1}^n y_j \geq -1; \\ x_i &\geq 0, \quad i = 1, 2, \dots, m; \\ y_j &\geq 0, \quad j = 1, 2, \dots, n. \end{aligned}$$

### Problem 3.2.6

$$\text{Maximize } \left\{ \sum_{i=1}^m \left( \sum_{j=1}^n (a_{ij}^L(0), a_{ij}^L(1), a_{ij}^R(1), a_{ij}^R(0)) y_j \right) x_i \right\}$$

Subject to

Constraints of Problem 3.2.5.

**Step 2:** Using the property,  $\sum_{i=1}^n \lambda (a_i^L(0), a_i^L(1), a_i^R(1), a_i^R(0)) =$

$$\left( \sum_{i=1}^m \lambda a_{ij}^L(0), \sum_{i=1}^m \lambda a_{ij}^L(1), \sum_{i=1}^m \lambda a_{ij}^R(1), \sum_{i=1}^m \lambda a_{ij}^R(0) \right), \lambda \geq 0, \text{ Problem 3.2.5 and Problem 3.2.6 can be}$$

transformed into Problem 3.2.7 and Problem 3.2.8 respectively.

### Problem 3.2.7

$$\text{Minimize } \left\{ \sum_{j=1}^n \left( \sum_{i=1}^m x_i a_{ij}^L(0), \sum_{i=1}^m x_i a_{ij}^L(1), \sum_{i=1}^m x_i a_{ij}^R(1), \sum_{i=1}^m x_i a_{ij}^R(0) \right) y_j \right\}$$

Subject to

$$\left( \sum_{i=1}^m b_{il}^L(0) x_i, \sum_{i=1}^m b_{il}^L(1) x_i, \sum_{i=1}^m b_{il}^R(1) x_i, \sum_{i=1}^m b_{il}^R(0) x_i \right) \preceq (c_l^L(0), c_l^L(1), c_l^R(1), c_l^R(0)), \quad l = 1, 2, \dots, p;$$

$$\left( \sum_{j=1}^n e_{kj}^L(0)y_j, \sum_{j=1}^n e_{kj}^L(1)y_j, \sum_{j=1}^n e_{kj}^R(1)y_j, \sum_{j=1}^n e_{kj}^R(0)y_j \right) \succeq (d_k^L(0), d_k^L(1), d_k^R(1), d_k^R(0)), \quad k=1, 2, \dots, q;$$

$$\sum_{i=1}^m x_i \leq 1; \quad -\sum_{i=1}^m x_i \leq -1; \quad \sum_{j=1}^n y_j \geq 1; \quad -\sum_{j=1}^n y_j \geq -1;$$

$$x_i \geq 0, i=1, 2, \dots, m;$$

$$y_j \geq 0, j=1, 2, \dots, n.$$

### Problem 3.2.8

$$\text{Maximize} \left\{ \sum_{i=1}^m \left( \sum_{j=1}^n y_j a_{ij}^L(0), \sum_{j=1}^n y_j a_{ij}^L(1), \sum_{j=1}^n y_j a_{ij}^R(1), \sum_{j=1}^n y_j a_{ij}^R(0) \right) x_i \right\}$$

Subject to

Constraints of Problem 3.2.7.

**Step 3:** Using the relation  $(a^L(0), a^L(1), a^R(1), a^R(0)) \preceq (b^L(0), b^L(1), b^R(1), b^R(0)) \Rightarrow a^L(0) \leq$

$$b^L(0), a^L(1) \leq b^L(1), a^R(1) \leq b^R(1), a^R(0) \leq b^R(0) \Rightarrow \lambda_1 a^L(0) + \lambda_2 a^L(1) + \lambda_3 a^R(1) + \lambda_4 a^R(0)$$

$$\leq \lambda_1 b^L(0) + \lambda_2 b^L(1) + \lambda_3 b^R(1) + \lambda_4 b^R(0), \text{ where } \lambda_1, \lambda_2, \lambda_3, \lambda_4 \geq 0 \text{ and } \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 1,$$

Problem 3.2.7 and Problem 3.2.8 can be transformed into Problem 3.2.9 and Problem 3.2.10 respectively.

### Problem 3.2.9

$$\text{Minimize} \left\{ \sum_{j=1}^n \left( \lambda_1 \sum_{i=1}^m a_{ij}^L(0)x_i + \lambda_2 \sum_{i=1}^m a_{ij}^L(1)x_i + \lambda_3 \sum_{i=1}^m a_{ij}^R(1)x_i + \lambda_4 \sum_{i=1}^m a_{ij}^R(0)x_i \right) y_j \right\}$$

Subject to

$$\lambda_1 \sum_{i=1}^m b_{il}^L(0)x_i + \lambda_2 \sum_{i=1}^m b_{il}^L(1)x_i + \lambda_3 \sum_{i=1}^m b_{il}^R(1)x_i + \lambda_4 \sum_{i=1}^m b_{il}^R(0)x_i \\ \leq \lambda_1 c_l^L(0) + \lambda_2 c_l^L(1) + \lambda_3 c_l^R(1) + \lambda_4 c_l^R(0), \quad l=1, 2, \dots, p;$$

$$\lambda_1 \sum_{j=1}^n e_{kj}^L(0)y_j + \lambda_2 \sum_{j=1}^n e_{kj}^L(1)y_j + \lambda_3 \sum_{j=1}^n e_{kj}^R(1)y_j + \lambda_4 \sum_{j=1}^n e_{kj}^R(0)y_j \\ \geq \lambda_1 d_k^L(0) + \lambda_2 d_k^L(1) + \lambda_3 d_k^R(1) + \lambda_4 d_k^R(0), \quad k=1, 2, \dots, q;$$

$$\sum_{i=1}^m x_i \leq 1; \quad -\sum_{i=1}^m x_i \leq -1; \quad \sum_{j=1}^n y_j \geq 1; \quad -\sum_{j=1}^n y_j \geq -1;$$

$$x_i \geq 0, i = 1, 2, \dots, m;$$

$$y_j \geq 0, j = 1, 2, \dots, n.$$

**Problem 3.2.10**

$$\text{Maximize} \left\{ \sum_{i=1}^m \left( \lambda_1 \sum_{j=1}^n a_{ij}^L(0) y_j + \lambda_2 \sum_{j=1}^n a_{ij}^L(1) y_j + \lambda_3 \sum_{j=1}^n a_{ij}^R(1) y_j + \lambda_4 \sum_{j=1}^n a_{ij}^R(0) y_j \right) x_i \right\}$$

Subject to

Constraints of Problem 3.2.9.

**Step 4:** The dual of Problem 3.2.9 by considering only  $y_j, j = 1, 2, \dots, n$  as decision variables

and dual of Problem 3.2.10 by considering only  $x_i, i = 1, 2, \dots, m$  as decision variables are

Problem 3.2.11 and Problem 3.2.12 respectively.

**Problem 3.2.11**

$$\text{Maximize} \left\{ \lambda_1 \sum_{k=1}^q d_k^L(0) z_k + \lambda_2 \sum_{k=1}^q d_k^L(1) z_k + \lambda_3 \sum_{k=1}^q d_k^R(1) z_k + \lambda_4 \sum_{k=1}^q d_k^R(0) z_k + z_{q+1} - z_{q+2} \right\}$$

Subject to

$$\lambda_1 \sum_{k=1}^q e_{kj}^L(0) z_k + \lambda_2 \sum_{k=1}^q e_{kj}^L(1) z_k + \lambda_3 \sum_{k=1}^q e_{kj}^R(1) z_k + \lambda_4 \sum_{k=1}^q e_{kj}^R(0) z_k + z_{q+1} - z_{q+2} \leq$$

$$\lambda_1 \sum_{i=1}^m a_{ij}^L(0) x_i + \lambda_2 \sum_{i=1}^m a_{ij}^L(1) x_i + \lambda_3 \sum_{i=1}^m a_{ij}^R(1) x_i + \lambda_4 \sum_{i=1}^m a_{ij}^R(0) x_i, j = 1, 2, \dots, n;$$

$$\lambda_1 \sum_{i=1}^m b_{il}^L(0) x_i + \lambda_2 \sum_{i=1}^m b_{il}^L(1) x_i + \lambda_3 \sum_{i=1}^m b_{il}^R(1) x_i + \lambda_4 \sum_{i=1}^m b_{il}^R(0) x_i \leq$$

$$\lambda_1 c_l^L(0) + \lambda_2 c_l^L(1) + \lambda_3 c_l^R(1) + \lambda_4 c_l^R(0), l = 1, 2, \dots, p;$$

$$\sum_{i=1}^m x_i \leq 1; \quad -\sum_{i=1}^m x_i \leq -1;$$

$$x_i \geq 0, i = 1, 2, \dots, m;$$

$$z_k \geq 0, k = 1, 2, \dots, q + 2.$$

**Problem 3.2.12**

$$\text{Minimize} \left\{ \lambda_1 \sum_{l=1}^p c_l^L(0) t_l + \lambda_2 \sum_{l=1}^p c_l^L(1) t_l + \lambda_3 \sum_{l=1}^p c_l^R(1) t_l + \lambda_4 \sum_{l=1}^p c_l^R(0) t_l + t_{p+1} - t_{p+2} \right\}$$

Subject to

$$\begin{aligned}
& \lambda_1 \sum_{l=1}^p b_{il}^L(0)t_l + \lambda_2 \sum_{l=1}^p b_{il}^L(1)t_l + \lambda_3 \sum_{l=1}^p b_{il}^R(1)t_l + \lambda_4 \sum_{l=1}^p b_{il}^R(0)t_l + t_{p+1} - t_{p+2} \geq \\
& \quad \lambda_1 \sum_{j=1}^n a_{ij}^L(0)y_j + \lambda_2 \sum_{j=1}^n a_{ij}^L(1)y_j + \lambda_3 \sum_{j=1}^n a_{ij}^R(1)y_j + \lambda_4 \sum_{j=1}^n a_{ij}^R(0)y_j, i=1,2,\dots,m \\
& \lambda_1 \sum_{j=1}^n e_{kj}^L(0)y_j + \lambda_2 \sum_{j=1}^n e_{kj}^L(1)y_j + \lambda_3 \sum_{j=1}^n e_{kj}^R(1)y_j + \lambda_4 \sum_{j=1}^n e_{kj}^R(0)y_j \geq \\
& \quad \lambda_1 d_k^L(0) + \lambda_2 d_k^L(1) + \lambda_3 d_k^R(1) + \lambda_4 d_k^R(0), k=1,2,\dots,q; \\
& \sum_{j=1}^n y_j \geq 1; \quad -\sum_{j=1}^n y_j \geq -1; \\
& t_l \geq 0, l=1,2,\dots,p+2; \\
& y_j \geq 0, j=1,2,\dots,n.
\end{aligned}$$

**Step 5:** Using the relation,  $\lambda_1 a^L(0) + \lambda_2 a^L(1) + \lambda_3 a^R(1) + \lambda_4 a^R(0) \geq \lambda_1 b^L(0) + \lambda_2 b^L(1) + \lambda_3 b^R(1) +$

$\lambda_4 b^R(0) \Rightarrow (a^L(0), a^L(1), a^R(1), a^R(0)) \succeq (b^L(0), b^L(1), b^R(1), b^R(0))$ , Problem 3.2.11 and

Problem 3.2.12 can be transformed into Problem 3.2.13 and Problem 3.2.14 respectively.

### Problem 3.2.13

$$\text{Maximize} \left\{ \left( \sum_{k=1}^q d_k^L(0)z_k, \sum_{k=1}^q d_k^L(1)z_k, \sum_{k=1}^q d_k^R(1)z_k, \sum_{k=1}^q d_k^R(0)z_k \right) + z_{q+1} - z_{q+2} \right\}$$

Subject to

$$\begin{aligned}
& \left( \sum_{k=1}^q e_{kj}^L(0)z_k, \sum_{k=1}^q e_{kj}^L(1)z_k, \sum_{k=1}^q e_{kj}^R(1)z_k, \sum_{k=1}^q e_{kj}^R(0)z_k \right) + z_{q+1} - z_{q+2} \preceq \\
& \quad \left( \sum_{i=1}^m a_{ij}^L(0)x_i, \sum_{i=1}^m a_{ij}^L(1)x_i, \sum_{i=1}^m a_{ij}^R(1)x_i, \sum_{i=1}^m a_{ij}^R(0)x_i \right), j=1,2,\dots,n; \\
& \left( \sum_{i=1}^m b_{il}^L(0)x_i, \sum_{i=1}^m b_{il}^L(1)x_i, \sum_{i=1}^m b_{il}^R(1)x_i, \sum_{i=1}^m b_{il}^R(0)x_i \right) \preceq (c_l^L(0), c_l^L(1), c_l^R(1), c_l^R(0)), l=1,2,\dots,p; \\
& \sum_{i=1}^m x_i \leq 1; \quad -\sum_{i=1}^m x_i \leq -1; \\
& x_i \geq 0, i=1,2,\dots,m; \\
& z_k \geq 0, k=1,2,\dots,q+2.
\end{aligned}$$

### Problem 3.2.14

$$\text{Minimize} \left\{ \left( \sum_{l=1}^p c_l^L(0)t_l, \sum_{l=1}^p c_l^L(1)t_l, \sum_{l=1}^p c_l^R(1)t_l, \sum_{l=1}^p c_l^R(0)t_l \right) + t_{p+1} - t_{p+2} \right\}$$

Subject to

$$\begin{aligned}
& \left( \sum_{l=1}^p b_{il}^L(0)t_l, \sum_{l=1}^p b_{il}^L(1)t_l, \sum_{l=1}^p b_{il}^R(1)t_l, \sum_{l=1}^p b_{il}^R(0)t_l \right) + t_{p+1} - t_{p+2} \succeq \\
& \quad \left( \sum_{j=1}^n a_{ij}^L(0)y_j, \sum_{j=1}^n a_{ij}^L(1)y_j, \sum_{j=1}^n a_{ij}^R(1)y_j, \sum_{j=1}^n a_{ij}^R(0)y_j \right), i = 1, 2, \dots, m \\
& \left( \sum_{j=1}^n e_{kj}^L(0)y_j, \sum_{j=1}^n e_{kj}^L(1)y_j, \sum_{j=1}^n e_{kj}^R(1)y_j, \sum_{j=1}^n e_{kj}^R(0)y_j \right) \succeq (d_k^L(0), d_k^L(1), d_k^R(1), d_k^R(0)), k = 1, 2, \dots, q; \\
& \sum_{j=1}^n y_j \geq 1; \quad -\sum_{j=1}^n y_j \geq -1; \\
& t_l \geq 0, l = 1, 2, \dots, p+2; \\
& y_j \geq 0, j = 1, 2, \dots, n.
\end{aligned}$$

### 3.3 Literature review of constrained matrix games with fuzzy payoffs

In this section, a brief review of the methods, proposed in the literature for solving constrained matrix games with fuzzy payoffs, is presented.

Li and Cheng [74] solved the Problem 3.2.11 and Problem 3.2.12 to find minimum expected gain of Player I and maximum expected loss of Player II respectively.

Li and Hong [75,76] split the Problem 3.2.11 into Problem 3.3.1, Problem 3.3.2, Problem 3.3.3 and Problem 3.3.4 and obtained the minimum expected gain of Player I by using the optimal values of these problems as well as split the Problem 3.2.12 into Problem 3.3.5, Problem 3.3.6, Problem 3.3.7 and Problem 3.3.8 and obtained the maximum expected loss of Player I by using the optimal values of these problems.

#### Problem 3.3.1

$$\text{Maximize} \left\{ \sum_{k=1}^q d_k^L(0)z_k + z_{q+1} - z_{q+2} \right\}$$

Subject to

$$\begin{aligned}
& \sum_{k=1}^q e_{kj}^L(0)z_k + z_{q+1} - z_{q+2} \leq \\
& \sum_{i=1}^m a_{ij}^L(0)x_i, \quad j = 1, 2, \dots, n;
\end{aligned}$$

#### Problem 3.3.2

$$\text{Maximize} \left\{ \sum_{k=1}^q d_k^L(1)z_k + z_{q+1} - z_{q+2} \right\}$$

Subject to

$$\begin{aligned}
& \sum_{k=1}^q e_{kj}^L(1)z_k + z_{q+1} - z_{q+2} \leq \\
& \sum_{i=1}^m a_{ij}^L(1)x_i, \quad j = 1, 2, \dots, n;
\end{aligned}$$

$$\sum_{i=1}^m b_{il}^L(0)x_i \leq c_l^L(0), \quad l=1,2,\dots,p;$$

$$\sum_{i=1}^m x_i \leq 1; -\sum_{i=1}^m x_i \leq -1;$$

$$x_i \geq 0, \quad i=1,2,\dots,m;$$

$$z_k \geq 0, \quad k=1,2,\dots,q+2;$$

### Problem 3.3.3

$$\text{Maximize} \left\{ \sum_{k=1}^q d_k^R(1)z_k + z_{q+1} - z_{q+2} \right\}$$

Subject to

$$\sum_{k=1}^q e_{kj}^R(1)z_k + z_{q+1} - z_{q+2} \leq$$

$$\sum_{i=1}^m a_{ij}^R(1)x_i, \quad j=1,2,\dots,n;$$

$$\sum_{i=1}^m b_{il}^R(1)x_i \leq c_l^R(1), \quad l=1,2,\dots,p;$$

$$\sum_{i=1}^m x_i \leq 1; -\sum_{i=1}^m x_i \leq -1;$$

$$x_i \geq 0, \quad i=1,2,\dots,m;$$

$$z_k \geq 0, \quad k=1,2,\dots,q+2;$$

### Problem 3.3.5

$$\text{Minimize} \left\{ \sum_{l=1}^p c_l^L(0)t_l + t_{p+1} - t_{p+2} \right\}$$

Subject to

$$\sum_{l=1}^p b_{il}^L(0)t_l + t_{p+1} - t_{p+2} \geq$$

$$\sum_{j=1}^n a_{ij}^L(0)y_j, \quad i=1,2,\dots,m;$$

$$\sum_{j=1}^n e_{kj}^L(0)y_j \geq d_k^L(0), \quad k=1,2,\dots,q;$$

$$\sum_{j=1}^n y_j \geq 1; -\sum_{j=1}^n y_j \geq -1;$$

$$t_l \geq 0, \quad l=1,2,\dots,p+2;$$

$$\sum_{i=1}^m b_{il}^L(1)x_i \leq c_l^L(1), \quad l=1,2,\dots,p;$$

$$\sum_{i=1}^m x_i \leq 1; -\sum_{i=1}^m x_i \leq -1;$$

$$x_i \geq 0, \quad i=1,2,\dots,m;$$

$$z_k \geq 0, \quad k=1,2,\dots,q+2;$$

### Problem 3.3.4

$$\text{Maximize} \left\{ \sum_{k=1}^q d_k^R(0)z_k + z_{q+1} - z_{q+2} \right\}$$

Subject to

$$\sum_{k=1}^q e_{kj}^R(0)z_k + z_{q+1} - z_{q+2} \leq$$

$$\sum_{i=1}^m a_{ij}^R(0)x_i, \quad j=1,2,\dots,n;$$

$$\sum_{i=1}^m b_{il}^R(0)x_i \leq c_l^R(0), \quad l=1,2,\dots,p;$$

$$\sum_{i=1}^m x_i \leq 1; -\sum_{i=1}^m x_i \leq -1;$$

$$x_i \geq 0, \quad i=1,2,\dots,m;$$

$$z_k \geq 0, \quad k=1,2,\dots,q+2;$$

### Problem 3.3.6

$$\text{Minimize} \left\{ \sum_{l=1}^p c_l^L(1)t_l + t_{p+1} - t_{p+2} \right\}$$

Subject to

$$\sum_{l=1}^p b_{il}^L(1)t_l + t_{p+1} - t_{p+2} \geq$$

$$\sum_{j=1}^n a_{ij}^L(1)y_j, \quad i=1,2,\dots,m;$$

$$\sum_{j=1}^n e_{kj}^L(1)y_j \geq d_k^L(1), \quad k=1,2,\dots,q;$$

$$\sum_{j=1}^n y_j \geq 1; -\sum_{j=1}^n y_j \geq -1;$$

$$t_l \geq 0, \quad l=1,2,\dots,p+2;$$

$$y_j \geq 0, j = 1, 2, \dots, n.$$

### Problem 3.3.7

$$\text{Minimize } \left\{ \sum_{l=1}^p c_l^R(1) t_l + t_{p+1} - t_{p+2} \right\}$$

Subject to

$$\sum_{l=1}^p b_{il}^R(1) t_l + t_{p+1} - t_{p+2} \geq \sum_{j=1}^n a_{ij}^R(1) y_j, i = 1, 2, \dots, m;$$

$$\sum_{j=1}^n e_{kj}^R(1) y_j \geq d_k^R(1), k = 1, 2, \dots, q;$$

$$\sum_{j=1}^n y_j \geq 1; -\sum_{j=1}^n y_j \geq -1;$$

$$t_l \geq 0, l = 1, 2, \dots, p+2;$$

$$y_j \geq 0, j = 1, 2, \dots, n.$$

$$y_j \geq 0, j = 1, 2, \dots, n.$$

### Problem 3.3.8

$$\text{Minimize } \left\{ \sum_{l=1}^p c_l^R(0) t_l + t_{p+1} - t_{p+2} \right\}$$

Subject to

$$\sum_{l=1}^p b_{il}^R(0) t_l + t_{p+1} - t_{p+2} \geq \sum_{j=1}^n a_{ij}^R(0) y_j, i = 1, 2, \dots, m;$$

$$\sum_{j=1}^n e_{kj}^R(0) y_j \geq d_k^R(0), k = 1, 2, \dots, q;$$

$$\sum_{j=1}^n y_j \geq 1; -\sum_{j=1}^n y_j \geq -1;$$

$$t_l \geq 0, l = 1, 2, \dots, p+2;$$

$$y_j \geq 0, j = 1, 2, \dots, n.$$

### 3.4 Flaws of the existing methods

In this section, shortcomings of the existing method [75,76] are pointed out.

1. If  $a^L(0), a^L(1), a^R(1), a^R(0), b^L(0), b^L(1), b^R(1)$  and  $b^R(0)$  are real numbers then

$$\lambda_1 a^L(0) + \lambda_2 a^L(1) + \lambda_3 a^R(1) + \lambda_4 a^R(0) \geq \lambda_1 b^L(0) + \lambda_2 b^L(1) + \lambda_3 b^R(1) + \lambda_4 b^R(0), \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 1,$$

$$\lambda_1, \lambda_2, \lambda_3, \lambda_4 \geq 0 \nRightarrow a^L(0) \geq b^L(0), a^L(1) \geq b^L(1), a^R(1) \geq b^R(1), a^R(0) \geq b^R(0) \text{ e.g., if } a^L(0) = 1,$$

$$a^L(1) = 2, a^R(1) = 3, a^R(0) = 4, b^L(0) = 0, b^L(1) = 3, b^R(1) = 5, b^R(0) = 10, \lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \frac{1}{4}$$

$$\text{then } \lambda_1 a^L(0) + \lambda_2 a^L(1) + \lambda_3 a^R(1) + \lambda_4 a^R(0) = \frac{10}{4} \text{ and } \lambda_1 b^L(0) + \lambda_2 b^L(1) + \lambda_3 b^R(1) + \lambda_4 b^R(0) = \frac{18}{4}$$

$$\Rightarrow \lambda_1 b^L(0) + \lambda_2 b^L(1) + \lambda_3 b^R(1) + \lambda_4 b^R(0) > \lambda_1 a^L(0) + \lambda_2 a^L(1) + \lambda_3 a^R(1) + \lambda_4 a^R(0) \text{ but}$$

$$b^L(0) < a^L(0), b^L(1) < a^L(1), b^R(1) < a^R(1), b^R(0) < a^R(0).$$

Similarly, if  $a^L(0) = 1, a^L(1) = 2, a^R(1) = 3, a^R(0) = 5, b^L(0) = 0, b^L(1) = 1, b^R(1) = 4, b^R(0) = 6,$

$\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \frac{1}{4}$  then  $\lambda_1 a^L(0) + \lambda_2 a^L(1) + \lambda_3 a^R(1) + \lambda_4 a^R(0) = \lambda_1 b^L(0) + \lambda_2 b^L(1) + \lambda_3 b^R(1) + \lambda_4 b^R(0) = \frac{11}{4}$ . While,  $b^L(0) < a^L(0)$ ,  $b^L(1) < a^L(1)$ ,  $b^R(1) > a^R(1)$ ,  $b^R(0) > a^R(0)$ . However, Li and Hong [75,76] have used the mathematically incorrect assumption  $\lambda_1 a^L(0) + \lambda_2 a^L(1) + \lambda_3 a^R(1) + \lambda_4 a^R(0) \geq \lambda_1 b^L(0) + \lambda_2 b^L(1) + \lambda_3 b^R(1) + \lambda_4 b^R(0)$ ;  $\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 1$ ;  $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \geq 0$   $\Rightarrow a^L(0) \geq b^L(0)$ ,  $a^L(1) \geq b^L(1)$ ,  $a^R(1) \geq b^R(1)$ ,  $a^R(0) \geq b^R(0)$  for transforming Problem 3.2.11 and Problem 3.2.12 into Problem 3.3.1, Problem 3.3.2, Problem 3.3.3, Problem 3.3.4 and Problem 3.3.5, Problem 3.3.6, Problem 3.3.7, Problem 3.3.8 respectively. Therefore, the existing method [74-76], in which the minimum expected gain of Player I obtained by splitting 3.2.11 into Problem 3.3.1, Problem 3.3.2, Problem 3.3.3, Problem 3.3.4 as well as maximum expected loss of Player II obtained by splitting Problem 3.3.12 into Problem 3.3.5, Problem 3.3.6, Problem 3.3.7, Problem 3.3.8 respectively, is not valid.

### 3.5 Proposed Vaishnavi method

In the existing methods [75,76], it is assumed that if  $\tilde{a} = (a^L(0), a^L(1), a^R(1), a^R(0))$  and  $\tilde{b} = (b^L(0), b^L(1), b^R(1), b^R(0))$  are two trapezoidal fuzzy numbers then  $\tilde{a} \succeq \tilde{b}$  if  $a^L(0) \geq b^L(0)$ ,  $a^L(1) \geq b^L(1)$ ,  $a^R(1) \geq b^R(1)$ ,  $a^R(0) \geq b^R(0)$ . In this section, a new method (named as Vaishnavi method), on the basis of this comparing method, is proposed to find the minimum expected gain of Player I, maximum expected loss of Player II and their corresponding optimal strategies.

#### 3.5.1 Minimum expected gain of Player I

Using the comparing method,  $\tilde{a} \preceq \tilde{b}$  if  $a^L(0) \leq b^L(0)$ ,  $a^L(1) \leq b^L(1)$ ,  $a^R(1) \leq b^R(1)$ ,  $a^R(0) \leq b^R(0)$ , the minimum expected gain of Player I and the corresponding optimal strategies can be obtained as follows:

**Step 1:** Find  $(y_1^*, y_2^*, \dots, y_n^*) \in Y$  such that the value of  $\sum_{j=1}^n \sum_{i=1}^m x_i (a_{ij}^L(0), a_{ij}^L(1), a_{ij}^R(1), a_{ij}^R(0)) y_j$

is minimum for all  $(x_1, x_2, \dots, x_m) \in X$  i.e., find the optimal solution of Problem 3.2.5.

**Step 2:** Using the property,  $\sum_{i=1}^n \lambda (a_i^L(0), a_i^L(1), a_i^R(1), a_i^R(0)) =$

$\left( \sum_{i=1}^n \lambda a_{ij}^L(0), \sum_{i=1}^n \lambda a_{ij}^L(1), \sum_{i=1}^n \lambda a_{ij}^R(1), \sum_{i=1}^n \lambda a_{ij}^R(0) \right), \lambda \geq 0$ , the Problem 3.2.5 can be transformed into

Problem 3.2.7.

**Step 3:** Using the property  $(a^L(0), a^L(1), a^R(1), a^R(0)) \preceq (b^L(0), b^L(1), b^R(1), b^R(0)) \Rightarrow a^L(0) \leq$

$b^L(0), a^L(1) \leq b^L(1), a^R(1) \leq b^R(1), a^R(0) \leq b^R(0)$ , the Problem 3.2.7 can be transformed into

Problem 3.5.1.

### Problem 3.5.1

Minimize  $\left\{ \left( \sum_{j=1}^n \sum_{i=1}^m x_i a_{ij}^L(0) y_j, \sum_{j=1}^n \sum_{i=1}^m x_i a_{ij}^L(1) y_j, \sum_{j=1}^n \sum_{i=1}^m x_i a_{ij}^R(1) y_j, \sum_{j=1}^n \sum_{i=1}^m x_i a_{ij}^R(0) y_j \right) \right\}$

Subject to

$$\sum_{i=1}^m b_{il}^L(0) x_i \leq c_l^L(0), l = 1, 2, \dots, p; \quad \sum_{i=1}^m b_{il}^L(1) x_i \leq c_l^L(1), l = 1, 2, \dots, p;$$

$$\sum_{i=1}^m b_{il}^R(1) x_i \leq c_l^R(1), l = 1, 2, \dots, p; \quad \sum_{i=1}^m b_{il}^R(0) x_i \leq c_l^R(0), l = 1, 2, \dots, p;$$

$$\sum_{j=1}^n e_{kj}^L(0) y_j \geq d_k^L(0), k = 1, 2, \dots, q; \quad \sum_{j=1}^n e_{kj}^L(1) y_j \geq d_k^L(1), k = 1, 2, \dots, q;$$

$$\sum_{j=1}^n e_{kj}^R(1) y_j \geq d_k^R(1), k = 1, 2, \dots, q; \quad \sum_{j=1}^n e_{kj}^R(0) y_j \geq d_k^R(0), k = 1, 2, \dots, q;$$

$$\sum_{i=1}^m x_i \leq 1; \quad -\sum_{i=1}^m x_i \leq -1; \quad \sum_{j=1}^n y_j \geq 1; \quad -\sum_{j=1}^n y_j \geq -1;$$

$$x_i \geq 0, i = 1, 2, \dots, m; \quad y_j \geq 0, j = 1, 2, \dots, n.$$

**Step 4:** According to comparing method, to find optimal solution  $\{y_j^*, j = 1, 2, \dots, n\}$  of

Problem 3.5.1 such that the value of

$\left( \sum_{j=1}^n \sum_{i=1}^m x_i a_{ij}^L(0) y_j, \sum_{j=1}^n \sum_{i=1}^m x_i a_{ij}^L(1) y_j, \sum_{j=1}^n \sum_{i=1}^m x_i a_{ij}^R(1) y_j, \sum_{j=1}^n \sum_{i=1}^m x_i a_{ij}^R(0) y_j \right)$  is minimum for all

$(x_1, x_2, \dots, x_m) \in X$  is equivalent to find  $\{y_j^*, j=1, 2, \dots, n\}$  such that value of

$\sum_{j=1}^n \sum_{i=1}^m x_i a_{ij}^L(0) y_j, \sum_{j=1}^n \sum_{i=1}^m x_i a_{ij}^L(1) y_j, \sum_{j=1}^n \sum_{i=1}^m x_i a_{ij}^R(1) y_j, \sum_{j=1}^n \sum_{i=1}^m x_i a_{ij}^R(0) y_j$  is minimum for all

$(x_1, x_2, \dots, x_m) \in X$  or if it is not possible to find  $\{y_j^*, j=1, 2, \dots, n\}$  such that value of

$\sum_{j=1}^n \sum_{i=1}^m x_i a_{ij}^L(0) y_j, \sum_{j=1}^n \sum_{i=1}^m x_i a_{ij}^L(1) y_j, \sum_{j=1}^n \sum_{i=1}^m x_i a_{ij}^R(1) y_j, \sum_{j=1}^n \sum_{i=1}^m x_i a_{ij}^R(0) y_j$  is minimum for all

$(x_1, x_2, \dots, x_m) \in X$  then it is equivalent to find  $\{y_{j1}^*, j=1, 2, \dots, n\}$  such that value of

$\sum_{j=1}^n \sum_{i=1}^m x_{i1} a_{ij}^L(0) y_{j1}$  is minimum for all  $(x_{11}, x_{21}, \dots, x_{m1}) \in X$  but value of

$\sum_{j=1}^n \sum_{i=1}^m x_{i1} a_{ij}^L(1) y_{j1}$  and/or  $\sum_{j=1}^n \sum_{i=1}^m x_{i1} a_{ij}^R(1) y_{j1}$  and/or  $\sum_{j=1}^n \sum_{i=1}^m x_{i1} a_{ij}^R(0) y_{j1}$  is not minimum and to find

$\{y_{j2}^*, j=1, 2, \dots, n\}$  such that value of  $\sum_{j=1}^n \sum_{i=1}^m x_{i2} a_{ij}^L(1) y_{j2}$  is minimum for all

$(x_{12}, x_{22}, \dots, x_{m2}) \in X$  but value of  $\sum_{j=1}^n \sum_{i=1}^m x_{i2} a_{ij}^L(0) y_{j2}$  and/or  $\sum_{j=1}^n \sum_{i=1}^m x_{i2} a_{ij}^R(1) y_{j2}$  and/or

$\sum_{j=1}^n \sum_{i=1}^m x_{i2} a_{ij}^R(0) y_{j2}$  is not minimum and to find  $\{y_{j3}^*, j=1, 2, \dots, n\}$  such that value of

$\sum_{j=1}^n \sum_{i=1}^m x_{i3} a_{ij}^R(1) y_{j3}$  is minimum for all  $(x_{13}, x_{23}, \dots, x_{m3}) \in X$  but value of

$\sum_{j=1}^n \sum_{i=1}^m x_{i3} a_{ij}^L(0) y_{j3}$  and/or  $\sum_{j=1}^n \sum_{i=1}^m x_{i3} a_{ij}^L(1) y_{j3}$  and/or  $\sum_{j=1}^n \sum_{i=1}^m x_{i3} a_{ij}^R(0) y_{j3}$  is not minimum and to

find  $\{y_{j4}^*, j=1, 2, \dots, n\}$  such that value of  $\sum_{j=1}^n \sum_{i=1}^m x_{i4} a_{ij}^R(0) y_{j4}$  is minimum for all

$(x_{14}, x_{24}, \dots, x_{m4}) \in X$  but value of  $\sum_{j=1}^n \sum_{i=1}^m x_{i4} a_{ij}^L(0) y_{j4}$  and/or  $\sum_{j=1}^n \sum_{i=1}^m x_{i4} a_{ij}^L(1) y_{j4}$  and/or

$\sum_{j=1}^n \sum_{i=1}^m x_{i4} a_{ij}^R(1) y_{j4}$  is not minimum i.e., to find the optimal solution

$\{y_{j1}^*, j=1, 2, \dots, n\}, \{y_{j2}^*, j=1, 2, \dots, n\}, \{y_{j3}^*, j=1, 2, \dots, n\}, \{y_{j4}^*, j=1, 2, \dots, n\}$  of Problem

3.5.2, Problem 3.5.3, Problem 3.5.4, Problem 3.5.5 respectively.

### Problem 3.5.2

$$\text{Minimize } \left\{ \sum_{j=1}^n \sum_{i=1}^m x_{i1} a_{ij}^L(0) y_{j1} \right\}$$

Subject to

$$\begin{aligned} \sum_{i=1}^m b_{il}^L(0) x_{i1} &\leq c_l^L(0), \quad l=1, 2, \dots, p; & \sum_{i=1}^m b_{il}^L(1) x_{i1} &\leq c_l^L(1), \quad l=1, 2, \dots, p; \\ \sum_{i=1}^m b_{il}^R(1) x_{i1} &\leq c_l^R(1), \quad l=1, 2, \dots, p; & \sum_{i=1}^m b_{il}^R(0) x_{i1} &\leq c_l^R(0), \quad l=1, 2, \dots, p; \\ \sum_{j=1}^n e_{kj}^L(0) y_{j1} &\geq d_k^L(0), \quad k=1, 2, \dots, q; & \sum_{j=1}^n e_{kj}^L(1) y_{j1} &\geq d_k^L(1), \quad k=1, 2, \dots, q; \\ \sum_{j=1}^n e_{kj}^R(1) y_{j1} &\geq d_k^R(1), \quad k=1, 2, \dots, q; & \sum_{j=1}^n e_{kj}^R(0) y_{j1} &\geq d_k^R(0), \quad k=1, 2, \dots, q; \\ \sum_{i=1}^m x_{i1} &\leq 1; & -\sum_{i=1}^m x_{i1} &\leq -1; & \sum_{j=1}^n y_{j1} &\geq 1; & -\sum_{j=1}^n y_{j1} &\geq -1; \\ x_{i1} &\geq 0, \quad i=1, 2, \dots, m; & y_{j1} &\geq 0, \quad j=1, 2, \dots, n. \end{aligned}$$

### Problem 3.5.3

$$\text{Minimize } \left\{ \sum_{j=1}^n \sum_{i=1}^m x_{i2} a_{ij}^L(1) y_{j2} \right\}$$

Subject to

$$\begin{aligned} \sum_{i=1}^m b_{il}^L(0) x_{i2} &\leq c_l^L(0), \quad l=1, 2, \dots, p; & \sum_{i=1}^m b_{il}^L(1) x_{i2} &\leq c_l^L(1), \quad l=1, 2, \dots, p; \\ \sum_{i=1}^m b_{il}^R(1) x_{i2} &\leq c_l^R(1), \quad l=1, 2, \dots, p; & \sum_{i=1}^m b_{il}^R(0) x_{i2} &\leq c_l^R(0), \quad l=1, 2, \dots, p; \\ \sum_{j=1}^n e_{kj}^L(0) y_{j2} &\geq d_k^L(0), \quad k=1, 2, \dots, q; & \sum_{j=1}^n e_{kj}^L(1) y_{j2} &\geq d_k^L(1), \quad k=1, 2, \dots, q; \\ \sum_{j=1}^n e_{kj}^R(1) y_{j2} &\geq d_k^R(1), \quad k=1, 2, \dots, q; & \sum_{j=1}^n e_{kj}^R(0) y_{j2} &\geq d_k^R(0), \quad k=1, 2, \dots, q; \end{aligned}$$

$$\sum_{i=1}^m x_{i2} \leq 1; \quad -\sum_{i=1}^m x_{i2} \leq -1; \quad \sum_{j=1}^n y_{j2} \geq 1; \quad -\sum_{j=1}^n y_{j2} \geq -1;$$

$$x_{i2} \geq 0, i = 1, 2, \dots, m; \quad y_{j2} \geq 0, j = 1, 2, \dots, n.$$

**Problem 3.5.4**

$$\text{Minimize } \left\{ \sum_{j=1}^n \sum_{i=1}^m x_{i3} a_{ij}^R(1) y_{j3} \right\}$$

Subject to

$$\sum_{i=1}^m b_{il}^L(0) x_{i3} \leq c_l^L(0), \quad l = 1, 2, \dots, p; \quad \sum_{i=1}^m b_{il}^L(1) x_{i3} \leq c_l^L(1), \quad l = 1, 2, \dots, p;$$

$$\sum_{i=1}^m b_{il}^R(1) x_{i3} \leq c_l^R(1), \quad l = 1, 2, \dots, p; \quad \sum_{i=1}^m b_{il}^R(0) x_{i3} \leq c_l^R(0), \quad l = 1, 2, \dots, p;$$

$$\sum_{j=1}^n e_{kj}^L(0) y_{j3} \geq d_k^L(0), \quad k = 1, 2, \dots, q; \quad \sum_{j=1}^n e_{kj}^L(1) y_{j3} \geq d_k^L(1), \quad k = 1, 2, \dots, q;$$

$$\sum_{j=1}^n e_{kj}^R(1) y_{j3} \geq d_k^R(1), \quad k = 1, 2, \dots, q; \quad \sum_{j=1}^n e_{kj}^R(0) y_{j3} \geq d_k^R(0), \quad k = 1, 2, \dots, q;$$

$$\sum_{i=1}^m x_{i3} \leq 1; \quad -\sum_{i=1}^m x_{i3} \leq -1; \quad \sum_{j=1}^n y_{j3} \geq 1; \quad -\sum_{j=1}^n y_{j3} \geq -1;$$

$$x_{i3} \geq 0, i = 1, 2, \dots, m; \quad y_{j3} \geq 0, j = 1, 2, \dots, n.$$

**Problem 3.5.5**

$$\text{Minimize } \left\{ \sum_{j=1}^n \sum_{i=1}^m x_{i4} a_{ij}^R(0) y_{j4} \right\}$$

Subject to

$$\sum_{i=1}^m b_{il}^L(0) x_{i4} \leq c_l^L(0), \quad l = 1, 2, \dots, p; \quad \sum_{i=1}^m b_{il}^L(1) x_{i4} \leq c_l^L(1), \quad l = 1, 2, \dots, p;$$

$$\sum_{i=1}^m b_{il}^R(1) x_{i4} \leq c_l^R(1), \quad l = 1, 2, \dots, p; \quad \sum_{i=1}^m b_{il}^R(0) x_{i4} \leq c_l^R(0), \quad l = 1, 2, \dots, p;$$

$$\sum_{j=1}^n e_{kj}^L(0) y_{j4} \geq d_k^L(0), \quad k = 1, 2, \dots, q; \quad \sum_{j=1}^n e_{kj}^L(1) y_{j4} \geq d_k^L(1), \quad k = 1, 2, \dots, q;$$

$$\sum_{j=1}^n e_{kj}^R(1) y_{j4} \geq d_k^R(1), \quad k = 1, 2, \dots, q; \quad \sum_{j=1}^n e_{kj}^R(0) y_{j4} \geq d_k^R(0), \quad k = 1, 2, \dots, q;$$

$$\sum_{i=1}^m x_{i4} \leq 1; \quad -\sum_{i=1}^m x_{i4} \leq -1; \quad \sum_{j=1}^n y_{j4} \geq 1; \quad -\sum_{j=1}^n y_{j4} \geq -1;$$

$$x_{i4} \geq 0, i=1,2,\dots,m; \quad y_{j4} \geq 0, j=1,2,\dots,n.$$

**Step 5:** Since, in Problem 3.5.2, Problem 3.5.3, Problem 3.5.4 and Problem 3.5.5 only  $y_{j1}, y_{j2}, y_{j3}$  and  $y_{j4}$  respectively have been considered as decision variables. So, Problem 3.5.2, Problem 3.5.3, Problem 3.5.4 and Problem 3.5.5 are linear programming problems and hence, the optimal value of Problem 3.5.2, Problem 3.5.3, Problem 3.5.4 and Problem 3.5.5 will be equal to its optimal value of corresponding dual problem i.e., Problem 3.5.6, Problem 3.5.7, Problem 3.5.8 and Problem 3.5.9 respectively.

**Problem 3.5.6**

$$\text{Maximize} \left\{ \sum_{k=1}^q d_k^L(0) z_{k1} + z_{(q+1)1} - z_{(q+2)1} \right\}$$

Subject to

$$\sum_{k=1}^q e_{kj}^L(0) z_{k1} + z_{(q+1)1} - z_{(q+2)1} \leq \sum_{i=1}^m a_{ij}^L(0) x_{i1}, \quad j=1,2,\dots,n;$$

$$\sum_{i=1}^m b_{il}^L(0) x_{i1} \leq c_l^L(0), \quad l=1,2,\dots,p;$$

$$\sum_{i=1}^m x_{i1} \leq 1; \quad -\sum_{i=1}^m x_{i1} \leq -1;$$

$$x_{i1} \geq 0, \quad i=1,2,\dots,m;$$

$$z_{k1} \geq 0, \quad k=1,2,\dots,q+2;$$

**Problem 3.5.8**

$$\text{Maximize} \left\{ \sum_{k=1}^q d_k^R(1) z_{k3} + z_{(q+1)3} - z_{(q+2)3} \right\}$$

Subject to

$$\sum_{k=1}^q e_{kj}^R(1) z_{k3} + z_{(q+1)3} - z_{(q+2)3} \leq \sum_{i=1}^m a_{ij}^R(1) x_{i3}, \quad j=1,2,\dots,n;$$

**Problem 3.5.7**

$$\text{Maximize} \left\{ \sum_{k=1}^q d_k^L(1) z_{k2} + z_{(q+1)2} - z_{(q+2)2} \right\}$$

Subject to

$$\sum_{k=1}^q e_{kj}^L(1) z_{k2} + z_{(q+1)2} - z_{(q+2)2} \leq \sum_{i=1}^m a_{ij}^L(1) x_{i2}, \quad j=1,2,\dots,n;$$

$$\sum_{i=1}^m b_{il}^L(1) x_{i2} \leq c_l^L(1), \quad l=1,2,\dots,p;$$

$$\sum_{i=1}^m x_{i2} \leq 1; \quad -\sum_{i=1}^m x_{i2} \leq -1;$$

$$x_{i2} \geq 0, \quad i=1,2,\dots,m;$$

$$z_{k2} \geq 0, \quad k=1,2,\dots,q+2;$$

**Problem 3.5.9**

$$\text{Maximize} \left\{ \sum_{k=1}^q d_k^R(0) z_{k4} + z_{(q+1)4} - z_{(q+2)4} \right\}$$

Subject to

$$\sum_{k=1}^q e_{kj}^R(0) z_{k4} + z_{(q+1)4} - z_{(q+2)4} \leq \sum_{i=1}^m a_{ij}^R(0) x_{i4}, \quad j=1,2,\dots,n;$$

$$\sum_{i=1}^m b_{il}^R(1)x_{i3} \leq c_l^R(1), l = 1, 2, \dots, p;$$

$$\sum_{i=1}^m x_{i3} \leq 1; -\sum_{i=1}^m x_{i3} \leq -1;$$

$$x_{i3} \geq 0, i = 1, 2, \dots, m;$$

$$z_{k3} \geq 0, k = 1, 2, \dots, q + 2;$$

$$\sum_{i=1}^m b_{il}^R(0)x_{i4} \leq c_l^R(0), l = 1, 2, \dots, p;$$

$$\sum_{i=1}^m x_{i4} \leq 1; -\sum_{i=1}^m x_{i4} \leq -1;$$

$$x_{i4} \geq 0, i = 1, 2, \dots, m;$$

$$z_{k4} \geq 0, k = 1, 2, \dots, q + 2;$$

**Step 6:** Substitute the optimal solution  $\{x_{i1}^*, i = 1, 2, \dots, m\}, \{x_{i2}^*, i = 1, 2, \dots, m\},$

$\{x_{i3}^*, i = 1, 2, \dots, m\}$  and  $\{x_{i4}^*, i = 1, 2, \dots, m\}$  of Problem 3.5.6, Problem 3.5.7, Problem 3.5.8 and

Problem 3.5.9 in Problem 3.5.2, Problem 3.5.3, Problem 3.5.4 and Problem 3.5.5 respectively

and find all the alternative basic optimal solutions  $\{y_{j1}^{g*}, g = 1, 2, \dots, u\},$

$\{y_{j2}^{r*}, r = 1, 2, \dots, f\}, \{y_{j3}^{s*}, s = 1, 2, \dots, h\}$  and  $\{y_{j4}^{t*}, t = 1, 2, \dots, w\}$  of Problem 3.5.2, Problem

3.5.3, Problem 3.5.4 and Problem 3.5.5 respectively.

**Step 7:** Find

$$\text{minimum} \left\{ \begin{array}{l} \left( \sum_{j=1}^n \sum_{i=1}^m x_{i1}^* a_{ij}^L(0) y_{j1}^{g*}, \sum_{j=1}^n \sum_{i=1}^m x_{i1}^* a_{ij}^L(1) y_{j1}^{g*}, \sum_{j=1}^n \sum_{i=1}^m x_{i1}^* a_{ij}^R(1) y_{j1}^{g*}, \sum_{j=1}^n \sum_{i=1}^m x_{i1}^* a_{ij}^R(0) y_{j1}^{g*} \right), g = 1, 2, \dots, u \\ \left( \sum_{j=1}^n \sum_{i=1}^m x_{i2}^* a_{ij}^L(0) y_{j2}^{r*}, \sum_{j=1}^n \sum_{i=1}^m x_{i2}^* a_{ij}^L(1) y_{j2}^{r*}, \sum_{j=1}^n \sum_{i=1}^m x_{i2}^* a_{ij}^R(1) y_{j2}^{r*}, \sum_{j=1}^n \sum_{i=1}^m x_{i2}^* a_{ij}^R(0) y_{j2}^{r*} \right), r = 1, 2, \dots, f \\ \left( \sum_{j=1}^n \sum_{i=1}^m x_{i3}^* a_{ij}^L(0) y_{j3}^{s*}, \sum_{j=1}^n \sum_{i=1}^m x_{i3}^* a_{ij}^L(1) y_{j3}^{s*}, \sum_{j=1}^n \sum_{i=1}^m x_{i3}^* a_{ij}^R(1) y_{j3}^{s*}, \sum_{j=1}^n \sum_{i=1}^m x_{i3}^* a_{ij}^R(0) y_{j3}^{s*} \right), s = 1, 2, \dots, h \\ \left( \sum_{j=1}^n \sum_{i=1}^m x_{i4}^* a_{ij}^L(0) y_{j4}^{t*}, \sum_{j=1}^n \sum_{i=1}^m x_{i4}^* a_{ij}^L(1) y_{j4}^{t*}, \sum_{j=1}^n \sum_{i=1}^m x_{i4}^* a_{ij}^R(1) y_{j4}^{t*}, \sum_{j=1}^n \sum_{i=1}^m x_{i4}^* a_{ij}^R(0) y_{j4}^{t*} \right), t = 1, 2, \dots, w \end{array} \right\}.$$

**Step 8:** All the trapezoidal fuzzy numbers which will be minimum, represents minimum

expected gain of Player I and the optimal strategies for all such minimum which will be

obtained corresponding to

$$\left( \sum_{j=1}^n \sum_{i=1}^m x_{i1}^* a_{ij}^L(0) y_{j1}^{g*}, \sum_{j=1}^n \sum_{i=1}^m x_{i1}^* a_{ij}^L(1) y_{j1}^{g*}, \sum_{j=1}^n \sum_{i=1}^m x_{i1}^* a_{ij}^R(1) y_{j1}^{g*}, \sum_{j=1}^n \sum_{i=1}^m x_{i1}^* a_{ij}^R(0) y_{j1}^{g*} \right), g = 1, 2, \dots, u,$$

$$\left( \sum_{j=1}^n \sum_{i=1}^m x_{i2}^* a_{ij}^L(0) y_{j2}^{r*}, \sum_{j=1}^n \sum_{i=1}^m x_{i2}^* a_{ij}^L(1) y_{j2}^{r*}, \sum_{j=1}^n \sum_{i=1}^m x_{i2}^* a_{ij}^R(1) y_{j2}^{r*}, \sum_{j=1}^n \sum_{i=1}^m x_{i2}^* a_{ij}^R(0) y_{j2}^{r*} \right), r = 1, 2, \dots, f,$$

$$\left( \sum_{j=1}^n \sum_{i=1}^m x_{i3}^* a_{ij}^L(0) y_{j3}^{s*}, \sum_{j=1}^n \sum_{i=1}^m x_{i3}^* a_{ij}^L(1) y_{j3}^{s*}, \sum_{j=1}^n \sum_{i=1}^m x_{i3}^* a_{ij}^R(1) y_{j3}^{s*}, \sum_{j=1}^n \sum_{i=1}^m x_{i3}^* a_{ij}^R(0) y_{j3}^{s*} \right), s = 1, 2, \dots, h,$$

$$\left( \sum_{j=1}^n \sum_{i=1}^m x_{i4}^* a_{ij}^L(0) y_{j4}^{t*}, \sum_{j=1}^n \sum_{i=1}^m x_{i4}^* a_{ij}^L(1) y_{j4}^{t*}, \sum_{j=1}^n \sum_{i=1}^m x_{i4}^* a_{ij}^R(1) y_{j4}^{t*}, \sum_{j=1}^n \sum_{i=1}^m x_{i4}^* a_{ij}^R(0) y_{j4}^{t*} \right), t = 1, 2, \dots, w, \text{ will}$$

be  $\{x_{i1}^*, i = 1, 2, \dots, m\}, \{x_{i2}^*, i = 1, 2, \dots, m\}, \{x_{i3}^*, i = 1, 2, \dots, m\}$  and  $\{x_{i4}^*, i = 1, 2, \dots, m\}$  respectively,

which is optimal solution of Problem 3.5.6, Problem 3.5.7, Problem 3.5.8 and Problem 3.5.9 respectively.

### 3.5.2 Maximum expected loss of Player II

Using the comparing method,  $\tilde{a} \succeq \tilde{b}$  if  $a^L(0) \geq b^L(0), a^L(1) \geq b^L(1), a^R(1) \geq b^R(1),$

$a^R(0) \geq b^R(0)$ , the maximum expected loss of Player II and the corresponding optimal strategies can be obtained as follows:

**Step 1:** Find  $(x_1^*, x_2^*, \dots, x_m^*) \in X$  such that the value of  $\sum_{j=1}^n \sum_{i=1}^m x_i (a_{ij}^L(0), a_{ij}^L(1), a_{ij}^R(1), a_{ij}^R(0)) y_j$

is maximum for all  $(y_1, y_2, \dots, y_n) \in Y$  i.e., find the optimal solution of Problem 3.2.6.

**Step 2:** Using the property,  $\sum_{i=1}^n \lambda (a_i^L(0), a_i^L(1), a_i^R(1), a_i^R(0)) =$

$$\left( \sum_{i=1}^n \lambda a_{ij}^L(0), \sum_{i=1}^n \lambda a_{ij}^L(1), \sum_{i=1}^n \lambda a_{ij}^R(1), \sum_{i=1}^n \lambda a_{ij}^R(0) \right), \lambda_i \geq 0, , \text{ the Problem 3.2.6 can be transformed}$$

into Problem 3.2.8.

