ANALYSIS AND DESIGN OF FRACTIONAL ORDER DIGITAL DIFFERENTIATOR

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Submitted by

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DECLARATION

I hereby declare that the work which is being presented in the dissertation entitled, **"Analysis & Design of Fractional Order Digital Differentiator"** in partial fulfilment of the requirement for the award of degree of Master of Engineering in Electronics and Communication Engineering Department of Thapar University, Patiala, is an authentic record of my own work carried out under the supervision of **Dr. Sanjay Kumar** (Assistant Professor), ECED and refers other researcher's work which are duly listed in the reference section.

The matter presented in this dissertation has not been submitted in any other University/Institute for the award of degree.

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This is certified that the above statement made by the student is correct to the best of my knowledge and belief.

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(Avinash Kumar Dubey)
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ABSTRACT

Digital differentiator is an important signal processing tool. It is found in various applications like biomedical image enhancement and in high frequency radars. There are various techniques to design a digital differentiator and lot of modifications has done in this area to design several useful differentiators. But there is still scope of improvement in terms of parameter ‘optimization’. The design problem of differentiators is a challenging task. Therefore, there is strong inspiration to make design process easy and efficient.

In this dissertation, the design of first and second order differentiator is studied. Next, the concept of higher order differentiator and fractional order differentiator is investigated. The study of Finite Impulse Response (FIR), digital differentiator is done and different orders of differentiator are compared with the ideal differentiator and calculated the error is calculated with different orders.

Secondly, a thorough study has been done on Fractional Order Savitzky Golay Digital Differentiator (FOSGDD). Here we have studied the method of design and then implemented this method for the calculation of derivative of a sinusoidal signal and chirp signal. The proposed method is compared with several popular methods and their root mean square error and computation time is computed. Also, the frequency response characteristic of the proposed method is studied by varying order n and moving window coefficient I.

A new method for the calculation of fractional order derivative using power function and least square method is also proposed. This method is imposed for the calculation of fractional order derivative and this method is compared with other popular method and its magnitude response and error characteristics are computed.

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LIST OF ABBREVIATIONS

FIR	Finite Impulse Response
IIR	Infinite Impulse Response
RL	Riemann and Liouville
GL	Grunwald and Letnikov
GA	Genetic Algorithm
FD	Fractional order Differentiator
DD	Digital Differentiator
LPF	Low Pass Filter
DHT	Digital Hilbert Transformer
FOID	Fractional Order Integrator and Differentiator
AFOD	Adjustable Fractional Order Differentiator
MATLAB	Matrix Laboratory
FOSGDD	Fractional Order Savitzky Golay Digital Differentiator
RMSE	Root Mean Square Error

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Chapter 1

Introduction

1.1 Concept of Differentiators

A differentiator is a circuit that is designed such that the output of the circuit is approximately directly proportional to the rate of change (the time derivative) of the input [1]. Differentiators are used to perform a differentiation operation on discrete-time signals. They can be used in biomedical engineering and motion analysis and several other applications. There has been some research to develop different kinds of differentiators: full-band differentiators, low pass differentiators and differentiators for midband frequencies. Full-band differentiators cause noise amplification in a digital differentiation process. Low pass differentiators can be used in order to avoid this undesirable phenomenon [9]. Some of these designs are narrowband designs for frequencies around $\frac{\pi}{2}$. FIR differentiators can be designed by using FIR filter Types 3 and 4. The ideal differentiator has a frequency response of the form:

$$H(e^{j\omega}) = j\omega, \qquad |\omega| \le \pi \tag{1.1}$$

and the ideal low pass differentiator has a frequency response of the form:

$$H(e^{j\omega}) = \begin{cases} j\omega, & |\omega| \le \omega_c \\ 0, & \omega_c \le |\omega| \le \pi \end{cases} \tag{1.2}$$

where ω_c is the cutoff frequency.

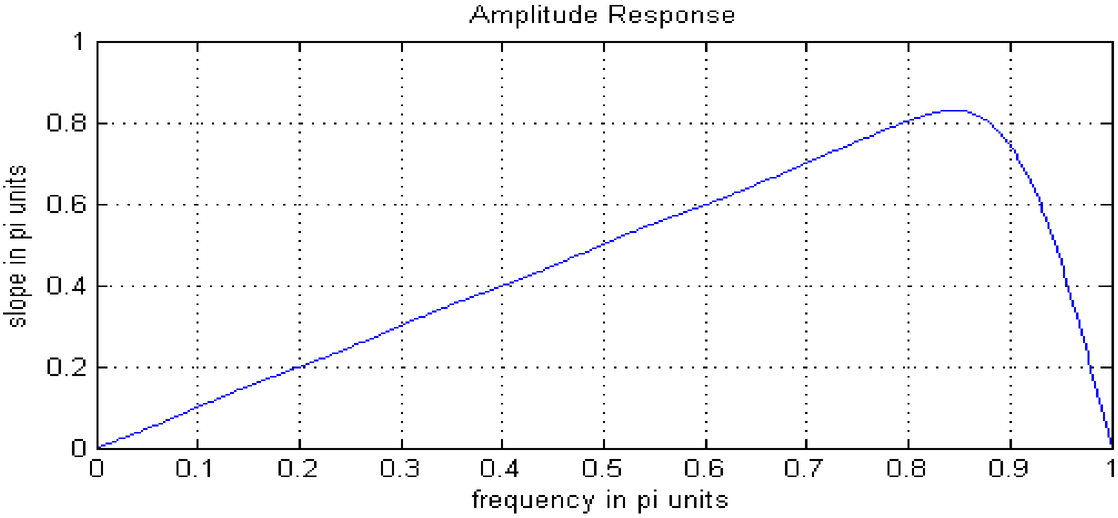

Figure 1.1: Frequency response of differentiator using Hamming window.

The ideal transfer function is of infinite length and non-causal and has to be truncated or approximated. The approximation of the impulse response can be done e.g. by using polynomials. The differentiator transfer function has an increasing slope on the pass band. The low pass differentiators can be designed with several methods. Linear programming can be used as the FIR differentiator optimization tool like in FIR filters under consideration in this thesis. The FIR differentiator optimization using linear programming can be stated as a problem of minimizing

$$\max_{\omega\epsilon|0,1|}|H(\omega)-\omega|\,\delta_p \tag{1.3}$$

where the $H(\omega)$ is a zero-phase frequency response, ω is an angular frequency and δ_p is pass band ripple.

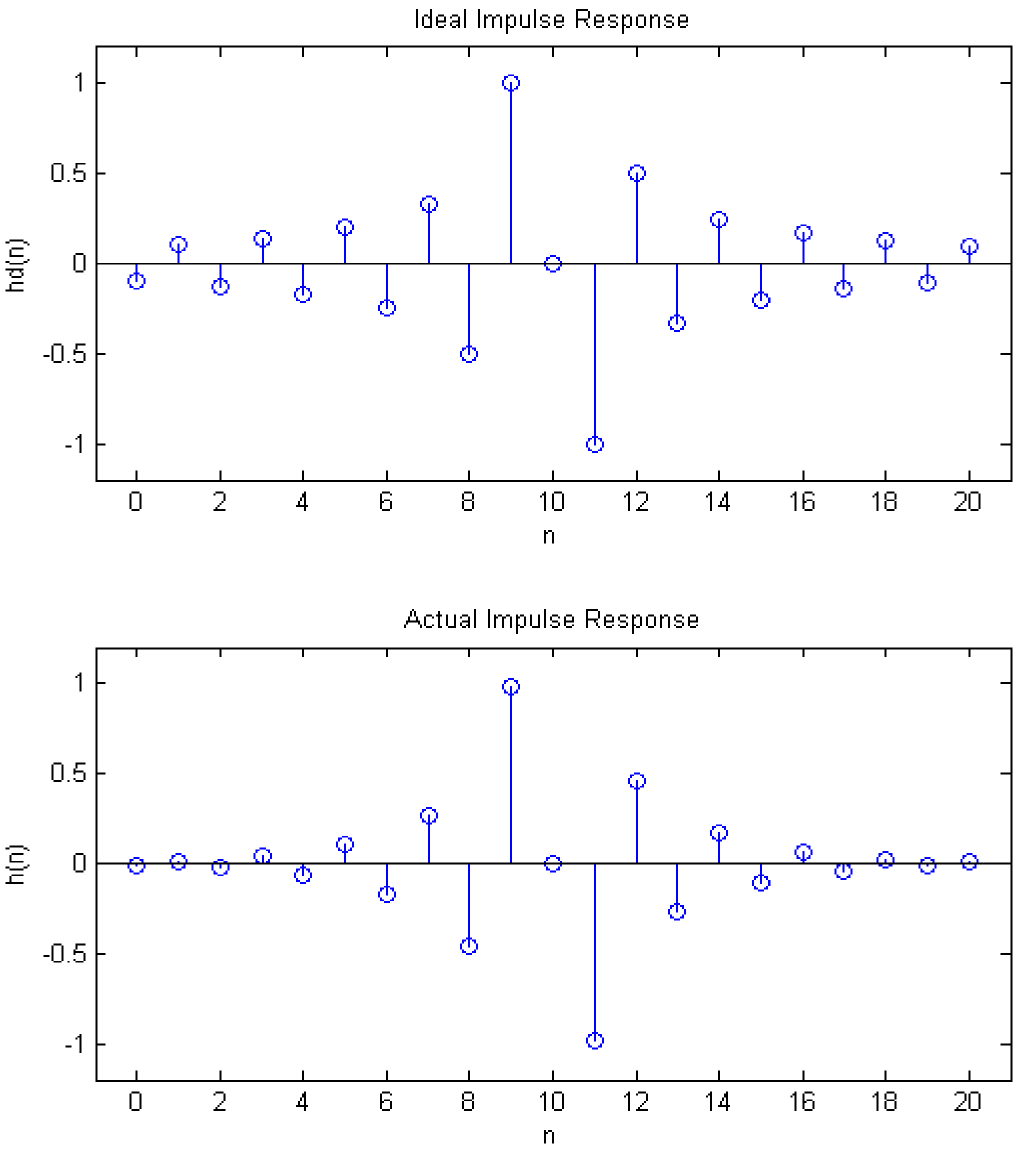

Figure 1.2: Ideal and actual impulse response of differentiator.

In this section, the design of an ideal differentiator by using Hamming window is designed and the ideal and actual impulse response is also plotted.

1.2 Different methods for design

1. Taylor series approximation is broadly used to derive digital differentiator. They are basically in the form of central approximations, such as [2]

$$y(n) = \int_{k=-N}^{N} c(k)x(n-k) \tag{1.4}$$

N is the order of Taylor series approximation, we can calculate coefficients as:

$$c(k) = -c(k) = \frac{(-1)^{k+1}N!^2}{K(N-k)!\,(N+k)!}, \qquad \mathrm{c}(0) = 0 \tag{1.5}$$

2. FIR type differentiators

For the FIR type differentiators $y(n)$ can be written as: [2]

$$y(n) = \int_{k=0}^{N} c(k)x(n-N) \tag{1.6}$$

3. IIR Type Digital Differentiators

An IIR type digital differentiator can be obtained from a digital integrator. We can use numerous integrators to obtain the digital differentiator, such as Tustin Integrator, Simpson’s integrator and Tick Integrator. By using these integrators we can design the differentiators with different order. An ideal integrator is defined by the following transfer function [2]

$$H_1(j\omega) = \frac{1}{j\omega} \tag{1.7}$$

Rectangular Integrator

$$H_1(Z) = \frac{zT}{z-1} \tag{1.8}$$

Tustin or Trapezoidal integrator

$$H_2(Z) = \frac{(z+1)T}{2(z-1)} \tag{1.9}$$

1.3 First and Second Order Differentiators

1.3.1 First Order Al-Alaoui Digital Differentiator

It can be obtained by the interpolation of rectangular interpolator so, [2]

$$H_{AL}(z) = \alpha H_1(z) + (1-\alpha)H_2(z) \tag{1.10}$$

where, α lies between 0 and 1, for $\alpha = \frac{3}{4}$ above equation becomes

$$H_{AL}(z) = \frac{3}{4}H_1(z) + \frac{1}{4}H_2(z) \tag{1.11}$$

Substituting $H_1(z)$ and $H_2(z)$ in above equation and solving

$$H_{AL}(z) = \frac{T(z+7)}{8(z-1)} \tag{1.12}$$

Reflecting the zero *z* = -7 with its reciprocal -1/7, and compensating the magnitude results in a minimum phase digital integrator with the transfer function

$$H_{AL}(z) = \frac{7T(z+1/7)}{8(z-1)} \tag{1.13}$$

Inverting the above transfer function gives Al-Alaoui's stabilized IIR differentiator of the first order

$$H_{AL}(z) = \frac{8(z-1)}{7T(z+1/7)} \tag{1.14}$$

1.3.2 Second order Al-Alaoui Digital Differentiator

Tick integrator has two real pole at $z = \pm 1$ whose transfer function is given by

$$H_{AL}(z) = \frac{T(0.385z^2 + 1.2832z + 0.3584)}{(z^2-1)} \tag{1.15}$$

By following the above mentioned procedure, the following transfer function obtained for the digital differentiator is,

$$H_4(z) = \frac{0.852(z^2-1)}{(z^2 + 0.611z + 1)} \tag{1.16}$$

The transfer function of the Simpson's integrator is:

$$H_3(z) = \frac{T(z^2+4z+1)}{3(z^2-1)} \tag{1.17}$$

The corresponding transfer function of the digital differentiator will be:

$$G_3(z) = \frac{0.8038(z^2 - 1)}{T(z^2 + 0.5358z + 0.0718)} \tag{1.18}$$

By interpolating the Simpson and trapezoidal digital integrators the following hybrid digital integrator is obtained [2].

$$H_{AL2}(z) = \alpha H_3(z) + (1 - \alpha)H_2(z) \tag{1.19}$$

Substituting the expressions of Simpson's and the trapezoidal integrators, the following is the expression for the new digital integrator [2].

$$H_{AL2}(z) = \frac{T(3 - \alpha)(z^2 + \frac{2(s + \alpha)z}{s - \alpha} + 1)}{6(z^2 - 1)} \tag{1.20}$$

For, $\alpha = 0.6$

$$H_{AL2}(z) = \frac{0.4T(z^2 + 2.5z + 1)}{(z^2 - 1)} \tag{1.21}$$

Following the above mentioned procedure, the digital differentiator of second order will be obtained:

$$H_{AL2}(z) = \frac{1.25(z^2 - 1)}{T(z^2 + z + 0.25)} \tag{1.22}$$

1.4 Higher and Fractional Order Differentiators

Differentiation and integration are usually regarded as discrete operations, in the sense that we differentiate or integrate a function once, twice, or any whole number of times. However, in some circumstances it's useful to evaluate a fractional derivative.

