

**ARITHMETIC OPERATIONS USING SIGMOIDAL
FUNCTION UNDER FUZZY ENVIRONMENT AND IT'S
APPLICATION TO DECISION MAKING**

A Thesis

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by

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CANDIDATE'S DECLARATION

I hereby certify that the work which is being presented in the thesis entitled "Arithmetic operations using Sigmoidal function under fuzzy environment and its application to decision making" in partial fulfillment of the requirement for the award of degree of Master of Science, School of Mathematics (SoM), Thapar University, Patiala is an authentic record of my own carried out under the supervision of Dr. Harish Garg, Assistant Professor, SoM, Thapar University Patiala.

The matter presented in this thesis has not been submitted by me for the award of any other degree of this or any other institute.

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Abstract

Decision making problems are the important parts of modern decision theory due to the rapid development of economic and social uncertainties. Today's, decision maker wants to attain more than one goal in selecting the course of action while satisfying the constraints dictated by environmental processes and resources. But the increasing complexity of the society has brought new problems that involve very large number of variables. The decision maker may be subjective and uncertain about his level of preference due to incomplete information or knowledge, inherent complexity and uncertainty within the decision environment, lack of an appropriate measure or scale. Thus, if the assessment values are known to have various types of vagueness/imprecision or subjectiveness, then the classical decision making techniques are not useful for such problems. To cope with such situation, a fuzzy set theory is one of the successful and widely used theory for dealing with the uncertainties and hence play an important role in the areas of engineering and management disciplines. As compared to other research domains, the fuzzy arithmetic gained great interest in scientific areas such as decision problems, reliability analysis, optimization etc. In order to perform operations on fuzzy observations, fuzzy numbers came into existence.

The objective of this work is divided into two folds. In the first fold, generalized sigmoidal fuzzy number has been introduced and based on it various arithmetic operations are executed for analyzing the system performance. For this, data related to the system are handled with the help of nonlinear sigmoidal type membership function, instead of linear or parabolic fuzzy numbers. Based on it, an arithmetic operations have been performed in the form of fuzzy membership functions by using the concept of distribution and complementary distribution functions. The advantage of the proposed approach is that it gives compressed range of prediction and they do not need the computation of α –

cut of the fuzzy number. Thus, results obtained by using fuzzy numbers are practically much better than those obtained by classical methods. During the second fold, an analysis has been conducted under fuzzy environment for depicting the best type of fuzzy number, namely linear, parabolic and sigmoidal fuzzy numbers, under the different set of criteria. Finally, based on the collected information in the form of decision matrix all the individual decision makers' opinion for rating the candidate are aggregated using the fuzzy positive and negative ideal solutions.

The present thesis is organized into five chapters which are briefly summarized as follows:

A brief account of the related work of various authors in the evaluation of arithmetic operations, membership functions, decision makers etc., by using conventional, fuzzy and optimization techniques is presented in the first chapter. In **Chapter 2**, the basic and preliminaries related to the arithmetic operation and to be used in the subsequent chapters are given.

Chapter 3 presents a concept of generalized sigmoidal fuzzy number. Also, the definition of trapezoidal sigmoidal fuzzy number and their arithmetic operations between two generalized sigmoidal fuzzy numbers are introduced. Various arithmetic operations, such as addition, subtraction, multiplication, inverse, division etc are studied by using the concept of the distribution and complementary distribution functions. The operations have been validated through some elementary applications and results are compared with that of α -cut method and shows the supremacy of the result.

In **Chapter 4**, the decision making technique has been introduced for computing the best type of the fuzzy number out of linear, parabolic and sigmoidal fuzzy numbers under the different set of criteria. Since handling the uncertainty in the data is the big task for the decision maker, so results computed in Chapter 3 has been used here in the form of fuzzy decision matrix and then aggregated by using fuzzy positive and negative ideal solutions. Finally, a closeness coefficient is defined for each alternative, to determine the rankings of all alternatives.

Chapter 5 deals with the overall concluding observations of this study and a brief discussion on the scope for future work.

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Patiala

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(Pavneet Kaur)

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Chapter 1

Introduction

With the advancement in science and technology, complex and non-linear systems are coming into existence and increased automation is the need of the hour. For the control of such highly complex and non-linear systems the conventional approaches fail miserably because of their limitations. As earlier the systems were designed only to accept precise and accurate data. However, in certain systems it is not possible to get the accurate data. Moreover, a traditional set includes or do not include an individual element; there is no other case than true or false. Thus, in this light, Zadeh [18] extended this notion by allowing partial membership, i.e. it allows intermediate values between the conventional evaluations such as *yes/no*, *true/false*, *etc.* However, we could find that fuzzy logic may obtain different simulated efficiency and performance while adopting various forms of fuzzy arithmetic in the form of a triangular fuzzy numbers [6]. In the framework of fuzzy arithmetic [13] various operations as, e.g., addition, subtraction, etc., are realized. These operations are made with the use of Zadeh's possibilistic extension principle or its new, improved, and also possibilistic version proposed by Klir [15], which takes into account the so-called requisite constraints. Arithmetic operations are also performed under the assumption which was introduced by Zadeh [20] that the membership function of a fuzzy set is of a possibilistic character and that each element of the universal set, with a non-zero membership grade, belongs to a fuzzy set. Now in these days, we experience many applications which perform computation using ambiguous data. In all such cases, the imprecise data from the measuring instruments are generally expressed in the form of intervals, and hence suitable mathematical operations are performed over these intervals

to obtain a reliable data of the measurements. This type of computation is called *interval arithmetic or interval analysis*. For the past few years, various authors [2, 3, 5, 9, 14] have addressed the computation of fuzzy arithmetic operations under the different environment using linear (triangular or trapezoidal) membership functions. A well known extension principle and α -cut methods are used for their computation in which former one is directly considering the membership functions while the latter one is to deal with the α -cut sets without considering the membership functions. Both the methods have their own limitations, such as it is not always possible to compute the α -cut of a fuzzy numbers and hence their approaches are quite restricted. Chutia et al. [4] developed a method of finding membership function for functions of triangular fuzzy variable from the concept of credibility theory and a method for computation of basic arithmetical operations of fuzzy variables is forwarded. For more details about the applications and fuzzy arithmetic operation, we may refer to [2, 5, 7, 8, 10, 11, 14, 16, 17] and their corresponding references related to different fields. Moreover, it is quite clear that there exists a large amount of uncertainties during the computation when linear membership functions have been taken. Therefore, there is a need of suitable methodology which will handle this problem and compute the arithmetic operations in a fuzzy environment. So, in this study, instead of using these methods, we compute the membership functions by using distribution and complementary distribution functions.

Multi-criteria/attribute decision making is one of the process for finding the optimal alternative from all the feasible alternatives according to some criteria or attributes. Technique for order preference by similarity to an ideal solution (TOPSIS), known as a classical multiple attribute decision making (MADM) method, has been developed by Hwang and Yoon [12] for solving the MADM problem. It is based on the idea that the chosen alternative should have the shortest distance from the positive ideal solution, and, on the other side, the farthest distance from the negative ideal solution. Traditionally, it has been generally assumed that all the information which access the alternative in terms of criteria and their corresponding weights are expressed in the form of crisp numbers. But it has been widely recognized that most of the decisions in the real-life situations are taken in the environment where the goals and constraints are generally imprecise or vague

in nature and hence cannot estimate his preference with an exact numerical value. This is due to the fact that most of the information given by the decision-makers should be under time pressure and lack of knowledge or data. Moreover, it may be that the decision makers have limited information processing capacities. Therefore, the analysis conducted under such circumstances are not ideal and hence does not tell the exact information to the system analyst. To cope with such situation, fuzzy set theory [18] has been widely used for handling the uncertainties and vagueness of the data.

1.1 Objective of the Thesis

The arithmetic operations on fuzzy quantities are widely used in the literature. There are two well-known additions of fuzzy quantities in vector space that have been adopted in the literature. One is based on the extension principle by directly considering the membership functions, and another one uses the α -level sets without considering the membership functions. Here, in this thesis, instead of using these methods for computing the membership functions, we compute the membership functions by using distribution and complementary distribution functions. Thus, the objective of the presented work is to compute the various arithmetic operations of fuzzy set theory by utilizing the available, imprecise and vague data. Fuzzy set theory has been used for handling the uncertainties in the data and then analyze the various arithmetic operations in the form of fuzzy membership functions using generalized sigmoidal fuzzy numbers. The concept of distribution and complementary distribution functions have been used for computing these membership functions. The advantage of this approach is that it gives a compressed range of prediction and it does not involve the computation of α -cuts for finding the membership functions. Apart from their computation of the membership functions, method for computing the best alternative under the given set of different criteria has been presented here for finding the best alternative under the different set of criteria.

1.2 Structure of the Thesis

The present thesis is organized into five chapters including the present one that contains mainly the literature review. The rest of the chapters are described below:

In **Chapter 2**, the basic and preliminaries related to the arithmetic operations and to be used in the subsequent chapters are given.

Chapter 3 presents a concept of generalized sigmoidal fuzzy number. Also, the definition of trapezoidal sigmoidal fuzzy number and arithmetic operations between two generalized sigmoidal fuzzy numbers are introduced. Various arithmetic operations, such as addition, subtraction, multiplication, inverse, division etc are studied by using the concept of the distribution and complementary distribution functions. The operations have been validated through some elementary applications and results are compared with that of α -cut method and shows the supremacy of the result.

In **Chapter 4**, the decision making technique has been introduced for computing the best type of the fuzzy number out of linear, parabolic and sigmoidal fuzzy numbers under the different set of criteria. Since handling the uncertainty in the data is the big task for the decision maker, so in order to that the result computed in Chapter 3 has been used here in the form of fuzzy decision matrix for analyzing the best candidate. The collected information are aggregated by using fuzzy positive and negative ideal solutions. Finally, a closeness coefficient is defined for each alternative, to determine the rankings of all alternatives.

Chapter 5 deals with the overall concluding observations of this study and a brief discussion on the scope for future work.

Chapter 2

Preliminaries

This chapter presents some of the fundamental definitions and mathematical theory for fuzzy set theory. The focus is on defining the fuzzy set, α -cuts, convex and normal fuzzy set, fuzzy numbers and sigmoidal fuzzy numbers.

2.1 Fuzzy set theory

Zadeh extended and generalized the concept of crisp set by allowing the partial membership i.e. between 0 and 1 and named as Fuzzy sets [18]. In it, he defines the degree of membership function which takes values between 0 and 1 and represents degree of belongingness to that set which is denoted by $[0, 1]$, where 0 represent the “no” and 1 represent “yes”. Thus, based on it, a fuzzy set has been defined as a mapping from universe of discourse U to $[0, 1]$ i.e. $\tilde{A} = \{(x, \mu_{\tilde{A}}(x) \mid x \in U\}$ where $\mu_{\tilde{A}}(x)$ is the degree of membership of x in fuzzy set \tilde{A} . Clearly $\mu_{\tilde{A}}(x) \in [0, 1]$.

2.2 Membership functions

Membership function defines the fuzziness in a fuzzy set irrespective of the elements in the set, which are discrete or continuous. For a fuzzy set \tilde{A} a membership function, denoted by $\mu_{\tilde{A}}(\cdot)$ maps U to the membership space M , i.e. $\mu_{\tilde{A}} : U \rightarrow M$. The membership value ranges in the interval $[0, 1]$ i.e. the range of the membership function is a subset of the non-negative real numbers whose supremum is finite.

The three main basic features involved in characterizing membership function are the following[18].

- (i) **Core:** The core of a membership function for some fuzzy set \tilde{A} is defined as that region of universe that is characterized by complete membership in the set \tilde{A} . The core has elements x of the universe such that

$$\mu_{\tilde{A}}(x) = 1 \quad (2.2.1)$$

The core of a fuzzy set may be an empty set.

- (ii) **Support:** The support of a membership function for a fuzzy set \tilde{A} is defined as that region of universe that is characterized by a nonzero membership in the set \tilde{A} . The support comprises elements x of the universe such that

$$\mu_{\tilde{A}}(x) > 0 \quad (2.2.2)$$

- (iii) **Boundary:** The boundary of a membership function for a fuzzy set \tilde{A} is defined as the region of universe that contains a nonzero but not a complete membership. In other words the boundary comprises those elements x of the universe such that

$$0 < \mu_{\tilde{A}}(x) < 1 \quad (2.2.3)$$

The boundary elements are those which possess partial membership in the fuzzy set \tilde{A} .

2.3 α - cuts

α -cut is one of the most significant and extensively used concept in fuzzy set theory which was introduced by Zadeh [19]. When we want to exhibit an element $x \in X$ that typically belongs to a fuzzy set \tilde{A} , we may demand that its membership value be greater than some threshold $\alpha \in [0, 1]$.

For a fuzzy set \tilde{A} ,

$$A_{\alpha} = \{x \mid \mu_{\tilde{A}}(x) > \alpha\}; \quad \alpha \in [0, 1]$$

$$A_{\alpha} = \{x \mid \mu_{\tilde{A}}(x) \geq \alpha\}; \quad \alpha \in [0, 1]$$

are called strong α - cut and weak α - cut respectively.

2.4 Convex fuzzy set

A fuzzy set $\tilde{A} = \langle (x, \mu_{\tilde{A}}(x)) \mid x \in U \rangle$ is said to be convex fuzzy set if the following inequality has been hold for $x_1, x_2 \in X$

$$\mu_{\tilde{A}}(\lambda x_1 + (1 - \lambda)x_2) \geq \min[\mu_{\tilde{A}}(x_1), \mu_{\tilde{A}}(x_2)]$$

If above inequality does not hold then that it is said to be non-convex fuzzy set.