**Step 3:** Using the property,  $(a^L(0), a^L(1), a^R(1), a^R(0)) \succeq (b^L(0), b^L(1), b^R(1), b^R(0)) \Rightarrow a^L(0) \geq$

$b^L(0), a^L(1) \geq b^L(1), a^R(1) \geq b^R(1), a^R(0) \geq b^R(0)$ , the Problem 3.2.8 can be transformed into

Problem 3.5.10.

#### Problem 3.5.10

$$\text{Maximize} \left\{ \left( \sum_{j=1}^n \sum_{i=1}^m x_i a_{ij}^L(0) y_j, \sum_{j=1}^n \sum_{i=1}^m x_i a_{ij}^L(1) y_j, \sum_{j=1}^n \sum_{i=1}^m x_i a_{ij}^R(1) y_j, \sum_{j=1}^n \sum_{i=1}^m x_i a_{ij}^R(0) y_j \right) \right\}$$

Subject to

$$\begin{aligned}
& \sum_{i=1}^m b_{il}^L(0)x_i \leq c_l^L(0), l=1,2,\dots,p; & \sum_{i=1}^m b_{il}^L(1)x_i \leq c_l^L(1), l=1,2,\dots,p; \\
& \sum_{i=1}^m b_{il}^R(1)x_i \leq c_l^R(1), l=1,2,\dots,p; & \sum_{i=1}^m b_{il}^R(0)x_i \leq c_l^R(0), l=1,2,\dots,p; \\
& \sum_{j=1}^n e_{kj}^L(0)y_j \geq d_k^L(0), k=1,2,\dots,q; & \sum_{j=1}^n e_{kj}^L(1)y_j \geq d_k^L(1), k=1,2,\dots,q; \\
& \sum_{j=1}^n e_{kj}^R(1)y_j \geq d_k^R(1), k=1,2,\dots,q; & \sum_{j=1}^n e_{kj}^R(0)y_j \geq d_k^R(0), k=1,2,\dots,q; \\
& \sum_{i=1}^m x_i \leq 1; & -\sum_{i=1}^m x_i \leq -1; & \sum_{j=1}^n y_j \geq 1; & -\sum_{j=1}^n y_j \geq -1; \\
& x_i \geq 0, i=1,2,\dots,m; & y_j \geq 0, j=1,2,\dots,n.
\end{aligned}$$

**Step 4:** According to comparing method, to find optimal solution  $\{x_i^*, i=1,2,\dots,m\}$  of Problem

3.5.10 such that the value of

$$\left( \sum_{j=1}^n \sum_{i=1}^m x_i a_{ij}^L(0)y_j, \sum_{j=1}^n \sum_{i=1}^m x_i a_{ij}^L(1)y_j, \sum_{j=1}^n \sum_{i=1}^m x_i a_{ij}^R(1)y_j, \sum_{j=1}^n \sum_{i=1}^m x_i a_{ij}^R(0)y_j \right) \text{ is maximum for all}$$

$(y_1, y_2, \dots, y_n) \in Y$  is equivalent to find  $\{x_i^*, i=1,2,\dots,m\}$  such that value of

$$\sum_{j=1}^n \sum_{i=1}^m x_i a_{ij}^L(0)y_j, \sum_{j=1}^n \sum_{i=1}^m x_i a_{ij}^L(1)y_j, \sum_{j=1}^n \sum_{i=1}^m x_i a_{ij}^R(1)y_j, \sum_{j=1}^n \sum_{i=1}^m x_i a_{ij}^R(0)y_j \text{ is maximum for all}$$

$(y_1, y_2, \dots, y_n) \in Y$  or if it is not possible to find  $\{x_i^*, i=1,2,\dots,m\}$  such that value of

$$\sum_{j=1}^n \sum_{i=1}^m x_i a_{ij}^L(0)y_j, \sum_{j=1}^n \sum_{i=1}^m x_i a_{ij}^L(1)y_j, \sum_{j=1}^n \sum_{i=1}^m x_i a_{ij}^R(1)y_j, \sum_{j=1}^n \sum_{i=1}^m x_i a_{ij}^R(0)y_j \text{ is maximum for all}$$

$(y_1, y_2, \dots, y_n) \in Y$  then it is equivalent to find  $\{x_{i1}^*, i=1,2,\dots,m\}$  such that value of

$$\sum_{j=1}^n \sum_{i=1}^m x_{i1} a_{ij}^L(0)y_{j1} \text{ is minimum for all } (y_{11}, y_{21}, \dots, y_{n1}) \in Y \text{ but value of}$$

$$\sum_{j=1}^n \sum_{i=1}^m x_{i1} a_{ij}^L(1)y_{j1} \text{ and/or } \sum_{j=1}^n \sum_{i=1}^m x_{i1} a_{ij}^R(1)y_{j1} \text{ and/or } \sum_{j=1}^n \sum_{i=1}^m x_{i1} a_{ij}^R(0)y_{j1} \text{ is not maximum and to}$$

find  $\{x_{i2}^*, i=1, 2, \dots, m\}$  such that value of  $\sum_{j=1}^n \sum_{i=1}^m x_{i2} a_{ij}^L(1) y_{j2}$  is maximum for all

$(y_{12}, y_{22}, \dots, y_{n2}) \in Y$  but value of  $\sum_{j=1}^n \sum_{i=1}^m x_{i2} a_{ij}^L(0) y_{j2}$  and/or  $\sum_{j=1}^n \sum_{i=1}^m x_{i2} a_{ij}^R(1) y_{j2}$

and/or  $\sum_{j=1}^n \sum_{i=1}^m x_{i2} a_{ij}^R(0) y_{j2}$  is not maximum and to find  $\{x_{i3}^*, i=1, 2, \dots, m\}$  such that value of

$\sum_{j=1}^n \sum_{i=1}^m x_{i3} a_{ij}^R(1) y_{j3}$  is maximum for all  $(y_{13}, y_{23}, \dots, y_{n3}) \in Y$  but value of

$\sum_{j=1}^n \sum_{i=1}^m x_{i3} a_{ij}^L(0) y_{j3}$  and/or  $\sum_{j=1}^n \sum_{i=1}^m x_{i3} a_{ij}^L(1) y_{j3}$  and/or  $\sum_{j=1}^n \sum_{i=1}^m x_{i3} a_{ij}^R(0) y_{j3}$  is not maximum and to find

$\{x_{i4}^*, i=1, 2, \dots, m\}$  such that value of  $\sum_{j=1}^n \sum_{i=1}^m x_{i4} a_{ij}^R(0) y_{j4}$  is maximum for all

$(y_{14}, y_{24}, \dots, y_{n4}) \in Y$  but value of  $\sum_{j=1}^n \sum_{i=1}^m x_{i4} a_{ij}^L(0) y_{j4}$  and/or  $\sum_{j=1}^n \sum_{i=1}^m x_{i4} a_{ij}^L(1) y_{j4}$  and/or

$\sum_{j=1}^n \sum_{i=1}^m x_{i4} a_{ij}^R(1) y_{j4}$  is not maximum i.e., to find the optimal solution

$\{x_{i1}^*, i=1, 2, \dots, m\}, \{x_{i2}^*, i=1, 2, \dots, m\}, \{x_{i3}^*, i=1, 2, \dots, m\}, \{x_{i4}^*, i=1, 2, \dots, m\}$  of Problem

3.5.11, Problem 3.5.12, Problem 3.5.13, Problem 3.5.14 respectively.

### Problem 3.5.11

$$\text{Maximize } \left\{ \sum_{j=1}^n \sum_{i=1}^m x_{i1} a_{ij}^L(0) y_{j1} \right\}$$

Subject to

$$\sum_{i=1}^m b_{il}^L(0) x_{i1} \leq c_l^L(0), \quad l=1, 2, \dots, p; \quad \sum_{i=1}^m b_{il}^L(1) x_{i1} \leq c_l^L(1), \quad l=1, 2, \dots, p;$$

$$\sum_{i=1}^m b_{il}^R(1) x_{i1} \leq c_l^R(1), \quad l=1, 2, \dots, p; \quad \sum_{i=1}^m b_{il}^R(0) x_{i1} \leq c_l^R(0), \quad l=1, 2, \dots, p;$$

$$\sum_{j=1}^n e_{kj}^L(0) y_{j1} \geq d_k^L(0), \quad k=1, 2, \dots, q; \quad \sum_{j=1}^n e_{kj}^L(1) y_{j1} \geq d_k^L(1), \quad k=1, 2, \dots, q;$$

$$\begin{aligned} \sum_{j=1}^n e_{kj}^R(1)y_{j1} &\geq d_k^R(1), \quad k=1,2,\dots,q; & \sum_{j=1}^n e_{kj}^R(0)y_{j1} &\geq d_k^R(0), \quad k=1,2,\dots,q; \\ \sum_{i=1}^m x_{i1} &\leq 1; & -\sum_{i=1}^m x_{i1} &\leq -1; & \sum_{j=1}^n y_{j1} &\geq 1; & -\sum_{j=1}^n y_{j1} &\geq -1; \\ x_{i1} &\geq 0, i=1,2,\dots,m; & y_{j1} &\geq 0, j=1,2,\dots,n. \end{aligned}$$

**Problem 3.5.12**

$$\text{Maximize } \left\{ \sum_{j=1}^n \sum_{i=1}^m x_{i2} a_{ij}^L(0) y_{j2} \right\}$$

Subject to

$$\begin{aligned} \sum_{i=1}^m b_{il}^L(0)x_{i2} &\leq c_l^L(0), \quad l=1,2,\dots,p; & \sum_{i=1}^m b_{il}^L(1)x_{i2} &\leq c_l^L(1), \quad l=1,2,\dots,p; \\ \sum_{i=1}^m b_{il}^R(1)x_{i2} &\leq c_l^R(1), \quad l=1,2,\dots,p; & \sum_{i=1}^m b_{il}^R(0)x_{i2} &\leq c_l^R(0), \quad l=1,2,\dots,p; \\ \sum_{j=1}^n e_{kj}^L(0)y_{j2} &\geq d_k^L(0), \quad k=1,2,\dots,q; & \sum_{j=1}^n e_{kj}^L(1)y_{j2} &\geq d_k^L(1), \quad k=1,2,\dots,q; \\ \sum_{j=1}^n e_{kj}^R(1)y_{j2} &\geq d_k^R(1), \quad k=1,2,\dots,q; & \sum_{j=1}^n e_{kj}^R(0)y_{j2} &\geq d_k^R(0), \quad k=1,2,\dots,q; \\ \sum_{i=1}^m x_{i2} &\leq 1; & -\sum_{i=1}^m x_{i2} &\leq -1; & \sum_{j=1}^n y_{j2} &\geq 1; & -\sum_{j=1}^n y_{j2} &\geq -1; \\ x_{i2} &\geq 0, i=1,2,\dots,m; & y_{j2} &\geq 0, j=1,2,\dots,n. \end{aligned}$$

**Problem 3.5.13**

$$\text{Maximize } \left\{ \sum_{j=1}^n \sum_{i=1}^m x_{i3} a_{ij}^L(0) y_{j3} \right\}$$

Subject to

$$\begin{aligned} \sum_{i=1}^m b_{il}^L(0)x_{i3} &\leq c_l^L(0), \quad l=1,2,\dots,p; & \sum_{i=1}^m b_{il}^L(1)x_{i3} &\leq c_l^L(1), \quad l=1,2,\dots,p; \\ \sum_{i=1}^m b_{il}^R(1)x_{i3} &\leq c_l^R(1), \quad l=1,2,\dots,p; & \sum_{i=1}^m b_{il}^R(0)x_{i3} &\leq c_l^R(0), \quad l=1,2,\dots,p; \\ \sum_{j=1}^n e_{kj}^L(0)y_{j3} &\geq d_k^L(0), \quad k=1,2,\dots,q; & \sum_{j=1}^n e_{kj}^L(1)y_{j3} &\geq d_k^L(1), \quad k=1,2,\dots,q; \\ \sum_{j=1}^n e_{kj}^R(1)y_{j3} &\geq d_k^R(1), \quad k=1,2,\dots,q; & \sum_{j=1}^n e_{kj}^R(0)y_{j3} &\geq d_k^R(0), \quad k=1,2,\dots,q; \end{aligned}$$

$$\sum_{i=1}^m x_{i3} \leq 1; \quad -\sum_{i=1}^m x_{i3} \leq -1; \quad \sum_{j=1}^n y_{j3} \geq 1; \quad -\sum_{j=1}^n y_{j3} \geq -1;$$

$$x_{i3} \geq 0, i = 1, 2, \dots, m; \quad y_{j3} \geq 0, j = 1, 2, \dots, n.$$

**Problem 3.5.14**

$$\text{Maximize } \left\{ \sum_{j=1}^n \sum_{i=1}^m x_{i4} a_{ij}^L(0) y_{j4} \right\}$$

Subject to

$$\sum_{i=1}^m b_{il}^L(0) x_{i4} \leq c_l^L(0), \quad l = 1, 2, \dots, p; \quad \sum_{i=1}^m b_{il}^L(1) x_{i4} \leq c_l^L(1), \quad l = 1, 2, \dots, p;$$

$$\sum_{i=1}^m b_{il}^R(1) x_{i4} \leq c_l^R(1), \quad l = 1, 2, \dots, p; \quad \sum_{i=1}^m b_{il}^R(0) x_{i4} \leq c_l^R(0), \quad l = 1, 2, \dots, p;$$

$$\sum_{j=1}^n e_{kj}^L(0) y_{j4} \geq d_k^L(0), \quad k = 1, 2, \dots, q; \quad \sum_{j=1}^n e_{kj}^L(1) y_{j4} \geq d_k^L(1), \quad k = 1, 2, \dots, q;$$

$$\sum_{j=1}^n e_{kj}^R(1) y_{j4} \geq d_k^R(1), \quad k = 1, 2, \dots, q; \quad \sum_{j=1}^n e_{kj}^R(0) y_{j4} \geq d_k^R(0), \quad k = 1, 2, \dots, q;$$

$$\sum_{i=1}^m x_{i4} \leq 1; \quad -\sum_{i=1}^m x_{i4} \leq -1; \quad \sum_{j=1}^n y_{j4} \geq 1; \quad -\sum_{j=1}^n y_{j4} \geq -1;$$

$$x_{i4} \geq 0, i = 1, 2, \dots, m; \quad y_{j4} \geq 0, j = 1, 2, \dots, n.$$

**Step 5:** Since, in Problem 3.5.11, Problem 3.5.12, Problem 3.5.13 and Problem 3.5.14 only  $x_{i1}, x_{i2}, x_{i3}$  and  $x_{i4}$  respectively have been considered as decision variables. So, Problem 3.5.11, Problem 3.5.12, Problem 3.5.13 and Problem 3.5.14 are linear programming problems and hence, the optimal value of Problem 3.5.11, Problem 3.5.12, Problem 3.5.13 and Problem 3.5.14 will be equal to optimal value of its corresponding dual problem i.e., Problem 3.5.15, Problem 3.5.16, Problem 3.5.17 and Problem 3.5.18 respectively.

**Problem 3.5.15**

$$\text{Minimize } \left\{ \sum_{l=1}^p c_l^L(0) t_{l1} + t_{(p+1)1} - t_{(p+2)1} \right\}$$

Subject to

**Problem 3.5.16**

$$\text{Minimize } \left\{ \sum_{l=1}^p c_l^L(1) t_{l2} + t_{(p+1)2} - t_{(p+2)2} \right\}$$

Subject to

$$\sum_{l=1}^p b_{il}^L(0)t_{l1} + t_{(p+1)1} - t_{(p+2)1} \geq$$

$$\sum_{j=1}^n a_{ij}^L(0)y_{j1}, i = 1, 2, \dots, m;$$

$$\sum_{j=1}^n e_{kl}^L(0)y_{j1} \geq d_k^L(0), k = 1, 2, \dots, q;$$

$$\sum_{j=1}^n y_{j1} \leq 1; -\sum_{j=1}^n y_{j1} \leq -1;$$

$$y_{j1} \geq 0, j = 1, 2, \dots, n;$$

$$t_{l1} \geq 0, l = 1, 2, \dots, p+2;$$

**Problem 3.5.17**

$$\text{Minimize } \left\{ \sum_{l=1}^p c_l^L(0)t_{l3} + t_{(p+1)3} - t_{(p+2)3} \right\}$$

Subject to

$$\sum_{l=1}^p b_{il}^R(1)t_{l3} + t_{(p+1)3} - t_{(p+2)3} \geq$$

$$\sum_{j=1}^n a_{ij}^R(1)y_{j3}, i = 1, 2, \dots, m;$$

$$\sum_{j=1}^n e_{kl}^R(1)y_{j3} \geq d_k^R(1), k = 1, 2, \dots, q;$$

$$\sum_{j=1}^n y_{j3} \leq 1; -\sum_{j=1}^n y_{j3} \leq -1;$$

$$y_{j3} \geq 0, j = 1, 2, \dots, n;$$

$$t_{l3} \geq 0, l = 1, 2, \dots, p+2;$$

**Step 6:** Substitute the optimal solution  $\{y_{j1}^*, j = 1, 2, \dots, n\}, \{y_{j2}^*, j = 1, 2, \dots, n\},$

$\{y_{j3}^*, j = 1, 2, \dots, n\}$  and  $\{y_{j4}^*, j = 1, 2, \dots, n\}$  of Problem 3.5.15, Problem 3.5.16, Problem

3.5.17 and Problem 3.5.18 in Problem 3.5.11, Problem 3.5.12, Problem 3.5.13 and Problem

3.5.14 respectively and find all the alternative basic optimal solutions  $\{x_{i1}^{k*}, g = 1, 2, \dots, u\},$

$\{x_{i2}^{r*}, r = 1, 2, \dots, f\}, \{x_{i3}^{s*}, i = 1, 2, \dots, h\}$  and  $\{x_{i4}^{t*}, t = 1, 2, \dots, w\}$  of Problem 3.5.11, Problem

$$\sum_{l=1}^p b_{il}^L(1)t_{l2} + t_{(p+1)2} - t_{(p+2)2} \geq$$

$$\sum_{j=1}^n a_{ij}^L(1)y_{j2}, i = 1, 2, \dots, m;$$

$$\sum_{j=1}^n e_{kl}^L(1)y_{j2} \geq d_k^L(1), k = 1, 2, \dots, q;$$

$$\sum_{j=1}^n y_{j2} \leq 1; -\sum_{j=1}^n y_{j2} \leq -1;$$

$$y_{j2} \geq 0, j = 1, 2, \dots, n;$$

$$t_{l2} \geq 0, l = 1, 2, \dots, p+2;$$

**Problem 3.5.18**

$$\text{Minimize } \left\{ \sum_{l=1}^p c_l^L(0)t_{l4} + t_{(p+1)4} - t_{(p+2)4} \right\}$$

Subject to

$$\sum_{l=1}^p b_{il}^R(0)t_{l4} + t_{(p+1)4} - t_{(p+2)4} \geq$$

$$\sum_{j=1}^n a_{ij}^R(0)y_{j4}, i = 1, 2, \dots, m;$$

$$\sum_{j=1}^n e_{kl}^R(0)y_{j4} \geq d_k^R(0), k = 1, 2, \dots, q;$$

$$\sum_{j=1}^n y_{j4} \leq 1; -\sum_{j=1}^n y_{j4} \leq -1;$$

$$y_{j4} \geq 0, j = 1, 2, \dots, n;$$

$$t_{l4} \geq 0, l = 1, 2, \dots, p+2;$$

3.5.12, Problem 3.5.13 and Problem 3.5.14 respectively.

**Step 7:** Find

$$\text{maximum} \left\{ \begin{array}{l} \left( \sum_{j=1}^n \sum_{i=1}^m x_{i1}^{g*} a_{ij}^L(0) y_{j1}^*, \sum_{j=1}^n \sum_{i=1}^m x_{i1}^{g*} a_{ij}^L(1) y_{j1}^*, \sum_{j=1}^n \sum_{i=1}^m x_{i1}^{g*} a_{ij}^R(1) y_{j1}^*, \sum_{j=1}^n \sum_{i=1}^m x_{i1}^{g*} a_{ij}^R(0) y_{j1}^* \right), g = 1, 2, \dots, u \\ \left( \sum_{j=1}^n \sum_{i=1}^m x_{i2}^{r*} a_{ij}^L(0) y_{j2}^*, \sum_{j=1}^n \sum_{i=1}^m x_{i2}^{r*} a_{ij}^L(1) y_{j2}^*, \sum_{j=1}^n \sum_{i=1}^m x_{i2}^{r*} a_{ij}^R(1) y_{j2}^*, \sum_{j=1}^n \sum_{i=1}^m x_{i2}^{r*} a_{ij}^R(0) y_{j2}^* \right), r = 1, 2, \dots, f \\ \left( \sum_{j=1}^n \sum_{i=1}^m x_{i3}^{s*} a_{ij}^L(0) y_{j3}^*, \sum_{j=1}^n \sum_{i=1}^m x_{i3}^{s*} a_{ij}^L(1) y_{j3}^*, \sum_{j=1}^n \sum_{i=1}^m x_{i3}^{s*} a_{ij}^R(1) y_{j3}^*, \sum_{j=1}^n \sum_{i=1}^m x_{i3}^{s*} a_{ij}^R(0) y_{j3}^* \right), s = 1, 2, \dots, h \\ \left( \sum_{j=1}^n \sum_{i=1}^m x_{i4}^{t*} a_{ij}^L(0) y_{j4}^*, \sum_{j=1}^n \sum_{i=1}^m x_{i4}^{t*} a_{ij}^L(1) y_{j4}^*, \sum_{j=1}^n \sum_{i=1}^m x_{i4}^{t*} a_{ij}^R(1) y_{j4}^*, \sum_{j=1}^n \sum_{i=1}^m x_{i4}^{t*} a_{ij}^R(0) y_{j4}^* \right), t = 1, 2, \dots, w \end{array} \right\}.$$

**Step 8:** All the trapezoidal fuzzy numbers which will be maximum, represents maximum expected loss of Player II and the optimal strategies for all such minimum which will be obtained corresponding to

$$\begin{aligned} & \left( \sum_{j=1}^n \sum_{i=1}^m x_{i1}^{g*} a_{ij}^L(0) y_{j1}^*, \sum_{j=1}^n \sum_{i=1}^m x_{i1}^{g*} a_{ij}^L(1) y_{j1}^*, \sum_{j=1}^n \sum_{i=1}^m x_{i1}^{g*} a_{ij}^R(1) y_{j1}^*, \sum_{j=1}^n \sum_{i=1}^m x_{i1}^{g*} a_{ij}^R(0) y_{j1}^* \right), g = 1, 2, \dots, u, \\ & \left( \sum_{j=1}^n \sum_{i=1}^m x_{i2}^{r*} a_{ij}^L(0) y_{j2}^*, \sum_{j=1}^n \sum_{i=1}^m x_{i2}^{r*} a_{ij}^L(1) y_{j2}^*, \sum_{j=1}^n \sum_{i=1}^m x_{i2}^{r*} a_{ij}^R(1) y_{j2}^*, \sum_{j=1}^n \sum_{i=1}^m x_{i2}^{r*} a_{ij}^R(0) y_{j2}^* \right), r = 1, 2, \dots, f, \\ & \left( \sum_{j=1}^n \sum_{i=1}^m x_{i3}^{s*} a_{ij}^L(0) y_{j3}^*, \sum_{j=1}^n \sum_{i=1}^m x_{i3}^{s*} a_{ij}^L(1) y_{j3}^*, \sum_{j=1}^n \sum_{i=1}^m x_{i3}^{s*} a_{ij}^R(1) y_{j3}^*, \sum_{j=1}^n \sum_{i=1}^m x_{i3}^{s*} a_{ij}^R(0) y_{j3}^* \right), s = 1, 2, \dots, h, \\ & \left( \sum_{j=1}^n \sum_{i=1}^m x_{i4}^{t*} a_{ij}^L(0) y_{j4}^*, \sum_{j=1}^n \sum_{i=1}^m x_{i4}^{t*} a_{ij}^L(1) y_{j4}^*, \sum_{j=1}^n \sum_{i=1}^m x_{i4}^{t*} a_{ij}^R(1) y_{j4}^*, \sum_{j=1}^n \sum_{i=1}^m x_{i4}^{t*} a_{ij}^R(0) y_{j4}^* \right), t = 1, 2, \dots, w, \text{ will} \end{aligned}$$

be  $\{y_{j1}^*, j = 1, 2, \dots, n\}, \{y_{j2}^*, j = 1, 2, \dots, n\}, \{y_{j3}^*, j = 1, 2, \dots, n\}$  and  $\{y_{j4}^*, j = 1, 2, \dots, n\}$ , which

is optimal solution of Problem 3.5.15, Problem 3.5.16, Problem 3.5.17 and Problem 3.5.18 respectively.

### 3.6 Numerical examples

In this section, some existing numerical examples have been solved by proposed Vaishnavi method.

#### 3.6.1 Existing numerical example considered by Li & Hong

In this section, constrained matrix games with fuzzy payoffs

$$\tilde{A} = \begin{bmatrix} (27, 29, 32, 35) & (-25, -19, -18, -17) \\ (-11, -10, -8, -5) & \left(35, 40, \frac{81}{2}, 41\right) \end{bmatrix}, X = \left\{ (x_1, x_2) \mid \begin{array}{l} x_1, x_2 \geq 0; x_1 + x_2 \leq 1; -x_1 - x_2 \leq -1; \\ 80x_1 + 50x_2 \leq 60 \end{array} \right\}$$

and  $Y = \{y_1, y_2 \mid y_1, y_2 \geq 0; y_1 + y_2 \geq 1; -y_1 - y_2 \geq -1; -40y_1 - 70y_2 \geq -50\}$ , chosen by Li and Hong [75], is solved by the proposed Vaishnavi method.

### 3.6.1.1 Minimum expected gain of Player I

Using the proposed Vaishnavi method minimum expected gain of Player I and corresponding optimal strategies, can be obtained as follows:

**Step 1:** Find  $(y_1^*, y_2^*) \in Y$  such that the value of

$$\left( (27, 29, 32, 35)x_1y_1 + (-25, -19, -18, -17)x_1y_2 + (-11, -10, -8, -5)x_2y_1 + \left(35, 40, \frac{81}{2}, 41\right)x_2y_2 \right)$$

is minimum for all  $(x_1, x_2) \in X$  i.e., find the optimal solution of Problem 3.6.1.

#### Problem 3.6.1

Minimize

$$\left\{ (27, 29, 32, 35)x_1y_1 + (-25, -19, -18, -17)x_1y_2 + (-11, -10, -8, -5)x_2y_1 + \left(35, 40, \frac{81}{2}, 41\right)x_2y_2 \right\}$$

Subject to

$$80x_1 + 50x_2 \leq 60; \quad -40y_1 - 70y_2 \geq -50;$$

$$x_1 + x_2 \leq 1; \quad -x_1 - x_2 \leq -1; \quad y_1 + y_2 \geq 1; \quad -y_1 - y_2 \geq -1;$$

$$x_1, x_2 \geq 0; y_1, y_2 \geq 0.$$

**Step 2:** Using the property,  $\sum_{i=1}^n \lambda(a_i^L(0), a_i^L(1), a_i^R(1), a_i^R(0)) =$

$$\left( \sum_{i=1}^n \lambda a_{ij}^L(0), \sum_{i=1}^n \lambda a_{ij}^L(1), \sum_{i=1}^n \lambda a_{ij}^R(1), \sum_{i=1}^n \lambda a_{ij}^R(0) \right), \lambda \geq 0, , \text{ the Problem 3.6.1 can be transformed}$$

into Problem 3.6.2.

**Problem 3.6.2**

$$\text{Minimize } \left\{ \begin{pmatrix} 27x_1y_1 - 25x_1y_2 - 11x_2y_1 + 35x_2y_2, 29x_1y_1 - 19x_1y_2 - 10x_2y_1 + 40x_2y_2, \\ 32x_1y_1 - 18x_1y_2 - 8x_2y_1 + \frac{81}{2}x_2y_2, 35x_1y_1 - 17x_1y_2 - 5x_2y_1 + 41x_2y_2 \end{pmatrix} \right\}$$

Subject to

$$80x_1 + 50x_2 \leq 60; \quad -40y_1 - 70y_2 \geq -50;$$

$$x_1 + x_2 = 1; \quad y_1 + y_2 = 1;$$

$$x_1, x_2 \geq 0; y_1, y_2 \geq 0.$$

**Step 3:** According to comparing method, to find optimal solution  $\{y_j^*, j=1, 2\}$  of Problem 3.6.2

$$\text{such that the value of } \left\{ \begin{pmatrix} 27x_1y_1 - 25x_1y_2 - 11x_2y_1 + 35x_2y_2, 29x_1y_1 - 19x_1y_2 - 10x_2y_1 + 40x_2y_2, \\ 32x_1y_1 - 18x_1y_2 - 8x_2y_1 + \frac{81}{2}x_2y_2, 35x_1y_1 - 17x_1y_2 - 5x_2y_1 + 41x_2y_2 \end{pmatrix} \right\} \text{ is}$$

minimum for all  $(x_1, x_2) \in X$  is equivalent to find  $\{y_j^*, j=1, 2\}$  such that value of

$$27x_1y_1 - 25x_1y_2 - 11x_2y_1 + 35x_2y_2; 29x_1y_1 - 19x_1y_2 - 10x_2y_1 + 40x_2y_2; 32x_1y_1 - 18x_1y_2 - 8x_2y_1 + \frac{81}{2}x_2y_2$$

and  $35x_1y_1 - 17x_1y_2 - 5x_2y_1 + 41x_2y_2$  is minimum for all  $(x_1, x_2) \in X$  or to find  $\{y_{j1}^*, j=1, 2\}$

such that value of  $27x_{11}y_{11} - 25x_{11}y_{21} - 11x_{21}y_{11} + 35x_{21}y_{21}$  is minimum for all  $(x_{11}, x_{21}) \in X$  but

value of  $29x_{11}y_{11} - 19x_{11}y_{21} - 10x_{21}y_{11} + 40x_{21}y_{21}$  and/or  $32x_{11}y_{11} - 18x_{11}y_{21} - 8x_{21}y_{11} + \frac{81}{2}x_{21}y_{21}$

and/or  $35x_{11}y_{11} - 17x_{11}y_{21} - 5x_{21}y_{11} + 41x_{21}y_{21}$  is not minimum and to find  $\{y_{j2}^*, j=1, 2\}$  such

that value of  $29x_{12}y_{12} - 19x_{12}y_{22} - 10x_{22}y_{12} + 40x_{22}y_{22}$  is minimum for all  $(x_{12}, x_{22}) \in X$  but value

of  $27x_{12}y_{12} - 25x_{12}y_{22} - 11x_{22}y_{12} + 35x_{22}y_{22}$  and/or

$$32x_{12}y_{12} - 18x_{12}y_{22} - 8x_{22}y_{12} + \frac{81}{2}x_{22}y_{22} \text{ and/or } 35x_{12}y_{12} - 17x_{12}y_{22} - 5x_{22}y_{12} + 41x_{22}y_{22} \text{ is not}$$

minimum and to find  $\{y_{j3}^*, j=1,2\}$  such that value of  $32x_{13}y_{13} - 18x_{13}y_{23} - 8x_{23}y_{13} + \frac{81}{2}x_{23}y_{23}$  is minimum for all  $(x_{13}, x_{23}) \in X$  but value of  $27x_{13}y_{13} - 25x_{13}y_{23} - 11x_{23}y_{13} + 35x_{23}y_{23}$  and/or  $29x_{13}y_{13} - 19x_{13}y_{23} - 10x_{23}y_{13} + 40x_{23}y_{23}$  and/or  $35x_{13}y_{13} - 17x_{13}y_{23} - 5x_{23}y_{13} + 41x_{23}y_{23}$  is not minimum and to find  $\{y_{j4}^*, j=1,2\}$  such that value of  $35x_{14}y_{14} - 17x_{14}y_{24} - 5x_{24}y_{14} + 41x_{24}y_{24}$  is minimum for all  $(x_{13}, x_{23}) \in X$  but value of  $27x_{14}y_{14} - 25x_{14}y_{24} - 11x_{24}y_{14} + 35x_{24}y_{24}$  and/or  $32x_{14}y_{14} - 18x_{14}y_{24} - 8x_{24}y_{14} + \frac{81}{2}x_{24}y_{24}$  and/or  $29x_{14}y_{14} - 19x_{14}y_{24} - 10x_{24}y_{14} + 40x_{24}y_{24}$  is not minimum i.e., to find the optimal solution  $\{y_{j1}^*, j=1,2\}, \{y_{j2}^*, j=1,2\}, \{y_{j3}^*, j=1,2\}, \{y_{j4}^*, j=1,2\}$  of Problem 3.6.3, Problem 3.6.4, Problem 3.6.5, Problem 3.6.6 respectively.

### Problem 3.6.3

Minimize  $\{27x_{11}y_{11} - 25x_{11}y_{21} - 11x_{21}y_{11} + 35x_{21}y_{21}\}$

Subject to

$$80x_{11} + 50x_{21} \leq 60; \quad -40y_{11} - 70y_{21} \geq -50;$$

$$x_{11} + x_{21} = 1; \quad y_{11} + y_{21} = 1;$$

$$x_{11}, x_{21} \geq 0; y_{11}, y_{21} \geq 0.$$

### Problem 3.6.4

Minimize  $\{29x_{12}y_{12} - 19x_{12}y_{22} - 10x_{22}y_{12} + 40x_{22}y_{22}\}$

Subject to

$$80x_{12} + 50x_{22} \leq 60; \quad -40y_{12} - 70y_{22} \geq -50;$$

$$x_{12} + x_{22} = 1; \quad y_{12} + y_{22} = 1; x_{12}, x_{22} \geq 0; y_{12}, y_{22} \geq 0.$$

### Problem 3.6.5

Minimize  $\left\{32x_{13}y_{13} - 18x_{13}y_{23} - 8x_{23}y_{13} + \frac{81}{2}x_{23}y_{23}\right\}$

Subject to

$$80x_{13} + 50x_{23} \leq 60; \quad -40y_{13} - 70y_{23} \geq -50;$$

$$x_{13} + x_{23} = 1; \quad y_{13} + y_{23} = 1;$$

$$x_{13}, x_{23} \geq 0; y_{13}, y_{23} \geq 0.$$

### Problem 3.6.6

$$\text{Minimize } \{35x_{14}y_{14} - 17x_{14}y_{24} - 5x_{24}y_{14} + 41x_{24}y_{24}\}$$

Subject to

$$80x_{14} + 50x_{24} \leq 60; \quad -40y_{14} - 70y_{24} \geq -50;$$

$$x_{14} + x_{24} = 1; \quad y_{14} + y_{24} = 1;$$

$$x_{14}, x_{24} \geq 0; y_{14}, y_{24} \geq 0.$$

**Step 4:** Since, in Problem 3.6.3, Problem 3.6.4, Problem 3.6.5 and Problem 3.6.6 only  $y_{j1}, y_{j2}, y_{j3}$  and  $y_{j4}$  respectively have been considered as decision variables. So, Problem 3.6.3, Problem 3.6.4, Problem 3.6.5 and Problem 3.6.6 are linear programming problems and hence, the optimal value of Problem 3.6.3, Problem 3.6.4, Problem 3.6.5 and Problem 3.6.6 will be equal to optimal value of its corresponding dual problem i.e., Problem 3.6.7, Problem 3.6.8, Problem 3.6.9 and Problem 3.6.10 respectively.

### Problem 3.6.7

$$\text{Maximize } \{-50z_{11} + z_{21}\}$$

Subject to

$$-40z_{11} + z_{21} \leq 27x_{11} - 11x_{21};$$

$$-70z_{11} + z_{21} \leq -25x_{11} + 35x_{21};$$

$$80x_{11} + 50x_{21} \leq 60;$$

$$x_{11} + x_{21} = 1;$$

$$x_{11}, x_{21} \geq 0, z_{11} \geq 0.$$

### Problem 3.6.9

$$\text{Maximize } \{-50z_{13} + z_{23}\}$$

Subject to

### Problem 3.6.8

$$\text{Maximize } \{-50z_{12} + z_{22}\}$$

Subject to

$$-40z_{12} + z_{22} \leq 29x_{12} - 10x_{22};$$

$$-70z_{12} + z_{22} \leq -19x_{12} + 40x_{22};$$

$$80x_{12} + 50x_{22} \leq 60;$$

$$x_{12} + x_{22} = 1;$$

$$x_{12}, x_{22} \geq 0, z_{12} \geq 0.$$

### Problem 3.6.10

$$\text{Maximize } \{-50z_{14} + z_{24}\}$$

Subject to

$$-40z_{13} + z_{23} \leq 32x_{13} - 8x_{23};$$

$$-70z_{13} + z_{23} \leq -18x_{13} + \frac{81}{2}x_{23};$$

$$80x_{13} + 50x_{23} \leq 60;$$

$$x_{13} + x_{23} = 1;$$

$$x_{13}, x_{23} \geq 0, z_{13} \geq 0.$$

$$-40z_{14} + z_{24} \leq 35x_{14} - 5x_{24};$$

$$-70z_{14} + z_{24} \leq -17x_{14} + 41x_{24};$$

$$80x_{14} + 50x_{24} \leq 60;$$

$$x_{14} + x_{24} = 1;$$

$$x_{14}, x_{24} \geq 0, z_{14} \geq 0.$$

**Step 5:** Substituting the optimal solution  $\left\{x_{11}^* = \frac{1}{3}, x_{21}^* = \frac{2}{3}\right\}, \left\{x_{12}^* = \frac{1}{3}, x_{22}^* = \frac{2}{3}\right\},$

$\left\{x_{13}^* = \frac{1}{3}, x_{23}^* = \frac{2}{3}\right\}$  and  $\left\{x_{14}^* = \frac{1}{3}, x_{24}^* = \frac{2}{3}\right\}$  in Problem 3.6.3, Problem 3.6.4, Problem 3.6.5 and

Problem 3.6.6 respectively, the obtained optimal solution of Problem 3.6.3, Problem 3.6.4,

Problem 3.6.5 and Problem 3.6.6 is  $\{y_{11}^* = 1, y_{21}^* = 0\}, \{y_{12}^* = 1, y_{22}^* = 0\}, \{y_{13}^* = 1, y_{23}^* = 0\}$  and

$\{y_{14}^* = 1, y_{24}^* = 0\}$  respectively.

**Step 6:** Now, minimum

$$\left[ \begin{aligned} &\left( (27, 29, 32, 35)x_{11}^*y_{11}^* + (-25, -19, -18, -17)x_{11}^*y_{21}^* + (-11, -10, -8, -5)x_{21}^*y_{11}^* + \left(35, 40, \frac{81}{2}, 41\right)x_{21}^*y_{21}^* \right), \\ &\left( (27, 29, 32, 35)x_{12}^*y_{12}^* + (-25, -19, -18, -17)x_{12}^*y_{22}^* + (-11, -10, -8, -5)x_{22}^*y_{12}^* + \left(35, 40, \frac{81}{2}, 41\right)x_{22}^*y_{22}^* \right), \\ &\left( (27, 29, 32, 35)x_{13}^*y_{13}^* + (-25, -19, -18, -17)x_{13}^*y_{23}^* + (-11, -10, -8, -5)x_{23}^*y_{13}^* + \left(35, 40, \frac{81}{2}, 41\right)x_{23}^*y_{23}^* \right), \\ &\left( (27, 29, 32, 35)x_{14}^*y_{14}^* + (-25, -19, -18, -17)x_{14}^*y_{24}^* + (-11, -10, -8, -5)x_{24}^*y_{14}^* + \left(35, 40, \frac{81}{2}, 41\right)x_{24}^*y_{24}^* \right) \end{aligned} \right]$$

$$= \text{minimum} \left\{ \left( \frac{5}{3}, \frac{9}{3}, \frac{16}{3}, \frac{25}{3} \right), \left( \frac{5}{3}, \frac{9}{3}, \frac{16}{3}, \frac{25}{3} \right), \left( \frac{5}{3}, \frac{9}{3}, \frac{16}{3}, \frac{25}{3} \right), \left( \frac{5}{3}, \frac{9}{3}, \frac{16}{3}, \frac{25}{3} \right) \right\} = \left( \frac{5}{3}, \frac{9}{3}, \frac{16}{3}, \frac{25}{3} \right).$$

**Step 7:** The minimum expected gain of Player I is  $\left( \frac{5}{3}, \frac{9}{3}, \frac{16}{3}, \frac{25}{3} \right)$  and corresponding optimal

strategies are  $\left\{x_{11}^* = \frac{1}{3}, x_{21}^* = \frac{2}{3}\right\} = \left\{x_{12}^* = \frac{1}{3}, x_{22}^* = \frac{2}{3}\right\} = \left\{x_{13}^* = \frac{1}{3}, x_{23}^* = \frac{2}{3}\right\} = \left\{x_{14}^* = \frac{1}{3}, x_{24}^* = \frac{2}{3}\right\}.$

### 3.6.1.2 Maximum expected loss of Player II

Using the proposed Vaishnavi method maximum expected loss of Player II and

corresponding optimal strategies, can be obtained as follows:

**Step 1:** Find  $(x_1^*, x_2^*) \in X$  such that the value of

$$\left( (27, 29, 32, 35)x_1y_1 + (-25, -19, -18, -17)x_1y_2 + (-11, -10, -8, -5)x_2y_1 + \left( 35, 40, \frac{81}{2}, 41 \right)x_2y_2 \right)$$

is maximum for all  $(y_1, y_2) \in Y$  i.e., find the optimal solution of Problem 3.6.11.

**Problem 3.6.11**

Maximize

$$\left\{ (27, 29, 32, 35)x_1y_1 + (-25, -19, -18, -17)x_1y_2 + (-11, -10, -8, -5)x_2y_1 + \left( 35, 40, \frac{81}{2}, 41 \right)x_2y_2 \right\}$$

Subject to

$$80x_1 + 50x_2 \leq 60; \quad -40y_1 - 70y_2 \geq -50;$$

$$x_1 + x_2 \leq 1; \quad -x_1 - x_2 \leq -1; \quad y_1 + y_2 \geq 1; \quad -y_1 - y_2 \geq -1;$$

$$x_1, x_2 \geq 0; \quad y_1, y_2 \geq 0.$$

**Step 2:** Using the property,  $\sum_{i=1}^n \lambda (a_i^L(0), a_i^L(1), a_i^R(1), a_i^R(0)) =$

$$\left( \sum_{i=1}^n \lambda a_{ij}^L(0), \sum_{i=1}^n \lambda a_{ij}^L(1), \sum_{i=1}^n \lambda a_{ij}^R(1), \sum_{i=1}^n \lambda a_{ij}^R(0) \right), \lambda \geq 0, , \text{ the Problem 3.6.11 can be transformed}$$

into Problem 3.6.12.

**Problem 3.6.12**

$$\text{Maximize } \left\{ \left( 27x_1y_1 - 25x_1y_2 - 11x_2y_1 + 35x_2y_2, 29x_1y_1 - 19x_1y_2 - 10x_2y_1 + 40x_2y_2, \right. \right. \\ \left. \left. 32x_1y_1 - 18x_1y_2 - 8x_2y_1 + \frac{81}{2}x_2y_2, 35x_1y_1 - 17x_1y_2 - 5x_2y_1 + 41x_2y_2 \right) \right\}$$

Subject to

$$80x_1 + 50x_2 \leq 60; \quad -40y_1 - 70y_2 \geq -50;$$

$$x_1 + x_2 = 1; \quad y_1 + y_2 = 1; \quad x_1, x_2 \geq 0; \quad y_1, y_2 \geq 0.$$

**Step 3:** Find the optimal solution  $\{x_{i1}^*, i=1,2\}, \{x_{i2}^*, i=1,2\}, \{x_{i3}^*, i=1,2\}$ , and  $\{x_{i4}^*, i=1,2\}$  of Problem 3.6.13, Problem 3.6.14, Problem 3.6.15 and Problem 3.6.16 respectively.

**Problem 3.6.13**

$$\text{Maximize } \{27x_{11}y_{11} - 25x_{11}y_{21} - 11x_{21}y_{11} + 35x_{21}y_{21}\}$$

Subject to

$$80x_{11} + 50x_{21} \leq 60; \quad -40y_{11} - 70y_{21} \geq -50;$$

$$x_{11} + x_{21} = 1; \quad y_{11} + y_{21} = 1;$$

$$x_{11}, x_{21} \geq 0; y_{11}, y_{21} \geq 0.$$

**Problem 3.6.14**

$$\text{Maximize } \{29x_{12}y_{12} - 19x_{12}y_{22} - 10x_{22}y_{12} + 40x_{22}y_{22}\}$$

Subject to

$$80x_{12} + 50x_{22} \leq 60; \quad -40y_{12} - 70y_{22} \geq -50;$$

$$x_{12} + x_{22} = 1; \quad y_{12} + y_{22} = 1;$$

$$x_{12}, x_{22} \geq 0; y_{12}, y_{22} \geq 0.$$

**Problem 3.6.15**

$$\text{Maximize } \left\{ 32x_{13}y_{13} - 18x_{13}y_{23} - 8x_{23}y_{13} + \frac{81}{2}x_{23}y_{23} \right\}$$

Subject to

$$80x_{13} + 50x_{23} \leq 60; \quad -40y_{13} - 70y_{23} \geq -50;$$

$$x_{13} + x_{23} = 1; \quad y_{13} + y_{23} = 1;$$

$$x_{13}, x_{23} \geq 0; y_{13}, y_{23} \geq 0.$$

**Problem 3.6.16**

$$\text{Maximize } \{35x_{14}y_{14} - 17x_{14}y_{24} - 5x_{24}y_{14} + 41x_{24}y_{24}\}$$

Subject to

$$80x_{14} + 50x_{24} \leq 60; \quad -40y_{14} - 70y_{24} \geq -50;$$

$$x_{14} + x_{24} = 1; \quad y_{14} + y_{24} = 1;$$

$$x_{14}, x_{24} \geq 0; y_{14}, y_{24} \geq 0.$$

**Step 4:** Since, in Problem 3.6.13, Problem 3.6.14, Problem 3.6.15 and Problem 3.6.16 only  $x_{i1}, x_{i2}, x_{i3}$  and  $x_{i4}$  respectively have been considered as decision variables. So, 3.6.13, Problem 3.6.14, Problem 3.6.15 and Problem 3.6.16 are linear programming problems and hence, the optimal value of Problem 3.6.13, Problem 3.6.14, Problem 3.6.15 and Problem 3.6.16 will be equal to corresponding its dual problem Problem 3.6.17, Problem 3.6.18, Problem 3.6.19 and Problem 3.6.20 respectively.

**Problem 3.6.17**

$$\text{Minimize } \{60t_{11} + t_{21}\}$$

Subject to

$$80t_{11} + t_{21} \geq 27y_{11} - 25y_{21};$$

$$50t_{11} + t_{21} \geq -11y_{11} + 35y_{21};$$

$$-40y_{11} - 70y_{21} \geq -50;$$

$$y_{11} + y_{21} = 1;$$

$$y_{11}, y_{21} \geq 0, t_{11} \geq 0.$$

**Problem 3.6.19**

$$\text{Maximize } \{60t_{13} + t_{23}\}$$

Subject to

$$80t_{13} + t_{23} \geq 32y_{13} - 18y_{23};$$

$$50t_{13} + t_{23} \geq -8y_{13} + \frac{81}{2}y_{23};$$

$$-40y_{13} - 70y_{23} \geq -50;$$

$$y_{13} + y_{23} = 1;$$

$$y_{13}, y_{23} \geq 0, t_{13} \geq 0.$$

**Problem 3.6.18**

$$\text{Minimize } \{60t_{12} + t_{22}\}$$

Subject to

$$80t_{12} + t_{22} \geq 29y_{12} - 19y_{22};$$

$$50t_{12} + t_{22} \geq -10y_{12} + 40y_{22};$$

$$-40y_{12} - 70y_{22} \geq -50;$$

$$y_{12} + y_{22} = 1;$$

$$y_{12}, y_{22} \geq 0, t_{12} \geq 0.$$

**Problem 3.6.10**

$$\text{Maximize } \{60t_{14} + t_{24}\}$$

Subject to

$$80t_{14} + t_{24} \geq 35y_{14} - 17y_{24};$$

$$50t_{14} + t_{24} \geq -5y_{14} + 41y_{24};$$

$$-40y_{14} - 70y_{24} \geq -50;$$

$$y_{14} + y_{24} = 1;$$

$$y_{14}, y_{24} \geq 0, t_{14} \geq 0.$$

**Step 5:** Substituting the optimal solution  $\{y_{11}^* = 1, y_{21}^* = 0\}, \{y_{12}^* = 1, y_{22}^* = 0\}, \{y_{13}^* = 1, y_{23}^* = 0\}$

and  $\{y_{14}^* = 1, y_{24}^* = 0\}$  in Problem 3.6.13, Problem 3.6.14, Problem 3.6.15 and Problem 3.6.16

respectively, the obtained optimal solutions of Problem 3.6.13, Problem 3.6.14, Problem 3.6.15

and Problem 3.6.16 is  $\left\{x_{11}^* = \frac{1}{3}, x_{21}^* = \frac{2}{3}\right\}, \left\{x_{12}^* = \frac{1}{3}, x_{22}^* = \frac{2}{3}\right\}, \left\{x_{13}^* = \frac{1}{3}, x_{23}^* = \frac{2}{3}\right\}$  and

$\left\{x_{14}^* = \frac{1}{3}, x_{24}^* = \frac{2}{3}\right\}$  respectively.

**Step 6:** Now, maximum

$$\begin{aligned} & \left[ \begin{aligned} & \left( (27, 29, 32, 35)x_{11}^*y_{11}^* + (-25, -19, -18, -17)x_{11}^*y_{21}^* + (-11, -10, -8, -5)x_{21}^*y_{11}^* + \left(35, 40, \frac{81}{2}, 41\right)x_{21}^*y_{21}^* \right), \\ & \left( (27, 29, 32, 35)x_{12}^*y_{12}^* + (-25, -19, -18, -17)x_{12}^*y_{22}^* + (-11, -10, -8, -5)x_{22}^*y_{12}^* + \left(35, 40, \frac{81}{2}, 41\right)x_{22}^*y_{22}^* \right), \\ & \left( (27, 29, 32, 35)x_{13}^*y_{13}^* + (-25, -19, -18, -17)x_{13}^*y_{23}^* + (-11, -10, -8, -5)x_{23}^*y_{13}^* + \left(35, 40, \frac{81}{2}, 41\right)x_{23}^*y_{23}^* \right), \\ & \left( (27, 29, 32, 35)x_{14}^*y_{14}^* + (-25, -19, -18, -17)x_{14}^*y_{24}^* + (-11, -10, -8, -5)x_{24}^*y_{14}^* + \left(35, 40, \frac{81}{2}, 41\right)x_{24}^*y_{24}^* \right) \end{aligned} \right] \\ & = \text{maximum} \left\{ \left( \frac{5}{3}, \frac{9}{3}, \frac{16}{3}, \frac{25}{3} \right), \left( \frac{5}{3}, \frac{9}{3}, \frac{16}{3}, \frac{25}{3} \right), \left( \frac{5}{3}, \frac{9}{3}, \frac{16}{3}, \frac{25}{3} \right), \left( \frac{5}{3}, \frac{9}{3}, \frac{16}{3}, \frac{25}{3} \right) \right\} = \left( \frac{5}{3}, \frac{9}{3}, \frac{16}{3}, \frac{25}{3} \right). \end{aligned}$$

**Step 7:** The maximum expected loss of Player II is  $\left( \frac{5}{3}, \frac{9}{3}, \frac{16}{3}, \frac{25}{3} \right)$  and corresponding optimal

strategies are  $\{y_{11}^* = 1, y_{21}^* = 0\} = \{y_{12}^* = 1, y_{22}^* = 0\} = \{y_{13}^* = 1, y_{23}^* = 0\} = \{y_{14}^* = 1, y_{24}^* = 0\}$ .

### 3.6.2 Existing numerical example considered by Li and Cheng

In this section, constrained matrix games with fuzzy payoffs

$$\tilde{A} = \begin{bmatrix} (18, 20, 23) & (-21, -18, -16) \\ (-33, -32, -27) & (38, 40, 43) \end{bmatrix}, X = \left\{ (x_1, x_2) \mid x_1, x_2 \geq 0; x_1 + x_2 \leq 1; -x_1 - x_2 \leq -1 \right\} \text{ and}$$

$$Y = \left\{ (y_1, y_2) \mid y_1, y_2 \geq 0; y_1 + y_2 \geq 1; -y_1 - y_2 \geq -1; (-48, -40, -35)y_1 + (-79, -70, -65)y_2 \succeq (-60, -52, -50) \right\},$$

chosen by Li and Cheng [74], is solved by the proposed Vaishnavi method.

### 3.6.2.1 Minimum expected gain of Player I

Using the proposed Vaishnavi method minimum expected gain of Player I and corresponding optimal strategies, can be obtained as follows:

**Step 1:** Find  $(y_1^*, y_2^*) \in Y$  such that the value of

$((18, 20, 23)x_1y_1 + (-21, -18, -16)x_1y_2 + (-33, -32, -27)x_2y_1 + (38, 40, 43)x_2y_2)$  is minimum

for all  $(x_1, x_2) \in X$  i.e., find the optimal solution of Problem 3.6.21.