The fractional order calculus is a 300-years-old topic; the theory of fractional-order derivative was developed usually in the nineteenth century [36]. Recent books provide a good source of references on fractional calculus. However, applying fractional-order calculus to dynamic systems control is just a recent focus of interest. The fractional order of differentiation and integration is useful in control system applications. Fractional Calculus is generalization of ordinary differentiation and integration to non-integer order i.e. taking real number powers of differentiation operator [37].

$$D^v f(x) = \frac{d^v f(x)}{dx^v} \tag{1.23}$$

If v is a natural number then the case is called higher integer order differentiation. Positive real number corresponds to fractional order differentiation. The historical developments culminated in two calculi which are based on the work of Riemann and Liouville (RL) at the one side and on the work of Grunwald and Letnikov (GL) on the other. The classical form of fractional calculus is given by the Riemann-Liouville integral. It is given as follows: [38]

$${a^D}_t^{-\alpha} f(t) = {a^I}_t^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} f(\tau)\, d\tau \tag{1.24}$$

The important step in digital implementation of IIR fractional order differentiators is the discretization of the fractional-order differentiators s^r . The half-order numerical differentiator is expressed in higher order terms by using continuous fraction expansion. The discretize half order differentiator is cascaded with third order low-pass chebyshev filter resulting in linear phase low-pass IIR digital differentiator. The fractional order differentiator gives better performance to the real-time signal.

1.5 Digital Differentiator Applications

- In radar and sonar, the velocity and acceleration are computed from the position measurements using differentiation. Velocity is estimated by first order differentiation and acceleration by second order [36].
- The rate of liquid flow in a tank (which may be part of a chemical plant) is estimated from the derivative of the measured liquid level.
- In biomedical investigations, it is often necessary to obtain the first and higher order derivatives of the biomedical data, especially at low frequency ranges. For example in QRS complex detection in ECG [2].
- For geo-physical data processing, derivatives of the observation samples are usually needed for midband frequencies of the spectrum. Maximally flat differentiators near half Nyquist frequency are useful for this operation [9].
- The derivatives at high frequencies are useful for solving the problems of image restoration and image texture enhancement (to detect various features, like an edge, for example, of an object in the picture) [2].

- The use of derivatives of various signals in control engineering (in auto-follow, servomechanism, robotics, artificial eye etc.) is also well known.
- Fractional dimension is used to measure some real-world data such as coastline, clouds, dust in the air, and network of neurons in the body. The fractional dimension has been applied widely to pattern recognition and classification. Fractional Order Differentiators are used to exploit such real world issues. Fractional Order Differentiators are also used in bar code readers [36].

1.6 Organization of Thesis

This dissertation is organized as follows

Chapter 1	contains a brief introduction of the digital differentiators and their applications.
Chapter 2	contains the literature review of papers related to the work. It contains the literature review of papers used in the designing of different types of FIR, IIR and Fractional order differentiators.
Chapter 3	contains the design procedure of various FIR and IIR filters and digital differentiators which are studied during this thesis.
Chapter 4	in this chapter work which is done in this thesis is explained to design fractional order differentiator and to validate their accuracy.
Chapter 5	concludes the dissertation with future scope of the work.

Chapter 2
Literature Survey

In this chapter, the literature survey of the Digital Differentiators and their design method has been presented.

Krishna *et al.* [2]: an effort is made to study the change in phase angle of digital differentiators with the help of application of fractional delay. The use of digital differentiators for the recognition of edges in an image, QRS detection in an ECG signal is explained. A study on the design of digital differentiators defines that, type III FIR differentiators have the integral nature in frequency response of approaching zero at Nyquist-frequency. To extend the recital of type III FIR filters in the higher frequency bands, one has to improve the filter taps, with the proven efficiency of these differentiators they can be engaged in hardware using Verilog.

Medlin *et al.* [3]: the new differentiators have linear phase and are maximally precise at the center of the differentiation band. Their design is based on a minimization technique for the integrated square error of the frequency response, over designated approximation bands. The closed-form solution for the filter coefficients is attained by the method of Lagrange multipliers. The addition of stop bands in the design process is also discussed. This technique has also been effectively used by the authors for the design of optimal low pass differentiators.

Zhou *et al.* [5]: this paper presents in detail the optimal design approach of high-order digital differentiator based on the algorithm of neural networks. The main idea is to minimize the sum of the square errors among the amplitude response of the ideal differentiator and that of the designed by training the weight vector of neural networks, then locating the impulse response of digital differentiator. The convergence theorem of the neural-network algorithm is presented and proved, and the optimal design approach is presented by examples of high-order digital differentiator. Since the method is not involved in operation of inverse matrix, it solves the difficult problem effectively on designing high order digital differentiator. The algorithm is not only appropriate for designing high order digital differentiator, but also for designing low order digital differentiator.

Samadi *et al.* [8]: a characterization of polynomial signals in the r domain is introduced, and. It is revealed that for a given member of the class, all polynomial signals of a certain degree pass through the filter unaltered after a possibly fractional delay. It is then proved that using appropriate maximally at filters from the family, it is possible to up sample and then fractionally delay any polynomial signal by a factor of 2 in an exact manner.

Selesnick *et al.* [9]: this paper describes the design of type III and type IV linear-phase finite-impulse response (FIR) low-pass digital differentiators according to the maximally flat criterion. We present a two-term recursive formula that enables the simple stable computation of the impulse response coefficients. The same recursive formula is valid for both Type III and Type IV solutions.

Selesnick *et al.* [10]: this paper describes a simple formulation for the no iterative design of narrowband FIR linear-phase low pass digital differentiators. The frequency responses of the filters are flat around dc and have equally spaced nulls in the stop band. The design problem is formulated so as to avoid the complexity or ill-conditioning of standard methods for the design of similar filters when those methods are used to design narrow-band filters with long impulse responses.

Samadi *et al.* [11]: a discrete-time fractional-order differentiator is modeled as a finite-impulse response (FIR) system. The system yields fractional-order derivatives of Riemann-Liouville type for a uniformly sampled polynomial signal. The computation of the output signal is based on the additive combination of the weighted outputs of cascaded first-order digital differentiators. For differentiators of fractional order with a terminal value equal to zero, the weights are time-varying**.**

Ngo *et al.* [12]: this brief presents a general theory of the Newton–Cotes digital integrators which is derived by applying the –transform technique to the closed-form Newton–Cotes integration formula. Based on this developed theory, a new wideband third-order trapezoidal digital integrator is found to be a class of trapezoidal digital integrators. The novel wideband third-order trapezoidal integrator accurately approximates the ideal integrator over the whole Nyquist frequency range and compares favorably with existing integrators. Using the new design of the wide band third-order trapezoidal integrator, a new wideband digital differentiator has been designed. The novel wideband digital differentiator approximates the ideal differentiator reasonably well over the whole Nyquist frequency range and compares favorably with the existing differentiators.

Alaoui *et al.* [41]: a novel class of IIR (infinite impulse response) digital integrators and differentiators is developed. A class of digital integrators is first derived from a class of numerical integration rules. A class of digital differentiators is subsequently obtained by inverting the transfer functions of the obtained integrators and stabilizing the resulting transfer functions together with magnitude compensation if necessary. Simulated annealing is applied to optimize some of the obtained integrators and differentiators.

Tahmasbi *et al.* [16]: in this paper a novel approach is proposed for approximating Parks-McClellan low-pass differentiators using optimized low-order IIR filters. Indeed, a suitable IIR filter is designed for approximating Parks- McClellan Low pass differentiator using modified Al-Alaoui's method, and then denominator polynomial coefficients of resulting transfer function optimized by Genetic algorithm. A suitable fitness function is defined to optimize both magnitude and phase responses; moreover, appropriate weighting coefficients and GA parameters are reported for several cut-off frequencies. It is shown that the order-4 proposed low-pass differentiators yield a frequency response which is almost equal to order-30 Parks-McClellan low-pass differentiators.

Krishna [17]: a first order s to z transforms is designed by interpolating Backward and Al-Alaoui Transforms. The proposed transform is used for the design of fractional order differentiators using indirect discretization. The results reveal the efficacy of the proposed transform. Design of fractional order digital differentiators using indirect discretization technique has been presented. The rational approximation for the fractional order operator is calculated using Continued fraction expansion and is digitized using novel s to z transforms. Novel s to z transform is obtained by the interpolation of Bilinear and Al-Alaoui transform. The Magnitude response of fractional order differentiators obtained by using proposed transform is closer to ideal one compared to Al-Alaoui transform.

Yang *et al*. [18]: a new feedback-based methodology for the implementation of a fractional-order differentiator (FD) is described in this paper. The differentiator can be based on a standard definition of the fractional calculus, such as the Riemann-Liouville or Grunwald-Letnikov definitions. Some methods by which the FD functions can be approximated using a DSP-based implementation (either FIR or IIR) are described, In the new implementation, a classical IIR FD and a fractional-order integrator are combined using a proportional feedback loop. This structure is found to improve the performance of the FD, in terms of both its frequency-domain and time-domain characteristics.

Dutta Roy *et al.* [19]: Interrelationships between the digital differentiator (DD), the digital Hilbert transformer (DHT), and the half-band low-pass filter (1/2-LPF) have been brought out. A number of important properties, confirming the close proximity of these filters, are highlighted. Theoretical results have been substantiated by transforming relative error DD's to equiripple DHT's and equiripple 1 /2-LPF's. It has been shown that the relative error digital differentiators can easily be transformed to equiripple digital Hilbert transformers and equiripple half-band low-pass filters. Relations connecting their impulse responses and also their frequency responses have been brought out.

Antoniou [15]: a procedure which can be used to design digital differentiators satisfying prescribed specifications was introduced. The procedure is based on the Fourier series method for the design of no recursive digital filters, and uses the Kaiser Window function for the minimization of the amplitude of Gibbs oscillations. The approach is used to design a number of differentiators assuming various prescribed bandwidths and various prescribed in band errors. It is also used to design a wideband high-precision differentiator. The proposed method is compared briefly with that of McClellan and Parks, based on the theory of weighted Chebyshev approximation.

Rabiner *et al.* [39]: developed relative error technique to design wideband differentiators using Remez optimization procedure. Observations made are such as the smaller the bandwidth, the faster the decrease of peak relative error with increasing order of differentiator (N). Larger the value of N faster is the decrease of the peak relative error with decreasing bandwidth. Also it is established that differentiators with even values of N have peak relative errors which are approximately one to two orders of magnitude smaller than identical bandwidth differentiators with odd values of N. Optimal, maximally accurate digital differentiators (DDS) are derived for the low frequency range.

Kumar *et al.* [21]: Exact coefficients used in the proposed DDs can be readily computed from explicit formulas, whereas the optimal (minimal RE) DDs require and optimization program to derive the coefficients. The lower the frequency of differentiation, the better is the performance of the proposed differentiators, making them suitable for many typical applications

Kumar *et al*. [22]: Digital differentiators which are maximally linear at spot frequency $\omega = \pi/p$, where p is a positive integer, were proposed. The suggested differentiator, besides giving zero phase error over entire set of frequencies, can achieve very high accuracy in the magnitude response over a given frequency.

Alaoui [5]: developed a novel class of stable, minimum phase, second-order, IIR digital differentiators. It is obtained by inverting the transfer functions of a class of second-order integrators, stabilizing the resulting transfer functions, and compensating their magnitudes. The responses of second-order integrators are obtained by interpolating the traditional Simpson and trapezoidal integrators. The resulting integrators have a perfect -90 degrees phase over the Nyquist interval and could better approximate the ideal magnitude response than either of the two traditional integrators. In addition to the above two integrators, the Tick integrator is also a member of the class. The resulting integrators and differentiators extend the frequency range of operation beyond that possible by using either of the two traditional integrators. The low order and high accuracy of the filters developed in this article make them attractive for real time applications.

Alaoui [7]: a novel approach to designing recursive stable digital differentiators is discussed. A four step design procedure is presented. The procedure consists of obtaining or designing an integrator and then modifying its transfer function approximately to obtain a stable differentiator. As an example a second order recursive differentiator is developed in this text.

Alaoui [14]: a novel digital integrator and a novel digital differentiator were presented. Both the integrator and the differentiator are of first order and thus eminently suitable for real-time applications. Both have an almost linear phase. The integrator is obtained by interpolating two popular digital integration techniques, the rectangular and the trapezoidal rules. The resulting integrator outperforms both the rectangular and trapezoidal integrators in range and frequency. The new differentiator is obtained by taking the inverse of the transfer function of the integrator. The effective range of the differentiator is about 0.8 of the Nyquist frequency.

Tseng *et al.* [27]: a novel approach to the design of digital FIR differentiators is presented. The differentiator designed has linear phase and has zero derivatives at zero frequency. The design is based on the maximization of signal-to-noise ratio (SNR) at the output of the differentiator. The optimal filter coefficients have been obtained from the generalized eigenvector associated with the maximum eigenvalue of a pair of symmetric matrices. Estimation of the time derivative of polynomial signal, sinusoidal signal and handwritten Chinese signature is used to demonstrate that the proposed method provides better accuracy and higher SNR than the conventional differentiator method (Eigen filter).

Alaoui [13]: A novel approach to design approximately linear phase infinite-impulse-response (IIR) digital filters in the pass band region was introduced. The proposed approach yields digital IIR filters whose numerators represent linear phase finite-impulse-response (FIR) filters. As an example, low-pass IIR differentiators have been introduced in the paper. The range and high-frequency suppression of the proposed low-pass differentiators are comparable to those obtained by higher order FIR low-pass differentiators. In addition, the differentiators exhibit almost linear phases in the pass band regions. These results are compared with Salesnick's non recursive differentiator's performance.

Chen *et al.* [29]: optimal design of higher order digital differentiators in parallel sense was studied. Conventionally, using parallel error criterion for this design problem results in a nonlinear optimization problem since the corresponding objective function contains an absolute error function. The authors have first reformulated the design problem as a linear programming problem in frequency domain. To avoid the requirement of huge computation, an algorithm is proposed based on modification of Karmarkar's algorithm. This leads to very efficient procedure for the considered design problem. Simulations show better performance as compared to Chebyshev error criteria.

Mollova *et al.* [31]: proposes a new, simple analytic closed-form relation for least squares design of higher-order differentiators. Using this approach, solving a system of linear equations for full band differentiators is avoided. Numerical and graphical results are given in the paper for illustration. The design method using Eigen filter approach is based on the computation of an eigenvector of an appropriate real, symmetric, and positive-definite matrix. The elements of this matrix are usually evaluated by very time-consuming numerical integration.