2.5 Normal fuzzy set

A fuzzy set is said to be normal fuzzy set if there exist at least one element in the universal set U such that their corresponding membership function is unity.

2.6 Fuzzy number

A convex, normal membership function on the real line \mathbb{R} is called a fuzzy number i.e., if its membership function is piecewise continuous and there exist at least one $x_0 \in U$ such that $\mu_{\tilde{A}}(x_0) = 1$. The corresponding membership function defined on $[a, b] \neq \emptyset$ is given as

$$\mu_{\tilde{A}}(x) = \begin{cases} f(x) & ; \quad x \in (-\infty, a) \\ 1 & ; \quad x = [a, b] \\ g(x) & ; \quad x \in (b, \infty) \\ 0 & ; \quad \text{otherwise} \end{cases}$$

where f and g are monotonic, continuous from the right and left, nondecreasing and nonincreasing functions such that $f(x) = 0$ for $x \in (-\infty, \omega_1)$ and $g(x) = 0$ for $x \in (\omega_2, \infty)$.

2.7 Sigmoidal functions

A measurable function $\psi : \mathbb{R} \rightarrow \mathbb{R}$ is called “a sigmoidal function” whenever

$$\lim_{x \rightarrow -\infty} \psi(x) = 0 \quad \text{and} \quad \lim_{x \rightarrow +\infty} \psi(x) = 1$$

These functions are usually S -shaped curves and is very useful to handle the uncertainties in the data. As non-linearity is avoided in the linear membership function and hence it

is difficult to select the solution of the problem. Moreover, a triangular and trapezoidal membership functions are an approximate form of a sigmoidal function. Therefore, sigmoidal function is much much profitable to handle the decision maker goals during their implementation than the linear ones. For this, a standard sigmoidal function

$$\psi(a) = \frac{1}{1 + e^{-a}}$$

is considered by taking the domain of this function as $[-5, 5]$ and hence their corresponding membership function for $x \leq m$ and $x \geq M$, respectively are defined as

$$\mu(x) = \begin{cases} 1 & ; \quad x \leq m \\ \frac{\psi(5) - \psi\left(\left\{x - \frac{M+m}{2}\right\} \frac{10}{M-m}\right)}{\psi(5) - \psi(-5)} & ; \quad m \leq x \leq M \\ 0 & ; \quad x \geq M \end{cases}$$

and

$$\mu(x) = \begin{cases} 1 & ; \quad x \geq M \\ \frac{\psi\left(\left\{x - \frac{M+m}{2}\right\} \frac{10}{M-m}\right) - \psi(-5)}{\psi(5) - \psi(-5)} & ; \quad m \leq x \leq M \\ 0 & ; \quad x \leq m \end{cases}$$

Here m and M represents the lower and upper limits of the function and $(M + m)/2$ be its crossover point.

2.8 Generalized fuzzy number

A fuzzy number $\tilde{A} = \langle (a_1, a_2, a_3; \omega) \mid a_i \in \mathbb{R} \rangle$, is said to be generalized fuzzy number if there corresponding membership function $\mu_{\tilde{A}}(x) : \mathbb{R} \rightarrow [0, 1]$ satisfies the following properties.

- (i) It is continuous.
- (ii) It is zero for all $x \in (-\infty, a_1] \cup [a_3, \infty)$.
- (iii) It is strictly increasing on $[a_1, a_2]$ and strictly decreasing on $[a_2, a_3]$.

(iv) $\mu_{\tilde{A}}(x) = \omega$ for all $x = a_2$ where $0 < \omega \leq 1$.

If $\omega = 1$ then it is said to be normal fuzzy number else it is generalized fuzzy number. A fuzzy number $\tilde{A} = (a_1, a_2, a_3; \omega)$, is called

- a generalized linear fuzzy number if

$$\mu_{\tilde{A}}(x) = \begin{cases} \omega \left(\frac{x - a_1}{a_2 - a_1} \right) & ; \text{ if } a_1 \leq x < a_2 \\ \omega & ; \text{ if } x = a_2 \\ \omega \left(\frac{a_3 - x}{a_3 - a_2} \right) & ; \text{ if } a_2 \leq x < a_3 \\ 0 & ; \text{ if otherwise} \end{cases}$$

- a generalized parabolic fuzzy number if

$$\mu_{\tilde{A}}(x) = \begin{cases} \omega \left(\frac{x - a_1}{a_2 - a_1} \right)^2 & ; \text{ } a_1 \leq x < a_2 \\ \omega & ; \text{ } x = a_2 \\ \omega \left(\frac{a_3 - x}{a_3 - a_2} \right)^2 & ; \text{ } a_2 \leq x < a_3 \\ 0 & ; \text{ otherwise} \end{cases}$$

- a generalized sigmoidal fuzzy number if

$$\mu_{\tilde{A}}(x) = \begin{cases} \omega \left(\frac{\psi\left(\left\{x - \frac{a_1 + a_2}{2}\right\} \frac{10}{a_2 - a_1}\right) - \psi(-5)}{\psi(5) - \psi(-5)} \right) & ; \text{ } a_1 \leq x < a_2 \\ \omega & ; \text{ } x = a_2 \\ \omega \left(\frac{\psi(5) - \psi\left(\left\{x - \frac{a_2 + a_3}{2}\right\} \frac{10}{a_3 - a_2}\right)}{\psi(5) - \psi(-5)} \right) & ; \text{ } a_2 \leq x \leq a_3 \\ 0 & ; \text{ otherwise} \end{cases}$$

Chapter 3

Arithmetic operations on the Sigmoidal fuzzy numbers

The work present in this chapter is an attempt to give an alternative approach for computing the membership functions of the various arithmetic operations in fuzzy environment. The uncertainties which are present in the data are handled with the help of defining the non-linear namely sigmoidal membership functions. The major advantages of using these distribution functions are that they do not need the computation of α -cuts and hence the method is quite useful in those cases where it is difficult to compute the α -cut of the fuzzy numbers. The operations have been validated through some elementary applications and illustrated with their defuzzified values. Finally results are compared with the α -cut method by taking linear and parabolic membership functions and shows the supremacy of the result.

3.1 Membership function for function of a fuzzy variable

Consider a fuzzy variable $\tilde{X} = (a_1, a_2, a_3; \omega)$ with height of the variable is ω and membership function is defined as below

$$\mu_X(x) = \begin{cases} \omega L_1(x) & \text{if } a_1 \leq x < a_2 \\ \omega & \text{if } x = a_2 \\ \omega R_1(x) & \text{if } a_2 < x \leq a_3 \end{cases}$$

where $L_1(x)$ and $R_1(x)$ are the nondecreasing and nonincreasing functions of x respectively.

Let $F(X) = [F(a_1), F(a_2), F(a_3); F(\omega)]$ be the fuzzy variable of the function $F(X)$. In order to find the fuzzy membership of $F(X)$, let $z = F(x), x \in X$ or $x = \zeta(z)$. Hence the density functions for the distribution functions $L_1(x)$ and $R_1(x)$ are obtained as

$$\begin{aligned} f_1(x) &= \frac{d}{dx}(L_1(x)) = \eta_1(z) \quad \text{at } x = \zeta_1(z) \\ g_1(x) &= \frac{d}{dx}(R_1(x)) = \eta_2(z) \quad \text{at } x = \zeta_2(z) \end{aligned}$$

Now, let,

$$\frac{dx}{dz} = \frac{d}{dz}(\zeta_1(z)) = m_1(z) \quad ; \quad \frac{dx}{dz} = \frac{d}{dz}(\zeta_2(z)) = m_2(z)$$

Then the distribution function for $F(x)$ would be given as

$$\int_{F(a_1)}^x \eta_1(z)m_1(z)dz \quad ; \quad F(a_1) \leq x \leq F(a_2)$$

while their complementary distribution function would be given as

$$\int_x^{F(a_3)} \eta_2(z)m_2(z)dz \quad ; \quad F(a_2) \leq x \leq F(a_3)$$

Hence, the membership function for the fuzzy variable function $F(x)$ is given by

$$\mu_{F(x)}(x) = \begin{cases} F(\omega) \int_{F(a_1)}^x \eta_1(z)m_1(z)dz & ; \quad F(a_1) \leq x < F(a_2) \\ F(\omega) & ; \quad x = F(a_2) \\ F(\omega) \int_x^{F(a_3)} \eta_2(z)m_2(z)dz & ; \quad F(a_2) < x \leq F(a_3) \\ 0 & ; \quad \text{otherwise} \end{cases}$$

In order to evaluate the fuzzy arithmetic for the sigmoidal fuzzy number, consider the two sigmoidal fuzzy numbers $X = [a_1, a_2, a_3; \omega_1]$ and $Y = [b_1, b_2, b_3; \omega_2]$ where ω_1, ω_2 represents the degree of their membership functions in crisp environment. Based on it, their corresponding membership functions are defined as

$$\mu_X(x) = \begin{cases} \omega_1 L_1(x) & \text{if } a_1 \leq x < a_2 \\ \omega_1 & \text{if } x = a_2 \\ \omega_1 R_1(x) & \text{if } a_2 < x \leq a_3 \end{cases} \quad (3.1.1)$$

and

$$\mu_Y(y) = \begin{cases} \omega_2 L_1(y) & \text{if } b_1 \leq y < b_2 \\ \omega_2 & \text{if } y = b_2 \\ \omega_2 R_1(y) & \text{if } b_2 < y \leq b_3 \end{cases} \quad (3.1.2)$$

where

$$L_1(x) = \frac{\psi(\{x - \frac{a_1+a_2}{2}\} \frac{10}{a_2-a_1}) - \psi(-5)}{\psi(5) - \psi(-5)} \quad \text{and} \quad L_1(y) = \frac{\psi(\{y - \frac{b_1+b_2}{2}\} \frac{10}{b_2-b_1}) - \psi(-5)}{\psi(5) - \psi(-5)}$$

are the left distribution functions, while

$$R_1(x) = \frac{\psi(5) - \psi(\{x - \frac{a_2+a_3}{2}\} \frac{10}{a_3-a_2})}{\psi(5) - \psi(-5)} \quad \text{and} \quad R_1(y) = \frac{\psi(5) - \psi(\{y - \frac{b_2+b_3}{2}\} \frac{10}{b_3-b_2})}{\psi(5) - \psi(-5)}$$

are the right distribution functions of X and Y respectively.

In order to find the distribution functions of their corresponding arithmetic operations, we start with equating $L_1(x)$ with $L_1(y)$ and $R_1(x)$ with $R_1(y)$ and we obtain $y = \phi_1(x)$ and $y = \phi_2(x)$ respectively, where $\phi_1(x) = \frac{b_1+b_2}{2} + \frac{b_2-b_1}{a_2-a_1}(x - \frac{a_1+a_2}{2})$, $\phi_2(x) = \frac{b_2+b_3}{2} + \frac{b_3-b_2}{a_3-a_2}(x - \frac{a_2+a_3}{2})$. Let Z be the resultant of the arithmetic operations of X and Y . Then at $y = \phi_1(x)$ and $y = \phi_2(x)$ we get $x = \zeta_1(z)$ and $x = \zeta_2(z)$ respectively. Based on these distribution functions, we get the probability density function corresponding to the distribution and complementary distribution function as

$$f_1(x) = \frac{d}{dx}(L_1(x)) = \eta_1(z) \quad \text{at } x = \zeta_1(z)$$

$$g_1(x) = \frac{d}{dx}(R_1(x)) = \eta_2(z) \quad \text{at } x = \zeta_2(z)$$

Also,

$$\frac{dx}{dz} = \frac{d}{dz}(\zeta_1(z)) = m_1(z)$$

$$\frac{dx}{dz} = \frac{d}{dz}(\zeta_2(z)) = m_2(z)$$

Hence, the distribution function for fuzzy variable $F(z)$ where $F(z) = [z_1, z_2, z_3; \omega]$, $\omega = \min(\omega_1, \omega_2)$ are

$$\mu_{F(z)}(x) = \begin{cases} \omega \int_{z_1}^x \eta_1(z) m_1(z) dz & \text{if } z_1 \leq x < z_2 \\ \omega & \text{if } x = z_2 \\ \omega \int_x^{z_3} \eta_2(z) m_2(z) dz & \text{if } z_2 < x \leq z_3 \end{cases}$$

Based on these functions, we obtain the membership functions of various operators such as addition, subtraction, multiplication, inverse etc.

