Exact Travelling Wave Solution of Some Nonlinear Partial Differential Equations

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under
the guidance of

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to the

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CONTENTS

List of Figures

CERTIFICATE

I hereby certify that the work presented in the thesis entitled "Exact Travelling Wave Solutions of Some Nonlinear Partial Differential Equations " *which is being presented for the award of degree of master of science, School of Mathematics and Computer Application, Thapar University, Patiala is an authentic record of my own work carried out under the supervision of Dr. Rajesh Kumar Gupta.*

The matter presented in this thesis has not been submitted for the award of any other degree of this or any other university.

Bhajan Lal

(*Bhajan Lal*)

This is to certify that the above statement made by the candidate is correct and true to the best of my knowledge.

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I would like to thanks my parents for their unconditional support and encouragement, I have relied throughout my time at University. The expectations of my parents, from me have motivated me to proceed with this work and their blessings have given me the strength to manage all the things for my thesis work. Also my friend **Shantu** and BB helped me a lot, whenever I was in difficulty.

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ABSTRACT

Exact solutions to nonlinear partial differential equations play an important role for understanding of qualitative as well as quantitative features of many phenomena and processes. Exact solutions visually demonstrate and make it possible to understand the mechanism of complex nonlinear effects.

The thesis entitled "**Exact Travelling Wave Solutions of Some Nonlinear Partial Differential Equations**" is an attempt to obtain the exact solutions of some nonlinear partial differential equations. The thesis has been divided into six chapters. The brief outline of the research work presented chapter wise in the thesis is as follows:

First chapter is introductory in nature, in this chapter, definition of nonlinear differential equations and basic concepts are discussed. A brief summary of literature available on the subject and summary of the work presented in the thesis also appears in this chapter. In the second chapter, methodology of $\left(\frac{G'}{G}\right)$-expansion method and modified $\left(\frac{G'}{G}\right)$-expansion method have been presented.

In the third chapter, sinh-Gordon equation has been solved by using $\left(\frac{G'}{G}\right)$-expansion method. We have successfully derived two type of travelling wave solution in term of hyperbola and trigonometric functions for the generalized sinh-Gorden equation by using the $\left(\frac{G'}{G}\right)$-expansion method.

The fourth chapter comprises Huber's equation with solved by $\left(\frac{G'}{G}\right)$-expansion method and in fifth chapter we have obtained exact travelling wave solution of ZK-BBM equation by $\left(\frac{G'}{G}\right)$-expansion method.

In the sixth chapter Boussinesq equation has been solved with the modified $\left(\frac{G'}{G}\right)$-expansion method. We have successfully derived travelling wave solution in term of trigonometric functions for the generalized Boussinesq equation by using the modified $\left(\frac{G'}{G}\right)$-expansion method.

It is worth to mention that all the solutions reported in this thesis are new and authenticity of the solutions is checked by Maple software.

CHAPTER – 1

INTRODUCTION

A differential equation is a mathematical equation for an unknown function of one or several variables that relates the values of the function itself and its derivatives of various orders. Differential equations play a prominent role in engineering, physics, economics, and other disciplines. Differential equations arise in many areas of science and technology, specifically, whenever a deterministic relation involving some continuously varying quantities (modeled by functions) and their rates of change in space and/or time (expressed as derivatives) is known.

In other words differential equation is an equation $f(x, y, \frac{dy}{dx}, \ldots\ldots\ldots \frac{d^n y}{dx^n}) = 0$, which expresses a relation between independent and dependent variables and their derivatives of any order.

There are two types of differential equations:

1. Ordinary differential equations.
2. Partial differential equations.

Ordinary differential equation is an equation involving only one independent variable.

Example:

$$\frac{d^3 y}{dx^3} + \frac{dy}{dx} + y = 7,$$

Order of an ordinary differential equation is the highest order derivative present in the equation.

Example:

$$\frac{d^3 y}{dx^3} + y = 7, \quad Order = 3$$

Degree of an ordinary differential equation is the exponent of the derivative of the highest order in the differential equation after the equation is free from radicals and fractions in its derivatives.

Example:

$$\left(\frac{dy}{dx}\right)^3 - y = 21,\ Degree \ = 3.$$

Linear differential equation is an equation if the unknown function and its derivative, which occur in the equation, occur only in the first degree and are not multiplied together. Otherwise, the differential equation is said to be non-linear.

Example:

$$\frac{d^3y}{dx^3} + y = 0, \text{ is linear.}$$

$$\frac{dy}{dx} + \left[2 + \left(\frac{d^2y}{dx^2}\right)^4\right]^{\frac{7}{2}} = y, \text{ is nonlinear}$$

Partial differential equation is an equation involving one dependent variable and two or more independent variables.

Examples:

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = u.$$

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = z.$$

Order of the partial differential equation is the order of the highest partial derivatives in the equation.

Example:

$$\frac{\partial^2 u}{\partial x \partial y} = \left(\frac{\partial^2 u}{\partial z^2}\right)^3, \text{Order=2.}$$

Degree of a partial differential equation is the degree of the highest order derivative occurring in the equation.

$$\frac{\partial^2 u}{\partial x \partial y} = \left(\frac{\partial^2 u}{\partial z^2} \right)^3, \text{ degree=3.}$$

1.1 Historical Background

Differential equations have been a major branch of pure and applied mathematics since their beginning in the mid 17th century. The study of differential equations is almost as old as that of the calculus itself. While their history has been well studied, it remains a vital field of on-going investigation, with the emergence of new connections with other parts of mathematics.

Differential equations began with Leibniz, the Bernoulli brothers and others from the 1680s, not long after Newton's 'fluxional equations' in the 1670s. Newton discovered a method of infinite series and the calculus in 1665-66. In 1671 he wrote an account of his theory of "fluxions," a fluxion being a derivative of a "fluent," the name Newton gave to his dependent variables. Newton discussed "fluxional equations," or, as they are now called, differential equations. Newton and Leibniz were the first to explain an "algorithmic process" for differentiation and integration techniques. Leibniz discovered the inverse relationship between the area and derivative by utilizing his definition of the differential. Most 18th-century developments consolidated the Leibnizian tradition, extending its multi-variate form, thus leading to partial differential equations. Generalization of isoperimetrical problems led to the calculus of variations.

1.2 Nonlinear equations and exact solutions

An equation in which one or more terms have a degree 2 or higher is called nonlinear equation. A nonlinear system of equations contains at least one nonlinear equation. In maths linear generally means "simple" and nonlinear means "complicated". The theory for solving linear equations is very well developed because linear equations are simple enough to be solveable. Non-linear equations can usually not be solved exactly and are the subject of much on-going research.

Nonlinear equations are of the interest to the engineers, physicists and mathematicians because most physical systems are inherently nonlinear in nature. Nonlinear equations are difficult to solve and give rise to interesting phenomena such as chaos. The weather is famously nonlinear, where simple changes in one part of the system produce complex effects throughout.

The study of nonlinear differential equations has not only provided information about the phenomenon but has, in fact, helped in making more precise some of the concepts and theories developed in the last century mathematics. The standard strategies adopted to get the solutions of nonlinear partial differential equations (PDEs) to date are following:
(i). Linearize the given set of nonlinear equations by invoking certain physical assumptions.
(ii). Numerical integration of the equations under appropriate boundary conditions.

Exact travelling wave solutions of nonlinear partial differential equations play an important role in the proper understanding of qualitative features of many phenomena and processes in various areas of natural science. These solutions can be used to verify the consistencies and estimate errors of various numerical, asymptotic and approximate analytical method.