Pei *et al.* [44]: proposed a simple analytic closed-form formula to compute these matrix elements very efficiently. Hence, the Eigen filter approach for differentiators becomes much easier and more accurate than before and design time is reduced greatly for designing long filters.

Pei *et al.* [43]: proposed a fractional delay filter, an integer-order differ integrator, a fractional Hilbert transformer and a fractional differ integrator. Through the time-domain analysis on the desired input and output signals of a linear time-invariant system, a set of linear equations are derived, which can be solved to obtain the coefficients of the desired filter. It is also showed that the difference between the desired output signal and the actual output of the system can be represented as the convolution of the derivative of the input signal and the Peano kernel. Design examples are illustrated to show the performance of each proposed filter. This method provides full band differentiator design whose performance for a given order is better than previous designs.

Tseng [23]: a new method of the design of a fractional order FIR differentiator is invented. First, the fractional derivative of power function is defined. Then, the impulse response of fractional order differentiator is obtained by solving linear equations of Vander-monde form. Finally, one example is used to demonstrate that the fractional derivatives of digital signals are easily computed by using proposed filtering technique. This paper proposes easy recursive formulas to design fractional order differentiators with low error as compared to previous methods, differentiators of order 1, 1.5 and 2 are designed in one dimensional case.

Tseng *et al.* [24]: computation of fractional derivative using the Fourier transform and a digital FIR differentiator is investigated. First, the Cauchy integral formula is generalized to define the fractional derivative of functions. Then the fractional differentiation property of the Fourier transform of functions is presented. Using this property, the fractional derivative of a function can be computed in the frequency domain. A least-squares method to design the fractional order digital differentiator is designed next. When a signal passes through the designed differentiator, the output will be its fractional derivative. One design example is included to illustrate the effectiveness of this approach. Finally, the designed fractional order differentiator is used to generate a random fractal process which is better than the process obtained by the conventional method.

Tseng *et al.* [26]: the design problems of variable fractional order integrator and differentiator (FOID) are investigated. First, the transfer function of FOID is obtained by taking fractional power of the transfer function of conventional first order integrator and differentiator. Then, to implement this irrational transfer function, the logarithm and Taylor series expansion are used to get a realizable approximated rational function. The proposed implementation structure is similar to the conventional Farrow structure of fractional sample delay filter. Next, the proposed approach is applied to design fractional rectangular integrator, fractional trapezoidal integrator, fractional Simpson integrator, fractional Al-Alaoui differentiator and fractional maximally flat differentiator. Finally, design examples are demonstrated to illustrate the performance of the proposed design method.

Tseng *et al.* [25]: the design and implementation structures of adjustable fractional order differentiator (AFOD) are discussed. First, the series expansion of ideal frequency response is used to transform the design of AFOD into the designs of log differentiators with various orders. Then, conventional FIR filter design method is applied to design log differentiators. The proposed method is flexible because the AFOD can be designed by considering the trade-off among the storage requirement of filter coefficients, implementation complexity and delay of filter. Finally, several numerical examples are shown to illustrate the effectiveness of the proposed design approach.

Samadi *et al.* [8]: a discrete-time fractional-order differentiator being modelled as a finite-impulse response (FIR) system is explored. The system yields fractional-order derivatives of Riemann-Liouville type for a uniformly sampled polynomial signal. The computation of the output signal is based on the additive combination of the weighted outputs of N cascaded first-order digital differentiators. For differentiators of fractional order with a terminal value equal to zero, the weights are time-varying. The weights are obtained in a closed form involving the Stirling numbers of the first kind. The system tends to a time-invariant integer-order differentiator when the order of the derivative tends to an integer value. It yields exact fractional- or integer-order derivatives of a sampled polynomial signal of a certain order.

Zhao et al. [33]: a new method for the design of fractional order differentiator was proposed. Firstly, a fractional order differentiator (FOD) of power digital signal is defined in the frequency domain. Secondly, a FIR filter is chosen to approximate to the ideal digital FOD under the weighting mean square error (MMSE) sense of the frequency response. Finally, design example and fractional derivative simulation are given and the advantages of the proposed method are illustrated.

Tseng *et al.* [28]: the radial basis function based design of fractional order digital differentiator is investigated. The radial basis function interpolation method is described.

Then, the non-integer delay sample estimation of discrete-time sequence is derived by using the radial basis function interpolation approach. Next, the Grunewald– Letnikov derivative and non-integer delay sample estimation are applied to obtain the transfer function of fractional order digital differentiator. The fractional order digital differentiator designed has better approximation of ideal frequency response. The applications in digital image sharpening and parameter estimation of fractional noise process are studied to demonstrate the usefulness of this new design methodology.

Chen *et al.* [48]: presented a new infinite impulse response (IIR) type digital fractional order differentiator (DFOD). This differentiator is proposed by using a new family of 2^{nd} order digital differentiators expressed in the second-order IIR filter form. The integer 2^{nd} order digital differentiators are obtained by the stable inversion of the weighted sum of Simpson integration rule and the trapezoidal integration rule. The distinguishing point of the proposed DFOD lies in an additional tuning knob to compromise the high-frequency approximation accuracy.

Chapter 3

Digital Filter and Differentiator Design

3.1 Introduction to Digital Filter:

A filter is a system or network that selectively changes the wave shape, amplitude frequency and phase frequency characteristics in an anticipated manner. The objective of filtering is to remove unwanted signal, to improve quality of signal and to separate to combined signals, [30].

A digital filter is basically a mathematical operation that can be implemented in hardware or software that can operate on a digital input signal to produce a digital output signal, for the purpose of obtaining a filtering objective. The basic structure of a digital filter is shown below

Figure 3.1: Basic structure of a Digital filter [1].

The digital filter is a discrete system, and it can do a series of mathematic processing to the input signal, and therefore obtain the desired information from the input signal. The transfer function for a linear, time-invariant, digital filter is usually expressed as

$$H(z) = \frac{\sum_{j=0}^{M} b_j z^{-j}}{1 + \sum_{i=1}^{N} a_i j^{-i}} \tag{3.1}$$

where a_i and b_j are coefficients of the filter in Z-transform.

There are many kinds of digital filters, and also many different ways to classify them. According to their function, the FIR filters can be classified into four categories, which are low pass filter, high pass filter, band pass filter, and band stop filter. According to the impulse response, there are usually two types of digital filters, which are finite impulse response (FIR) filters and infinite impulse response (IIR) filters. According to the formula above, if a_i is always zero, then it is a FIR filter, otherwise, if there is at least one none-zero a_i, then it is an IIR filter. Usually we need three basic arithmetic units to design a digital filter, which are the adder, the delay, and the multiplier.

The following are several steps of designing a digital filter [35]:

1. Make sure of the property of a digital filter according to the given requirements.
2. Use a discrete linear time-invariant system function to approach to the properties.
3. Make use of algorithms to design the system function.
4. Use a computer simulation or hardware to achieve it.

3.2 FIR Filter

The finite impulse response filter is one of the most basic elements in a digital signal processing system, and it can guarantee a strict linear phase frequency with any kind of amplitude frequency characteristics. Besides, the unit impulse response is finite; therefore FIR filters are stable system. The FIR filter has many applications in various fields like image processing, telecommunication and so on [30], [35]:

The system function of FIR filter is

$$H(z) = \sum_{n=0}^{L-1} h[n]\, z^{-n} \tag{3.2}$$

where L is the length of filter, and h[n] is impulse response.

FIR filters are simple and robust way of obtaining a digital filter. These filters are inherently stable when implemented non recursively and free of limit cycles. In these filters it is easy to attain linear phase. These filters are low sensitive to quantization effects as compared to many IIR filters.

3.2.1 Window Function

In this method, a truncated ideal lowpass filter with a certain bandwidth is generated, and then we use a chosen window to get certain stopband attenuation. The length of filter L can be adjusted to meet a specified roll-off rate in the transition band. Let’s start with windowed, truncated lowpass filters, and then other kind of filters, like highpass, bandpass, and bandstop filters can also be achieved by several techniques [30].

Any finite-length of the ideal lowpass impulse response may be considered as the product of the infinite-length lowpass impulse response and a window function W, which has a finite number of contiguous nonzero-valued samples

$$b = \frac{sin\omega_c[n-M]}{\pi[n-M]} W_L[n-M] \tag{3.3}$$

where L is the window length, M=(L-1)/2, 0≤ n ≤ L-1, and $W_L[n]$ is generally a function $F_E[n]$ which has even symmetry about *M* defined as [30]:

$$W_L[n] = \begin{cases} F_E[n] & 0 \ll n \ll L-1 \\ 0 & otherwise \end{cases} \tag{3.4}$$

The result is a finite-length or truncated lowpass filter.

We now discuss some of the basic information on standard windows.

(1) The simplest window is the rectangular window $R[n]$, which is defined as

$$R[n] = \begin{cases} 1 & 0 \ll n \ll L-1 \\ 0 & otherwise \end{cases} \tag{3.5}$$

(2) The Hanning window is described as

$$R[n] = \begin{cases} 0.5 - 0.5\cos\left(\frac{2\pi n}{L-1}\right) & 0 \ll n \ll L-1 \\ 0 & otherwise \end{cases} \tag{3.6}$$

(3) The Hamming window is described as

$$R[n] = \begin{cases} 0.54 - 0.46\cos\left(\frac{2\pi n}{L-1}\right) & 0 \ll n \ll L-1 \\ 0 & otherwise \end{cases} \tag{3.7}$$

(4) The Blackman window is described as

$$R[n] = \begin{cases} 0.42 - 0.5\cos\left(\frac{2\pi n}{L-1}\right) + 0.08\cos\left(\frac{4\pi n}{L-1}\right) & 0 \ll n \ll L-1 \\ 0 & otherwise \end{cases} \tag{3.8}$$

(5) The Kaiser window is described by the formula

$$R[n] = \begin{cases} I_0\left(\beta\sqrt{1-\left(\frac{n-M}{M}\right)^2}\right) & 0 \ll n \ll L-1 \\ 0 & otherwise \end{cases} \tag{3.9}$$

For n=0:1 :*L*-1, where *L* is the window length, *M*= (*L*-1)/2, and I_0 represents the modified Bessel function of the first kind.

Table 3.1 Window Functions [30].

Name	Approx L	Exact L	Min Stop band (att) dB
Rectangular	$4\pi/\omega_t$	$1.8\pi/\omega_t$	21
Hanning	$8\pi/\omega_t$	$6.2\pi/\omega_t$	44
Hamming	$8\pi/\omega_t$	$6.6\pi/\omega_t$	53
Blackmann	$12\pi/\omega_t$	$11\pi/\omega_t$	74

6.2 IIR Filter

The infinite impulse response (IIR) filter has recursive structure, and it has a feedback loop. The precision of amplitude frequency characteristic is very high and IIR filters are not linear phase. Compared with FIR digital filters, IIR digital filters can achieve much better performance under the same set of design specifications. However, IIR filter designs face more challenges due to the presence of the denominator [30], [35].

The techniques of invariant impulse response, matched- z transformation and bilinear transformation are widely used to achieve an IIR digital filter from a given analog filter. These design techniques are straightforward, and can naturally guarantee the stability of obtained IIR digital filters. However, these techniques can only be applied to transform standard analog filters, such as lowpass, highpass, bandpass and bandstop filters, into digital counterparts. Nowadays, IIR filter designs can be performed directly on the discrete time or frequency domain. If only the magnitude response is of concern, an IIR filter design problem can be simplified to some extent, since the stability can always be achieved by flipping the poles outside the unit circle into the inside without changing the magnitude response of the obtained IIR digital filter [46]. So far, the design for magnitude response approximation has been widely studied. One of most often used techniques is to approximate the squared ideal magnitude response by $H(z)H(z)^{-1}$. This is mainly because in the form of squared magnitude, the design problem can be simplified to a quasi-convex optimization problem.

If phase (or group delay) responses are also under consideration, IIR filter design problems become more complicated. As in FIR filter design problems, the WLS and minimax criteria are also widely used in practical IIR filter designs. The major difficulties we encounter in designing are as follows:

1. Since the poles of an IIR digital filter can be anywhere in the z plane, in general, IIR filter design problems are nonconvex optimization problems. Accordingly, there exist many local optima on error performance surfaces, and globally optimal solutions cannot be definitely achieved or even verified.

2. If phase (or group delay) responses are also of concern, stability constraints must be incorporated in design procedures. However, when the denominator order M is larger than 2, the stability domain cannot be expressed as a convex set with respect to denominator coefficients q, [30]:

3.3.1 IIR filter types

Digital IIR filter designs come from the classical analog designs and include the following filter types [30].

- Butterworth filters
- Chebyshev filters
- Chebyshev II filters, also known as inverse Chebyshev and Type II Chebyshev filters
- Elliptic filters, also known as Cauer filters

The IIR filter designs differ in the sharpness of the transition between the passband and the stopband and where they exhibit their various characteristics—in the passband or the stopband.

3.3.2 Butterworth Filters

Butterworth filters have the following characteristics:

- Smooth response at all frequencies
- Monotonic decrease from the specified cut-off frequencies
- Maximal flatness, with the ideal response of unity in the passband and zero in the stopband
- Half-power frequency, or 3 dB down frequency, that corresponds to the specified cut-off frequencies.

The transfer function for Butterworth filter is given by

$$B(\omega) = \frac{1}{[1 + \left({}^{\omega}/_{\omega_0}\right)^{2n}]^{\frac{1}{2}}} \tag{3.10}$$

where n is the order of filter.

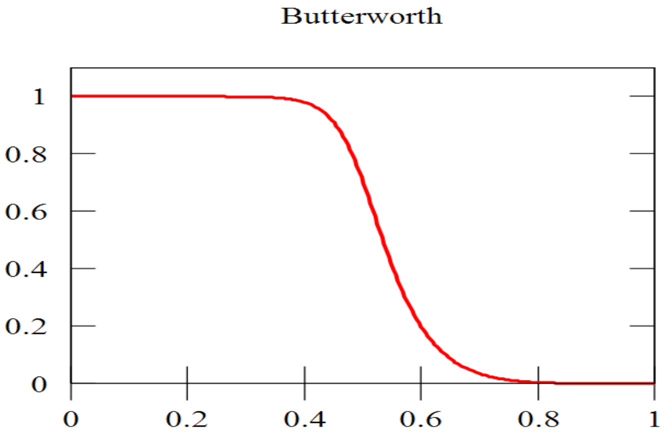

Figure 3.2: Response of Butterworth Filter.