#### Problem 3.6.21

Minimize  $\{(18, 20, 23)x_1y_1 + (-21, -18, -16)x_1y_2 + (-33, -32, -27)x_2y_1 + (38, 40, 43)x_2y_2\}$

Subject to

$(70, 80, 88)x_1 + (44, 50, 54)x_2 \preceq (61, 67, 74); (-48, -40, -35)y_1 + (-79, -70, -65)y_2 \succeq (-60, -52, -50);$   
 $x_1 + x_2 \leq 1; -x_1 - x_2 \leq -1; y_1 + y_2 \geq 1; -y_1 - y_2 \geq -1;$   
 $x_1, x_2 \geq 0; y_1, y_2 \geq 0.$

**Step 2:** Using the property,  $\sum_{i=1}^n \lambda(a_i^L(0), a_i^L(1), a_i^R(1), a_i^R(0)) =$

$\left( \sum_{i=1}^n \lambda a_{ij}^L(0), \sum_{i=1}^n \lambda a_{ij}^L(1), \sum_{i=1}^n \lambda a_{ij}^R(1), \sum_{i=1}^n \lambda a_{ij}^R(0) \right), \lambda \geq 0, ,$  the Problem 3.6.21 can be transformed

into Problem 3.6.22.

#### Problem 3.6.22

Minimize  $\left\{ \begin{pmatrix} 18x_1y_1 - 21x_1y_2 - 33x_2y_1 + 38x_2y_2, 20x_1y_1 - 18x_1y_2 - 32x_2y_1 + 40x_2y_2, \\ 23x_1y_1 - 16x_1y_2 - 27x_2y_1 + 43x_2y_2 \end{pmatrix} \right\}$

Subject to

$(70x_1 + 44x_2, 80x_1 + 50x_2, 88x_1 + 54x_2) \preceq (61, 67, 74);$   
 $(-48y_1 - 79y_2, -40y_1 - 70y_2, -35y_1 - 65y_2) \succeq (-60, -52, -50);$   
 $x_1 + x_2 \leq 1; -x_1 - x_2 \leq -1; y_1 + y_2 \geq 1; -y_1 - y_2 \geq -1; x_1, x_2 \geq 0; y_1, y_2 \geq 0.$

**Step 3:** Using the property  $(a^L(0), a^L(1), a^R(1), a^R(0)) \preceq (b^L(0), b^L(1), b^R(1), b^R(0)) \Rightarrow a^L(0) \leq$

$b^L(0), a^L(1) \leq b^L(1), a^R(1) \leq b^R(1), a^R(0) \leq b^R(0)$ , the Problem 3.6.22 can be transformed into Problem 3.6.23.

**Problem 3.6.23**

$$\text{Minimize } \left\{ \begin{pmatrix} 18x_1y_1 - 21x_1y_2 - 33x_2y_1 + 38x_2y_2, 20x_1y_1 - 18x_1y_2 - 32x_2y_1, 40x_2y_2, \\ 23x_1y_1 - 16x_1y_2 - 27x_2y_1 + 43x_2y_2 \end{pmatrix} \right\}$$

Subject to

$$\begin{aligned} 70x_1 + 44x_2 &\leq 61; & 80x_1 + 50x_2 &\leq 67; & 88x_1 + 54x_2 &\leq 74; \\ -48y_1 - 79y_2 &\geq -60; & -40y_1 - 70y_2 &\geq -52; & -35y_1 - 65y_2 &\geq -50; \\ x_1 + x_2 &= 1; & y_1 + y_2 &= 1; \\ x_1, x_2 &\geq 0; & y_1, y_2 &\geq 0. \end{aligned}$$

**Step 4:** Find the optimal solution  $\{y_{j1}^*, j=1,2\}, \{y_{j2}^*, j=1,2\}$  and  $\{y_{j3}^*, j=1,2\}$  of Problem

3.6.24, Problem 3.6.25 and Problem 3.6.26 respectively.

**Problem 3.6.24**

$$\text{Minimize } \{18x_{11}y_{11} - 21x_{11}y_{21} - 33x_{21}y_{11} + 38x_{21}y_{21}\}$$

Subject to

$$\begin{aligned} 70x_{11} + 44x_{21} &\leq 61; & 80x_{11} + 50x_{21} &\leq 67; & 88x_{11} + 54x_{21} &\leq 74; \\ -48y_{11} - 79y_{21} &\geq -60; & -40y_{11} - 70y_{21} &\geq -52; & -35y_{11} - 65y_{21} &\geq -50; \\ x_{11} + x_{21} &= 1; & y_{11} + y_{21} &= 1; \\ x_{11}, x_{21} &\geq 0; & y_{11}, y_{21} &\geq 0. \end{aligned}$$

**Problem 3.6.25**

$$\text{Minimize } \{20x_{12}y_{12} - 18x_{12}y_{22} - 32x_{22}y_{12} + 40x_{22}y_{22}\}$$

Subject to

$$\begin{aligned}
70x_{12} + 44x_{22} &\leq 61; & 80x_{12} + 50x_{22} &\leq 67; & 88x_{21} + 54x_{22} &\leq 74; \\
-48y_{12} - 79y_{22} &\geq -60; & -40y_{12} - 70y_{22} &\geq -52; & -35y_{12} - 65y_{22} &\geq -50; \\
x_{12} + x_{22} &= 1; & y_{12} + y_{22} &= 1; \\
x_{12}, x_{22} &\geq 0; & y_{12}, y_{22} &\geq 0.
\end{aligned}$$

### Problem 3.6.26

$$\text{Minimize } \{23x_{13}y_{13} - 16x_{13}y_{23} - 27x_{23}y_{13} + 43x_{23}y_{23}\}$$

Subject to

$$\begin{aligned}
70x_{13} + 44x_{23} &\leq 61; & 80x_{13} + 50x_{23} &\leq 67; & 88x_{13} + 54x_{23} &\leq 74; \\
-48y_{13} - 79y_{23} &\geq -60; & -40y_{13} - 70y_{23} &\geq -52; & -35y_{13} - 65y_{23} &\geq -50; \\
x_{13} + x_{23} &= 1; & y_{13} + y_{23} &= 1; \\
x_{13}, x_{23} &\geq 0; & y_{13}, y_{23} &\geq 0.
\end{aligned}$$

**Step 5:** Since, in Problem 3.6.24, Problem 3.6.25 and Problem 3.6.26 only  $y_{j1}, y_{j2}$  and  $y_{j3}$  respectively have been considered as decision variables. So, Problem 3.6.24, Problem 3.6.25 and Problem 3.6.26 are linear programming problems and hence, the optimal value of Problem 3.6.24, Problem 3.6.25 and Problem 3.6.26 will be equal to optimal value of its corresponding optimal value of dual problem i.e., Problem 3.6.27, Problem 3.6.28, and Problem 3.6.29 respectively.

### Problem 3.6.27

$$\text{Maximize } \{-60z_{11} - 52z_{21} - 50z_{31} + z_{41}\}$$

Subject to

$$\begin{aligned}
-48z_{11} - 40z_{21} - 35z_{31} + z_{41} &\leq 18x_{11} - 33x_{21}; \\
-79z_{11} - 70z_{21} - 65z_{31} + z_{41} &\leq -21x_{11} + 38x_{21}; \\
70x_{11} + 44x_{21} &\leq 61; \\
80x_{11} + 50x_{21} &\leq 67; \\
88x_{11} + 54x_{21} &\leq 74; \\
x_{11} + x_{21} &= 1; \\
x_{11}, x_{21} &\geq 0, & z_{11}, z_{21}, z_{31} &\geq 0.
\end{aligned}$$

### Problem 3.6.28

$$\text{Maximize } \{-60z_{12} - 52z_{22} - 50z_{32} + z_{42}\}$$

Subject to

$$\begin{aligned}
-48z_{12} - 40z_{22} - 35z_{32} + z_{42} &\leq 20x_{12} - 32x_{22}; \\
-79z_{12} - 70z_{22} - 65z_{32} + z_{42} &\leq -18x_{12} + 40x_{22}; \\
70x_{12} + 44x_{22} &\leq 61; \\
80x_{12} + 50x_{22} &\leq 67; \\
88x_{12} + 54x_{22} &\leq 74; \\
x_{12} + x_{22} &= 1; \\
x_{12}, x_{22} &\geq 0, & z_{12}, z_{22}, z_{32} &\geq 0.
\end{aligned}$$

### Problem 3.6.29

$$\text{Maximize } \{-60z_{13} - 52z_{23} - 50z_{33} + z_{43}\}$$

Subject to

$$\begin{aligned} -48z_{13} - 40z_{23} - 35z_{33} + z_{43} &\leq 23x_{13} - 27x_{23}; \\ -79z_{13} - 70z_{23} - 65z_{33} + z_{43} &\leq -16x_{13} + 43x_{23}; \\ 70x_{13} + 44x_{23} &\leq 61; \\ 80x_{13} + 50x_{23} &\leq 67; \\ 88x_{13} + 54x_{23} &\leq 74; \\ x_{13} + x_{23} &= 1; \\ x_{13}, x_{23} &\geq 0, \quad z_{13}, z_{23}, z_{33} \geq 0. \end{aligned}$$

**Step 6:** Substituting the optimal solution  $\left\{x_{11}^* = \frac{17}{30}, x_{21}^* = \frac{13}{30}\right\}, \left\{x_{12}^* = \frac{17}{30}, x_{22}^* = \frac{13}{30}\right\}$  and

$\left\{x_{13}^* = \frac{17}{30}, x_{23}^* = \frac{13}{30}\right\}$  in Problem 3.6.24, Problem 3.6.25 and Problem 3.6.26 respectively, the

obtained optimal solution of Problem 3.6.24, Problem 3.6.25 and Problem 3.6.26 is

$\{y_{11}^* = 1, y_{21}^* = 0\}, \{y_{12}^* = 1, y_{22}^* = 0\}$  and  $\{y_{13}^* = 1, y_{23}^* = 0\}$  respectively.

**Step 7:**

$$\begin{aligned} &\text{minimum} \left\{ \begin{aligned} &\left( (18, 20, 23)x_{11}^*y_{11}^* + (-21, -18, -16)x_{11}^*y_{21}^* + (-33, -32, -27)x_{21}^*y_{11}^* + (38, 40, 43)x_{21}^*y_{21}^* \right), \\ &\left( (18, 20, 23)x_{12}^*y_{12}^* + (-21, -18, -16)x_{12}^*y_{22}^* + (-33, -32, -27)x_{22}^*y_{12}^* + (38, 40, 43)x_{22}^*y_{22}^* \right), \\ &\left( (18, 20, 23)x_{13}^*y_{13}^* + (-21, -18, -16)x_{13}^*y_{23}^* + (-33, -32, -27)x_{23}^*y_{13}^* + (38, 40, 43)x_{23}^*y_{23}^* \right) \end{aligned} \right\} \\ &= \text{minimum} \left\{ \left( \frac{-41}{10}, \frac{-38}{15}, \frac{4}{3} \right), \left( \frac{-41}{10}, \frac{-38}{15}, \frac{4}{3} \right), \left( \frac{-41}{10}, \frac{-38}{15}, \frac{4}{3} \right) \right\} = \left( \frac{-41}{10}, \frac{-38}{15}, \frac{4}{3} \right). \end{aligned}$$

**Step 8:** The minimum expected gain of Player I is  $\left( \frac{-41}{10}, \frac{-38}{15}, \frac{4}{3} \right)$  and corresponding optimal

strategies are  $\left\{x_{11}^* = \frac{17}{30}, x_{21}^* = \frac{13}{30}\right\} = \left\{x_{12}^* = \frac{17}{30}, x_{22}^* = \frac{13}{30}\right\} = \left\{x_{13}^* = \frac{17}{30}, x_{23}^* = \frac{13}{30}\right\}.$

### 3.6.2.2 Maximum expected loss of Player II

Using the proposed Vaishnavi method maximum expected loss of Player II and corresponding optimal strategies, can be obtained as follows:

**Step 1:** Find  $(x_1^*, x_2^*) \in X$  such that the value of

$$((18, 20, 23)x_1y_1 + (-21, -18, -16)x_1y_2 + (-33, -32, -27)x_2y_1 + (38, 40, 43)x_2y_2) \text{ is maximum}$$

for all  $(y_1, y_2) \in Y$  i.e., find the optimal solution of Problem 3.6.30.

**Problem 3.6.30**

$$\text{Maximize } \{(18, 20, 23)x_1y_1 + (-21, -18, -16)x_1y_2 + (-33, -32, -27)x_2y_1 + (38, 40, 43)x_2y_2\}$$

Subject to

$$(70, 80, 88)x_1 + (44, 50, 54)x_2 \preceq (61, 67, 74); \quad (-48, -40, -35)y_1 + (-79, -70, -65)y_2 \succeq (-60, -52, -50);$$

$$x_1 + x_2 \leq 1; \quad -x_1 - x_2 \leq -1; \quad y_1 + y_2 \geq 1; \quad -y_1 - y_2 \geq -1;$$

$$x_1, x_2 \geq 0; y_1, y_2 \geq 0.$$

**Step 2:** Using the property,  $\sum_{i=1}^n \lambda(a_i^L(0), a_i^L(1), a_i^R(1), a_i^R(0)) =$

$$\left( \sum_{i=1}^n \lambda a_{ij}^L(0), \sum_{i=1}^n \lambda a_{ij}^L(1), \sum_{i=1}^n \lambda a_{ij}^R(1), \sum_{i=1}^n \lambda a_{ij}^R(0) \right), \lambda \geq 0, \text{ the Problem 3.6.30 can be transformed}$$

into Problem 3.6.31.

**Problem 3.6.31**

$$\text{Maximize } \left\{ \begin{pmatrix} 18x_1y_1 - 21x_1y_2 - 33x_2y_1 + 38x_2y_2, 20x_1y_1 - 18x_1y_2 - 32x_2y_1, 40x_2y_2, \\ 23x_1y_1 - 16x_1y_2 - 27x_2y_1 + 43x_2y_2 \end{pmatrix} \right\}$$

Subject to

$$(70x_1 + 44x_2, 80x_1 + 50x_2, 88x_1 + 54x_2) \preceq (61, 67, 74);$$

$$(-48y_1 - 79y_2, -40y_1 - 70y_2, -35y_1 - 65y_2) \succeq (-60, -52, -50);$$

$$x_1 + x_2 \leq 1; \quad -x_1 - x_2 \leq -1; \quad y_1 + y_2 \geq 1; \quad -y_1 - y_2 \geq -1;$$

$$x_1, x_2 \geq 0; y_1, y_2 \geq 0.$$

**Step 3:** Using the property,  $(a^L(0), a^L(1), a^R(1), a^R(0)) \succeq (b^L(0), b^L(1), b^R(1), b^R(0)) \Rightarrow a^L(0) \geq$

$$b^L(0), a^L(1) \geq b^L(1), a^R(1) \geq b^R(1), a^R(0) \geq b^R(0), \text{ the Problem 3.6.31 can be transformed into}$$

Problem 3.6.32.

**Problem 3.6.32**

$$\text{Maximize } \left\{ \begin{pmatrix} 18x_1y_1 - 21x_1y_2 - 33x_2y_1 + 38x_2y_2, & 20x_1y_1 - 18x_1y_2 - 32x_2y_1 & 40x_2y_2, \\ 23x_1y_1 - 16x_1y_2 - 27x_2y_1 + 43x_2y_2 \end{pmatrix} \right\}$$

Subject to

$$\begin{aligned} 70x_1 + 44x_2 &\leq 61; & 80x_1 + 50x_2 &\leq 67; & 88x_1 + 54x_2 &\leq 74; \\ -48y_1 - 79y_2 &\geq -60; & -40y_1 - 70y_2 &\geq -52; & -35y_1 - 65y_2 &\geq -50; \\ x_1 + x_2 &= 1; & y_1 + y_2 &= 1; \\ x_1, x_2 &\geq 0; & y_1, y_2 &\geq 0. \end{aligned}$$

**Step 4:** Find the optimal solution  $\{x_{i1}^*, i=1, 2\}, \{x_{i2}^*, i=1, 2\}$  and  $\{x_{i3}^*, i=1, 2\}$  of Problem 3.6.33, Problem 3.6.34 and Problem 3.6.35 respectively.

**Problem 3.6.33**

$$\text{Maximize } \{18x_{11}y_{11} - 21x_{11}y_{21} - 33x_{21}y_{11} + 38x_{21}y_{21}\}$$

Subject to

$$\begin{aligned} 70x_{11} + 44x_{21} &\leq 61; & 80x_{11} + 50x_{21} &\leq 67; & 88x_{11} + 54x_{21} &\leq 74; \\ -48y_{11} - 79y_{21} &\geq -60; & -40y_{11} - 70y_{21} &\geq -52; & -35y_{11} - 65y_{21} &\geq -50; \\ x_{11} + x_{21} &= 1; & y_{11} + y_{21} &= 1; \\ x_{11}, x_{21} &\geq 0; & y_{11}, y_{21} &\geq 0. \end{aligned}$$

**Problem 3.6.34**

$$\text{Maximize } \{20x_{12}y_{12} - 18x_{12}y_{22} - 32x_{22}y_{12} + 40x_{22}y_{22}\}$$

Subject to

$$\begin{aligned} 70x_{12} + 44x_{22} &\leq 61; & 80x_{12} + 50x_{22} &\leq 67; & 88x_{12} + 54x_{22} &\leq 74; \\ -48y_{12} - 79y_{22} &\geq -60; & -40y_{12} - 70y_{22} &\geq -52; & -35y_{12} - 65y_{22} &\geq -50; \\ x_{12} + x_{22} &= 1; & y_{12} + y_{22} &= 1; \\ x_{12}, x_{22} &\geq 0; & y_{12}, y_{22} &\geq 0. \end{aligned}$$

**Problem 3.6.35**

$$\text{Maximize } \{23x_{13}y_{13} - 16x_{13}y_{23} - 27x_{23}y_{13} + 43x_{23}y_{23}\}$$

Subject to

$$\begin{aligned}
70x_{13} + 44x_{23} &\leq 61; & 80x_{13} + 50x_{23} &\leq 67; & 88x_{13} + 54x_{23} &\leq 74; \\
-48y_{13} - 79y_{23} &\geq -60; & -40y_{13} - 70y_{23} &\geq -52; & -35y_{13} - 65y_{23} &\geq -50; \\
x_{13} + x_{23} &= 1; & y_{13} + y_{23} &= 1; \\
x_{13}, x_{23} &\geq 0; & y_{13}, y_{23} &\geq 0.
\end{aligned}$$

**Step 5:** Since, in Problem 3.6.33, Problem 3.6.34 and Problem 3.6.35 only  $x_{i1}, x_{i2}$  and  $x_{i3}$  respectively have been considered as decision variables. So, 3.6.33, Problem 3.6.34 and Problem 3.6.35 are linear programming problems and hence, the optimal value of Problem 3.6.33, Problem 3.6.34 and Problem 3.6.35 will be equal to optimal value of its corresponding dual problem i.e., Problem 3.6.36, Problem 3.6.37, and Problem 3.6.38 respectively.

**Problem 3.6.36**

$$\text{Minimize } \{61t_{11} + 67t_{21} + 74t_{31} + t_{41}\}$$

Subject to

$$\begin{aligned}
70t_{11} + 80t_{21} + 88t_{31} + t_{41} &\geq 18y_{11} - 21y_{21}; \\
44t_{11} + 50t_{21} + 54t_{31} + t_{41} &\geq -33y_{11} + 38y_{21}; \\
-48y_{11} - 79y_{21} &\geq -60; \\
-40y_{11} - 70y_{21} &\geq -52; \\
-35y_{11} - 65y_{21} &\geq -50; \\
y_{11} + y_{21} &= 1; \\
y_{11}, y_{21} &\geq 0, \quad t_{11}, t_{21}, t_{31} \geq 0.
\end{aligned}$$

**Problem 3.6.37**

$$\text{Minimize } \{61t_{12} + 67t_{22} + 74t_{32} + t_{42}\}$$

Subject to

$$\begin{aligned}
70t_{12} + 80t_{22} + 88t_{32} + t_{42} &\geq 20y_{12} - 18y_{22}; \\
44t_{12} + 50t_{22} + 54t_{32} + t_{42} &\geq -32y_{11} + 40y_{21}; \\
-48y_{12} - 79y_{22} &\geq -60; \\
-40y_{12} - 70y_{22} &\geq -52; \\
-35y_{12} - 65y_{22} &\geq -50; \\
y_{12} + y_{22} &= 1; \\
y_{12}, y_{22} &\geq 0, \quad t_{12}, t_{22}, t_{32} \geq 0.
\end{aligned}$$

**Problem 3.6.38**

$$\text{Minimize } \{61t_{13} + 67t_{23} + 74t_{33} + t_{43}\}$$

Subject to

$$\begin{aligned}
70t_{13} + 80t_{23} + 88t_{33} + t_{43} &\geq 23y_{13} - 16y_{23}; & 44t_{13} + 50t_{23} + 54t_{33} + t_{43} &\geq -27y_{13} + 43y_{23}; \\
-48y_{13} - 79y_{23} &\geq -60; & -40y_{13} - 70y_{23} &\geq -52; \\
-35y_{13} - 65y_{23} &\geq -50; & y_{13} + y_{23} &= 1; \\
y_{13}, y_{23} &\geq 0, & t_{13}, t_{23}, t_{33} &\geq 0.
\end{aligned}$$

**Step 6:** Substituting the optimal solution  $\{y_{11}^* = 1, y_{21}^* = 0\}, \{y_{12}^* = 1, y_{22}^* = 0\}, \{y_{13}^* = 1, y_{23}^* = 0\}$

and  $\{y_{14}^* = 1, y_{24}^* = 0\}$  of Problem 3.6.36 Problem 3.6.37 and Problem 3.6.38 in Problem 3.6.33, Problem 3.6.34 and Problem 3.6.35 respectively, the obtained optimal solutions of Problem 3.6.33, Problem 3.6.34 and Problem 3.6.35 is  $\left\{x_{11}^* = \frac{17}{30}, x_{21}^* = \frac{13}{30}\right\}, \left\{x_{12}^* = \frac{17}{30}, x_{22}^* = \frac{13}{30}\right\}$  and  $\left\{x_{13}^* = \frac{17}{30}, x_{23}^* = \frac{13}{30}\right\}$  respectively.

**Step 7:** Now,

$$\begin{aligned} & \text{maximum} \left\{ \begin{aligned} & \left( (18, 20, 23)x_{11}^*y_{11}^* + (-21, -18, -16)x_{11}^*y_{21}^* + (-33, -32, -27)x_{21}^*y_{11}^* + (38, 40, 43)x_{21}^*y_{21}^* \right), \\ & \left( (18, 20, 23)x_{12}^*y_{12}^* + (-21, -18, -16)x_{12}^*y_{22}^* + (-33, -32, -27)x_{22}^*y_{12}^* + (38, 40, 43)x_{22}^*y_{22}^* \right), \\ & \left( (18, 20, 23)x_{13}^*y_{13}^* + (-21, -18, -16)x_{13}^*y_{23}^* + (-33, -32, -27)x_{23}^*y_{13}^* + (38, 40, 43)x_{23}^*y_{23}^* \right) \end{aligned} \right\} \\ & = \text{maximum} \left\{ \left( \frac{-41}{10}, \frac{-38}{15}, \frac{4}{3} \right), \left( \frac{-41}{10}, \frac{-38}{15}, \frac{4}{3} \right), \left( \frac{-41}{10}, \frac{-38}{15}, \frac{4}{3} \right) \right\} = \left( \frac{-41}{10}, \frac{-38}{15}, \frac{4}{3} \right). \end{aligned}$$

**Step 8:** The maximum expected loss of Player II is  $\left( \frac{-41}{10}, \frac{-38}{15}, \frac{4}{3} \right)$  and corresponding optimal strategies are  $\{y_{11}^* = 1, y_{21}^* = 0\} = \{y_{12}^* = 1, y_{22}^* = 0\} = \{y_{13}^* = 1, y_{23}^* = 0\}$ .

### 3.7 Conclusion

On the basis of present study, it can be concluded that some mathematically incorrect assumptions have been considered in the existing methods [64,74-76] for solving constrained matrix games with fuzzy payoffs. Therefore, it is not genuine to use these methods. Furthermore, to resolve flaws of the existing methods [75,76], a new method (named as Vaishnavi method) is proposed for solving constrained matrix games with fuzzy payoffs.



# Chapter 4

## Ambika methods for solving matrix games with intuitionistic fuzzy payoffs<sup>§</sup>

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In this chapter, flaws of the existing methods [78,95-97] for solving matrix games with intuitionistic fuzzy payoffs (matrix games in which payoffs are represented by intuitionistic fuzzy numbers) are pointed out. To resolve these flaws, new methods (named as Ambika methods) are also proposed to obtain the optimal strategies as well as minimum expected gain of Player I and maximum expected loss of Player II for matrix games with intuitionistic fuzzy payoffs. To illustrate proposed Ambika methods, some existing numerical problems of matrix games with intuitionistic fuzzy payoffs are solved by proposed Ambika methods.

### 4.1 Matrix games with intuitionistic fuzzy payoffs

Fuzzy sets are designed to handle uncertainties by attributing a degree, called the membership degree, to which an object belongs to a set. The degree to which it does not belong to the same set is taken as one minus the membership degree, and is termed as the non-membership degree. However, in real decision making problems there are instances where not only the degree to which an object belongs to a set is known but in addition a degree to which the same object does not belong to the set is also known. For example, when several experts evaluate a product, they can give arguments for the product being ‘good’ as well as arguments for the product being ‘bad’. The ‘goodness’ and ‘badness’ of the product can be quantified in terms of pre-decided thresholds. The arguments in favor of ‘good’ describe the membership degree that the product belongs to a ‘set of good products’, while the arguments in favor of

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‘badness’ describe the non-membership degree of the product not being in the ‘set of good products’.

An important point to observe is that an expert may have his/her own reservations in classifying the object in one of these two categories. In other words, it amounts to say that the two degrees do not necessarily add up to one. Atanassov [11] proposed an interesting generalization of fuzzy sets called intuitionistic fuzzy sets to capture this aspect of human behavior. Intuitionistic fuzzy sets are characterized by two membership functions, one for the degree of belongingness and the other for the degree of non-belongingness. These membership functions are defined such that for each element of the universe the sum of their degrees is less than or equal to one rather than being one as in classical fuzzy set theory.

Bustince and Burillo [24] pointed out that notion of intuitionistic fuzzy set is same as that of vague set [47] i.e., intuitionistic fuzzy set and vague sets are same only their representations is different. The only difference between the representation of intuitionistic fuzzy set and vague set is that in the intuitionistic fuzzy set, the degree of non-membership is mentioned while, in the vague set instead of degree of non-membership i.e.,  $1 - (\text{degree of non-membership})$  is mentioned [33]. In the literature [1,72,78,80,95-97], such games in which payoffs are represented by intuitionistic fuzzy number (vague set) are named as matrix games with intuitionistic fuzzy payoffs.

## 4.2 Preliminaries

In this section, some basic definitions and arithmetic operations of vague sets (intuitionistic fuzzy sets) are presented [33].

### 4.2.1 Basic definitions

In this section, some basic definitions are presented.

**Definition 4.1** A vague set  $\tilde{A} = \left\{ \left\langle (x; \mu_{\tilde{A}}(x), 1 - \nu_{\tilde{A}}(x)) \right\rangle \mid x \in X \right\}$ , defined on the universal set  $X$ ,

is characterized by a truth membership function  $\mu_{\tilde{A}}, \mu_{\tilde{A}} : X \rightarrow [0,1]$  and complement of a false membership function  $(1-\nu_{\tilde{A}}), (1-\nu_{\tilde{A}}) : X \rightarrow [0,1]$ . The values  $\mu_{\tilde{A}}(x)$  and  $1-\nu_{\tilde{A}}(x)$  represents the degree of membership and complement of non-membership for  $x \in X$  and always satisfies the condition  $\mu_{\tilde{A}}(x) \leq 1-\nu_{\tilde{A}}(x) \forall x \in X$ . The value  $(1-\mu_{\tilde{A}}(x)-\nu_{\tilde{A}}(x))$  represents the degree of hesitation for  $x \in X$ .

**Definition 4.2** Let,  $\tilde{A}$  be a vague set then  $A(\alpha, \beta) = \{x \in X \mid \mu_{\tilde{A}}(x) \geq \alpha, (1-\nu_{\tilde{A}}(x)) \geq \beta\}$ ,  $\alpha, \beta \in [0,1]$  is said to be an  $(\alpha, \beta)$ -cut of  $\tilde{A}$ .

**Definition 4.3** A vague set  $\tilde{A}$ , defined on the universal set of real numbers  $\mathbb{R}$ , denoted as  $\tilde{A} = \langle (a^L(0), a(1), a^R(0)); w_{\tilde{A}}, 1-u_{\tilde{A}} \rangle$ , where  $a^L(0) \leq a(1) \leq a^R(0)$  and  $w_{\tilde{A}} \leq 1-u_{\tilde{A}}$ , is said to be a triangular vague set if the membership function,  $\mu_{\tilde{A}}$  and complement of non-membership function,  $1-\nu_{\tilde{A}}$  are given as

$$\mu_{\tilde{A}}(x) = \begin{cases} \frac{w_{\tilde{A}}(x-a^L(0))}{a(1)-a^L(0)}, & a^L(0) \leq x < a(1) \\ w_{\tilde{A}}, & x = a(1) \\ \frac{w_{\tilde{A}}(x-a^R(0))}{a(1)-a^R(0)}, & a(1) < x \leq a^R(0) \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad 1-\nu_{\tilde{A}}(x) = \begin{cases} \frac{(1-u_{\tilde{A}})(x-a^L(0))}{a(1)-a^L(0)}, & a^L(0) \leq x < a(1) \\ (1-u_{\tilde{A}}), & x = a(1) \\ \frac{(1-u_{\tilde{A}})(x-a^R(0))}{a(1)-a^R(0)}, & a(1) < x \leq a^R(0) \\ 0, & \text{otherwise} \end{cases}$$

where,  $w_{\tilde{A}} = \supremum \{\mu_{\tilde{A}}(x) : x \in \mathbb{R}\}$  and  $1-u_{\tilde{A}} = \supremum \{1-\nu_{\tilde{A}}(x) : x \in \mathbb{R}\}$ .

**Definition 4.4** Let,  $\tilde{A} = \langle (a^L(0), a(1), a^R(0)); w_{\tilde{A}}, 1-u_{\tilde{A}} \rangle$  be a triangular vague set then  $(\alpha, \beta)$ -cut of  $\tilde{A}$ ,  $\alpha \in [0, w_{\tilde{A}}], \beta \in [0, 1-u_{\tilde{A}}]$ , for the triangular vague set  $\tilde{A}$  can be defined as

$$\tilde{A}(\alpha, \beta) = \left\{ \left[ a^L(0) + (a(1)-a^L(0)) \frac{\alpha}{w_{\tilde{A}}}, a^R(0) - (a^R(0)-a(1)) \frac{\alpha}{w_{\tilde{A}}} \right]; \right.$$

$$\left[ a^L(0) + (a(1) - a^L(0)) \frac{\beta}{1 - u_{\tilde{A}}}, a^R(0) - (a^R(0) - a(1)) \frac{\beta}{1 - u_{\tilde{A}}} \right].$$

**Definition 4.5** A vague set  $\tilde{A}$ , defined on the universal set of real numbers  $\mathbb{R}$ , denoted as

$$\tilde{A} = \langle (a^L(0), a^L(1), a^R(1), a^R(0)); w_{\tilde{A}}, 1 - u_{\tilde{A}} \rangle, \quad \text{where} \quad a^L(0) \leq a^L(1) \leq a^R(1) \leq a^R(0) \quad \text{and}$$

$w_{\tilde{A}} \leq 1 - u_{\tilde{A}}$ , is said to trapezoidal vague set if the membership function,  $\mu_{\tilde{A}}$  and the

complement of non-membership function,  $1 - \nu_{\tilde{A}}$  are given as

$$\mu_{\tilde{A}}(x) = \begin{cases} \frac{w_{\tilde{A}}(x - a^L(0))}{a^L(1) - a^L(0)}, & a^L(0) \leq x < a^L(1) \\ w_{\tilde{A}}, & a^L(1) \leq x \leq a^R(1) \\ \frac{w_{\tilde{A}}(x - a^R(0))}{a^R(1) - a^R(0)}, & a^R(1) < x \leq a^R(0) \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad 1 - \nu_{\tilde{A}}(x) = \begin{cases} \frac{(1 - u_{\tilde{A}})(x - a^L(0))}{a^L(1) - a^L(0)}, & a^L(0) \leq x < a^L(1) \\ (1 - u_{\tilde{A}}), & a^L(1) \leq x \leq a^R(1) \\ \frac{(1 - u_{\tilde{A}})(x - a^R(0))}{a^R(1) - a^R(0)}, & a^R(1) < x \leq a^R(0) \\ 0, & \text{otherwise} \end{cases}$$

where,  $w_{\tilde{A}} = \sup\{\mu_{\tilde{A}}(x) : x \in \mathbb{R}\}$  and  $1 - u_{\tilde{A}} = \sup\{1 - \nu_{\tilde{A}}(x) : x \in \mathbb{R}\}$ .

**Definition 4.6** Let,  $\tilde{A} = \langle (a^L(0), a^L(1), a^R(1), a^R(0)); w_{\tilde{A}}, 1 - u_{\tilde{A}} \rangle$  be a trapezoidal vague set then

$(\alpha, \beta)$ -cut of  $\tilde{A}$ ,  $\alpha \in [0, w_{\tilde{A}}]$ ,  $\beta \in [0, 1 - u_{\tilde{A}}]$ , for the trapezoidal vague set  $\tilde{A}$  can be defined

as 
$$\tilde{A}(\alpha, \beta) = \left\{ \left[ a^L(0) + (a^L(1) - a^L(0)) \frac{\alpha}{w_{\tilde{A}}}, a^R(0) - (a^R(0) - a^R(1)) \frac{\alpha}{w_{\tilde{A}}} \right]; \right.$$

$$\left. \left[ a^L(0) + (a^L(1) - a^L(0)) \frac{\beta}{1 - u_{\tilde{A}}}, a^R(0) - (a^R(0) - a^R(1)) \frac{\beta}{1 - u_{\tilde{A}}} \right] \right\}.$$

**Remark 4.1:** If in the triangular vague set  $\tilde{A} = \langle (a^L(0), a(1), a^R(0)); w_{\tilde{A}}, 1 - u_{\tilde{A}} \rangle$ ,  $1 - u_{\tilde{A}}$  is

replaced by  $u_{\tilde{A}}$  then  $\tilde{A} = \langle (a^L(0), a(1), a^R(0)); w_{\tilde{A}}, u_{\tilde{A}} \rangle$  is called triangular intuitionistic fuzzy

number.

**Remark 4.2:** If in the trapezoidal vague set  $\tilde{A} = \langle (a^L(0), a^L(1), a^R(1), a^R(0)); w_{\tilde{A}}, 1 - u_{\tilde{A}} \rangle$ ,  $1 - u_{\tilde{A}}$

is replaced by  $u_{\tilde{A}}$  then  $\tilde{A} = \langle (a^L(0), a^L(1), a^R(1), a^R(0)); w_{\tilde{A}}, u_{\tilde{A}} \rangle$  is called trapezoidal intuitionistic fuzzy number.

#### 4.2.2 Arithmetic operations over trapezoidal vague sets

In this section, arithmetic operations over trapezoidal vague sets, defined on universal set of real numbers  $\mathbb{R}$ , are presented.

Let,

$$\tilde{A} = \langle (a^L(0), a^L(1), a^R(1), a^R(0)); w_{\tilde{A}}, 1 - u_{\tilde{A}} \rangle \text{ and } \tilde{B} = \langle (b^L(0), b^L(1), b^R(1), b^R(0)); w_{\tilde{B}}, 1 - u_{\tilde{B}} \rangle \text{ be}$$

two trapezoidal vague sets (trapezoidal intuitionistic fuzzy numbers) then

$$\begin{aligned} \text{(i)} \quad \tilde{A} + \tilde{B} &= \langle (a^L(0) + b^L(0), a^L(1) + b^L(1), a^R(1) + b^R(1), a^R(0) + b^R(0)); \min\{w_{\tilde{A}}, w_{\tilde{B}}\}, \min\{1 - u_{\tilde{A}}, 1 - u_{\tilde{B}}\} \rangle \\ \text{(ii)} \quad \tilde{A} - \tilde{B} &= \langle (a^L(0) - b^R(0), a^L(1) - b^R(1), a^R(1) - b^L(1), a^R(0) - b^L(0)); \min\{w_{\tilde{A}}, w_{\tilde{B}}\}, \min\{1 - u_{\tilde{A}}, 1 - u_{\tilde{B}}\} \rangle \\ \text{(iii)} \quad \lambda \tilde{A} &= \begin{cases} \langle (\lambda a^L(0), \lambda a^L(1), \lambda a^R(1), \lambda a^R(0)); w_{\tilde{A}}, 1 - u_{\tilde{A}} \rangle, \lambda \geq 0 \\ \langle (\lambda a^R(0), \lambda a^R(1), \lambda a^L(1), \lambda a^L(0)); w_{\tilde{A}}, 1 - u_{\tilde{A}} \rangle, \lambda < 0. \end{cases} \end{aligned}$$

**Remark 4.3:** If in the arithmetic operations presented in Section 4.2.2,  $\min\{1 - u_{\tilde{A}}, 1 - u_{\tilde{B}}\}$  is replaced by  $\max\{u_{\tilde{A}}, u_{\tilde{B}}\}$  then these arithmetic operations will be arithmetic operations over trapezoidal intuitionistic fuzzy numbers.

#### 4.3 Existing mathematical formulation of matrix games with intuitionistic fuzzy payoffs

In the literature, Problem 4.3.1 and Problem 4.3.2 are used to obtain the minimum expected gain of Player I and maximum expected loss of Player II as well as their corresponding optimal strategies. These problems can be obtained by using the same procedure, discussed in Section 1.6.1 of Chapter 1, by replacing the intervals  $[a_{ij}^L, a_{ij}^R]$ ,  $[v^L, v^R]$  and  $[\omega^L, \omega^R]$  with triangular/trapezoidal intuitionistic fuzzy numbers  $\tilde{a}_{ij}$ ,  $\tilde{v}$  and  $\tilde{\omega}$  respectively.

##### Problem 4.3.1

Maximize  $\{\tilde{v}\}$

##### Problem 4.3.2

Minimize  $\{\tilde{\omega}\}$

Subject to

$$\begin{aligned} \sum_{i=1}^m \tilde{a}_{ij} x_i &\succeq \tilde{v}, \quad j = 1, 2, \dots, n; \\ \sum_{i=1}^m x_i &= 1; \\ x_i &\geq 0, \quad i = 1, 2, \dots, m. \end{aligned}$$

Subject to

$$\begin{aligned} \sum_{j=1}^n \tilde{a}_{ij} y_j &\preceq \tilde{w}, \quad i = 1, 2, \dots, m; \\ \sum_{j=1}^n y_j &= 1; \\ y_j &\geq 0, \quad j = 1, 2, \dots, n. \end{aligned}$$

#### 4.4 Literature review of matrix games with intuitionistic fuzzy payoffs

In this section, a brief review of the methods, proposed in the literature in last ten years for solving matrix games with intuitionistic fuzzy payoffs (matrix games in which payoffs are either represented by triangular intuitionistic numbers or trapezoidal intuitionistic fuzzy numbers) is presented.

Nan et al. [95] proposed a method to find the minimum expected gain of Player I, maximum expected loss of Player II and their corresponding strategies for matrix games with triangular intuitionistic fuzzy payoffs. In this method, firstly authors have transformed Problem 4.3.1 and Problem 4.3.2 into Problem 4.4.1 and Problem 4.4.2 respectively then it is claimed that the optimal solution  $\{x_i, i = 1, 2, \dots, m\}$  and optimal value of Problem 4.4.3 represents the optimal strategies and minimum expected gain of Player I as well as the optimal solution  $\{y_j, j = 1, 2, \dots, n\}$  and optimal value of Problem 4.4.5 represents the optimal strategies and maximum expected loss of Player II respectively.

##### Problem 4.4.1

$$\text{Maximize } \left\{ \min_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}} \{w_{\tilde{a}_{ij}}\} \left( \frac{v^L(0) + 2v(1) + v^R(0)}{4} \right), 1 - \max_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}} \{u_{\tilde{a}_{ij}}\} \left( \frac{v^L(0) + 2v(1) + v^R(0)}{4} \right) \right\}$$

Subject to

$$\sum_{i=1}^m \frac{\min_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}} \{w_{\tilde{a}_{ij}}\} \left( \frac{v^L(0) + 2a_{ij}(1) + a_{ij}^R(0)}{4} \right)}{4} x_i \geq \min_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}} \{w_{\tilde{a}_{ij}}\} \left( \frac{v^L(0) + 2v(1) + v^R(0)}{4} \right), \quad j = 1, 2, \dots, n;$$

$$\sum_{i=1}^m \frac{\left(1 - \max_{1 \leq i \leq m} \{u_{\tilde{a}_{ij}}\}\right) \left(a_{ij}^L(0) + 2a_{ij}(1) + a_{ij}^R(0)\right)}{4} x_i \geq 1 - \max_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}} \{u_{\tilde{a}_{ij}}\} \left( \frac{v^L(0) + 2v(1) + v^R(0)}{4} \right), \quad j = 1, 2, \dots, n;$$

$$v^L(0) \leq v(1); v(1) \leq v^R(0);$$

$$\sum_{i=1}^m x_i = 1; x_i \geq 0, \quad i = 1, 2, \dots, m.$$

#### Problem 4.4.2

$$\text{Minimize} \left\{ \min_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}} \{w_{\tilde{a}_{ij}}\} \left( \frac{\omega^L(0) + 2\omega(1) + \omega^R(0)}{4} \right), 1 - \max_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}} \{u_{\tilde{a}_{ij}}\} \left( \frac{\omega^L(0) + 2\omega(1) + \omega^R(0)}{4} \right) \right\}$$

Subject to

$$\sum_{j=1}^n \frac{\min_{1 \leq j \leq n} \{w_{\tilde{a}_{ij}}\} \left( a_{ij}^L(0) + 2a_{ij}(1) + a_{ij}^R(0) \right)}{4} y_j \leq \min_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}} \{w_{\tilde{a}_{ij}}\} \left( \frac{\omega^L(0) + 2\omega(1) + \omega^R(0)}{4} \right), \quad i = 1, 2, \dots, m;$$

$$\sum_{j=1}^n \frac{\left(1 - \max_{1 \leq j \leq n} \{u_{\tilde{a}_{ij}}\}\right) \left(a_{ij}^L(0) + 2a_{ij}(1) + a_{ij}^R(0)\right)}{4} y_j \leq 1 - \max_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}} \{u_{\tilde{a}_{ij}}\} \left( \frac{\omega^L(0) + 2\omega(1) + \omega^R(0)}{4} \right), \quad i = 1, 2, \dots, m;$$

$$\omega^L(0) \leq \omega(1); \omega(1) \leq \omega^R(0);$$

$$\sum_{j=1}^n y_j = 1; y_j \geq 0, \quad j = 1, 2, \dots, n.$$

#### Problem 4.4.3

$$\text{Maximize} \{v_2\}$$

Subject to

$$\sum_{i=1}^m \frac{\min_{1 \leq i \leq m} \{w_{\tilde{a}_{ij}}\} \left( a_{ij}^L(0) + 2a_{ij}(1) + a_{ij}^R(0) \right)}{4} x_i \geq v_1, \quad j = 1, 2, \dots, n;$$

$$\sum_{i=1}^m \frac{\left(1 - \max_{1 \leq i \leq m} \{u_{\tilde{a}_{ij}}\}\right) \left(a_{ij}^L(0) + 2a_{ij}(1) + a_{ij}^R(0)\right)}{4} x_i \geq v_2, \quad j = 1, 2, \dots, n;$$

$$v_2 \geq v_1; \quad v_1 \geq v_1^0; \quad v_2 \geq v_2^0;$$

$$\sum_{i=1}^m x_i = 1; x_i \geq 0, \quad i = 1, 2, \dots, m.$$

where,  $v_1^0$  and  $v_2^0$  are optimal solution of Problem 4.4.4.

#### Problem 4.4.4

Maximize  $\{\nu_1\}$

Subject to

$$\sum_{i=1}^m \frac{\min_{1 \leq i \leq m} \{w_{\tilde{a}_{ij}}\} (a_{ij}^L(0) + 2a_{ij}(1) + a_{ij}^R(0))}{4} x_i \geq \nu_1, \quad j = 1, 2, \dots, n;$$

$$\sum_{i=1}^m \frac{(1 - \max_{1 \leq i \leq m} \{u_{\tilde{a}_{ij}}\}) (a_{ij}^L(0) + 2a_{ij}(1) + a_{ij}^R(0))}{4} x_i \geq \nu_2, \quad j = 1, 2, \dots, n;$$

$$\sum_{i=1}^m x_i = 1; x_i \geq 0, \quad i = 1, 2, \dots, m.$$

$$\text{where, } \nu_1 = \min_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}} \{w_{\tilde{a}_{ij}}\} \left( \frac{\nu^L(0) + 2\nu(1) + \nu^R(0)}{4} \right), \quad \nu_2 = 1 - \max_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}} \{u_{\tilde{a}_{ij}}\} \left( \frac{\nu^L(0) + 2\nu(1) + \nu^R(0)}{4} \right).$$

#### Problem 4.4.5

Minimize  $\{\omega_2\}$

Subject to

$$\sum_{j=1}^n \frac{\min_{1 \leq j \leq n} \{w_{\tilde{a}_{ij}}\} (a_{ij}^L(0) + 2a_{ij}(1) + a_{ij}^R(0))}{4} y_j \leq \omega_1, \quad i = 1, 2, \dots, m;$$

$$\sum_{j=1}^n \frac{(1 - \max_{1 \leq j \leq n} \{u_{\tilde{a}_{ij}}\}) (a_{ij}^L(0) + 2a_{ij}(1) + a_{ij}^R(0))}{4} y_j \leq \omega_2, \quad i = 1, 2, \dots, m;$$

$$\omega_2 \geq \omega_1; \omega_1 \leq \omega_1^0; \omega_2 \leq \omega_2^0$$

$$\sum_{j=1}^n y_j = 1; y_j \geq 0, \quad j = 1, 2, \dots, n.$$

where,  $\omega_1^0$  and  $\omega_2^0$  are optimal solution of Problem 4.4.6.

#### Problem 4.4.6

Minimize  $\{\omega_1\}$

Subject to

$$\sum_{j=1}^n \frac{\min_{1 \leq j \leq n} \{w_{\tilde{a}_{ij}}\} (a_{ij}^L(0) + 2a_{ij}(1) + a_{ij}^R(0))}{4} y_j \leq \omega_1, i = 1, 2, \dots, m;$$

$$\sum_{j=1}^n \frac{\left(1 - \max_{1 \leq j \leq n} \{u_{\tilde{a}_{ij}}\}\right) (a_{ij}^L(0) + 2a_{ij}(1) + a_{ij}^R(0))}{4} y_j \leq \omega_2, i = 1, 2, \dots, m;$$

$$\sum_{j=1}^n y_j = 1; y_j \geq 0, j = 1, 2, \dots, n.$$

$$\text{where, } \omega_1 = \min_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}} \{w_{\tilde{a}_{ij}}\} \left( \frac{\omega^L(0) + 2\omega(1) + \omega^R(0)}{4} \right),$$

$$\omega_2 = \left( 1 - \max_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}} \{u_{\tilde{a}_{ij}}\} \right) \left( \frac{\omega^L(0) + 2\omega(1) + \omega^R(0)}{4} \right) \text{ and } \lambda \in [0, 1].$$

Li et al. [78] proposed a method to find the minimum expected gain of Player I, maximum expected loss of Player II and their corresponding optimal strategies for matrix games with triangular intuitionistic fuzzy payoffs. In this method, firstly authors have transformed Problem 4.3.1 and Problem 4.3.2 into Problem 4.4.1 and Problem 4.4.2 respectively. Then, it is claimed that the optimal solution  $\{x_i, i = 1, 2, \dots, m\}$  and optimal value of Problem 4.4.7 represents the optimal strategies and minimum expected gain of Player I respectively as well as optimal solution  $\{y_j, j = 1, 2, \dots, n\}$  and optimal value of Problem 4.4.8 represents the optimal strategies and maximum expected loss of Player II respectively.

#### **Problem 4.4.7**

$$\text{Maximize } \{\lambda v_1 + (1 - \lambda)v_2\}$$

Subject to

Constraints of Problem 4.4.1,

$$\text{where, } v_1 = \min_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}} \{w_{\tilde{a}_{ij}}\} \left( \frac{v^L(0) + 2v(1) + v^R(0)}{4} \right), \quad v_2 = \left( 1 - \max_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}} \{u_{\tilde{a}_{ij}}\} \right) \left( \frac{v^L(0) + 2v(1) + v^R(0)}{4} \right)$$

and  $\lambda \in [0, 1]$ .

**Problem 4.4.8**

Minimize  $\{\lambda\omega_1 + (1-\lambda)\omega_2\}$

Subject to

Constraints of Problem 4.4.2

$$\text{where, } \omega_1 = \min_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}} \left\{ w_{\tilde{a}_{ij}} \right\} \left( \frac{\omega^L(0) + 2\omega(1) + \omega^R(0)}{4} \right),$$

$$\omega_2 = \left( 1 - \max_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}} \left\{ u_{\tilde{a}_{ij}} \right\} \right) \left( \frac{\omega^L(0) + 2\omega(1) + \omega^R(0)}{4} \right) \text{ and } \lambda \in [0,1].$$

Nan et al. [96] proposed a method to find the minimum expected gain of Player I, maximum expected loss of Player II and corresponding optimal strategies for matrix games with triangular intuitionistic fuzzy payoffs. In this method, firstly the authors have transformed Problem 4.3.1 and Problem 4.3.2 into Problem 4.4.9 and Problem 4.4.10 respectively. Then, it is claimed that optimal solution  $\{x_i, i=1,2,\dots,m\}$  and optimal value of Problem 4.4.11 represents the optimal strategies and minimum expected gain of Player I respectively as well as optimal solution  $\{y_j, j=1,2,\dots,n\}$  and optimal value of Problem 4.4.13 represents the optimal strategies and maximum expected loss of Player II respectively.

**Problem 4.4.9**

Maximize  $\{V_1\}$ , Minimize  $\{A_1\}$

Subject to

$$\sum_{i=1}^m \left( a_{ij}^L(0) + 4a_{ij}(1) + a_{ij}^R(0) \right) \left[ \lambda \min_{1 \leq i \leq m} \left\{ w_{\tilde{a}_{ij}} \right\}^2 + (1-\lambda) \left( 1 - \max_{1 \leq i \leq m} \left\{ u_{\tilde{a}_{ij}} \right\} \right)^2 \right] \frac{x_i}{6} \geq V_1, \quad j=1,2,\dots,n;$$

$$\sum_{i=1}^m \left( a_{ij}^R(0) - a_{ij}^L(0) \right) \left[ \lambda \min_{1 \leq i \leq m} \left\{ w_{\tilde{a}_{ij}} \right\}^2 + (1-\lambda) \left( 1 - \max_{1 \leq i \leq m} \left\{ u_{\tilde{a}_{ij}} \right\} \right)^2 \right] \frac{x_i}{6} \leq A_1, \quad j=1,2,\dots,n;$$

$$\sum_{i=1}^m x_i = 1; x_i \geq 0, \quad i=1,2,\dots,m.$$

$$\text{where, } V_1 = \left( \frac{\nu^L(0) + 4\nu(1) + \nu^R(0)}{6} \right) \left( \lambda \min_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}} \{w_{\tilde{a}_{ij}}\}^2 + (1-\lambda) \left( 1 - \max_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}} \{u_{\tilde{a}_{ij}}\} \right)^2 \right) \text{ and}$$

$$A_1 = \left( \frac{\nu^R(0) - \nu^L(0)}{6} \right) \left( \lambda \min_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}} \{w_{\tilde{a}_{ij}}\}^2 + (1-\lambda) \left( 1 - \max_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}} \{u_{\tilde{a}_{ij}}\} \right)^2 \right).$$

**Problem 4.4.10**

Minimize  $\{V_2\}$ , Maximize  $\{A_2\}$

Subject to

$$\begin{aligned} \sum_{j=1}^n (a_{ij}^L(0) + 4a_{ij}(1) + a_{ij}^R(0)) \left[ \lambda \min_{1 \leq j \leq n} \{w_{\tilde{a}_{ij}}\}^2 + (1-\lambda) \left( 1 - \max_{1 \leq j \leq n} \{u_{\tilde{a}_{ij}}\} \right)^2 \right] \frac{y_j}{6} &\leq V_2, \quad i = 1, 2, \dots, m; \\ \sum_{j=1}^n (a_{ij}^R(0) - a_{ij}^L(0)) \left[ \lambda \min_{1 \leq j \leq n} \{w_{\tilde{a}_{ij}}\}^2 + (1-\lambda) \left( 1 - \max_{1 \leq j \leq n} \{u_{\tilde{a}_{ij}}\} \right)^2 \right] \frac{y_j}{6} &\geq A_2, \quad i = 1, 2, \dots, m; \quad \text{where,} \\ \sum_{j=1}^n y_j &= 1; y_j \geq 0, \quad j = 1, 2, \dots, n. \end{aligned}$$

$$V_2 = \left( \frac{\omega^L(0) + 4\omega(1) + \omega^R(0)}{6} \right) \left( \lambda \min_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}} \{w_{\tilde{a}_{ij}}\}^2 + (1-\lambda) \left( 1 - \max_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}} \{u_{\tilde{a}_{ij}}\} \right)^2 \right) \text{ and}$$

$$A_2 = \left( \frac{\omega^R(0) - \omega^L(0)}{6} \right) \left( \lambda \min_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}} \{w_{\tilde{a}_{ij}}\}^2 + (1-\lambda) \left( 1 - \max_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}} \{u_{\tilde{a}_{ij}}\} \right)^2 \right).$$

**Problem 4.4.11**

Minimize  $\{A_1\}$

Subject to

Constraints of Problem 4.4.9 with the additional constraints

$$V_1 \geq V_1^0; \quad A_1 \leq A_1^0.$$

where,  $V_1^0$  and  $A_1^0$  are optimal solution of Problem 4.4.12.