Some examples of fields using differential equations in their analysis include:

1. Solid mechanics & motion
2. Heat transfer & energy balances
3. Vibrational dynamics & seismology
4. Aerodynamics & fluid dynamics
5. Electronics & circuit design
6. Population dynamics & biological systems
7. Climatology and environmental analysis

8. Options trading & economics

1.3 Literature review

Nonlinear complex physics phenomena are related to nonlinear partial differential equations (NLPDEs) which are involved in many fields from physics to biology, chemistry, mechanics, etc. As mathematical model of the phenomena, the investigation of the exact solutions of nonlinear partial differential equations (NLPDEs) will help us to understand these phenomena better.

Exact travelling wave solutions of NLPDEs play a very important role in nonlinear science, especially in nonlinear physical science, since they can provide much physical information's and more insight into the physical aspects of the problems and thus lead to further applications. Much work has been done over the years on the subject of obtaining the analytical solutions to the partial differential equations. Especially with the development of computer algebra, the simple and direct methods such as tanh-method for obtaining exact analytical solutions of partial differential equations have drawn a lot of interest.

The search for exact solutions of nonlinear equations has been in more interest in the recent years because of the availability of the symbolic computation Mathematica or Maple. These programs allow us to perform some complicated and differential calculations on a computer. Physical structures of the exact solutions are important to give more insight into the physical aspects of the nonlinear equations applicable for a considerably number of nonlinear systems. One of the most exciting advances of the nonlinear and theoretical physics has been development of methods to look for exact solutions for nonlinear partial differential equations.

Due to the increasing interest for obtaining the exact solutions of the NLPDEs, a wide range of methods are now available for obtaining the exact solutions such as:

(1). Backlund transformation [15].

(2). Hyperbolic functions expansion method [16, 17]

(3). sine-cosine method [39].

(4). Jacobi elliptic functions expansion method [20].

(5). Inverse scattering method [2].

(6). $(\frac{G'}{G})$ —expansion method [6, 7, 37].

(7). Modified $(\frac{G'}{G})$ —expansion method [18], etc.

The investigation of the travelling wave solutions of NPDEs plays an important role in the study of nonlinear physical phenomena. In recent years, the exact solutions of nonlinear PDEs have been investigated by many outhers who are interested in nonlinear physical phenomena. Many powerful methods have been presented by those authors such as the homogeneous balance method [5,25], the hyperbolic tangent expansion method [21, 32], the trial function method [14], the tanh-method [1, 9, 29, 34], the nonlinear transform method [12], the Backlund transform [19, 23], the Hirota's bilinear method [10, 11], the generalized Riccati equation [30], the Weierstrass elliptic function method [22], the theta function method [4, 8], the sine-cosine method [33], the complex hyperbolic function method [38], and so on.

In the present thesis, we shall use a recently introduced method which is called the $\left(\frac{G'}{G}\right)$-expansion method [35, 36]. This method is firstly proposed by the Chinese mathematicians Wang et al. [26] for which the travelling wave solutions of the nonlinear evolution equations are obtained. This method has been extended to solve difference-differential equations. The improved $\left(\frac{G'}{G}\right)$-expansion method also has been used in. Miao [18] have obtained the exact travelling wave solutions of some nonlinear PDEs using an extended $(\frac{G'}{G})$ —expansion method. The main idea of this method is that the travelling wave solutions of nonlinear equations can be expressed by a polynomial in $\left(\frac{G'}{G}\right)$, where $G=G(\xi)$ satisfies the second order linear ordinary differential equation $G''+\lambda G'+\mu G=0$, where $\xi=x-Vt$, and λ, μ and V are constants. The degree of this polynomial can be determined by considering the homogeneous balance between the highest order derivative and the nonlinear terms appearing in the given nonlinear equations. The coefficients of this polynomial can be obtained by solving a set of algebraic equation`s resulted from the process of using the proposed method. So the $\left(\frac{G'}{G}\right)$-expansion method will play an important role in expressing the travelling wave solutions of the (1+1) -dimensional dispersive long wave equations.

CHAPTER - 2

METHODOLOGY

2.1 $(\frac{G^{'}}{G})-$*expansion method*

$\left(\frac{G'}{G}\right)$-expansion method [7,37] is firstly proposed by the Chinese mathematicians Wang et al. [26] for which the travelling wave solutions of the nonlinear evolution equations are obtained. This method has been extended to solve difference-differential equations.

In this section we describe the $(\frac{G^{'}}{G})-$expansion method for finding travelling wave solutions of nonlinear evolution equations. Suppose that a nonlinear equation, say in two independent variables x and t is given by

$$P(u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}, \dots\dots\dots \quad \dots\dots) = 0, \tag{2.1.1}$$

where $u = u(x,t)$ is an unknown function, P is a polynomial in $u = u(x,t)$ and its various partial derivatives, in which the highest order derivatives and nonlinear terms are involved. In the following we give the main steps of the $(\frac{G^{'}}{G})-$expansion method.

Step 1. Combining the independent variables x and t into one variables $\xi = x - Vt$, we suppose that

$$u(x,t) = u(\xi), \xi = x - Vt. \tag{2.1.2}$$

The travelling wave variables (2.1.2) permits us reducing equation (2.1.1) to an ordinary differential equation (ODE)

$$P(u, -Vu', u', V^2u'', -Vu'', u'', \dots\dots) = 0. \tag{2.1.3}$$

Step 2. Suppose that the solution of ODE (2.1.3) can be expressed by a polynomial in $(\frac{G^{'}}{G})$ as follows:

$$.u(\xi) = \alpha_m (\frac{G'}{G})^m + \ldots, \tag{2.1.4}$$

where $G = G(\xi)$ satisfies the second order linear ordinary differential equation (LODE) in the form

$$G'' + \lambda G' + \mu G = 0. \tag{2.1.5}$$

α_m ……. λ and μ are constants to be determined later, $\alpha_m \neq 0$, the unwritten part in (2.1.4) is also a polynomial in $(\frac{G'}{G})$, but the degree of which is generally equal to or less than m-1. The positive integer m can be determined by considering the homogenous balance the highest order derivatives and nonlinear terms appearing ODE (2.1.3).

Step 3. By substituting (2.1.4) into equation (2.1.3) and using second order LODE (2.1.5), collecting all terms with the same order of $(\frac{G'}{G})$ together, the left –hand side of equation (2.1.3) is converted into another polynomial in $(\frac{G'}{G})$. Equating each coefficient of this polynomial to zero, yields a set of algebraic for $\alpha_m, \ldots\ldots V, \lambda, \mu$

Step 4. The constants $\alpha_m, \ldots\ldots V, \lambda, \mu$ can be obtained by solving the algebraic in step 3. Since the general solutions of the second order LODE(2.1.5) have been well known for us, than substituting $\alpha_m, \ldots\ldots V$ and the general solutions of equation (2.1.5) into (2.1.4) we have travelling wave solutions of the nonlinear evolution equation (2.1.1).

2.2 Modified $(\frac{G'}{G})$ –expansion method

In this section we describe the modified $(\frac{G^{'}}{G})$ —expansion method [18] for finding travelling wave solutions of nonlinear evolution equations. Suppose that a nonlinear equation, say in two independent variables x and t is given by

$$P(u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}, ...) = 0, \tag{2.2.1}$$

where $u = u(x,t)$ is an unknown function, P in a polynomial in $u = u(x,t)$ and its various partial derivatives, in which the highest order derivatives and nonlinear terms are involved. In the following we give the main steps of the $(\frac{G^{'}}{G})$ —expansion method.