As shown in Figure 3.2, after specifying the cut-off frequency of a Butterworth filter, MATLAB sets the steepness of the transition proportional to the filter order. Higher order Butterworth filters approach the ideal lowpass filter response. Butterworth filters do not always provide a good approximation of the ideal filter response because of the slow rolloff between the passband and the stopband.

3.3.3 Chebyshev Filters

Chebyshev filters have the following characteristics:

- Minimization of peak error in the passband
- Equiripple magnitude response in the passband
- Monotonically decreasing magnitude response in the stopband
- Sharper rolloff than Butterworth filters

Compared to a Butterworth filter, a Chebyshev filter can achieve a sharper transition between the passband and the stopband with a lower order filter. The sharp transition between the passband and the stopband of a Chebyshev filter produces smaller absolute errors and faster execution speeds than a Butterworth filter.

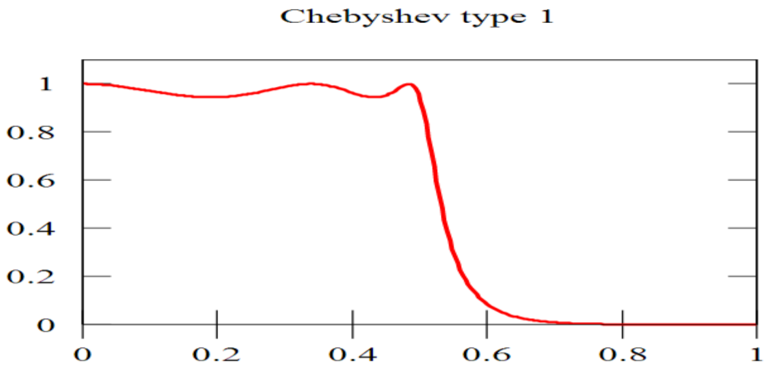

Figure 3.3: Response of Chebyshev Filter.

Figure 3.3 shows the frequency response of a lowpass Chebyshev filter. In Figure 3.3, the maximum tolerable error constrains the equiripple response in the passband. Also, the sharp rolloff appears in the stopband. The frequency response of the filter is given by

$$|H(\Omega)|^2 = \left(1 + \varepsilon^2 T_n^2 \left({}^{\omega}/_{\omega_p}\right)\right)^{-1} \tag{3.11}$$

Where ε is a parameter of the filter related to ripple present in the passband and $T_n(x)$ is the N^{th}- order Chebyshev polynomial defined as

$$T_n = \cos(N cos^{-1} x) \qquad when\ |x| \leq 1 \tag{3.12}$$

$$T_n = \cos(N cosh^{-1} x) \qquad when\ |x| \geq 1 \tag{3.13}$$

3.3.4 Chebyshev II filters or Inverse Chebyshev filters

Chebyshev II filters have the following characteristics:

- Minimization of peak error in the stopband
- Equiripple magnitude response in the stopband
- Monotonically decreasing magnitude response in the passband
- Sharper rolloff than Butterworth filters

Chebyshev II filters are similar to Chebyshev filters. However, Chebyshev II filters differ from Chebyshev filters in the following ways:

- Chebyshev II filters minimize peak error in the stopband instead of the passband. Minimizing peak error in the stopband instead of the passband is an advantage of Chebyshev II filters over Chebyshev filters. Chebyshev II filters have an equiripple magnitude response in the stopband instead of the passband.
- Chebyshev II filters have a monotonically decreasing magnitude response in the passband instead of the stopband

In Figure 3.4, the maximum tolerable error constrains the equiripple response in the stopband. Also, the smooth monotonic rolloff appears in the stopband. Chebyshev II filters have the same advantage over Butterworth filters that Chebyshev filters have a sharper transition between the passband and the stopband with a lower order filter, resulting in a smaller absolute error and faster execution speed.

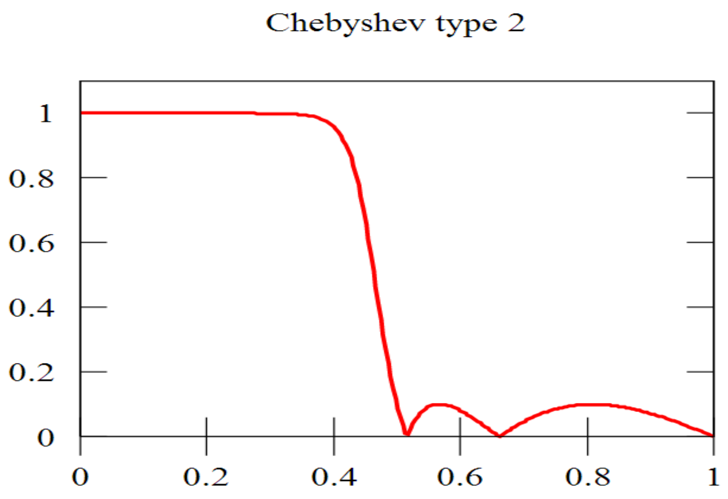

Figure 3.4: Response of Inverse Chebyshev filter.

3.3.5 Elliptic Filters

Elliptic filters have the following characteristics [35].

- Minimization of peak error in the passband and the stopband
- Equiripples in the passband and the stopband

Compared with the same order Butterworth or Chebyshev filters, the elliptic filters provide the sharpest transition between the passband and the stopband, which accounts for their widespread use.

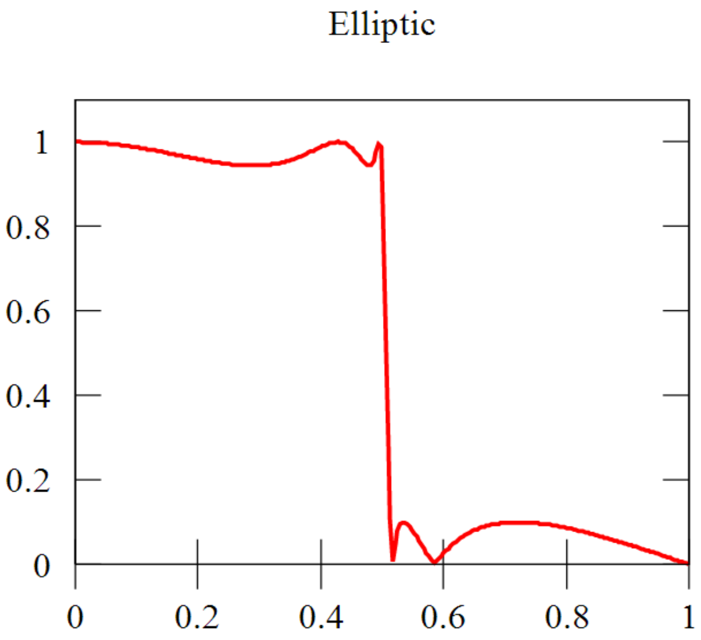

Figure 3.5: Response of Elliptic Filter.

In Figure 3.5, the same maximum tolerable error constrains the ripple in both the passband and the stopband. Also, even low-order elliptic filters have a sharp transition edge. The transfer function is given by

$$|H(\Omega)|^2 = \left(1 + \varepsilon^2 U_n^2\left({}^{\Omega}/_{\Omega_c}\right)\right)^{-1} \tag{3.14}$$

where $U_n(x)$ is the Jacobian elliptic function of order N and ε is a constant related to passband ripple. They provide a realization with the lowest order for a particular set of conditions.

3.4 Comparison of FIR and IIR filter

(1) Under the same conditions as in the technical indicators, output of the IIR filter has feedback to input, so it can meet the requirements better than FIR. The storage units are less than those of IIR, the number of calculations is also less, and it’s more economical [30], [35].

(2) The phase of FIR filter is strictly linear, while the IIR filter is not. The better the selectivity of IIR filter is, the more serious the nonlinearity of the phase will be.

(3) The FIR filter is non-recursive structure, finite precision arithmetic error is very small. While IIR filter is recursive structure, and parasitic oscillation may occur in the operation of IIR filter.

(4) Fast Fourier Transformation can be used in FIR filter, while IIR cannot.

(5) The IIR filter can make use of the formulas, data and tables of the analog filter, and only a small amount of calculation. While FIR filter design may always make use of the computer to calculate, and the order of FIR filter could be large to meet the design specifications.

3.5 Selesnick’s Method

This method describes the design of Type III and Type IV linear-phase finite-impulse response (FIR) low-pass digital differentiators according to the maximally flat criterion. A two-term recursive formula that enables the simple stable computation of the impulse response coefficients was introduced by Salesnick. The same recursive formula is valid for both Type III and Type IV solution. The derivation of the solution will depend on a transformation that maps polynomials on the real interval [0,1] to polynomials on the upper half of the unit circle. Let k denote the number of zeroes a transfer function has at $z = -1$. A Type III transfer function always has an odd number of zeros at $z = -1$. In terms of K we have $K = 2M + 1$ for type III transfer function and $K = 2M$ for type IV transfer function [9].

Let K denote the number of zeros a transfer function has at $Z = -1$. A Type IV transfer function always has an even number of zeros at $Z = -1$;

The transfer function of maximally flat linear phase low pass digital differentiator is given by

$$H(Z) = \left(\frac{1 - Z^{-1}}{2}\right)\left(\frac{1 - Z^{-1}}{2}\right)^{k} Z^{-L} \sum_{n=0}^{L} c(n) \left(\frac{-Z + 2 - Z^{-1}}{2}\right)^{n} \tag{3.15}$$

$$c(n) = \frac{(8n^2 + 4Kn - 10n - K + 3)c(n-1) - (2n + K - 3)2c(n-2)}{2n(2n+1)} \tag{3.16}$$

Magnitude response of maximally flat low pass differentiators designed using the Salesnick’s method is given in Figure. It shows the frequency response for a family of Type III low-pass differentiators of length $N = 29$, where K is varied from 1 to 24 in increments of 4, and where $L = (N - K)/2 - 1$.Where K is number of zeros present at $Z = -1$. As it can be seen as K is increased from 1 to 24 with keeping length of filter 29, the differentiator turns from full band to narrow band low pass differentiator. It would be interesting to find any other way to design low pass differentiator of same order with better transition characteristics. So by using recursive equations given by Salesnick Low pass differentiator with variable pass band can be designed. Figure illustrates error plot of the filter, error is in fewer than 2% for the pass band but the filter does not show sharp transitions [9].

When K is even we obtain a Type IV transfer function, when K is odd we obtain a Type III transfer function. In either case, the length of the impulse response is $N = K + 2L + 2$. Notice that $c(n)$ does not depend on L; rather L determines how many values of $c(n)$ are needed.

This method describes the design of low-pass linear-phase FIR digital differentiators according to the maximally flat criterion. The solutions cannot be obtained from a low-pass filter as in the case of a full-band differentiator. The algorithms for automatic sum simplification described in were used to obtain a simple two-term recurrence relation for computing the coefficients of the impulse response. There are several possible extensions to the problem described in this paper. For example, the extension of the recursive formulas to the case where the maximally flat approximation to the ideal differentiator is performed not at $\omega = 0$ but at another frequency ω_0. For the full-band differentiator, solutions are given. This type of solution is relevant when the signal is centered on a known frequency (as in radar using Doppler tracking). Another remaining question is the existence of low-complexity structures for maximally flat differentiators, of the kind described by Samadi and Nishihara, for maximally flat low-pass filters. Those structures are multiplier less and have a regular structure.

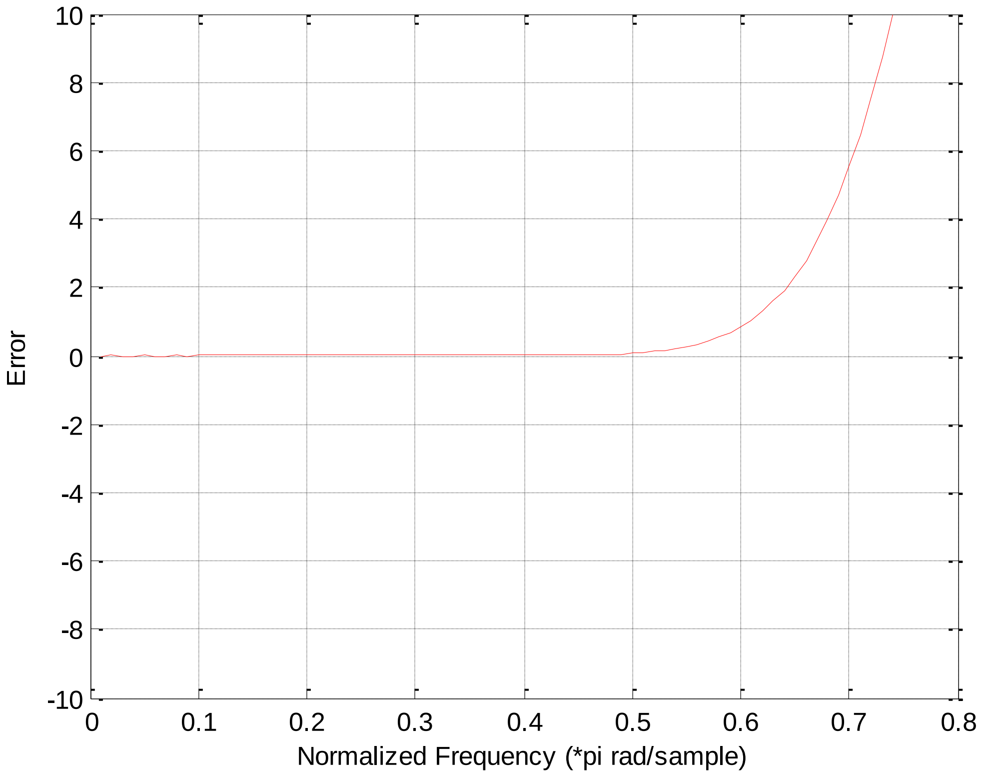

Figure 3.6: Error response of Salesnick method with different order

The order of minimax relative error digital differentiator becomes very large for extremely low relative error in the low frequency range this report proposes an alternative technique to achieve the same performance with much low order. The proposed DD is derived from the maximally flat non recursive low pass digital filters. A value of relative error in the range -160 db to -200 db is obtainable for the narrow band of frequencies using filters of order N ranging from N=3 to N=19.

Maximally accurate Digital Differentiator (DD) has been derived for low frequency range. Coefficients used in DD can be calculated from the explicit formulas. The proposed method describes that the lower the frequency of differentiation the better is the performance of proposed differentiator. This makes it suitable for many practical applications.