Lemma 3.1.1. *If $L_1(x)$ be the left distribution function of the sigmoidal fuzzy membership function such that*

$$L_1(x) = \frac{\psi(\{x - \frac{a_1+a_2}{2}\} \frac{10}{a_2-a_1}) - \psi(-5)}{\psi(5) - \psi(-5)}$$

then

$$\frac{d}{dx}(L_1) = \frac{10}{(a_2 - a_1)(\psi(5) - \psi(-5))} \left(\psi(\{x - \frac{a_1 + a_2}{2}\} \frac{10}{a_2 - a_1}) \right) \left(1 - \psi(\{x - \frac{a_1 + a_2}{2}\} \frac{10}{a_2 - a_1}) \right)$$

Proof. As

$$\begin{aligned} \psi(ax + b) &= \frac{1}{1 + \exp(-(ax + b))} \\ \Rightarrow \frac{d}{dx}(\psi(ax + b)) &= \frac{-a}{(1 + \exp(-(ax + b)))^2} (-\exp(-(ax + b))) \\ &= a\psi(ax + b)(1 - \psi(ax + b)) \\ \therefore \frac{d}{dx}(L_1) &= \frac{1}{\psi(5) - \psi(-5)} \frac{d}{dx} \left(\psi(\{x - \frac{a_1 + a_2}{2}\} \frac{10}{a_2 - a_1}) \right) \\ &= \frac{10}{(a_2 - a_1)(\psi(5) - \psi(-5))} \left(\psi(\{x - \frac{a_1 + a_2}{2}\} \frac{10}{a_2 - a_1}) \right) \left(1 - \psi(\{x - \frac{a_1 + a_2}{2}\} \frac{10}{a_2 - a_1}) \right) \end{aligned} \quad \square$$

Lemma 3.1.2. *If $\psi(ax + b) = \frac{1}{1 + \exp(-(ax + b))}$ be represent the sigmoidal function then*

$$\int \psi(ax + b)(1 - \psi(ax + b))dx = \frac{1}{a}\psi(ax + b)$$

Proof. As

$$\begin{aligned} \psi(ax + b) &= \frac{1}{1 + \exp(-(ax + b))} \\ \therefore \psi(ax + b)(1 - \psi(ax + b)) &= \frac{\exp(-(ax + b))}{(1 + \exp(-(ax + b)))^2} \\ \Rightarrow \int \psi(ax + b)(1 - \psi(ax + b))dx &= \int \frac{\exp(-(ax + b))}{(1 + \exp(-(ax + b)))^2} dx \\ &= \frac{1}{(a)(1 + \exp(-(ax + b)))} \\ &= \frac{1}{a}\psi(ax + b) \end{aligned} \quad \square$$

Theorem 3.1.1. *If X and Y be the two sigmoidal fuzzy number over the universe U whose membership function are defined in Eq. (3.1.1) and (3.1.2) respectively then the fuzzy variable $Z = X + Y$ is also a sigmoidal fuzzy number with membership function*

$$\mu_Z(x) = \begin{cases} \omega \frac{\psi\left(\left\{x - \frac{a_1 + a_2 + b_1 + b_2}{2}\right\} \frac{10}{(a_2 + b_2 - a_1 - b_1)}\right)^{-\psi(-5)}}{\psi(5) - \psi(-5)} & ; \quad a_1 + b_1 \leq x < a_2 + b_2 \\ \omega & ; \quad x = a_2 + b_2 \\ \omega \frac{\psi(5) - \psi\left(\left\{x - \frac{a_2 + a_3 + b_2 + b_3}{2}\right\} \frac{10}{(a_3 + b_3 - a_2 - b_2)}\right)}{\psi(5) - \psi(-5)} & ; \quad a_2 + b_2 < x \leq a_3 + b_3 \end{cases}$$

Proof. Consider two sigmoidal fuzzy numbers X and Y whose membership functions are defined in Eq. (3.1.1) and (3.1.2) respectively. For addition of these fuzzy sigmoidal numbers we get the fuzzy number $Z = X + Y = [a_1 + b_1, a_2 + b_2, a_3 + b_3]$. Now, let $z = x + y$ we get $z = x + \phi_1(x)$ and $z = x + \phi_2(x)$ which implies that $x = \zeta_1(z)$ and $x = \zeta_2(z)$ where

$$x = \zeta_1(z) = \frac{(a_2 - a_1)z - (b_1 a_2 - a_1 b_2)}{a_2 + b_2 - a_1 - b_1}$$

Hence,

$$m_1(z) = \frac{d(\zeta_1(z))}{dz} = \frac{a_2 - a_1}{a_2 - a_1 + b_2 - b_1}$$

Also, from Lemma I, we have

$$\begin{aligned} \eta_1(z) &= \frac{d}{dx}(L_1) \quad \text{at } x = \zeta_1(z) \\ &= \frac{10}{(a_2 - a_1)(\psi(5) - \psi(-5))} \left(\psi\left(\left\{x - \frac{a_1 + a_2}{2}\right\} \frac{10}{a_2 - a_1}\right) \right) \left(1 - \psi\left(\left\{x - \frac{a_1 + a_2}{2}\right\} \frac{10}{a_2 - a_1}\right) \right) \Big|_{x=\zeta_1(z)} \end{aligned}$$

Now, at $x = \zeta_1(z) = \frac{(a_2 - a_1)z - (b_1 a_2 - a_1 b_2)}{a_2 + b_2 - a_1 - b_1}$, we have

$$\begin{aligned} &\psi\left(\left\{x - \frac{a_1 + a_2}{2}\right\} \frac{10}{a_2 - a_1}\right) \\ &= \psi\left(\left(\frac{(a_2 - a_1)z - (b_1 a_2 - a_1 b_2)}{a_2 + b_2 - a_1 - b_1} - \frac{a_1 + a_2}{2}\right) \frac{10}{a_2 - a_1}\right) \\ &= \psi\left(\left(\frac{2(a_2 - a_1)z - 2(b_1 a_2 - a_1 b_2) - (a_1 + a_2)(a_2 + b_2 - a_1 - b_1)}{2(a_2 + b_2 - a_1 - b_1)}\right) \frac{10}{a_2 - a_1}\right) \\ &= \psi\left(\left(\frac{2(a_2 - a_1)z + (a_1 + a_2)(a_1 - a_2) + b_2(a_1 - a_2) + b_1(a_1 - a_2)}{2(a_2 + b_2 - a_1 - b_1)}\right) \frac{10}{a_2 - a_1}\right) \\ &= \psi\left(\left(\frac{2z - (a_1 + a_2 + b_1 + b_2)}{2(a_2 + b_2 - a_1 - b_1)}\right) 10\right) \\ &= \psi\left(\left(z - \frac{a_1 + a_2 + b_1 + b_2}{2}\right) \frac{10}{(a_2 + b_2 - a_1 - b_1)}\right) \end{aligned}$$

Thus,

$$\eta_1(z) = \frac{10}{(a_2 - a_1)(\psi(5) - \psi(-5))} \psi\left(\left(z - \frac{a_1 + a_2 + b_1 + b_2}{2}\right) \frac{10}{(a_2 + b_2 - a_1 - b_1)}\right) \times \left[1 - \psi\left(\left(z - \frac{a_1 + a_2 + b_1 + b_2}{2}\right) \frac{10}{(a_2 + b_2 - a_1 - b_1)}\right)\right]$$

Therefore,

$$\begin{aligned} L_1(z) &= \int_{a_1+b_1}^x m_1(z)\eta_1(z)dz \\ &= \int_{a_1+b_1}^x \frac{10(a_2 - a_1)}{(a_2 - a_1 + b_2 - b_1)(a_2 - a_1)(\psi(5) - \psi(-5))} [\psi(\cdot)(1 - \psi(\cdot))] dz \\ &= \frac{1}{(\psi(5) - \psi(-5))} \left[\psi\left(\left(x - \frac{a_1 + a_2 + b_1 + b_2}{2}\right) \frac{10}{(a_2 + b_2 - a_1 - b_1)}\right) \right]_{a_1+b_1}^x \\ &= \frac{1}{(\psi(5) - \psi(-5))} \left[\psi\left(x - \left\{\frac{a_1 + a_2 + b_1 + b_2}{2}\right\} \frac{10}{(a_2 + b_2 - a_1 - b_1)}\right) - \psi\left(\left(a_1 + b_1 - \frac{a_1 + a_2 + b_1 + b_2}{2}\right) \frac{10}{(a_2 + b_2 - a_1 - b_1)}\right) \right] \\ &= \frac{1}{(\psi(5) - \psi(-5))} \left[\psi\left(x - \left\{\frac{a_1 + a_2 + b_1 + b_2}{2}\right\} \frac{10}{(a_2 + b_2 - a_1 - b_1)}\right) - \psi(-5) \right] \end{aligned}$$

Thus, left sided distribution function for the fuzzy number $Z = X + Y$ is

$$L_1(z) = \frac{\psi\left(x - \left\{\frac{a_1 + a_2 + b_1 + b_2}{2}\right\} \frac{10}{(a_2 + b_2 - a_1 - b_1)}\right) - \psi(-5)}{\psi(5) - \psi(-5)}$$

Similarly, if $y = \phi_2(x)$ then $z = x + y$ becomes $x = \zeta_2(z)$ where

$$\zeta_2(z) = \frac{(a_3 - a_2)z - (b_2a_3 - a_2b_3)}{(a_3 - a_2) + (b_3 - b_2)}$$

Thus,

$$m_2(z) = \frac{d(\zeta_2(z))}{dz} = \frac{a_3 - a_2}{(a_3 - a_2) + (b_3 - b_2)}$$

and,

$$\begin{aligned} \eta_2(z) &= \frac{d}{dx}(R_1(x)) \quad \text{at } x = \zeta_2(z) \\ &= \frac{-10}{(a_3 - a_2)(\psi(5) - \psi(-5))} \left(\psi\left(\left\{x - \frac{a_2 + a_3}{2}\right\} \frac{10}{a_3 - a_2}\right) \right) \left(1 - \psi\left(\left\{x - \frac{a_2 + a_3}{2}\right\} \frac{10}{a_3 - a_2}\right) \right) \end{aligned}$$

At $x = \zeta_2(z) = \frac{(a_3 - a_2)z - (b_2 a_3 - a_2 b_3)}{(a_3 - a_2) + (b_3 - b_2)}$, we have

$$\begin{aligned}
& \psi\left(\left\{x - \frac{a_2 + a_3}{2}\right\} \frac{10}{a_3 - a_2}\right) \\
&= \psi\left(\left\{\frac{(a_3 - a_2)z - (b_2 a_3 - a_2 b_3)}{(a_3 - a_2) + (b_3 - b_2)} - \frac{a_2 + a_3}{2}\right\} \frac{10}{a_3 - a_2}\right) \\
&= \psi\left(\frac{2(a_3 - a_2)z - 2(b_2 a_3 - a_2 b_3) - (a_2 + a_3)(a_3 + b_3 - a_2 - b_2)}{2(a_3 + b_3 - a_2 - b_2)} \frac{10}{a_3 - a_2}\right) \\
&= \psi\left(\frac{2(a_3 - a_2)z + (a_2 + a_3)(a_2 - a_3) + b_2(a_2 - a_3) + b_3(a_2 - a_3)}{2(a_3 + b_3 - a_2 - b_2)} \frac{10}{a_3 - a_2}\right) \\
&= \psi\left(\left(\frac{2z - (a_2 + a_3 + b_2 + b_3)}{2(a_3 + b_3 - a_2 - b_2)}\right) 10\right) \\
&= \psi\left(\left(z - \frac{a_3 + a_2 + b_2 + b_3}{2}\right) \frac{10}{a_3 + b_3 - a_2 - b_2}\right)
\end{aligned}$$

Thus,

$$\begin{aligned}
\eta_2(z) &= \frac{10}{(a_3 - a_2)(\psi(5) - \psi(-5))} \psi\left(\left(z - \frac{a_3 + a_2 + b_2 + b_3}{2}\right) \frac{10}{a_3 + b_3 - a_2 - b_2}\right) \times \\
&\quad \left[1 - \psi\left(\left(z - \frac{a_3 + a_2 + b_2 + b_3}{2}\right) \frac{10}{a_3 + b_3 - a_2 - b_2}\right)\right]
\end{aligned}$$

Therefore,

$$\begin{aligned}
R_1(z) &= \int_x^{a_3 + b_3} \eta_2(z) m_2(z) dz \\
&= \int_x^{a_3 + b_3} \frac{10(a_3 - a_2)}{(a_3 - a_2 + b_3 - b_2)(a_3 - a_2)(\psi(5) - \psi(-5))} [\psi(\cdot)(1 - \psi(\cdot))] dz \\
&= \frac{1}{(\psi(5) - \psi(-5))} \left[\psi\left(z - \frac{a_3 + a_2 + b_2 + b_3}{2}\right) \frac{10}{a_3 + b_3 - a_2 - b_2} \right]_x^{a_3 + b_3} \\
&= \frac{1}{(\psi(5) - \psi(-5))} \left[\psi\left(\left(a_3 + b_3 - \frac{a_3 + a_2 + b_2 + b_3}{2}\right) \frac{10}{a_3 + b_3 - a_2 - b_2}\right) - \right. \\
&\quad \left. \psi\left(\left(x - \frac{a_3 + a_2 + b_2 + b_3}{2}\right) \frac{10}{a_3 + b_3 - a_2 - b_2}\right) \right] \\
&= \frac{\psi(5) - \psi\left(\left\{x - \frac{a_3 + a_2 + b_2 + b_3}{2}\right\} \frac{10}{a_3 + b_3 - a_2 - b_2}\right)}{(\psi(5) - \psi(-5))}
\end{aligned}$$

Thus, right sided distribution function for the fuzzy number $Z = X + Y$ is

$$R_1(z) = \frac{\psi(5) - \psi\left(\left\{x - \frac{a_3 + a_2 + b_2 + b_3}{2}\right\} \frac{10}{a_3 + b_3 - a_2 - b_2}\right)}{(\psi(5) - \psi(-5))}$$

Hence, the membership function of the fuzzy variables $Z = X + Y$ is given by

$$\mu_Z(x) = \begin{cases} \omega \frac{\psi\left(\left\{x - \frac{a_1 + a_2 + b_1 + b_2}{2}\right\} \frac{10}{(a_2 + b_2 - a_1 - b_1)}\right) - \psi(-5)}{\psi(5) - \psi(-5)} & ; \quad a_1 + b_1 \leq x < a_2 + b_2 \\ \omega & ; \quad x = a_2 + b_2 \\ \omega \frac{\psi(5) - \psi\left(\left\{x - \frac{a_2 + a_3 + b_2 + b_3}{2}\right\} \frac{10}{(a_3 + b_3 - a_2 - b_2)}\right)}{\psi(5) - \psi(-5)} & ; \quad a_2 + b_2 < x \leq a_3 + b_3 \end{cases}$$