Step 1. Combining the independent variables x and t into one variables $\xi = x - Vt$, we suppose that

$$u(x,t) = u(\xi), \xi = x - Vt\ . \tag{2.2.2}$$

The travelling wave variables (2.2.2) permits us reducing equation (2.2.1) to an ODE for $u = u(\xi)$

$$P(u, -Vu', u', V^2u'', -Vu'', u'', ...) = 0\ . \tag{2.2.3}$$

Step 2. Suppose that the solution of ODE (2.2.3) can be

$$u(\xi) = a_0 + \sum \left[a_i \left(\frac{G'(\xi)}{G(\xi)} \right)^i + b_i \left(\frac{G'(\xi)}{G(\xi)} \right)^{-i} \right], \tag{2.2.4}$$

where $a_{0,} a_i$ and b_i are constants and the positive integer m are can be determined by considering the homogeneous balance the highest order derivatives and highest order nonlinear appearing in ODE (2.2.3). The function $G(\xi)$ is the solution of the auxiliary LODE

$$G''(\xi) + \mu G(\xi) = 0, \tag{2.2.5}$$

where μ is constant to be determined.

Step 3. Substituting the solution of equation (2.2.4) together with equation (2.2.5) into equation (2.2.3) yields an algebraic equation involving powers of $(\frac{G'}{G})$. Equating the coefficients of each power of $(\frac{G'}{G})$ to zero gives a system of algebraic equations for $a_{0,}, a_i, b_i, \boldsymbol{\mu}$ *and* V .

Step 4. Since the general solutions of (2.2.5) have been well known for us, then substituting a_0, a_i, b_i, V and the general solutions of (2.2.5) into (2.2.4), we obtain the travelling wave solutions of NLPDE (2.2.1).

CHAPTER – 3

Sinh-GORDEN EQUATION

3.1 Introduction

The **sine–Gordon equation** [27] is a nonlinear hyperbolic partial differential equation in (1 + 1) dimensions involving the D'Alembert operator and the sine of the unknown function. It was originally considered in the nineteenth century in the course of study of surfaces of constant negative curvature. This equation attracted a lot of attention in the 1970s due to the presence of solutions.

The sine–Gordon equation

$$\varphi_{uv} = \sin \varphi$$

was first considered in the nineteenth century in the course of investigation of surfaces of constant Gaussian curvature $K = -1$, also called pseudo spherical surfaces. Choose a coordinate system for such a surface in which the coordinate mesh $u = const,\ v = const$ is given by the asymptotic lines parameterized with respect to the arc length. The first fundamental form of the surface in these coordinates has a special form

$$ds^2 = du^2 + 2\cos\varphi\, du\, dv + dv^2,$$

where φ expresses the angle between the asymptotic lines, and for the second fundamental form, $L = N = 0$. Then the Codazzi-Mainardi equation expressing a compatibility condition between the first and second fundamental forms results in the sine–Gordon equation. The study of this equation and of the associated transformations of pseudo spherical surfaces in the 19th century by Bianchi and Bäcklund led to the discovery of Bäcklund transformations.

3.2 $\left(\frac{G'}{G}\right)$-expansion method

The sinh-Gordon equation

$$u_{xt} = \sinh(u) \tag{3.2.1}$$

appeared in many branches of non linear science [31], which has been important model equation and played an important role in physics.

Abdul-Majid Wazwaz [27] studied the following generalized Sinh-Gorden equation:

$$u_{tt} = au_{xx} - b\sinh(nu) \tag{3.2.2}$$

where a, b are constant and n is a positive integer .

By using the $\left(\frac{G'}{G}\right)$-expansion method as mentioned in chapter-2 (sec-2.1), we will derive families of exact travelling wave solutions of equation (3.2.2).

Now let

$$u(x,t) = u(\xi) = u(x - ct), \tag{3.2.3}$$

where c is the wave speed. Substituing the above travelling wave transformation into (3.2.2), we get following ordinary differential equation (ODE)

$$du_{\xi\xi} + b\sinh(nu) = 0, \tag{3.2.4}$$

where $d = c^2 - a$.

Appling the transformation $v = e^{nu}$ to equation (3.2.4), we can get

$$\sinh(nu) = \frac{v - v^{-1}}{2},\ \cosh(nu) = \frac{v + v^{-1}}{2},\ u = \frac{1}{n}\operatorname{arccos}h\frac{v + v^{-1}}{2},\ u^{"} = \frac{v^{"}v - v^{'2}}{nv^2}.$$

Equation (3.2.4) becomes

$$2d(v^{"}v - v^{'2}) + bnv^3 - bnv = 0. \tag{3.2.5}$$

Suppose that the solution of ODE (3.2.5) can be expressed by a polynomial in $\left(\frac{G'}{G}\right)$ as follows:

$$v(\xi)=\alpha_m(\frac{G'}{G})^m+\ldots\ , \tag{3.2.6}$$

where $\left(\frac{G'}{G}\right)$ satisfies the second order LODE in the form

$$G''+\lambda G'+\mu G=0. \tag{3.2.7}$$

By using (3.2.6) and (3.2.7) It is easily derived that

$$\begin{aligned} v^3 &= \alpha_m{}^3(\frac{G'}{G})^{3m}+\ldots\ldots\ldots\ \ldots\ldots\ldots\ . \\ v' &= -m\alpha_m(\frac{G'}{G})^{m+1}+\ldots\ldots\ldots\ \ldots\ldots \\ v'' &= m(m+1)\alpha_m(\frac{G'}{G})^{m+2}+\ldots\ldots\ldots\ \ldots\ldots \end{aligned} \tag{3.2.8}$$

Considering the homogeneous balance between $v''v$ and v^3 in equation (3.2.5), we required that m=2.

So we can write (3.2.6) as

$$v(\xi)=\alpha_2(\frac{G'}{G})^2+\alpha_1(\frac{G'}{G})\ +\alpha_0;\alpha_2\neq 0 \tag{3.2.9}$$

and there for

$$v^3(\xi)=(\alpha_2(\frac{G'}{G})^2+\alpha_1(\frac{G'}{G})\ +\alpha_0)^3. \tag{3.2.10}$$

By using (3.2.8) and (3.2.9) It is derived that

$$v''(\xi) = 6\alpha_2(\frac{G'}{G})^4 + (2\alpha_1 10\alpha_2\lambda)(\frac{G'}{G})^3 + (8\alpha_2\mu + 3\alpha_1\lambda + 4\alpha_2\lambda^2)(\frac{G'}{G})^2 + \\ (6\alpha_2\lambda\mu + 2\alpha_1\mu + \alpha_1\lambda^2)(\frac{G'}{G}) + 2\alpha_2\mu^2 + \alpha_1\lambda\mu. \qquad (3.2.11)$$