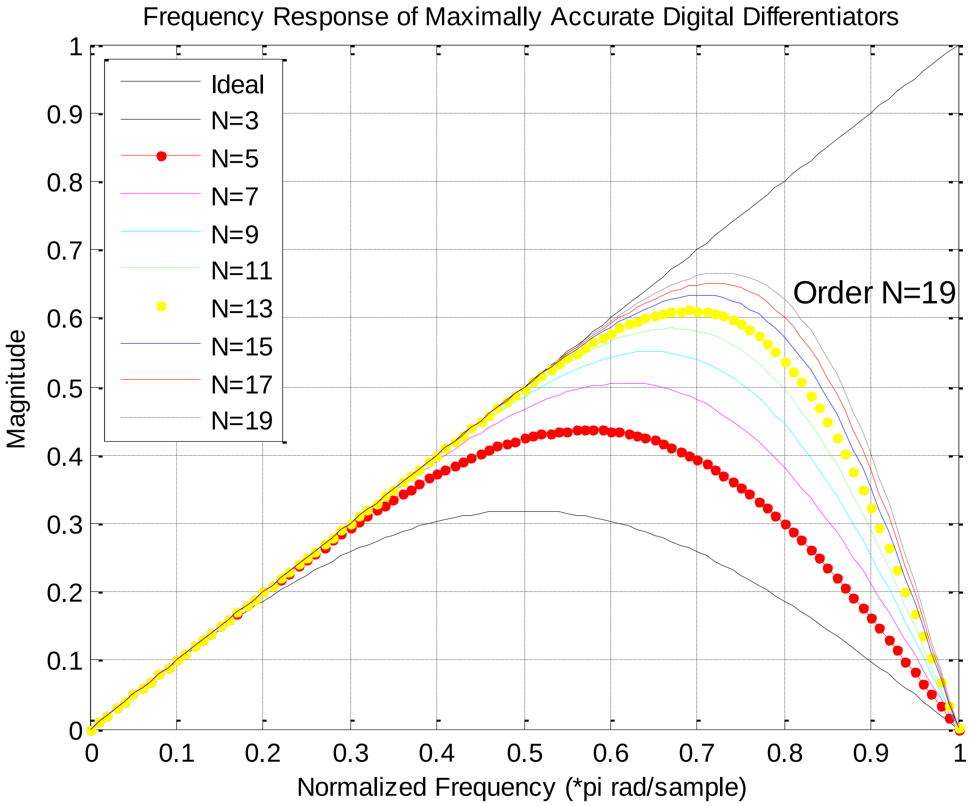

Figure 3.7: Magnitude response of Salesnick method.

The waveform for the relative error is shown above. We have calculated the relative error by comparing the differentiators with different order with the Ideal DD [9], [10].

The length of impulse response of FIR differentiator is critical in many applications. Therefore, efficient use of number of coefficients is necessary. It can be achieved by improving frequency magnitude response available to us from current techniques. First, this report describes the design of linear phase higher order case of FIR low pass digital differentiators. The formula to compute impulse response coefficients is derived using Fourier integral.

Chapter 4
Fractional Order Differentiator

Fractional calculus is a 300 year old topic and a lot of work has been done in the area of fractional calculus. For past three centuries, this subject was studied by several mathematicians and only in last few years, this has been utilized in several applied fields of engineering, science and economics. However recent attempt is on to have definition of fractional derivative as local operator specifically to fractal science theory [36]. In the recent years, fractional calculus received great attention in many engineering fields. Fractional order digital differentiator is an important topic in the area of fractional calculus, because it computes fractional derivative of the digital signal without known function.

The main reason of interest in this field is that there are some systems present in nature whose response can be accurately studied with the help of fractional derivative. Fractional order basically signifies us that order of system is a non-integer, for the n^{th} derivative of a function $f(x)$ can be written as

$$D^n f(x) = \frac{d^n f(x)}{dx^n} \tag{4.1}$$

If the value of n is a non-integer, $n = \frac{1}{2}$ and $f(x) = x^m$, using Gamma function, we can write derivative of $f(x)$ as

$$\frac{d^{1/2} f(x)}{dx^{1/2}} = \frac{\Gamma(m+1)}{\Gamma(m - \frac{1}{2} + 1)} x^{m-1/2} \tag{4.2}$$

so, by using gamma function, we can calculate the fractional order derivative of a function with integer order. In the field of fractional calculus several methods have been implemented for the calculation of fractional order derivative of the signal [36].

In this chapter, we have visualized the method of Savitzky-Golay to enhance the performance of the signal which is contaminated with noise. Various other methods are also proposed for the efficient smoothening of the signal and to calculate its fractional order derivative. But these differentiators are not appropriate to compute the fractional order derivative of the contaminated signal, genetic algorithm method is also used but it is not efficient because of its large computation time. Savitzky-Golay filter is a regression technique, which can estimate the integer order derivative of contaminated signal but not for fractional order.

In order to overcome this problem, Fractional Order Savitzky-Golay Differentiator (FOSGDD) with the help of least square method and Riemann-Liouville technique is proposed, which can estimate the fractional order derivative of the contaminated signal. Further in this chapter, we have applied different signals to verify the proposed work. The computation time is also calculated to specify the efficiency of the method [49].

We have applied chirp signal to this differentiator because chirp signal and fractional order differentiator can be efficiently used in the area of radar and sonar. Chirp signals are mostly observed in the sonar and radar. Chirp signals are interchangeably used with sweep signal. It has other applications also such as in spread spectrum communication, so there are enormous uses of applying chirp signal to fractional differentiator to the chirp signal [1].

4.1 Design Method

We want to smooth the given uniformly sampled signal and to estimate its dn^{th} order derivative using a filtering window of size K, for the Savitzky-Golay filter dn, is a non-negative integer in which $n < K$ so that we can compute least square polynomial [49].

$$f_n(i) = \sum_{k=0}^{n} a_k i^k \tag{4.3}$$

where $f_n(i)$ is a polynomial function with degree n, a_k is the k^{th} coefficient to fit the given signal. If we want to accurately calculate the coefficient a_k, least square method will be used, for better implementation (4.3) can be written in matrix form as

$$Z = XA + \varepsilon \tag{4.4}$$

where $Z = [z_1, z_2 \dots\dots z_K]^T$ denotes the measured signal points in the filtering window, $A = [a_1, a_2 \dots\dots a_n]^T$ implies coefficient vector of polynomial function, ε is the estimation error and X is a $K \times (n+1)$ Vander-monde matrix, written as

$$X = \begin{bmatrix} 1 & 1^1 & \dots & 1^n \\ 1 & 2^1 & \cdots & 2^n \\ \vdots & \vdots & \vdots & \vdots \\ 1 & I^1 & \dots & I^n \end{bmatrix} \tag{4.5}$$

We can obtain the coefficients of best fit polynomial by minimizing the sum of squared errors between the actual data and fitting points, hence we can obtain

$$A = (X^T X)^{-1} X^T Z \tag{4.6}$$

Using (4.6) we will get estimation of the given signal by

$$\bar{Z} = XA = (X^T X)^{-1} X^T Z = MZ \tag{4.7}$$

Where M denotes the moving window coefficient matrix, which we will use for the smoothening of the given signal. If we want to smooth the k^{th} value of the signal, we will use k^{th} coefficient of M to implement [49].

Integer order derivative of (4.3) can be given as

$$\frac{d^n f_n(i)}{di^n} = n!\, b_n \tag{4.8}$$

Hence, the dn^{th} order derivative of the given signal can be estimated by

$$\begin{aligned} \bar{Z}^{(dn)} &= X_i^{(dn)} A = M_i^{(dn)} Z \\ &= [\underbrace{0, \dots, 0}_{dn}, dn!, \dots C_n^{n-dn} i^{n-dn}](X^T X)^{-1} X^T Z \end{aligned} \tag{4.9}$$

Here, $\bar{Z}^{(dn)}$ denotes the dn^{th} order derivative of the i^{th} point $X_i^{(dn)}$, denotes the coefficient of the dn^{th} derivative and $M_i^{(dn)}$ denotes the moving window's coefficients. When dn is equal to zero, (4.9) is equivalent to (4.7) and can be used for the smoothening of the given signal.

Further in this dissertation, Riemann-Liouville definition is used for the purpose of generalization of Savitzky-Golay differentiator from integer order to fractional order.

$${}_0D_x^{\alpha} f(x) = \frac{1}{\Gamma(y-\alpha)} \frac{dy}{dxy} \int_0^x (x-t)^{y-\alpha-1} f(t)dt \tag{4.10}$$

where $0 \leq y-1 < \alpha < y$, and $\Gamma(y-\alpha)$ is the Gamma Function. Assuming the signal $f(x) = x^k, k \geq 0, x \geq 0$ then the above equation becomes.

$$ {}_0D_x^{\alpha} f(x) = \frac{\Gamma(k+1)}{\Gamma(k+1-\alpha)} x^{k-\alpha} \tag{4.11} $$

If α is integer, then the fractional order differentiation is equivalent to integer order. In the proposed method, operation is linear and the rule of linearity follows. Using these properties final result can be written as

$$ Z_i^{(\alpha)} = X_i^{(\alpha)} A = M_i^{(\alpha)} Z = a(X^T X)^{-1} X^T Z \tag{4.12} $$

Where a is given by

$$ a = \frac{1}{\Gamma(1-\alpha)} i^{-\alpha}, \frac{1}{\Gamma(2-\alpha)} i^{1-\alpha}, \frac{\Gamma(3)}{\Gamma(3-\alpha)} i^{2-\alpha} \ldots\ldots \frac{\Gamma(n+1)}{\Gamma(n+1-\alpha)} i^{n-\alpha} \tag{4.13} $$

as we have observed from the previous equations that matrix X is the key to affect the computation time. We will face problems in computation, if K is too large, so to efficiently overcome this problem, we will increase the sampling interval of the observed signal [49].

4.2 Simulation Results:

In this section, the accuracy of the proposed algorithm will be validated. First we will compute the coefficients for the moving window's weights $M_i^{(\alpha)}$ of the i^{th} point x_i for given $\alpha = 0.5, n = 3, K = 9$ and sampling interval $\theta = 1$. The results are given in Table.4.1 for the given signal $z = x_i^3$. For the verification of given results in the function $z = x_i^3$ using $M_i^{(\alpha)}$, if $x_i = 5$ we can write $(\Gamma(4)/\Gamma(4-0.5))\, x_i^{3-0.5} = 100.9253$, which is same as result presented in last column of Table.4.1.

Table 4.1: Moving window's weights of the FOSGDD with $\boldsymbol{\alpha = 0.5, n = 3}$ for different $\boldsymbol{x_i's}$.

$M_i^{(0.5)}$ (x_i)	1	2	3	4	5	6	7	8	9	x_i^3 derivative
1	0.192	0.151	0.114	0.080	0.050	0.024	0.001	-0.01	-0.03	1.80
2	-0.062	0.031	0.095	0.131	0.139	0.118	0.069	-0.00	-0.11	10.21
3	-0.159	-0.02	0.075	0.136	0.160	0.147	0.096	0.008	-0.11	28.14
4	-0.187	-0.05	0.049	0.115	0.147	0.144	0.106	0.033	-0.07	57.77
5	-0.170	-0.06	0.018	0.076	0.110	0.118	0.102	0.062	-0.00	100.92
6	-0.121	-0.06	-0.01	0.022	0.053	0.076	0.090	0.096	0.093	159.20
7	-0.046	-0.05	-0.05	-0.04	-0.01	0.020	0.071	0.134	0.209	234.05
8	0.051	-0.04	-0.10	-0.12	-0.10	-0.04	0.045	0.175	0.343	326.81
9	0.167	-0.02	-0.14	-0.20	-0.20	-0.12	0.013	0.220	0.493	438.71

In this section, we will compute moving window coefficient for different values of α, $x_i = 4, n = 2, K = 9$ and $\theta = 1$, the results are shown in Table.4.2. From the results, it is verified that this method is efficient for the calculation of fractional order derivative of signal. The results presented in Table.4.2 are same as result calculated by (4.11).

Table 4.2: Moving window's weights of the FOSGDD with different $\boldsymbol{\alpha's}$, given$\boldsymbol{x_i} = \mathbf{4}$ and $\boldsymbol{n} = \mathbf{3}$.

$M_i^{(\alpha)}$ (α_i)	1	2	3	4	5	6	7	8	9	x_i^3 (derivative)
0	-0.090	0.060	0.168	0.233	0.255	0.233	0.168	0.060	-0.09	64.000
0.1	-0.133	0.018	0.128	0.198	0.226	0.212	0.158	0.062	-0.07	63.081
0.2	-0.160	-0.01	0.094	0.164	0.196	0.190	0.146	0.063	-0.05	61.995
0.3	-0.174	-0.03	0.064	0.132	0.166	0.166	0.132	0.063	-0.03	60.744
0.4	-0.177	-0.05	0.039	0.103	0.137	0.142	0.117	0.063	-0.02	59.335
0.5	-0.170	-0.06	0.018	0.076	0.110	0.118	0.102	0.062	-0.00	57.773
0.6	-0.157	-0.06	0.002	0.052	0.083	0.095	0.087	0.060	0.014	56.066
0.7	-0.138	-0.06	-0.01	0.031	0.059	0.073	0.073	0.058	0.030	54.224
0.8	-0.116	-0.06	-0.02	0.012	0.037	0.052	0.058	0.056	0.044	52.258
0.9	-0.091	-0.05	-0.02	-0.00	0.017	0.033	0.045	0.053	0.056	50.179
1	-0.066	-0.05	-0.03	-0.01	-0.00	0.016	0.033	0.050	0.066	48.000

Next the proposed differentiator is used for the purpose of smoothening of given signal $z(t) = \sin(4t)$ contaminated with uniformly distributed random noise. We will take $K = 1000, n = 14, \theta = 5$ and α is changing from 0.1 to 0.9 at an interval of 0.2. The result of smoothen signal is shown in Figure.4.1 Solid curve denotes the contaminated signal and others show the result of different α values. From the result, it is verified that proposed differentiator has good smoothening performance for the contaminated signal.