□

Theorem 3.1.2. *If X be a sigmoidal fuzzy number and $z = kx$ be the transformation then kX is also a sigmoidal fuzzy number given by*

$$kX = \begin{cases} (ka_1, ka_2, ka_3; \omega_1) & \text{if } k > 0 \\ (ka_3, ka_2, ka_1; \omega_1) & \text{if } k < 0 \end{cases}$$

Proof. Using the transformation $z = kx$, we get $x = z/k$ and hence $x = \zeta(z) = z/k$. Thus $|\frac{dx}{dz}| = \frac{1}{k} = m(z)$. Therefore, at $x = \zeta(z)$, we have

$$\psi(z) = \psi\left(\left\{\frac{z}{k} - \frac{a_1 + a_2}{2}\right\} \frac{10}{(a_2 - a_1)}\right)$$

Also, from Lemma I, we have

$$\begin{aligned} \eta_1(z) &= \frac{d}{dx}(L_1) \quad \text{at } x = \zeta_1(z) = \frac{z}{k} \\ &= \frac{10}{(a_2 - a_1)(\psi(5) - \psi(-5))} \left(\psi\left(\left\{\frac{z}{k} - \frac{a_1 + a_2}{2}\right\} \frac{10}{a_2 - a_1}\right) \right) \left(1 - \psi\left(\left\{\frac{z}{k} - \frac{a_1 + a_2}{2}\right\} \frac{10}{a_2 - a_1}\right) \right) \end{aligned}$$

Hence,

$$\begin{aligned} L_1(z) &= \int_{ka_1}^x \eta_1(z) m(z) dz \\ &= \int_{ka_1}^x \frac{10}{k(a_2 - a_1)(\psi(5) - \psi(-5))} \left(\psi\left(\left\{\frac{z}{k} - \frac{a_1 + a_2}{2}\right\} \frac{10}{a_2 - a_1}\right) \right) \times \\ &\quad \left(1 - \psi\left(\left\{\frac{z}{k} - \frac{a_1 + a_2}{2}\right\} \frac{10}{a_2 - a_1}\right) \right) dz \\ &= \frac{\psi\left(\left(x - \frac{ka_1 + ka_2}{2}\right) \frac{10}{(ka_2 - ka_1)}\right) - \psi(-5)}{\psi(5) - \psi(-5)} \end{aligned}$$

Similarly, for non-complementary distribution function $R_1(x)$, we have

$$\begin{aligned} \eta_2(z) &= \frac{d}{dx}(R_1(x)) \quad \text{at } x = \zeta(z) = \frac{z}{k} \\ &= \frac{10}{(a_3 - a_2)(\psi(5) - \psi(-5))} \left(\psi\left(\left\{\frac{z}{k} - \frac{a_2 + a_3}{2}\right\} \frac{10}{a_3 - a_2}\right) \right) \left(1 - \psi\left(\left\{\frac{z}{k} - \frac{a_2 + a_3}{2}\right\} \frac{10}{a_3 - a_2}\right) \right) \end{aligned}$$

and hence

$$\begin{aligned}
R_1(z) &= \int_x^{ka_3} \eta_2(z)m(z)dz \\
&= \int_x^{ka_3} \frac{10}{(a_3 - a_2)(\psi(5) - \psi(-5))} \left(\psi\left(\left\{\frac{z}{k} - \frac{a_2 + a_3}{2}\right\} \frac{10}{a_3 - a_2}\right) \right) \times \\
&\quad \left(1 - \psi\left(\left\{\frac{z}{k} - \frac{a_2 + a_3}{2}\right\} \frac{10}{a_3 - a_2}\right) \right) \\
&= \frac{\psi(5) - \psi\left(\left\{x - \frac{ka_2 + ka_3}{2}\right\} \frac{10}{(ka_3 - ka_2)}\right)}{\psi(5) - \psi(-5)}
\end{aligned}$$

Thus, the membership function of the fuzzy number kX , $k > 0$ is given by

$$\mu_{kX}(x) = \begin{cases} \frac{\psi\left(\left(x - \frac{ka_1 + ka_2}{2}\right) \frac{10}{(ka_2 - ka_1)}\right) - \psi(-5)}{\omega_1 \frac{\psi(5) - \psi(-5)}{\psi(5) - \psi(-5)}} & ; \quad ka_1 \leq x \leq ka_2 \\ \omega_1 & ; \quad x = ka_2 \\ \frac{\psi(5) - \psi\left(\left(x - \frac{ka_2 + ka_3}{2}\right) \frac{10}{(ka_3 - ka_2)}\right)}{\omega_1 \frac{\psi(5) - \psi(-5)}{\psi(5) - \psi(-5)}} & ; \quad ka_2 \leq x \leq ka_3 \end{cases}$$

Similarly, for $k < 0$, the membership functions for the fuzzy number kX is given by

$$\mu_{kX}(x) = \begin{cases} \frac{\psi\left(\left(x - \frac{ka_2 + ka_3}{2}\right) \frac{10}{(ka_2 - ka_3)}\right) - \psi(-5)}{\omega_1 \frac{\psi(5) - \psi(-5)}{\psi(5) - \psi(-5)}} & ; \quad ka_3 \leq x \leq ka_2 \\ \omega_1 & ; \quad x = ka_2 \\ \frac{\psi(5) - \psi\left(\left(x - \frac{ka_1 + ka_2}{2}\right) \frac{10}{(ka_1 - ka_2)}\right)}{\omega_1 \frac{\psi(5) - \psi(-5)}{\psi(5) - \psi(-5)}} & ; \quad ka_2 \leq x \leq ka_1 \end{cases}$$

□

Theorem 3.1.3. *If X and Y be the two sigmoidal membership function over the universe U then the fuzzy variable $Z = X - Y$ is also a sigmoidal fuzzy number whose membership function is given by*

$$\mu_Z(x) = \begin{cases} \frac{\psi\left(\left(x - \frac{a_1 + a_2 - b_2 - b_3}{2}\right) \frac{10}{(a_2 - b_2 - a_1 + b_3)}\right) - \psi(-5)}{\omega \frac{\psi(5) - \psi(-5)}{\psi(5) - \psi(-5)}} & ; \quad a_1 - b_3 \leq x < a_2 - b_2 \\ \omega & ; \quad x = a_2 - b_2 \\ \frac{\psi(5) - \psi\left(\left(x - \frac{a_2 + a_3 - b_2 - b_1}{2}\right) \frac{10}{(a_3 - b_1 - a_2 + b_2)}\right)}{\omega \frac{\psi(5) - \psi(-5)}{\psi(5) - \psi(-5)}} & ; \quad a_2 - b_2 < x \leq a_3 - b_1 \end{cases}$$

Proof. The proof is trivial by using the addition and scalar multiplication ($k = -1 < 0$) of two sigmoidal fuzzy numbers. \square

Theorem 3.1.4. *If X and Y be the two sigmoidal membership function over the universe U then the fuzzy variable $Z = X \cdot Y$ is also a sigmoidal fuzzy number whose membership function is given by*

$$\mu_{XY}(x) = \begin{cases} \omega \frac{\psi(\xi_1) - \psi(-5)}{\psi(5) - \psi(-5)} & ; \quad a_1b_1 \leq x \leq a_2b_2 \\ \omega & ; \quad x = a_2b_2 \\ \omega \frac{\psi(5) - \psi(\xi_2)}{\psi(5) - \psi(-5)} & ; \quad a_2b_2 \leq x \leq a_3b_3 \end{cases}$$

where $\xi_1 = 10 \left(\frac{-B_1 + \sqrt{B_1^2 - 4A_1(C_1 - x) - A_1}}{2A_1} \right)$, $\xi_2 = 10 \left(\frac{-B_2 - \sqrt{B_2^2 - 4A_2(C_2 - x) + A_2}}{2A_2} \right)$, $A_1 = (a_2 - a_1)(b_2 - b_1)$, $B_1 = a_1(b_2 - b_1) + b_1(a_2 - a_1)$, $C_1 = a_1b_1$, $A_2 = (a_3 - a_2)(b_3 - b_2)$, $B_2 = -a_3(b_3 - b_2) - b_3(a_3 - a_2)$ and $C_2 = a_3b_3$

Proof. As the sigmoidal membership functions of X and Y are given in Eq. (3.1.1) and (3.1.2) respectively, thus, in order to find the membership function of the fuzzy number $Z = XY$, we equate $L_1(x)$ with $L_1(y)$ and $R_1(x)$ with $R_2(y)$ and get $y = \phi_1(x)$ and $y = \phi_2(x)$ respectively, where $\phi_1(x) = \frac{b_1 + b_2}{2} + \frac{b_2 - b_1}{a_2 - a_1} \left(x - \frac{a_1 + a_2}{2} \right)$, $\phi_2(x) = \frac{b_2 + b_3}{2} + \frac{b_3 - b_2}{a_3 - a_2} \left(x - \frac{a_2 + a_3}{2} \right)$.

Therefore at $y = \phi_1(x)$, $z = xy$ becomes

$$x = \frac{a_1b_2 - a_2b_1 \pm \sqrt{(a_1b_2 - a_2b_1)^2 + 4(a_2 - a_1)(b_2 - b_1)z}}{2(b_2 - b_1)} = \zeta_1(z) \quad (\text{say})$$

Hence,

$$\begin{aligned} m_1(z) &= \left| \frac{dx}{dz} \right|_{x=\zeta_1(z)} \\ &= \frac{4(a_2 - a_1)(b_2 - b_1)}{4(b_2 - b_1)\sqrt{(a_1b_2 - a_2b_1)^2 + 4(a_2 - a_1)(b_2 - b_1)z}} \\ &= \frac{a_2 - a_1}{\sqrt{(a_1b_2 - a_2b_1)^2 + 4(a_2 - a_1)(b_2 - b_1)z}} \\ &= \frac{a_2 - a_1}{\sqrt{B_1^2 - 4A_1(C_1 - z)}} \end{aligned}$$

where $A_1 = (a_2 - a_1)(b_2 - b_1)$; $B_1 = a_1(b_2 - b_1) + b_1(a_2 - a_1)$; $C_1 = a_1b_1$

Also,

$$\begin{aligned} \eta_1(z) &= \frac{d}{dx}(L_1) \quad \text{at } x = \zeta_1(z) \\ &= \frac{1}{(\psi(5) - \psi(-5))} \frac{d}{dx} \left(\psi \left(\left\{ x - \frac{a_1 + a_2}{2} \right\} \frac{10}{(a_2 - a_1)} \right) \right) \\ &= \frac{10}{(a_2 - a_1)(\psi(5) - \psi(-5))} \psi \left(\left\{ x - \frac{a_1 + a_2}{2} \right\} \frac{10}{(a_2 - a_1)} \right) \left(1 - \psi \left(\left\{ x - \frac{a_1 + a_2}{2} \right\} \frac{10}{(a_2 - a_1)} \right) \right) \end{aligned}$$

At

$$\begin{aligned}
x = \zeta_1(z) &= \frac{a_1 b_2 - a_2 b_1 \pm \sqrt{(a_1 b_2 - a_2 b_1)^2 + 4(a_2 - a_1)(b_2 - b_1)z}}{2(b_2 - b_1)} \\
&\psi\left(\left\{x - \frac{a_1 + a_2}{2}\right\} \frac{10}{(a_2 - a_1)}\right) \\
&= \psi\left(\left\{\frac{a_1 b_2 - a_2 b_1 \pm \sqrt{(a_1 b_2 - a_2 b_1)^2 + 4(a_2 - a_1)(b_2 - b_1)z}}{2(b_2 - b_1)} - \frac{a_1 + a_2}{2}\right\} \frac{10}{a_2 - a_1}\right) \\
&= \psi\left(\left\{\frac{a_1 b_2 - a_2 b_1 \pm \sqrt{B_1^2 - 4A_1(C_1 - z)}}{2(b_2 - b_1)} - \frac{a_1 + a_2}{2}\right\} \frac{10}{a_2 - a_1}\right) \\
&= \psi\left(\left\{\frac{a_1 b_1 - a_2 b_2 \pm \sqrt{B_1^2 - 4A_1(C_1 - z)}}{2(b_2 - b_1)}\right\} \frac{10}{a_2 - a_1}\right)
\end{aligned}$$

Hence,

$$\begin{aligned}
\eta_1(z) &= \frac{10}{(a_2 - a_1)(\psi(5) - \psi(-5))} \psi\left(\left\{\frac{a_1 b_1 - a_2 b_2 \pm \sqrt{B_1^2 - 4A_1(C_1 - z)}}{2(b_2 - b_1)}\right\} \frac{10}{a_2 - a_1}\right) \times \\
&\quad \left[1 - \psi\left(\left\{\frac{a_1 b_1 - a_2 b_2 \pm \sqrt{B_1^2 - 4A_1(C_1 - z)}}{2(b_2 - b_1)}\right\} \frac{10}{a_2 - a_1}\right)\right]
\end{aligned}$$