By using (3.2.9) to (3.2.11) into equation (3.2.5) and collecting all terms with the same power of $\left(\frac{G'}{G}\right)$ together, the left hand side of equation (3.2.5) is converted into another polynomial in $\left(\frac{G'}{G}\right)$. Equating each coefficient of this polynomial to zero, yields a set of simultaneous algebraic equations for $\alpha_1, \alpha_2, \alpha_0$ and d as follow:

$$4d\alpha_2^2 + bn\alpha_2^3 = 0$$

$$8d(-2\alpha_2\lambda - \alpha_1)\alpha_2 + 3bn\alpha_1\alpha_2^2 + 2d(2\alpha_1 + 10\alpha_2\lambda)\alpha_2 + 12d\alpha_2\alpha_1 = 0$$

$$bn\left(\alpha_0\alpha_2^2 + 2\alpha_1^2\alpha_2 + \alpha_2 + \alpha_2(2\alpha_0\alpha_2 + \alpha_1^2)\right) + 2d(8\alpha_2\mu + 3\alpha_1\lambda + 4\alpha_2\lambda^2) + \\ 2d(2\alpha_1 + 10\alpha_2\lambda)\alpha_1 + 12d\alpha_2\alpha_0 - 2d(-4(-2\alpha_2\mu - \alpha_1\lambda)\alpha_2 + (-2\alpha_2\lambda - \alpha_1)^2) = 0$$

$$bn(4\alpha_1\alpha_2\alpha_0 + \alpha_1(2\alpha_0\alpha_2 + \alpha_1^2)) + 2d(6\alpha_2\mu\lambda + 2\alpha_1\mu + \alpha_1\lambda^2)\alpha_2 \\ + 2d(8\alpha_2\mu + 3\alpha_1\lambda + 4\alpha_2\lambda^2)\alpha_1 + 2d(2\alpha_1 + 10\alpha_2\lambda)\alpha_0 \\ - 2d(4\alpha_1\mu\alpha_2 + 2(-2\alpha_2\mu - \alpha_1\lambda)(-2\alpha_2\lambda - \alpha_1)) = 0$$

$$bn(\alpha_0(2\alpha_0\alpha_1 + \alpha_1^2) + 2\alpha_1^2\alpha_0 + \alpha_2\alpha_0^2) - bn\alpha_2 + 2d(2\alpha_2\mu^2 + \alpha_1\lambda\mu)\alpha_2 \\ + 2d(6\alpha_2\lambda\mu + 2\alpha_1\mu + \alpha_1\lambda^2)\alpha_1 - 2d(8\alpha_2\mu + 3\alpha_1\lambda + 4\alpha_2\lambda^2)\alpha_0 \\ - 2d(-2\alpha_1\mu(-2\alpha_2\lambda - \alpha_1) + (-2\alpha_2\mu - \alpha_1\lambda)^2) = 0$$

$$3bn\alpha_0^2\alpha_1 - bn\alpha_1 + 2d(2\alpha_2\mu^2 + \alpha_1\lambda\mu)\alpha_1 + 2d(6\alpha_2\lambda\mu + 2\alpha_1\mu + \alpha_1\lambda^2)\alpha_0 \\ + 4d\alpha_1\mu(-2\alpha_2\mu - \alpha_1\lambda) = 0$$

$$2d(2\alpha_2\mu^2 + \alpha_1\lambda\mu)\alpha_0 + bn\alpha_0^3 - 2d\alpha_1^2\mu^2 - bn\alpha_0 = 0. \qquad (3.2.12)$$

Solving the algebraic equations above, yields solution

$$\alpha_0 = \frac{\lambda^2}{4\mu-\lambda^2}, \alpha_1 = \frac{4\lambda}{4\mu-\lambda^2}, \alpha_2 = \frac{4}{4\mu-\lambda^2}.d = -\frac{6n}{4\mu-\lambda^2}, \tag{3.2.13}$$

where λ, μ and n are arbitrary constants.

By using (3.2.13) expression (3.2.9) can be written as

$$v(\xi) = \frac{4}{4\mu-\lambda^2}(\frac{G'}{G})^2 + \frac{4\lambda}{4\mu-\lambda^2}(\frac{G'}{G}) + \frac{\lambda^2}{4\mu-\lambda^2}, \tag{3.2.14}$$

where $\xi = x - t\sqrt{-\frac{6n}{4\mu-\lambda^2}+a}$.

Equation (3.2.14) is a solution of equation (3.2.5). Substituting the general solution of equation (3.2.7) into equation (3.2.14) we have two types of travelling wave solution of equation (3.2.2)

Case 1: When $\lambda^2 - 4\mu > 0$

$$u(\xi) = \frac{1}{n}\log\frac{\left(C_1^2\cosh(K)^2 - C_1^2 + C_1\cosh(K)C_2\sinh(K) + C_2^2\cosh(K)^2\right)}{\left(C_1\cosh(K) + C_2\sinh(K)\right)},$$

where $K = \frac{\sqrt{\lambda^2-4\mu}\xi}{2}$ and

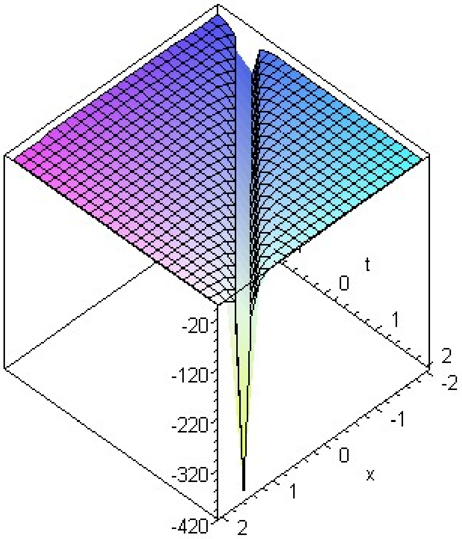

Fig. 3(a) Trigonometric function solution for Sinh-Gordon equation.

$\lambda = 2\sqrt{2}, \mu = 1, n = 1, C_1 = 1, C_2 = 2, c = 1$

$\xi = x - t\sqrt{-\dfrac{6n}{4\mu - \lambda^2} + a}$, C_1 and C_2 are arbitrary constants.

Case 2: When $\lambda^2 - 4\mu < 0$

$$u(\xi) = \frac{1}{n}\log\left(\frac{-C_1^2 + C_1^2\cos(K)^2 + 2C_1\cos(K)C_2\sin(K) - C_2^2\cos(K)^2}{C_1^2\cos(K)^2 + 2C_1\cos(K)C_2\sin(K) + C_2^2 - C_2^2\cos(K)^2}\right)$$

where $K = \frac{1}{2}\sqrt{-\lambda^2 + 4\mu}\xi$ and

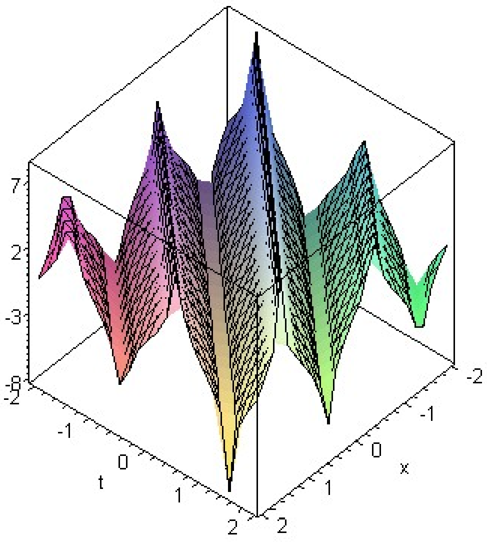

Fig. 3(b) Hyperbolic function solution for Sinh-Gordon equation.
$\lambda = 1, \mu = 2\sqrt{2}, n = 1, C_1 = 1, C_2 = 1, c = 1$

$\xi = x - t\sqrt{\frac{6n}{4\mu - \lambda^2} + a}$, C_1 and C_2 are arbitrary constants.