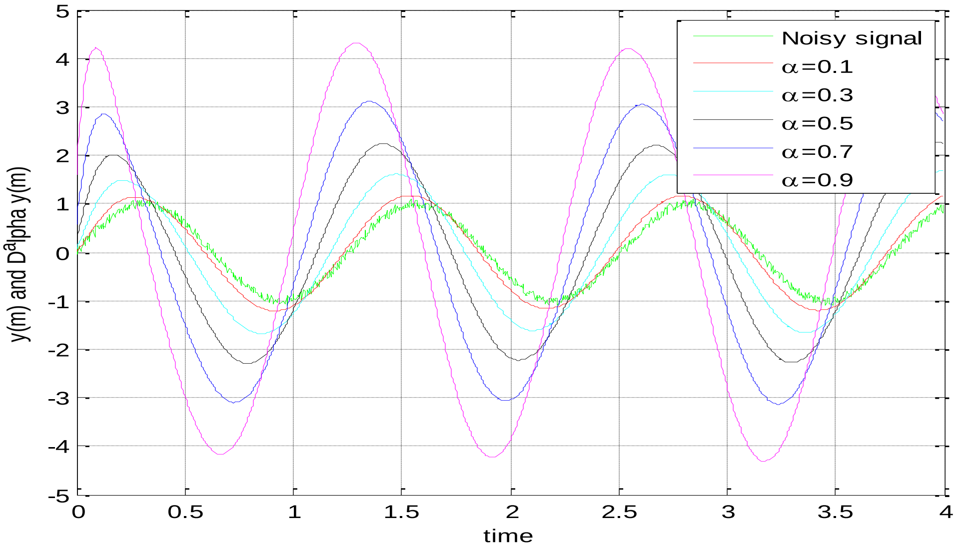

Figure 4.1: Smoothening of curve for different fractional order derivative.

In this experiment, the proposed FOSGDD is compared with several popular DFODs, such as Oustaloup's method, Tustin's method, Euler's method, Al-Alaoui method, Simpson's method, and a New IIR method. Here, we set $\alpha = 0.5$, $I = 201$, $n = 9$, and $\theta = 2$. Figure. 4.2 shows the comparison of the different methods. The given signal is obtained by sampling from $y(m) = e^{-m}\sin(3m + 1)$ at an interval of 0.02.

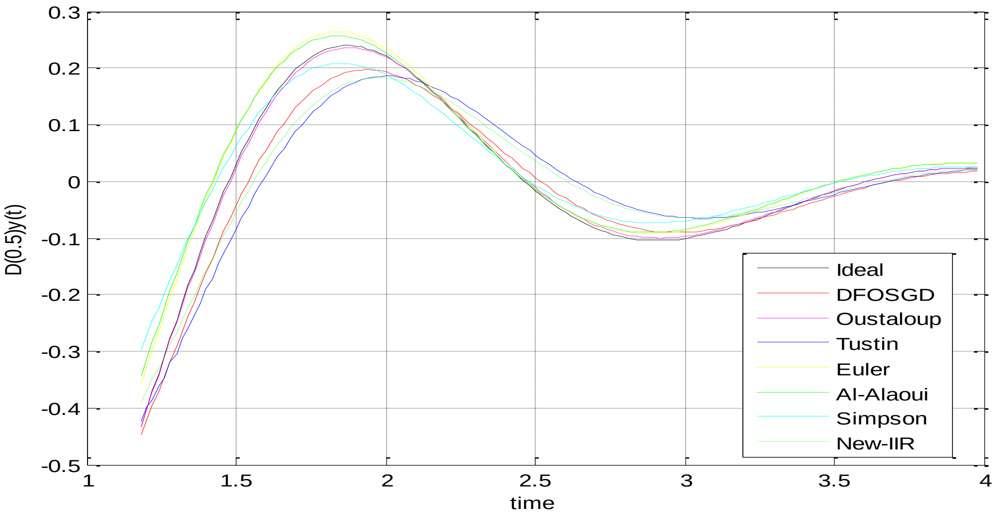

Figure 4.2: Comparison of proposed method with several popular method with given sinusoidal signal,

In the figure, the solid curve is the ideal curve, the dashed curve is estimated by the proposed FOSGDD, and the other curves are estimated by the different methods. From the comparison, we can see that the performances of the FOSGDD are better than the other methods. We also give the root-mean-square (RMS) error between the estimation and the ideal value for α = 0.1, 0.3, 0.5, 0.7, 0.9, and the results are listed in the first row of Table 4.3.

From the error comparison, it can be seen that our proposed FOSGDD is just a little bit worse than the Oustaloup method, and the difference between them is not more than 0.01. It is the main reason that the initial portion results in the estimation error. Furthermore, comparison of the different methods for the noisy signal of SNR = 24 dB is shown in Figure. 4.3 to evaluate the robustness of the different methods.

From the figure, it can be seen that only the DFOSGD can accurately estimate the half-order derivative of the noisy signal. Meanwhile, the RMS error comparison data for α = 0.1, 0.3, 0.5, 0.7, 0.9 are listed in the second row of Table 4.3. From the table, it can be seen that the proposed DFOSGD is much more robust than the other methods.

From these results, we can conclude that the proposed DFOSGD not only accurately estimate the different fractional order derivatives of the noise-free signal, but can also obtain much better results than other methods for the contaminated signal.

In addition, the computation complexity is greatly improved because of using convolution, instead of complex mathematical deduction. Table 4.4 shows the computation time of the different methods in this experiment. Comparing with the other methods, it can be seen that the proposed FOSGDD can be easily and quickly used.

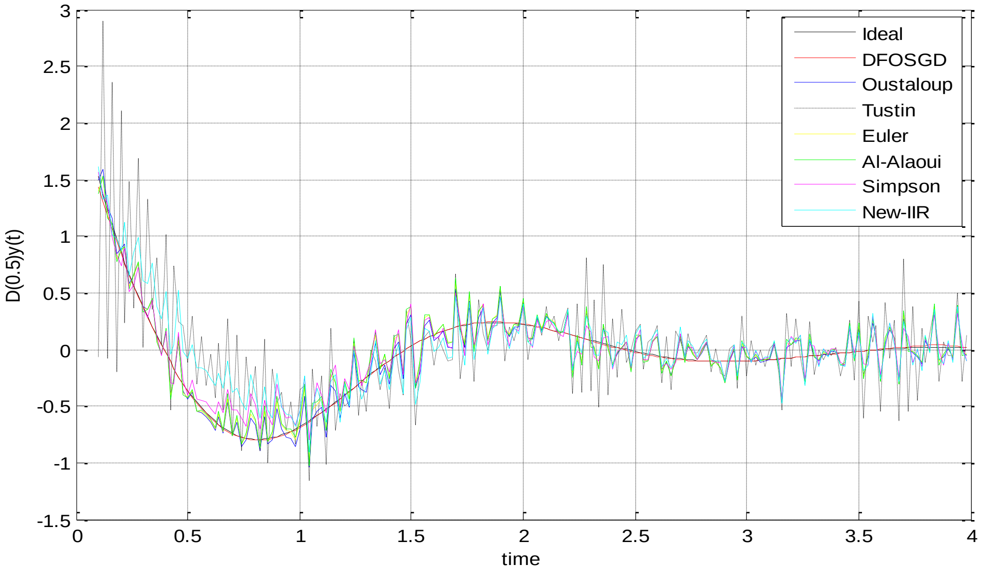

Figure 4.3: Comparison of proposed method with several popular methods with noise added signal.

In this section we will apply chirp signal to the proposed differentiator, applied signal can be written as

$$Z = \cos(f_i + 2\pi(f_1 t \frac{(f_2 - f_1)}{2t_1} t^2) \tag{4.14}$$

where $f_i = 0$, $f_1 = 0, f_2 = 50, t_1 = 2,$ t=sampling interval

Here, we will compare, performance of the purposed differentiator with several popular methods, such as Tustin's method, Oustaloup's method, Euler method, Al-Alaoui Method, Simpson's method and a New IIR method, where the order of differentiation α is taken 0.5, $K = 200, n = 9$. Figure. 4.4 shows the comparison of different methods.

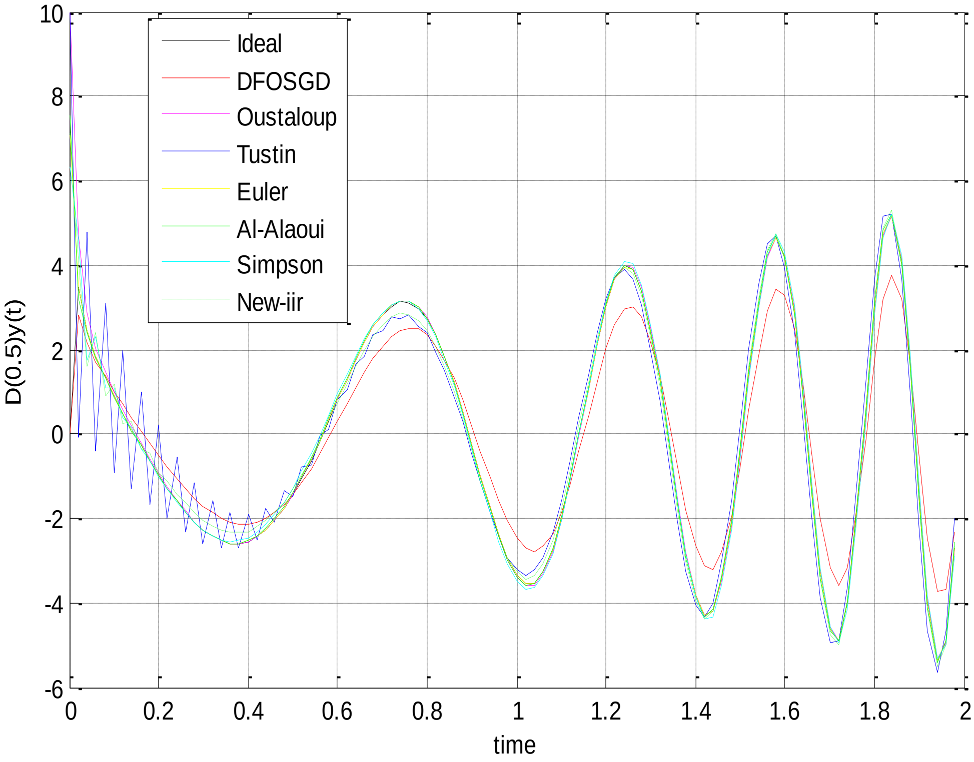

Figure 4.4: Comparison of several popular methods with noise free chirp signal.

In the Figure.4.4 the curve is in plane line is ideal curve and the curve is in asterisk is of proposed method and the rest are of different methods, from the figure it is clear that the performance of FOSGDD is better than other methods and the RMS error of different methods is also calculated with varied fractional orders from 0.3 to 0.9 at an interval of 0.2. From the results it is clear that the FOSGDD method is efficiently smoothening the signal and calculating its RMS error From Table.4.3 we can analyze that response of FOSGDD is far better than other methods.

Table 4.3: RMS error comparison, for different methods, noise free and contaminated signal.

	Noise free chirp signal				Noisy chirp signal with random noise			
Method	α=0.3	α=0.5	α=0.7	α=0.9	α=0.3	α=0.5	α=0.7	α=0.9
FOSGDD	**0.0211**	**0.1616**	**0.4411**	**0.0053**	**0.0030**	**0.1674**	**0.9239**	**0.0053**
Tustin	0.0435	0.2141	1.1689	1.4289	0.3069	0.5850	3.4289	5.7724
Euler	0.0408	0.3189	0.8821	0.0105	0.1442	0.7935	2.2098	0.0262
AlAlaoui	0.0417	0.3524	0.9491	0.0130	0.1569	0.9265	2.4785	0.0358
Simpson	0.0412	0.2850	0.9041	0.0089	0.1275	0.6588	2.2951	0.0196
New IIR	0.0420	0.3234	0.9873	0.0113	0.1438	0.8112	2.6288	0.0292

In the error comparison, we have analyzed that, Proposed FOSGDD is better than all other methods in the both, noise free and contaminated chirp signal experiment. In the noise free case, we have observed that other methods are equally effective with a comparatively high RMSE error but when it comes to comparison between noisy signals, responses of all the other methods are below average with a high error rate. This method is more robust than any other method and it can be easily and quickly be used.

Furthermore, comparison of different methods for efficient smoothening of noisy signal is performed. Result is shown in Figure.4.5 In this experiment we have applied signal added with random noise to the differentiator. From the results, it is visible that only proposed method is efficient for computation of derivative of chirp signal.

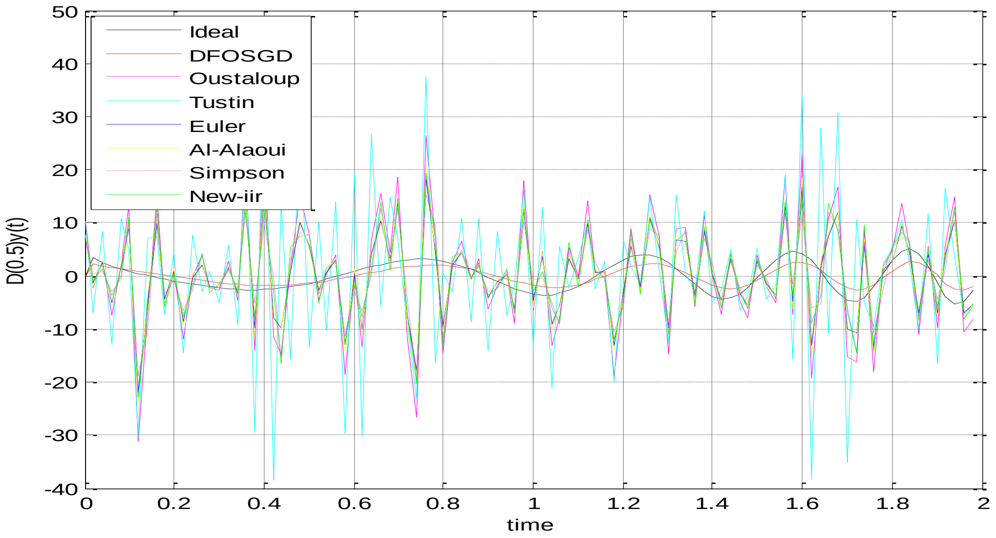

Figure 4.5: Comparison of several popular methods for the noisy chirp signal.

The signal is in normal line is ideal performance and in asterisk is proposed method and rest are of different methods. From the results it is clear that noise is totally removed and from the table it is verified that this method has low RMS error as compared to other methods. For the calculation of root mean square error, we can write as

$$RMSE = \sqrt{\frac{\sum_{i=1}^{n}(\bar{Z} - Z)^2}{n}} \tag{4.15}$$

where, $\bar{Z} =$ estimated signal response, $Z =$ ideal signal response and $n =$ number of samples

By calculating the RMSE, it is verified that the response of differentiator is better than the other methods, so that it is the only differentiator of its type which will be used for the purpose of smoothening and computation of fractional derivative.

Table 4.4: Computation time of different methods.

Method	FOSGDD	Tustin	Euler	Al-Alaoui	Simpson	New IIR
Computation Time(sec)	**0.000067**	0.050387	0.006281	0.006170	0.005385	0.005213

In Table 4.4, computation time of different methods is computed and it shows that FOSGDD consumes very less time in computation. This method consumes very less time for the computation of derivative of the signal, so it will be efficiently used in many practical applications.