Therefore

$$\begin{aligned}
L_1(z) &= \int_{a_1 b_1}^x m_1(z) \eta_1(z) dz \\
&= \int_{a_1 b_1}^x \frac{10}{(\psi(5) - \psi(-5)) \sqrt{B_1^2 - 4A_1(C_1 - z)}} \psi(\cdot) (1 - \psi(\cdot)) dz \\
&= \frac{1}{(\psi(5) - \psi(-5))} \left[\psi\left(\left\{\frac{a_1 b_1 - a_2 b_2 \pm \sqrt{B_1^2 - 4A_1(C_1 - z)}}{2(b_2 - b_1)}\right\} \frac{10}{a_2 - a_1}\right) \right]_{a_1 b_1}^x \\
&= \frac{1}{(\psi(5) - \psi(-5))} \left[\psi\left(\left\{\frac{a_1 b_1 - a_2 b_2 \pm \sqrt{(B_1)^2 - 4A_1(C_1 - x)}}{2(b_2 - b_1)(a_2 - a_1)}\right\} 10\right) - \right. \\
&\quad \left. \psi\left(\left\{\frac{a_1 b_1 - a_2 b_2 \pm \sqrt{B_1^2 - 4A_1(C_1 - a_1 b_1)}}{2(b_2 - b_1)(a_2 - a_1)}\right\} 10\right) \right] \\
&= \frac{\psi\left(\left\{\frac{a_1 b_1 - a_2 b_2 \pm \sqrt{B_1^2 - 4A_1(C_1 - x)}}{2(b_2 - b_1)(a_2 - a_1)}\right\} 10\right) - \psi(-5)}{\psi(5) - \psi(-5)} \\
&= \frac{\psi(\xi_1) - \psi(-5)}{\psi(5) - \psi(-5)}
\end{aligned}$$

where $\xi_1 = 10 \left(\frac{-B_1 + \sqrt{B_1^2 - 4A_1(C_1 - x)} - A_1}{2A_1} \right)$

Similarly, by taking

$$A_2 = (a_3 - a_2)(b_3 - b_2) \quad ; \quad B_2 = -a_3(b_3 - b_2) - b_3(a_3 - a_2) \quad ; \quad C_2 = a_3 b_3$$

we get the right membership function of $Z = XY$ as

$$R_1(z) = \int_x^{a_3 b_3} \eta_2(z) m_2(z) dz = \frac{\psi(5) - \psi(\xi_2)}{\psi(5) - \psi(-5)}$$

where $\xi_2 = 10 \left(\frac{-B_2 - \sqrt{B_2^2 - 4A_2(C_2 - x)} + A_2}{2A_2} \right)$

Hence, the membership function of the fuzzy number $Z = XY$ is given by

$$\mu_{X \cdot Y}(x) = \begin{cases} \omega \frac{\psi(\xi_1) - \psi(-5)}{\psi(5) - \psi(-5)} & ; \quad a_1 b_1 \leq x \leq a_2 b_2 \\ \omega & ; \quad x = a_2 b_2 \\ \omega \frac{\psi(5) - \psi(\xi_2)}{\psi(5) - \psi(-5)} & ; \quad a_2 b_2 \leq x \leq a_3 b_3 \end{cases}$$

where $\xi_1 = 10 \left(\frac{-B_1 + \sqrt{B_1^2 - 4A_1(C_1 - x)} - A_1}{2A_1} \right)$ and $\xi_2 = 10 \left(\frac{-B_2 - \sqrt{B_2^2 - 4A_2(C_2 - x)} + A_2}{2A_2} \right)$ □

Theorem 3.1.5. *If fuzzy number X represent the sigmoidal membership function given in Eq. (3.1.1) then the inverse of X i.e $X^{-1} = [a_3^{-1}, a_2^{-1}, a_1^{-1}; \omega_1]$ is also a sigmoidal fuzzy number whose membership function is*

$$\mu_{X^{-1}}(x) = \begin{cases} \omega_1 \frac{\psi(5) - \psi \left(\left\{ \frac{1}{x} - \frac{a_2 + a_3}{2} \right\} \frac{10}{(a_3 - a_2)} \right)}{\psi(5) - \psi(-5)} & ; \quad a_3^{-1} \leq x \leq a_2^{-1} \\ \omega_1 & ; \quad x = a_2^{-1} \\ \omega_1 \frac{\psi \left(\left\{ \frac{1}{x} - \frac{a_1 + a_2}{2} \right\} \frac{10}{(a_2 - a_1)} \right) - \psi(-5)}{\psi(5) - \psi(-5)} & ; \quad a_2^{-1} \leq x \leq a_1^{-1} \end{cases}$$

Proof. Consider a fuzzy number $X = [a_1, a_2, a_3; \omega_1]$ with membership function given in Eq. (3.1.1). Let $z = \frac{1}{x}$ so that $\left| \frac{dx}{dz} \right| = \frac{1}{z^2}$. Therefore for X^{-1} we have

$$\begin{aligned} R_1(x) &= \int_x^{a_1^{-1}} \eta_1(z) m(z) dz \\ &= \frac{10}{(\psi(5) - \psi(-5))(a_2 - a_1)} \int_x^{a_1^{-1}} \psi \left(\left(\frac{1}{z} - \frac{a_1 + a_2}{2} \right) \frac{10}{(a_2 - a_1)} \right) \times \\ &\quad \left(1 - \psi \left(\left(\frac{1}{z} - \frac{a_1 + a_2}{2} \right) \frac{10}{(a_2 - a_1)} \right) \right) \left(\frac{1}{z^2} \right) dz \\ &= \frac{\psi \left(\left(\frac{1}{x} - \frac{a_1 + a_2}{2} \right) \frac{10}{(a_2 - a_1)} \right) - \psi(-5)}{\psi(5) - \psi(-5)} \end{aligned}$$

and

$$\begin{aligned}
L_1(x) &= \int_{a_3^{-1}}^x \eta_2(z)m(z)dz \\
&= \frac{10}{(\psi(5) - \psi(-5))(a_3 - a_2)} \int_{a_3^{-1}}^x \psi\left(\left(\frac{1}{z} - \frac{a_2 + a_3}{2}\right)\frac{10}{(a_3 - a_2)}\right) \times \\
&\quad \left(1 - \psi\left(\left(\frac{1}{z} - \frac{a_2 + a_3}{2}\right)\frac{10}{(a_3 - a_2)}\right)\right) \left(\frac{1}{z^2}\right) dz \\
&= \frac{\psi(5) - \psi\left(\left(\frac{1}{x} - \frac{a_2 + a_3}{2}\right)\frac{10}{(a_3 - a_2)}\right)}{\psi(5) - \psi(-5)}
\end{aligned}$$

Thus, based on these distribution functions, fuzzy membership function of X^{-1} are

$$\mu_{X^{-1}}(x) = \begin{cases} \omega_1 \frac{\psi(5) - \psi\left(\left\{\frac{1}{x} - \frac{a_2 + a_3}{2}\right\}\frac{10}{(a_3 - a_2)}\right)}{\psi(5) - \psi(-5)} & ; \quad a_3^{-1} \leq x \leq a_2^{-1} \\ \omega_1 & ; \quad x = a_2^{-1} \\ \omega_1 \frac{\psi\left(\left\{\frac{1}{x} - \frac{a_1 + a_2}{2}\right\}\frac{10}{(a_2 - a_1)}\right) - \psi(-5)}{\psi(5) - \psi(-5)} & ; \quad a_2^{-1} \leq x \leq a_1^{-1} \end{cases}$$

□

Theorem 3.1.6. *If X and Y be the two sigmoidal fuzzy numbers over the universe U then, for $0 \notin Y$, the fuzzy variable $Z = \frac{X}{Y} = X \cdot Y^{-1}$ is also a sigmoidal fuzzy number.*

Proof. By using the Theorem 3.1.5 and Theorem 3.1.4, we get the membership function of $Z = X \cdot Y^{-1}$ □

3.2 Illustrative examples

The above methodology for computing the membership functions of various arithmetic operations has been illustrated through a numerical examples as given below.

3.2.1 Addition of two numbers

Let $X = [1, 2, 4; 1]$ and $Y = [3, 5, 6; 1]$ be two sigmoidal fuzzy numbers with membership functions as

$$\mu_X(x) = \begin{cases} \frac{\psi(\{x-1.5\}10) - \psi(-5)}{\psi(5) - \psi(-5)} & ; \quad 1 \leq x \leq 2 \\ 1 & ; \quad x = 2 \\ \frac{\psi(5) - \psi(\{x-3\}5)}{\psi(5) - \psi(-5)} & ; \quad 2 \leq x \leq 4 \end{cases} \quad \text{and} \quad \mu_Y(y) = \begin{cases} \frac{\psi(\{y-4\}5) - \psi(-5)}{\psi(5) - \psi(-5)} & ; \quad 3 \leq y \leq 5 \\ 1 & ; \quad y = 5 \\ \frac{\psi(5) - \psi(\{y-5.5\}10)}{\psi(5) - \psi(-5)} & ; \quad 5 \leq y \leq 6 \end{cases}$$

In order to evaluate the degree of membership of $X + Y$ we start with the equating of the distribution and complementary distribution functions and hence we get, $y = 2x + 1 = \phi_1(x)$ and $y = \frac{x+8}{2} = \phi_2(x)$. Therefore, for $Z = X + Y$, we get

$$x = \frac{z-1}{3} = \zeta_1(x), m_1(z) = \frac{1}{3}, \eta_1(z) = \frac{10}{\psi(5) - \psi(-5)} \psi\left(\left\{z - 5.5\right\} \frac{10}{3}\right) [1 - \psi\left(\left\{z - 5.5\right\} \frac{10}{3}\right)]$$

$$x = \frac{2z-8}{3} = \zeta_2(x), m_2(z) = \frac{2}{3}, \eta_2(z) = \frac{5}{\psi(5) - \psi(-5)} \psi\left(\left\{z - \frac{17}{2}\right\} \frac{10}{3}\right) [1 - \psi\left(\left\{z - \frac{17}{2}\right\} \frac{10}{3}\right)]$$

.

Thus, the distribution function of the fuzzy variable $Z = X + Y$ would now be given as

$$\int_4^x \eta_1(z) m_1(z) dz = \frac{\psi\left(\left\{x - \frac{11}{2}\right\} \frac{10}{3}\right) - \psi(-5)}{\psi(5) - \psi(-5)} \quad ; \quad 4 \leq x \leq 7$$

and the complementary distribution function is

$$\int_x^{10} \eta_2(z) m_2(z) dz = \frac{\psi(5) - \psi\left(\left\{x - \frac{17}{2}\right\} \frac{10}{3}\right)}{\psi(5) - \psi(-5)} \quad ; \quad 7 \leq x \leq 10$$

Hence, the membership function of $X + Y$ is

$$\mu_Z(x) = \begin{cases} \frac{\psi\left(\left\{x - \frac{11}{2}\right\} \frac{10}{3}\right) - \psi(-5)}{\psi(5) - \psi(-5)} & ; \quad 4 \leq x < 7 \\ 1 & ; \quad x = 7 \\ \frac{\psi(5) - \psi\left(\left\{x - \frac{17}{2}\right\} \frac{10}{3}\right)}{\psi(5) - \psi(-5)} & ; \quad 7 < x \leq 10 \end{cases}$$

The obtained results are depicted graphically in Fig. 3.1 along with the existing results, crisp, linear and parabolic which is explained as below.

- The results computed by the traditional or crisp methodology is shown in Fig. 3.1 with ‘crisp’ legend shows that their results are independent of the degree of uncertainties and hence remain constant and unsuitable for the decision makers.
- The results computed by taking the linear membership function have wide range of uncertainties as given in Fig. 3.1 and thus their results are not so practical.

- As the results computed by taking the parabolic as well as sigmoidal membership functions have less range of uncertainties as compared to linear membership functions and hence take a sound decision for the decision maker where uncertainties play a dominant role during the analysis. Moreover, it has been analyzed that the results computed by the sigmoidal membership function act as a sandwich between the linear and parabolic membership functions at different levels of significance, shown in Table 3.1.

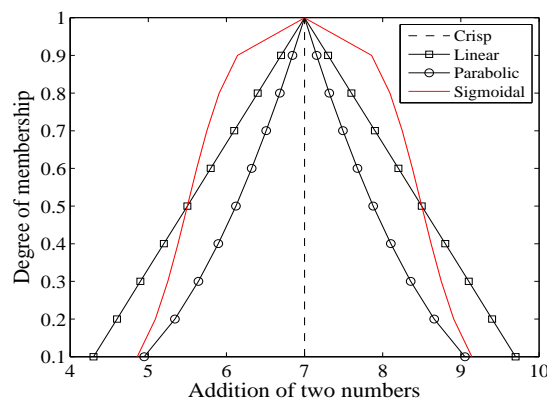


Figure 3.1: Membership function of Addition of two numbers

Table 3.1: Comparison of results for Addition of two numbers

α	Left membership function			Right membership functions		
	Linear [2]	Parabolic [1]	Sigmoidal	Linear [2]	Parabolic [1]	Sigmoidal
0	4.0000	4.0000	4.0000	10.0000	10.0000	10.0000
0.2	4.6000	5.3416	5.0916	9.4000	8.6584	8.9084
0.4	5.2000	5.8974	5.3800	8.8000	8.1026	8.6200
0.6	5.8000	6.3238	5.6200	8.2000	7.6762	8.3800
0.8	6.4000	6.6833	5.9084	7.6000	7.3167	8.0916
1.0	7.0000	7.0000	7.0000	7.0000	7.0000	7.0000

3.2.2 Length of the Rod

Let length of a rod is a sigmoidal fuzzy number $\tilde{A} = (12cm, 13.5cm, 15cm; 0.8)$. If the length $\tilde{B} = (5cm, 6.5cm, 8cm; 0.7)$, a sigmoidal fuzzy number, is cut off from this rod then the remaining length of the rod \tilde{C} is $\tilde{A} - \tilde{B}$.