3.3 Summary

In this chapter, we have successfully derived two type of travelling wave solution in term of hyperbola and trigonometric functions for the generalized sinh-Gorden equation by using the $\left(\frac{G'}{G}\right)$-expansion method.

CHAPTER-4
HUBER EQUATION

4.1 Introduction

The scaled NPDE [13] in (1+1) dimension under consideration is given by :

$$\frac{\partial^2 u}{\partial t^2}+\frac{\partial^4 u}{\partial^3 x\partial t}+3\frac{\partial^2 u}{\partial^2 x}\frac{\partial^2 u}{\partial x\partial t}=0 \qquad (4.1.1)$$
$$u=u(x,t)$$

where $u=u(x,t)$ can be field describing a wave propagation depending on time t and x mean local coordinates. In paper [13], Huber calculates solitony class solution of above equation by using tanh-method. Exact class of solution of solution in the form of hyperbolic function and additional ellipse function of form of first kind result. In next section we will find exact solution of equation (4.1.1) by using $\left(\frac{G'}{G}\right)$- expansion method.

4.2 $\left(\frac{G'}{G}\right)$ expansion method

By using the $\left(\frac{G'}{G}\right)$ -expansion method, the derived families of exact travelling wave solutions of equations (4.1.1).

Now let

$$u(x,t)=u(\xi)=u(x-ct), \qquad (4.2.1)$$

where c is the wave speed. Substituting the above travelling wave transformation into (4.1.1), we have

$$\frac{\partial^4 u}{\partial \xi^4}+3\left(\frac{\partial^2 u}{\partial \xi^2}\right)-c\frac{\partial^2 u}{\partial \xi^2}=0. \qquad (4.2.2)$$

Putting $h = \frac{\partial^2 u}{\partial \xi^2}$ as new dependent variables we have the following nODE of the second order

$$\frac{\partial^2 h}{\partial \xi^2} + 3h^2 - ch = 0 . \tag{4.2.3}$$

Suppose that the solution of ODE can be expressed by a polynomial in $\left(\frac{G'}{G}\right)$ as follows:

$$h(\xi) = \alpha_m (\frac{G^{'}}{G})^m + ... , \tag{4.2.4}$$

where $G = G(\xi)$ satisfies the second order LODE in the form

$$G^{''} + \lambda G^{'} + \mu G = 0 . \tag{4.2.5}$$

By using (4.2.4) and (4.2.5) It is easily derived that

$$\begin{aligned} h^3 &= \alpha_m{}^3 (\frac{G^{'}}{G})^{3m} + ... \\ h^{'} &= -m\alpha_m (\frac{G^{'}}{G})^{m+1} + ... \\ h^{''} &= m(m+1)\alpha_m (\frac{G^{'}}{G})^{m+2} + ... \end{aligned} \tag{4.2.6}$$

Considering the homogeneous balance between $h''h$ and h^3 in equation (4.2.3), we required that m=2.

So we can write (4.2.4) as

$$h(\xi) = \alpha_2 (\frac{G^{'}}{G})^2 + \alpha_1 (\frac{G^{'}}{G}) \ + \alpha_0 ; \alpha_2 \neq 0 \tag{4.2.7}$$

and there for

$$h^2(\xi) = (\alpha_2 (\frac{G'}{G})^2 + \alpha_1 (\frac{G'}{G}) \ \ + \alpha_0)^2 . \tag{4.2.8}$$

By using (4.2.7) and (4.2.5) It is derived that

$$\begin{aligned} h''(\xi) &= 6\alpha_2 (\frac{G'}{G})^4 + (2\alpha_1 10\alpha_2 \lambda)(\frac{G'}{G})^3 + (8\alpha_2 \mu + 3\alpha_1 \lambda + 4\alpha_2 \lambda^2)(\frac{G'}{G})^2 + \\ &(6\alpha_2 \lambda\mu + 2\alpha_1 \mu + \alpha_1 \lambda^2)(\frac{G'}{G}) + 2\alpha_2 \mu^2 + \alpha_1 \lambda\mu \end{aligned} \tag{4.2.9}$$

By using (4.2.7) to (4.2.9) into equation (4.2.3) and collecting all terms with the same power of $\left(\frac{G'}{G}\right)$ together, the left hand side of equation (4.2.4) is converted into another polynomial in $\left(\frac{G'}{G}\right)$. Equating each coefficient of this polynomial to zero, yields a set of simultaneous algebraic equation for $\alpha_{1,} \alpha_2, \alpha_0$ and c as follow:

$$\begin{aligned} &3\alpha_2^2 + 6\alpha_2 = 0 \\ &2\alpha_1 + 10\alpha_2 \lambda + 6\alpha_1 \alpha_2 = 0 \\ &8\alpha_2 \mu + 3\alpha_1 \lambda + 4\alpha_2 \lambda^2 - c\alpha_2 + 6\alpha_0 \alpha_2 + 3\alpha_1^2 = 0 \\ &-c\alpha_1 + 6\alpha_2 \lambda\mu + 2\alpha_1 \mu + \alpha_1 \lambda^2 + 6\alpha_0 \alpha_1 = 0 \\ &2\alpha_2 \mu^2 + \alpha_1 \lambda\mu + 3\alpha_0^2 - c\alpha_0 = 0; \end{aligned} \tag{4.2.10}$$

Solving the algebraic equations above, yields

$$\alpha_0 = -2\mu, \alpha_1 = 2\lambda, \alpha_2 = -2, c = -4\mu + \lambda^2, \tag{4.2.11}$$

where λ, μ and are arbitrary constants.
By using (4.2.11), expression (4.2.7) can be written as

$$h(\xi) = -2\left(\frac{G'}{G}\right)^2 + 2\lambda\left(\frac{G'}{G}\right) - 2\mu, \tag{4.2.12}$$

where $\xi = x-(-4\mu+\lambda^2)t$.

Equation (4.2.12) with $h=\dfrac{\partial^2 u}{\partial \xi^2}$ is a solution of equation (4.2.2). Substituting the general solution of equation (4.2.5) into equation (4.2.12) we have two types of travelling wave solution of equation (4.1.1)

Case 1: When $\lambda^2-4\mu<0$

$$u(x,t)=\iint h(\xi)d\xi^2=-\frac{1}{2}\iint\frac{-C_1^2\lambda^2-\lambda^2C_2^2+4C_1^2\mu+4\mu C_2^2}{C_1^2\cos(\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi)^2+2C_1\cos(\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi)C_2\sin(\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi)+C_2^2\cos(}$$

where $\xi = x-(-4\mu+\lambda^2)t$ and C_1 and C_2 are arbitrary constants.

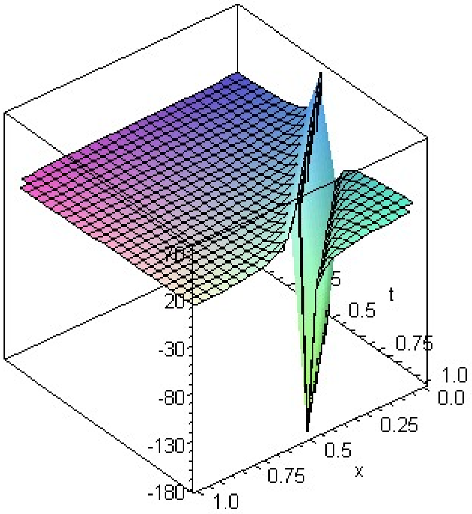

Fig. 4(a) Trigonometric function solution for Huber equation.
$\lambda=1,\mu=2\sqrt{2},n=1,C_1=1,C_2=1,c=1$

Case 2: When $\lambda^2-4\mu>0$

$$u(x,t)=\iint h(\xi)d\xi^2=-\frac{1}{2}\iint\frac{\lambda^2C_2^2+4C_1^2\mu-C_1^2\lambda^2-4\mu C_2^2}{\left(C_1\cosh(\frac{1}{2}\sqrt{-4\mu+\lambda^2}\xi)+C_2\sinh(\frac{1}{2}\sqrt{-4\mu+\lambda^2}\xi)\right)^2}d\xi d\xi$$

where $\xi=x-(-4\mu+\lambda^2)t$ and C_1 and C_2 are arbitrary constants.