In this experiment, we analyze the proposed FOSGDD in the frequency domain to reveal its nature. Figure 4.6 shows the frequency response of the FOSGDD with the different parameters *I* and *n*. In the figures, the ideal frequency response curve is given to compare with the result of the FOSGDD. It can be seen that the proposed FOSGDD can be considered as a fractional order low-pass differentiator, which keeps the low frequency and decays the high frequency. In general, the noise exists in the high-frequency portion of the signal.

Thus, the FOSGDD can be used to accurately estimate the fractional order derivative of the contaminated signal. Figure. 4.6 shows the frequency response of the FOSGDD with the different *I*'s when n = 14 and α = 0.5. It can be seen that the cutoff frequency of the

FOSGDD decreases along with the parameter *I* increases, and the low-pass performance of the FOSGDD enhances when *I* increases.

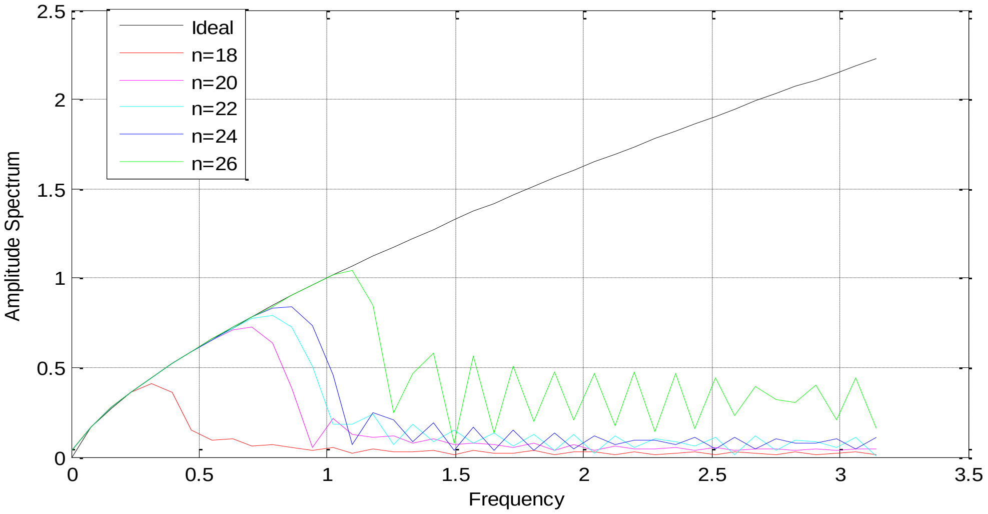

Figure 4.6: Frequency response of proposed method with varying order *n*.

The main reason is that the robustness of the FOSGDD enhances when the amount of input data increases. Thus, we should choose a larger parameter *I* to enhance the robustness of the differentiator in application.

Figure 4.7 shows the frequency response of the FOSGDD with the different *n*'s when $I = 51$ and $\alpha = 0.5$. It can be seen that the cutoff frequency of the FOSGDD increases along with the parameter *n* increases, and the low-pass performance of the FOSGDD gets worse when *n* increases. The main reason is that the polynomial fitting precision increases when *n* increases, and the robustness of the differentiator gets worse along with *n* increases.

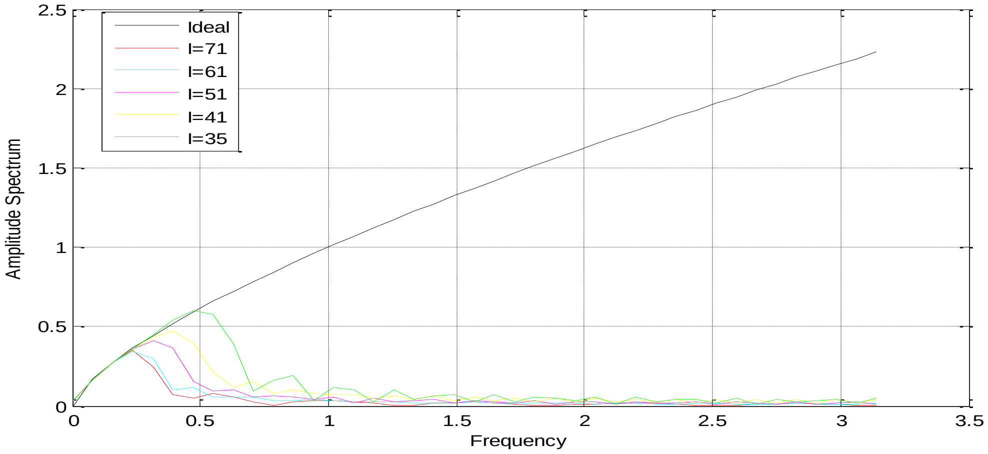

Figure 4.7: Frequency response of proposed method with varying order *I*.

4.3 Computation of fractional Derivative using power function and least square method

From the last few times, the concept of fractional calculus has been used in many applications of signal processing. The exclusive feature of fractional calculus is its capability to generalization of integral and differential operators to non-integer order. The generalized continuous integral-differential operator in is as follows [36]:

$$ {}_{a}D_{t}^{\alpha} = \begin{cases} \dfrac{d^{\alpha}}{dt^{\alpha}}, & \alpha > 0 \\ 1, & \alpha = 0 \\ \displaystyle\int_{a}^{t} (d\tau)^{\alpha}, & \alpha < 0 \end{cases} \tag{4.16} $$

where ${}_{a}D_{t}^{\alpha}$ denotes integral-differential operator to compute the α^{th} order fractional differentiation and integration of the input signal with respect to t and α is the primary condition of the operation. Some of the standard definitions for this integral-differential operation are Riemann-Liouville, Grunewald- Letnikov and the Caputo definitions etc. In this dissertation, the Grunwald-Letnikov definition for the fractional order calculation is used which is as follows [36]:

$$ {}_{a}D_{t}^{\alpha} s(t) = \lim_{\Delta \to 0} \sum_{k=0}^{\alpha} \frac{(-1)^{k} C_{k}^{\alpha}}{\Delta^{\alpha}} s(t - k\Delta) \tag{4.17} $$

where C_{k}^{α} is the binomial coefficient. The value of C_{k}^{α} is given by using the relation in between Eulers, Gamma function and factorial, which is defined as

$$ C_{k}^{\alpha} = \binom{\alpha}{k} = \frac{\Gamma(\alpha + 1)}{\Gamma(k + 1)\Gamma(\alpha - k + 1)} $$

$$ = \begin{cases} 1 & k = 1 \\ \dfrac{\alpha(\alpha - 1)(\alpha - 2) \ldots (\alpha - k + 1)}{1.2.3..k} \end{cases} \tag{4.18} $$

where, $\Gamma(.)$ is the gamma function. The outcome of fractional derivative depends on the bound of the operator a. A common value for this bound is $a = 0$. Based on this operator the derivative of power function t^{r} is

$$ {}_{0}D_{t}^{\alpha} t^{r} = \frac{\Gamma(r + 1)}{\Gamma(r - \alpha + 1)} t^{r-\alpha} \tag{4.19} $$

If $s(t)$ is a given function in terms of power series expansion, its fractional order derivative can be calculated using equation (4.19). The fractional order derivative of a digital signal is calculated by relating the discrete time samples of the signal to continuous time signal $s(t)$. Any given function $s(t)$ can be represented in the polynomial of t using Taylor series expansion [23], as

$$s(t) = \sum_{r=0}^{\infty} a_r t^r \tag{4.20}$$

where, $a_r = \frac{D^r s(0)}{r!}$ For t=0, the α^{th} order fractional derivative of s(t) is given as

$$ {}_0D_t^{\alpha} s(t) = \sum_{r=0}^{\infty} a_r D_t^{\alpha} t^r \tag{4.21}$$

$$= \sum_{r=0}^{\alpha} a_r \frac{\Gamma(r+1)}{\Gamma(r-\alpha+1)} t^{r-\alpha} \tag{4.22}$$

Now, assume that $t = nT$, where T is sampling period. $Z(n) = s(nT)$ is the sampling of $s(t)$ at $t = n$. Let power function $s(n) = n^r, 0 \leq n \leq M-1$ and its fractional derivative $D_t^{\alpha} s(n)$ is given as

$$D^{\alpha} s(n) = \sum_{r=0}^{\infty} a_r \frac{\Gamma(r+1)}{\Gamma(r-\alpha+1)} t^{r-\alpha} \tag{4.23}$$

If the signal $s(n)$ is delayed by a value I its fractional derivative $D_t^{\alpha} s(n-1)$ is given as

$$D^{\alpha} s(n-I) = \sum_{r=0}^{\infty} a_r \frac{\Gamma(r+1)}{\Gamma(r-\alpha+1)} (n-I)^{r-\alpha} \tag{4.24}$$

The above equation shows the desired response of fractional order differentiator. In the following section, this result will be used to estimate the FIR filter output.

4.3 Least square design method

In this section, we will use the results of $D^{\alpha} s(n)$ to compute the transfer function of fractional order differentiator, whose frequency response estimates the ideal frequency response of fractional order differentiator in (4.16). The transfer function of digital FIR filter can be written as

$$H(z) = \sum_{q=0}^{N} h(q) \tag{4.25}$$

We have to design a FIR filter *H*(*z*) which is a digital differentiator with filter coefficients *h*(*q*). When the signal *s*(*n*) passes through *N*th order FIR filter *H*(*z*), its output *y*(*n*) is given by

$$y(n) = \sum_{k=0}^{N} h(k)s(n-k) \tag{4.26}$$

since

$$s(n) = \sum_{r=0}^{\infty} a_r n^r \tag{4.27}$$

$$s(n-k) = \sum_{r=0}^{\infty} a_r (n-k)^r \tag{4.28}$$

$$y(n) = \sum_{r=0}^{\infty} a_r \sum_{k=0}^{N} h(k)\,(n-k)^r \tag{4.29}$$

To achieve the α^{th} order fractional derivative of *s*(*n*), compute the filter coefficients *h*(*q*) such that filter output *y*(*n*) is identical to the delayed fractional derivative $D^{\alpha}s(n-I)$, that is

$$D^{\alpha}s(n-I) = h(n) * s(n) \tag{4.30}$$

$$D^{\alpha}s(n-I) = \sum_{r=0}^{\infty} a^r \sum_{k=0}^{N} h(k)\,(n-k)^r \tag{4.31}$$

From eq. (4.24) and (4.31), we obtain

$$\sum_{r=0}^{\infty} a_r \frac{\Gamma(r+1)}{\Gamma(r-\alpha+1)}(n-I)^{r-\alpha} = \sum_{r=0}^{\infty} a^r \sum_{k=0}^{N} h(k)\,(n-k)^r \tag{4.32}$$

The comparison between the desired response and FIR filter output gives the error function *e*(*n*), which can be written as

$$e(n)=\sum_{r=0}^{\infty}a_r\frac{\Gamma(r+1)}{\Gamma(r-\alpha+1)}(n-I)^{r-\alpha}-\sum_{r=0}^{\infty}a^r\sum_{k=0}^{N}h(k)\,(n-k)^r \tag{4.33}$$

Using least square technique, the function to be minimized is

$$\varepsilon=\sum_{n=0}^{M-1}e^2(n) \tag{4.34}$$

$$=\sum_{n=0}^{M-1}.\left[\sum_{r=0}^{\infty}a_r\frac{\Gamma(r+1)}{\Gamma(r-\alpha+1)}(n-I)^{r-\alpha}-\sum_{r=0}^{\infty}a^r\sum_{k=0}^{N}h(k)\,(n-k)^r\right]^2$$

$$=\sum_{n=0}^{M-1}.\left[\sum_{r=0}^{\infty}a_rC(r,\alpha)(n-I)^{r-\alpha}-\sum_{r=0}^{\infty}a^r\sum_{k=0}^{N}h(k)\,(n-k)^r\right]^2$$

where,

$$C(r,\alpha)=\frac{\Gamma(r+1)}{\Gamma(r-\alpha+1)} \tag{4.35}$$

$$\varepsilon=\sum_{r=0}^{N}\sum_{n=0}^{M-1}\left[a_rC(r,\alpha)(n-I)^{r-\alpha}-\sum_{k=0}^{N}a_rh(k)(n-k)^r\right]^2 \tag{4.36}$$

To reduce the least-square error, derivative of above equation must be zero, according to the optimization theory, which can be written as

$$\frac{\partial\varepsilon}{\partial h(m)}=0,\qquad 0\le m\le N \tag{4.37}$$

$$\sum_{r=0}^{N}\sum_{n=0}^{M-1}\left[a_rC(r,\alpha)(n-I)^{r-\alpha}-\sum_{k=0}^{N}a_rh(k)(n-k)^r\right]^2=0 \tag{4.38}$$

$$\sum_{k=0}^{N}h(k)\left[\sum_{r=0}^{N}\sum_{n=0}^{M-1}a_r(n-m)^r(n-k)^r\right.$$
$$\left.=\sum_{r=0}^{N}\sum_{n=0}^{M-1}a_rC(r,\alpha)(n-I)^{r-\alpha}(n-m)^r\right] \tag{4.39}$$

$$\sum_{k=0}^{N} R(m,k)h(k) = T(m,\alpha) \quad 0 \leq m \leq N \tag{4.40}$$

where

$$R(m,k) = \sum_{r=0}^{N}\sum_{n=0}^{M-1} a_r (n-m)^r (n-k)^r \tag{4.41}$$

$$T(m,\alpha) = \sum_{r=0}^{N}\sum_{n=0}^{M-1} a_r C(r,\alpha)(n-I)^{r-\alpha}(n-m)^r \tag{4.42}$$

Solving (4.40) gives the filter coefficient $h(k)$. In the next section, two examples are used to evaluate the performance of least square fractional order differentiator.

4.4 Experiments and Analysis

In this section, the results attained for fractional order differentiator based on the least square technique are discussed. To validate the effectiveness of the least square method, two examples are solved. The main purpose of this design method is to decrease the effect of error between the ideal and the desired response of fractional order differentiator. To estimate the performance of the least square fractional order differentiator, the integral square error function of frequency response can be written as

$$\varepsilon_m = \int_{0}^{\pi} \left|H\left(e^{j\omega}\right) - H\left(e^{-j\omega}\right)\right|^2 \tag{4.43}$$

The error is calculated in the frequency range $[0,\pi]$. The Grunwald-Letnikov definition was used with range of frequency $\omega \epsilon [0,\pi]$ for the calculation of fractional order derivative.