The sigmoidal membership function corresponding to sigmoidal fuzzy numbers \tilde{A} and \tilde{B} are defined as below

$$\mu_{\tilde{A}}(x) = \begin{cases} 0.8 \left(\frac{\psi(\{x - \frac{25.5}{2}\}_{\frac{10}{1.5}}) - \psi(-5)}{\psi(5) - \psi(-5)} \right) & ; \quad 12 \leq x < 13.5 \\ 0.8 & ; \quad x = 13.5 \\ 0.8 \left(\frac{\psi(5) - \psi(\{x - \frac{28.5}{2}\}_{\frac{10}{1.5}})}{\psi(5) - \psi(-5)} \right) & ; \quad 13.5 < x \leq 15 \end{cases}$$

$$\mu_{\tilde{B}}(y) = \begin{cases} 0.7 \left(\frac{\psi(\{y - \frac{11.5}{2}\}_{\frac{10}{1.5}}) - \psi(-5)}{\psi(5) - \psi(-5)} \right) & ; \quad 5 \leq y < 6.5 \\ 0.7 & ; \quad y = 6.5 \\ 0.7 \left(\frac{\psi(5) - \psi(\{x - \frac{14.5}{2}\}_{\frac{10}{1.5}})}{\psi(5) - \psi(-5)} \right) & ; \quad 6.5 < y \leq 8 \end{cases}$$

Now $\tilde{B} = (-8cm, -6.5cm, -5cm; 0.7)$ be a negative of fuzzy number \tilde{B} , then the corresponding membership function is given by

$$\mu_{-\tilde{B}}(y) = \begin{cases} 0.7 \left(\frac{\psi(\{x + \frac{14.5}{2}\}_{\frac{10}{1.5}}) - \psi(-5)}{\psi(5) - \psi(-5)} \right) & ; \quad -8 \leq x < -6.5 \\ 0.7 & ; \quad x = -6.5 \\ 0.7 \left(\frac{\psi(5) - \psi(\{x + \frac{11.5}{2}\}_{\frac{10}{1.5}})}{\psi(5) - \psi(-5)} \right) & ; \quad -6.5 < x \leq -5 \end{cases}$$

Hence, using the property of the addition of the two sigmoidal fuzzy numbers, the membership functions of the remaining length of the rod is a sigmoidal fuzzy number \tilde{C} and is given by

$$\mu_{\tilde{C}}(x) = \begin{cases} 0.7 \left(\frac{\psi(\{x - \frac{11}{2}\}_{\frac{10}{3}}) - \psi(-5)}{\psi(5) - \psi(-5)} \right) & ; \quad 4 \leq x < 7 \\ 0.7 & ; \quad x = 7 \\ 0.7 \left(\frac{\psi(5) - \psi(\{x - \frac{17}{2}\}_{\frac{10}{3}})}{\psi(5) - \psi(-5)} \right) & ; \quad 7 < x \leq 10 \end{cases}$$

From above, we conclude that the length of the rod lies between 4cm and 10cm and there are 70% probabilities that the length takes the value 7cm. The corresponding membership values are plotted in Fig. 3.2 at different levels of significance while their corresponding values are summarized in Table 3.2 along with the results of linear and parabolic.

Table 3.2: Comparison of results for Subtraction of two numbers

α	Left membership function			Right membership functions		
	Linear [2]	Parabolic [1]	Sigmoidal	Linear [2]	Parabolic [1]	Sigmoidal
0	4.0000	4.0000	4.0000	10.0000	10.0000	10.0000
0.1	4.4286	5.1339	4.9740	9.5714	8.8661	9.0260
0.2	4.8571	5.6036	5.2293	9.1429	8.3964	8.7707
0.3	5.2857	5.9640	5.4149	8.7143	8.0360	8.5851
0.4	5.7143	6.2678	5.5851	8.2857	7.7322	8.4149
0.5	6.1429	6.5355	5.7707	7.8571	7.4645	8.2293
0.6	6.5714	6.7775	6.0260	7.4286	7.2225	7.9740
0.7	7.0000	7.0000	7.0000	7.0000	7.0000	7.0000

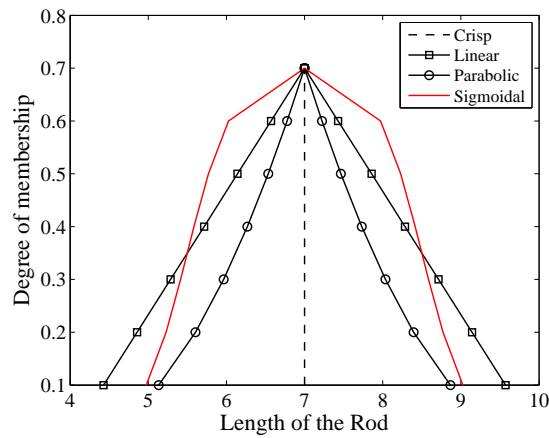


Figure 3.2: Membership function of Subtraction of two numbers

3.2.3 Area of the rectangle

Let length and breath of a rectangle are two sigmoidal fuzzy numbers given by $\tilde{A} = (1cm, 2cm, 4cm; 0.75)$ and $\tilde{B} = (3cm, 5cm, 6cm; 0.85)$, then area \tilde{C} of the rectangle is $\tilde{C} = \tilde{A} \cdot \tilde{B}$.

In order to evaluate the membership functions of \tilde{C} we equate the distribution and complementary distribution functions respectively of \tilde{A} and \tilde{B} and hence we get $y = \phi_1(x) = 2x + 1$ and $y = \phi_2(x) = \frac{x+8}{2}$. Now for $Z = A \cdot B$ we get

$$\zeta_1(z) = \frac{-1 + \sqrt{1 + 8z}}{4}, \quad m_1(z) = \frac{1}{\sqrt{1 + 8z}}$$

$$\eta_1(z) = \left(\frac{10}{\psi(5) - \psi(-5)} \right) \psi \left(\left\{ \frac{-7 + \sqrt{1 + 8z}}{4} \right\} 10 \right) \left[1 - \psi \left(\left\{ \frac{-7 + \sqrt{1 + 8z}}{4} \right\} 10 \right) \right]$$

$$\begin{aligned}\zeta_2(z) &= -4 + \sqrt{16 + 2z}, \quad m_2(z) = \frac{1}{\sqrt{16 + 2z}} \\ \eta_2(z) &= \left(\frac{5}{\psi(5) - \psi(-5)} \right) \psi \left(\{-7 + \sqrt{16 + 2z}\}5 \right) \left[1 - \psi \left(\{-7 + \sqrt{16 + 2z}\}5 \right) \right]\end{aligned}$$

Therefore, the distribution function of the fuzzy variable \tilde{C} is given by

$$\begin{aligned}& \int_3^x \eta_1(z) m_1(z) dz \\ &= \int_3^x \left(\frac{10}{\psi(5) - \psi(-5)} \right) \psi \left(\left\{ \frac{-7 + \sqrt{1 + 8z}}{4} \right\} 10 \right) \left[1 - \psi \left(\left\{ \frac{-7 + \sqrt{1 + 8z}}{4} \right\} 10 \right) \right] \left(\frac{1}{\sqrt{1 + 8z}} \right) dz \\ &= \left(\frac{\psi \left(\left\{ \frac{-7 + \sqrt{1 + 8x}}{4} \right\} 10 \right)}{\psi(5) - \psi(-5)} \right)_3^x \\ &= \frac{\psi \left(\left\{ \frac{-7 + \sqrt{1 + 8x}}{4} \right\} 10 \right) - \psi(-5)}{\psi(5) - \psi(-5)} \quad ; \quad 3 \leq x \leq 10\end{aligned}$$

and complementary distribution function is given by

$$\begin{aligned}& \int_x^{24} \eta_2(z) m_2(z) dz \\ &= \int_x^{24} \left(\frac{5}{\psi(5) - \psi(-5)} \right) \psi \left(\{-7 + \sqrt{16 + 2z}\}5 \right) \left[1 - \psi \left(\{-7 + \sqrt{16 + 2z}\}5 \right) \right] \left(\frac{1}{\sqrt{16 + 2z}} \right) dz \\ &= \left(\frac{\psi \left(\{-7 + \sqrt{16 + 2z}\}5 \right)}{\psi(5) - \psi(-5)} \right)_x^{24} \\ &= \frac{\psi(5) - \psi \left(\{-7 + \sqrt{16 + 2x}\}5 \right)}{\psi(5) - \psi(-5)} \quad ; \quad 10 \leq x \leq 24\end{aligned}$$

Hence, the membership function of the area of rectangle is given as

$$\mu_{\tilde{C}}(x) = \begin{cases} 0.75 \left(\frac{\psi \left(\left\{ \frac{-7 + \sqrt{1 + 8x}}{4} \right\} 10 \right) - \psi(-5)}{\psi(5) - \psi(-5)} \right) & ; \quad 3 \leq x < 10 \\ 0.75 & ; \quad x = 10 \\ 0.75 \left(\frac{\psi(5) - \psi \left(\{-7 + \sqrt{16 + 2x}\}5 \right)}{\psi(5) - \psi(-5)} \right) & ; \quad 10 < x \leq 24 \end{cases}$$

The variation of their membership functions corresponding to crisp, linear, parabolic and sigmoidal membership functions are shown in Fig. 3.3. It has been concluded from this figure that the resultant fuzzy number is a sigmoidal type which is increasing at a nonlinear rate of $\frac{10\psi \left(\left\{ \frac{-7 + \sqrt{1 + 8x}}{4} \right\} 10 \right) [1 - \psi \left(\left\{ \frac{-7 + \sqrt{1 + 8x}}{4} \right\} 10 \right)]}{(\psi(5) - \psi(-5))\sqrt{1 + 8x}}$ from 3 to 10 cm² and then decreases from 10

to 24cm^2 with nonlinear decreasing rate $\frac{5\psi(\{-7+\sqrt{16+2x}\}5)[1-\psi(\{-7+\sqrt{16+2x}\}5)]}{(\psi(5)-\psi(-5))\sqrt{16+2x}}$, while these changes are given as $\frac{\sqrt{1+8x}-5}{2\sqrt{1+8x}}$ and $\frac{8-\sqrt{16+2x}}{2\sqrt{16+2x}}$ for the parabolic function. Also, there is a 75% probability that the area of the rectangle is 10cm^2 . The area of the rectangle at different level of significance by linear, parabolic and sigmoidal membership functions are summarized in Table 3.3.

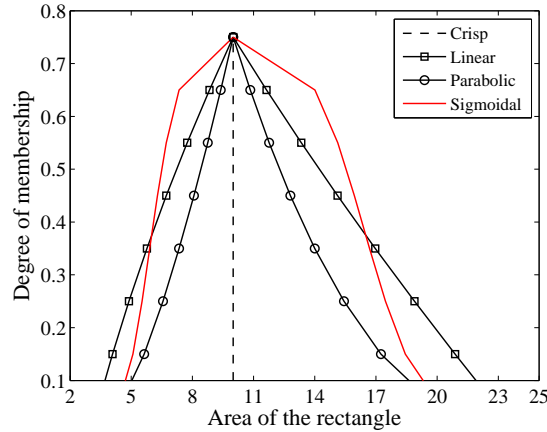


Figure 3.3: Membership function of Area of the rectangle

Table 3.3: Comparison of results for Multiplication of two numbers

α	Left membership function			Right membership functions		
	Linear [2]	Parabolic [1]	Sigmoidal	Linear [2]	Parabolic [1]	Sigmoidal
0	3.0000	3.0000	3.0000	24.0000	24.0000	24.0000
0.15	4.0800	5.6361	5.0841	20.8800	17.2446	18.4430
0.25	4.8889	6.5534	5.5311	18.8889	15.4291	17.4657
0.35	5.7689	7.3490	5.9081	16.9689	14.0033	16.6848
0.45	6.7200	8.0730	6.2831	15.1200	12.8065	15.9434
0.55	7.7422	8.7484	6.7168	13.3422	11.7651	15.1259
0.65	8.8356	9.3881	7.3480	11.6356	10.8381	14.0050
0.75	10.0000	10.0000	10.0000	10.0000	10.0000	10.0000

3.2.4 Length of the rectangle

Let area and breadth of the rectangle be given as a sigmoidal fuzzy numbers $\tilde{A} = (1\text{cm}^2, 2\text{cm}^2, 4\text{cm}^2; 0.75)$ and $\tilde{B} = (3\text{cm}, 5\text{cm}, 6\text{cm}; 0.85)$ respectively, the length of the rectangle is given by $\tilde{A}(\div)\tilde{B}$ or $\tilde{A}(\cdot)\tilde{B}^{-1}$.

Now based on the membership function of \tilde{B} , we obtain the membership function of

$\tilde{B}^{-1} = (6^{-1}, 5^{-1}, 3^{-1}; 0.85)$ as

$$\mu_{\tilde{B}^{-1}}(y) = \begin{cases} 0.85 \left(\frac{\psi(5) - \psi(\{\frac{1}{y} - \frac{11}{2}\}10)}{\psi(5) - \psi(-5)} \right) & ; \quad 6^{-1} \leq y < 5^{-1} \\ 0.85 & ; \quad y = 5^{-1} \\ 0.85 \left(\frac{\psi(\{\frac{1}{y} - \frac{8}{2}\}5) - \psi(-5)}{\psi(5) - \psi(-5)} \right) & ; \quad 5^{-1} < y \leq 3^{-1} \end{cases}$$

Hence, the membership function of the length of the rectangle is obtained by multiplying the two sigmoidal fuzzy numbers \tilde{A} and \tilde{B}^{-1} . It has been concluded from their membership function that there is a 75% probability that the length of the rectangle is 0.4cm while its range is $[\frac{1}{6}, \frac{4}{3}]$. The complete variation of their membership values at different level of membership values are plotted in Fig. 3.4. From this figure, it has been observed that the length of the rectangle is increased from $\frac{1}{6}$ to $\frac{2}{5}$ with a nonlinear rate while decreases from $\frac{2}{5}$ to $\frac{4}{3}$ with a nonlinear rate. Thus the shape of this function is concave-convex function instead of linear one and hence depicts the actual behavior of the system. The complete variety of these membership functions in the degree of satisfaction α are given in Table 3.4 along with the values by taking the linear and parabolic membership functions. Moreover, the defuzzified values obtained by the proposed approach are 0.5605 while for linear and parabolic functions are computed as 0.6103 and 0.5414 respectively.