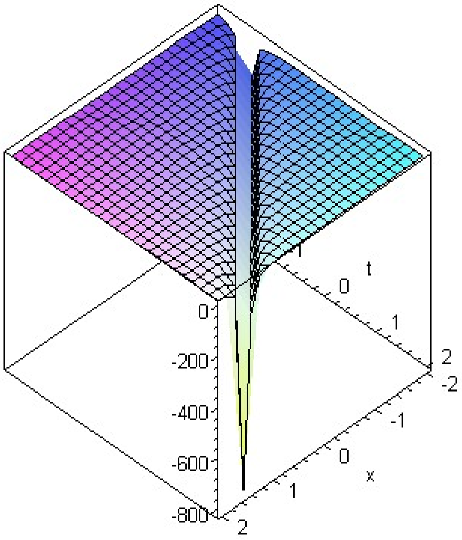

Fig. 4(b) Hyperbolic function solution for Huber equation.

$\lambda=2\sqrt{2},\mu=1,n=1,C_1=1,C_2=2,c=1$

4.3 Summary

In this chapter, we have successfully derived two type of travelling wave solution in term of hyperbolic and trigonometric functions for the generalized Huber equation by using the $\left(\frac{G'}{G}\right)$-expansion method

CHAPTER- 5

(2 + 1) DIMENSIONAL ZK–BBM EQUATION

5.1 Introduction

A variety of exact solutions for the (2 + 1) dimensional ZK–BBM equation [28]

$$u_t + u_x - a(u^2)_x + (bu_{xt} - ku_{xt})_x = 0 \tag{5.1.1}$$

where a, b and k are constants, are developed by means of the tanh method and the sine–cosine methods. Generalized forms of the ZK–BBM equation are studied. The $\left(\frac{G'}{G}\right)$-expansion method is reliable to derive trigonometric and hyperbolic solutions of ZK-BBM equation.

5.2 $\left(\frac{G'}{G}\right)$ expansion method

Using transformation

$$\begin{aligned} u(\xi) &= u(x + y - ct) \\ \xi &= x + y - ct \end{aligned} \tag{5.2.1}$$

permits us converting equation (5.1.1) into an ODE for $u = u(\xi)$ as follows

$$u'(1-c) - 2auu' + cu'''(b-k) = 0 \tag{5.2.2}$$

Integrating it with respect to ξ once yields

$$V+u(1-c)-au^2+cu''(b-k)=0, \tag{5.2.3}$$

where V is an intergration constant that is to be determined later.

Suppose that the solution of ODE (5.2.3) can be expressed by a polynomials in $\left(\frac{G'}{G}\right)$ as follows

$$u(\xi)=\alpha_m(\frac{G'}{G})^m+....., \tag{5.2.4}$$

where $G=G(\xi)$ satisfies the second order LODE in the form

$$G''+\lambda G'+\mu G=0. \tag{5.2.5}$$

By using (5.2.4) and (5.2.5), It is easily derived that

$$\begin{aligned} u^2 &= \alpha_m{}^2(\frac{G'}{G})^{2m}..... \\ u' &= -m\alpha_m(\frac{G'}{G})^{m+1}+.... \\ u'' &= m(m+1)\alpha_m(\frac{G'}{G})^{m+2}.... \end{aligned} \tag{5.2.6}$$

Considering the homogeneous balance between u'' and u^2 in equation (5.2.3) based on (5.2.6), we required that m=2.

So we can write (5.2.4) as

$$u(\xi)=\alpha_2(\frac{G'}{G})^2+\alpha_1(\frac{G'}{G})\ +\alpha_0;\alpha_2\neq 0 \tag{5.2.7}$$

By using (5.2.7) and (5.2.5), It is derived that

$$u''(\xi)=6\alpha_2(\frac{G'}{G})^4+(2\alpha_1 10\alpha_2\lambda)(\frac{G'}{G})^3+(8\alpha_2\mu+3\alpha_1\lambda+4\alpha_2\lambda^2)(\frac{G'}{G})^2+ \\ (6\alpha_2\lambda\mu+2\alpha_1\mu+\alpha_1\lambda^2)(\frac{G'}{G})+2\alpha_2\mu^2+\alpha_1\lambda\mu. \qquad (5.2.8)$$

By using (5.2.6) to (5.2.8) into equation (5.2.3) and collecting all terms with the same power of $\left(\frac{G'}{G}\right)$ together, the left hand side of equation (5.2.3) is converted into another polynomial in $\left(\frac{G'}{G}\right)$. Equating each coefficient of this polynomial to zero, yields a set of simultaneous algebraic equation for $\alpha_1, \alpha_2, \alpha_0$,c and V as follow:

$$\begin{aligned}
&-a\alpha_2^2+bc\alpha_2(b-k)=0\\
&c(2\alpha_1+10\alpha_2\lambda)(b-k)-2a\alpha_1\alpha_2=0\\
&\alpha_2(1-c)+c(8\alpha_2\mu+3\alpha_1\lambda+4\alpha_2\lambda^2)(b-k)-a(2\alpha_0\alpha_2+\alpha_1^2)=0\\
&\alpha_1(1-c)+c(b\alpha_2\lambda\mu+2\alpha_1\lambda+4\alpha_2\lambda^2)(b-k)-2a\alpha_0\alpha_1=0\\
&V+\alpha_0(1-c)-a\alpha_0^2+c(2\alpha_2\mu^2\alpha_1\lambda\mu)(b-k)=0.
\end{aligned} \qquad (5.2.9)$$

Solving the algebraic equation above yields

$$\alpha_2=\frac{bc(b-k)}{a},\alpha_1=\frac{b\lambda c(b-k)}{a},\alpha_0=\frac{b\lambda^2c+8bc\mu-k\lambda^2c+1-c-8kc\mu}{2a},$$

$$V=\frac{\begin{array}{l}(2c-8b^2\lambda^2c^2\mu-8k^2\lambda^2c^2\mu+16b\lambda^2c^2\mu k+16c^2k^2\mu^2+16c^2k^2\mu^2-32bc^2k\mu^2-2bkc^2\lambda^4\\ +b^2\lambda^4c^2+k^2\lambda^4c^2-1-c^2)\end{array}}{4a}.$$

…….(5.2.10)

Substitute (5.2.10) into (5.2.7), we get

$$u(\xi)=\frac{bc(b-k)}{a}(\frac{G'}{G})^2+\frac{b\lambda c(b-k)}{a}(\frac{G'}{G})+\frac{b\lambda^2c+8bc\mu-k\lambda^2c+1-c-8kc\mu}{2a},$$

(5.2.11)

where λ, μ, b, c and k are arbitrary constants.