**Problem 4.4.12**

Maximize  $\{V_1\}$

Subject to

Constraints of Problem 4.4.9.

**Problem 4.4.13**

Maximize  $\{A_2\}$

Subject to

Constraints of Problem 4.4.10 with the additional constraints

$$V_2 \leq V_2^0; A_2 \geq A_2^0.$$

where,  $V_2^0$  and  $A_2^0$  are optimal solution of Problem 4.4.14.

**Problem 4.4.14**

Minimize  $\{V_2\}$

Subject to

Constraints of Problem 4.4.10.

Nan et al. [97] proposed a method to find the minimum expected gain of Player I, maximum expected loss of Player II and their corresponding optimal strategies for matrix games with trapezoidal intuitionistic fuzzy payoffs. In this method, firstly the authors have transformed Problem 4.3.1 and Problem 4.3.2 into Problem 4.4.15 and Problem 4.4.16 respectively.

**Problem 4.4.15**

Maximize  $\{V_1\}$ , Minimize  $\{A_1\}$

Subject to

$$\sum_{i=1}^m \left( a_{ij}^L(0) + 2a_{ij}^L(1) + 2a_{ij}^R(1) + a_{ij}^R(0) \right) \left[ (1-\lambda) \min_{1 \leq i \leq m} \{w_{\tilde{a}_{ij}}\} + \lambda \left( 1 - \max_{1 \leq i \leq m} \{u_{\tilde{a}_{ij}}\} \right) \right] \frac{x_i}{6} \geq V_1, \quad j=1, 2, \dots, n;$$

$$\sum_{i=1}^m \left( a_{ij}^R(0) - a_{ij}^L(0) + 2a_{ij}^R(1) - 2a_{ij}^L(1) \right) \left[ \lambda \min_{1 \leq i \leq m} \{w_{\tilde{a}_{ij}}\} + (1-\lambda) \left( 1 - \max_{1 \leq i \leq m} \{u_{\tilde{a}_{ij}}\} \right) \right] \frac{x_i}{6} \leq A_1, \quad j=1, 2, \dots, n;$$

$$\sum_{i=1}^m x_i = 1; x_i \geq 0, \quad i=1, 2, \dots, m.$$

$$\text{where,} \quad V_1 = \left( \frac{v^L(0) + 2v^L(1) + 2v^R(1) + v^R(0)}{6} \right) \left[ (1-\lambda) \min_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}} \{w_{\tilde{a}_{ij}}\} + \lambda \left( 1 - \max_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}} \{u_{\tilde{a}_{ij}}\} \right) \right] \quad \text{and}$$

$$A_1 = \left( \frac{v^R(0) - v^L(0) + 2v^R(1) - 2v^L(1)}{3} \right) \left[ \lambda \min_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}} \{w_{\tilde{a}_{ij}}\} + (1-\lambda) \left( 1 - \max_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}} \{u_{\tilde{a}_{ij}}\} \right) \right] \dots$$

#### Problem 4.4.16

Minimize  $\{V_2\}$ , Maximize  $\{A_2\}$

Subject to

$$\sum_{j=1}^n \left( a_{ij}^L(0) + 2a_{ij}^L(1) + 2a_{ij}^R(1) + a_{ij}^R(0) \right) \left[ (1-\lambda) \min_{1 \leq j \leq n} \{w_{\tilde{a}_{ij}}\} + \lambda \left( 1 - \max_{1 \leq j \leq n} \{u_{\tilde{a}_{ij}}\} \right) \right] \frac{y_j}{6} \leq V_2, \quad i=1, 2, \dots, m;$$

$$\sum_{j=1}^n \left( a_{ij}^R(0) - a_{ij}^L(0) + 2a_{ij}^R(1) - 2a_{ij}^L(1) \right) \left[ \lambda \min_{1 \leq j \leq n} \{w_{\tilde{a}_{ij}}\} + (1-\lambda) \left( 1 - \max_{1 \leq j \leq n} \{u_{\tilde{a}_{ij}}\} \right) \right] \frac{y_j}{6} \geq A_2, \quad i=1, 2, \dots, m;$$

$$\sum_{j=1}^n y_j = 1; y_j \geq 0, \quad j=1, 2, \dots, n.$$

$$\text{where,} \quad V_2 = \left( \frac{\omega^L(0) + 2\omega^L(1) + 2\omega^R(1) + \omega^R(0)}{6} \right) \left[ (1-\lambda) \min_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}} \{w_{\tilde{a}_{ij}}\} + \lambda \left( 1 - \max_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}} \{u_{\tilde{a}_{ij}}\} \right) \right] \quad \text{and}$$

$$A_2 = \left( \frac{\omega^R(0) - \omega^L(0) + 2\omega^R(1) - 2\omega^L(1)}{3} \right) \left[ \lambda \min_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}} \{w_{\tilde{a}_{ij}}\} + (1-\lambda) \left( 1 - \max_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}} \{u_{\tilde{a}_{ij}}\} \right) \right] \dots$$

Then, it is claimed that optimal solution  $\{x_i, i=1, 2, \dots, m\}$  and optimal value of Problem 4.4.17 represents the optimal strategies and minimum expected gain of Player I respectively as well as optimal solution  $\{y_j, j=1, 2, \dots, n\}$  and optimal value of Problem 4.4.18 represents the optimal strategies and maximum expected loss of Player II respectively.

**Problem 4.4.17**

Maximize  $\{V_1 - A_1\}$

Subject to

Constraints of Problem 4.4.15.

**Problem 4.4.18**

Minimize  $\{V_2 - A_2\}$

Subject to

Constraints of Problem 4.4.16.

Seikh et al. [111] proposed a method to find the minimum expected gain of Player I, maximum expected loss of Player II and their corresponding optimal strategies for matrix games with triangular intuitionistic fuzzy payoffs. In this method, the authors have transformed Problem 4.3.1 and Problem 4.3.2 into Problem 4.4.19 and Problem 4.4.20 respectively.

**Problem 4.4.19**

Maximize  $\left\{ \frac{\nu_1 + \nu_2}{2} \right\}$

Subject to

$$\left( \frac{1}{2} \min_{1 \leq i \leq m} \{w_{\tilde{a}_{ij}}\} \sum_{i=1}^m \frac{a_{ij}^L(0) + 4a_{ij}(1) + a_{ij}^R(0)}{6w_{\tilde{a}_{ij}}} x_i - \frac{1}{2} \min_{1 \leq i \leq m} \{1 - u_{\tilde{a}_{ij}}\} \frac{a_{ij}^R(0) - a_{ij}^L(0)}{3(1 - u_{\tilde{a}_{ij}})} x_i + \right. \\ \left. \frac{1}{2} \min_{1 \leq i \leq m} \{1 - u_{\tilde{a}_{ij}}\} \sum_{i=1}^m \frac{a_{ij}^L(0) + 4a_{ij}(1) + a_{ij}^R(0)}{6(1 - u_{\tilde{a}_{ij}})} x_i - \frac{1}{2} \min_{1 \leq i \leq m} \{w_{\tilde{a}_{ij}}\} \sum_{i=1}^m \frac{a_{ij}^R(0) - a_{ij}^L(0)}{3w_{\tilde{a}_{ij}}} x_i \right) \geq \frac{\nu_1 + \nu_2}{2}, \quad j = 1, 2, \dots, n;$$

$$\nu_1 \leq \nu_2;$$

$$\sum_{i=1}^m x_i = 1; x_i \geq 0, i = 1, 2, \dots, m.$$

$$\text{where, } \nu_1 = \frac{\nu^L(0) + 4\nu(1) + \nu^R(0)}{6} w_{\tilde{\nu}} - \frac{\nu^R(0) - \nu^L(0)}{3} w_{\tilde{\nu}} \text{ and}$$

$$\nu_2 = \frac{\nu^L(0) + 4\nu(1) + \nu^R(0)}{6} (1 - u_{\tilde{\nu}}) - \frac{\nu^R(0) - \nu^L(0)}{3} (1 - u_{\tilde{\nu}}).$$

### Problem 4.4.20

$$\text{Minimize } \left\{ \frac{\omega_1 + \omega_2}{2} \right\}$$

Subject to

$$\left( \frac{1}{2} \min_{1 \leq j \leq n} \{w_{\tilde{a}_{ij}}\} \sum_{j=1}^n \frac{a_{ij}^L(0) + 4a_{ij}(1) + a_{ij}^R(0)}{6w_{\tilde{a}_{ij}}} y_j - \frac{1}{2} \min_{1 \leq j \leq n} \{1 - u_{\tilde{a}_{ij}}\} \frac{a_{ij}^R(0) - a_{ij}^L(0)}{3(1 - u_{\tilde{a}_{ij}})} y_j + \right. \\ \left. \frac{1}{2} \min_{1 \leq j \leq n} \{1 - u_{\tilde{a}_{ij}}\} \sum_{j=1}^n \frac{a_{ij}^L(0) + 4a_{ij}(1) + a_{ij}^R(0)}{6(1 - u_{\tilde{a}_{ij}})} y_j - \frac{1}{2} \min_{1 \leq j \leq n} \{w_{\tilde{a}_{ij}}\} \sum_{j=1}^n \frac{a_{ij}^R(0) - a_{ij}^L(0)}{3w_{\tilde{a}_{ij}}} y_j \right) \leq \frac{\omega_1 + \omega_2}{2}, \quad i = 1, 2, \dots, m;$$

$$\omega_1 \leq \omega_2;$$

$$\sum_{j=1}^n y_j = 1; y_j \geq 0, j = 1, 2, \dots, n.$$

$$\text{where, } \omega_1 = \frac{\omega^L(0) + 4\omega(1) + \omega^R(0)}{6} w_{\tilde{\omega}} - \frac{\omega^R(0) - \omega^L(0)}{3} w_{\tilde{\omega}} \text{ and}$$

$$\omega_2 = \frac{\omega^L(0) + 4\omega(1) + \omega^R(0)}{6} (1 - u_{\tilde{\omega}}) - \frac{\omega^R(0) - \omega^L(0)}{3} (1 - u_{\tilde{\omega}}).$$

Then, it is claimed that optimal solution  $\{x_i, i = 1, 2, \dots, m\}$  of Problem 4.4.19 represents the optimal strategies of Player I as well as optimal solution  $\{y_j, j = 1, 2, \dots, n\}$  of Problem 4.4.22 represents the optimal strategies of Player II and the value of game is

$$\left\langle \sum_{j=1}^n \sum_{i=1}^m x_i (a_{ij}^L(0), a_{ij}(1), a_{ij}^R(0)) y_j, \min_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}} \{w_{\tilde{a}_{ij}}\}, \max_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}} \{u_{\tilde{a}_{ij}}\} \right\rangle.$$

### 4.5 Flaws of the existing methods

In this section, flaws of the existing methods [78,95-97] are pointed out.

1. In the existing method [95], Problem 4.4.1 is solved to obtain the minimum expected gain by Player, is obtained by assuming that  $\tilde{a} \preceq \tilde{b}$  if  $S_\mu(\tilde{a}) \leq S_\mu(\tilde{b})$  and  $S_\nu(\tilde{a}) \leq S_\nu(\tilde{b})$ , which contradicts the claim of the authors [95, Definition 7] that  $\tilde{a} \preceq \tilde{b}$  if  $S_\mu(\tilde{a}) < S_\mu(\tilde{b})$  or if  $S_\mu(\tilde{a}) = S_\mu(\tilde{b})$  then  $S_\nu(\tilde{a}) < S_\nu(\tilde{b})$ . Similarly, the authors [95] have obtained the Problem 4.4.2

by assuming that  $\tilde{a} \succeq \tilde{b}$  if  $S_\mu(\tilde{a}) \geq S_\mu(\tilde{b})$  and  $S_\nu(\tilde{a}) \geq S_\nu(\tilde{b})$ , which contradicts the claim of the authors [95, Definition 7] that  $\tilde{a} \succeq \tilde{b}$  if  $S_\mu(\tilde{a}) > S_\mu(\tilde{b})$  or if  $S_\mu(\tilde{a}) = S_\mu(\tilde{b})$  then  $S_\nu(\tilde{a}) > S_\nu(\tilde{b})$

. where,  $S_\mu(\tilde{a}) = w_{\tilde{a}} \frac{a^L(0) + 2a(1) + a^R(0)}{4}$  and

$$S_\nu(\tilde{a}) = (1 - u_{\tilde{a}}) \frac{a^L(0) + 2a(1) + a^R(0)}{4}.$$

2. In the existing method [78], authors have assumed if  $\tilde{a} = \langle (a^L(0), a(1), a^R(0)), w_{\tilde{a}}, u_{\tilde{a}} \rangle$  and  $\tilde{b} = \langle (b^L(0), b(1), b^R(0)), w_{\tilde{b}}, u_{\tilde{b}} \rangle$  are two triangular vague sets (triangular intuitionistic fuzzy numbers) then (i)  $\tilde{a} \preceq \tilde{b}$  if  $S_\mu(\tilde{a}) \leq S_\mu(\tilde{b})$  and  $S_\nu(\tilde{a}) \leq S_\nu(\tilde{b})$ . (ii)  $\tilde{a} = \tilde{b}$  if  $S_\mu(\tilde{a}) = S_\mu(\tilde{b})$  and  $S_\nu(\tilde{a}) = S_\nu(\tilde{b})$ .

It is obvious from it that to find the minimum expected gain of Player I of those matrix games in which payoffs are represented by triangular intuitionistic fuzzy numbers i.e.,

$$\text{minimum} \left\{ \sum_{j=1}^n \sum_{i=1}^m x_i \langle (a_{ij}^L(0), a_{ij}(1), a_{ij}^R(0)), w_{\tilde{a}_{ij}}, u_{\tilde{a}_{ij}} \rangle y_j \mid \sum_{i=1}^m x_i = 1, \sum_{j=1}^n y_j = 1, x_i \in X, i = 1, 2, \dots, m, \right. \\ \left. y_j \in Y, j = 1, 2, \dots, n \right\},$$

there is need to find such  $y_j^* \in Y$  corresponding to which value of

$$\sum_{j=1}^n \sum_{i=1}^m x_i \langle (a_{ij}^L(0), a_{ij}(1), a_{ij}^R(0)), w_{\tilde{a}_{ij}}, u_{\tilde{a}_{ij}} \rangle y_j \text{ is minimum } \forall (x_1, x_2, \dots, x_m) \in X \text{ i.e., there is need}$$

to find such  $y_j^* \in Y$  corresponding to which value of

$$S_\mu \left( \sum_{j=1}^n \sum_{i=1}^m x_i \langle (a_{ij}^L(0), a_{ij}(1), a_{ij}^R(0)), w_{\tilde{a}_{ij}}, u_{\tilde{a}_{ij}} \rangle y_j \right) \text{ as well as}$$

$$S_\nu \left( \sum_{j=1}^n \sum_{i=1}^m x_i \langle (a_{ij}^L(0), a_{ij}(1), a_{ij}^R(0)), w_{\tilde{a}_{ij}}, u_{\tilde{a}_{ij}} \rangle y_j \right) \forall (x_1, x_2, \dots, x_m) \in X \text{ will be minimum i.e.,}$$

there is need to find the optimal solution  $\{y_j^*\}$  of Problem 4.5.1.

### Problem 4.5.1

$$\text{Minimize} \left\{ S_{\mu} \left( \sum_{j=1}^n \sum_{i=1}^m x_i \left\langle (a_{ij}^L(0), a_{ij}(1), a_{ij}^R(0)), w_{\tilde{a}_{ij}}, u_{\tilde{a}_{ij}} \right\rangle y_j \right) \right\}$$

$$\text{Minimize} \left\{ S_{\nu} \left( \sum_{j=1}^n \sum_{i=1}^m x_i \left\langle (a_{ij}^L(0), a_{ij}(1), a_{ij}^R(0)), w_{\tilde{a}_{ij}}, u_{\tilde{a}_{ij}} \right\rangle y_j \right) \right\}$$

Subject to

$$\sum_{i=1}^m x_i = 1; x_i \geq 0, i = 1, 2, \dots, m; \sum_{j=1}^n y_j = 1; y_j \geq 0, j = 1, 2, \dots, n.$$

Since, Problem 4.5.1 is a multi-objective problem. So, optimal solution of Problem 4.5.1 can be obtained by transforming into a single objective problem i.e., Problem 4.5.2 to find the optimal solution of Problem 4.5.1 is equivalent to find the optimal solution of Problem 4.5.2.

### Problem 4.5.2

$$\text{Minimize} \left\{ \lambda \left( S_{\mu} \left( \sum_{j=1}^n \sum_{i=1}^m x_i \left\langle (a_{ij}^L(0), a_{ij}(1), a_{ij}^R(0)), w_{\tilde{a}_{ij}}, u_{\tilde{a}_{ij}} \right\rangle y_j \right) \right) + \right. \\ \left. (1-\lambda) \left( S_{\nu} \left( \sum_{j=1}^n \sum_{i=1}^m x_i \left\langle (a_{ij}^L(0), a_{ij}(1), a_{ij}^R(0)), w_{\tilde{a}_{ij}}, u_{\tilde{a}_{ij}} \right\rangle y_j \right) \right) \right\}$$

Subject to

$$\sum_{i=1}^m x_i = 1; x_i \geq 0, i = 1, 2, \dots, m; \sum_{j=1}^n y_j = 1; y_j \geq 0, j = 1, 2, \dots, n.$$

Since, in Problem 4.5.2 only  $y_j$  are considered as decision variables so Problem 4.5.2 is a linear programming problem and hence, the optimal value of Problem 4.5.2 is equal to the optimal value of its corresponding dual problem i.e., Problem 4.5.3.

### Problem 4.5.3

$$\text{Maximize} \left\{ \lambda S_{\mu}(\tilde{v}) + (1-\lambda) S_{\nu}(\tilde{v}) \right\}$$

Subject to

$$\left( \begin{array}{l} \lambda \left( S_{\mu} \left( \sum_{i=1}^m x_i \left\langle \left( a_{ij}^L(0), a_{ij}(1), a_{ij}^R(0) \right), \min_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}} \{ w_{\tilde{a}_{ij}} \}, \max_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}} \{ u_{\tilde{a}_{ij}} \} \right\rangle \right) \right) + \\ (1-\lambda) \left( S_{\nu} \left( \sum_{i=1}^m x_i \left\langle \left( a_{ij}^L(0), a_{ij}(1), a_{ij}^R(0) \right), \min_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}} \{ w_{\tilde{a}_{ij}} \}, \max_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}} \{ u_{\tilde{a}_{ij}} \} \right\rangle \right) \right) \end{array} \right) \geq \lambda S_{\mu}(\tilde{v}) + (1-\lambda) S_{\nu}(\tilde{v}), j = 1, 2, \dots, n;$$

$$\sum_{i=1}^m x_i = 1; x_i \geq 0, i = 1, 2, \dots, m.$$

Since,  $\lambda x_1 + (1-\lambda)x_2 \geq \lambda y_1 + (1-\lambda)y_2 \not\Rightarrow x_1 \geq y_1$  and  $x_2 \geq y_2$ . So, Problem 4.5.3

cannot be transformed into Problem 4.5.4.

#### Problem 4.5.4

Maximize  $\{ \lambda S_{\mu}(\tilde{v}) + (1-\lambda) S_{\nu}(\tilde{v}) \}$

Subject to

$$S_{\mu} \left( \sum_{i=1}^m x_i \left\langle \left( a_{ij}^L(0), a_{ij}(1), a_{ij}^R(0) \right), \min_{1 \leq i \leq m} \{ w_{\tilde{a}_{ij}} \}, \max_{1 \leq i \leq m} \{ u_{\tilde{a}_{ij}} \} \right\rangle \right) \geq S_{\mu}(\tilde{v}), j = 1, 2, \dots, n;$$

$$S_{\nu} \left( \sum_{i=1}^m x_i \left\langle \left( a_{ij}^L(0), a_{ij}(1), a_{ij}^R(0) \right), \min_{1 \leq i \leq m} \{ w_{\tilde{a}_{ij}} \}, \max_{1 \leq i \leq m} \{ u_{\tilde{a}_{ij}} \} \right\rangle \right) \geq S_{\nu}(\tilde{v}), j = 1, 2, \dots, n;$$

$$\sum_{i=1}^m x_i = 1; x_i \geq 0, i = 1, 2, \dots, m.$$

While, in the existing method [78], authors have transformed Problem 4.5.3 into Problem 4.5.4, which is mathematically incorrect. Similarly, in the existing method [78], wrong mathematical assumption is considered to find the maximum expected loss of Player II.

3. In the existing method [94], Problem 4.4.11 is solved to obtain the minimum expected gain of Player I which is obtained by assuming that  $\tilde{a} \preceq \tilde{b}$  if  $V_{\lambda}(\tilde{a}) \leq V_{\lambda}(\tilde{b})$  and  $A_{\lambda}(\tilde{a}) \geq A_{\lambda}(\tilde{b})$ , which contradicts the claim of the authors [96, Definition 7] that  $\tilde{a} \preceq \tilde{b}$  if  $V_{\lambda}(\tilde{a}) < V_{\lambda}(\tilde{b})$  or if  $V_{\lambda}(\tilde{a}) = V_{\lambda}(\tilde{b})$  then  $A_{\lambda}(\tilde{a}) > A_{\lambda}(\tilde{b})$ . Similarly, the authors have obtained the Problem 4.4.15 by

assuming that if  $\tilde{a} \geq \tilde{b}$   $V_\lambda(\tilde{a}) \geq V_\lambda(\tilde{b})$  and  $A_\lambda(\tilde{a}) \leq A_\lambda(\tilde{b})$ . This contradicts the claim of the authors [96, Definition 7] that  $\tilde{a} \succeq \tilde{b}$  if  $V_\lambda(\tilde{a}) > V_\lambda(\tilde{b})$  or if  $V_\lambda(\tilde{a}) = V_\lambda(\tilde{b})$  then  $A_\lambda(\tilde{a}) < A_\lambda(\tilde{b})$

where,  $V_\lambda(\tilde{a}) = \frac{(\lambda w_{\tilde{a}}^2 + (1-\lambda)(1-u_{\tilde{a}})^2)(a^L(0) + 2a(1) + a^R(0))}{6}$  and

$$A_\lambda(\tilde{a}) = \frac{(\lambda w_{\tilde{a}}^2 + (1-\lambda)(1-u_{\tilde{a}})^2)(a^R(0) - a^L(0))}{6}.$$

4. In the existing method [97], authors have assumed that if  $\tilde{a} = \langle (a^L(0), a^L(1), a^R(1), a^R(0)), w_{\tilde{a}}, u_{\tilde{a}} \rangle$  and  $\tilde{b} = \langle (b^L(0), b^L(1), b^R(1), b^R(0)), w_{\tilde{b}}, u_{\tilde{b}} \rangle$  are two trapezoidal intuitionistic fuzzy numbers then  $\tilde{a} \preceq \tilde{b}$  if  $V_\lambda(\tilde{a}) \leq V_\lambda(\tilde{b})$  and  $A_\lambda(\tilde{a}) \geq A_\lambda(\tilde{b})$ , where,

$$V_\lambda(\tilde{a}) = \frac{(\lambda(1-u_{\tilde{a}}) + (1-\lambda)w_{\tilde{a}})(a^L(0) + 2a^L(1) + 2a^R(1) + a^R(0))}{6},$$

$$A_\lambda(\tilde{a}) = \frac{(\lambda w_{\tilde{a}} + (1-\lambda)(1-u_{\tilde{a}}))(a^R(0) - a^L(0) + 2(a^R(1) - a^L(1)))}{3}.$$

It is obvious from it that to find the minimum expected gain of Player I of those matrix games in which payoffs are represented by trapezoidal intuitionistic fuzzy numbers i.e.,

$$\text{minimum} \left\{ \sum_{j=1}^n \sum_{i=1}^m x_i \langle (a_{ij}^L(0), a_{ij}^L(1), a_{ij}^R(1), a_{ij}^R(0)), w_{\tilde{a}_{ij}}, u_{\tilde{a}_{ij}} \rangle y_j \mid \sum_{i=1}^n x_i = 1, \sum_{j=1}^m y_j = 1, x_i \geq 0, \right. \\ \left. i = 1, 2, \dots, m, y_j \geq 0, j = 1, 2, \dots, n \right\},$$

there is need to find such  $y_j^*$  corresponding to which  $\sum_{j=1}^n \sum_{i=1}^m x_i \langle (a_{ij}^L(0), a_{ij}^L(1), a_{ij}^R(1), a_{ij}^R(0)), w_{\tilde{a}_{ij}}, u_{\tilde{a}_{ij}} \rangle y_j$  is minimum  $\forall (x_1, x_2, \dots, x_m) \in X$  i.e., there is need to find such  $\{y_j^*\}$  corresponding to

which  $V_\lambda \left( \sum_{j=1}^n \sum_{i=1}^m x_i \langle (a_{ij}^L(0), a_{ij}^L(1), a_{ij}^R(1), a_{ij}^R(0)), w_{\tilde{a}_{ij}}, u_{\tilde{a}_{ij}} \rangle y_j \right)$  is minimum and value of

$$A_\lambda \left( \sum_{j=1}^n \sum_{i=1}^m x_i \langle (a_{ij}^L(0), a_{ij}^L(1), a_{ij}^R(1), a_{ij}^R(0)), w_{\tilde{a}_{ij}}, u_{\tilde{a}_{ij}} \rangle y_j \right) \text{ is maximum } \forall (x_1, x_2, \dots, x_m) \in X \text{ i.e.,}$$

to find the optimal solution  $\{y_j^*\}$  of Problem 4.5.5.

**Problem 4.5.5**

$$\text{Minimize} \left\{ V_\lambda \left( \sum_{j=1}^n \sum_{i=1}^m x_i \left\langle (a_{ij}^L(0), a_{ij}^L(1), a_{ij}^R(1), a_{ij}^R(0)), w_{\tilde{a}_{ij}}, u_{\tilde{a}_{ij}} \right\rangle y_j \right) \right\}$$

$$\text{Maximize} \left\{ A_\lambda \left( \sum_{j=1}^n \sum_{i=1}^m x_i \left\langle (a_{ij}^L(0), a_{ij}^L(1), a_{ij}^R(1), a_{ij}^R(0)), w_{\tilde{a}_{ij}}, u_{\tilde{a}_{ij}} \right\rangle y_j \right) \right\}$$

Subject to

$$\sum_{i=1}^m x_i = 1; x_i \geq 0, i = 1, 2, \dots, m; \sum_{j=1}^n y_j = 1; y_j \geq 0, j = 1, 2, \dots, n.$$

Furthermore, to find the optimal solution of Problem 4.5.5, it is equivalent to find the optimal solution of Problem 4.5.6

**Problem 4.5.6**

$$\text{Minimize} \left\{ \begin{array}{l} V_\lambda \left( \sum_{j=1}^n \sum_{i=1}^m x_i \left\langle (a_{ij}^L(0), a_{ij}^L(1), a_{ij}^R(1), a_{ij}^R(0)), w_{\tilde{a}_{ij}}, u_{\tilde{a}_{ij}} \right\rangle y_j \right) \\ -A_\lambda \left( \sum_{j=1}^n \sum_{i=1}^m x_i \left\langle (a_{ij}^L(0), a_{ij}^L(1), a_{ij}^R(1), a_{ij}^R(0)), w_{\tilde{a}_{ij}}, u_{\tilde{a}_{ij}} \right\rangle y_j \right) \end{array} \right\}$$

Subject to

$$\sum_{i=1}^m x_i = 1; x_i \geq 0, i = 1, 2, \dots, m; \sum_{j=1}^n y_j = 1; y_j \geq 0, j = 1, 2, \dots, n.$$

Since, in Problem 4.5.6, only  $y_j$  has been considered as decision variable. So, Problem 4.5.6 is a linear programming problem and hence, the optimal value of Problem 4.5.6 will be equal to optimal value of its dual problem i.e., Problem 4.5.7.

**Problem 4.5.7**

$$\text{Maximize} \{V_\lambda(\tilde{v}) - A_\lambda(\tilde{v})\}$$

Subject to

$$\left( V_{\lambda} \left( \sum_{i=1}^m x_i \left\langle \left( a_{ij}^L(0), a_{ij}^L(1), a_{ij}^R(1), a_{ij}^R(0) \right), \min_{1 \leq i \leq m} \{ w_{\tilde{a}_{ij}} \}, \max_{1 \leq i \leq m} \{ u_{\tilde{a}_{ij}} \} \right\rangle \right) - A_{\lambda} \left( \sum_{i=1}^m x_i \left\langle \left( a_{ij}^L(0), a_{ij}^L(1), a_{ij}^R(1), a_{ij}^R(0) \right), \min_{1 \leq i \leq m} \{ w_{\tilde{a}_{ij}} \}, \max_{1 \leq i \leq m} \{ u_{\tilde{a}_{ij}} \} \right\rangle \right) \right) \geq V_{\lambda}(\tilde{v}) - A_{\lambda}(\tilde{v}), j = 1, 2, \dots, n$$

$$\sum_{i=1}^m x_i = 1; x_i \geq 0, i = 1, 2, \dots, m.$$

Since,  $x_1 - x_2 \geq y_1 - y_2 \not\Rightarrow x_1 \geq y_1$  and  $x_2 \leq y_2$  so Problem 4.5.7 cannot be transformed into Problem 4.5.8.

#### Problem 4.5.8

$$\text{Maximize } \{V_{\lambda}(\tilde{v}) - A_{\lambda}(\tilde{v})\}$$

Subject to

$$V_{\lambda} \left( \sum_{i=1}^m x_i \left\langle \left( a_{ij}^L(0), a_{ij}^L(1), a_{ij}^R(1), a_{ij}^R(0) \right), \min_{1 \leq i \leq m} \{ w_{\tilde{a}_{ij}} \}, \max_{1 \leq i \leq m} \{ u_{\tilde{a}_{ij}} \} \right\rangle \right) \geq V_{\lambda}(\tilde{v}), j = 1, 2, \dots, n$$

$$A_{\lambda} \left( \sum_{i=1}^m x_i \left\langle \left( a_{ij}^L(0), a_{ij}^L(1), a_{ij}^R(1), a_{ij}^R(0) \right), \min_{1 \leq i \leq m} \{ w_{\tilde{a}_{ij}} \}, \max_{1 \leq i \leq m} \{ u_{\tilde{a}_{ij}} \} \right\rangle \right) \leq A_{\lambda}(\tilde{v}), j = 1, 2, \dots, n$$

$$\sum_{i=1}^m x_i = 1; x_i \geq 0, i = 1, 2, \dots, m.$$

While, in the existing method [97], authors have transformed Problem 4.5.7 into Problem 4.5.8, which is mathematically incorrect. Similarly, this wrong assumptions has been considered to obtain the maximum expected loss of Player II.

#### 4.6 Proposed Ambika methods

It is obvious from Section 4.5 that different methods for comparing triangular/trapezoidal intuitionistic fuzzy numbers are used in published papers [78,95-97]. In this section, new methods (named as Ambika methods), corresponding to different existing methods, are proposed for finding the minimum expected gain of Player I, maximum expected loss of Player II and their corresponding optimal strategies of matrix games with triangular/trapezoidal intuitionistic fuzzy payoffs.

#### 4.6.1 Ambika method-I

In the existing method [95], it is assumed that if  $\tilde{a} = \langle (a^L(0), a(1), a^R(0)), w_{\tilde{a}}, u_{\tilde{a}} \rangle$  and  $\tilde{b} = \langle (b^L(0), b(1), b^R(0)), w_{\tilde{b}}, u_{\tilde{b}} \rangle$  are two triangular intuitionistic fuzzy numbers then  $\tilde{a} \preceq \tilde{b}$  if  $S_{\mu}(\tilde{a}) \leq S_{\mu}(\tilde{b})$  and if  $S_{\mu}(\tilde{a}) = S_{\mu}(\tilde{b})$  then  $S_{\nu}(\tilde{a}) \leq S_{\nu}(\tilde{b})$  where,

$$S_{\mu}(\tilde{a}) = \frac{w_{\tilde{a}}(a^L(0) + 2a(1) + a^R(0))}{4} \text{ and } S_{\nu}(\tilde{a}) = \frac{(1 - u_{\tilde{a}})(a^L(0) + 2a(1) + a^R(0))}{4}.$$

In this section, a new method, based on this existing method, is proposed to find the minimum expected gain of Player I, maximum expected loss of Player II and their corresponding optimal strategies.

##### 4.6.1.1 Minimum expected gain of Player I

Using the comparing method,  $\tilde{a} \preceq \tilde{b}$  if  $S_{\mu}(\tilde{a}) \leq S_{\mu}(\tilde{b})$  and if  $S_{\mu}(\tilde{a}) = S_{\mu}(\tilde{b})$  then  $S_{\nu}(\tilde{a}) \leq S_{\nu}(\tilde{b})$ , the minimum expected gain of Player I and the corresponding optimal strategies can be obtained as follows:

**Step 1:** According to comparing method, to find the optimal solution  $\{y_1^*, y_2^*, \dots, y_n^*\} \in Y$

such that value of  $\sum_{j=1}^n \sum_{i=1}^m x_i \langle (a_{ij}^L(0), a_{ij}(1), a_{ij}^R(0)), w_{\tilde{a}_{ij}}, u_{\tilde{a}_{ij}} \rangle y_j$  is minimum for all

$(x_1, x_2, \dots, x_m) \in X$ , firstly it is equivalent to find  $\{y_1^*, y_2^*, \dots, y_n^*\} \in Y$  such that value of

$S_{\mu} \left( \sum_{j=1}^n \sum_{i=1}^m x_i \langle (a_{ij}^L(0), a_{ij}(1), a_{ij}^R(0)), w_{\tilde{a}_{ij}}, u_{\tilde{a}_{ij}} \rangle y_j \right)$ , is minimum for all  $(x_1, x_2, \dots, x_m) \in X$  i.e.,

find optimal solution  $\{y_j^*, j = 1, 2, \dots, n\}$  of Problem 4.6.1.

##### Problem 4.6.1

Minimize  $\left\{ S_{\mu} \left( \sum_{j=1}^n \sum_{i=1}^m x_i \langle (a_{ij}^L(0), a_{ij}(1), a_{ij}^R(0)), w_{\tilde{a}_{ij}}, u_{\tilde{a}_{ij}} \rangle y_j \right) \right\}$

Subject to

$$\sum_{i=1}^m x_i = 1; x_i \geq 0, i = 1, 2, \dots, m; \sum_{j=1}^n y_j = 1; y_j \geq 0, j = 1, 2, \dots, n.$$

**Step 2:** Using the property,  $\lambda \langle (a_i^L(0), a_i(1), a_i^R(0)), w_{\tilde{a}_i}, u_{\tilde{a}_i} \rangle =$

$$\langle (\lambda a_i^L(0), \lambda a_i(1), \lambda a_i^R(0)), w_{\tilde{a}_i}, u_{\tilde{a}_i} \rangle \text{ and } \sum_{i=1}^n \left( \langle (a_i^L(0), a_i(1), a_i^R(0)), w_{\tilde{a}_i}, u_{\tilde{a}_i} \rangle \right) =$$

$$\left( \left\langle \left( \sum_{i=1}^n a_i^L(0), \sum_{i=1}^n a_i(1), \sum_{i=1}^n a_i^R(0) \right), \min_{1 \leq i \leq n} \{w_{\tilde{a}_i}\}, \max_{1 \leq i \leq n} \{u_{\tilde{a}_i}\} \right\rangle \right), \text{ Problem 4.6.1 can be transformed}$$

into Problem 4.6.2.

### Problem 4.6.2

$$\text{Minimize } \left\{ S_{\mu} \left( \left\langle \left( \sum_{j=1}^n \sum_{i=1}^m x_i a_{ij}^L(0) y_j, \sum_{j=1}^n \sum_{i=1}^m x_i a_{ij}(1) y_j, \sum_{j=1}^n \sum_{i=1}^m x_i a_{ij}^R(0) y_j \right), \min_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}} \{w_{\tilde{a}_{ij}}\}, \max_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}} \{u_{\tilde{a}_{ij}}\} \right\rangle \right) \right\}$$

Subject to

$$\sum_{i=1}^m x_i = 1; x_i \geq 0, i = 1, 2, \dots, m; \sum_{j=1}^n y_j = 1; y_j \geq 0, j = 1, 2, \dots, n.$$

**Step 3:** Since, in Problem 4.6.2, only  $y_j$  have been considered as decision variables. So,

Problem 4.6.2 is a linear programming problem and hence, the optimal value of Problem 4.6.2

will be equal to optimal value of its corresponding dual problem i.e., Problem 4.6.3.

### Problem 4.6.3

$$\text{Maximize } \{v\}$$

Subject to

$$S_{\mu} \left( \sum_{i=1}^m \left\langle (a_{ij}^L(0), a_{ij}(1), a_{ij}^R(0)), \min_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}} \{w_{\tilde{a}_{ij}}\}, \max_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}} \{u_{\tilde{a}_{ij}}\} \right\rangle x_i \right) \geq v, j = 1, 2, \dots, n;$$

$$\sum_{i=1}^m x_i = 1; x_i \geq 0, i = 1, 2, \dots, m.$$

**Step 4:** Find the optimal solution of Problem 4.6.3.

**Step 5:** Substitute the optimal solution  $\{x_i^*, i = 1, 2, \dots, m\}$  of Problem 4.6.3 in Problem 4.6.1 and then find the optimal solution  $\{y_j^*, j = 1, 2, \dots, n\}$  of Problem 4.6.1.

**Case 1:** If there exist a unique optimal solution  $\{y_j^*, j = 1, 2, \dots, n\}$  of Problem 4.6.1 then the minimum expected gain of Player I is  $\sum_{j=1}^n \sum_{i=1}^m x_i^* \langle (a_{ij}^L(0), a_{ij}(1), a_{ij}^R(0)), w_{\tilde{a}_{ij}}, u_{\tilde{a}_{ij}} \rangle y_j^*$  and the corresponding optimal strategy for Player I will be  $\{x_i^*, i = 1, 2, \dots, m\}$ , which is optimal solution of Problem 4.6.2.

**Case 2:** If there exist more than one basic optimal solution  $\{y_{j1}^*, j = 1, 2, \dots, n\}$ ,  $\{y_{j2}^*, j = 1, 2, \dots, n\}, \dots, \{y_{jp}^*, j = 1, 2, \dots, n\}$  of Problem 4.6.1 then

$$\text{find minimum} \left\{ S_\nu \left( \sum_{j=1}^n \sum_{i=1}^m x_i^* \langle (a_{ij}^L(0), a_{ij}(1), a_{ij}^R(0)), w_{\tilde{a}_{ij}}, u_{\tilde{a}_{ij}} \rangle y_{js}^* \right), s = 1, 2, \dots, p \right\}.$$

If  $S_\nu \left( \sum_{j=1}^n \sum_{i=1}^m x_i^* \langle (a_{ij}^L(0), a_{ij}(1), a_{ij}^R(0)), w_{\tilde{a}_{ij}}, u_{\tilde{a}_{ij}} \rangle y_{jr}^* \right)$  represents the minimum value then

minimum expected gain of Player I is  $\sum_{j=1}^n \sum_{i=1}^m x_i^* \langle (a_{ij}^L(0), a_{ij}(1), a_{ij}^R(0)), w_{\tilde{a}_{ij}}, u_{\tilde{a}_{ij}} \rangle y_{jr}^*$  and the corresponding optimal strategy for Player I will be  $\{x_i^*, i = 1, 2, \dots, m\}$ , which is optimal solution of Problem 4.6.2.

#### 4.6.1.2 Maximum expected loss of Player II

Using the comparing method,  $\tilde{a} \succeq \tilde{b}$  if  $S_\mu(\tilde{a}) > S_\mu(\tilde{b})$  and if  $S_\mu(\tilde{a}) = S_\mu(\tilde{b})$  then

$S_\nu(\tilde{a}) \geq S_\nu(\tilde{b})$ , the maximum expected loss of Player II and the corresponding optimal strategies can be obtained as follows:

**Step 1:** According to comparing method, to find the optimal solution  $\{x_i^*, i = 1, 2, \dots, m\}$  such

that value of  $\sum_{j=1}^n \sum_{i=1}^m x_i \langle (a_{ij}^L(0), a_{ij}(1), a_{ij}^R(0)), w_{\tilde{a}_{ij}}, u_{\tilde{a}_{ij}} \rangle y_j$  is maximum, it is equivalent to find

$(x_1^*, x_2^*, \dots, x_m^*) \in X$  such that value of  $S_\mu \left( \sum_{j=1}^n \sum_{i=1}^m x_i \langle (a_{ij}^L(0), a_{ij}(1), a_{ij}^R(0)), w_{\tilde{a}_{ij}}, u_{\tilde{a}_{ij}} \rangle y_j \right)$ , is

maximum  $\forall (y_1, y_2, \dots, y_n) \in Y$  i.e., find optimal solution  $\{x_i^*, i = 1, 2, \dots, m\}$  of Problem 4.6.4.

#### Problem 4.6.4

$$\text{Maximize} \left\{ S_\mu \left( \sum_{j=1}^n \sum_{i=1}^m x_i \langle (a_{ij}^L(0), a_{ij}(1), a_{ij}^R(0)), w_{\tilde{a}_{ij}}, u_{\tilde{a}_{ij}} \rangle y_j \right) \right\}$$

Subject to

$$\sum_{i=1}^m x_i = 1; x_i \geq 0, i = 1, 2, \dots, m; \sum_{j=1}^n y_j = 1; y_j \geq 0, j = 1, 2, \dots, n.$$

**Step 2:** Using the property  $\lambda \langle (a_i^L(0), a_i(1), a_i^R(0)), w_{\tilde{a}_i}, u_{\tilde{a}_i} \rangle =$

$$\langle (\lambda a_i^L(0), \lambda a_i(1), \lambda a_i^R(0)), w_{\tilde{a}_i}, u_{\tilde{a}_i} \rangle, \lambda \geq 0 \text{ and } \sum_{i=1}^n \left( \langle (a_i^L(0), a_i(1), a_i^R(0)), w_{\tilde{a}_i}, u_{\tilde{a}_i} \rangle \right) =$$

$$\left( \left\langle \left( \sum_{i=1}^n a_i^L(0), \sum_{i=1}^n a_i(1), \sum_{i=1}^n a_i^R(0) \right), \min_{1 \leq i \leq n} \{w_{\tilde{a}_i}\}, \max_{1 \leq i \leq n} \{u_{\tilde{a}_i}\} \right\rangle \right), \text{ Problem 4.6.4 can be transformed}$$

into Problem 4.6.5.

#### Problem 4.6.5

Maximize

$$\left\{ S_\mu \left( \left\langle \left( \sum_{j=1}^n \sum_{i=1}^m x_i a_{ij}^L(0) y_j, \sum_{j=1}^n \sum_{i=1}^m x_i a_{ij}(1) y_j, \sum_{j=1}^n \sum_{i=1}^m x_i a_{ij}^R(0) y_j \right), \min_{1 \leq i \leq m} \{w_{\tilde{a}_{ij}}\}, \max_{1 \leq i \leq m} \{u_{\tilde{a}_{ij}}\} \right\rangle \right) \right\}$$

Subject to

$$\sum_{i=1}^m x_i = 1; x_i \geq 0, i = 1, 2, \dots, m; \sum_{j=1}^n y_j = 1; y_j \geq 0, j = 1, 2, \dots, n.$$

**Step 3:** Since, in Problem 4.6.5, only  $x_i, i=1,2,\dots,m$  have been considered as decision variables. So, Problem 4.6.5 is a linear programming problem and hence, the optimal value of Problem 4.6.5 will be equal to optimal value of its corresponding dual problem i.e., Problem 4.6.6.

**Problem 4.6.6**

Minimize  $\{\omega\}$

Subject to

$$S_{\mu} \left( \sum_{j=1}^n \left\langle \left( a_{ij}^L(0), a_{ij}(1), a_{ij}^R(0) \right), \min_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}} \{w_{\tilde{a}_{ij}}\}, \max_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}} \{u_{\tilde{a}_{ij}}\} \right\rangle y_j \right) \leq \omega, i=1,2,\dots,m;$$

$$\sum_{j=1}^n y_j = 1; y_j \geq 0, j=1,2,\dots,n.$$

**Step 4:** Find the optimal solution of Problem 4.6.6.

**Step 5:** Substitute the optimal solution  $\{y_j^*, j=1,2,\dots,n\}$  of Problem 4.6.6 in Problem 4.6.5 and then find the optimal solution  $\{x_i^*, i=1,2,\dots,m\}$  of Problem 4.6.5.

**Case 1:** If there exist a unique optimal solution  $\{x_i^*, i=1,2,\dots,m\}$  of Problem 4.6.5 then the

maximum expected loss of Player II is  $\sum_{j=1}^n \sum_{i=1}^m x_i^* \left\langle \left( a_{ij}^L(0), a_{ij}(1), a_{ij}^R(0) \right), w_{\tilde{a}_{ij}}, u_{\tilde{a}_{ij}} \right\rangle y_j^*$  and the

corresponding optimal strategy for Player II will be  $\{y_j^*, j=1,2,\dots,n\}$ , which is optimal solution of Problem 4.6.6.

**Case 2:** If there exist more than one basic optimal solution  $\{x_{i_1}^*, i=1,2,\dots,m\}$ ,

$\{x_{i_2}^*, i=1,2,\dots,m\}, \dots, \{x_{i_q}^*, i=1,2,\dots,m\}$  of Problem 4.6.5 then find maximum

$$\left\{ S_{\nu} \left( \sum_{j=1}^n \sum_{i=1}^m x_{is}^* \left\langle \left( a_{ij}^L(0), a_{ij}(1), a_{ij}^R(0) \right), w_{\tilde{a}_{ij}}, u_{\tilde{a}_{ij}} \right\rangle y_j^* \right), s=1,2,\dots,q \right\}.$$

If  $S_\nu \left( \sum_{j=1}^n \sum_{i=1}^m x_{it}^* \left\langle (a_{ij}^L(0), a_{ij}(1), a_{ij}^R(0)), w_{\tilde{a}_{ij}}, u_{\tilde{a}_{ij}} \right\rangle y_j^* \right)$  represents the maximum value then

maximum expected loss of Player II is  $\sum_{j=1}^n \sum_{i=1}^m x_{it}^* \left\langle (a_{ij}^L(0), a_{ij}(1), a_{ij}^R(0)), w_{\tilde{a}_{ij}}, u_{\tilde{a}_{ij}} \right\rangle y_j^*$  and the

corresponding optimal strategy for Player II will be  $\{y_j^*, j=1, 2, \dots, n\}$ , which is optimal solution of Problem 4.6.6.

#### 4.6.2 Ambika method-II

In the existing method [78], authors have assumed if  $\tilde{a} = \left\langle (a^L(0), a(1), a^R(0)), w_{\tilde{a}}, u_{\tilde{a}} \right\rangle$

and  $\tilde{b} = \left\langle (b^L(0), b(1), b^R(0)), w_{\tilde{b}}, u_{\tilde{b}} \right\rangle$  are two triangular intuitionistic fuzzy numbers then

$\tilde{a} \preceq \tilde{b}$  if  $S_\mu(\tilde{a}) \leq S_\mu(\tilde{b})$  and  $S_\nu(\tilde{a}) \leq S_\nu(\tilde{b})$ , where,  $S_\mu(\tilde{a}) = \frac{w_{\tilde{a}}(a^L(0) + 2a(1) + a^R(0))}{4}$  and

$$S_\nu(\tilde{a}) = \frac{(1 - u_{\tilde{a}})(a^L(0) + 2a(1) + a^R(0))}{4}.$$

In this section, a new method, based on this existing method [78], is proposed to find the minimum expected gain of Player I, maximum expected loss of Player II and their corresponding optimal strategies.

##### 4.6.2.1 Minimum expected gain of Player I

Using the comparing method,  $\tilde{a} \preceq \tilde{b}$  if  $S_\mu(\tilde{a}) \leq S_\mu(\tilde{b})$  and  $S_\nu(\tilde{a}) \leq S_\nu(\tilde{b})$ , the minimum expected gain of Player I and the corresponding optimal strategies can be obtained as follows:

**Step 1:** According to comparing method, to find the optimal solution  $\{y_1^*, y_2^*, \dots, y_n^*\}$  such that

value of  $\sum_{j=1}^n \sum_{i=1}^m \left\langle (a_{ij}^L(0)x_i y_j, a_{ij}^L(1)x_i y_j, a_{ij}^R(1)x_i y_j, a_{ij}^R(0)x_i y_j), w_{\tilde{a}_{ij}}, u_{\tilde{a}_{ij}} \right\rangle$  is minimum for all

$(x_1, x_2, \dots, x_m) \in X$  is equivalent to find  $\{y_1^*, y_2^*, \dots, y_n^*\}$  such that values of

$$S_\mu \left( \sum_{j=1}^n \sum_{i=1}^m x_i \left\langle (a_{ij}^L(0), a_{ij}(1), a_{ij}^R(0)), w_{\tilde{a}_{ij}}, u_{\tilde{a}_{ij}} \right\rangle y_j \right) \text{ and}$$

$S_\nu \left( \sum_{j=1}^n \sum_{i=1}^m x_i \left\langle (a_{ij}^L(0), a_{ij}(1), a_{ij}^R(0)), w_{\tilde{a}_{ij}}, u_{\tilde{a}_{ij}} \right\rangle y_j \right)$  is minimum for all  $(x_1, x_2, \dots, x_m) \in X$  or if it

is not possible to find  $\{y_1^*, y_2^*, \dots, y_n^*\}$  for which the value of

$$S_\mu \left( \sum_{j=1}^n \sum_{i=1}^m x_i \left\langle (a_{ij}^L(0), a_{ij}(1), a_{ij}^R(0)), w_{\tilde{a}_{ij}}, u_{\tilde{a}_{ij}} \right\rangle y_j \right) \text{ and}$$

$$S_\nu \left( \sum_{j=1}^n \sum_{i=1}^m x_i \left\langle (a_{ij}^L(0), a_{ij}(1), a_{ij}^R(0)), w_{\tilde{a}_{ij}}, u_{\tilde{a}_{ij}} \right\rangle y_j \right) \text{ is minimum then find such } \{y_{j1}^*, j=1, 2, \dots, n\}$$

for which the value of  $S_\mu \left( \sum_{j=1}^n \sum_{i=1}^m x_{i1} \left\langle (a_{ij}^L(0), a_{ij}(1), a_{ij}^R(0)), w_{\tilde{a}_{ij}}, u_{\tilde{a}_{ij}} \right\rangle y_{j1} \right)$  is minimum but value

of  $S_\nu \left( \sum_{j=1}^n \sum_{i=1}^m x_{i1} \left\langle (a_{ij}^L(0), a_{ij}(1), a_{ij}^R(0)), w_{\tilde{a}_{ij}}, u_{\tilde{a}_{ij}} \right\rangle y_{j1} \right)$  is not minimum for all

$(x_{11}, x_{21}, \dots, x_{m1}) \in X$  and find such  $\{y_{j2}^*, j=1, 2, \dots, n\}$  for which the value of

$$S_\nu \left( \sum_{j=1}^n \sum_{i=1}^m x_{i2} \left\langle (a_{ij}^L(0), a_{ij}(1), a_{ij}^R(0)), w_{\tilde{a}_{ij}}, u_{\tilde{a}_{ij}} \right\rangle y_{j2} \right) \text{ is minimum but value of}$$

$$S_\mu \left( \sum_{j=1}^n \sum_{i=1}^m x_{i2} \left\langle (a_{ij}^L(0), a_{ij}(1), a_{ij}^R(0)), w_{\tilde{a}_{ij}}, u_{\tilde{a}_{ij}} \right\rangle y_{j2} \right) \text{ is not minimum for all } (x_{12}, x_{22}, \dots, x_{m2}) \in X$$

i.e., find optimal solution  $\{y_{j1}^*, j=1, 2, \dots, n\}$  and  $\{y_{j2}^*, j=1, 2, \dots, n\}$  of Problem 4.6.7 and

Problem 4.6.8 respectively.

#### Problem 4.6.7

$$\text{Minimize } \left\{ S_\mu \left( \sum_{j=1}^n \sum_{i=1}^m x_{i1} \left\langle (a_{ij}^L(0), a_{ij}(1), a_{ij}^R(0)), w_{\tilde{a}_{ij}}, u_{\tilde{a}_{ij}} \right\rangle y_{j1} \right) \right\}$$

Subject to

$$\sum_{i=1}^m x_{i1} = 1; x_{i1} \geq 0, i=1, 2, \dots, m; \sum_{j=1}^n y_{j1} = 1; y_{j1} \geq 0, j=1, 2, \dots, n.$$

#### Problem 4.6.8

$$\text{Minimize} \left\{ S_\nu \left( \sum_{j=1}^n \sum_{i=1}^m x_{i2} \left\langle \left( a_{ij}^L(0), a_{ij}(1), a_{ij}^R(0) \right), w_{\tilde{a}_{ij}}, u_{\tilde{a}_{ij}} \right\rangle y_{j2} \right) \right\}$$

Subject to

$$\sum_{i=1}^m x_{i2} = 1; x_{i2} \geq 0, i = 1, 2, \dots, m; \sum_{j=1}^n y_{j2} = 1; y_{j2} \geq 0, j = 1, 2, \dots, n.$$

**Step 2:** Using the property  $\lambda \left\langle \left( a_i^L(0), a_i(1), a_i^R(0) \right), w_{\tilde{a}_i}, u_{\tilde{a}_i} \right\rangle =$

$$\left\langle \left( \lambda a_i^L(0), \lambda a_i(1), \lambda a_i^R(0) \right), w_{\tilde{a}_i}, u_{\tilde{a}_i} \right\rangle, \lambda \geq 0 \text{ and } \sum_{i=1}^n \left( \left\langle \left( a_i^L(0), a_i(1), a_i^R(0) \right), w_{\tilde{a}_i}, u_{\tilde{a}_i} \right\rangle \right) =$$

$$\left( \left\langle \left( \sum_{i=1}^n a_i^L(0), \sum_{i=1}^n a_i(1), \sum_{i=1}^n a_i^R(0) \right), \min_{1 \leq i \leq n} \{ w_{\tilde{a}_i} \}, \max_{1 \leq i \leq n} \{ u_{\tilde{a}_i} \} \right\rangle \right), \text{ Problem 4.6.7 and Problem 4.6.8 can}$$

be transformed into Problem 4.6.9 and Problem 4.6.10 respectively.