Experiment 1: In this experiment, the design of fractional order differentiator has been given for m = 3, N = 2, M = 3, delay I = 6 and order α = 0.5, the fractional order derivative of the polynomial signal has been calculated. For α = 0.5, (4.42) can be written as

$$T(3,0.5) = \sum_{r=0}^{2}\sum_{n=0}^{2} a_r C(r,0.5)(n-6)^{0.5}(n-3)^r \tag{4.44}$$

The above coefficients $C(r,0.5)$ are given by

$$C(r, 0.5) = \frac{\Gamma(r + 1)}{\Gamma(r + 0.5)} \tag{4.45}$$

Substituting eq. (4.44) into eq. (4.43), we get

$$T(3,0.5) = \frac{a_0\Gamma(1)}{\Gamma(0.5)}[(-6)^{-0.5} + (-5)^{-0.5} + (-4)^{-0.5}]$$

$$+\frac{a_1\Gamma(2)}{\Gamma(1.5)}[-3(-6)^{0.5} - 2(-5)^{0.5} - (-4)^{0.5}]$$

$$+\frac{a_2\Gamma(3)}{\Gamma(2.5)}[9(-6)^{1.5} - 2(-5)^{1.5} - (-4)^{1.5}] \tag{4.46}$$

For the given m = 3, N = 2, M = 3, delay I = 6 and order α = 0:5, (4.41) can be written as

$$R(3, k) = \sum_{r=0}^{2}\sum_{n=0}^{2} a_r(n - 3)^r(n - k)^r \tag{4.47}$$

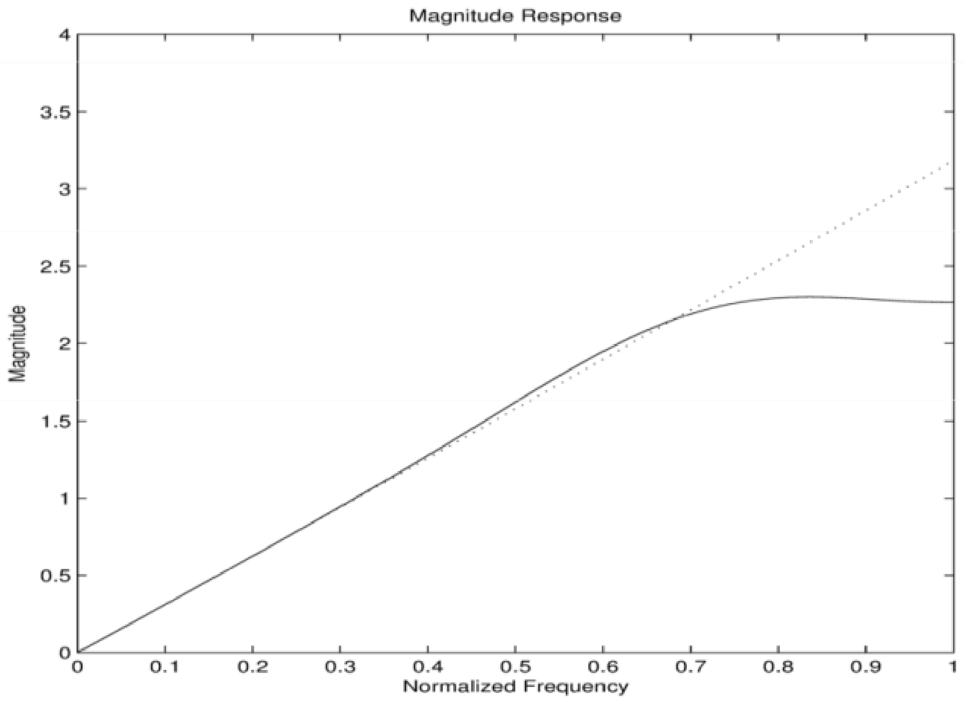

Figure 4.8: Magnitude responses of the fractional order FIR differentiators for α = 0:5. The solid lines are the designed magnitude responses and dotted lines are ideal responses.

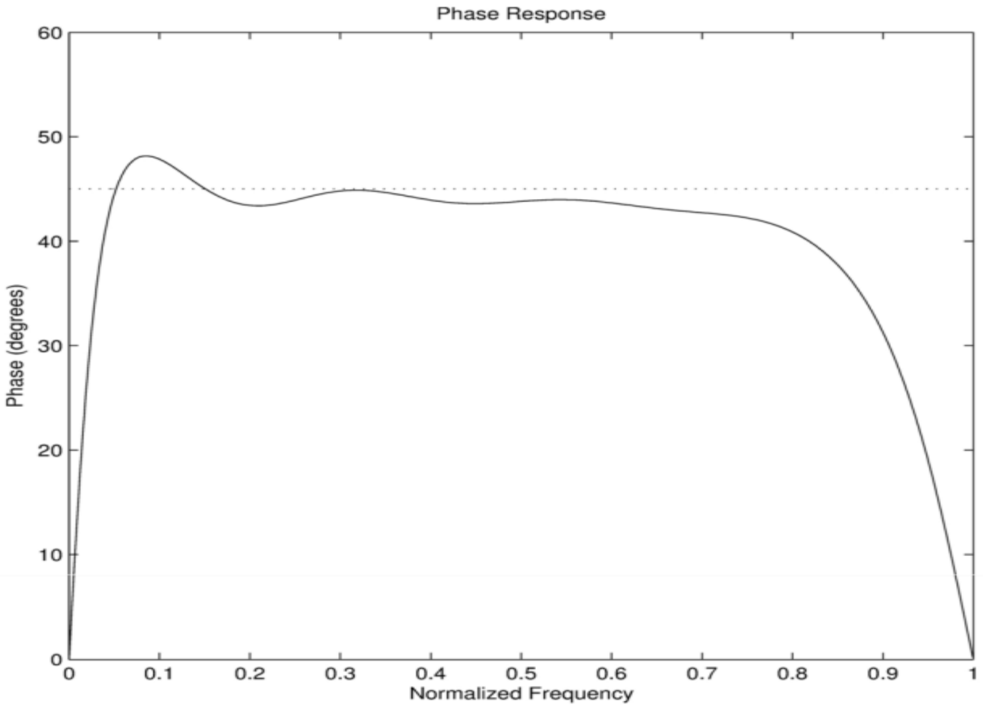

Figure 4.9: Phase responses of the fractional order FIR differentiators for α = 05. The solid line is the designed phase responses and dotted lines are ideal responses.

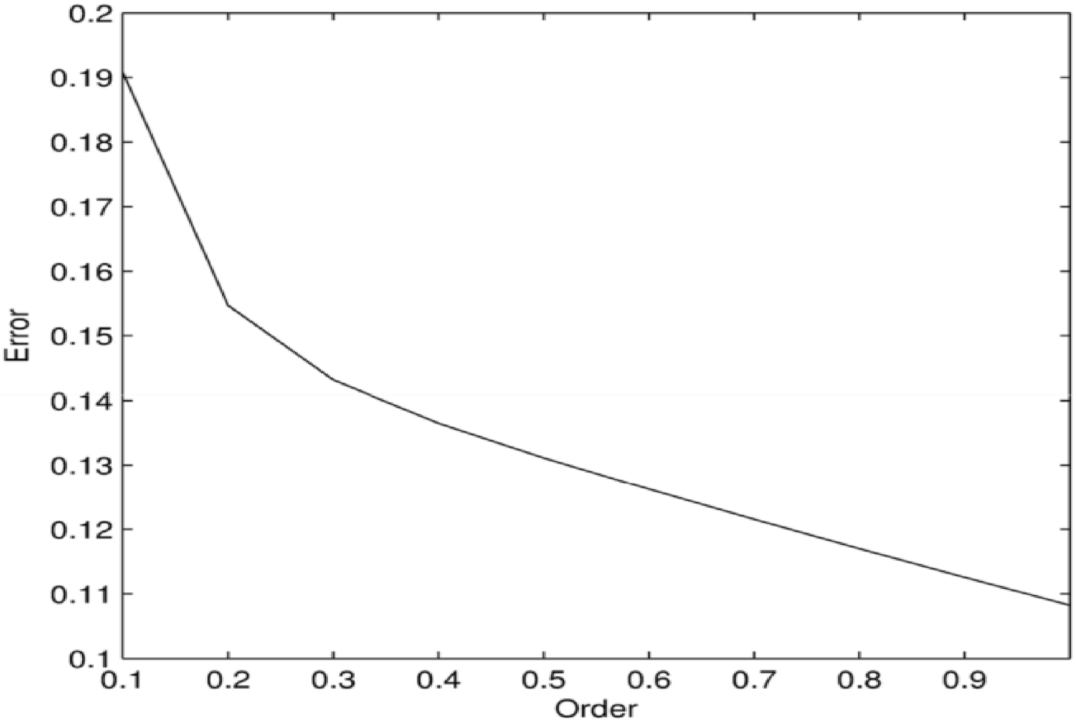

Figure 4.10: The integral squared error for the fractional order differentiator H(z) with order α in experiment 1.

Figure. 4.8 illustrates the magnitude response of the fractional order differentiator of polynomial signal with α = 0:5. The dotted line is the ideal magnitude response ω^{α}. Approximation errors can be decrease by selecting higher value of *N* and *M*. Figure. 4.9 shows the phase response of the designed fractional order differentiators. The dotted line is the ideal phase response 90α. It can be observed that the fractional order α must be selected large enough to minimize the objective error function. Figure. 4.10 shows the error curve of the projected fractional order differentiator.

Experiment 2: The design Example as given in [23], where N = 10, M = 11, delay I = 5, and order α = 1; 1:5; 2 is repeated with power function based least square method for the designing of fractional order differentiator. Figure. 4.11 shows the magnitude response of designed fractional order differentiator and designed example in [23]. The fractional order derivative of the specified polynomial signal can be accurately calculated using proposed method.

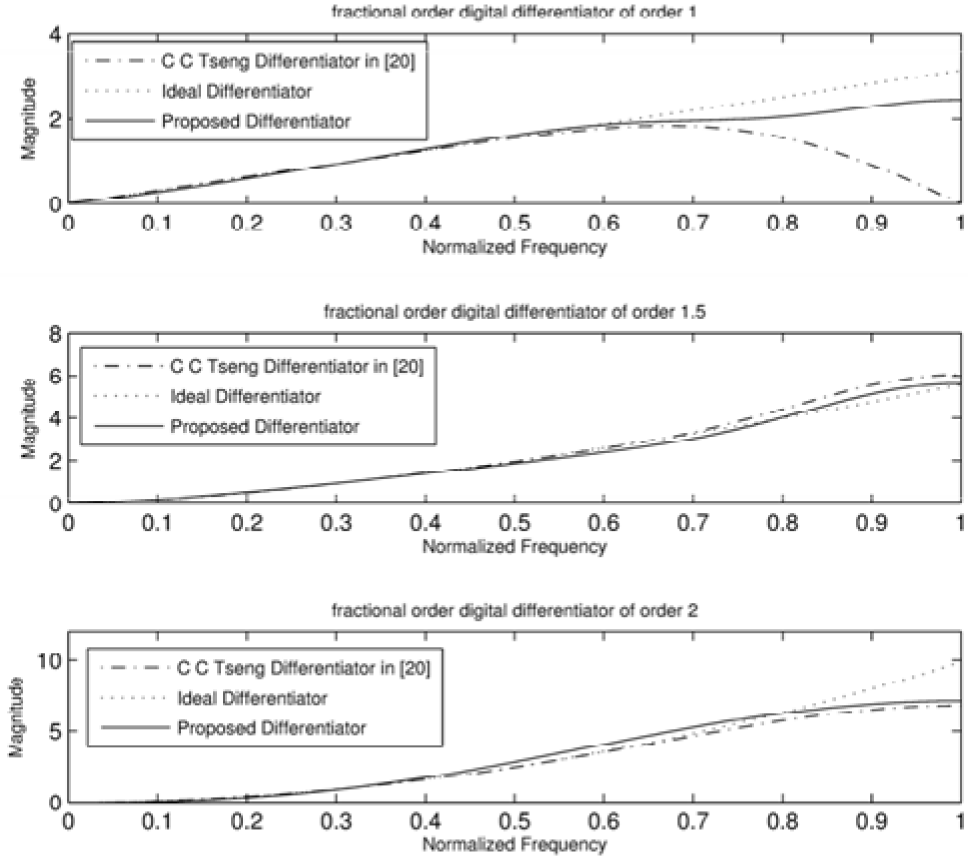

Figure 4.11: Magnitude responses of the designed fractional order FIR differentiators (a) order α=1 (b) order α=1.5 (c) order α=2.

Chapter 5
Conclusion & Future Work

5.1 Conclusion

The dissertation focuses on a comprehensive study of digital differentiators. As we have studied and found that digital differentiators have a vast involvement in modern engineering. Digital differentiators are applicable in low frequency bio medical equipment's to high frequency radars. Also the differentiators are useful in signature verification and image processing.

In this dissertation, we have found that the length of impulse response of digital differentiators is a major issue and lot of work has been done to improve the length of differentiators and to improve the coefficients of the differentiators. First we have explained the design of digital filters, that are basically FIR and IIR filters, then the study of digital differentiator of first and second order is done. Next we have studied the design of digital differentiator with Selesnick method and calculated the error response of the method by increasing the order of differentiator.

Fractional calculus is also an important field in the area of discrete signal processing. The entire modern signal processing applications has a requirement of fractional derivative of signal. Various researchers have studied about the design of fractional order differentiators. In this dissertation, first the design of fractional order Savitzky Golay differentiator is studied. In this design method we have computed the derivative of given signal by using moving window coefficient. Then the frequency response of proposed method is studied by varying order n and moving window coefficient I. Then sinusoidal signal and chirp signal is applied to the proposed method and the response of the obtained signal is compared with several popular methods. In comparison to other methods, we have found that proposed method is efficient than other methods. Its computation time is very less as compared to other methods and its root mean square error is almost negligible. This method is only method which can efficiently compute the derivative of noisy signal.

Further, another method for calculation of fractional order derivative of the signal is studied, which uses power function and least square function. To validate the efficiency of this method two examples have been described and then this method is also compared with the method of C.C Tseng and it is found that proposed method is much more efficient than the C.C Tseng method.

5.2 Future Scope of Work

Higher order low pass differentiators could well be used in applications like higher derivative based low frequency biomedical signal processing at a lower computational cost, finding applications in different domains. It is interesting to seek other possible applications of the proposed filter design method in the future.

In this dissertation, we have applied sinusoidal and chirp signal to the FOSGDD in the future this technique can be executed for signature verification and bio medical image enhancement.

Publications

1. Avinash Kumar Dubey, “Analysis of chirp signal with Fractional Order Savitzky-Golay Digital Differentiator,” *International Journal of Advance Research in Electrical Electronics and Instrumentation Engineering*, vol. 3, no.6 pp.10012-10018, 2014.

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