Equation (5.2.11) is a solution of equation (5.2.3). Substituting the general solution of equation (5.2.5) into equation (5.2.11), we have two types of travelling wave solution of equation (5.2.1)

Case 1: When $\lambda^2-4\mu>0$

$$u(\xi)=\frac{bc(b-k)}{a}(\frac{G'}{G})^2+\frac{b\lambda c(b-k)}{a}(\frac{G'}{G})+\frac{b\lambda^2c+8bc\mu-k\lambda^2c+1-c-8kc\mu}{2a},$$

where $$\left(\frac{G'}{G}\right)=-\frac{\lambda}{2}+\frac{1}{2}\frac{\left(C_1\sinh\left(\frac{\sqrt{\lambda^2-4\mu}\xi}{2}\right)+C_2\cosh\left(\frac{\sqrt{\lambda^2-4\mu}\xi}{2}\right)\right)\sqrt{\lambda^2-4\mu}\xi}{C_1\cosh\left(\frac{\sqrt{\lambda^2-4\mu}\xi}{2}\right)+C_2\sinh\left(\frac{\sqrt{\lambda^2-4\mu}\xi}{2}\right)}$$

where $\xi=x+y-ct$ and C_1 and C_2 are arbitrary constants.

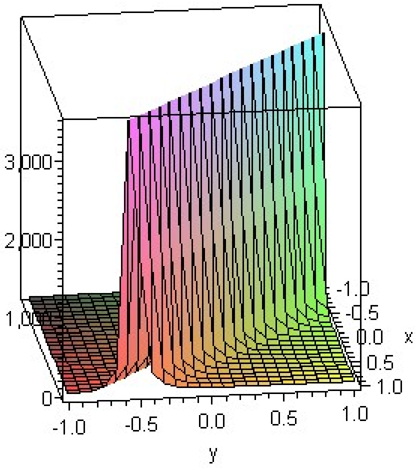

Fig. 5(b) Hyperbolic function solution for ZK-BBM equation.

$$\lambda = 2\sqrt{2}, \mu = 1, a = 1.5, k = 1, t = 1, b = 2, n = 1, C_1 = 1, C_2 = 2, c = 1$$

Case 2: When $\lambda^2 - 4\mu < 0$

$$u(\xi) = \frac{bc(b-k)}{a}(\frac{G'}{G})^2 + \frac{b\lambda c(b-k)}{a}(\frac{G'}{G}) + \frac{b\lambda^2 c + 8bc\mu - k\lambda^2 c + 1 - c - 8kc\mu}{2a};$$

where $\frac{G'(\xi)}{G(\xi)} = -\frac{1}{2}\lambda + \frac{1}{2}\frac{\left(-C_1\sin(\frac{1}{2}\sqrt{-\lambda^2+4\mu}\xi) + C_2\cos(\frac{1}{2}\sqrt{-\lambda^2+4\mu}\xi)\right)\sqrt{-\lambda^2+4\mu}}{C_1\cos(\frac{1}{2}\sqrt{-\lambda^2+4\mu}\xi)_2\sin(\frac{1}{2}\sqrt{-\lambda^2+4\mu}\xi)}$.

where $\xi = x + y - ct$ and C_1 and C_2 are arbitrary constants.

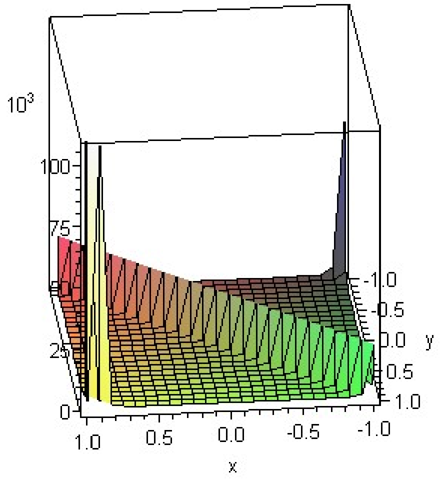

Fig. 5(a) Trigonometric function solution for ZK-BBM equation.
$\lambda = 1, \mu = 2\sqrt{2}, a = 1.5, k = 1, t = 3, b = 2, n = 1, C_1 = 1, C_2 = 2, c = 4$

5.3 Summary

In this chapter, we have successfully derived two type of travelling wave solution in term of hyperbola and trigonometric functions for the generalized ZK-BBM equation by using the $\left(\frac{G'}{G}\right)$-expansion method.

CHAPTER 6

BOUSSINESQ EQUATION

6.1 Introduction

In fluid dynamics, the Boussinesq approximation [21, 24] for water waves is an approximation valid for weakly non-linear and fairly long waves. The approximation is named after Joseph Boussinesq, who first derived them in response to the observation by John Scott Russell of the wave of translation (also known as solitary wave or soliton). The 1872 paper of Boussinesq introduced the equations now known as the Boussinesq equations.

\`The Boussinesq approximation for water waves takes into account the vertical structure of the horizontal and vertical flow velocity. This results in non linear partial differential equations, called Boussinesq-type equations, which incorporate frequency dispersion (as opposite to the shallow water equations, which are not frequency-dispersive). In coastal engineering, Boussinesq-type equations

$$u_{tt} - u_{xx} - u_{xxxx} - 3(u^2)_{xx} = 0 \qquad (6.1.1)$$

are frequently used in computer models for the simulation of water waves in shallow seas and harbours.

6.2 Modified $\left(\frac{G'}{G}\right)$ expansion method

We now consider the Boussinesq equation in the form (6.1.1). In what follows, we study the travelling wave solutions to equation (6.1.1). Substituting

$$u(x,t) = u(\xi) = u(x - vt) \qquad (6.2.1)$$

into equation (6.2.1). Where v is the wave speed. Substituting the above travelling wave transformation and integrating twice, we have

$$(v^2 - 1)u - u'' - 3u^2 + c = 0 \qquad (6.2.2)$$

where c is the integration constant, and the first integrating constant is taken to zero. Suppose that the solutions of the ODE (6.2.2) can be expressed by a polynomial in $\left(\frac{G'}{G}\right)$ as follows:

$$u(\xi) = \alpha_0 + \sum_{i=1}^{m}\left(\alpha_i\left(\frac{G'}{G}\right)^i + \beta_i\left(\frac{G'}{G}\right)^i\right), \qquad (6.2.3)$$

where $G = G(\xi)$, satisfies the second order LODE in the form

$$G''(\xi) + \mu G(\xi) = 0\,. \qquad (6.2.4)$$

Considering the homogeneous balance between u^2 *and* u'' in equation (6.2.2) based on ODE we get m=2.