#### Problem 4.6.9

Minimize

$$\left\{ S_\mu \left( \left\langle \left( \sum_{j=1}^n \sum_{i=1}^m x_{i1} a_{ij}^L(0) y_{j1}, \sum_{j=1}^n \sum_{i=1}^m x_{i1} a_{ij}(1) y_{j1}, \sum_{j=1}^n \sum_{i=1}^m x_{i1} a_{ij}^R(0) y_{j1} \right), \min_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}} \{ w_{\tilde{a}_{ij}} \}, \max_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}} \{ u_{\tilde{a}_{ij}} \} \right\rangle \right) \right\}$$

Subject to

$$\sum_{i=1}^m x_{i1} = 1; x_{i1} \geq 0, i = 1, 2, \dots, m; \sum_{j=1}^n y_{j1} = 1; y_{j1} \geq 0, j = 1, 2, \dots, n.$$

#### Problem 4.6.10

Minimize

$$\left\{ S_\nu \left( \left\langle \left( \sum_{j=1}^n \sum_{i=1}^m x_{i2} a_{ij}^L(0) y_{j2}, \sum_{j=1}^n \sum_{i=1}^m x_{i2} a_{ij}(1) y_{j2}, \sum_{j=1}^n \sum_{i=1}^m x_{i2} a_{ij}^R(0) y_{j2} \right), \min_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}} \{ w_{\tilde{a}_{ij}} \}, \max_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}} \{ u_{\tilde{a}_{ij}} \} \right\rangle \right) \right\}$$

Subject to

$$\sum_{i=1}^m x_{i2} = 1; x_{i2} \geq 0, i = 1, 2, \dots, m; \sum_{j=1}^n y_{j2} = 1; y_{j2} \geq 0, j = 1, 2, \dots, n.$$

**Step 3:** Since, in Problem 4.6.9 and Problem 4.6.10, only  $y_{j1}$  and  $y_{j2}$  have been considered as decision variables. So, Problem 4.6.9 and Problem 4.6.10 are linear programming problems and hence, the optimal value of Problem 4.6.9 and Problem 4.6.10 will be equal to optimal value of its corresponding dual problem i.e., Problem 4.6.11 and Problem 4.6.12 respectively.

**Problem 4.6.11**

Maximize  $\{v_1\}$

Subject to

$$S_{\mu} \left( \sum_{i=1}^m \left\langle \left( a_{ij}^L(0), a_{ij}(1), a_{ij}^R(0) \right), \min_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}} \{w_{\tilde{a}_{ij}}\}, \max_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}} \{u_{\tilde{a}_{ij}}\} \right\rangle x_{i1} \right) \geq v_1, j = 1, 2, \dots, n;$$

$$\sum_{i=1}^m x_{i1} = 1; x_{i1} \geq 0, i = 1, 2, \dots, m.$$

**Problem 4.6.12**

Maximize  $\{v_2\}$

Subject to

$$S_{\nu} \left( \sum_{i=1}^m \left\langle \left( a_{ij}^L(0), a_{ij}(1), a_{ij}^R(0) \right), \min_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}} \{w_{\tilde{a}_{ij}}\}, \max_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}} \{u_{\tilde{a}_{ij}}\} \right\rangle x_{i2} \right) \geq v_2, j = 1, 2, \dots, n;$$

$$\sum_{i=1}^m x_{i2} = 1; x_{i2} \geq 0, i = 1, 2, \dots, m.$$

**Step 4:** Find the optimal solution  $\{x_{i1}^*\}$  and  $\{x_{i2}^*\}$  of Problem 4.6.11 and Problem 4.6.12 respectively.

**Step 5:** Substitute the optimal solution  $\{x_{i1}^*, i = 1, 2, \dots, m\}$  and  $\{x_{i2}^*, i = 1, 2, \dots, m\}$  of Problem 4.6.11 and Problem 4.6.12 in Problem 4.6.9 and Problem 4.6.10 respectively and then find all the alternative basic optimal solutions  $\{y_{j1}^{1*}, j = 1, 2, \dots, n\}, \{y_{j1}^{2*}, j = 1, 2, \dots, n\}, \dots,$

$\{y_{j1}^{r*}, j=1, 2, \dots, n\}$  and  $\{y_{j2}^{1*}, j=1, 2, \dots, n\}, \{y_{j2}^{2*}, j=1, 2, \dots, n\}, \dots, \{y_{j2}^{s*}, j=1, 2, \dots, n\}$  of

Problem 4.6.9 and Problem 4.6.10 respectively.

**Step 6:** Find minimum 
$$\left\{ \begin{array}{l} \sum_{j=1}^n \sum_{i=1}^m x_{i1}^* \left\langle (a_{ij}^L(0), a_{ij}(1), a_{ij}^R(0)), w_{\tilde{a}_{ij}}, u_{\tilde{a}_{ij}} \right\rangle y_{j1}^{h*}, h=1, 2, \dots, r, \\ \sum_{j=1}^n \sum_{i=1}^m x_{i2}^* \left\langle (a_{ij}^L(0), a_{ij}(1), a_{ij}^R(0)), w_{\tilde{a}_{ij}}, u_{\tilde{a}_{ij}} \right\rangle y_{j2}^{k*}, k=1, 2, \dots, s \end{array} \right\}.$$

**Step 7:** All the triangular intuitionistic fuzzy numbers which will be minimum, will represent the minimum expected gain of Player I. Also, for all such minimum which will be obtained

corresponding to  $\sum_{j=1}^n \sum_{i=1}^m x_{i1}^* \left\langle (a_{ij}^L(0), a_{ij}(1), a_{ij}^R(0)), w_{\tilde{a}_{ij}}, u_{\tilde{a}_{ij}} \right\rangle y_{j1}^{h*}, h=1, 2, \dots, r$  the optimal strategy

will be  $\{x_{i1}^*, i=1, 2, \dots, m\}$ , which is optimal solution of Problem 4.6.11 and for all such

minimum which will be obtained corresponding to

$\sum_{j=1}^n \sum_{i=1}^m x_{i2}^* \left\langle (a_{ij}^L(0), a_{ij}(1), a_{ij}^R(0)), w_{\tilde{a}_{ij}}, u_{\tilde{a}_{ij}} \right\rangle y_{j2}^{k*}, k=1, 2, \dots, s$  the optimal strategy will be

$\{x_{i2}^*, i=1, 2, \dots, m\}$ , which is optimal solution of Problem 4.6.12.

#### 4.6.2.2 Maximum expected loss of Player II

Using the comparing method,  $\tilde{a} \succeq \tilde{b}$  if  $S_\mu(\tilde{a}) \geq S_\mu(\tilde{b})$  and  $S_\nu(\tilde{a}) \geq S_\nu(\tilde{b})$ , the maximum expected loss of Player II and the corresponding optimal strategies can be obtained as follows:

**Step 1:** According to comparing method, to find the optimal solution  $\{x_1^*, x_2^*, \dots, x_m^*\}$  of such

that value of  $\sum_{j=1}^n \sum_{i=1}^m \left\langle (a_{ij}^L(0)x_i y_j, a_{ij}^L(1)x_i y_j, a_{ij}^R(1)x_i y_j, a_{ij}^R(0)x_i y_j), w_{\tilde{a}_{ij}}, u_{\tilde{a}_{ij}} \right\rangle$  is maximum for all

$(y_1, y_2, \dots, y_n) \in Y$  is equivalent to find  $\{x_1^*, x_2^*, \dots, x_m^*\}$  such that values of

$$S_\mu \left( \sum_{j=1}^n \sum_{i=1}^m x_i \left\langle (a_{ij}^L(0), a_{ij}(1), a_{ij}^R(0)), w_{\tilde{a}_{ij}}, u_{\tilde{a}_{ij}} \right\rangle y_j \right) \text{ and}$$

$S_\nu \left( \sum_{j=1}^n \sum_{i=1}^m x_i \left\langle (a_{ij}^L(0), a_{ij}(1), a_{ij}^R(0)), w_{\tilde{a}_{ij}}, u_{\tilde{a}_{ij}} \right\rangle y_j \right)$  is maximum for all  $(y_1, y_2, \dots, y_n) \in Y$  or if it is

not possible to find  $\{x_1^*, x_2^*, \dots, x_m^*\}$  for which the value of

$$S_\mu \left( \sum_{j=1}^n \sum_{i=1}^m x_i \left\langle (a_{ij}^L(0), a_{ij}(1), a_{ij}^R(0)), w_{\tilde{a}_{ij}}, u_{\tilde{a}_{ij}} \right\rangle y_j \right) \text{ and}$$

$$S_\nu \left( \sum_{j=1}^n \sum_{i=1}^m x_i \left\langle (a_{ij}^L(0), a_{ij}(1), a_{ij}^R(0)), w_{\tilde{a}_{ij}}, u_{\tilde{a}_{ij}} \right\rangle y_j \right) \text{ is maximum then find such } \{x_{i1}^*, i=1, 2, \dots, m\}$$

for which the value of  $S_\mu \left( \sum_{j=1}^n \sum_{i=1}^m x_{i1} \left\langle (a_{ij}^L(0), a_{ij}(1), a_{ij}^R(0)), w_{\tilde{a}_{ij}}, u_{\tilde{a}_{ij}} \right\rangle y_{j1} \right)$  is maximum but value

of  $S_\nu \left( \sum_{j=1}^n \sum_{i=1}^m x_{i1} \left\langle (a_{ij}^L(0), a_{ij}(1), a_{ij}^R(0)), w_{\tilde{a}_{ij}}, u_{\tilde{a}_{ij}} \right\rangle y_{j1} \right)$  is not maximum for all

$(y_{11}, y_{21}, \dots, y_{n1}) \in Y$  and find such  $\{x_{i2}^*, i=1, 2, \dots, m\}$  for which the value of

$$S_\nu \left( \sum_{j=1}^n \sum_{i=1}^m x_{i2} \left\langle (a_{ij}^L(0), a_{ij}(1), a_{ij}^R(0)), w_{\tilde{a}_{ij}}, u_{\tilde{a}_{ij}} \right\rangle y_{j2} \right) \text{ is maximum but value of}$$

$$S_\mu \left( \sum_{j=1}^n \sum_{i=1}^m x_{i2} \left\langle (a_{ij}^L(0), a_{ij}(1), a_{ij}^R(0)), w_{\tilde{a}_{ij}}, u_{\tilde{a}_{ij}} \right\rangle y_{j2} \right) \text{ is not maximum for all } (y_{12}, y_{22}, \dots, y_{n2}) \in Y$$

i.e., find optimal solution  $\{x_{i1}^*, i=1, 2, \dots, m\}$  and  $\{x_{i2}^*, i=1, 2, \dots, m\}$  of Problem 4.6.13 and

Problem 4.6.14 respectively.

#### Problem 4.6.13

$$\text{Maximize } \left\{ S_\mu \left( \sum_{j=1}^n \sum_{i=1}^m x_{i1} \left\langle (a_{ij}^L(0), a_{ij}(1), a_{ij}^R(0)), w_{\tilde{a}_{ij}}, u_{\tilde{a}_{ij}} \right\rangle y_{j1} \right) \right\}$$

Subject to

$$\sum_{i=1}^m x_{i1} = 1; x_{i1} \geq 0, i=1, 2, \dots, m; \sum_{j=1}^n y_{j1} = 1; y_{j1} \geq 0, j=1, 2, \dots, n.$$

#### Problem 4.6.14

$$\text{Maximize} \left\{ S_\nu \left( \sum_{j=1}^n \sum_{i=1}^m x_{i2} \left\langle \left( a_{ij}^L(0), a_{ij}(1), a_{ij}^R(0) \right), w_{\tilde{a}_{ij}}, u_{\tilde{a}_{ij}} \right\rangle y_{j2} \right) \right\}$$

Subject to

$$\sum_{i=1}^m x_{i2} = 1; x_{i2} \geq 0, i = 1, 2, \dots, m; \sum_{j=1}^n y_{j2} = 1; y_{j2} \geq 0, j = 1, 2, \dots, n.$$

**Step 2:** Using the property  $\lambda \left\langle \left( a_i^L(0), a_i(1), a_i^R(0) \right), w_{\tilde{a}_i}, u_{\tilde{a}_i} \right\rangle =$

$$\left\langle \left( \lambda a_i^L(0), \lambda a_i(1), \lambda a_i^R(0) \right), w_{\tilde{a}_i}, u_{\tilde{a}_i} \right\rangle, \lambda \geq 0 \text{ and } \sum_{i=1}^n \left( \left\langle \left( a_i^L(0), a_i(1), a_i^R(0) \right), w_{\tilde{a}_i}, u_{\tilde{a}_i} \right\rangle \right) =$$

$$\left( \left\langle \left( \sum_{i=1}^n a_i^L(0), \sum_{i=1}^n a_i(1), \sum_{i=1}^n a_i^R(0) \right), \min_{1 \leq i \leq n} \{ w_{\tilde{a}_i} \}, \max_{1 \leq i \leq n} \{ u_{\tilde{a}_i} \} \right\rangle \right), \text{ Problem 4.6.13 and Problem 4.6.14}$$

can be transformed into Problem 4.6.15 and Problem 4.6.16 respectively.

#### Problem 4.6.15

Maximize

$$\left\{ S_\mu \left( \left\langle \left( \sum_{j=1}^n \sum_{i=1}^m x_{i1} a_{ij}^L(0) y_{j1}, \sum_{j=1}^n \sum_{i=1}^m x_{i1} a_{ij}(1) y_{j1}, \sum_{j=1}^n \sum_{i=1}^m x_{i1} a_{ij}^R(0) y_{j1} \right), \min_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}} \{ w_{\tilde{a}_{ij}} \}, \max_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}} \{ u_{\tilde{a}_{ij}} \} \right\rangle \right) \right\}$$

Subject to

$$\sum_{i=1}^m x_{i1} = 1; x_{i1} \geq 0, i = 1, 2, \dots, m; \sum_{j=1}^n y_{j1} = 1; y_{j1} \geq 0, j = 1, 2, \dots, n.$$

#### Problem 4.6.16

Maximize

$$\left\{ S_\nu \left( \left\langle \left( \sum_{j=1}^n \sum_{i=1}^m x_{i2} a_{ij}^L(0) y_{j2}, \sum_{j=1}^n \sum_{i=1}^m x_{i2} a_{ij}(1) y_{j2}, \sum_{j=1}^n \sum_{i=1}^m x_{i2} a_{ij}^R(0) y_{j2} \right), \min_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}} \{ w_{\tilde{a}_{ij}} \}, \max_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}} \{ u_{\tilde{a}_{ij}} \} \right\rangle \right) \right\}$$

Subject to

$$\sum_{i=1}^m x_{i2} = 1; x_{i2} \geq 0, i = 1, 2, \dots, m; \sum_{j=1}^n y_{j2} = 1; y_{j2} \geq 0, j = 1, 2, \dots, n.$$

**Step 3:** Since, in Problem 4.6.15 and Problem 4.6.16, only  $x_{i1}$  and  $x_{i2}$  have been considered as decision variables. So, Problem 4.6.15 and Problem 4.6.16 are linear programming problems and hence, the optimal value of Problem 4.6.15 and Problem 4.6.16 will be equal to optimal value of its corresponding dual problem i.e., Problem 4.6.17 and Problem 4.6.18 respectively.

**Problem 4.6.17**

Minimize  $\{\omega_1\}$

Subject to

$$S_{\mu} \left( \sum_{j=1}^n \left\langle \left( a_{ij}^L(0), a_{ij}(1), a_{ij}^R(0) \right), \min_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}} \{w_{\tilde{a}_{ij}}\}, \max_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}} \{u_{\tilde{a}_{ij}}\} \right\rangle y_{j1} \right) \leq \omega_1, i = 1, 2, \dots, m;$$

$$\sum_{j=1}^n y_{j1} = 1; y_{j1} \geq 0, j = 1, 2, \dots, n.$$

**Problem 4.6.18**

Minimize  $\{\omega_2\}$

Subject to

$$S_{\nu} \left( \sum_{j=1}^n \left\langle \left( a_{ij}^L(0), a_{ij}(1), a_{ij}^R(0) \right), \min_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}} \{w_{\tilde{a}_{ij}}\}, \max_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}} \{u_{\tilde{a}_{ij}}\} \right\rangle y_{j2} \right) \leq \omega_2, i = 1, 2, \dots, m;$$

$$\sum_{j=1}^n y_{j2} = 1; y_{j2} \geq 0, j = 1, 2, \dots, n.$$

**Step 4:** Find the optimal solution  $\{y_{j1}^*\}$  and  $\{y_{j2}^*\}$  of Problem 4.6.17 and Problem 4.6.18 respectively.

**Step 5:** Substitute the optimal solution  $\{y_{j1}^*, j = 1, 2, \dots, n\}$  and  $\{y_{j2}^*, j = 1, 2, \dots, n\}$  of Problem 4.6.17 and Problem 4.6.18 in Problem 4.6.15 and Problem 4.6.16 respectively and then find the alternative basic optimal solutions  $\{x_{i1}^*, i = 1, 2, \dots, m\}, \{x_{i1}^{2*}, i = 1, 2, \dots, m\}, \dots,$   
 $\{x_{i1}^{q*}, i = 1, 2, \dots, m\}$  and  $\{x_{i2}^*, i = 1, 2, \dots, m\}, \{x_{i2}^{2*}, i = 1, 2, \dots, m\}, \dots, \{x_{i2}^{l*}, i = 1, 2, \dots, m\}$  of Problem 4.6.15 and Problem 4.6.16 respectively.

**Step 6:** Find maximum  $\left\{ \begin{array}{l} \sum_{j=1}^n \sum_{i=1}^m x_{i1}^{t*} \left\langle (a_{ij}^L(0), a_{ij}(1), a_{ij}^R(0)), w_{\tilde{a}_{ij}}, u_{\tilde{a}_{ij}} \right\rangle y_{j1}^*, t=1, 2, \dots, q, \\ \sum_{j=1}^n \sum_{i=1}^m x_{i2}^{w*} \left\langle (a_{ij}^L(0), a_{ij}(1), a_{ij}^R(0)), w_{\tilde{a}_{ij}}, u_{\tilde{a}_{ij}} \right\rangle y_{j2}^*, w=1, 2, \dots, l \end{array} \right\}.$

**Step 7:** All the triangular intuitionistic fuzzy numbers which will be maximum, will represent the maximum expected loss of Player II. Also, for all such maximum which will be obtained

corresponding to  $\sum_{j=1}^n \sum_{i=1}^m x_{i1}^{t*} \left\langle (a_{ij}^L(0), a_{ij}(1), a_{ij}^R(0)), w_{\tilde{a}_{ij}}, u_{\tilde{a}_{ij}} \right\rangle y_{j1}^*, t=1, 2, \dots, q$  the optimal strategy

will be  $\{y_{j1}^*, j=1, 2, \dots, n\}$ , which is optimal solution of Problem 4.6.17 and for all such

maximum which will be obtained corresponding to

$\sum_{j=1}^n \sum_{i=1}^m x_{i2}^{w*} \left\langle (a_{ij}^L(0), a_{ij}(1), a_{ij}^R(0)), w_{\tilde{a}_{ij}}, u_{\tilde{a}_{ij}} \right\rangle y_{j2}^*, w=1, 2, \dots, l$  the optimal strategy will be

$\{y_{j2}^*, j=1, 2, \dots, n\}$ , which is optimal solution of Problem 4.6.18.

### 4.6.3 Ambika method-III

In the existing method [96], it is assumed that if  $\tilde{a} = \langle (a^L(0), a(1), a^R(0)), w_{\tilde{a}}, u_{\tilde{a}} \rangle$  and

$\tilde{b} = \langle (b^L(0), b(1), b^R(0)), w_{\tilde{b}}, u_{\tilde{b}} \rangle$  are two triangular intuitionistic fuzzy numbers then  $\tilde{a} \preceq \tilde{b}$  if

$V_{\lambda}(\tilde{a}) \leq V_{\lambda}(\tilde{b})$  and if  $V_{\lambda}(\tilde{a}) = V_{\lambda}(\tilde{b})$  then  $A_{\lambda}(\tilde{a}) \geq A_{\lambda}(\tilde{b})$ , where,

$$V_{\lambda}(\tilde{a}) = \left( \lambda w_{\tilde{a}}^2 + (1-\lambda)(1-u_{\tilde{a}})^2 \right) \left( \frac{a^L(0) + a(1) + a^R(0)}{6} \right) \text{ and}$$

$$A_{\lambda}(\tilde{a}) = \left( \lambda w_{\tilde{a}}^2 + (1-\lambda)(1-u_{\tilde{a}})^2 \right) \left( \frac{a^R(0) - a^L(0)}{6} \right).$$

In this section, a new method, based on this existing method, is proposed to find the minimum expected gain of Player I, maximum expected loss of Player II and their corresponding optimal strategies.

#### 4.6.3.1 Minimum expected gain of Player I

Using the comparing method,  $\tilde{a} \preceq \tilde{b}$  if  $V_\lambda(\tilde{a}) \leq V_\lambda(\tilde{b})$  and if  $V_\lambda(\tilde{a}) = V_\lambda(\tilde{b})$  then  $A_\lambda(\tilde{a}) \geq A_\lambda(\tilde{b})$ , the minimum expected gain of Player I and the corresponding optimal strategies can be obtained as follows:

**Step 1:** According to comparing method, to find the optimal solution  $\{y_1^*, y_2^*, \dots, y_n^*\} \in Y$  such

that value of  $\sum_{j=1}^n \sum_{i=1}^m x_i \langle (a_{ij}^L(0), a_{ij}(1), a_{ij}^R(0)), w_{\tilde{a}_{ij}}, u_{\tilde{a}_{ij}} \rangle y_j$  is minimum for all  $(x_1, x_2, \dots, x_m) \in X$

, firstly it is equivalent to find  $\{y_j^*, j=1, 2, \dots, n\} \in Y$  such that value of

$V_\lambda \left( \sum_{j=1}^n \sum_{i=1}^m x_i \langle (a_{ij}^L(0), a_{ij}(1), a_{ij}^R(0)), w_{\tilde{a}_{ij}}, u_{\tilde{a}_{ij}} \rangle y_j \right)$ , is minimum for all  $(x_1, x_2, \dots, x_m) \in X$  i.e.,

find optimal solution  $\{y_j^*, j=1, 2, \dots, n\} \in Y$  of Problem 4.6.19.

**Problem 4.6.19**

$$\text{Minimize} \left\{ V_\lambda \left( \sum_{j=1}^n \sum_{i=1}^m x_i \langle (a_{ij}^L(0), a_{ij}(1), a_{ij}^R(0)), w_{\tilde{a}_{ij}}, u_{\tilde{a}_{ij}} \rangle y_j \right) \right\}$$

Subject to

$$\sum_{i=1}^m x_i = 1; x_i \geq 0, i=1, 2, \dots, m; \sum_{j=1}^n y_j = 1; y_j \geq 0, j=1, 2, \dots, n.$$

**Step 2:** Using the property  $\lambda \langle (a_i^L(0), a_i(1), a_i^R(0)), w_{\tilde{a}_i}, u_{\tilde{a}_i} \rangle =$

$$\langle (\lambda a_i^L(0), \lambda a_i(1), \lambda a_i^R(0)), w_{\tilde{a}_i}, u_{\tilde{a}_i} \rangle, \lambda \geq 0 \text{ and } \sum_{i=1}^n \left( \langle (a_i^L(0), a_i(1), a_i^R(0)), w_{\tilde{a}_i}, u_{\tilde{a}_i} \rangle \right) =$$

$$\left( \left\langle \left( \sum_{i=1}^n a_i^L(0), \sum_{i=1}^n a_i(1), \sum_{i=1}^n a_i^R(0) \right), \min_{1 \leq i \leq n} \{w_{\tilde{a}_i}\}, \max_{1 \leq i \leq n} \{u_{\tilde{a}_i}\} \right\rangle \right), \text{ Problem 4.6.19 can be transformed}$$

into Problem 4.6.20.

**Problem 4.6.20**

$$\text{Minimize } \left\{ V_{\lambda} \left( \left\langle \left( \sum_{j=1}^n \sum_{i=1}^m x_i a_{ij}^L(0) y_j, \sum_{j=1}^n \sum_{i=1}^m x_i a_{ij}(1) y_j, \sum_{j=1}^n \sum_{i=1}^m x_i a_{ij}^R(0) y_j \right), \min_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}} \{ w_{\tilde{a}_{ij}} \}, \max_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}} \{ u_{\tilde{a}_{ij}} \} \right\rangle \right) \right\}$$

Subject to

$$\sum_{i=1}^m x_i = 1; x_i \geq 0, i = 1, 2, \dots, m; \sum_{j=1}^n y_j = 1; y_j \geq 0, j = 1, 2, \dots, n.$$

**Step 3:** Since, in Problem 4.6.20, only  $y_j$  has been considered as decision variables. So, Problem 4.6.20 is a linear programming problem and hence, the optimal value of Problem 4.6.20 will be equal to optimal value of its corresponding dual problem i.e., Problem 4.6.21.

**Problem 4.6.21**

Maximize  $\{v\}$

Subject to

$$V_{\lambda} \left( \sum_{i=1}^m \left\langle \left( a_{ij}^L(0), a_{ij}(1), a_{ij}^R(0) \right) x_i, \min_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}} \{ w_{\tilde{a}_{ij}} \}, \max_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}} \{ u_{\tilde{a}_{ij}} \} \right\rangle \right) \geq v, j = 1, 2, \dots, n;$$

$$\sum_{i=1}^m x_i = 1; x_i \geq 0, i = 1, 2, \dots, m.$$

**Step 4:** Find the optimal solution of Problem 4.6.21.

**Step 5:** Substitute the optimal solution  $\{x_i^*, i = 1, 2, \dots, m\}$  of Problem 4.6.21 in Problem 4.6.20 and then find the optimal solution  $\{y_j^*, j = 1, 2, \dots, n\}$  of Problem 4.6.20.

**Case 1:** If there exist a unique optimal solution  $\{y_j^*, j = 1, 2, \dots, n\}$  of Problem 4.6.20 then the minimum expected gain of Player I is  $\sum_{j=1}^n \sum_{i=1}^m x_i^* \left\langle \left( a_{ij}^L(0), a_{ij}(1), a_{ij}^R(0) \right), w_{\tilde{a}_{ij}}, u_{\tilde{a}_{ij}} \right\rangle y_j^*$  and the corresponding optimal strategy for Player I will be  $\{x_i^*, i = 1, 2, \dots, m\}$ , which is optimal solution of Problem 4.6.21.

**Case 2:** If there exist more than one basic optimal solutions  $\{y_{j1}^*\}, \{y_{j2}^*\}, \dots, \{y_{jp}^*\}$  of Problem

4.6.20 then find maximum  $\left\{ A_\lambda \left( \sum_{j=1}^n \sum_{i=1}^m x_i^* \left\langle (a_{ij}^L(0), a_{ij}(1), a_{ij}^R(0)), w_{\tilde{a}_{ij}}, u_{\tilde{a}_{ij}} \right\rangle y_{js}^* \right), s=1, 2, \dots, p \right\}.$

If  $A_\lambda \left( \sum_{j=1}^n \sum_{i=1}^m x_i^* \left\langle (a_{ij}^L(0), a_{ij}(1), a_{ij}^R(0)), w_{\tilde{a}_{ij}}, u_{\tilde{a}_{ij}} \right\rangle y_{jr}^* \right)$  represents the maximum value then

minimum expected gain of Player I is  $\sum_{j=1}^n \sum_{i=1}^m x_i^* \left\langle (a_{ij}^L(0), a_{ij}(1), a_{ij}^R(0)), w_{\tilde{a}_{ij}}, u_{\tilde{a}_{ij}} \right\rangle y_{jr}^*$  and the

corresponding optimal strategy for Player I will be  $\{x_i^*, i=1, 2, \dots, m\}$ , which is optimal solution of Problem 4.6.21.

#### 4.6.3.2 Maximum expected loss of Player II

Using the comparing method,  $\tilde{a} \succeq \tilde{b}$  if  $V_\lambda(\tilde{a}) > V_\lambda(\tilde{b})$  and if  $V_\lambda(\tilde{a}) = V_\lambda(\tilde{b})$  then  $A_\lambda(\tilde{a}) < A_\lambda(\tilde{b})$ , the maximum expected loss of Player II and the corresponding optimal strategies can be obtained as follows:

**Step 1:** According to comparing method, to find the optimal solution  $\{x_i^*, i=1, 2, \dots, m\}$  such

that value of  $\sum_{j=1}^n \sum_{i=1}^m x_i \left\langle (a_{ij}^L(0), a_{ij}(1), a_{ij}^R(0)), w_{\tilde{a}_{ij}}, u_{\tilde{a}_{ij}} \right\rangle y_j$  is maximum, firstly it is equivalent to

find  $\{x_i^*, i=1, 2, \dots, m\} \in X$  such that value of  $V_\lambda \left( \sum_{j=1}^n \sum_{i=1}^m x_i \left\langle (a_{ij}^L(0), a_{ij}(1), a_{ij}^R(0)), w_{\tilde{a}_{ij}}, u_{\tilde{a}_{ij}} \right\rangle y_j \right)$

is maximum for all  $\forall (y_1, y_2, \dots, y_n) \in Y$  i.e.,

find optimal solution  $\{x_i^*, i=1, 2, \dots, m\} \in X$  of Problem 4.6.22.

#### Problem 4.6.22

Maximize  $\left\{ V_\lambda \left( \sum_{j=1}^n \sum_{i=1}^m x_i \left\langle (a_{ij}^L(0), a_{ij}(1), a_{ij}^R(0)), w_{\tilde{a}_{ij}}, u_{\tilde{a}_{ij}} \right\rangle y_j \right) \right\}$

Subject to

$$\sum_{i=1}^m x_i = 1; x_i \geq 0, i = 1, 2, \dots, m; \sum_{j=1}^n y_j = 1; y_j \geq 0, j = 1, 2, \dots, n.$$

**Step 2:** Using the property  $\lambda \langle (a_i^L(0), a_i(1), a_i^R(0)), w_{\tilde{a}_i}, u_{\tilde{a}_i} \rangle =$

$$\langle (\lambda a_i^L(0), \lambda a_i(1), \lambda a_i^R(0)), w_{\tilde{a}_i}, u_{\tilde{a}_i} \rangle, \lambda \geq 0 \text{ and } \sum_{i=1}^n \left( \langle (a_i^L(0), a_i(1), a_i^R(0)), w_{\tilde{a}_i}, u_{\tilde{a}_i} \rangle \right) =$$

$$\left( \left\langle \left( \sum_{i=1}^n a_i^L(0), \sum_{i=1}^n a_i(1), \sum_{i=1}^n a_i^R(0) \right), \min_{1 \leq i \leq n} \{w_{\tilde{a}_i}\}, \max_{1 \leq i \leq n} \{u_{\tilde{a}_i}\} \right\rangle \right), \text{ Problem 4.6.22 can be transformed}$$

into Problem 4.6.23.

### Problem 4.6.23

Maximize

$$\left\{ V_{\lambda} \left( \left\langle \left( \sum_{j=1}^n \sum_{i=1}^m x_i a_{ij}^L(0) y_j, \sum_{j=1}^n \sum_{i=1}^m x_i a_{ij}(1) y_j, \sum_{j=1}^n \sum_{i=1}^m x_i a_{ij}^R(0) y_j \right), \min_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}} \{w_{\tilde{a}_{ij}}\}, \max_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}} \{u_{\tilde{a}_{ij}}\} \right\rangle \right) \right\}$$

Subject to

$$\sum_{i=1}^m x_i = 1; x_i \geq 0, i = 1, 2, \dots, m; \sum_{j=1}^n y_j = 1; y_j \geq 0, j = 1, 2, \dots, n.$$

**Step 3:** Since, in Problem 4.6.23, only  $\{x_i^*, i = 1, 2, \dots, m\} \in X$  has been considered as decision

variables. So, Problem 4.6.23 is a linear programming problem and hence, the optimal value

of Problem 4.6.23 will be equal to optimal value of its corresponding dual problem i.e.,

Problem 4.6.24.

### Problem 4.6.24

Minimize  $\{\omega\}$

Subject to

$$V_\lambda \left( \sum_{j=1}^n \left\langle \left( a_{ij}^L(0)y_j, a_{ij}(1), a_{ij}^R(0) \right), \min_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}} \{ w_{\tilde{a}_{ij}} \}, \max_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}} \{ u_{\tilde{a}_{ij}} \} \right\rangle y_j \right) \leq \omega, i = 1, 2, \dots, m;$$

$$\sum_{j=1}^n y_j = 1; y_j \geq 0, j = 1, 2, \dots, n.$$

**Step 4:** Find the optimal solution of Problem 4.6.24.

**Step 5:** Substitute the optimal solution  $\{y_j^*, j = 1, 2, \dots, n\}$  of Problem 4.6.24 in Problem 4.6.23

and then find the optimal solution  $\{x_i^*, i = 1, 2, \dots, m\} \in X$  of Problem 4.6.23.

**Case 1:** If there exist a unique optimal solution  $\{x_i^*, i = 1, 2, \dots, m\} \in X$  of Problem 4.6.23 then

the maximum expected loss of Player II is  $\sum_{j=1}^n \sum_{i=1}^m x_i^* \left\langle \left( a_{ij}^L(0), a_{ij}(1), a_{ij}^R(0) \right), w_{\tilde{a}_{ij}}, u_{\tilde{a}_{ij}} \right\rangle y_j^*$  and the

corresponding optimal strategy for Player II will be  $\{y_j^*, j = 1, 2, \dots, n\}$ , which is optimal solution of Problem 4.6.24.

**Case 2:** If there exist more than one basic optimal solution

$\{x_{i1}^*, i = 1, 2, \dots, m\}, \{x_{i2}^*, i = 1, 2, \dots, m\}, \dots, \{x_{iq}^*, i = 1, 2, \dots, m\}$  of Problem 4.6.23 then

find minimum  $\left\{ A_\lambda \left( \sum_{j=1}^n \sum_{i=1}^m x_{is}^* \left\langle \left( a_{ij}^L(0), a_{ij}(1), a_{ij}^R(0) \right), w_{\tilde{a}_{ij}}, u_{\tilde{a}_{ij}} \right\rangle y_j^* \right), s = 1, 2, \dots, q \right\}.$

If  $A_\lambda \left( \sum_{j=1}^n \sum_{i=1}^m x_{it}^* \left\langle \left( a_{ij}^L(0), a_{ij}(1), a_{ij}^R(0) \right), w_{\tilde{a}_{ij}}, u_{\tilde{a}_{ij}} \right\rangle y_j^* \right)$  represents the minimum value then

maximum expected loss of Player II is  $\sum_{j=1}^n \sum_{i=1}^m x_{it}^* \left\langle \left( a_{ij}^L(0), a_{ij}(1), a_{ij}^R(0) \right), w_{\tilde{a}_{ij}}, u_{\tilde{a}_{ij}} \right\rangle y_j^*$  and the

corresponding optimal strategy for Player II will be  $\{y_j^*, j = 1, 2, \dots, n\}$ , which is optimal solution of Problem 4.6.24.

#### 4.6.4 Ambika method-IV

In the existing method [97], authors have assumed if

$\tilde{a} = \langle (a^L(0), a^L(1), a^R(1), a^R(0)), w_{\tilde{a}}, u_{\tilde{a}} \rangle$  and  $\tilde{b} = \langle (b^L(0), b^L(1), b^R(1), b^R(0)), w_{\tilde{b}}, u_{\tilde{b}} \rangle$  are two

trapezoidal intuitionistic fuzzy numbers then  $\tilde{a} \preceq \tilde{b}$  if  $V_{\lambda}(\tilde{a}) \leq V_{\lambda}(\tilde{b})$  and  $A_{\lambda}(\tilde{a}) \geq A_{\lambda}(\tilde{b})$ , where,

$$V_{\lambda}(\tilde{a}) = (\lambda(1 - u_{\tilde{a}}) + (1 - \lambda)w_{\tilde{a}}) \left( \frac{a^L(0) + 2a^L(1) + 2a^R(1) + a^R(0)}{6} \right) \text{ and}$$

$$A_{\lambda}(\tilde{a}) = (\lambda w_{\tilde{a}} + (1 - \lambda)(1 - u_{\tilde{a}})) \left( \frac{a^R(0) - a^L(0) + 2a^R(1) - 2a^L(1)}{3} \right). \text{ In this section, a new}$$

method, based on this existing method, is proposed to find the minimum expected gain of Player I, maximum expected loss of Player II and their corresponding optimal strategies.

#### 4.6.4.1 Minimum expected gain of Player I

Using the comparing method,  $\tilde{a} \preceq \tilde{b}$  if  $V_{\lambda}(\tilde{a}) \leq V_{\lambda}(\tilde{b})$  and  $A_{\lambda}(\tilde{a}) \geq A_{\lambda}(\tilde{b})$ , the minimum expected gain of Player I and the corresponding optimal strategies can be obtained as follows:

**Step 1:** According to comparing method, to find the optimal solution  $\{y_1^*, y_2^*, \dots, y_n^*\}$  such

that value of  $\sum_{j=1}^n \sum_{i=1}^m \langle (a_{ij}^L(0)x_i y_j, a_{ij}^L(1)x_i y_j, a_{ij}^R(1)x_i y_j, a_{ij}^R(0)x_i y_j), w_{\tilde{a}_{ij}}, u_{\tilde{a}_{ij}} \rangle$  is minimum for all

$(x_1, x_2, \dots, x_m) \in X$  is equivalent to find  $\{y_1^*, y_2^*, \dots, y_n^*\}$  such that value of

$$V_{\lambda} \left( \sum_{j=1}^n \sum_{i=1}^m x_i \langle (a_{ij}^L(0), a_{ij}^L(1), a_{ij}^R(0)), w_{\tilde{a}_{ij}}, u_{\tilde{a}_{ij}} \rangle y_j \right) \text{ is minimum and}$$

$$A_{\lambda} \left( \sum_{j=1}^n \sum_{i=1}^m x_i \langle (a_{ij}^L(0), a_{ij}^L(1), a_{ij}^R(0)), w_{\tilde{a}_{ij}}, u_{\tilde{a}_{ij}} \rangle y_j \right) \text{ is maximum for all } (x_1, x_2, \dots, x_m) \in X \text{ or if}$$

it is not possible to find  $\{y_1^*, y_2^*, \dots, y_n^*\}$  for which the value of

$$V_{\lambda} \left( \sum_{j=1}^n \sum_{i=1}^m x_i \langle (a_{ij}^L(0), a_{ij}^L(1), a_{ij}^R(0)), w_{\tilde{a}_{ij}}, u_{\tilde{a}_{ij}} \rangle y_j \right) \text{ is minimum and}$$

$$A_{\lambda} \left( \sum_{j=1}^n \sum_{i=1}^m x_i \langle (a_{ij}^L(0), a_{ij}^L(1), a_{ij}^R(0)), w_{\tilde{a}_{ij}}, u_{\tilde{a}_{ij}} \rangle y_j \right) \text{ is maximum then find such}$$

$\{y_{j1}^*, j=1, 2, \dots, n\}$  for which the value of  $V_\lambda \left( \sum_{j=1}^n \sum_{i=1}^m x_{i1} \langle (a_{ij}^L(0), a_{ij}(1), a_{ij}^R(0)), w_{\tilde{a}_{ij}}, u_{\tilde{a}_{ij}} \rangle y_{j1} \right)$  is

minimum but value of  $A_\lambda \left( \sum_{j=1}^n \sum_{i=1}^m x_{i1} \langle (a_{ij}^L(0), a_{ij}(1), a_{ij}^R(0)), w_{\tilde{a}_{ij}}, u_{\tilde{a}_{ij}} \rangle y_{j1} \right)$  is not maximum for

all  $(x_{11}, x_{21}, \dots, x_{m1}) \in X$  and find such  $\{y_{j2}^*, j=1, 2, \dots, n\}$  for which the value of

$A_\lambda \left( \sum_{j=1}^n \sum_{i=1}^m x_{i2} \langle (a_{ij}^L(0), a_{ij}(1), a_{ij}^R(0)), w_{\tilde{a}_{ij}}, u_{\tilde{a}_{ij}} \rangle y_{j2} \right)$  is maximum but value of

$V_\lambda \left( \sum_{j=1}^n \sum_{i=1}^m x_{i2} \langle (a_{ij}^L(0), a_{ij}(1), a_{ij}^R(0)), w_{\tilde{a}_{ij}}, u_{\tilde{a}_{ij}} \rangle y_{j2} \right)$  is not minimum for all  $(x_{12}, x_{22}, \dots, x_{m2}) \in X$

i.e., find optimal solution  $\{y_{j1}^*, j=1, 2, \dots, n\}$  and  $\{y_{j2}^*, j=1, 2, \dots, n\}$  of Problem 4.6.25 and

Problem 4.6.26 respectively.

#### Problem 4.6.25

$$\text{Minimize} \left\{ V_\lambda \left( \sum_{j=1}^n \sum_{i=1}^m x_{i1} \langle (a_{ij}^L(0), a_{ij}^L(1), a_{ij}^R(1), a_{ij}^R(0)), w_{\tilde{a}_{ij}}, u_{\tilde{a}_{ij}} \rangle y_{j1} \right) \right\}$$

Subject to

$$\sum_{i=1}^m x_{i1} = 1; x_{i1} \geq 0, i=1, 2, \dots, m; \sum_{j=1}^n y_{j1} = 1; y_{j1} \geq 0, j=1, 2, \dots, n.$$

#### Problem 4.6.26

$$\text{Maximize} \left\{ A_\lambda \left( \sum_{j=1}^n \sum_{i=1}^m x_{i2} \langle (a_{ij}^L(0), a_{ij}^L(1), a_{ij}^R(1), a_{ij}^R(0)), w_{\tilde{a}_{ij}}, u_{\tilde{a}_{ij}} \rangle y_{j2} \right) \right\}$$

Subject to

$$\sum_{i=1}^m x_{i2} = 1; x_{i2} \geq 0, i=1, 2, \dots, m; \sum_{j=1}^n y_{j2} = 1; y_{j2} \geq 0, j=1, 2, \dots, n.$$

**Step 2:** Using the property  $\lambda \langle (a_i^L(0), a_i(1), a_i^R(0)), w_{\tilde{a}_i}, u_{\tilde{a}_i} \rangle =$

$$\left\langle \left( \lambda a_i^L(0), \lambda a_i(1), \lambda a_i^R(0) \right), w_{\tilde{a}_i}, u_{\tilde{a}_i} \right\rangle, \lambda \geq 0 \text{ and } \sum_{i=1}^n \left( \left\langle \left( a_i^L(0), a_i(1), a_i^R(0) \right), w_{\tilde{a}_i}, u_{\tilde{a}_i} \right\rangle \right) =$$

$$\left( \left\langle \left( \sum_{i=1}^n a_i^L(0), \sum_{i=1}^n a_i(1), \sum_{i=1}^n a_i^R(0) \right), \min_{1 \leq i \leq n} \{w_{\tilde{a}_i}\}, \max_{1 \leq i \leq n} \{u_{\tilde{a}_i}\} \right\rangle \right), \text{ Problem 4.6.25 and Problem 4.6.26}$$

can be transformed into Problem 4.6.27 and Problem 4.6.28 respectively.

#### Problem 4.6.27

Minimize

$$\left\{ V_{\lambda} \left( \left\langle \left( \sum_{j=1}^n \sum_{i=1}^m x_{i1} a_{ij}^L(0) y_{j1}, \sum_{j=1}^n \sum_{i=1}^m x_{i1} a_{ij}(1) y_{j1}, \sum_{j=1}^n \sum_{i=1}^m x_{i1} a_{ij}^R(0) y_{j1} \right), \min_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}} \{w_{\tilde{a}_{ij}}\}, \max_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}} \{u_{\tilde{a}_{ij}}\} \right\rangle \right) \right\}$$

Subject to

$$\sum_{i=1}^m x_{i1} = 1; x_{i1} \geq 0, i = 1, 2, \dots, m; \sum_{j=1}^n y_{j1} = 1; y_{j1} \geq 0, j = 1, 2, \dots, n.$$

#### Problem 4.6.28

Maximize

$$\left\{ A_{\lambda} \left( \left\langle \left( \sum_{j=1}^n \sum_{i=1}^m x_{i2} a_{ij}^L(0) y_{j2}, \sum_{j=1}^n \sum_{i=1}^m x_{i2} a_{ij}(1) y_{j2}, \sum_{j=1}^n \sum_{i=1}^m x_{i2} a_{ij}^R(0) y_{j2} \right), \min_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}} \{w_{\tilde{a}_{ij}}\}, \max_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}} \{u_{\tilde{a}_{ij}}\} \right\rangle \right) \right\}$$

Subject to

$$\sum_{i=1}^m x_{i2} = 1; x_{i2} \geq 0, i = 1, 2, \dots, m; \sum_{j=1}^n y_{j2} = 1; y_{j2} \geq 0, j = 1, 2, \dots, n.$$

**Step 3:** Since, in Problem 4.6.27 and Problem 4.6.28, only  $\{y_{j1}^*, j = 1, 2, \dots, n\} \in Y$  and

$\{y_{j2}^*, j = 1, 2, \dots, n\} \in Y$  have been considered as decision variables. So, Problem 4.6.27 and

Problem 4.6.28 are linear programming problems and hence, the optimal value of Problem

4.6.27 and Problem 4.6.28 will be equal to optimal value its corresponding dual problem i.e.,

Problem 4.6.29 and Problem 4.6.30 respectively.

#### Problem 4.6.29

Maximize  $\{v_1\}$

Subject to

$$V_\lambda \left( \sum_{i=1}^m \left\langle \left( a_{ij}^L(0), a_{ij}^L(1), a_{ij}^R(1), a_{ij}^R(0) \right), \min_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}} \{w_{\tilde{a}_{ij}}\}, \max_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}} \{u_{\tilde{a}_{ij}}\} \right\rangle x_{i1} \right) \geq v_1, j = 1, 2, \dots, n;$$

$$\sum_{i=1}^m x_{i1} = 1; x_{i1} \geq 0, i = 1, 2, \dots, m.$$

**Problem 4.6.30**

Minimize  $\{v_2\}$

Subject to

$$A_\lambda \left( \sum_{i=1}^m \left\langle \left( a_{ij}^L(0), a_{ij}^L(1), a_{ij}^R(1), a_{ij}^R(0) \right), \min_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}} \{w_{\tilde{a}_{ij}}\}, \max_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}} \{u_{\tilde{a}_{ij}}\} \right\rangle x_{i2} \right) \leq v_2, j = 1, 2, \dots, n;$$

$$\sum_{i=1}^m x_{i2} = 1; x_{i2} \geq 0, i = 1, 2, \dots, m.$$

**Step 4:** Find the optimal solution  $\{x_{i1}^*, i = 1, 2, \dots, m\}$  and  $\{x_{i2}^*, i = 1, 2, \dots, m\}$  of Problem 4.6.29

and Problem 4.6.30 respectively.

**Step 5:** Substitute the optimal solution  $\{x_{i1}^*, i = 1, 2, \dots, m\}$  and  $\{x_{i2}^*, i = 1, 2, \dots, m\}$  of Problem

4.6.29 and Problem 4.6.30 in Problem 4.6.27 and Problem 4.6.28 respectively and then find all

the alternative basic optimal solutions  $\{y_{j1}^{1*}, j = 1, 2, \dots, n\}, \{y_{j1}^{2*}, j = 1, 2, \dots, n\}, \dots,$

$\{y_{j1}^{r*}, j = 1, 2, \dots, n\}$  and  $\{y_{j2}^{1*}, j = 1, 2, \dots, n\}, \{y_{j2}^{2*}, j = 1, 2, \dots, n\}, \dots, \{y_{j2}^{s*}, j = 1, 2, \dots, n\}$  of

Problem 4.6.27 and Problem 4.6.28 respectively.

**Step 6:** Find minimum  $\left\{ \begin{aligned} & \sum_{j=1}^n \sum_{i=1}^m x_{i1}^* \left\langle \left( a_{ij}^L(0), a_{ij}^L(1), a_{ij}^R(1), a_{ij}^R(0) \right), w_{\tilde{a}_{ij}}, u_{\tilde{a}_{ij}} \right\rangle y_{j1}^{h*}, h = 1, 2, \dots, r, \\ & \sum_{j=1}^n \sum_{i=1}^m x_{i2}^* \left\langle \left( a_{ij}^L(0), a_{ij}^L(1), a_{ij}^R(1), a_{ij}^R(0) \right), w_{\tilde{a}_{ij}}, u_{\tilde{a}_{ij}} \right\rangle y_{j2}^{k*}, k = 1, 2, \dots, s \end{aligned} \right\}.$

**Step 7:** All the trapezoidal intuitionistic fuzzy numbers which will be minimum, will represent

the minimum expected gain of Player I. Also, for all such minimum which will be obtained corresponding to  $\sum_{j=1}^n \sum_{i=1}^m x_{i1}^* \left\langle \left( a_{ij}^L(0), a_{ij}^L(1), a_{ij}^R(1), a_{ij}^R(0) \right), w_{\tilde{a}_{ij}}, u_{\tilde{a}_{ij}} \right\rangle y_{j1}^{h*}, h=1, 2, \dots, r$  the optimal strategy will be  $\{x_{i1}^*, i=1, 2, \dots, m\}$ , which is optimal solution of Problem 4.6.29 and for all such minimum which will be obtained corresponding to  $\sum_{j=1}^n \sum_{i=1}^m x_{i2}^* \left\langle \left( a_{ij}^L(0), a_{ij}^L(1), a_{ij}^R(1), a_{ij}^R(0) \right), w_{\tilde{a}_{ij}}, u_{\tilde{a}_{ij}} \right\rangle y_{j2}^{k*}, k=1, 2, \dots, s$  the optimal strategy will be  $\{x_{i2}^*, i=1, 2, \dots, m\}$ , which is optimal solution of Problem 4.6.30

#### 4.6.4.2 Maximum expected loss of Player II

Using the comparing method,  $\tilde{a} \succeq \tilde{b}$  if  $V_\lambda(\tilde{a}) \geq V_\lambda(\tilde{b})$  and  $A_\lambda(\tilde{a}) \leq A_\lambda(\tilde{b})$ , the maximum expected loss of Player II and the corresponding optimal strategies can be obtained as follows:

**Step 1:** According to comparing method, to find the optimal solution  $\{x_1^*, x_2^*, \dots, x_m^*\}$  of Problem 4.3.2 such that value of  $\sum_{j=1}^n \sum_{i=1}^m \left\langle \left( a_{ij}^L(0)x_i y_j, a_{ij}^L(1)x_i y_j, a_{ij}^R(1)x_i y_j, a_{ij}^R(0)x_i y_j \right), w_{\tilde{a}_{ij}}, u_{\tilde{a}_{ij}} \right\rangle$

is maximum for all  $(y_1, y_2, \dots, y_n) \in Y$  is equivalent to find  $\{x_1^*, x_2^*, \dots, x_m^*\}$  such that value of

$V_\lambda \left( \sum_{j=1}^n \sum_{i=1}^m x_i \left\langle \left( a_{ij}^L(0), a_{ij}^L(1), a_{ij}^R(0) \right), w_{\tilde{a}_{ij}}, u_{\tilde{a}_{ij}} \right\rangle y_j \right)$  is maximum and

$A_\lambda \left( \sum_{j=1}^n \sum_{i=1}^m x_i \left\langle \left( a_{ij}^L(0), a_{ij}^L(1), a_{ij}^R(0) \right), w_{\tilde{a}_{ij}}, u_{\tilde{a}_{ij}} \right\rangle y_j \right)$  is minimum for all  $(y_1, y_2, \dots, y_n) \in Y$  or if it

is not possible to find  $\{x_1^*, x_2^*, \dots, x_m^*\}$  for which the value of

$V_\lambda \left( \sum_{j=1}^n \sum_{i=1}^m x_i \left\langle \left( a_{ij}^L(0), a_{ij}^L(1), a_{ij}^R(0) \right), w_{\tilde{a}_{ij}}, u_{\tilde{a}_{ij}} \right\rangle y_j \right)$  is maximum and

$A_\lambda \left( \sum_{j=1}^n \sum_{i=1}^m x_i \left\langle \left( a_{ij}^L(0), a_{ij}^L(1), a_{ij}^R(0) \right), w_{\tilde{a}_{ij}}, u_{\tilde{a}_{ij}} \right\rangle y_j \right)$  is minimum for all  $(y_1, y_2, \dots, y_n) \in Y$  then

find such  $\{x_{i1}^*, i=1, 2, \dots, m\}$  for which the value of

$V_\lambda \left( \sum_{j=1}^n \sum_{i=1}^m x_{i1} \left\langle (a_{ij}^L(0), a_{ij}(1), a_{ij}^R(0)), w_{\tilde{a}_{ij}}, u_{\tilde{a}_{ij}} \right\rangle y_{j1} \right)$  is maximum but value of

$A_\lambda \left( \sum_{j=1}^n \sum_{i=1}^m x_{i1} \left\langle (a_{ij}^L(0), a_{ij}(1), a_{ij}^R(0)), w_{\tilde{a}_{ij}}, u_{\tilde{a}_{ij}} \right\rangle y_{j1} \right)$  is not minimum for all  $(y_{11}, y_{21}, \dots, y_{n1}) \in Y$

and find such  $\{x_{i2}^*, i = 1, 2, \dots, m\}$  for which the value of

$A_\lambda \left( \sum_{j=1}^n \sum_{i=1}^m x_{i2} \left\langle (a_{ij}^L(0), a_{ij}(1), a_{ij}^R(0)), w_{\tilde{a}_{ij}}, u_{\tilde{a}_{ij}} \right\rangle y_{j2} \right)$  is maximum but value of

$V_\lambda \left( \sum_{j=1}^n \sum_{i=1}^m x_{i2} \left\langle (a_{ij}^L(0), a_{ij}(1), a_{ij}^R(0)), w_{\tilde{a}_{ij}}, u_{\tilde{a}_{ij}} \right\rangle y_{j2} \right)$  is not minimum for all  $(y_{12}, y_{22}, \dots, y_{n2}) \in Y$

i.e., to find optimal solution  $\{y_{j1}^*, j = 1, 2, \dots, n\}$  and  $\{y_{j2}^*, j = 1, 2, \dots, n\}$  of Problem 4.6.31 and

Problem 4.6.32 respectively

#### Problem 4.6.31

$$\text{Maximize} \left\{ V_\lambda \left( \sum_{j=1}^n \sum_{i=1}^m x_{i1} \left\langle (a_{ij}^L(0), a_{ij}^L(1), a_{ij}^R(1), a_{ij}^R(0)), w_{\tilde{a}_{ij}}, u_{\tilde{a}_{ij}} \right\rangle y_{j1} \right) \right\}$$

Subject to

$$\sum_{i=1}^m x_{i1} = 1; x_{i1} \geq 0, i = 1, 2, \dots, m; \sum_{j=1}^n y_{j1} = 1; y_{j1} \geq 0, j = 1, 2, \dots, n.$$

#### Problem 4.6.32

$$\text{Minimize} \left\{ A_\lambda \left( \sum_{j=1}^n \sum_{i=1}^m x_{i2} \left\langle (a_{ij}^L(0), a_{ij}^L(1), a_{ij}^R(1), a_{ij}^R(0)), w_{\tilde{a}_{ij}}, u_{\tilde{a}_{ij}} \right\rangle y_{j2} \right) \right\}$$

Subject to

$$\sum_{i=1}^m x_{i2} = 1; x_{i2} \geq 0, i = 1, 2, \dots, m; \sum_{j=1}^n y_{j2} = 1; y_{j2} \geq 0, j = 1, 2, \dots, n.$$

**Step 2:** Using the property  $\lambda \left\langle (a_i^L(0), a_i(1), a_i^R(0)), w_{\tilde{a}_i}, u_{\tilde{a}_i} \right\rangle =$

$$\left\langle \left( \lambda a_i^L(0), \lambda a_i(1), \lambda a_i^R(0) \right), w_{\tilde{a}_i}, u_{\tilde{a}_i} \right\rangle, \lambda \geq 0 \text{ and } \sum_{i=1}^n \left( \left\langle \left( a_i^L(0), a_i(1), a_i^R(0) \right), w_{\tilde{a}_i}, u_{\tilde{a}_i} \right\rangle \right) =$$

$$\left( \left\langle \left( \sum_{i=1}^n a_i^L(0), \sum_{i=1}^n a_i(1), \sum_{i=1}^n a_i^R(0) \right), \min_{1 \leq i \leq n} \{w_{\tilde{a}_i}\}, \max_{1 \leq i \leq n} \{u_{\tilde{a}_i}\} \right\rangle \right), \text{ Problem 4.6.31 and Problem 4.6.32}$$

can be transformed into Problem 4.6.33 and Problem 4.6.34 respectively.