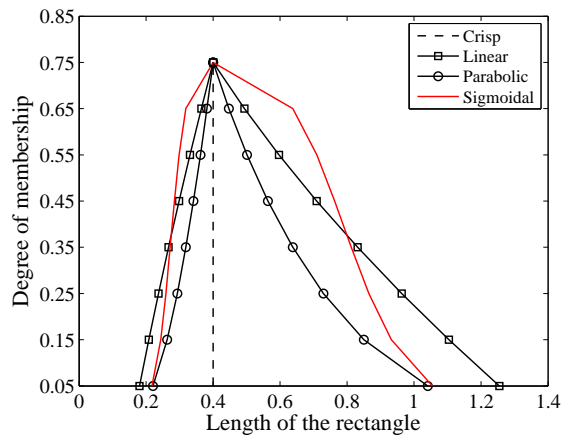


Figure 3.4: Membership function of Length of the rectangle

Table 3.4: Comparison of results for Inverse

α	Left membership function			Right membership functions		
	Linear [2]	Parabolic [1]	Sigmoidal	Linear [2]	Parabolic [1]	Sigmoidal
0	0.1667	0.1667	0.1667	1.3333	1.3333	1.3333
0.15	0.2080	0.2628	0.2439	1.1040	0.8500	0.9320
0.25	0.2370	0.2932	0.2592	0.9630	0.7294	0.8650
0.35	0.2673	0.3188	0.2719	0.8314	0.6380	0.8123
0.45	0.2987	0.3416	0.2844	0.7093	0.5638	0.7631
0.55	0.3313	0.3624	0.2986	0.5967	0.5013	0.7097
0.65	0.3650	0.3817	0.3188	0.4936	0.4473	0.6381
0.75	0.4000	0.4000	0.4000	0.4000	0.4000	0.4000

3.2.5 Perimeter of the rectangle

Consider the length and breadth of the rectangle as a sigmoidal fuzzy numbers $\tilde{A} = (12cm, 13.5cm, 14cm; 0.9)$ and $\tilde{B} = (6cm, 7.5cm, 9cm; 0.8)$, respectively, then perimeter \tilde{C} of the rectangle is given by $2[\tilde{A} + \tilde{B}]$. The membership function corresponding to sigmoidal number of \tilde{A} and \tilde{B} are

$$\mu_{\tilde{A}}(x) = \begin{cases} 0.9 \left(\frac{\psi(\{x - \frac{25.5}{2}\}_{\frac{10}{1.5}}) - \psi(-5)}{\psi(5) - \psi(-5)} \right) & ; \quad 12 \leq x < 13.5 \\ 0.9 & ; \quad x = 13.5 \\ 0.9 \left(\frac{\psi(5) - \psi(\{x - \frac{27.5}{2}\}_{\frac{10}{0.5}})}{\psi(5) - \psi(-5)} \right) & ; \quad 13.5 < x \leq 14 \end{cases}$$

and

$$\mu_{\tilde{B}}(y) = \begin{cases} 0.8 \left(\frac{\psi(\{y - \frac{13.5}{2}\}_{\frac{10}{1.5}}) - \psi(-5)}{\psi(5) - \psi(-5)} \right) & ; \quad 6 \leq y < 7.5 \\ 0.8 & ; \quad y = 7.5 \\ 0.8 \left(\frac{\psi(5) - \psi(\{y - \frac{16.5}{2}\}_{\frac{10}{1.5}})}{\psi(5) - \psi(-5)} \right) & ; \quad 7.5 < y \leq 9 \end{cases}$$

Now, by the property of addition and scalar multiplication of two sigmoidal numbers, the membership functions of the perimeter \tilde{C} of the rectangle is given as

$$\mu_{\tilde{C}}(x) = \begin{cases} 0.8 \left(\frac{\psi(\{x - \frac{78}{2}\}_{\frac{10}{6}}) - \psi(-5)}{\psi(5) - \psi(-5)} \right) & ; \quad 36 \leq x \leq 42 \\ 0.8 & ; \quad x = 42 \\ 0.8 \left(\frac{\psi(5) - \psi(\{x - \frac{88}{2}\}_{\frac{10}{4}})}{\psi(5) - \psi(-5)} \right) & ; \quad 42 \leq x \leq 46 \end{cases}$$

The results corresponding to this membership function have been summarized in Fig. 3.5 for the different level of significance. From these results it has been seen that the 80%

of the probability to get the perimeter of the rectangle 42cm, while it is increasing by a nonlinear rate from 36cm to 42cm and then decreases with a nonlinear rate from 42cm to 46cm. The corresponding values by taking linear, parabolic and sigmoidal functions are summarized in Table 3.5. The defuzzified values of these approaches are computed by COG method and are given as 41.3333, 41.4741 and 41.4214 respectively for linear, parabolic and sigmoidal membership functions. From these values, it has been observed that defuzzified value by sigmoidal approach is nearer to the crisp value than the other values.

Table 3.5: Comparison of results for Perimeter of the rectangle

α	Left membership function			Right membership functions		
	Linear [2]	Parabolic [1]	Sigmoidal	Linear [2]	Parabolic [1]	Sigmoidal
0	36.0000	36.0000	36.0000	46.0000	46.0000	46.0000
0.1	36.7500	38.1213	37.8595	45.5000	44.5858	44.7603
0.2	37.5000	39.0000	38.3515	45.0000	44.0000	44.4323
0.3	38.2500	39.6742	38.6978	44.5000	43.5505	44.2015
0.4	39.0000	40.2426	39.0000	44.0000	43.1716	44.0000
0.5	39.7500	40.7434	39.3022	43.5000	42.8377	43.7985
0.6	40.5000	41.1962	39.6485	43.0000	42.5359	43.5677
0.7	41.2500	41.6125	40.1405	42.5000	42.2583	43.2397
0.8	42.0000	42.0000	42.0000	42.0000	42.0000	42.0000

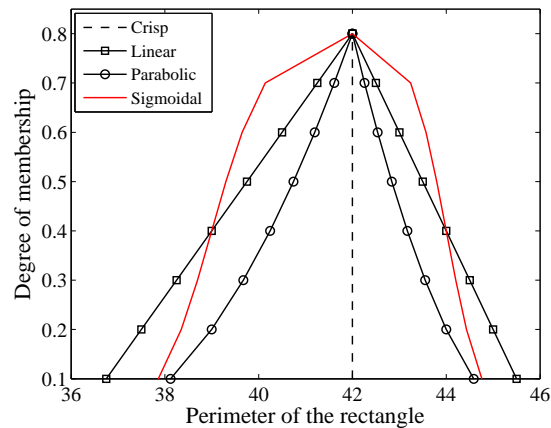


Figure 3.5: Membership function of Perimeter of the rectangle

3.3 Conclusion

The objective of this chapter is to address the various arithmetic operations by using the nonlinear logistic sigmoidal function. The major advantage of this function is that they define both the concavity and convexity behavior of the system. Since, in our day-today life, it is difficult to represent or collect the data in a precise way due to human error or other unavoidable factors and hence difficult to tell the correct behavior of the data. For handling this, a non-linear logistic sigmoidal function has been taken in assessing the effects of the uncertainties in the data. Based on these sigmoidal input numbers, various arithmetic operations such as addition, scalar multiplication, subtraction, multiplication, inverse is computed using the concept of complementary and non-complementary distribution functions. These operations are more useful and easy to apply in a situation where the computation of α - cut method fails or difficult to apply. The validity of the method has been evaluated by solving some problems on mensuration using generalized sigmoidal fuzzy numbers and compare their results with the linear and parabolic fuzzy numbers. From the computed results it has been observed that the system analyst may use their results in increasing the performance of the system and may change their target goals with the proposed results rather than existing results.

Chapter 4

Application of Fuzzy TOPSIS for finding the best fuzzy number

Technique for order preference by similarity to an ideal solution (TOPSIS), known as a classical multiple attribute decision making (MADM) method, has been developed by Hwang and Yoon [12] for solving the MADM problem. It is based on the idea that the chosen alternative should have the shortest distance from the positive ideal solution, and, on the other side, the farthest distance from the negative ideal solution. Here, in this chapter the best type of fuzzy number has been identified under the different set of criteria by using the fuzzy TOPSIS approach.

4.1 TOPSIS method

Suppose a multi-attribute decision making problem has the set of m alternatives namely $A = \{A_1, A_2, \dots, A_m\}$ and decision maker will choose the best one from A according to a criterion set $G = \{G_1, G_2, \dots, G_n\}$ which include n criteria. Let there be a prioritization between the attributes expressed by the linear ordering $G_1 \succ G_2 \succ \dots \succ G_n$ (indicating that the attribute G_i has a higher priority than G_j , if $i < j$). The decision matrix is denoted by $X = (x_{ij})_{m \times n}$. Let $W = (w_1, w_2, \dots, w_n)$ be the relative weight vector about the attributes. Then the TOPSIS method with incomplete weight information can be summarized as follows.

Step 1: *Normalize the decision matrix* $X = (x_{ij})_{m \times n}$: There are many methods in existing literatures for standardized the fuzzy matrix. These methods are summarized by

the following equation.

$$r_{ij} = \frac{x_{ij}}{\sqrt{\sum_{k=1}^m x_{kj}^2}}, \quad i = 1, 2, \dots, m; j = 1, 2, \dots, n \quad (4.1.1)$$

We can also standardize the decision matrix by the following method. If G_i is a type of benefit, then

$$r_{ij} = \frac{x_{ij}}{\sum_{k=1}^m x_{kj}}, \quad i = 1, 2, \dots, m; j = 1, 2, \dots, n \quad (4.1.2)$$

If G_i is a type of cost, then

$$r_{ij} = \frac{\frac{1}{x_{ij}}}{\sum_{k=1}^m \frac{1}{x_{kj}}}, \quad i = 1, 2, \dots, m; j = 1, 2, \dots, n \quad (4.1.3)$$

where r_{ij} is the normalize attribute value.

Step 2: Calculate the weighted normalize decision matrix $V = (v_{ij})_{m \times n}$, where

$$v_{ij} = w_j r_{ij}, \quad i = 1, 2, \dots, m; j = 1, 2, \dots, n$$

where w_j is the relative weight of the j^{th} attribute, and $\sum_{j=1}^n w_j = 1$.

Step 3: Find the positive and negative-ideal solutions:

$$A^+ = \{v_1^+, v_2^+, \dots, v_n^+\} = \{\max_j v_{ij} | j \in B\} \quad \text{or} \quad \{\min_j v_{ij} | j \in C\}$$

$$A^- = \{v_1^-, v_2^-, \dots, v_n^-\} = \{\min_j v_{ij} | j \in B\} \quad \text{or} \quad \{\max_j v_{ij} | j \in C\}$$

where B and C are the sets of benefit attribute and cost attributes, respectively.

Step 4: Calculate the Euclidean distance of each alternative from the positive ideal solution (PIS) and negative-ideal solution (NIS), respectively:

$$D_i^+ = \sqrt{\sum_{j=1}^n (v_{ij} - v_j^+)^2}, \quad i = 1, 2, \dots, m$$

$$D_i^- = \sqrt{\sum_{j=1}^n (v_{ij} - v_j^-)^2}, \quad i = 1, 2, \dots, m$$

Step 5: *Calculate the relative closeness coefficient (CC)*: Finally, relative closeness coefficient of each alternative with respect to ideal solutions is computed by using following expression and rank the preference order of all alternatives.

$$C_i = \frac{D_i^-}{D_i^- + D_i^+}, \quad i = 1, 2, \dots, m$$

The larger value of relative closeness coefficient indicates that an alternative is closer to PIS and farther from NIS simultaneously. Therefore, the ranking order of all the alternatives can be determined according to the descending order of relative coefficient values. The most preferred alternative is the one with the highest value.

4.2 Proposed method

Consider a multi-attribute decision making problem which consists of m different alternatives and decision maker will choose best among them under the set of different n criteria. The evaluated values of the alternatives $X = (x_{ij})_{m \times n}$ are expressed in terms of fuzzy numbers where $x_{ij} = [x_{ij}^L, x_{ij}^R]$, where x_{ij}^L and x_{ij}^U are the lower and upper bound of the number respectively. Let $W = (w_1, w_2, \dots, w_n)$ be the weight vector which is uncertain. Then the following is the procedure for computing the best alternative among the existing

Step 1: *Normalize the decision matrix*: In order to avoid the complicated normalization formula we used in classical TOPSIS, the linear scale transformation is used here to transform the various criteria scales into a comparable scale. Therefore, we can obtain the normalized fuzzy decision matrix denoted by $R = (r_{ij})_{m \times n}$, where $r_{ij} = [r_{ij}^L, r_{ij}^R]$

$$\widetilde{r}_{ij} = \begin{cases} \frac{x_{ij}}{\max_i x_{ij}}, & j \in B \\ \frac{\min_i x_{ij}}{x_{ij}}, & j \in C \end{cases} \quad (4.2.1)$$

where B and C are the set of benefit criteria and cost criteria, respectively, and

Step 2: *Calculate the weighted normalize decision matrix* $V = (v_{ij})_{m \times n}$, where

$$v_{ij} = w_j r_{ij}, \quad i = 1, 2, \dots, m; j = 1, 2, \dots, n$$

where w_j is the relative weight of the j^{th} attribute, and $\sum_{j=1}^n w_j = 1$.