So we can write (6.2.3)

$$u(\xi)=\alpha_0+\alpha_1\left(\frac{G'}{G}\right)^1+\alpha_2\left(\frac{G'}{G}\right)^2+\beta_1\left(\frac{G'}{G}\right)^{-1}+\beta_2\left(\frac{G'}{G}\right)^{-2} \qquad (6.2.5)$$

and there for

$$u'(\xi)=\alpha_1(-\mu-\left(\frac{G'}{G}\right)^2)+2\alpha_2\left(\frac{G'}{G}\right)\left((-\mu-\left(\frac{G'}{G}\right)^2)\right)-\beta_1\left(\frac{G'}{G}\right)^{-2}\left((-\mu-\left(\frac{G'}{G}\right)^2)\right)-2\beta_2\left(\frac{G'}{G}\right)^{-3}\left(\left((-\mu-\left(\frac{G'}{G}\right)^2)\right)\right)$$

$$(6.2.6)$$

and

$$u''(\xi)=-2\left(\frac{G'}{G}\right)\left((-\mu-\left(\frac{G'}{G}\right)^2)\right)\left(\alpha_1+2\alpha_2\left(\frac{G'}{G}\right)-\beta\left(\frac{G'}{G}\right)^{-2}-2\beta_2\left(\frac{G'}{G}\right)^{-3}\right)+$$
$$\left((-\mu-\left(\frac{G'}{G}\right)^2)\right)\left(2\alpha_2\left((-\mu-\left(\frac{G'}{G}\right)^2)\right)+2\beta_1\left((-\mu-\left(\frac{G'}{G}\right)^2)\right)\left(\frac{G'}{G}\right)^{-3}+6\beta_2\left(\frac{G'}{G}\right)^{-4}\left((-\mu-\left(\frac{G'}{G}\right)^2)\right)\right)$$

$$\ldots.(6.2.7)$$

By using (6.2.5) to (6.2.7) into equation (6.2.2) and collecting all terms with the same power of $\left(\frac{G'}{G}\right)$ together , the left hand side of equation (6.2.2) is converted into another polynomial in $\left(\frac{G'}{G}\right)$, equating each coefficient of this polynomial to zero, yields a set of simultaneous algebraic equations for $\alpha_{1,}\alpha_2,\alpha_0,\beta_1,\beta_2$ and c as follow

$$6\alpha_2 + 3\alpha_2^2 = 0$$
$$2\alpha_1 + 6\alpha_1\alpha_2 = 0$$
$$6\alpha_0\alpha_2 - v^2\alpha_2 + \alpha_2 + 8\mu\alpha_2 + 3\alpha_1^2 = 0$$
$$\alpha_1 + 2\mu\alpha_1 - v^2\alpha_1 + 6\alpha_2\beta_1 + 6\alpha_0\alpha_1 = 0$$
$$6\alpha_2\beta_2 + 3\alpha_0^2 + \alpha_0 + 2\beta_2 + 2\mu^2\alpha_2 - v^2\alpha_0 + 6\alpha_1\beta_1 - c = 0$$
$$2\mu\beta_1 - v^2\beta_1 + 6\alpha_1\beta_2 + 6\alpha_0\beta_1 + \beta_1 = 0$$
$$8\mu\beta_2 - v^2\beta_1 + 3\beta_1^2 + 6\alpha_0\beta_2\beta_2 = 0$$
$$6\beta_1\beta_2 + 2\mu^2\beta_1 = 0$$
$$6\mu^2\beta_2 + 3\beta_2^3 = 0.$$

Solving the algebraic equation above, yields

$$\alpha_0 = \frac{-4}{3}\mu + \frac{1}{6}v^2 - \frac{1}{6}, \alpha_2 = -2, \beta_1 = 0, \beta_2 = -2\mu^2, c = \frac{64}{3}\mu^2 - \frac{1}{12}v^4 + \frac{1}{6}v^2 - \frac{1}{12}. \tag{6.2.8}$$

Substituting the value of $\alpha_0, \alpha_1, \alpha_2, \beta_1, \beta_2$ in (6.2.5), we get

$$u(\xi) = \frac{-4}{3}\mu + \frac{1}{6}v^2 - \frac{1}{6} - 2\left(\frac{G'}{G}\right)^2 - 2\mu^2\left(\frac{G'}{G}\right)^{-2} \tag{6.2.9}$$

Equation (6.2.9) is a solution of equation (6.2.2). Substituting the general solution of equation (6.2.4) into equation (6.2.9) we have travelling wave solution of equation (6.2.3)

$$u(\xi) = \frac{-4}{3}\mu + \frac{1}{6}v^2 - \frac{1}{6} - 2\left[\frac{\sqrt{\mu}(C1\cos(\sqrt{\mu}\xi) - C2\sin(\sqrt{\mu}\xi))}{C1\sin(\sqrt{\mu}\xi) + C2\cos(\sqrt{\mu}\xi)}\right]^2 - 2\mu^2\left[\frac{\sqrt{\mu}(C1\cos(\sqrt{\mu}\xi) - C2\sin(\sqrt{\mu}\xi))}{C1\sin(\sqrt{\mu}\xi) + C2\cos(\sqrt{\mu}\xi)}\right]$$

where $\xi = x - Vt$ and C_1 and C_2 are arbitrary constants.

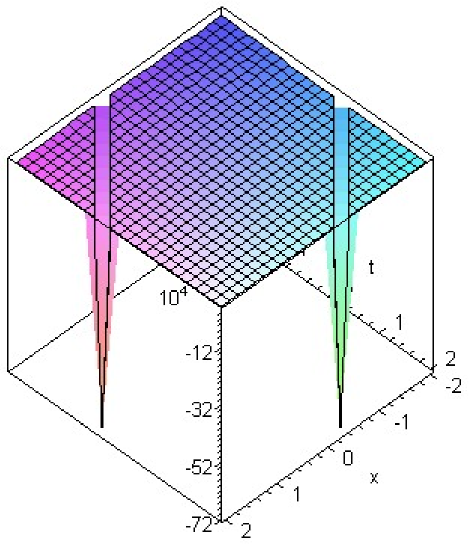

Fig. 6(a) Trigonometric function solution for Boussinesq equation.

$b = 2\sqrt{2}, \mu = 1, k = 1, v = 1, c = 1, C1 = 1, C2 = 1$

6.3 Summary

In this chapter, we have successfully derived travelling wave solution in term of trigonometric functions for the generalized Boussinesq equation by using the modified $\left(\frac{G'}{G}\right)$-expansion method.

Conclusion

In the present thesis, we have used recently introduced method which is called the $\left(\frac{G'}{G}\right)$-expansion method. This method is firstly proposed by which the travelling wave solutions of NPDEs are obtained. The main idea of this method is that the travelling wave solutions of nonlinear equations can be expressed by a polynomial in $\left(\frac{G'}{G}\right)$, where $G = G(\xi)$ satisfies the second order linear ordinary differential equation $G'' + \lambda G' + \mu G = 0$; where $\xi = x - Vt$, where λ, μ and V are constants. The degree of this polynomial can be determined by considering the homogeneous balance between the highest order derivatives and the nonlinear terms appearing in the given nonlinear equations. The coefficients of this polynomial can be obtained by solving a set of algebraic equations resulted from the process of using the proposed method.

From our results obtained in this thesis, we conclude that the $\left(\frac{G'}{G}\right)$- expansion method and modified $\left(\frac{G'}{G}\right)$- expansion method are powerful, effective and convenient. The performance of this method is reliable, simple and gives many new solutions. The $\left(\frac{G'}{G}\right)$ -expansion method has more advantages: It is direct and concise. It is also a standard and computerizable method which allows us to solve complicated nonlinear evolution equations in mathematical physics. Finally, the solutions of the proposed nonlinear evolution equations in this paper have many potential applications in physics and engineering.

We have successfully derived two type of travelling wave solution in term of hyperbola and trigonometric functions for the generalized sinh-Gorden equation in chapter third, Huber's equation in chapter fourth and ZK-BBM equation in chapter fifth by using the $\left(\frac{G'}{G}\right)$ -expansion method.

In the sixth chapter Boussinesq equation has been solved with the modified $\left(\frac{G'}{G}\right)$-expansion method. We have successfully derived travelling wave solution in term of trigonometric functions for the generalized Boussinesq equation by using the modified $\left(\frac{G'}{G}\right)$-expansion method.

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