### Problem 4.6.33

Maximize

$$\left\{ V_{\lambda} \left( \left\langle \left( \sum_{j=1}^n \sum_{i=1}^m x_{i1} a_{ij}^L(0) y_{j1}, \sum_{j=1}^n \sum_{i=1}^m x_{i1} a_{ij}(1) y_{j1}, \sum_{j=1}^n \sum_{i=1}^m x_{i1} a_{ij}^R(0) y_{j1} \right), \min_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}} \{w_{\tilde{a}_{ij}}\}, \max_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}} \{u_{\tilde{a}_{ij}}\} \right\rangle \right) \right\}$$

Subject to

$$\sum_{i=1}^m x_{i1} = 1; x_{i1} \geq 0, i = 1, 2, \dots, m; \sum_{j=1}^n y_{j1} = 1; y_{j1} \geq 0, j = 1, 2, \dots, n.$$

### Problem 4.6.34

Minimize

$$\left\{ A_{\lambda} \left( \left\langle \left( \sum_{j=1}^n \sum_{i=1}^m x_{i2} a_{ij}^L(0) y_{j2}, \sum_{j=1}^n \sum_{i=1}^m x_{i2} a_{ij}(1) y_{j2}, \sum_{j=1}^n \sum_{i=1}^m x_{i2} a_{ij}^R(0) y_{j2} \right), \min_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}} \{w_{\tilde{a}_{ij}}\}, \max_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}} \{u_{\tilde{a}_{ij}}\} \right\rangle \right) \right\}$$

Subject to

$$\sum_{i=1}^m x_{i2} = 1; x_{i2} \geq 0, i = 1, 2, \dots, m; \sum_{j=1}^n y_{j2} = 1; y_{j2} \geq 0, j = 1, 2, \dots, n.$$

**Step 3:** Since, in Problem 4.6.33 and Problem 4.6.34, only  $x_{i1}, i = 1, 2, \dots, m$  and  $x_{i2}, i = 1, 2, \dots, m$  have been considered as variable. So, Problem 4.6.33 and Problem 4.6.34 are linear programming problems and hence, the optimal value of Problem 4.6.33 and Problem 4.6.34 will be equal to optimal value of its corresponding dual problem i.e., Problem 4.6.35 and Problem 4.6.36 respectively.

### Problem 4.6.35

Minimize  $\{\omega_1\}$

Subject to

$$V_\lambda \left( \sum_{j=1}^n \left\langle \left( a_{ij}^L(0), a_{ij}^L(1), a_{ij}^R(1), a_{ij}^R(0) \right), \min_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}} \{w_{\tilde{a}_{ij}}\}, \max_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}} \{u_{\tilde{a}_{ij}}\} \right\rangle y_{j1} \right) \leq \omega_1, i = 1, 2, \dots, m;$$

$$\sum_{j=1}^n y_{j1} = 1; y_{j1} \geq 0, j = 1, 2, \dots, n.$$

**Problem 4.6.36**

Maximize  $\{\nu_2\}$

Subject to

$$A_\lambda \left( \sum_{j=1}^n \left\langle \left( a_{ij}^L(0), a_{ij}^L(1), a_{ij}^R(1), a_{ij}^R(0) \right), \min_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}} \{w_{\tilde{a}_{ij}}\}, \max_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}} \{u_{\tilde{a}_{ij}}\} \right\rangle y_{j2} \right) \geq \omega_2, i = 1, 2, \dots, m;$$

$$\sum_{j=1}^n y_{j2} = 1; y_{j2} \geq 0, j = 1, 2, \dots, n.$$

**Step 4:** Find the optimal solution  $\{y_{j1}^*, j = 1, 2, \dots, n\}$  and  $\{y_{j2}^*, j = 1, 2, \dots, n\}$  of Problem 4.6.35 and Problem 4.6.36 respectively.

**Step 5:** Substitute the optimal solution  $\{y_{j1}^*, j = 1, 2, \dots, n\}$  and  $\{y_{j1}^*, j = 1, 2, \dots, n\}$  of Problem 4.6.35 and Problem 4.6.36 in Problem 4.6.33 and Problem 4.6.34 respectively and then find the optimal solution  $\{x_{i1}^{1*}, i = 1, 2, \dots, m\}, \{x_{i1}^{2*}, i = 1, 2, \dots, m\}, \dots, \{x_{i1}^{q*}, i = 1, 2, \dots, m\}$  and  $\{x_{i2}^{1*}, i = 1, 2, \dots, m\}, \{x_{i2}^{2*}, i = 1, 2, \dots, m\}, \dots, \{x_{i2}^{l*}, i = 1, 2, \dots, m\}$  of Problem 4.6.33 and Problem 4.6.34 respectively.

**Step 6:** Find maximum  $\left\{ \begin{aligned} &\sum_{j=1}^n \sum_{i=1}^m x_{i1}^{t*} \left\langle \left( a_{ij}^L(0), a_{ij}^L(1), a_{ij}^R(1), a_{ij}^R(0) \right), w_{\tilde{a}_{ij}}, u_{\tilde{a}_{ij}} \right\rangle y_{j1}^*, t = 1, 2, \dots, q \\ &\sum_{j=1}^n \sum_{i=1}^m x_{i2}^{w*} \left\langle \left( a_{ij}^L(0), a_{ij}^L(1), a_{ij}^R(1), a_{ij}^R(0) \right), w_{\tilde{a}_{ij}}, u_{\tilde{a}_{ij}} \right\rangle y_{j2}^*, w = 1, 2, \dots, l \end{aligned} \right\}.$

**Step 7:** All the trapezoidal intuitionistic fuzzy numbers which will be maximum, will represent the maximum expected loss of Player II. Also, for all such maximum which will be obtained

corresponding to  $\sum_{j=1}^n \sum_{i=1}^m x_{i1}^{t*} \langle (a_{ij}^L(0), a_{ij}^L(1), a_{ij}^R(1), a_{ij}^R(0)), w_{\tilde{a}_{ij}}, u_{\tilde{a}_{ij}} \rangle y_{j1}^*, t=1, 2, \dots, q$  the optimal

strategy will be  $\{y_{j1}^*, j=1, 2, \dots, n\}$ , which is optimal solution of Problem 4.6.35 and for all

such maximum which will be obtained corresponding to

$\sum_{j=1}^n \sum_{i=1}^m x_{i2}^{w*} \langle (a_{ij}^L(0), a_{ij}^L(1), a_{ij}^R(1), a_{ij}^R(0)), w_{\tilde{a}_{ij}}, u_{\tilde{a}_{ij}} \rangle y_{j2}^*, w=1, 2, \dots, l$  the optimal strategy will be

$\{y_{j2}^*, j=1, 2, \dots, n\}$ , which is optimal solution of Problem 4.6.36.

## 4.7 Numerical examples

In this section, some existing numerical examples are solved by proposed Ambika methods.

### 4.7.1 Existing numerical example considered by Nan et al.

In this section, matrix games with intuitionistic fuzzy payoffs

$$\tilde{A} = \begin{bmatrix} \langle (175, 180, 190); 0.6, 0.2 \rangle & \langle (150, 156, 158); 0.6, 0.2 \rangle \\ \langle (80, 90, 100); 0.9, 0.1 \rangle & \langle (175, 180, 190); 0.6, 0.2 \rangle \end{bmatrix}, \text{ chosen by Nan et al. [95], is solved}$$

by the proposed Ambika method-I.

#### 4.7.1.1 Minimum expected gain of Player I

Using the proposed Ambika method-I minimum expected gain of Player I and corresponding optimal strategies, can be obtained as follows:

**Step 1:** Find  $\{y_j^*, j=1, 2\} \in Y$  such that value of  $S_\mu(\langle (175, 180, 190); 0.6, 0.2 \rangle x_1 y_1 + \langle (150, 156, 158); 0.6, 0.2 \rangle x_1 y_2 + \langle (80, 90, 100); 0.9, 0.1 \rangle x_2 y_1 + \langle (175, 180, 190); 0.6, 0.2 \rangle x_2 y_2)$ , is minimum for all  $(x_1, x_2) \in X$  i.e., find optimal solution  $\{y_1^*, y_2^*\}$  of Problem 4.7.1.

#### Problem 4.7.1

$$\text{Minimize} \left\{ S_\mu \left( \langle (175, 180, 190); 0.6, 0.2 \rangle x_1 y_1 + \langle (150, 156, 158); 0.6, 0.2 \rangle x_1 y_2 + \langle (80, 90, 100); 0.9, 0.1 \rangle x_2 y_1 + \langle (175, 180, 190); 0.6, 0.2 \rangle x_2 y_2 \right) \right\}$$

Subject to

$$x_1 + x_2 = 1; x_1, x_2 \geq 0; y_1 + y_2 = 1; y_1, y_2 \geq 0.$$

**Step 2:** Using the property  $\lambda \langle (a_i^L(0), a_i(1), a_i^R(0)), w_{\tilde{a}_i}, u_{\tilde{a}_i} \rangle$

$$= \langle (\lambda a_i^L(0), \lambda a_i(1), \lambda a_i^R(0)), w_{\tilde{a}_i}, u_{\tilde{a}_i} \rangle, \lambda \geq 0 \text{ and}$$

$$\sum_{i=1}^n \left( \langle (a_i^L(0), a_i(1), a_i^R(0)), w_{\tilde{a}_i}, u_{\tilde{a}_i} \rangle \right) = \left\langle \left( \sum_{i=1}^n a_i^L(0), \sum_{i=1}^n a_i(1), \sum_{i=1}^n a_i^R(0) \right), \min_{1 \leq i \leq n} \{w_{\tilde{a}_i}\}, \max_{1 \leq i \leq n} \{u_{\tilde{a}_i}\} \right\rangle,$$

Problem 4.7.1 can be transformed into Problem 4.7.2.

### Problem 4.7.2

Minimize

$$\left\{ S_{\mu} \left( \left\langle \left( \begin{array}{l} 175x_1y_1 + 150x_1y_2 + 80x_2y_1 + 175x_2y_2, 180x_1y_1 + 156x_1y_2 + 90x_2y_1 + 180x_2y_2, \\ 190x_1y_1 + 158x_1y_2 + 100x_2y_1 + 190x_2y_2 \end{array} \right), 0.6, 0.2 \right\rangle \right) \right\}$$

Subject to

$$x_1 + x_2 = 1; x_1, x_2 \geq 0; y_1 + y_2 = 1; y_1, y_2 \geq 0.$$

**Step 3:** Since, in Problem 4.7.2, only  $y_j$  has been considered as decision variables. So,

Problem 4.7.2 is a linear programming problem and hence, the optimal value of Problem 4.7.2 will be equal to optimal value of its corresponding dual problem i.e., Problem 4.7.3.

### Problem 4.7.3

Maximize  $\{\nu\}$

Subject to

$$S_{\mu} \left( \langle (175x_1 + 80x_2, 180x_1 + 90x_2, 190x_1 + 100x_2); 0.6, 0.2 \rangle \right) \geq \nu;$$

$$S_{\mu} \left( \langle (150x_1 + 175x_2, 156x_1 + 180x_2, 158x_1 + 190x_2); 0.6, 0.2 \rangle \right) \geq \nu;$$

$$x_1 + x_2 = 1; x_1, x_2 \geq 0.$$

**Step 4:** The optimal solution of Problem 4.6.3 is  $\left\{ x_1^* = \frac{73}{94}, x_2^* = \frac{21}{94} \right\}.$

**Step 5:** Substituting the optimal solution  $\left\{x_1^* = \frac{73}{94}, x_2^* = \frac{21}{94}\right\}$  of Problem 4.7.3 in Problem

4.7.1, the obtained alternative basic optimal solutions of Problem 4.7.1 are  $\{y_1^{1*} = 1, y_2^{1*} = 0\}$

and  $\{y_1^{2*} = 0, y_2^{2*} = 1\}$ .

Since, there exists more than one optimal solution  $\{y_1^{1*} = 1, y_2^{1*} = 0\}$  and  $\{y_1^{2*} = 0, y_2^{2*} = 1\}$  of

Problem 4.7.1 so find minimum

$$\begin{aligned} & \left\{ S_v \left( \left\langle \begin{pmatrix} 175x_1^*y_1^{1*} + 150x_1^1y_2^{1*} + 80x_2^*y_1^{1*} + 175x_2^*y_2^{1*}, 180x_1^*y_1^{1*} + 156x_1^*y_2^{1*} + 90x_2^*y_1^{1*} + 180x_2^*y_2^{1*}, \\ 190x_1^*y_1^{1*} + 158x_1^*y_2^{1*} + 100x_2^*y_1^{1*} + 190x_2^*y_2^{1*} \end{pmatrix}; 0.6, 0.2 \right\rangle \right), \right. \\ & \left. S_v \left( \left\langle \begin{pmatrix} 175x_1^*y_1^{2*} + 150x_1^2y_2^{2*} + 80x_2^*y_1^{2*} + 175x_2^*y_2^{2*}, 180x_1^*y_1^{2*} + 156x_1^*y_2^{2*} + 90x_2^*y_1^{2*} + 180x_2^*y_2^{2*}, \\ 190x_1^*y_1^{2*} + 158x_1^*y_2^{2*} + 100x_2^*y_1^{2*} + 190x_2^*y_2^{2*} \end{pmatrix}; 0.6, 0.2 \right\rangle \right) \right\} \\ & = \text{minimum} \left\{ \frac{12097}{94}, \frac{12097}{94} \right\} = \frac{12097}{94}. \end{aligned}$$

Since,

$$S_v \left( \left\langle \begin{pmatrix} 175x_1^*y_1^{1*} + 150x_1^1y_2^{1*} + 80x_2^*y_1^{1*} + 175x_2^*y_2^{1*}, 180x_1^*y_1^{1*} + 156x_1^*y_2^{1*} + 90x_2^*y_1^{1*} + 180x_2^*y_2^{1*}, \\ 190x_1^*y_1^{1*} + 158x_1^*y_2^{1*} + 100x_2^*y_1^{1*} + 190x_2^*y_2^{1*} \end{pmatrix}; 0.6, 0.2 \right\rangle \right)$$

and

$$S_v \left( \left\langle \begin{pmatrix} 175x_1^*y_1^{1*} + 150x_1^1y_2^{1*} + 80x_2^*y_1^{1*} + 175x_2^*y_2^{1*}, 180x_1^*y_1^{1*} + 156x_1^*y_2^{1*} + 90x_2^*y_1^{1*} + 180x_2^*y_2^{1*}, \\ 190x_1^*y_1^{1*} + 158x_1^*y_2^{1*} + 100x_2^*y_1^{1*} + 190x_2^*y_2^{1*} \end{pmatrix}; 0.6, 0.2 \right\rangle \right)$$

both represents the minimum value so minimum expected gain of Player I is

$$\begin{aligned} & \left\langle \begin{pmatrix} 175x_1^*y_1^{1*} + 150x_1^1y_2^{1*} + 80x_2^*y_1^{1*} + 175x_2^*y_2^{1*}, 180x_1^*y_1^{1*} + 156x_1^*y_2^{1*} + 90x_2^*y_1^{1*} + 180x_2^*y_2^{1*}, \\ 190x_1^*y_1^{1*} + 158x_1^*y_2^{1*} + 100x_2^*y_1^{1*} + 190x_2^*y_2^{1*} \end{pmatrix}; 0.6, 0.2 \right\rangle \\ & = \left\langle \left( \frac{14455}{94}, \frac{7517}{47}, \frac{7985}{47} \right); 0.6, 0.2 \right\rangle \text{ and} \end{aligned}$$

$$\left\langle \begin{pmatrix} 175x_1^*y_1^{2*} + 150x_1^2y_2^{1*} + 80x_2^*y_1^{2*} + 175x_2^*y_2^{2*}, 180x_1^*y_1^{2*} + 156x_1^*y_2^{2*} + 90x_2^*y_1^{2*} + 180x_2^*y_2^{2*} \\ 190x_1^*y_1^{2*} + 158x_1^*y_2^{2*} + 100x_2^*y_1^{2*} + 190x_2^*y_2^{2*} \end{pmatrix}; 0.6, 0.2 \right\rangle$$

$$= \left\langle \left( \frac{14625}{94}, \frac{7584}{47}, \frac{7762}{47} \right); 0.6, 0.2 \right\rangle \text{ respectively and the corresponding optimal strategy for}$$

Player I will be  $\left\{ x_1^* = \frac{73}{94}, x_2^* = \frac{21}{94} \right\}$ , which is optimal solution of Problem 4.7.2.

#### 4.7.1.2 Maximum expected loss of Player II

Using proposed Ambika method-I, the maximum expected loss of Player II and the corresponding optimal strategies can be obtained as follows:

**Step 1:** Find  $x_i^*, i = 1, 2$  such that value of  $S_\mu \left( \langle (175, 180, 190); 0.6, 0.2 \rangle x_1 y_1 + \langle (150, \right.$   
 $\left. \langle 156, 158 \rangle; 0.6, 0.2 \rangle x_1 y_2 + \langle (80, 90, 100); 0.9, 0.1 \rangle x_2 y_1 + \langle (175, 180, 190); 0.6, 0.2 \rangle x_2 y_2 \right)$ , is  
maximum  $\forall (y_1, y_2) \in Y$  i.e., find optimal solution  $\{x_i^*, i = 1, 2\}$  of Problem 4.7.4.

#### Problem 4.7.4

$$\text{Maximize } \left\{ S_\mu \left( \langle (175, 180, 190); 0.6, 0.2 \rangle x_1 y_1 + \langle (150, 156, 158); 0.6, 0.2 \rangle x_1 y_2 + \right. \right. \\ \left. \left. \langle (80, 90, 100); 0.9, 0.1 \rangle x_2 y_1 + \langle (175, 180, 190); 0.6, 0.2 \rangle x_2 y_2 \right) \right\}$$

Subject to

$$x_1 + x_2 = 1; x_1, x_2 \geq 0; y_1 + y_2 = 1; y_1, y_2 \geq 0.$$

**Step 2:** Using the property  $\lambda \langle (a_i^L(0), a_i(1), a_i^R(0)), w_{\tilde{a}_i}, u_{\tilde{a}_i} \rangle =$

$$\langle (\lambda a_i^L(0), \lambda a_i(1), \lambda a_i^R(0)), w_{\tilde{a}_i}, u_{\tilde{a}_i} \rangle, \lambda \geq 0 \text{ and } \sum_{i=1}^n \left( \langle (a_i^L(0), a_i(1), a_i^R(0)), w_{\tilde{a}_i}, u_{\tilde{a}_i} \rangle \right) =$$

$$\left( \left\langle \left( \sum_{i=1}^n a_i^L(0), \sum_{i=1}^n a_i(1), \sum_{i=1}^n a_i^R(0) \right), \min_{1 \leq i \leq n} \{w_{\tilde{a}_i}\}, \max_{1 \leq i \leq n} \{u_{\tilde{a}_i}\} \right\rangle \right), \text{ Problem 4.7.4 can be transformed}$$

into Problem 4.7.5.

#### Problem 4.7.5

Maximize

$$\left\{ S_{\mu} \left( \left\langle \begin{pmatrix} 175x_1y_1 + 150x_1y_2 + 80x_2y_1 + 175x_2y_2, 180x_1y_1 + 156x_1y_2 + 90x_2y_1 + 180x_2y_2 \\ 190x_1y_1 + 158x_1y_2 + 100x_2y_1 + 190x_2y_2 \end{pmatrix}; 0.6, 0.2 \right\rangle \right) \right\}$$

Subject to

$$x_1 + x_2 = 1; x_1, x_2 \geq 0; y_1 + y_2 = 1; y_1, y_2 \geq 0.$$

**Step 3:** Since, in Problem 4.7.5, only  $x_i$  has been considered as decision variables. So, Problem 4.7.5 is a linear programming problem and hence, the optimal value of Problem 4.7.5 will be equal to optimal value of its corresponding dual problem i.e., Problem 4.7.6.

**Problem 4.7.6**

Minimize  $\{\omega\}$

Subject to

$$S_{\mu} \left( \left\langle (175y_1 + 150y_2, 180y_1 + 156y_2, 190y_1 + 158y_2); 0.6, 0.2 \right\rangle \right) \leq \omega;$$

$$S_{\mu} \left( \left\langle (80y_1 + 175y_2, 90y_1 + 180y_2, 100y_1 + 190y_2); 0.6, 0.2 \right\rangle \right) \leq \omega;$$

$$y_1 + y_2 = 1; y_1, y_2 \geq 0.$$

**Step 4:** The optimal solution of Problem 4.7.6 is  $\left\{ y_1^* = \frac{21}{94}, y_2^* = \frac{73}{94} \right\}$ .

**Step 5:** Substituting the optimal solution  $\left\{ y_1^* = \frac{21}{94}, y_2^* = \frac{73}{94} \right\}$  of Problem 4.7.6 in Problem

4.7.5, the obtained alternative basic optimal solutions of Problem 4.7.5 are  $\{x_1^{1*} = 1, x_2^{1*} = 0\}$

and  $\{x_1^{2*} = 0, x_2^{2*} = 1\}$ .

Since, there exist more than one optimal solution  $\{x_1^{1*} = 1, x_2^{1*} = 0\}$  and  $\{x_1^{2*} = 0, x_2^{2*} = 1\}$  of

Problem 4.7.5 so find maximum

$$\begin{aligned}
& \left\{ S_v \left( \left\langle \left( \begin{array}{l} 175x_1^{1*}y_1^* + 150x_1^{1*}y_2^* + 80x_2^{1*}y_1^* + 175x_2^{1*}y_2^*, 180x_1^{1*}y_1^* + 156x_1^{1*}y_2^* + 90x_2^{1*}y_1^* + 180x_2^{1*}y_2^* \\ 190x_1^{1*}y_1^* + 158x_1^{1*}y_2^* + 100x_2^{1*}y_1^* + 190x_2^{1*}y_2^* \end{array} \right); 0.6, 0.2 \right\rangle \right), \right. \\
& \left. S_v \left( \left\langle \left( \begin{array}{l} 175x_1^{2*}y_1^* + 150x_1^{2*}y_2^* + 80x_2^{2*}y_1^* + 175x_2^{2*}y_2^*, 180x_1^{2*}y_1^* + 156x_1^{2*}y_2^* + 90x_2^{2*}y_1^* + 180x_2^{2*}y_2^* \\ 190x_1^{2*}y_1^* + 158x_1^{2*}y_2^* + 100x_2^{2*}y_1^* + 190x_2^{2*}y_2^* \end{array} \right); 0.6, 0.2 \right\rangle \right) \right\} \\
& = \text{maximum} \left\{ \frac{12097}{94}, \frac{12097}{94} \right\} = \frac{12097}{94}.
\end{aligned}$$

Since,

$$S_v \left( \left\langle \left( \begin{array}{l} 175x_1^{1*}y_1^* + 150x_1^{1*}y_2^* + 80x_2^{1*}y_1^* + 175x_2^{1*}y_2^*, 180x_1^{1*}y_1^* + 156x_1^{1*}y_2^* + 90x_2^{1*}y_1^* + 180x_2^{1*}y_2^* \\ 190x_1^{1*}y_1^* + 158x_1^{1*}y_2^* + 100x_2^{1*}y_1^* + 190x_2^{1*}y_2^* \end{array} \right); 0.6, 0.2 \right\rangle \right)$$

and

$$S_v \left( \left\langle \left( \begin{array}{l} 175x_1^{2*}y_1^* + 150x_1^{2*}y_2^* + 80x_2^{2*}y_1^* + 175x_2^{2*}y_2^*, 180x_1^{2*}y_1^* + 156x_1^{2*}y_2^* + 90x_2^{2*}y_1^* + 180x_2^{2*}y_2^* \\ 190x_1^{2*}y_1^* + 158x_1^{2*}y_2^* + 100x_2^{2*}y_1^* + 190x_2^{2*}y_2^* \end{array} \right); 0.6, 0.2 \right\rangle \right)$$

both represents the maximum value so maximum expected loss of Player II is

$$\left\langle \left( \begin{array}{l} 175x_1^{1*}y_1^* + 150x_1^{1*}y_2^* + 80x_2^{1*}y_1^* + 175x_2^{1*}y_2^*, 180x_1^{1*}y_1^* + 156x_1^{1*}y_2^* + 90x_2^{1*}y_1^* + 180x_2^{1*}y_2^* \\ 190x_1^{1*}y_1^* + 158x_1^{1*}y_2^* + 100x_2^{1*}y_1^* + 190x_2^{1*}y_2^* \end{array} \right); 0.6, 0.2 \right\rangle$$

$$= \left\langle \left( \frac{14625}{94}, \frac{7584}{47}, \frac{7762}{47} \right); 0.6, 0.2 \right\rangle \text{ and}$$

$$\left\langle \left( \begin{array}{l} 175x_1^{2*}y_1^* + 150x_1^{2*}y_2^* + 80x_2^{2*}y_1^* + 175x_2^{2*}y_2^*, 180x_1^{2*}y_1^* + 156x_1^{2*}y_2^* + 90x_2^{2*}y_1^* + 180x_2^{2*}y_2^* \\ 190x_1^{2*}y_1^* + 158x_1^{2*}y_2^* + 100x_2^{2*}y_1^* + 190x_2^{2*}y_2^* \end{array} \right); 0.6, 0.2 \right\rangle$$

$$= \left\langle \left( \frac{14455}{94}, \frac{7517}{47}, \frac{7985}{47} \right); 0.6, 0.2 \right\rangle \text{ respectively and the corresponding optimal strategy for}$$

Player II will be  $\left\{ y_1^* = \frac{21}{94}, y_2^* = \frac{73}{94} \right\}$ , which is optimal solution of Problem 4.7.4.

#### 4.7.2 Existing numerical example considered by Li et al.

In this section, matrix games with intuitionistic fuzzy payoffs

$$\tilde{A} = \begin{bmatrix} \langle (175,180,190); 0.6, 0.2 \rangle & \langle (150,156,158); 0.6, 0.2 \rangle & \langle (80,90,100); 0.9, 0.1 \rangle \\ \langle (80,90,100); 0.9, 0.1 \rangle & \langle (175,180,190); 0.6, 0.2 \rangle & \langle (180,185,190); 0.5, 0.3 \rangle \\ \langle (180,185,190); 0.5, 0.3 \rangle & \langle (80,100,120); 0.7, 0.2 \rangle & \langle (150,160,170); 0.5, 0.1 \rangle \end{bmatrix}, \text{ chosen}$$

by Li et al. [78], is solved by the proposed Ambika method-II.

#### 4.7.2.1 Minimum expected gain of Player I

Using the proposed Ambika method -II, the minimum expected gain of Player I and the corresponding optimal strategies can be obtained as follows:

**Step 1:** Find  $\{y_{j1}^*, j = 1, 2, 3\} \in Y$  such that value of

$$S_\mu \left[ \begin{aligned} & \langle (175,180,190); 0.6, 0.2 \rangle x_{11}y_{11} + \langle (150,156,158); 0.6, 0.2 \rangle x_{11}y_{21} + \langle (80,90,100); 0.9, 0.1 \rangle x_{11}y_{31} + \\ & \langle (80,90,100); 0.9, 0.1 \rangle x_{21}y_{11} + \langle (175,180,190); 0.6, 0.2 \rangle x_{21}y_{21} + \langle (180,185,190); 0.5, 0.3 \rangle x_{21}y_{31} + \\ & \langle (180,185,190); 0.5, 0.3 \rangle x_{31}y_{11} + \langle (80,100,120); 0.7, 0.2 \rangle x_{31}y_{21} + \langle (150,160,170); 0.5, 0.1 \rangle x_{31}y_{31} \end{aligned} \right],$$

is minimum for all  $\{x_{i1}, i = 1, 2, 3\} \in X$  as well as find  $\{y_{j2}^*, j = 1, 2, 3\} \in Y$  such that value of

$$S_\nu \left[ \begin{aligned} & \langle (175,180,190); 0.6, 0.2 \rangle x_{12}y_{12} + \langle (150,156,158); 0.6, 0.2 \rangle x_{12}y_{22} + \langle (80,90,100); 0.9, 0.1 \rangle x_{12}y_{32} + \\ & \langle (80,90,100); 0.9, 0.1 \rangle x_{22}y_{12} + \langle (175,180,190); 0.6, 0.2 \rangle x_{22}y_{22} + \langle (180,185,190); 0.5, 0.3 \rangle x_{22}y_{32} + \\ & \langle (180,185,190); 0.5, 0.3 \rangle x_{32}y_{12} + \langle (80,100,120); 0.7, 0.2 \rangle x_{32}y_{22} + \langle (150,160,170); 0.5, 0.1 \rangle x_{32}y_{32} \end{aligned} \right]$$

, is minimum for all  $\{x_{i2}, i = 1, 2, 3\} \in X$  i.e., find optimal solution  $\{y_{j1}^*, j = 1, 2, 3\}$  and

$\{y_{j2}^*, j = 1, 2, 3\}$  of Problem 4.7.7 and Problem 4.7.8 respectively.

#### Problem 4.7.7

Minimize

$$\left\{ S_\mu \left[ \begin{aligned} & \langle (175,180,190); 0.6, 0.2 \rangle x_{11}y_{11} + \langle (150,156,158); 0.6, 0.2 \rangle x_{11}y_{21} + \langle (80,90,100); 0.9, 0.1 \rangle x_{11}y_{31} + \\ & \langle (80,90,100); 0.9, 0.1 \rangle x_{21}y_{11} + \langle (175,180,190); 0.6, 0.2 \rangle x_{21}y_{21} + \langle (180,185,190); 0.5, 0.3 \rangle x_{21}y_{31} + \\ & \langle (180,185,190); 0.5, 0.3 \rangle x_{31}y_{11} + \langle (80,100,120); 0.7, 0.2 \rangle x_{31}y_{21} + \langle (150,160,170); 0.5, 0.1 \rangle x_{31}y_{31} \end{aligned} \right] \right\}$$

Subject to

$$x_{11} + x_{21} + x_{31} = 1; x_{11}, x_{21}, x_{31} \geq 0; y_{11} + y_{21} + y_{31} = 1; y_{11}, y_{21}, y_{31} \geq 0.$$

**Problem 4.7.8**

Minimize

$$S_v \left( \begin{aligned} &\langle (175, 180, 190); 0.6, 0.2 \rangle x_{12} y_{12} + \langle (150, 156, 158); 0.6, 0.2 \rangle x_{12} y_{22} + \langle (80, 90, 100); 0.9, 0.1 \rangle x_{12} y_{32} + \\ &\langle (80, 90, 100); 0.9, 0.1 \rangle x_{22} y_{12} + \langle (175, 180, 190); 0.6, 0.2 \rangle x_{22} y_{22} + \langle (180, 185, 190); 0.5, 0.3 \rangle x_{22} y_{32} + \\ &\langle (180, 185, 190); 0.5, 0.3 \rangle x_{32} y_{12} + \langle (80, 100, 120); 0.7, 0.2 \rangle x_{32} y_{22} + \langle (150, 160, 170); 0.5, 0.1 \rangle x_{32} y_{32} \end{aligned} \right)$$

Subject to

$$x_{12} + x_{22} + x_{32} = 1; x_{12}, x_{22}, x_{32} \geq 0; y_{12} + y_{22} + y_{32} = 1; y_{12}, y_{22}, y_{32} \geq 0.$$

**Step 2:** Since, in Problem 4.7.7 and Problem 4.7.8, only  $y_{j1}$  and  $y_{j2}$  have been considered as decision variables. So, Problem 4.7.7 and Problem 4.7.8 are linear programming problems and hence, the optimal value of Problem 4.7.7 and Problem 4.7.8 will be equal to optimal value of its corresponding dual problem i.e., Problem 4.7.9 and Problem 4.7.10 respectively.

**Problem 4.7.9**Maximize  $\{v_1\}$ 

Subject to

$$\begin{aligned} S_\mu \left( \langle (175, 180, 190); 0.5, 0.3 \rangle x_{11} + \langle (80, 90, 100); 0.5, 0.3 \rangle x_{21} + \langle (180, 185, 190); 0.5, 0.3 \rangle x_{31} \right) &\geq v_1; \\ S_\mu \left( \langle (150, 156, 158); 0.5, 0.3 \rangle x_{11} + \langle (175, 180, 190); 0.5, 0.3 \rangle x_{21} + \langle (80, 100, 120); 0.5, 0.3 \rangle x_{31} \right) &\geq v_1 \\ S_\mu \left( \langle (80, 90, 100); 0.5, 0.3 \rangle x_{11} + \langle (180, 185, 190); 0.5, 0.3 \rangle x_{21} + \langle (150, 160, 170); 0.5, 0.3 \rangle x_{31} \right) &\geq v_1 \\ x_{11} + x_{21} + x_{31} &= 1; x_{11}, x_{21}, x_{31} \geq 0. \end{aligned}$$

**Problem 4.7.10**Maximize  $\{v_2\}$ 

Subject to

$$\begin{aligned}
S_\nu & \left( \langle (175, 180, 190); 0.5, 0.3 \rangle x_{12} + \langle (80, 90, 100); 0.5, 0.3 \rangle x_{22} + \langle (180, 185, 190); 0.5, 0.3 \rangle x_{32} \right) \geq \nu_2; \\
S_\nu & \left( \langle (150, 156, 158); 0.5, 0.3 \rangle x_{12} + \langle (175, 180, 190); 0.5, 0.3 \rangle x_{22} + \langle (80, 100, 120); 0.5, 0.3 \rangle x_{32} \right) \geq \nu_2 \\
S_\nu & \left( \langle (80, 90, 100); 0.5, 0.3 \rangle x_{12} + \langle (180, 185, 190); 0.5, 0.3 \rangle x_{22} + \langle (150, 160, 170); 0.5, 0.3 \rangle x_{32} \right) \geq \nu_2 \\
& x_{12} + x_{22} + x_{32} = 1; x_{12}, x_{22}, x_{32} \geq 0.
\end{aligned}$$

**Step 3:** The optimal solution of Problem 4.7.9 and Problem 4.7.10 is

$$\left\{ x_{11}^* = \frac{1236}{3995}, x_{21}^* = \frac{4544}{11985}, x_{31}^* = \frac{3733}{11985} \right\} \quad \text{and} \quad \left\{ x_{12}^* = \frac{1236}{3995}, x_{22}^* = \frac{4544}{11985}, x_{32}^* = \frac{3733}{11985} \right\}$$

respectively.

**Step 4:** Substituting the optimal solution  $\left\{ x_{11}^* = \frac{1236}{3995}, x_{21}^* = \frac{4544}{11985}, x_{31}^* = \frac{3733}{11985} \right\}$  and

$$\left\{ x_{12}^* = \frac{1236}{3995}, x_{22}^* = \frac{4544}{11985}, x_{32}^* = \frac{3733}{11985} \right\} \text{ of Problem 4.7.9 and Problem 4.7.10 in Problem 4.7.7}$$

and Problem 4.7.8 respectively, the alternative basic optimal solution of Problem 4.7.7 and

Problem 4.7.8  $\{y_{11}^1 = 1, y_{21}^1 = 0, y_{31}^1 = 0\}, \{y_{11}^2 = 0, y_{21}^2 = 1, y_{31}^2 = 0\}, \{y_{11}^3 = 0, y_{21}^3 = 0, y_{31}^3 = 1\}$  and

$$\{y_{12}^1 = 1, y_{22}^1 = 0, y_{32}^1 = 0\}, \{y_{12}^2 = 0, y_{22}^2 = 1, y_{32}^2 = 0\}, \{y_{12}^3 = 0, y_{22}^3 = 0, y_{32}^3 = 1\} \text{ respectively.}$$

**Step 5:** Now, minimum

$$\begin{aligned}
& \left[ \begin{aligned} & \langle (175, 180, 190); 0.6, 0.2 \rangle x_{11}^* y_{11}^{h*} + \langle (150, 156, 158); 0.6, 0.2 \rangle x_{11}^* y_{21}^{h*} + \langle (80, 90, 100); 0.9, 0.1 \rangle x_{11}^* y_{31}^{h*} + \\ & \langle (80, 90, 100); 0.9, 0.1 \rangle x_{21}^* y_{11}^{h*} + \langle (175, 180, 190); 0.6, 0.2 \rangle x_{21}^* y_{21}^{h*} + \langle (180, 185, 190); 0.5, 0.3 \rangle x_{21}^* y_{31}^{h*} + \\ & \langle (180, 185, 190); 0.5, 0.3 \rangle x_{31}^* y_{11}^{h*} + \langle (80, 100, 120); 0.7, 0.2 \rangle x_{31}^* y_{21}^{h*} + \langle (150, 160, 170); 0.5, 0.1 \rangle x_{31}^* y_{31}^{h*} \end{aligned} \right], \\
& h = 1, 2, 3, \\
& \left[ \begin{aligned} & \langle (175, 180, 190); 0.6, 0.2 \rangle x_{12}^* y_{12}^{s*} + \langle (150, 156, 158); 0.6, 0.2 \rangle x_{12}^* y_{22}^{s*} + \langle (80, 90, 100); 0.9, 0.1 \rangle x_{12}^* y_{32}^{s*} + \\ & \langle (80, 90, 100); 0.9, 0.1 \rangle x_{22}^* y_{12}^{s*} + \langle (175, 180, 190); 0.6, 0.2 \rangle x_{22}^* y_{22}^{s*} + \langle (180, 185, 190); 0.5, 0.3 \rangle x_{22}^* y_{32}^{s*} + \\ & \langle (180, 185, 190); 0.5, 0.3 \rangle x_{32}^* y_{12}^{s*} + \langle (80, 100, 120); 0.7, 0.2 \rangle x_{32}^* y_{22}^{s*} + \langle (150, 160, 170); 0.5, 0.1 \rangle x_{32}^* y_{32}^{s*} \end{aligned} \right], \\
& s = 1, 2, 3 \\
& = \left\langle \left( \frac{19816}{141}, \frac{353401}{2397}, \frac{124546}{799} \right); 0.5, 0.3 \right\rangle, \left\langle \left( \frac{330008}{2397}, \frac{1769668}{11985}, \frac{1897184}{11985} \right); 0.5, 0.3 \right\rangle, \\
& \left\langle \left( \frac{111634}{799}, \frac{354328}{2397}, \frac{373754}{2397} \right); 0.5, 0.3 \right\rangle.
\end{aligned}$$

**Step 6:** Since, all the triangular intuitionistic fuzzy numbers

$$\left\langle \left( \frac{19816}{141}, \frac{353401}{2397}, \frac{124546}{799} \right); 0.5, 0.3 \right\rangle, \left\langle \left( \frac{330008}{2397}, \frac{1769668}{11985}, \frac{1897184}{11985} \right); 0.5, 0.3 \right\rangle,$$

$$\left\langle \left( \frac{111634}{799}, \frac{354328}{2397}, \frac{373754}{2397} \right); 0.5, 0.3 \right\rangle \text{ represent the minimum value. So, all these triangular}$$

intuitionistic fuzzy numbers represent minimum expected gain of Player I and the

$$\text{corresponding optimal strategy is } \left\{ x_{11}^* = \frac{1236}{3995}, x_{21}^* = \frac{4544}{11985}, x_{31}^* = \frac{3733}{11985} \right\} =$$

$$\left\{ x_{12}^* = \frac{1236}{3995}, x_{22}^* = \frac{4544}{11985}, x_{32}^* = \frac{3733}{11985} \right\}.$$

#### 4.7.2.2 Maximum expected loss of Player II

Using the proposed Ambika method-II, the maximum expected loss of Player II and the corresponding optimal strategies can be obtained as follows:

**Step 1:** Find  $\{x_{i1}^*, i = 1, 2, 3\} \in X$  such that value of

$$S_{\mu} \left( \begin{aligned} &\langle (175, 180, 190); 0.6, 0.2 \rangle x_{11} y_{11} + \langle (150, 156, 158); 0.6, 0.2 \rangle x_{11} y_{21} + \langle (80, 90, 100); 0.9, 0.1 \rangle x_{11} y_{31} + \\ &\langle (80, 90, 100); 0.9, 0.1 \rangle x_{21} y_{11} + \langle (175, 180, 190); 0.6, 0.2 \rangle x_{21} y_{21} + \langle (180, 185, 190); 0.5, 0.3 \rangle x_{21} y_{31} + \\ &\langle (180, 185, 190); 0.5, 0.3 \rangle x_{31} y_{11} + \langle (80, 100, 120); 0.7, 0.2 \rangle x_{31} y_{21} + \langle (150, 160, 170); 0.5, 0.1 \rangle x_{31} y_{31} \end{aligned} \right),$$

is maximum for all  $\{y_{j1}, j = 1, 2, 3\} \in Y$  as well as find  $\{x_{i2}^*, i = 1, 2, 3\} \in X$  such that value of

$$S_{\nu} \left( \begin{aligned} &\langle (175, 180, 190); 0.6, 0.2 \rangle x_{12} y_{12} + \langle (150, 156, 158); 0.6, 0.2 \rangle x_{12} y_{22} + \langle (80, 90, 100); 0.9, 0.1 \rangle x_{12} y_{32} + \\ &\langle (80, 90, 100); 0.9, 0.1 \rangle x_{22} y_{12} + \langle (175, 180, 190); 0.6, 0.2 \rangle x_{22} y_{22} + \langle (180, 185, 190); 0.5, 0.3 \rangle x_{22} y_{32} + \\ &\langle (180, 185, 190); 0.5, 0.3 \rangle x_{32} y_{12} + \langle (80, 100, 120); 0.7, 0.2 \rangle x_{32} y_{22} + \langle (150, 160, 170); 0.5, 0.1 \rangle x_{32} y_{32} \end{aligned} \right),$$

is maximum for all  $\{y_{j2}, j = 1, 2, 3\} \in Y$  i.e., find optimal solution  $\{x_{i1}^*, i = 1, 2, 3\}$  and

$\{x_{i2}^*, i = 1, 2, 3\}$  of Problem 4.7.11 and Problem 4.7.12 respectively.

#### Problem 4.7.11

Maximize

$$\left\{ S_{\mu} \begin{pmatrix} \langle (175,180,190); 0.6, 0.2 \rangle x_{11} y_{11} + \langle (150,156,158); 0.6, 0.2 \rangle x_{11} y_{21} + \langle (80,90,100); 0.9, 0.1 \rangle x_{11} y_{31} + \\ \langle (80,90,100); 0.9, 0.1 \rangle x_{21} y_{11} + \langle (175,180,190); 0.6, 0.2 \rangle x_{21} y_{21} + \langle (180,185,190); 0.5, 0.3 \rangle x_{21} y_{31} + \\ \langle (180,185,190); 0.5, 0.3 \rangle x_{31} y_{11} + \langle (80,100,120); 0.7, 0.2 \rangle x_{31} y_{21} + \langle (150,160,170); 0.5, 0.1 \rangle x_{31} y_{31} \end{pmatrix} \right\}$$

Subject to

$$x_{11} + x_{21} + x_{31} = 1; x_{11}, x_{21}, x_{31} \geq 0; y_{11} + y_{21} + y_{31} = 1; y_{11}, y_{21}, y_{31} \geq 0.$$

#### Problem 4.7.12

Maximize

$$S_{\nu} \begin{pmatrix} \langle (175,180,190); 0.6, 0.2 \rangle x_{12} y_{12} + \langle (150,156,158); 0.6, 0.2 \rangle x_{12} y_{22} + \langle (80,90,100); 0.9, 0.1 \rangle x_{12} y_{32} + \\ \langle (80,90,100); 0.9, 0.1 \rangle x_{22} y_{12} + \langle (175,180,190); 0.6, 0.2 \rangle x_{22} y_{22} + \langle (180,185,190); 0.5, 0.3 \rangle x_{22} y_{32} + \\ \langle (180,185,190); 0.5, 0.3 \rangle x_{32} y_{12} + \langle (80,100,120); 0.7, 0.2 \rangle x_{32} y_{22} + \langle (150,160,170); 0.5, 0.1 \rangle x_{32} y_{32} \end{pmatrix}$$

Subject to

$$x_{12} + x_{22} + x_{32} = 1; x_{12}, x_{22}, x_{32} \geq 0; y_{12} + y_{22} + y_{32} = 1; y_{12}, y_{22}, y_{32} \geq 0.$$

**Step 2:** Since, in Problem 4.7.11 and Problem 4.7.12, only  $x_{i1}$  and  $x_{i2}$  have been considered as decision variables. So, Problem 4.7.11 and Problem 4.7.12 are linear programming problems and hence, the optimal value of Problem 4.7.11 and Problem 4.7.12 will be equal to optimal value of its corresponding dual problem i.e., Problem 4.7.13 and Problem 4.7.14 respectively.

#### Problem 4.7.13

Minimize  $\{\omega_1\}$

Subject to

$$\begin{aligned} S_{\mu} \left( \langle (175,180,190); 0.5, 0.3 \rangle y_{11} + \langle (150,156,158); 0.5, 0.3 \rangle y_{21} + \langle (80,90,100); 0.5, 0.3 \rangle y_{31} \right) &\leq \omega_1; \\ S_{\mu} \left( \langle (80,90,100); 0.5, 0.3 \rangle y_{11} + \langle (175,180,190); 0.5, 0.3 \rangle y_{21} + \langle (180,185,190); 0.5, 0.3 \rangle y_{31} \right) &\leq \omega_1; \\ S_{\mu} \left( \langle (180,185,190); 0.5, 0.3 \rangle y_{11} + \langle (80,100,120); 0.5, 0.3 \rangle y_{21} + \langle (150,160,170); 0.5, 0.3 \rangle y_{31} \right) &\leq \omega_1; \\ y_{11} + y_{21} + y_{31} &= 1; y_{11}, y_{21}, y_{31} \geq 0. \end{aligned}$$

#### Problem 4.7.14

Minimize  $\{\omega_2\}$

Subject to

$$\begin{aligned} S_v(\langle(175,180,190);0.5,0.3\rangle y_{12} + \langle(150,156,158);0.5,0.3\rangle y_{22} + \langle(80,90,100);0.5,0.3\rangle y_{32}) &\leq \omega_2; \\ S_v(\langle(80,90,100);0.5,0.3\rangle y_{12} + \langle(175,180,190);0.5,0.3\rangle y_{22} + \langle(180,185,190);0.5,0.3\rangle y_{32}) &\leq \omega_2; \\ S_v(\langle(180,185,190);0.5,0.3\rangle y_{12} + \langle(80,100,120);0.5,0.3\rangle y_{22} + \langle(150,160,170);0.5,0.3\rangle y_{32}) &\leq \omega_2; \\ y_{12} + y_{22} + y_{32} &= 1; y_{12}, y_{22}, y_{32} \geq 0. \end{aligned}$$

**Step 3:** The optimal solution of Problem 4.7.13 and Problem 4.7.14

$$\left\{ y_{11}^* = \frac{904}{2397}, y_{21}^* = \frac{4316}{11985}, y_{31}^* = \frac{67}{11985} \right\} \quad \text{and} \quad \left\{ y_{12}^* = \frac{904}{2397}, y_{22}^* = \frac{4316}{11985}, y_{32}^* = \frac{67}{11985} \right\}$$

respectively.

**Step 4:** Substituting the optimal solution  $\left\{ y_{11}^* = \frac{904}{2397}, y_{21}^* = \frac{4316}{11985}, y_{31}^* = \frac{67}{255} \right\}$  and

$$\left\{ y_{12}^* = \frac{904}{2397}, y_{22}^* = \frac{4316}{11985}, y_{32}^* = \frac{67}{255} \right\} \text{ of Problem 4.7.13 and Problem 4.7.10 in Problem 4.7.11}$$

and Problem 4.7.12 respectively, the basic optimal solutions of Problem 4.7.11 and Problem

4.7.12 are  $\{x_{11}^{1*} = 1, x_{21}^{1*} = 0, x_{31}^{1*} = 0\}, \{x_{11}^{2*} = 0, x_{21}^{2*} = 1, x_{31}^{2*} = 0\}, \{x_{11}^{3*} = 0, x_{21}^{3*} = 0, x_{31}^{3*} = 1\}$  and

$$\{x_{12}^{1*} = 1, x_{22}^{1*} = 0, x_{32}^{1*} = 0\}, \{x_{12}^{2*} = 0, x_{22}^{2*} = 1, x_{32}^{2*} = 0\}, \{x_{12}^{3*} = 0, x_{22}^{3*} = 0, x_{32}^{3*} = 1\} \text{ respectively.}$$

**Step 5:** Now, maximum

$$\begin{aligned}
 & \left\{ \begin{aligned} & \left\langle (175,180,190);0.6,0.2 \right\rangle x_{11}^{t*} y_{11}^* + \left\langle (150,156,158);0.6,0.2 \right\rangle x_{11}^{t*} y_{21}^* + \left\langle (80,90,100);0.9,0.1 \right\rangle x_{11}^{t*} y_{31}^* + \\ & \left\langle (80,90,100);0.9,0.1 \right\rangle x_{21}^{t*} y_{11}^* + \left\langle (175,180,190);0.6,0.2 \right\rangle x_{21}^{t*} y_{21}^* + \left\langle (180,185,190);0.5,0.3 \right\rangle x_{21}^{t*} y_{31}^* + \\ & \left\langle (180,185,190);0.5,0.3 \right\rangle x_{31}^{t*} y_{11}^* + \left\langle (80,100,120);0.7,0.2 \right\rangle x_{31}^{t*} y_{21}^* + \left\langle (150,160,170);0.5,0.1 \right\rangle x_{31}^{t*} y_{31}^* \end{aligned} \right\}, \\
 & t = 1, 2, 3, \\
 & \left\{ \begin{aligned} & \left\langle (175,180,190);0.6,0.2 \right\rangle x_{12}^{w*} y_{12}^* + \left\langle (150,156,158);0.6,0.2 \right\rangle x_{12}^{w*} y_{22}^* + \left\langle (80,90,100);0.9,0.1 \right\rangle x_{12}^{w*} y_{32}^* + \\ & \left\langle (80,90,100);0.9,0.1 \right\rangle x_{22}^{w*} y_{12}^* + \left\langle (175,180,190);0.6,0.2 \right\rangle x_{22}^{w*} y_{22}^* + \left\langle (180,185,190);0.5,0.3 \right\rangle x_{22}^{w*} y_{32}^* + \\ & \left\langle (180,185,190);0.5,0.3 \right\rangle x_{32}^{w*} y_{12}^* + \left\langle (80,100,120);0.7,0.2 \right\rangle x_{32}^{w*} y_{22}^* + \left\langle (150,160,170);0.5,0.1 \right\rangle x_{32}^{w*} y_{32}^* \end{aligned} \right\}, \\
 & w = 1, 2, 3 \\
 & = \left\langle \left( \frac{112688}{799}, \frac{590102}{3995}, \frac{1855628}{11985} \right), 0.5, 0.3 \right\rangle, \left\langle \left( \frac{112248}{799}, \frac{353249}{2397}, \frac{124690}{799} \right), 0.5, 0.3 \right\rangle, \\
 & \left\langle \left( \frac{326246}{2397}, \frac{354328}{2397}, \frac{127470}{799} \right), 0.5, 0.3 \right\rangle.
 \end{aligned}$$

**Step 6:** Since, all the triangular intuitionistic fuzzy numbers

$$\begin{aligned}
 & \left\langle \left( \frac{112688}{799}, \frac{590102}{3995}, \frac{1855628}{11985} \right), 0.5, 0.3 \right\rangle, \left\langle \left( \frac{112248}{799}, \frac{353249}{2397}, \frac{124690}{799} \right), 0.5, 0.3 \right\rangle, \\
 & \left\langle \left( \frac{326246}{2397}, \frac{354328}{2397}, \frac{127470}{799} \right), 0.5, 0.3 \right\rangle \text{ represent the maximum value. So, all these triangular}
 \end{aligned}$$

intuitionistic fuzzy numbers represent maximum expected loss of Player II and the

$$\text{corresponding optimal strategy is } \left\{ y_{11}^* = \frac{904}{2397}, y_{21}^* = \frac{4316}{11985}, y_{31}^* = \frac{67}{255} \right\} =$$

$$\left\{ y_{12}^* = \frac{904}{2397}, y_{22}^* = \frac{4316}{11985}, y_{32}^* = \frac{67}{255} \right\}.$$

#### 4.7.3 Existing numerical example considered by Nan et al.

In this section, matrix games with intuitionistic fuzzy payoffs

$$\tilde{A} = \begin{bmatrix} \left\langle (175,180,190);0.6,0.2 \right\rangle & \left\langle (150,156,158);0.6,0.1 \right\rangle \\ \left\langle (80,90,100);0.9,0.1 \right\rangle & \left\langle (175,180,190);0.6,0.2 \right\rangle \end{bmatrix}, \text{ chosen by Nan et al. [96], is solved}$$

by the proposed Ambika method-III.