Step 3: *Find the fuzzy positive and fuzzy negative-ideal solutions:*

$$A^+ = \{v_1^+, v_2^+, \dots, v_n^+\} = \{[\max_j v_{ij}^L, \max_j v_{ij}^R] | j \in B\} \quad \text{or} \quad \{[\min_j v_{ij}^L, \min_j v_{ij}^R] | j \in C\}$$

$$A^- = \{v_1^-, v_2^-, \dots, v_n^-\} = \{[\min_j v_{ij}^L, \min_j v_{ij}^R] | j \in B\} \quad \text{or} \quad \{[\max_j v_{ij}^L, \max_j v_{ij}^R] | j \in C\}$$

Step 4: *Calculate the Euclidean distance of each alternative from the fuzzy positive ideal solution (FPIS) and fuzzy negative-ideal solution (FNIS), respectively:*

$$D_i^+ = \sum_{j=1}^n d(v_{ij}, v_j^+), \quad i = 1, 2, \dots, m$$

$$D_i^- = \sum_{j=1}^n d(v_{ij}, v_j^-), \quad i = 1, 2, \dots, m$$

where

$$d(v_{ij}, v_j^+) = \sqrt{(v_{ij}^L - v_j^+)^2 + (v_{ij}^R - v_j^+)^2}, \quad i = 1, 2, \dots, m$$

$$d(v_{ij}, v_j^-) = \sqrt{(v_{ij}^L - v_j^-)^2 + (v_{ij}^R - v_j^-)^2}, \quad i = 1, 2, \dots, m$$

Step 5: *Calculate the relative closeness coefficient (CC):* Finally, relative closeness coefficient of each alternative with respect to ideal solutions is computed by using following expression and rank the preference order of all alternatives.

$$C_i = \frac{D_i^-}{D_i^- + D_i^+}, \quad i = 1, 2, \dots, m$$

4.3 Illustrative Example

The above approach has been illustrated by taking a case study as discussed in Chapter 3 for each illustration by taking a trapezoidal fuzzy numbers instead of triangular ones. The aim of this study is to make a decision of a panel who wants to choose the best type of membership functions, namely as linear, parabolic and sigmoidal type denoted by x_1, x_2 and x_3 respectively. The panel takes the decision according to the six criteria at the different level of α cuts. These three possible alternative x_i , ($i = 1, 2, 3$) are to be evaluated using the fuzzy information by the decision-maker under the above criteria.

4.3.1 Length of the Rod

Let length of a rod is a fuzzy number $\tilde{A} = (12cm, 13cm, 14cm, 15cm; 0.8)$. If the length $\tilde{B} = (5cm, 6cm, 7cm, 8cm; 0.7)$, a trapezoidal fuzzy number, is cut off from this rod then the remaining length of the rod \tilde{C} is $\tilde{A} - \tilde{B}$. Then the decision matrix corresponding to each α -cuts are summarized as below.

	$\alpha = 0$	$\alpha = 0.1$	$\alpha = 0.3$	$\alpha = 0.5$	$\alpha = 0.7$
x_1	(4.0000, 10.0000)	(4.2000, 9.8000)	(4.6000, 9.4000)	(5.0000, 9.0000)	(5.4000, 8.6000)
x_2	(4.0000, 10.0000)	(4.6325, 9.3675)	(5.0954, 8.9046)	(5.4142, 8.5858)	(5.6733, 8.3267)
x_3	(4.0000, 10.0000)	(4.5722, 9.4278)	(4.8331, 9.1669)	(5.0000, 9.0000)	(5.1669, 8.8331)

This decision matrix has been normalized by using Eq. (4.2.1) and get

$$R_{3 \times 6}(r_{ij}) = \begin{bmatrix} (0.7407, 0.8600) & (0.7778, 0.8776) & (0.8519, 0.9149) & (0.9259, 0.9556) & (1.0000, 1.0000) \\ (0.7051, 0.8327) & (0.8165, 0.8889) & (0.8981, 0.9351) & (0.9543, 0.9698) & (1.0000, 1.0000) \\ (0.7742, 0.8833) & (0.8849, 0.9369) & (0.9354, 0.9636) & (0.9677, 0.9815) & (1.0000, 1.0000) \end{bmatrix}$$

Now, define the fuzzy positive ideal solution (FPIS) and fuzzy negative ideal solution (FNIS) as $v^+ = [(1, 1), (1, 1), (1, 1), (1, 1), (1, 1)]$ and $v^- = [(0, 0), (0, 0), (0, 0), (0, 0), (0, 0)]$. Based on it, the distance of each alternative from its FPIS and FNIS are $D_1^+ = 0.4334$, $D_2^+ = 0.4226$, $D_3^+ = 0.2979$, $D_1^- = 2.8278$, $D_2^- = 2.8600$ and $D_3^- = 2.9569$. Thus, the evaluated result about the best membership function are computed by using the closeness coefficient $C_i = \frac{D_i^-}{D_i^+ + D_i^-}$ as $C_1 = 0.8671$, $C_2 = 0.8713$ and $C_3 = 0.9085$. Therefore ranking of the types of membership functions are $C_3 \succ C_2 \succ C_1$ and hence x_3 i.e. sigmoidal membership function is the most desirable one while x_1 i.e. linear is the least one. Thus, sigmoidal type membership function will be the best for decision maker for analyzing and depicting the analysis of the system.

4.3.2 Area of triangle

The aim of this problem is to choose the best alternative for finding the area of the triangle. Let $\tilde{A} = (2cm, 3cm, 4cm, 5cm; 0.8)$ and $\tilde{B} = (5cm, 6cm, 7cm, 8cm; 0.7)$ be the base side and height of the triangle in the form of trapezoidal fuzzy number then the area of the triangle becomes $\frac{1}{2}\tilde{A} \cdot \tilde{B}$. In order to do this, the normalized degree of membership functions at different level of α -cuts for these different types of membership functions are summarized as below.

	$\alpha = 0$	$\alpha = 0.1$	$\alpha = 0.3$	$\alpha = 0.5$	$\alpha = 0.7$
x_1	(0.6498, 0.7848)	(0.6959, 0.8109)	(0.79210, 0.8674)	(0.8934, 0.9301)	(1.0000, 1.0000)
x_2	(0.6040, 0.7456)	(0.7437, 0.8287)	(0.8537, 0.8988)	(0.9331, 0.9526)	(1.0000, 1.0000)
x_3	(0.6933, 0.8189)	(0.8378, 0.9008)	(0.9074, 0.9424)	(0.9532, 0.9705)	(1.0000, 1.0000)

Now, based on it, the distance of each alternative from its FPIS and FNIS are $D_1^+ = 0.6118$, $D_2^+ = 0.5958$, $D_3^+ = 0.4219$, $D_1^- = 2.6875$, $D_2^- = 2.7341$ and $D_3^- = 2.8682$. Thus, the evaluated result about the best membership function are computed by using the closeness coefficient $C_i = \frac{D_i^-}{D_i^+ + D_i^-}$ as $C_1 = 0.8146$, $C_2 = 0.8211$ and $C_3 = 0.8718$. Therefore ranking of the types of membership functions are $C_3 \succ C_2 \succ C_1$ and hence x_3 i.e. sigmoidal membership function is the most desirable one while x_1 i.e. linear is the least one.

4.3.3 Length of the rectangle

The aim of this problem is to choose the best alternative for finding the length of the rectangle so that wastage will be minimized. For this, assume that $\tilde{A} = (42cm^2, 45cm^2, 48cm^2, 51cm^2; 0.8)$ and $\tilde{B} = (3cm, 4cm, 5cm, 6cm; 0.9)$ be the area of rectangle and its breadth, respectively. In order to do this, the normalized degree of membership functions at different level of α -cuts for these different types of membership functions are summarized as below.

	$\alpha = 0$	$\alpha = 0.2$	$\alpha = 0.4$	$\alpha = 0.6$	$\alpha = 0.8$
x_1	(0.8198, 0.7523)	(0.8602, 0.8120)	(0.9035, 0.8732)	(0.9499, 0.9358)	(1.0000, 1.0000)
x_2	(0.7998, 0.7298)	(0.8919, 0.8613)	(0.9345, 0.9178)	(0.9692, 0.9621)	(1.0000, 1.0000)
x_3	(0.8551, 0.7942)	(0.9340, 0.9100)	(0.9566, 0.9414)	(0.9759, 0.9679)	(1.0000, 1.0000)

Now, based on it, the distance of each alternative from its FPIS and FNIS are $D_1^+ = 0.4251$, $D_2^+ = 0.3968$, $D_3^+ = 0.2877$, $D_1^- = 2.8274$, $D_2^- = 2.8792$ and $D_3^- = 2.9585$. Thus, the evaluated result about the best membership function are computed by using the closeness coefficient $C_i = \frac{D_i^-}{D_i^+ + D_i^-}$ as $C_1 = 0.8693$, $C_2 = 0.8789$ and $C_3 = 0.9114$. Therefore ranking of the types of membership functions are $C_3 \succ C_2 \succ C_1$ and hence x_3 i.e. sigmoidal membership function is the most desirable one while x_1 i.e. linear is the least one.

4.3.4 Perimeter of the rectangle

The aim of this problem is to choose the best alternative for finding the perimeter of the rectangle whose dimensions are in the form of trapezoidal fuzzy numbers as $\tilde{A} = (12cm, 13cm, 14cm, 15cm; 0.9)$ and $\tilde{B} = (6cm, 7cm, 8cm, 9cm; 0.8)$ respectively. Then the perimeter of the rectangle will be $2(\tilde{A} + \tilde{B})$. In order to do this, the normalized degree of membership functions at different level of α -cuts for these different types of membership functions are summarized as below.

	$\alpha = 0$	$\alpha = 0.2$	$\alpha = 0.4$	$\alpha = 0.6$	$\alpha = 0.8$
x_1	(0.9184, 0.9333)	(0.9388, 0.9492)	(0.9592, 0.9655)	(0.9796, 0.9825)	(1.0000, 1.0000)
x_2	(0.9096, 0.9255)	(0.9548, 0.9613)	(0.9735, 0.9770)	(0.9879, 0.9893)	(1.0000, 1.0000)
x_3	(0.9340, 0.9470)	(0.9717, 0.9766)	(0.9817, 0.9847)	(0.9900, 0.9916)	(1.0000, 1.0000)

Now, based on it, the distance of each alternative from its FPIS and FNIS are $D_1^+ = 0.1450$, $D_2^+ = 0.1370$, $D_3^+ = 0.0962$, $D_1^- = 3.0453$, $D_2^- = 3.0621$ and $D_3^- = 3.0926$. Thus, the evaluated result about the best membership function are computed by using the closeness coefficient $C_i = \frac{D_i^-}{D_i^+ + D_i^-}$ as $C_1 = 0.9546$, $C_2 = 0.9572$ and $C_3 = 0.9698$. Therefore ranking of the types of membership functions are $C_3 \succ C_2 \succ C_1$ and hence x_3 i.e. sigmoidal membership function is the most desirable one while x_1 i.e. linear is the least one.

4.4 Conclusion

The objective of this chapter is to present an efficient technique for finding the best fuzzy number for handling the uncertainties in the data. For this linear, parabolic and sigmoidal fuzzy numbers have been taken for illustration purposes. An approach for selecting the best alternative under the different criteria or selection is implemented. The preference of the various alternatives are represented in terms of interval number in the form of left and right cuts at different levels of α -cuts. Based on their decision maker information, all the individual decision makers' opinion for rating the candidate are aggregated using the fuzzy positive and negative ideal solutions. A ranking of the different alternatives has been done by using the closeness coefficient and based on that the best alternative is selected. From the study, it has been concluded that the sigmoidal fuzzy number has handled the data more accurately than the other fuzzy number and hence the decision is more reliable and efficient in a more profitable way.

Chapter 5

Summary and Future Scope

The chapter presents a comprehensive summary of the research contributions made during the period of this thesis. It also outlines the managerial implications for the implementation of recommendations. Finally the scope for future work has been outlined.

5.1 Summary of the work

The conclusion made from the work presented in this thesis are summarized below:

- (i) The research work presented in this thesis is to present an alternative approach for finding the membership functions without using α -cuts. For it, distribution and complementary distribution functions have been proposed by taking a non-linear sigmoidal type membership function, for handling the uncertainties in the data, to compute the various arithmetic operations in fuzzy environment. From the analysis, it has been concluded that the proposed sigmoidal type membership function arithmetic operations will suitably handle the uncertainty in the data and hence find an alternative ways for analyzing the performance of the system. Also the computed results have been compared with the existing techniques, namely by taking parabolic as well as linear type membership functions and found that it has a less range of uncertainties. Thus, this method is beneficial for those where computation of α -cut is difficult or even not possible.
- (ii) Apart from their computation of the membership functions, a strategy for finding the best type of fuzzy number under the different set of α -cuts are investigated. For

this, the preference of the various alternatives are represented in terms of interval number in the form of left and right cuts at different levels of α -cuts. Based on their decision maker information, all the individual decision makers' opinion for rating the candidate are aggregated using the fuzzy positive and negative ideal solutions. A ranking of the different alternatives has been done by using the closeness coefficient and based on that the best alternative is selected. From the study, it has been concluded that the sigmoidal fuzzy number has handled the data more accurately than the other fuzzy number and hence the decision is more reliable and efficient in a more profitable way.

5.2 Future scope of the work

The method for computing the various arithmetic operations using various fuzzy numbers can be extended in the following directions:

- (i) The computation of the fuzzy membership functions can be computed by formulating a nonlinear optimization model instead of using fuzzy arithmetic operations.
- (ii) The presented work done may be extended for a nonlinear fuzzy number such as exponential, hyperbolic etc.
- (iii) In our study, we have taken the constant data i.e. it follows the exponential distribution. In the future, we may try to extend the proposed approach for a time varying distribution function.
- (iv) The study, based on these arithmetic operations may be extended for the applications part in reliability optimization, resource allocation, facility planning and management, inventory control, network analysis and job shop scheduling.

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