##### 4.7.3.1 Minimum expected gain of Player I

Using the proposed Ambika method -III, the minimum expected gain of Player I and the corresponding optimal strategies can be obtained as follows:

**Step 1:** Find  $\{y_j^*, j=1,2\} \in Y$  such that value of

$$V_\lambda \left( \begin{array}{l} \langle (175,180,190); 0.6, 0.2 \rangle x_1 y_1 + \langle (150,156,158); 0.6, 0.1 \rangle x_1 y_2 + \langle (80,90,100); 0.9, 0.1 \rangle x_2 y_1 + \\ \langle (175,180,190); 0.6, 0.2 \rangle x_2 y_2 \end{array} \right)$$

is minimum for all  $(x_1, x_2, x_3) \in X$  i.e., find optimal solution  $\{y_j^*, j=1,2,3\} \in Y$  of Problem 4.7.15.

#### Problem 4.7.15

Minimize

$$\left\{ V_\lambda \left( \begin{array}{l} \langle (175,180,190); 0.6, 0.2 \rangle x_1 y_1 + \langle (150,156,158); 0.6, 0.1 \rangle x_1 y_2 + \langle (80,90,100); 0.9, 0.1 \rangle x_2 y_1 + \\ \langle (175,180,190); 0.6, 0.2 \rangle x_2 y_2 \end{array} \right) \right\}$$

Subject to

$$x_1 + x_2 = 1; x_1, x_2 \geq 0; y_1 + y_2 = 1; y_1, y_2 \geq 0.$$

**Step 2:** Since, in Problem 4.7.15, only  $y_j$  has been considered as decision variables. So, Problem 4.7.15 is a linear programming problem and hence, the optimal value of Problem 4.7.15 will be equal to optimal value of its corresponding dual problem i.e., Problem 4.7.16.

#### Problem 4.7.16

Maximize  $\{v\}$

Subject to

$$V_\lambda \left( \langle (175,180,190); 0.6, 0.2 \rangle x_1 + \langle (80,90,100); 0.6, 0.2 \rangle x_2 \right) \geq v;$$

$$V_\lambda \left( \langle (150,156,158); 0.6, 0.2 \rangle x_1 + \langle (175,180,190); 0.6, 0.2 \rangle x_2 \right) \geq v;$$

$$x_1 + x_2 = 1; x_1, x_2 \geq 0; y_1 + y_2 = 1; y_1, y_2 \geq 0.$$

**Step 3:** The optimal solution of Problem 4.7.16 is  $\left\{ x_1^* = \frac{545}{698}, x_2^* = \frac{153}{698} \right\}$ .

**Step 4:** Substituting the optimal solution  $\left\{x_1^* = \frac{545}{698}, x_2^* = \frac{153}{698}\right\}$  of Problem 4.7.16 in Problem

4.7.15, the basic optimal solutions of Problem 4.7.15 are  $\{y_1^{1*} = 1, y_2^{1*} = 0\}, \{y_1^{2*} = 0, y_2^{2*} = 1\}$ .

Since, there exist more than one optimal solution  $\{y_1^{1*} = 1, y_2^{1*} = 0\}, \{y_1^{2*} = 0, y_2^{2*} = 1\}$  of Problem 4.7.15 so find maximum

$$A_\lambda \left( \begin{aligned} &\langle (175, 180, 190); 0.6, 0.2 \rangle x_1^* y_1^{h*} + \langle (150, 156, 158); 0.6, 0.2 \rangle x_1^* y_2^{h*} + \langle (80, 90, 100); 0.6, 0.2 \rangle x_2^* y_2^{h*} + \\ &\langle (175, 180, 190); 0.6, 0.2 \rangle x_2^* y_2^{h*}, h=1, 2 \end{aligned} \right)$$

$$= \text{maximum} \left\{ \frac{2996}{1745}, \frac{5324}{5235} \right\} = \frac{2996}{1745}.$$

Since,  $\frac{2996}{1745}$  represents the maximum value so, minimum expected gain of Player I is

$$\langle (175, 180, 190); 0.6, 0.2 \rangle x_1^* y_1^{1*} + \langle (150, 156, 158); 0.6, 0.2 \rangle x_1^* y_2^{1*} + \langle (80, 90, 100); 0.6, 0.2 \rangle x_2^* y_2^{1*} +$$

$$\langle (175, 180, 190); 0.6, 0.2 \rangle x_2^* y_2^{1*} = \left\langle \left( \frac{107615}{698}, \frac{55935}{349}, \frac{59425}{349} \right), 0.6, 0.2 \right\rangle \text{ and the corresponding optimal}$$

strategy for Player I will be  $\left\{x_1^* = \frac{545}{698}, x_2^* = \frac{153}{698}\right\}$ , which is optimal solution of Problem

4.7.16.

#### 4.7.3.2 Maximum expected loss of Player II

Using the proposed Ambika method -III, the maximum expected loss of Player II and the corresponding optimal strategies can be obtained as follows:

**Step 1:** Find  $\{x_i^*, i=1, 2\} \in X$  such that value of

$$V_\lambda \left( \begin{aligned} &\langle (175, 180, 190); 0.6, 0.2 \rangle x_1 y_1 + \langle (150, 156, 158); 0.6, 0.1 \rangle x_1 y_2 + \langle (80, 90, 100); 0.9, 0.1 \rangle x_2 y_1 + \\ &\langle (175, 180, 190); 0.6, 0.2 \rangle x_2 y_2 \end{aligned} \right),$$

is maximum  $\forall (y_1, y_2) \in Y$  i.e., find optimal solution  $\{x_i^*, i=1, 2\} \in X$  of Problem 4.7.17.

#### Problem 4.7.17

Maximize

$$\left\{ V_{\lambda} \left( \langle (175, 180, 190); 0.6, 0.2 \rangle x_1 y_1 + \langle (150, 156, 158); 0.6, 0.1 \rangle x_1 y_2 + \langle (80, 90, 100); 0.9, 0.1 \rangle x_2 y_1 + \langle (175, 180, 190); 0.6, 0.2 \rangle x_2 y_2 \right) \right\}$$

Subject to

$$x_1 + x_2 = 1; x_1, x_2 \geq 0; y_1 + y_2 = 1; y_1, y_2 \geq 0.$$

**Step 2:** Since, in Problem 4.7.17, only  $\{x_i^*, i = 1, 2\} \in X$  has been considered as decision variables. So, Problem 4.7.17 is a linear programming problem and hence, the optimal value of Problem 4.7.17 will be equal to optimal value of its corresponding dual problem i.e., Problem 4.7.18.

#### Problem 4.7.18

Minimize  $\{\omega\}$

Subject to

$$V_{\lambda} \left( \langle (175, 180, 190); 0.6, 0.2 \rangle y_1 + \langle (150, 156, 158); 0.6, 0.2 \rangle y_2 \right) \leq \omega;$$

$$V_{\lambda} \left( \langle (80, 90, 100); 0.6, 0.2 \rangle y_1 + \langle (175, 180, 190); 0.6, 0.2 \rangle y_2 \right) \leq \omega;$$

$$y_1 + y_2 = 1; y_1, y_2 \geq 0.$$

**Step 3:** The optimal solution of Problem 4.7.18 is  $\left\{ y_1^* = \frac{153}{698}, y_2^* = \frac{545}{698} \right\}$ .

**Step 4:** Substituting the optimal solution  $\left\{ y_1^* = \frac{153}{698}, y_2^* = \frac{545}{698} \right\}$  of Problem 4.7.18 in Problem

4.7.17, the optimal basic solutions of Problem 4.7.17 are  $\{x_1^{1*} = 1, x_2^{1*} = 0\} \{x_1^{2*} = 0, x_2^{2*} = 1\}$ .

Since, there exist more than one optimal solution  $\{x_1^{1*} = 1, x_2^{1*} = 0\} \{x_1^{2*} = 0, x_2^{2*} = 1\}$  of

Problem 4.7.17 so find minimum

$$A_{\lambda} \left( \begin{aligned} &\langle (175,180,190);0.6,0.2 \rangle x_1^* y_1^* + \langle (150,156,158);0.6,0.2 \rangle x_1^* y_2^* + \langle (80,90,100);0.6,0.2 \rangle x_2^* y_2^* + \\ &\langle (175,180,190);0.6,0.2 \rangle x_2^* y_1^*, t=1,2 \end{aligned} \right)$$

$$= \text{minimum} \left\{ \frac{5324}{5235}, \frac{2996}{1745} \right\} = \frac{5324}{5235}.$$

Since,  $\frac{5324}{5235}$  represents the minimum value so, maximum expected loss of Player II is

$$\begin{aligned} &\langle (175,180,190);0.6,0.2 \rangle x_1^* y_1^* + \langle (150,156,158);0.6,0.2 \rangle x_1^* y_2^* + \langle (80,90,100);0.6,0.2 \rangle x_2^* y_2^* + \\ &\langle (175,180,190);0.6,0.2 \rangle x_2^* y_1^* = \left\langle \left( \frac{108525}{698}, \frac{56280}{349}, \frac{57590}{349} \right), 0.6, 0.2 \right\rangle \end{aligned}$$

and the corresponding optimal strategy for Player II will be  $\left\{ y_1^* = \frac{153}{698}, y_2^* = \frac{545}{698} \right\}$ , which is

optimal solution of Problem 4.7.17.

#### 4.7.4 Ambika method-IV

In this section, matrix games with intuitionistic fuzzy payoffs

$$\tilde{A} = \begin{bmatrix} \langle (175,180,180,190);0.6,0.3 \rangle & \langle (150,156,156,158);0.5,0.2 \rangle \\ \langle (125,128,132,140);0.9,0.1 \rangle & \langle (175,185,195,200);0.5,0.4 \rangle \end{bmatrix}, \text{ chosen by Nan et al. [97],}$$

is solved by the proposed Ambika method-IV.

##### 4.7.4.1 Minimum expected gain of Player I

Using Ambika method-IV, the minimum expected gain of Player I and the corresponding optimal strategies can be obtained as follows:

**Step 1:** Find  $\{y_{j1}^*, j=1,2\} \in Y$  such that value of

$$V_{\lambda} \left( \begin{aligned} &\langle (175,180,180,190);0.6,0.3 \rangle x_{11} y_{11} + \langle (150,156,156,158);0.5,0.2 \rangle x_{11} y_{21} + \\ &\langle (125,128,132,140);0.9,0.1 \rangle x_{21} y_{11} + \langle (175,185,195,200);0.5,0.4 \rangle x_{21} y_{21} \end{aligned} \right), \text{ is minimum f}$$

$\forall \{x_{i1}, i=1,2\} \in X$  as well as find  $\{y_{j2}^*, j=1,2,\dots,n\} \in Y$  such that value of

$$A_{\lambda} \left( \begin{aligned} &\langle (175,180,180,190);0.6,0.3 \rangle x_{12} y_{12} + \langle (150,156,156,158);0.5,0.2 \rangle x_{12} y_{22} + \\ &\langle (125,128,132,140);0.9,0.1 \rangle x_{22} y_{12} + \langle (175,185,195,200);0.5,0.4 \rangle x_{22} y_{22} \end{aligned} \right), \text{ is maximum}$$

$\forall \{x_{i2}, i=1,2\} \in X$  i.e., find optimal solution  $\{y_{j1}^*, j=1,2\} \in Y$  and  $\{y_{j2}^*, j=1,2\} \in Y$  of

Problem 4.7.19 and Problem 4.7.20 respectively.

**Problem 4.7.19**

$$\text{Minimize } \left\{ V_{\lambda} \left( \langle (175,180,180,190); 0.6, 0.3 \rangle x_{11} y_{11} + \langle (150,156,156,158); 0.5, 0.2 \rangle x_{11} y_{21} + \right. \right. \\ \left. \left. \langle (125,128,132,140); 0.9, 0.1 \rangle x_{21} y_{11} + \langle (175,185,195,200); 0.5, 0.4 \rangle x_{21} y_{21} \right) \right\}$$

Subject to

$$x_{11} + x_{21} = 1; x_{11}, x_{21} \geq 0; y_{11} + y_{21} = 1; y_{11}, y_{21} \geq 0.$$

**Problem 4.7.20**

$$\text{Maximize } \left\{ A_{\lambda} \left( \langle (175,180,180,190); 0.6, 0.3 \rangle x_{11} y_{11} + \langle (150,156,156,158); 0.5, 0.2 \rangle x_{11} y_{21} + \right. \right. \\ \left. \left. \langle (125,128,132,140); 0.9, 0.1 \rangle x_{21} y_{11} + \langle (175,185,195,200); 0.5, 0.4 \rangle x_{21} y_{21} \right) \right\}$$

Subject to

$$x_{12} + x_{22} = 1; x_{12}, x_{22} \geq 0; y_{12} + y_{22} = 1; y_{12}, y_{22} \geq 0.$$

**Step 2:** Since, in Problem 4.7.19 and Problem 4.7.20, only  $\{y_{j1}^*, j=1,2\} \in Y$  and

$\{y_{j2}^*, j=1,2\} \in Y$  have been considered as decision variables. So, Problem 4.7.19 and Problem

4.7.20 are linear programming problems and hence, the optimal value of Problem 4.7.19 and

Problem 4.7.20 will be equal to optimal value of its corresponding dual problem i.e., Problem

4.7.21 and Problem 4.7.22 respectively.

**Problem 4.7.21**

$$\text{Maximize } \{v_1\}$$

Subject to

$$V_{\lambda} \left( \langle (175,180,180,190); 0.5, 0.4 \rangle x_{11} + \langle (125,128,132,140); 0.5, 0.4 \rangle x_{21} \right) \geq v_1;$$

$$V_{\lambda} \left( \langle (150,156,156,158); 0.5, 0.4 \rangle x_{11} + \langle (175,185,195,200); 0.5, 0.4 \rangle x_{21} \right) \geq v_1;$$

$$x_{11} + x_{21} = 1; x_{11}, x_{21} \geq 0.$$

**Problem 4.7.22**

Minimize  $\{v_2\}$

Subject to

$$A_{\lambda} \left( \langle (175, 180, 180, 190); 0.5, 0.4 \rangle x_{12} + \langle (125, 128, 132, 140); 0.5, 0.4 \rangle x_{22} \right) \leq v_2;$$

$$A_{\lambda} \left( \langle (150, 156, 156, 158); 0.5, 0.4 \rangle x_{12} + \langle (175, 185, 195, 200); 0.5, 0.4 \rangle x_{22} \right) \leq v_2;$$

$$x_{12} + x_{22} = 1; x_{12}, x_{22} \geq 0.$$

**Step 3:** The optimal solution of Problem 4.7.21 and Problem 4.7.22 is  $\left\{ x_{11}^* = \frac{350}{503}, x_2^* = \frac{153}{503} \right\}$

and  $\{x_1^* = 1, x_2^* = 0\}$  respectively.

**Step 4:** Substituting the optimal solution  $\left\{ x_{11}^* = \frac{350}{503}, x_2^* = \frac{153}{503} \right\}$  and  $\{x_1^* = 1, x_2^* = 0\}$  of Problem

4.7.19 and Problem 4.7.20 in Problem 4.7.17 and Problem 4.7.18 respectively, the alternative

basic optimal solutions are  $\{y_{11}^{1*} = 1, y_{21}^{1*} = 0\}, \{y_{11}^{2*} = 0, y_{21}^{2*} = 1\}$  and  $\{y_{12}^{1*} = 1, y_{22}^{1*} = 0\}$  of Problem

4.7.17 and Problem 4.7.18 respectively.

**Step 5:** Now,

$$\begin{aligned} & \text{minimum} \left\{ \begin{aligned} & \langle (175, 180, 180, 190); 0.6, 0.3 \rangle x_{11}^* y_{11}^{h*} + \langle (150, 156, 156, 158); 0.5, 0.2 \rangle x_{11}^* y_{21}^{h*} + \\ & \langle (125, 128, 132, 140); 0.9, 0.1 \rangle x_{21}^* y_{11}^{h*} + \langle (175, 185, 195, 200); 0.5, 0.4 \rangle x_{21}^* y_{21}^{h*}, h = 1, 2 \\ & \langle (175, 180, 180, 190); 0.6, 0.3 \rangle x_{12}^* y_{12}^* + \langle (150, 156, 156, 158); 0.5, 0.2 \rangle x_{12}^* y_{22}^* + \\ & \langle (125, 128, 132, 140); 0.9, 0.1 \rangle x_{22}^* y_{12}^* + \langle (175, 185, 195, 200); 0.5, 0.4 \rangle x_{22}^* y_{22}^* \end{aligned} \right\}. \\ & = \left\langle \left( \frac{79275}{503}, \frac{82905}{503}, \frac{84435}{503}, \frac{85900}{503} \right), 0.5, 0.4 \right\rangle, \langle (175, 180, 180, 190), 0.5, 0.4 \rangle. \end{aligned}$$

**Step 6:** The trapezoidal intuitionistic fuzzy numbers

$$\left\langle \left( \frac{79275}{503}, \frac{82905}{503}, \frac{84435}{503}, \frac{85900}{503} \right), 0.5, 0.4 \right\rangle \text{ and } \langle (175, 180, 180, 190), 0.5, 0.4 \rangle \text{ represent the}$$

minimum expected gain of Player I and corresponding optimal strategies are

$$\left\{ x_{11}^* = \frac{350}{503}, x_2^* = \frac{153}{503} \right\} \text{ and } \{x_1^* = 1, x_2^* = 0\} \text{ respectively.}$$

#### 4.7.4.2 Maximum expected loss of Player II

Using Ambika method-IV, the maximum expected loss of Player II and the corresponding optimal strategies can be obtained as follows:

**Step 1:** Find  $\{x_{i1}^*, i=1, 2\} \in X$  such that value of

$$V_\lambda \left( \begin{array}{l} \langle (175, 180, 180, 190); 0.6, 0.3 \rangle x_{11} y_{11} + \langle (150, 156, 156, 158); 0.5, 0.2 \rangle x_{11} y_{21} + \\ \langle (125, 128, 132, 140); 0.9, 0.1 \rangle x_{21} y_{11} + \langle (175, 185, 195, 200); 0.5, 0.4 \rangle x_{21} y_{21} \end{array} \right), \text{ is maximum}$$

for all  $\{y_{j1}, j=1, 2\} \in Y$  as well as find  $\{x_{i2}^*, i=1, 2\} \in X$  such that value of

$$A_\lambda \left( \begin{array}{l} \langle (175, 180, 180, 190); 0.6, 0.3 \rangle x_{12} y_{12} + \langle (150, 156, 156, 158); 0.5, 0.2 \rangle x_{12} y_{22} + \\ \langle (125, 128, 132, 140); 0.9, 0.1 \rangle x_{22} y_{12} + \langle (175, 185, 195, 200); 0.5, 0.4 \rangle x_{22} y_{22} \end{array} \right), \text{ is minimum}$$

for all  $\{y_{j2}, j=1, 2\} \in Y$  i.e., find optimal solution  $\{y_{j1}^*, j=1, 2\}$  and  $\{y_{j2}^*, j=1, 2\}$  of Problem 4.7.23 and Problem 4.7.24 respectively.

##### Problem 4.7.23

$$\text{Maximize} \left\{ V_\lambda \left( \begin{array}{l} \langle (175, 180, 180, 190); 0.6, 0.3 \rangle x_{11} y_{11} + \langle (150, 156, 156, 158); 0.5, 0.2 \rangle x_{11} y_{21} + \\ \langle (125, 128, 132, 140); 0.9, 0.1 \rangle x_{21} y_{11} + \langle (175, 185, 195, 200); 0.5, 0.4 \rangle x_{21} y_{21} \end{array} \right) \right\}$$

Subject to

$$x_{11} + x_{21} = 1; x_{11}, x_{21} \geq 0; y_{11} + y_{21} = 1; y_{11}, y_{21} \geq 0.$$

##### Problem 4.7.24

$$\text{Minimize} \left\{ A_\lambda \left( \begin{array}{l} \langle (175, 180, 180, 190); 0.6, 0.3 \rangle x_{11} y_{11} + \langle (150, 156, 156, 158); 0.5, 0.2 \rangle x_{11} y_{21} + \\ \langle (125, 128, 132, 140); 0.9, 0.1 \rangle x_{21} y_{11} + \langle (175, 185, 195, 200); 0.5, 0.4 \rangle x_{21} y_{21} \end{array} \right) \right\}$$

Subject to

$$x_{12} + x_{22} = 1; x_{12}, x_{22} \geq 0; y_{12} + y_{22} = 1; y_{12}, y_{22} \geq 0.$$

**Step 2:** Since, in Problem 4.7.23 and Problem 4.7.24, only  $x_{i1}, i=1, 2$  and  $x_{i2}, i=1, 2$  have been considered as decision variables. So, Problem 4.7.23 and Problem 4.7.24 are linear

programming problems and hence, the optimal value of Problem 4.7.23 and Problem 4.7.24 will be equal to optimal value of its corresponding dual problem i.e., Problem 4.7.25 and Problem 4.7.26 respectively.

**Problem 4.7.25**

Minimize  $\{\omega_1\}$

Subject to

$$V_{\lambda} \left( \langle (175, 180, 180, 190); 0.5, 0.4 \rangle y_{11} + \langle (150, 156, 156, 158); 0.5, 0.4 \rangle y_{21} \right) \leq \omega_1;$$

$$V_{\lambda} \left( \langle (125, 128, 132, 140); 0.5, 0.4 \rangle y_{11} + \langle (175, 185, 195, 200); 0.5, 0.4 \rangle y_{21} \right) \leq \omega_1;$$

$$y_{11} + y_{21} = 1; y_{11}, y_{21} \geq 0.$$

**Problem 4.7.26**

Maximize  $\{\nu_2\}$

Subject to

$$A_{\lambda} \left( \langle (175, 180, 180, 190); 0.5, 0.4 \rangle y_{12} + \langle (150, 156, 156, 158); 0.5, 0.4 \rangle y_{22} \right) \geq \omega_2;$$

$$A_{\lambda} \left( \langle (125, 128, 132, 140); 0.5, 0.4 \rangle y_{12} + \langle (175, 185, 195, 200); 0.5, 0.4 \rangle y_{22} \right) \geq \omega_2;$$

$$y_{11} + y_{21} = 1; y_{11}, y_{21} \geq 0.$$

**Step 3:** The optimal solution of Problem 4.7.25 and Problem 4.7.26 is  $\left\{ y_{11}^* = \frac{203}{503}, y_{21}^* = \frac{300}{503} \right\}$

and  $\{y_{12}^* = 1, y_{22}^* = 0\}$  respectively.

**Step 4:** Substituting the optimal solution  $\left\{ y_{11}^* = \frac{203}{503}, y_{21}^* = \frac{300}{503} \right\}$  and  $\{y_{12}^* = 1, y_{22}^* = 0\}$  of

Problem 4.7.25 and Problem 4.7.26 in Problem 4.7.23 and Problem 4.7.24 respectively, the

basic optimal solutions of Problem 4.7.23 and Problem 4.7.24

$\{x_{11}^{1*} = 1, x_{21}^{1*} = 0\}, \{x_{11}^{2*} = 0, x_{21}^{2*} = 1\}$  and  $\{x_{12}^* = 1, x_{21}^* = 0\}$  respectively.

**Step 5:** Now,

$$\begin{aligned} & \text{maximum} \left\{ \begin{aligned} & \langle (175, 180, 180, 190); 0.6, 0.3 \rangle x_{11}^* y_{11}^* + \langle (150, 156, 156, 158); 0.5, 0.2 \rangle x_{11}^* y_{21}^* + \\ & \langle (125, 128, 132, 140); 0.9, 0.1 \rangle x_{21}^* y_{11}^* + \langle (175, 185, 195, 200); 0.5, 0.4 \rangle x_{21}^* y_{21}^*, t = 1, 2 \\ & \langle (175, 180, 180, 190); 0.6, 0.3 \rangle x_{12}^* y_{12}^* + \langle (150, 156, 156, 158); 0.5, 0.2 \rangle x_{12}^* y_{22}^* + \\ & \langle (125, 128, 132, 140); 0.9, 0.1 \rangle x_{22}^* y_{12}^* + \langle (175, 185, 195, 200); 0.5, 0.4 \rangle x_{22}^* y_{22}^* \end{aligned} \right\} \\ & = \left\langle \left( \frac{80525}{503}, \frac{83340}{503}, \frac{83340}{503}, \frac{85970}{503} \right), 0.5, 0.4 \right\rangle, \langle (175, 180, 180, 190), 0.5, 0.4 \rangle \end{aligned}$$

**Step 6:** The trapezoidal intuitionistic fuzzy numbers

$$= \left\langle \left( \frac{80525}{503}, \frac{83340}{503}, \frac{83340}{503}, \frac{85970}{503} \right), 0.5, 0.4 \right\rangle \text{ and } \langle (175, 180, 180, 190), 0.5, 0.4 \rangle \text{ represent the}$$

maximum expected loss of Player II and the corresponding optimal strategy is

$$\left\{ y_{11}^* = \frac{203}{503}, y_{21}^* = \frac{300}{503} \right\} \text{ and } \{ y_{12}^* = 1, y_{22}^* = 0 \}, \text{ which is optimal solution of Problem 4.7.25 and}$$

Problem 4.7.26 respectively.

#### 4.8 Conclusion

On the basis of present study, it can be concluded that some mathematically incorrect assumptions have been considered in the existing methods [78,95-97] for solving matrix games with intuitionistic fuzzy payoffs. Therefore, it is not desirable to use these methods. Furthermore, to resolve flaws of the existing methods [78,95-97], new methods (named as Ambika methods) is proposed for solving matrix games with intuitionistic fuzzy payoffs.

# Chapter 5

## Mehar method for solving bimatrix games with intuitionistic fuzzy payoffs\*\*

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Li and Yang [80] pointed out that there is no method in the literature for solving such bimatrix games or two person non-zero sum games (matrix games in which gain of one player is not equal to the loss of other player) in which payoffs are represented by intuitionistic fuzzy numbers and proposed a method for the same. In this chapter, it is pointed out that Li and Yang have considered some mathematically incorrect assumptions in their proposed method. For resolving the shortcomings of Li and Yang's method, a new method (named as Mehar method) is proposed. Also, the exact optimal solution of the numerical problem, solved by Li and Yang by their proposed method, is obtained by the proposed Mehar method.

### 5.1. The difference-index based ranking method

In this section, existing difference-index based ranking method [67], used by Li and Yang [80] for comparing trapezoidal intuitionistic fuzzy numbers, is presented [80, Section 2.3, Definition 8, pp. 4].

If  $\tilde{a}_1 = \langle (a_1^L(0), a_1^L(1), a_1^R(1), a_1^R(0)); w_{\tilde{a}_1}, u_{\tilde{a}_1} \rangle$  and  $\tilde{a}_2 = \langle (a_2^L(0), a_2^L(1), a_2^R(1), a_2^R(0)); w_{\tilde{a}_2}, u_{\tilde{a}_2} \rangle$  are two trapezoidal intuitionistic fuzzy numbers and  $\lambda \in [0,1]$ . Then,

- i.  $\tilde{a}_1 \succ \tilde{a}_2$  if and only if  $D_\lambda(\tilde{a}_1) > D_\lambda(\tilde{a}_2)$ .
- ii.  $\tilde{a}_1 \succeq \tilde{a}_2$  if and only if  $D_\lambda(\tilde{a}_1) \geq D_\lambda(\tilde{a}_2)$ .

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iii.  $\tilde{a}_1 \approx \tilde{a}_2$  if and only if  $D_\lambda(\tilde{a}_1) = D_\lambda(\tilde{a}_2)$ .

where,

$$\begin{aligned} D_\lambda(\tilde{a}_i) &= V_\lambda(\tilde{a}_i) - A_\lambda(\tilde{a}_i) \\ &= V_\lambda\left(\left\langle\left(a_i^L(0), a_i^L(1), a_i^R(1), a_i^R(0)\right); w_{\tilde{a}_i}, u_{\tilde{a}_i}\right\rangle\right) - A_\lambda\left(\left\langle\left(a_i^L(0), a_i^L(1), a_i^R(1), a_i^R(0)\right); w_{\tilde{a}_i}, u_{\tilde{a}_i}\right\rangle\right) \\ &= \left(\lambda(1 - u_{\tilde{a}_i}) + (1 - \lambda)(w_{\tilde{a}_i})\right) \left(\frac{a_i^L(0) + 2a_i^L(1) + 2a_i^R(1) + a_i^R(0)}{12}\right) - \\ &\quad \left(\lambda(w_{\tilde{a}_i}) + (1 - \lambda)(1 - u_{\tilde{a}_i})\right) \left(\frac{a_i^R(0) - a_i^L(0) + 2a_i^R(1) - 2a_i^L(1)}{12}\right). \end{aligned}$$

## 5.2 Maximum of trapezoidal intuitionistic fuzzy numbers

In this section, the method, used by Li and Yang [80], for finding the maximum of trapezoidal intuitionistic fuzzy numbers, is presented.

It is obvious from Section 5.1 that if

$\tilde{a}_i = \left\langle\left(a_i^L(0), a_i^L(1), a_i^R(1), a_i^R(0)\right); w_{\tilde{a}_i}, u_{\tilde{a}_i}\right\rangle$ ,  $i = 1, 2, \dots, n$  are  $n$  – trapezoidal intuitionistic fuzzy

numbers and  $\lambda \in [0, 1]$ . Then, maximum of these numbers can be obtained as follows:

**Step 1:** Find

$$\begin{aligned} D_\lambda(\tilde{a}_i) &= V_\lambda(\tilde{a}_i) - A_\lambda(\tilde{a}_i) \\ &= V_\lambda\left(\left\langle\left(a_i^L(0), a_i^L(1), a_i^R(1), a_i^R(0)\right); w_{\tilde{a}_i}, u_{\tilde{a}_i}\right\rangle\right) - A_\lambda\left(\left\langle\left(a_i^L(0), a_i^L(1), a_i^R(1), a_i^R(0)\right); w_{\tilde{a}_i}, u_{\tilde{a}_i}\right\rangle\right) \\ &= \left(\lambda(1 - u_{\tilde{a}_i}) + (1 - \lambda)(w_{\tilde{a}_i})\right) \left(\frac{a_i^L(0) + 2a_i^L(1) + 2a_i^R(1) + a_i^R(0)}{12}\right) - \\ &\quad \left(\lambda(w_{\tilde{a}_i}) + (1 - \lambda)(1 - u_{\tilde{a}_i})\right) \left(\frac{a_i^R(0) - a_i^L(0) + 2a_i^R(1) - 2a_i^L(1)}{12}\right). \end{aligned}$$

**Step 2:** Find  $\max_{1 \leq i \leq n} \{D_\lambda(\tilde{a}_i)\}$ .

**Step 3:** If  $\max_{1 \leq i \leq n} \{D_\lambda(\tilde{a}_i)\} = D_\lambda(\tilde{a}_p)$  (say). Then,  $\max_{1 \leq i \leq n} \{\tilde{a}_i\} = \tilde{a}_p$ .

## 5.3. Flaws in the existing mathematical formulation of bimatrix games with intuitionistic fuzzy payoffs

The aim of this section is to point out that Li and Yang [80] have used some

mathematically incorrect assumption to obtain the mathematical formulation of bimatrix games with intuitionistic fuzzy payoffs. To point out the same, it is necessary to explain the procedure followed by Li and Yang [80] to obtain this mathematical formulation. So, firstly the procedure followed by Li and Yang [80], to obtain the mathematical formulation of bimatrix games with intuitionistic fuzzy payoffs, is presented and then the mathematically incorrect assumptions, considered by Li and Yang [80], are pointed out.

### 5.3.1 Mathematical formulation of bimatrix games with intuitionistic fuzzy payoffs

Mangasarian and Stone [86] proved that a Nash equilibrium point (a pair of strategies where the objectives of both the players are fulfilled simultaneously) for bimatrix games (two person non-zero sum games) can be obtained by solving a quadratic programming problem. On the same direction, Li and Yang [80] obtained a quadratic Problem 5.3.1 to obtain a Nash equilibrium point for bimatrix games with intuitionistic fuzzy payoffs.

#### Problem 5.3.1 [80, equation 22, Section 3.2, pp. 5]

$$\text{Maximize} \left( \sum_{i=1}^m \sum_{j=1}^n x_i \left( D_{\lambda_1}(\tilde{a}_{ij}) + D_{\lambda_2}(\tilde{b}_{ij}) \right) y_j - u - v \right)$$

Subject to

$$\begin{aligned} \sum_{j=1}^n D_{\lambda_1}(\tilde{a}_{ij}) y_j - u &\leq 0, \quad i = 1, 2, \dots, m; & \sum_{i=1}^m D_{\lambda_2}(\tilde{b}_{ij}) x_i - v &\leq 0, \quad j = 1, 2, \dots, n; \\ \sum_{i=1}^m x_i &= 1; & \sum_{j=1}^n y_j &= 1; & x_i &\geq 0, \quad i = 1, 2, \dots, m; & y_j &\geq 0, \quad j = 1, 2, \dots, n. \end{aligned}$$

To point out the mathematically incorrect assumption considered by Li and Yang [80], it is necessary to explain the method, followed by Li and Yang [80], to obtain the mathematical formulation i.e. Problem 5.3.1 of bimatrix games with intuitionistic fuzzy payoffs. Therefore, in this section, the same is presented.

Let, Player I and Player II have mixed strategies as  $x_i, i = 1, 2, \dots, m$  and  $y_j, j = 1, 2, \dots, n$  respectively. Let,  $\tilde{A}_{m \times n} = (\tilde{a}_{ij})_{m \times n}$  and  $\tilde{B}_{m \times n} = (\tilde{b}_{ij})_{m \times n}$  be intuitionistic fuzzy

payoff matrices of Player I and Player II respectively. Player I maximizes profit over rows of fuzzy matrix  $\tilde{A}_{m \times n} = (\tilde{a}_{ij})_{m \times n}$  and Player II maximizes profit over columns of fuzzy matrix

$$\tilde{B}_{m \times n} = (\tilde{b}_{ij})_{m \times n}.$$

Therefore, the objective of Player I is to

$$\text{Maximize} \left( \sum_{i=1}^m \sum_{j=1}^n x_i \tilde{a}_{ij} y_j \right)$$

Subject to

$$\begin{aligned} \sum_{i=1}^m x_i &= 1, \\ x_i &\geq 0, \quad i = 1, 2, \dots, m. \end{aligned}$$

and

the objective of Player II is to

$$\text{Maximize} \left( \sum_{j=1}^n \sum_{i=1}^m x_i \tilde{b}_{ij} y_j \right)$$

Subject to

$$\begin{aligned} \sum_{j=1}^n y_j &= 1, \\ y_j &\geq 0, \quad j = 1, 2, \dots, n. \end{aligned}$$

Using Section 5.1, the objective of Player I is to

$$\text{Maximize} \left( D_{\lambda_1} \left( \sum_{i=1}^m \sum_{j=1}^n x_i \tilde{a}_{ij} y_j \right) \right)$$

Subject to

$$\begin{aligned} \sum_{i=1}^m x_i &= 1, \\ x_i &\geq 0, \quad i = 1, 2, \dots, m. \end{aligned}$$

and the objective of Player II is to

$$\text{Maximize} \left( D_{\lambda_2} \left( \sum_{j=1}^n \sum_{i=1}^m x_i \tilde{b}_{ij} y_j \right) \right)$$

Subject to

$$\sum_{j=1}^n y_j = 1,$$

$$y_j \geq 0, \quad j = 1, 2, \dots, n.$$

Li and Yang [80] assumed that to Maximize  $\left( D_{\lambda_1} \left( \sum_{i=1}^m \sum_{j=1}^n x_i \tilde{a}_{ij} y_j \right) \right)$  is equivalent to

Maximize  $\left( \sum_{i=1}^m \sum_{j=1}^n x_i D_{\lambda_1} (\tilde{a}_{ij}) y_j \right)$  and to Maximize  $\left( D_{\lambda_2} \left( \sum_{j=1}^n \sum_{i=1}^m x_i \tilde{b}_{ij} y_j \right) \right)$  is equivalent to

$$\text{Maximize} \left( \sum_{j=1}^n \sum_{i=1}^m x_i D_{\lambda_2} (\tilde{b}_{ij}) y_j \right).$$

Therefore, the objective of Player I is to

$$\text{Maximize} \left\{ \sum_{i=1}^m \sum_{j=1}^n x_i D_{\lambda_1} (\tilde{a}_{ij}) y_j \right\}$$

Subject to

$$\sum_{i=1}^m x_i = 1,$$

$$x_i \geq 0, \quad i = 1, 2, \dots, m.$$

and the objective of Player II is to

$$\text{Maximize} \left\{ \sum_{j=1}^n \sum_{i=1}^m x_i D_{\lambda_2} (\tilde{b}_{ij}) y_j \right\}$$

Subject to

$$\sum_{j=1}^n y_j = 1,$$

$$y_j \geq 0, \quad j = 1, 2, \dots, n.$$

According to Mangasarian and Stone [86], the point  $(x_i^0, y_j^0)$  will be a Nash equilibrium point (a pair of strategies  $x_i^0, y_j^0$  where the objectives of both the players are fulfilled simultaneously) if there exist real numbers  $u^0, v^0$  such that  $x_i^0, y_j^0, u^0, v^0$  satisfy the following conditions:

$$\begin{aligned} \sum_{i=1}^m \sum_{j=1}^n (x_i^0 D_{\lambda_1}(\tilde{a}_{ij}) y_j^0) - u^0 &= 0; & \sum_{j=1}^n \sum_{i=1}^m (x_i^0 D_{\lambda_2}(\tilde{b}_{ij}) y_j^0) - v^0 &= 0; \\ \sum_{j=1}^n (D_{\lambda_1}(\tilde{a}_{ij}) y_j^0) - u^0 &\leq 0, \quad i = 1, 2, \dots, m; & \sum_{i=1}^m (D_{\lambda_2}(\tilde{b}_{ij}) x_i^0) - v^0 &\leq 0, \quad j = 1, 2, \dots, n; \\ \sum_{i=1}^m x_i^0 &= 1; & \sum_{j=1}^n y_j^0 &= 1; & x_i^0 &\geq 0, \quad i = 1, 2, \dots, m; & y_j^0 &\geq 0, \quad j = 1, 2, \dots, n. \end{aligned}$$

Also, according to Mangasarian and Stone [86], the values of  $x_i^0, y_j^0, u^0, v^0$  which will satisfy the above conditions will be optimal solution of Problem 5.3.1.

### 5.3.2. Mathematically incorrect assumption considered by Li and Yang

It is obvious from Section 5.3.1 that to obtain the mathematical formulation i.e.,

Problem 5.3.1, Li and Yang [80] have assumed that to Maximize  $\left( D_{\lambda_1} \left( \sum_{i=1}^m \sum_{j=1}^n x_i \tilde{a}_{ij} y_j \right) \right)$  is

equivalent to Maximize  $\left( \sum_{i=1}^m \sum_{j=1}^n x_i D_{\lambda_1}(\tilde{a}_{ij}) y_j \right)$  and to Maximize  $\left( D_{\lambda_2} \left( \sum_{j=1}^n \sum_{i=1}^m x_i \tilde{b}_{ij} y_j \right) \right)$  is

equivalent to Maximize  $\left( \sum_{j=1}^n \sum_{i=1}^m x_i D_{\lambda_2}(\tilde{b}_{ij}) y_j \right)$ .

However,

$$\begin{aligned} & D_{\lambda} \left( \sum_{i=1}^n \tilde{a}_i \right) \\ &= D_{\lambda} \left( \sum_{i=1}^n \left\langle (a_i^L(0), a_i^L(1), a_i^R(1), a_i^R(0)); w_{\tilde{a}_i}, u_{\tilde{a}_i} \right\rangle \right) \\ &= D_{\lambda} \left( \left\langle \left( \sum_{i=1}^n a_i^L(0), \sum_{i=1}^n a_i^L(1), \sum_{i=1}^n a_i^R(1), \sum_{i=1}^n a_i^R(0) \right); \min_{1 \leq i \leq n} \{w_{\tilde{a}_i}\}, \max_{1 \leq i \leq n} \{u_{\tilde{a}_i}\} \right\rangle \right) \end{aligned}$$

$$\begin{aligned}
&= D_\lambda \left( \left\langle \left( \sum_{i=1}^n a_i^L(0), \sum_{i=1}^n a_i^L(1), \sum_{i=1}^n a_i^R(1), \sum_{i=1}^n a_i^R(0) \right); \min_{1 \leq i \leq n} \{w_{\tilde{a}_i}\}, \min_{1 \leq i \leq n} \{(1-u_{\tilde{a}_i})\} \right\rangle \right) \\
&= V_\lambda \left( \left\langle \left( \sum_{i=1}^n a_i^L(0), \sum_{i=1}^n a_i^L(1), \sum_{i=1}^n a_i^R(1), \sum_{i=1}^n a_i^R(0) \right); \min_{1 \leq i \leq n} \{w_{\tilde{a}_i}\}, \min_{1 \leq i \leq n} \{(1-u_{\tilde{a}_i})\} \right\rangle \right) - \\
&\quad A_\lambda \left( \left\langle \left( \sum_{i=1}^n a_i^L(0), \sum_{i=1}^n a_i^L(1), \sum_{i=1}^n a_i^R(1), \sum_{i=1}^n a_i^R(0) \right); \min_{1 \leq i \leq n} \{w_{\tilde{a}_i}\}, \min_{1 \leq i \leq n} \{(1-u_{\tilde{a}_i})\} \right\rangle \right) \\
&= \left( \lambda \min_{1 \leq i \leq n} \{(1-u_{\tilde{a}_i})\} + (1-\lambda) \min_{1 \leq i \leq n} \{w_{\tilde{a}_i}\} \right) \left( \frac{\sum_{i=1}^n a_i^L(0) + 2 \sum_{i=1}^n a_i^L(1) + 2 \sum_{i=1}^n a_i^R(1) + \sum_{i=1}^n a_i^R(0)}{12} \right) - \\
&\quad \left( \lambda \min_{1 \leq i \leq n} \{w_{\tilde{a}_i}\} + (1-\lambda) \min_{1 \leq i \leq n} \{(1-u_{\tilde{a}_i})\} \right) \left( \frac{\sum_{i=1}^n a_i^R(0) - \sum_{i=1}^n a_i^L(0) + 2 \sum_{i=1}^n a_i^R(1) - 2 \sum_{i=1}^n a_i^L(1)}{12} \right) \quad (5.1)
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{i=1}^n D_\lambda(\tilde{a}_i) \\
&= \sum_{i=1}^n D_\lambda \left( \left\langle (a_i^L(0), a_i^L(1), a_i^R(1), a_i^R(0)); w_{\tilde{a}_i}, u_{\tilde{a}_i} \right\rangle \right) \\
&= \sum_{i=1}^n \left( V_\lambda \left( \left\langle (a_i^L(0), a_i^L(1), a_i^R(1), a_i^R(0)); w_{\tilde{a}_i}, u_{\tilde{a}_i} \right\rangle \right) - A_\lambda \left( \left\langle (a_i^L(0), a_i^L(1), a_i^R(1), a_i^R(0)); w_{\tilde{a}_i}, u_{\tilde{a}_i} \right\rangle \right) \right) \\
&= \sum_{i=1}^n \left( \left( \left( \lambda(1-u_{\tilde{a}_i}) + (1-\lambda)w_{\tilde{a}_i} \right) \left( \frac{a_i^L(0) + 2a_i^L(1) + 2a_i^R(1) + a_i^R(0)}{12} \right) \right) - \right. \\
&\quad \left. \left( \left( \lambda w_{\tilde{a}_i} + (1-\lambda)(1-u_{\tilde{a}_i}) \right) \left( \frac{a_i^R(0) - a_i^L(0) + 2a_i^R(1) - 2a_i^L(1)}{12} \right) \right) \right) \quad (5.2)
\end{aligned}$$

It is obvious from equations (5.1) and (5.2) that  $D_\lambda \left( \sum_{i=1}^n \tilde{a}_i \right) \neq \sum_{i=1}^n D_\lambda(\tilde{a}_i)$ . Li and Yang

[80, Theorem 7, pp. 3] have also pointed out that  $D_\lambda \left( \sum_{i=1}^n \tilde{a}_i \right) = \sum_{i=1}^n D_\lambda(\tilde{a}_i)$  if and only if for

$w_{\tilde{a}_i}, i=1,2,\dots,n$  and  $u_{\tilde{a}_i}, i=1,2,\dots,n$ , the conditions  $w_{\tilde{a}_1} = w_{\tilde{a}_2} = \dots = w_{\tilde{a}_n}$  and  $u_{\tilde{a}_1} = u_{\tilde{a}_2} = \dots = u_{\tilde{a}_n}$  will be satisfied.

This clearly indicates that to Maximize  $\left( D_{\lambda_1} \left( \sum_{i=1}^m \sum_{j=1}^n x_i \tilde{a}_{ij} y_j \right) \right)$  is not equivalent to

Maximize  $\left( \sum_{i=1}^m \sum_{j=1}^n x_i D_{\lambda_1} (\tilde{a}_{ij}) y_j \right)$  and to Maximize  $\left( D_{\lambda_2} \left( \sum_{j=1}^n \sum_{i=1}^m x_i \tilde{b}_{ij} y_j \right) \right)$  is not equivalent to

Maximize  $\left( \sum_{j=1}^n \sum_{i=1}^m x_i D_{\lambda_2} (\tilde{b}_{ij}) y_j \right)$ .

However, Li and Yang [80] have considered the above mentioned mathematically incorrect assumption for obtaining the mathematical formulation i.e., Problem 5.3.1.

Therefore, the mathematical formulation i.e., Problem 5.3.1 and hence the method, proposed by Li and Yang [80] for solving bimatrix games with intuitionistic fuzzy payoffs based on the mathematical formulation i.e., Problem 5.3.1, is not valid.

#### 5.4. Exact solution of bimatrix games with intuitionistic fuzzy payoffs

Li and Yang [80] claimed that optimal solution  $\{x_i, y_j, u, v\}$  of any bimatrix games with intuitionistic fuzzy payoffs can be obtained by solving Problem 5.3.1. However, as discussed in Section 5.3.2 that the Problem 5.3.1 is not the exact mathematical formulation of bimatrix games with intuitionistic fuzzy payoffs. Therefore, it is not desirable to use Li and Yang's method [80] for solving bimatrix games with intuitionistic payoffs. In this section, the exact mathematical formulation of bimatrix games with intuitionistic fuzzy payoffs is proposed. Also, a new method (named as Mehar method) to find the exact solution of bimatrix games with intuitionistic fuzzy payoffs is proposed. The necessary and sufficient condition for the convergence of bimatrix games with intuitionistic fuzzy payoffs is also discussed.

### 5.4.1 Exact mathematical formulation of bimatrix games with intuitionistic fuzzy

#### payoffs

It can be easily verified from Section 5.1 that  $D_\lambda \left( \sum_{i=1}^n \tilde{a}_i \right) = \sum_{i=1}^n M_\lambda (\tilde{a}_i)$ ,

$$\text{where, } M_\lambda (\tilde{a}_i) = \left( \lambda \xi + (1 - \lambda) \kappa \right) \left( \frac{a_i^L(0) + 2a_i^L(1) + 2a_i^R(1) + a_i^R(0)}{12} \right) - \\ \left( \lambda \kappa + (1 - \lambda) \xi \right) \left( \frac{a_i^R(0) - a_i^L(0) + 2a_i^R(1) - 2a_i^L(1)}{12} \right),$$

$$\xi = \min_{1 \leq i \leq n} \left\{ \left( 1 - u_{\tilde{a}_i} \right) \right\}, \kappa = \min_{1 \leq i \leq n} \left\{ w_{\tilde{a}_i} \right\}.$$

Therefore, using the relation  $D_\lambda \left( \sum_{i=1}^n \tilde{a}_i \right) = \sum_{i=1}^n M_\lambda (\tilde{a}_i)$  in the procedure, for obtaining

mathematical formulation of bimatrix games with intuitionistic fuzzy payoffs described in Section 5.3.1, the exact mathematical formulation i.e., Problem 5.4.1 of bimatrix games with intuitionistic fuzzy payoffs is obtained.

#### Problem 5.4.1

$$\text{Maximize} \left( \sum_{i=1}^m \sum_{j=1}^n x_i \left( M_{\lambda_1} (\tilde{a}_{ij}) + M_{\lambda_2} (\tilde{b}_{ij}) \right) y_j - u - v \right)$$

Subject to

$$\sum_{j=1}^n M_{\lambda_1} (\tilde{a}_{ij}) y_j - u \leq 0, \quad i = 1, 2, \dots, m; \quad \sum_{i=1}^m M_{\lambda_2} (\tilde{b}_{ij}) x_i - v \leq 0, \quad j = 1, 2, \dots, n; \\ \sum_{i=1}^m x_i = 1; \quad \sum_{j=1}^n y_j = 1; \quad x_i \geq 0, \quad i = 1, 2, \dots, m; \quad y_j \geq 0, \quad j = 1, 2, \dots, n.$$

where

$$M_{\lambda_1} (\tilde{a}_{ij}) = \left( \lambda_1 \alpha + (1 - \lambda_1) \beta \right) \left( \frac{a_{ij}^L(0) + 2a_{ij}^L(1) + 2a_{ij}^R(1) + a_{ij}^R(0)}{12} \right) - \\ \left( \lambda_1 \beta + (1 - \lambda_1) \alpha \right) \left( \frac{a_{ij}^R(0) - a_{ij}^L(0) + 2a_{ij}^R(1) - 2a_{ij}^L(1)}{12} \right),$$

$$M_{\lambda_2}(\tilde{b}_{ij}) = (\lambda_2 \psi + (1 - \lambda_2) \tau) \left( \frac{b_{ij}^L(0) + 2b_{ij}^L(1) + 2b_{ij}^R(1) + b_{ij}^R(0)}{12} \right) - \\ (\lambda_2 \tau + (1 - \lambda_2) \psi) \left( \frac{b_{ij}^R(0) - b_{ij}^L(0) + 2b_{ij}^R(1) - 2b_{ij}^L(1)}{12} \right),$$

$$\alpha = \min_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}} \left\{ (1 - u_{\tilde{a}_{ij}}) \right\}, \beta = \min_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}} \left\{ w_{\tilde{a}_{ij}} \right\}, \psi = \min_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}} \left\{ (1 - u_{\tilde{b}_{ij}}) \right\}, \tau = \min_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}} \left\{ w_{\tilde{b}_{ij}} \right\}.$$

#### 5.4.2 Proposed Mehar method

In this section, a new method (named as Mehar method) is proposed for solving bimatrix games with intuitionistic fuzzy payoffs.

The steps of the proposed Mehar method are as follows:

**Step 1:** Find the values of

$$\alpha = \min_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}} \left\{ (1 - u_{\tilde{a}_{ij}}) \right\}, \beta = \min_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}} \left\{ w_{\tilde{a}_{ij}} \right\}, \psi = \min_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}} \left\{ (1 - u_{\tilde{b}_{ij}}) \right\}, \tau = \min_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}} \left\{ w_{\tilde{b}_{ij}} \right\}.$$

**Step 2:** Find the values of

$$M_{\lambda_1}(\tilde{a}_{ij}) = (\lambda_1 \alpha + (1 - \lambda_1) \beta) \left( \frac{a_{ij}^L(0) + 2a_{ij}^L(1) + 2a_{ij}^R(1) + a_{ij}^R(0)}{12} \right) - \\ (\lambda_1 \beta + (1 - \lambda_1) \alpha) \left( \frac{a_{ij}^R(0) - a_{ij}^L(0) + 2a_{ij}^R(1) - 2a_{ij}^L(1)}{12} \right), \\ M_{\lambda_2}(\tilde{b}_{ij}) = (\lambda_2 \psi + (1 - \lambda_2) \tau) \left( \frac{b_{ij}^L(0) + 2b_{ij}^L(1) + 2b_{ij}^R(1) + b_{ij}^R(0)}{12} \right) - \\ (\lambda_2 \tau + (1 - \lambda_2) \psi) \left( \frac{b_{ij}^R(0) - b_{ij}^L(0) + 2b_{ij}^R(1) - 2b_{ij}^L(1)}{12} \right).$$

**Step 3:** Put the values of  $M_{\lambda_1}(\tilde{a}_{ij})$  and  $M_{\lambda_2}(\tilde{b}_{ij})$ , obtained from Step 2, in Problem 5.4.1.

**Step 4:** Find the optimal solution  $\{x_i, y_j, u, v\}$  of the problem, obtained in Step 3, by choosing

$$\lambda_1 \in [0, 1] \text{ and } \lambda_2 \in [0, 1].$$

#### 5.4.3 Convergence of the proposed Mehar method

Mangasarian and Stone [86, Section II, pp. 349-350] proved that a bimatrix game with

crisp payoffs can be formulated into a crisp quadratic programming problem. Mangasarian and Stone [86, Section II, pp. 350-351] also proved that the necessary and sufficient condition for the convergence of a bimatrix game with crisp payoffs is that the global optimal value of the corresponding crisp quadratic programming problem is zero.

Following the procedure of Mangasarian and Stone [86, Section II, pp. 349-350] and replacing crisp payoffs  $a_{ij}$  and  $b_{ij}$  by  $D_{\lambda_1}(\tilde{a}_{ij})$  and  $D_{\lambda_2}(\tilde{b}_{ij})$  respectively, Li and Yang [80] proved that a bimatrix game with intuitionistic fuzzy payoffs can be formulated into a quadratic programming Problem 5.3.1. However, in Section 5.3.2, it is pointed out that the Problem 5.3.1 is not the exact mathematical formulation of a bimatrix game with intuitionistic fuzzy payoffs. Also, in Section 5.4.1, it is pointed out that the exact mathematical formulation of a bimatrix game with intuitionistic fuzzy payoffs can be obtained by following the procedure of Mangasarian and Stone [86, Section II, pp. 349-350] and replacing crisp payoffs  $a_{ij}$  and  $b_{ij}$  by  $M_{\lambda_1}(\tilde{a}_{ij})$  and  $M_{\lambda_2}(\tilde{b}_{ij})$  respectively instead of replacing  $a_{ij}$  and  $b_{ij}$  by  $D_{\lambda_1}(\tilde{a}_{ij})$  and  $D_{\lambda_2}(\tilde{b}_{ij})$  respectively.

Similarly, replacing crisp payoffs  $a_{ij}$  and  $b_{ij}$  by  $M_{\lambda_1}(\tilde{a}_{ij})$  and  $M_{\lambda_2}(\tilde{b}_{ij})$  respectively in the existing proof [86, Section II, pp. 350-351], it can be easily proved that the necessary and sufficient condition for the convergence of a bimatrix games with intuitionistic fuzzy payoffs is that the global optimal value of the corresponding mathematical quadratic programming Problem 5.4.1 is zero.

## 5.5. Numerical example

Li and Yang [80, Section 4, pp. 6-8] solved a commerce retailer's strategy choice problem to illustrate their proposed method by considering

$$\tilde{A} = \begin{bmatrix} \langle (50, 60, 70, 80), 0.8, 0.1 \rangle & \langle (30, 40, 70, 80), 0.4, 0.3 \rangle \\ \langle (20, 30, 40, 50), 0.5, 0.4 \rangle & \langle (40, 50, 60, 70), 0.6, 0.2 \rangle \end{bmatrix} \text{ and}$$

$$\tilde{B} = \begin{pmatrix} \langle (40, 50, 60, 70); 0.7, 0.1 \rangle & \langle (30, 40, 50, 60); 0.7, 0.2 \rangle \\ \langle (20, 30, 40, 50); 0.5, 0.3 \rangle & \langle (50, 60, 70, 80); 0.8, 0.1 \rangle \end{pmatrix} \text{ as intuitionistic fuzzy payoffs}$$

matrices of first and second commercial retailer respectively. However, as discussed in Section 5.3.2, that the existing method [80] is not valid. So, the intuitionistic fuzzy optimal solution of commerce retailer's strategy choice problem, obtained by Li and Yang [80], is not exact. Also, it can be easily verified that the value of objective function of Problem 5.3.1 corresponding to the existing optimal solution, obtained by Li and Yang [80] shown in Table 5.1 to Table 5.12, is not zero. While, according to Mangasarian and Stone [86], an optimal solution of Problem 5.3.1 will be Nash equilibrium if and only if the value of objective function of Problem 5.3.1 corresponding to that optimal solution is zero. This clearly indicates that the optimal solution of commerce retailer's strategy choice problem, obtained by Li and Yang [80] shown in Table 5.1 to Table 5.12, are actually not Nash equilibrium points.

In this section, the exact optimal solution (Nash equilibrium points) of this existing problem is obtained by the proposed Mehar method.

Using the proposed Mehar method, the exact optimal solution of this problem can be obtained as follows:

**Step 1:** Using Step 1 of proposed Mehar method

$$\alpha = \min \{(1-0.1), (1-0.3), (1-0.4), (1-0.2)\} = \min \{0.9, 0.7, 0.6, 0.8\} = 0.6,$$

$$\beta = \min \{0.8, 0.4, 0.5, 0.6\} = 0.4,$$

$$\psi = \min \{(1-0.1), (1-0.2), (1-0.3), (1-0.1)\} = \min \{0.9, 0.8, 0.7, 0.9\} = 0.7,$$

$$\tau = \min \{0.7, 0.7, 0.5, 0.8\} = 0.5.$$

**Step 2:** Using Step 2 of the proposed Mehar method,

$$M_{\lambda_1}(\tilde{a}_{11}) = M_{\lambda_1}(\langle (50, 60, 70, 80), 0.8, 0.1 \rangle) = (0.2\lambda_1 + 0.4)\left(\frac{390}{12}\right) - (0.6 - 0.2\lambda_1)\left(\frac{50}{12}\right)$$

$$M_{\lambda_1}(\tilde{a}_{12}) = M_{\lambda_1}(\langle (30, 40, 70, 80), 0.4, 0.3 \rangle) = (0.2\lambda_1 + 0.4)\left(\frac{330}{12}\right) - (0.6 - 0.2\lambda_1)\left(\frac{110}{12}\right)$$

$$\begin{aligned}
M_{\lambda_1}(\tilde{a}_{21}) &= M_{\lambda_1}(\langle(20, 30, 40, 50), 0.5, 0.4\rangle) = (0.2\lambda_1 + 0.4)\left(\frac{210}{12}\right) - (0.6 - 0.2\lambda_1)\left(\frac{50}{12}\right) \\
M_{\lambda_1}(\tilde{a}_{22}) &= M_{\lambda_1}(\langle(40, 50, 60, 70), 0.6, 0.2\rangle) = (0.2\lambda_1 + 0.4)\left(\frac{330}{12}\right) - (0.6 - 0.2\lambda_1)\left(\frac{50}{12}\right) \\
M_{\lambda_2}(\tilde{b}_{11}) &= M_{\lambda_2}(\langle(40, 50, 60, 70); 0.7, 0.1\rangle) = (0.2\lambda_1 + 0.5)\left(\frac{330}{12}\right) - (0.7 - 0.2\lambda_1)\left(\frac{50}{12}\right) \\
M_{\lambda_2}(\tilde{b}_{12}) &= M_{\lambda_2}(\langle(30, 40, 50, 60); 0.7, 0.2\rangle) = (0.2\lambda_1 + 0.5)\left(\frac{270}{12}\right) - (0.7 - 0.2\lambda_1)\left(\frac{50}{12}\right) \\
M_{\lambda_2}(\tilde{b}_{21}) &= M_{\lambda_2}(\langle(20, 30, 40, 50); 0.5, 0.3\rangle) = (0.2\lambda_1 + 0.5)\left(\frac{210}{12}\right) - (0.7 - 0.2\lambda_1)\left(\frac{50}{12}\right) \\
M_{\lambda_2}(\tilde{b}_{22}) &= M_{\lambda_2}(\langle(50, 60, 70, 80); 0.8, 0.1\rangle) = (0.2\lambda_1 + 0.5)\left(\frac{390}{12}\right) - (0.7 - 0.2\lambda_1)\left(\frac{50}{12}\right).
\end{aligned}$$

**Step 3:** Putting the values, obtained in Step 2, in Problem 5.4.1, then Problem 5.4.1 can be transformed into Problem 5.5.1.

### Problem 5.5.1

Maximize

$$\begin{pmatrix}
\left( (0.2\lambda_1 + 0.4)\left(\frac{390}{12}\right) - (0.6 - 0.2\lambda_1)\left(\frac{50}{12}\right) \right) y_1 z_1 + \left( (0.2\lambda_1 + 0.4)\left(\frac{330}{12}\right) - (0.6 - 0.2\lambda_1)\left(\frac{110}{12}\right) \right) y_1 z_2 + \\
\left( (0.2\lambda_1 + 0.4)\left(\frac{210}{12}\right) - (0.6 - 0.2\lambda_1)\left(\frac{50}{12}\right) \right) y_2 z_1 + \left( (0.2\lambda_1 + 0.4)\left(\frac{330}{12}\right) - (0.6 - 0.2\lambda_1)\left(\frac{50}{12}\right) \right) y_2 z_2 + \\
\left( (0.2\lambda_1 + 0.5)\left(\frac{330}{12}\right) - (0.7 - 0.2\lambda_1)\left(\frac{50}{12}\right) \right) y_1 z_1 + \left( (0.2\lambda_1 + 0.5)\left(\frac{270}{12}\right) - (0.7 - 0.2\lambda_1)\left(\frac{50}{12}\right) \right) y_1 z_2 + \\
\left( (0.2\lambda_1 + 0.5)\left(\frac{210}{12}\right) - (0.7 - 0.2\lambda_1)\left(\frac{50}{12}\right) \right) y_2 z_1 + \left( (0.2\lambda_1 + 0.5)\left(\frac{390}{12}\right) - (0.7 - 0.2\lambda_1)\left(\frac{50}{12}\right) \right) y_2 z_2 \\
-u - v
\end{pmatrix}$$

Subject to

$$\begin{aligned}
&\left( (0.2\lambda_1 + 0.4)\left(\frac{390}{12}\right) - (0.6 - 0.2\lambda_1)\left(\frac{50}{12}\right) \right) z_1 + \left( (0.2\lambda_1 + 0.4)\left(\frac{330}{12}\right) - (0.6 - 0.2\lambda_1)\left(\frac{110}{12}\right) \right) z_2 \leq u; \\
&\left( (0.2\lambda_1 + 0.4)\left(\frac{210}{12}\right) - (0.6 - 0.2\lambda_1)\left(\frac{50}{12}\right) \right) z_1 + \left( (0.2\lambda_1 + 0.4)\left(\frac{330}{12}\right) - (0.6 - 0.2\lambda_1)\left(\frac{50}{12}\right) \right) z_2 \leq u;
\end{aligned}$$

$$\begin{aligned} &\left( (0.2\lambda_1 + 0.5)\left(\frac{330}{12}\right) - (0.7 - 0.2\lambda_1)\left(\frac{50}{12}\right) \right) y_1 + \left( (0.2\lambda_1 + 0.5)\left(\frac{210}{12}\right) - (0.7 - 0.2\lambda_1)\left(\frac{50}{12}\right) \right) y_2 \leq v; \\ &\left( (0.2\lambda_1 + 0.5)\left(\frac{270}{12}\right) - (0.7 - 0.2\lambda_1)\left(\frac{50}{12}\right) \right) y_1 + \left( (0.2\lambda_1 + 0.5)\left(\frac{390}{12}\right) - (0.7 - 0.2\lambda_1)\left(\frac{50}{12}\right) \right) y_2 \leq v; \\ &y_1 + y_2 = 1; \\ &z_1 + z_2 = 1; \\ &y_1 \geq 0, y_2 \geq 0, z_1 \geq 0, z_2 \geq 0. \end{aligned}$$

**Step 4:** The exact optimal solution of commerce retailer's strategy choice problem with intuitionistic fuzzy payoffs obtained by solving Problem 5.5.1 as well as the incorrect optimal solution of the same problem obtained by Li and Yang [80] are shown in Table 5.1 to Table 5.12.

**Table 5.1:** Nash equilibrium values and corresponding strategies of first commerce retailer

| Parameters  |             | Li and Yang's method [80, Table 1, pp.7] |                  | Proposed Mehar method                   |                  |
|-------------|-------------|------------------------------------------|------------------|-----------------------------------------|------------------|
| $\lambda_1$ | $\lambda_2$ | $x^{*T}$                                 | $u^*(\lambda_1)$ | $x^{*T}$                                | $u^*(\lambda_1)$ |
| 0           | 0           | (0.484, 0.516)                           | 7.94             | $\left(\frac{3}{4}, \frac{1}{4}\right)$ | $\frac{15}{2}$   |
| 0           | 0.3         | (0.492, 0.508)                           | 7.94             | $\left(\frac{3}{4}, \frac{1}{4}\right)$ | $\frac{15}{2}$   |
| 0           | 0.5         | (0.498, 0.502)                           | 7.94             | $\left(\frac{3}{4}, \frac{1}{4}\right)$ | $\frac{15}{2}$   |
| 0           | 0.8         | (0.495, 0.505)                           | 7.94             | $\left(\frac{3}{4}, \frac{1}{4}\right)$ | $\frac{15}{2}$   |
| 0           | 1           | (0.490, 0.510)                           | 7.94             | $\left(\frac{3}{4}, \frac{1}{4}\right)$ | $\frac{15}{2}$   |

**Table 5.2:** Nash equilibrium values and corresponding strategies of second commerce retailer

| Parameters  |             | Li and Yang's method [80, Table1, pp.7] |                  | Proposed Mehar method                   |                  |
|-------------|-------------|-----------------------------------------|------------------|-----------------------------------------|------------------|
| $\lambda_1$ | $\lambda_2$ | $y^{*T}$                                | $v^*(\lambda_2)$ | $y^{*T}$                                | $v^*(\lambda_2)$ |
| 0           | 0           | (0.311,0.689)                           | 10.46            | $\left(\frac{1}{4}, \frac{3}{4}\right)$ | $\frac{115}{12}$ |
| 0           | 0.3         | (0.311,0.689)                           | 11.98            | $\left(\frac{1}{4}, \frac{3}{4}\right)$ | $\frac{34}{3}$   |
| 0           | 0.5         | (0.311,0.689)                           | 12.98            | $\left(\frac{1}{4}, \frac{3}{4}\right)$ | $\frac{25}{2}$   |
| 0           | 0.8         | (0.311,0.689)                           | 14.48            | $\left(\frac{1}{4}, \frac{3}{4}\right)$ | $\frac{57}{4}$   |
| 0           | 1           | (0.311,0.689)                           | 15.47            | $\left(\frac{1}{4}, \frac{3}{4}\right)$ | $\frac{185}{12}$ |

**Table 5.3:** Nash equilibrium values and corresponding strategies of first commerce retailer

| Parameters  |             | Li and Yang's method [80, Table 2, pp. 7] |                  | Proposed Mehar method                   |                  |
|-------------|-------------|-------------------------------------------|------------------|-----------------------------------------|------------------|
| $\lambda_1$ | $\lambda_2$ | $x^{*T}$                                  | $u^*(\lambda_1)$ | $x^{*T}$                                | $u^*(\lambda_1)$ |
| 0.5         | 0           | (0.484,0.516)                             | 11.19            | $\left(\frac{3}{4}, \frac{1}{4}\right)$ | $\frac{125}{12}$ |
| 0.5         | 0.3         | (0.492,0.508)                             | 11.19            | $\left(\frac{3}{4}, \frac{1}{4}\right)$ | $\frac{125}{12}$ |
| 0.5         | 0.5         | (0.498,0.502)                             | 11.19            | $\left(\frac{3}{4}, \frac{1}{4}\right)$ | $\frac{125}{12}$ |

|     |     |                |       |                                         |                  |
|-----|-----|----------------|-------|-----------------------------------------|------------------|
| 0.5 | 0.8 | (0.495, 0.505) | 11.19 | $\left(\frac{3}{4}, \frac{1}{4}\right)$ | $\frac{125}{12}$ |
| 0.5 | 1   | (0.490, 0.510) | 11.19 | $\left(\frac{3}{4}, \frac{1}{4}\right)$ | $\frac{125}{12}$ |

**Table 5.4:** Nash equilibrium values and corresponding strategies of second commerce retailer

| Parameters  |             | Li and Yang's method [80, Table 2, pp. 7] |                  | Proposed Mehar method                   |                  |
|-------------|-------------|-------------------------------------------|------------------|-----------------------------------------|------------------|
| $\lambda_1$ | $\lambda_2$ | $y^{*T}$                                  | $v^*(\lambda_2)$ | $y^{*T}$                                | $v^*(\lambda_2)$ |
| 0.5         | 0           | (0.266, 0.734)                            | 10.46            | $\left(\frac{1}{4}, \frac{3}{4}\right)$ | $\frac{115}{12}$ |
| 0.5         | 0.3         | (0.266, 0.734)                            | 11.98            | $\left(\frac{1}{4}, \frac{3}{4}\right)$ | $\frac{34}{3}$   |
| 0.5.5       | 0.5         | (0.266, 0.734)                            | 12.98            | $\left(\frac{1}{4}, \frac{3}{4}\right)$ | $\frac{25}{2}$   |
| 0.5         | 0.8         | (0.266, 0.734)                            | 14.48            | $\left(\frac{1}{4}, \frac{3}{4}\right)$ | $\frac{57}{4}$   |
| 0.5         | 1           | (0.266, 0.734)                            | 15.47            | $\left(\frac{1}{4}, \frac{3}{4}\right)$ | $\frac{185}{12}$ |

**Table 5.5:** Nash equilibrium values and corresponding strategies of first commerce retailer

| Parameters  |             | Li and Yang's method [80, Table 3, pp.8] |                  | Proposed Mehar method                   |                  |
|-------------|-------------|------------------------------------------|------------------|-----------------------------------------|------------------|
| $\lambda_1$ | $\lambda_2$ | $x^{*T}$                                 | $u^*(\lambda_1)$ | $x^{*T}$                                | $u^*(\lambda_1)$ |
| 1           | 0           | (0.484, 0.516)                           | 14.65            | $\left(\frac{3}{4}, \frac{1}{4}\right)$ | $\frac{40}{3}$   |

|   |     |                |       |                                         |                |
|---|-----|----------------|-------|-----------------------------------------|----------------|
| 1 | 0.3 | (0.492, 0.508) | 14.65 | $\left(\frac{3}{4}, \frac{1}{4}\right)$ | $\frac{40}{3}$ |
| 1 | 0.5 | (0.498, 0.502) | 14.65 | $\left(\frac{3}{4}, \frac{1}{4}\right)$ | $\frac{40}{3}$ |
| 1 | 0.8 | (0.495, 0.505) | 14.65 | $\left(\frac{3}{4}, \frac{1}{4}\right)$ | $\frac{40}{3}$ |
| 1 | 1   | (0.490, 0.510) | 14.65 | $\left(\frac{3}{4}, \frac{1}{4}\right)$ | $\frac{40}{3}$ |

**Table 5.6:** Nash equilibrium values and corresponding strategies of second commerce retailer

| Parameters  |             | Li and Yang's method [80, Table 3, pp. 8] |                  | Proposed Mehar method                   |                  |
|-------------|-------------|-------------------------------------------|------------------|-----------------------------------------|------------------|
| $\lambda_1$ | $\lambda_2$ | $y^{*T}$                                  | $v^*(\lambda_2)$ | $y^{*T}$                                | $v^*(\lambda_2)$ |
| 1           | 0           | (0.220, 0.780)                            | 10.46            | $\left(\frac{1}{4}, \frac{3}{4}\right)$ | $\frac{115}{12}$ |
| 1           | 0.3         | (0.220, 0.780)                            | 11.98            | $\left(\frac{1}{4}, \frac{3}{4}\right)$ | $\frac{34}{3}$   |
| 1           | 0.5         | (0.220, 0.780)                            | 12.98            | $\left(\frac{1}{4}, \frac{3}{4}\right)$ | $\frac{25}{2}$   |
| 1           | 0.8         | (0.220, 0.780)                            | 14.48            | $\left(\frac{1}{4}, \frac{3}{4}\right)$ | $\frac{57}{4}$   |
| 1           | 1           | (0.220, 0.780)                            | 15.47            | $\left(\frac{1}{4}, \frac{3}{4}\right)$ | $\frac{185}{12}$ |

**Table 5.7:** Nash equilibrium values and corresponding strategies of first commerce retailer

| Parameters  |             | Li and Yang's method [80, Table 4, pp. 8] |                  | Proposed Mehar method                   |                  |
|-------------|-------------|-------------------------------------------|------------------|-----------------------------------------|------------------|
| $\lambda_1$ | $\lambda_2$ | $x^{*T}$                                  | $u^*(\lambda_1)$ | $x^{*T}$                                | $u^*(\lambda_1)$ |
| 0           | 0           | (0.516, 0.484)                            | 7.94             | $\left(\frac{3}{4}, \frac{1}{4}\right)$ | $\frac{15}{2}$   |
| 0.3         | 0           | (0.516, 0.484)                            | 9.86             | $\left(\frac{3}{4}, \frac{1}{4}\right)$ | $\frac{37}{4}$   |
| 0.5         | 0           | (0.516, 0.484)                            | 11.19            | $\left(\frac{3}{4}, \frac{1}{4}\right)$ | $\frac{125}{12}$ |
| 0.8         | 0           | (0.516, 0.484)                            | 13.24            | $\left(\frac{3}{4}, \frac{1}{4}\right)$ | $\frac{73}{6}$   |
| 1           | 0           | (0.516, 0.484)                            | 14.65            | $\left(\frac{3}{4}, \frac{1}{4}\right)$ | $\frac{40}{3}$   |

**Table 5.8:** Nash equilibrium values and corresponding strategies of second commerce retailer

| Parameters  |             | Li and Yang's method [80, Table 4, pp. 8] |                  | Proposed Mehar method                   |                  |
|-------------|-------------|-------------------------------------------|------------------|-----------------------------------------|------------------|
| $\lambda_1$ | $\lambda_2$ | $y^{*T}$                                  | $v^*(\lambda_2)$ | $y^{*T}$                                | $v^*(\lambda_2)$ |
| 0           | 0           | (0.311, 0.689)                            | 10.46            | $\left(\frac{1}{4}, \frac{3}{4}\right)$ | $\frac{115}{12}$ |
| 0.3         | 0           | (0.284, 0.716)                            | 10.46            | $\left(\frac{1}{4}, \frac{3}{4}\right)$ | $\frac{115}{12}$ |
| 0.5         | 0           | (0.266, 0.734)                            | 10.46            | $\left(\frac{1}{4}, \frac{3}{4}\right)$ | $\frac{115}{12}$ |
| 0.8         | 0           | (0.239, 0.761)                            | 10.46            | $\left(\frac{1}{4}, \frac{3}{4}\right)$ | $\frac{115}{12}$ |

|   |   |                |       |                                         |                  |
|---|---|----------------|-------|-----------------------------------------|------------------|
| 1 | 0 | (0.220, 0.780) | 10.46 | $\left(\frac{1}{4}, \frac{3}{4}\right)$ | $\frac{115}{12}$ |
|---|---|----------------|-------|-----------------------------------------|------------------|

**Table 5.9:** Nash equilibrium values and corresponding strategies of first commerce retailer

| Parameters  |             | Li and Yang's method [80, Table 5, pp. 9] |                  | Proposed Mehar method                   |                  |
|-------------|-------------|-------------------------------------------|------------------|-----------------------------------------|------------------|
| $\lambda_1$ | $\lambda_2$ | $x^{*T}$                                  | $u^*(\lambda_1)$ | $x^{*T}$                                | $u^*(\lambda_1)$ |
| 0           | 0.5         | (0.502, 0.498)                            | 7.94             | $\left(\frac{3}{4}, \frac{1}{4}\right)$ | $\frac{15}{2}$   |
| 0.3         | 0.5         | (0.502, 0.498)                            | 9.86             | $\left(\frac{3}{4}, \frac{1}{4}\right)$ | $\frac{37}{4}$   |
| 0.5         | 0.5         | (0.502, 0.498)                            | 11.19            | $\left(\frac{3}{4}, \frac{1}{4}\right)$ | $\frac{125}{12}$ |
| 0.8         | 0.5         | (0.502, 0.498)                            | 13.24            | $\left(\frac{3}{4}, \frac{1}{4}\right)$ | $\frac{73}{6}$   |
| 1           | 0.5         | (0.502, 0.498)                            | 14.65            | $\left(\frac{3}{4}, \frac{1}{4}\right)$ | $\frac{40}{3}$   |

**Table 5.10:** Nash equilibrium values and corresponding strategies of second commerce retailer

| Parameters  |             | Li and Yang's method [80, Table 5, pp. 9] |                  | Proposed Mehar method                   |                  |
|-------------|-------------|-------------------------------------------|------------------|-----------------------------------------|------------------|
| $\lambda_1$ | $\lambda_2$ | $y^{*T}$                                  | $v^*(\lambda_2)$ | $y^{*T}$                                | $v^*(\lambda_2)$ |
| 0           | 0.5         | (0.311, 0.689)                            | 12.99            | $\left(\frac{1}{4}, \frac{3}{4}\right)$ | $\frac{25}{2}$   |
| 0.3         | 0.5         | (0.284, 0.716)                            | 12.99            | $\left(\frac{1}{4}, \frac{3}{4}\right)$ | $\frac{25}{2}$   |

|     |     |                |       |                                         |                |
|-----|-----|----------------|-------|-----------------------------------------|----------------|
| 0.5 | 0.5 | (0.266, 0.734) | 12.99 | $\left(\frac{1}{4}, \frac{3}{4}\right)$ | $\frac{25}{2}$ |
| 0.8 | 0.5 | (0.239, 0.761) | 12.99 | $\left(\frac{1}{4}, \frac{3}{4}\right)$ | $\frac{25}{2}$ |
| 1   | 0.5 | (0.220, 0.780) | 12.99 | $\left(\frac{1}{4}, \frac{3}{4}\right)$ | $\frac{25}{2}$ |

**Table 5.11:** Nash equilibrium values and corresponding strategies of first commerce retailer

| Parameters  |             | Li and Yang's method [80, Table 6, pp. 9] |                  | Proposed Mehar method                   |                  |
|-------------|-------------|-------------------------------------------|------------------|-----------------------------------------|------------------|
| $\lambda_1$ | $\lambda_2$ | $x^{*T}$                                  | $u^*(\lambda_1)$ | $x^{*T}$                                | $u^*(\lambda_1)$ |
| 0           | 1           | (0.490, 0.510)                            | 7.94             | $\left(\frac{3}{4}, \frac{1}{4}\right)$ | $\frac{15}{2}$   |
| 0.3         | 1           | (0.490, 0.510)                            | 9.86             | $\left(\frac{3}{4}, \frac{1}{4}\right)$ | $\frac{37}{4}$   |
| 0.5         | 1           | (0.490, 0.510)                            | 11.19            | $\left(\frac{3}{4}, \frac{1}{4}\right)$ | $\frac{125}{12}$ |
| 0.8         | 1           | (0.490, 0.510)                            | 13.24            | $\left(\frac{3}{4}, \frac{1}{4}\right)$ | $\frac{73}{6}$   |
| 1           | 1           | (0.490, 0.510)                            | 14.65            | $\left(\frac{3}{4}, \frac{1}{4}\right)$ | $\frac{40}{3}$   |

**Table 5.12:** Nash equilibrium values and corresponding strategies of second commerce retailer

| Parameters  |             | Li and Yang's method [80, Table 6, pp. 9] |                  | Proposed Mehar method |                  |
|-------------|-------------|-------------------------------------------|------------------|-----------------------|------------------|
| $\lambda_1$ | $\lambda_2$ | $y^{*T}$                                  | $v^*(\lambda_2)$ | $y^{*T}$              | $v^*(\lambda_2)$ |

|     |   |                  |       |                                         |                  |
|-----|---|------------------|-------|-----------------------------------------|------------------|
| 0   | 1 | $(0.311, 0.689)$ | 15.47 | $\left(\frac{1}{4}, \frac{3}{4}\right)$ | $\frac{185}{12}$ |
| 0.3 | 1 | $(0.284, 0.716)$ | 15.47 | $\left(\frac{1}{4}, \frac{3}{4}\right)$ | $\frac{185}{12}$ |
| 0.5 | 1 | $(0.266, 0.734)$ | 15.47 | $\left(\frac{1}{4}, \frac{3}{4}\right)$ | $\frac{185}{12}$ |
| 0.8 | 1 | $(0.239, 0.761)$ | 15.47 | $\left(\frac{1}{4}, \frac{3}{4}\right)$ | $\frac{185}{12}$ |
| 1   | 1 | $(0.220, 0.780)$ | 15.47 | $\left(\frac{1}{4}, \frac{3}{4}\right)$ | $\frac{185}{12}$ |

## 5.6 Conclusion

On the basis of the present study, it can be concluded that some mathematically incorrect assumptions are used in the existing method [80] and hence, it is not genuine to use the existing method [80] for solving such bimatrix games in which payoffs are represented by intuitionistic fuzzy numbers. Also, it can be concluded that in the Mehar method, proposed in this paper, no mathematically incorrect assumption is considered and hence, Mehar method should be used for solving these type of problems.



# Chapter 6

## Future scope

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The following research work may be considered as future research work:

1. In this thesis, only flaws in some existing methods for matrix games/constrained matrix games with interval/fuzzy/intuitionistic fuzzy payoffs are pointed out and to resolve flaws of these existing methods, new methods are proposed.

The flaws, pointed out in Chapter 1 to Chapter 5, of this thesis are also occurring in the existing methods [1-3,14-18,33,66,77,93,94,121-123] for solving matrix games with fuzzy payoffs/fuzzy goals/intuitionistic fuzzy goals. In future, it may be tried to resolve flaws of these existing methods [1-3,14-18,33,66,77,93,94,121-123].

2. In all the methods, proposed in this thesis, the existing methods for comparing intervals/fuzzy numbers/intuitionistic fuzzy numbers are used. It can be easily concluded that the maximum/minimum of intervals/fuzzy numbers/intuitionistic fuzzy numbers, obtained by using the existing methods, is not necessarily a unique interval/fuzzy number/intuitionistic fuzzy number. In future, it may be tried to propose such methods for comparing intervals/fuzzy numbers/intuitionistic fuzzy numbers on applying which the maximum/minimum of intervals/fuzzy numbers/intuitionistic fuzzy numbers is always a unique interval/fuzzy number/intuitionistic fuzzy number. Furthermore, new methods, on the basis of proposed comparing methods, can be proposed for solving cooperative and non-cooperative games with intervals/fuzzy/intuitionistic fuzzy payoffs.



# Bibliography

1. Aggarwal, A., Chandra, S., & Mehra, A. (2014). Solving matrix games with I-fuzzy payoffs: Pareto-optimal security strategies approach. *Fuzzy Information and Engineering*, 6, 167-192.
2. Aggarwal, A., Dubey, D., & Chandra, S. (2012). Application of Atanassov's I-fuzzy set theory to matrix games with fuzzy goals and fuzzy payoffs. *Fuzzy Information and Engineering*, 4, 401-414.
3. Aggarwal, A., Mehra, A., & Chandra, S. (2012). Application of linear programming with I-fuzzy sets to matrix games with I-fuzzy goals. *Fuzzy Optimization and Decision Making*, 11, 465-480.
4. Ahmad, I., Singh, D., & Dar, B.A. (2015). Optimality conditions in multiobjective programming problems with interval valued objective functions. *Control & Cybernetics*, 44, 19-45.
5. Akyar, H., & Akyar, E. (2011). A graphical method for solving interval matrix games. *Abstract and Applied Analysis*, 2011, 1-17.
6. Akyar, E., Akyar, H., & Duzce, S.A. (2011). Brown-Robinson method for interval matrix games. *Soft Computing*, 15, 2057-2064.
7. Allahviranloo, T., Lotfi, F.H., Kiasary, M.K., Kiani, N.A., & Alizadeh, L. (2008). Solving fully fuzzy linear programming problem by the ranking function. *Applied Mathematical Sciences*, 2, 19-32.
8. Angelov, P.P. (1997). Optimization in an intuitionistic fuzzy environment. *Fuzzy Sets and Systems*, 86, 299-306.

9. Aplak, H.S., & Turkbey, O. (2013). Fuzzy logic based game theory applications in multi-criteria decision making process. *Journal of Intelligent and Fuzzy Systems*, 25, 359-371.
10. Appadoo, S.S., Bector, C.R., & Bhatt, S.K. (2011). Possibilistic characterization of (m, n)-Trapezoidal fuzzy numbers with applications. *Journal of Interdisciplinary Mathematics*, 14, 347-372.
11. Atanassov, K.T. (1986). Intuitionistic fuzzy sets. *Fuzzy Sets and Systems*, 20, 87-96.
12. Aubin, J.P. (1981). Cooperative fuzzy game. *Mathematics of Operations Research*, 6, 1-13.
13. Bazarra, M.S., Jarvis, J.J., & Sherali, H.D. (1990). *Linear Programming and Network Flows* (2<sup>nd</sup> ed.). New York: John Willey & Sons.
14. Bector, C.R., & Chandra, S. (2002). On duality in linear programming under fuzzy environment. *Fuzzy Sets and Systems*, 125, 317-325.
15. Bector, C.R., & Chandra, S. (2005). *Fuzzy Mathematical Programming and Fuzzy Matrix Games*. Berlin: Springer.
16. Bector, C.R., Chandra, S., & Vijay, V. (2004). Bimatrix games with fuzzy payoffs and fuzzy goals. *Fuzzy Optimization and Decision Making*, 3, 327-344.
17. Bector, C.R., Chandra, S., & Vijay, V. (2004). Duality in linear programming with fuzzy parameters and matrix games with fuzzy pay-offs. *Fuzzy Sets and Systems*, 46, 253-269.
18. Bector, C.R., Chandra, S. & Vijay, V. (2004). Matrix games with fuzzy goals and fuzzy linear programming duality. *Fuzzy Optimization and Decision Making*, 3, 255-269.
19. Bellman, R.E., & Zadeh, L.A. (1970). Decision making in a fuzzy environment. *Management Science*, 17, 141-164.

20. Branzei, R., Dimitrov, D., & Tijs, S. (2003). Convex fuzzy games and participation monotonic allocation schemes. *Fuzzy Sets and Systems*, 139, 267-281.
21. Branzei, R., Dimitrov, D., & Tijs, S. (2004). Hypercubes and compromise values for cooperative fuzzy games. *European Journal of Operational Research*, 155, 733-740.
22. Buckley, J.J. (1984). Multiple goals non-cooperative conflict under uncertainty: A fuzzy set approach. *Fuzzy Sets and Systems*, 13, 107-124.
23. Buckley, J.J., & Jowers, L.J. (2008). *Monte Carlo Methods in Fuzzy Optimization*. Berlin: Springer.
24. Bustince, H., & Burillo, P. (1996). Vague sets are intuitionistic fuzzy sets. *Fuzzy Sets and Systems*, 79, 403-405.
25. Butnariu, D. (1978). Fuzzy games: A description of the concept. *Fuzzy Sets and Systems*, 1, 181-192.
26. Butnariu, D. (1980). Stability and Shapley value for n-persons fuzzy game. *Fuzzy Sets and Systems*, 4, 63-72.
27. Butnariu, D., & Kroupa, T. (2008). Shapley mappings and the cumulative value for n-person games with fuzzy coalitions. *European Journal of Operational Research*, 186, 288-299.
28. Campos, L. (1989). Fuzzy linear programming models to solve fuzzy matrix games. *Fuzzy Sets and Systems*, 32, 275-289.
29. Campos, L., & Gonzalez, A. (1991). Fuzzy matrix games considering the criteria of the players. *Kybernetes*, 20, 17-23.
30. Campos, L., Gonzalez, A., & Vila, M.A. (1992). On the use of the ranking function approach to solve fuzzy matrix games in a direct way. *Fuzzy Sets and Systems*, 49, 193-203.

31. Cevikel, A.C., & Ahlatcioglu, M. (2010). Solutions for fuzzy matrix games. *Computers and Mathematics with Applications*, 60, 399-410.
32. Chandra, S., & Aggarwal, A. (2015). On solving matrix games with pay-offs of triangular fuzzy numbers: Certain observations and generalizations. *European Journal of Operational Research*, 246, 575-581.
33. Chang, J.R., Chang, K.H., Liao, S.H., & Cheng, C.H. (2006). The reliability vague fault-tree analysis on weapon systems fault diagnosis. *Soft Computing*, 10, 531-542.
34. Charnes, A. (1953). Constrained games and linear programming. *Proceedings of the National Academy of Sciences of the United States of America*, 39, 639.
35. Chen, Y.W., & Larbani, M. (2006). Two-person zero-sum game approach for fuzzy multiple attribute decision making problems. *Fuzzy Sets and Systems*, 157, 34-51.
36. Clemente, M., Fernandez, F.R., & Puerto, J. (2011). Pareto-optimal security in matrix games with fuzzy payoffs. *Fuzzy Sets and Systems*, 176, 36-45.
37. Collins, W.D., & Hu, C.Y. (2008). Studying interval valued matrix games with fuzzy logic. *Soft Computing*, 12, 147-155.
38. Delgado, D., Kacprzyk, J., Verdegay, J.L., & Villa, M.A. (1994). *Fuzzy Optimization: Recent Advances*, Germany: Physica Verlag.
39. Diwekar, U. (2008). *Introduction to Applied Optimization* (2<sup>nd</sup> ed.). Springer-Verlag, Cambridge, MA.
40. Dresher, M. (1961). *Games of Strategy Theory and Applications*. New York: Prentice-Hall.
41. Dubey, D., & Mehra, A. (2013). A bipolar approach in fuzzy multi-objective linear programming. *Fuzzy Sets and Systems*, 246, 127-141.
42. Dubois, D., & Prade H. (1980). *Fuzzy Sets and Systems Theory and Applications*. New York: Academic Press.

43. Dutta, B., & Gupta, S.K. (2014). On Nash equilibrium strategy of two person zero sum games with trapezoidal fuzzy payoffs. *Fuzzy Information and Engineering*, 6, 299-314.
44. Ebrahimnejad, A., & Tavana, M. (2014). A novel method for solving linear programming problems with symmetric trapezoidal fuzzy numbers. *Applied Mathematical Modelling*, 38, 4388-4395.
45. Fan, M., Zou, P., Li, S.R., & Wu, C.C. (2014). A fast approach to bimatrix games with intuitionistic fuzzy payoffs. *The Scientific World Journal*, 2014, 1-6.
46. Garagic, D., & Cruz Jr, J.B. (2003). An approach to fuzzy non-cooperative Nash games. *Journal of Optimization Theory and Applications*, 118, 475-491.
47. Gau, W.L., & Buehrer, D.J. (1993). Vague sets, *IEEE Transactions on Systems, Man and Cybernetics*, 23, 29-41.
48. Gupta, A., Kumar, A., & Kaur, A. (2012). Mehar's method to find exact fuzzy optimal solution of unbalanced fully fuzzy multi-objective transportation problems. *Optimization Letters*, 6, 1737-1751.
49. Gupta, P., & Mehlawat, M.K. (2009). Bector-Chandra type duality in fuzzy linear programming with exponential membership functions. *Fuzzy Sets and Systems*, 160, 3290-3308.
50. Hadley, G. (1962). *Linear Programming*. Berkeley: Addison-Wesley, Reading, MA.
51. Hatami-Marbini, A., & Tavana, M. (2010). An extension of the linear programming method with fuzzy parameters. *International Journal of Mathematics in Operational Research*, 3, 44-55.
52. Hladik, M. (2010). Interval valued bimatrix games. *Kybernetika*, 46, 435-446.
53. Inuiguchi, M., Ichihashi, H., & Kume, Y. (1990). A solution algorithm for fuzzy linear programming with piecewise linear membership functions. *Fuzzy Sets and Systems*, 34, 15-31.

54. Kasana, H.S., & Kumar, K. D. (2013). *Introductory Operations Research: Theory and Applications*. Springer Science & Business Media.
55. Kaufmann, A., & Gupta, M. M. (1991). *Introduction to Fuzzy Arithmetic Theory and Applications*. Van Nostrand Publishing Co..
56. Kaufmann, A., & Gupta, M. M. (1988). *Fuzzy Mathematical Models in Engineering and Management Science*. New York, USA.
57. Kawaguchi, T., & Maruyama, Y. (1976). A note on minimax (maximin) programming. *Management Science*, 670-676.
58. Khan, M.A. (2013). Second-order duality for nondifferentiable minimax fractional programming problems with generalized convexity. *Journal of Inequalities and Applications*, 2013, 500.
59. Klir, G., & Yuan, B. (1995). *Fuzzy Sets and Fuzzy Logic*. New Jersey: Prentice Hall.
60. Lai, Y. J., & Hwang, C. L. (1992). *Fuzzy Mathematical Programming*. Springer Berlin Heidelberg.
61. Larbani, M. (2009). Solving bimatrix games with fuzzy payoffs by introducing Nature as a third player. *Fuzzy Sets and Systems*, 160, 657-666.
62. Larbani, M. (2009). Non cooperative fuzzy games in normal form: a survey. *Fuzzy Sets and Systems*, 160, 3184-3210.
63. Li, D.F. (1999). A fuzzy multiobjective programming approach to solve fuzzy matrix games. *Journal of Fuzzy Mathematics*, 7, 907-912.
64. Li, D.F. (1999). Fuzzy constrained matrix games with fuzzy payoffs. *The Journal of Fuzzy Mathematics*, 7, 907-912.
65. Li, D.F. (2008). Lexicographic method for matrix games with payoffs of triangular fuzzy numbers. *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, 16, 371-389.

66. Li, D.F. (2010). Mathematical-programming approach to matrix games with payoffs represented by Atanassov's interval-valued intuitionistic fuzzy sets. *IEEE Transactions on Fuzzy Systems*, 18, 1112-1128.
67. Li, D.F.(2010). A ratio ranking method for triangular intuitionistic fuzzy numbers and its application to MADM problems. *Computers and Mathematics with Applications* 60, 1557-1570.
68. Li, D.F. (2011). Linear programming approach to solve interval-valued matrix games. *Omega*, 39, 655-666.
69. Li, D.F. (2011). Notes on "Linear programming technique to solve two-person matrix games with interval pay-offs". *Asia-Pacific Journal of Operational Research*, 28, 705-737.
70. Li, D.F. (2012). A fast approach to compute fuzzy values of matrix games with payoffs of triangular fuzzy numbers. *European Journal of Operational Research*, 223, 421-429.
71. Li, D.F. (2013). An effective methodology for solving matrix games with fuzzy payoffs. *IEEE Transactions on Cybernetics*, 43, 610-621.
72. Li, D.F. (2014). *Decision and Game Theory in Management with Intuitionistic Fuzzy Sets*. Berlin: Springer-Verlag.
73. Li, D. F. (2015). *Linear Programming Models and Methods of Matrix Games with Payoffs of Triangular Fuzzy Numbers*. Berlin: Springer.
74. Li, D.F., & Cheng, C.T. (2002). Fuzzy multiobjective programming methods for fuzzy constrained matrix games with fuzzy numbers. *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, 10, 385-400.
75. Li, D.F., & Hong, F.X. (2012). Solving constrained matrix games with payoffs of triangular fuzzy numbers. *Computers and Mathematics with Applications*, 64, 432-446.

76. Li, D.F., & Hong, F.X. (2013). Alfa-cut based linear programming methodology for constrained matrix games with payoffs of trapezoidal fuzzy numbers. *Fuzzy Optimization and Decision Making*, 12, 191-213.
77. Li, D.F., & Nan, J. X. (2009). A nonlinear programming approach to matrix games with payoffs of Atanassov's intuitionistic fuzzy sets. *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, 17, 585-607.
78. Li, D.F., Nan, J. X., Tang, Z. P., Chen, K. J., Xiang, X. D., & Hong, F. X. (2012). A bi-objective programming approach to solve matrix games with payoffs of Atanassov's triangular intuitionistic fuzzy numbers. *Iranian Journal of Fuzzy Systems*, 9, 93-110.
79. Li, D.F., Nan, J.X., & Zhang, M.J. (2012). Interval programming models for matrix games with interval payoffs. *Optimization Methods and Software*, 27, 1–16.
80. Li, D.F., & Yang, J. (2013). A difference-index based ranking bilinear programming approach to solving bimatrix games with payoffs of trapezoidal intuitionistic fuzzy numbers. *Journal of Applied Mathematics*, 2013, 1-10.
81. Li, S.J., & Zhang, Q. (2008). The measure of interaction among players in games with fuzzy coalitions. *Fuzzy Sets and Systems*, 159, 119 -137.
82. Li, S.J., & Zhang, Q. (2009). A simplified expression of the Shapley function for fuzzy game. *European Journal of Operational Research*, 196, 234-245.
83. Liu, S.T., & Kao, C. (2007). Solution of fuzzy matrix games: An application of the extension principle, *International Journal of Intelligent Systems*, 22, 891-903.
84. Liu, S.T., & Kao, C. (2009). Matrix games with interval data. *Computers and Industrial Engineering*, 56, 1697-1700.
85. Maeda, T. (2003). On characterization of equilibrium strategy of two-person zero-sum games with fuzzy payoffs. *Fuzzy Sets and Systems*, 139, 283-296.

86. Mangasarian, O.L., & Stone, H. (1964). Two-person nonzero-sum games and quadratic programming. *Journal of Mathematical Analysis and Applications*, 9, 348-355.
87. Mare, M. (2000). Fuzzy coalitions structures. *Fuzzy Sets and Systems*, 114, 23-33.
88. Mathew, R., & Kaimal, M.R. (2004). A fuzzy approach to the  $2 \times 2$  games and an analysis of the game of chicken. *International Journal of Knowledge-based and Intelligent Engineering Systems*, 8, 181-188.
89. Mohanty, B.K. (1994). A procedure for measuring uncertainties due to lack of information in a fuzzy game theory problem. *International Journal of Systems Science*, 25, 2309-2317.
90. Mohanty, B.K., & Gupta, M. (2015). Product selection in E-Commerce under fuzzy environment: A MADM game theoretic model. *International Game Theory Review*, 17, 1540008.
91. Molina, E., & Tejada, J. (2002). The equalizer and the lexicographical solutions for cooperative fuzzy games: Characterization and properties. *Fuzzy Sets and Systems*, 125, 369-387.
92. Moore, R.E. (1979). *Methods and Applications of Interval Analysis*. Philadelphia: Siam.
93. Nan, J.X., & Li, D.F. (2013). Linear programming approach to matrix games with intuitionistic fuzzy goals. *International Journal of Computational Intelligence Systems*, 6, 186-197.
94. Nan, J.X., & Li, D.F. (2014). Linear programming technique for solving interval-valued constraint matrix games. *Journal of Industrial and Management Optimization*, 10, 1059-1070.

95. Nan, J.X., Li, D.F., & Zhang, M. J. (2010). A lexicographic method for matrix games with payoffs of triangular intuitionistic fuzzy numbers. *International Journal of Computational Intelligence Systems*, 3, 280-289.
96. Nan, J.X., Zhang, M.J., & Li, D.F. (2013). A methodology for matrix games with payoffs of triangular intuitionistic fuzzy number, *Journal of Intelligent and Fuzzy Systems*, 26, 2899-2912..
97. Nan, J.X., Zhang, M.J., & Li, D.F. (2013). Intuitionistic fuzzy programming models for matrix games with payoffs of trapezoidal intuitionistic fuzzy numbers. *International Journal of Fuzzy Systems*, 16, 444-456.
98. Nash, J. (1951). Non-cooperative games. *Annals of Mathematics*, 54, 286-295.
99. Nayak, P.K., & Pal, M. (2006). Solution of rectangular interval games using graphical method. *Tamsui Oxford Journal of Mathematical Sciences*, 22, 95-115.
100. Nayak, P.K., & Pal, M. (2009). Linear programming technique to solve two person matrix games with interval pay-offs. *Asia-Pacific Journal of Operational Research*, 26, 285-305.
101. Nayak, P.K., & Pal, M. (2010). Bimatrix games with intuitionistic fuzzy goals. *Iranian Journal of Fuzzy Systems*, 7, 65-79.
102. Nishizaki, I., & Sakawa, M. (2000). Equilibrium solutions in multiobjective bimatrix games with fuzzy pay offs and fuzzy goals. *Fuzzy Sets and Systems*, 111, 99-116.
103. Nishizaki, I., & Sakawa, M. (2000). Fuzzy cooperative games arising from linear production programming problems with fuzzy parameters. *Fuzzy Sets and Systems*, 114, 11-21.
104. Nishizaki, I., & Sakawa, M. (2000). Solutions based on fuzzy goals in fuzzy linear programming games. *Fuzzy Sets and Systems*, 111, 105-119.

105. Nishizaki, I., & Sakawa, M. (2001). *Fuzzy and Multiobjective Games for Conflict Resolution*. Heidelberg: Physica-Verleg.
106. Owen, G. (1982). *Game Theory* (2nd ed.). New York: Academic Press.
107. Peldschus, F., & Zavadskas, E.K. (2005). Fuzzy matrix games multi-criteria model for decision-making in engineering. *Informatica*, 16 107-120.
108. Rani, D., & Gulati, T.R. (2014). A new approach to solve unbalanced transportation problems in imprecise environment. *Journal of Transportation Security*, 7, 277-287.
109. Sakawa, M., & Nishizaki, I. (1994). A lexicographical solution concept in an n-person cooperative fuzzy game. *Fuzzy Sets and Systems*, 61, 265-275.
110. Sakawa, M., & Nishizaki, I. (1994), Maxmin solution for fuzzy multiobjective matrix games. *Fuzzy Sets and Systems* 67, 53-69.
111. Seikh, M.R., Nayak, P.K., & Pal, M. (2015). Matrix games with intuitionistic fuzzy payoffs. *Journal of Information & Optimization Sciences*, 36, 159-181.
112. Shashikhin, V.N. (2004). Antagonistic game with interval payoff function. *Cybernetics and System Analysis*, 40, 556-564.
113. Singh, S.K., & Yadav, S.P. (2015). Modeling and optimization of multi Objective non-linear programming problem in intuitionistic fuzzy environment. *Applied Mathematical Modelling*, 39, 4617-4629.
114. Song, Q., & Kandel, A. (1999). A fuzzy approach to strategic games. *IEEE Transactions on Fuzzy Systems*, 7, 634-642.
115. Stancu-Minasian, I. M., & Pop, B. (2003). On a fuzzy set approach to solving multiple objective linear fractional programming problem. *Fuzzy Sets and Systems*, 134, 397-405.

116. Tesfamariam, S., & Sadiq, R. (2006). Risk-based environmental decision-making using fuzzy analytic hierarchy process (F-AHP). *Stochastic Environmental Research and Risk Assessment*, 21, 35-50.
117. Tijs, S., Branzei, R., Ishiharac, S. & Muto, S. (2004). On cores and stable sets for fuzzy games. *Fuzzy Sets and Systems*, 146, 285-296.
118. Tsurumi, M., Tanino, T. & Inuiguchi, M. (2001). A Shapley function on a class of cooperative fuzzy games. *European Journal of Operational Research*, 129, 596-618.
119. Verma, M., Kumar, A., Singh, Y., & Abraham, A. (2013). Application of the weakest t-norm  $T_\omega$  based vague lambda-tau methodology for reliability analysis of gas turbine system. *Journal of Intelligent & Fuzzy Systems*.
120. Verma, A.K., Srividya, A., & Deka, B. C. (2005). Composite system reliability assessment using fuzzy linear programming. *Electric Power Systems Research*, 73, 143-149.
121. Vijay, V., Chandra, S., & Bector, C.R. (2004). Bimatrix games with fuzzy goals and fuzzy payoffs. *Fuzzy Optimization and Decision Making*, 3, 327-344.
122. Vijay, V., Chandra, S., & Bector, C.R. (2005). Matrix games with fuzzy goals and fuzzy pay offs, *Omega*, 33, 425-429.
123. Vijay, V., Mehra, A., & Chandra, S. (2007). Fuzzy matrix games via a fuzzy relation approach. *Fuzzy Optimization and Decision Making*, 6, 299-314.
124. Von Neumann, J. (1928). Zur theorie der Gesellschaftssoiele. *Mathematische Annalen*, 100, 295-320.
125. Von Neumann, J., & Morgenstern, O. (1944). *Theory of Games and Economic Behavior*. Princeton University Press.
126. Winston, W. L., & Goldberg, J. B. (2004). *Operations Research: Applications and Algorithms*. Boston: Duxbury Press.

127. Yang, S. X., Li, H., Meng, M. Q. H., & Liu, P. X. (2004). An embedded fuzzy controller for a behavior-based mobile robot with guaranteed performance. *IEEE Transactions on Fuzzy Systems*, 12, 436-446.
128. Zadeh, L. A. (1965). Fuzzy sets. *Information and Control*, 8, 338-353.
129. Zhang, Q., Yang, S. X., Mittal, G. S., & Yi, S. (2002). AE-automation and emerging technologies: prediction of performance indices and optimal parameters of rough rice drying using neural networks. *Biosystems Engineering*, 83, 281-290.
130. Zimmermann, H. J. (1991). *Fuzzy Set Theory and its Applications*. Dordrecht: Kluwer.
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